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A NOTE ON THE UNIQUENESS AND ATTRACTIVE BEHAVIOR OF SOLUTIONS FOR NONLINEAR VOLTERRA EQUATIONS

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ABSTRACT. In this paper we prove that positive solutions of some nonlinear Volterra integral equations must be locally bounded and global attractors of positive functions. These results complete previous results about the existence and uniqueness of solutions and their attractive behavior.

1. Introduction. This paper completes the study carried out in [**1**] about the properties of nontrivial solutions for nonlinear integral equation of Volterra type

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds. \tag{1}$$

Following [**1**], [**2**] this equation shall be referred to as equation (k, g) and the integral operator

$$Tf(x) = \int_0^x k(x-s)g(f(s))\,dx$$

shall be referred to as the *associated operator* to the equation (k, g). The basic general conditions imposed on the kernel k and the nonlinearity g are: k is a continuous positive function, defined on $\mathbf{R}^+$, such that

$$K(x) = \int_0^x k(s)\,ds$$

is a strictly increasing function; and g is a continuous strictly increasing function such that $g(0) = 0$ and $g'(x) > 0$ almost everywhere.

Note that the function 0 is a solution of (k, g), named the *trivial solution*, and horizontally translated solutions are solutions. We are

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neither interested in the trivial solution nor in translated solutions. So from now on, *solutions* means *nontrivial solutions*, i.e., positive solutions u such that $u \not\equiv 0$ on $(0,\delta]$ for any positive δ.

In [**1**] the uniqueness of locally bounded solutions was proved (see [**1**, Theorem 3.1]). It was also proved that those solutions are local attractors of all positive locally bounded and measurable functions (see [**1**, Corollary 3.2]), which means that for each positive locally bounded and measurable function f there is a $\delta > 0$ (∞ allowed) such that

$$\lim_{n\to\infty} T^n f(x) = u(x), \quad \forall\, x \in [0,\delta). \tag{2}$$

Our aim here is to complete these results by showing that

1. All possible solutions of equation (k,g) must be locally bounded. Hence, Theorem 3.1 in [**1**] is completed by Theorem 4.1.

2. Assuming (k,g) has a nontrivial solution, the solution is a global attractor (i.e., the interval $[0,\delta)$ in (2) is the domain of the solution independently of f). Hence, Corollary 3.2 in [**1**] is completed by Theorem 4.2.

2. On the uniqueness. The kernel k is continuous, so we can define the auxiliary continuous kernel

$$\bar{k}(x) = \begin{cases} 0 & x < 0, \\ \sup\{k(s) : s \in [0,x]\} & x \geq 0, \end{cases}$$

which is, moreover, increasing. Let $\overline{T}$ be the associated operator to the equation $(\bar{k},g)$. For any positive function f, we have that $Tf \leq \overline{T}f$. If f is a positive function, then $\overline{T}f$ is an increasing function and thus a locally bounded function. Let $0 < x < y$; since $\bar{k}(x-s) = 0$ for $s > x$ we have that

$$\begin{aligned} \overline{T}f(x) &= \int_0^x \bar{k}(x-s)g(f(s))\,ds = \int_0^y \bar{k}(x-s)g(f(x))\,ds \\ &\leq \int_0^x \bar{k}(y-s)g(f(s))\,ds = \overline{T}f(y). \end{aligned}$$

Now we can prove our first result:

Proposition 2.1. *All solutions of (k,g) are locally bounded functions.*

Proof. Consider u as a solution of (k,g). Then $u = Tu \leq \overline{T}u$, so u is bounded from above by an increasing function and therefore it is locally bounded. □

3. On the attractive behavior. In [**1**] it was proved that, if it exists, the solution u of the equation (k,g) is a local attractor. We shall prove that (2) holds on the domain of definition of u, for any measurable positive function f.

Let us recall that in [**4**] it has already been proved that if the kernel is an increasing function, then if it exists, the unique solution of equation (k,g) is a global attractor for all positive functions.

Let us consider the equation $(\bar{k},g)$. Since $u = Tu \leq \overline{T}u$, u is a subsolution of equation $(\bar{k},g)$. The existence of a subsolution implies the existence of a solution of the equation $(\bar{k},g)$ (for details see [**3**, Proof of Theorem 3.1]); and since the kernel $\bar{k}$ is increasing, applying [**4**, Theorem 1] shows that the unique solution $\bar{u}$ of the equation $(\bar{k},g)$ is a global attractor.

We have that $u \leq \bar{u}$. Note that T is a monotonous operator, so $u \leq T^n\bar{u}$ holds for all $n \in \mathbf{N}$; since $T\bar{u} \leq \overline{T}\bar{u} = \bar{u}$, the sequence $(T^n\bar{u})_{n\in\mathbf{N}}$ is a monotone decreasing sequence bounded from below by u. Since u is the unique solution of (1), and the limit of the sequence $(T^n\bar{u})_{n\in\mathbf{N}}$ is a solution of (k,g) (see [**3**, Proof of Theorem 3.1]), one has that

$$\lim_{n\to\infty} T^n\bar{u} = u.$$

In [**1**, Corollary 3.1] it was proved that for any positive measurable function f, such that $f \leq u$ and $f \neq 0$ almost everywhere, then $\lim_{n\to\infty} T^n f = u$ holds.

Lemma 3.1. *Let f be a positive measurable function such that $f \geq u$. Then*

$$\lim_{n\to\infty} T^n f = u.$$

Proof. Let us consider a positive measurable function f such that $f \geq u$. There are three possible cases:

1. $f \leq \bar{u}$. In this case, just by comparison, we have that

$$\lim_{n\to\infty} T^n f = u.$$

2. $f \geq \bar{u}$. In this case $T^n f \geq T^n \bar{u} \geq u$. On the other hand, $T^n f \leq \overline{T}^n f$. Since $\overline{T}^n f(x)$ converges to $\bar{u}(x)$, then the set of accumulation points of the sequence $(T^n f(x))_{n\in\mathbf{N}}$, denoted by $\Omega_f(x)$, is bounded from above by $\bar{u}(x)$ and from below by $u(x)$. Then $\Omega_f(x) = \{u(x)\}$, because $\Omega_f(x)$ is invariant by T and

$$u = \lim_{n\to\infty} T^n \bar{u}.$$

3. General case. Let us consider the auxiliary functions:

$$f_1(x) = \max\{f(x), \bar{u}(x)\}$$

$$f_2(x) = \min\{f(x), \bar{u}(x)\}.$$

Obviously $f_2 \leq \bar{u} \leq f_1$ and $f_2 \leq f \leq f_1$ hold. On the other hand, by the previous cases $T^n f_i$, $i = 1, 2$, converges to u. And the conclusion is obtained by a standard comparison argument. □

Thus, for continuous kernels the unique solution of (1), if it exists, is a global attractor.

Proposition 3.1. *Let f be a positive function such that $f \neq 0$ almost everywhere. Then*

$$\lim_{n\to\infty} T^n f = u.$$

Proof. It is a consequence of [**1**, Corollary 3] and Lemma 3.1. □

4. Final remarks. In this section we assume that the couple (k, g) satisfies the general conditions considered in [**1**], that is, the kernel k is a positive continuous function defined on $\mathbf{R}^+$ such that $K(x) = \int_0^x k(s)\,ds$ is a finite strictly increasing function, while the nonlinearity g is a positive, strictly increasing function defined on $\mathbf{R}^+$, such that $g(0) = 0$. Moreover, we assume that g transforms null sets into null sets and $\{x \leq 0 : g'(x) = 0\}$ is a null set. Null sets are referred to Lebesgue measure and measurable means Lebesgue measurable.

In [**1**] it was proved (see Theorem 3.1) that the equation (k, g) has at most one measurable and locally bounded positive solution. Considering Proposition 2.1, we immediately extend Theorem 3.1 in [**1**] to the following result

Theorem 4.1. *Equation* (k, g) *has at most one nontrivial solution.*

It was also proved [**1**, Corollary 3.2] that all positive, measurable and locally bounded functions on $\mathbf{R}^+$ are locally attracted by the solution. Considering Proposition 3.1, we extend Corollary 3.2 to the following result

Theorem 4.2. *All positive, measurable and locally bounded functions are globally attracted by the solution of* (k, g).

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