THE VANISHING OF LOW-DIMENSIONAL COHOMOLOGY GROUPS OF THE WITT AND THE VIRASORO ALGEBRA

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Abstract. A proof of the vanishing of the first and the third cohomology groups of the Witt algebra with values in the adjoint module is given. The proofs given in the present article are completely algebraic and independent of any underlying topology. They are a generalization of the ones provided by Schlichenmaier, who proved the vanishing of the second cohomology group using purely algebraic methods. In the case of the third cohomology group though, extra difficulties arise and the involved proofs are distinctly more complicated.

1. Introduction

The *Witt algebra* $\mathcal{W}$ is an infinite-dimensional, $\mathbb{Z}$-graded Lie algebra first introduced by Cartan in 1909 [1]. The Witt algebra and its universal central extension, the *Virasoro algebra*, are two of the most important infinite-dimensional Lie algebras, used in mathematics as well as in theoretical physics, see e.g. [23]. Therefore, knowledge of their cohomology groups is of outermost importance to a better understanding of central extensions, outer morphisms, deformations, obstructions, and so on.

In the present article, we consider algebraic cohomology with values in the adjoint module. The vanishing of the second cohomology group of the Witt and the Virasoro algebra with values in the adjoint module has been proved by Schlichenmaier in [29, 30] by using elementary algebraic methods; see also Fialowski [8]. In [33] by van den Hijligenberg and Kotchetkov, the vanishing of the second cohomology group with values in the adjoint module of the superalgebras $k(1), k^{+}(1)$ and of their central extensions was proved using algebraic methods. The aim of this article is to prove the vanishing of the first and the third cohomology groups with values in the adjoint module of the Witt algebra by merely algebraic means. We shall provide a generalization of the proof given in [29, 30] to the first and the third cohomology groups of the Witt algebra. As our goal is to consider arbitrary cohomology, we are dealing completely with algebraic cocycles, meaning we do not need to put any continuity constraints on them.

The results obtained for algebraic cohomology might be compared to those obtained for continuous cohomology. In fact, the Witt algebra is related to the group of diffeomorphisms of

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the unit circle S^1. It forms a dense subalgebra of the algebra $Vect(S^1)$ of all smooth vector fields on S^1. Based on results of Goncharova [20], Reshetnikov [26] and Tsujishita [32], the entire cohomology of the Lie algebra $Vect(S^1)$ with values in the adjoint module has been computed by Fialowski and Schlichenmaier in [10], yielding:

$$H^*(Vect(S^1), Vect(S^1)) = \{0\}\,.$$

As a first naive guess, one might be inclined to argue, by using density arguments, that this result implies the vanishing of the cohomology of the Witt algebra. However, this reasoning is not sound, as the argument using density is only valid when considering the sub-complex formed by the continuous cohomology of the Lie algebra. Instead, a direct proof for the vanishing of algebraic cohomology is needed and furnished here.

Although the proof of the vanishing of the third cohomology group with values in the adjoint module of the Witt algebra is rather involved, it consists of very simple, elementary algebraic manipulations. Only basic knowledge of the cohomology of Lie algebras is needed to understand the proofs in this article. The basic notions necessary to understand the proofs will be introduced in the following sections. Hence, this article is self-contained.

The article is organized as follows: in Section 2, we introduce the Witt algebra. In Section 3, we give some basic notions of the Chevalley-Eilenberg cohomology of Lie algebras, and internally graded Lie algebras in particular. This includes the interpretation of the first three cohomology groups with values in the adjoint module of a Lie algebra.

Section 4 includes the proof of the vanishing of the first cohomology group with values in the adjoint module of the Witt algebra $\mathcal{W}$, i.e. $H^1(\mathcal{W}, \mathcal{W}) = \{0\}$. The vanishing of this cohomology group is equivalent to the fact that all derivations of $\mathcal{W}$ are inner, which is a known fact. Nevertheless, for the sake of a coherent treatment of all cases, and also as some kind of warm-up for the higher cohomology case, we provide an elementary proof here.

Section 5 consists of the proof of the vanishing of the third cohomology group with values in the adjoint module of the Witt algebra, i.e. $H^3(\mathcal{W}, \mathcal{W}) = \{0\}$. In order to augment the readability of the proofs, all the details are provided and all the computations are given explicitly.

In the final section, Section 6, we announce some results obtained for the Virasoro algebra. More precisely, we prove the vanishing of the first cohomology group with values in the adjoint module of the Virasoro algebra, i.e. $H^1(\mathcal{V}, \mathcal{V}) = \{0\}$. The proof is not based on a direct computation as the one in Section 4, but rather uses the results obtained previously for the Witt algebra. The vanishing of the second cohomology group of the Virasoro algebra, i.e. $H^2(\mathcal{V}, \mathcal{V}) = \{0\}$, was shown in [30]. Concerning the third cohomology group $H^3(\mathcal{V}, \mathcal{V})$, numerical evidence indicates that it is one-dimensional. The algebraic proof is not yet completed, therefore we will not give details of this result in the present article. They will be presented in future work [4].

In the present article we only consider the first three cohomology groups of $\mathcal{W}$. These cohomology groups have a nice interpretation in terms of important Lie algebra objects such as outer derivations, deformations and obstructions. As a side-remark, note that on top of the obstructive viewpoint, the third cohomology group also comes with a more constructive viewpoint in terms of crossed modules, see Wagemann [34] for an explicit construction. Contrary to the first three cohomology groups, higher cohomology groups $H^k(\mathcal{W}, \mathcal{W})$ $k > 3$ do not come with such an easy interpretation in terms of known objects. Nonetheless, it would be interesting to have a similar elementary analysis also for higher cohomology groups. Unfortunately, purely algebraic proofs become increasingly complicated when considering higher cohomology groups. A direct comparison of the proofs provided for the first, second and third cohomology groups already suggests that a proof of the vanishing of e.g. $H^4(\mathcal{W},\mathcal{W})$ would be distinctly more intricate. In fact, although there are parallels between the three proofs, there is no immediate generalization when going from one cohomology group to the next cohomology group in the cohomology complex. Instead, each time new difficulties arise which cannot be anticipated when considering the precedent cohomology group. Still, we expect $H^k(\mathcal{W},\mathcal{W}) = \{0\}$ for all $k \geq 0$.

2. The Witt algebra

The Witt algebra can be decomposed into infinitely many one-dimensional homogeneous subspaces, i.e. $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$. Each subspace $\mathcal{W}_n$ is generated as a vector space over a field $\mathbb{K}$ with characteristic zero by one basis element e_n. The generators $\{e_n \mid n \in \mathbb{Z}\}$ of the Witt algebra satisfy the following Lie algebra structure equation:

$$[e_n, e_m] = (m-n)e_{n+m}\,, \qquad n, m \in \mathbb{Z}\,, \tag{1}$$

where $[\cdot,\cdot]$ is the Lie bracket, i.e. it satisfies the usual anti-commutativity $[e_n, e_m] = -[e_m, e_n]$ and the Jacobi identity. The Witt algebra is a $\mathbb{Z}$-graded Lie algebra with the degree of the elements of the homogeneous subspace $\mathcal{W}_n$ being defined by $deg(e_n) := n$. Obviously, the Lie product between elements of degree n and of degree m produces an element of degree $n+m$.

The grading of the Witt algebra is somewhat special as it is given by one of its own elements, namely $e_0 \in \mathcal{W}$. More precisely, the Witt structure equation (1) gives $[e_0, e_n] = n\ e_n$. Respectively, for the adjoint action on $\mathcal{W}$, we obtain:

$$ad_{e_0} := [e_0,\cdot] : \begin{array}{l} \mathcal{W} \to \mathcal{W} \\ e_n \to n\ e_n \end{array} .$$

In particular, e_n is an eigenvector of ad_{e_0} with eigenvalue n. The associated eigenspace corresponds to $\mathcal{W}_n$. In other words, the grading of the Witt algebra corresponds to the eigenspace decomposition of the ad_{e_0}-action on $\mathcal{W}$. The Witt algebra is thus an *internally* graded Lie algebra, as its grading is ensured by one of its own elements.

The Witt algebra is a perfect Lie algebra, i.e. $[\mathcal{W},\mathcal{W}] = \mathcal{W}$. A perfect Lie algebra admits a universal central extension, see e.g. the book by Weibel [35]. In general, the central extensions of a Lie algebra $\mathcal{L}$ are classified by $H^2(\mathcal{L},\mathbb{K})$. In the case of the Witt algebra, the second cohomology group with values in the trivial module is one-dimensional, i.e. $dim(H^2(\mathcal{W},\mathbb{K})) = 1$. This means that up to equivalence and rescaling, there is exactly one non-trivial central extension, which is the universal central extension of the Witt algebra.

The universal central extension of the Witt algebra is called *Virasoro* algebra $\mathcal{V}$. As a vector space, the Virasoro algebra is given as a direct sum $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$ generated by the basis elements $\hat{e}_n := (0, e_n)$, $n \in \mathbb{Z}$ and its one-dimensional center $t := (1, 0)$. These generators satisfy the following Lie structure equation:

$$[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - n)\delta_n^{-m} t, \qquad [\hat{e}_n, t] = [t,t] = 0\,,$$

for all $n, m \in \mathbb{Z}$ [1]. By defining $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) = 0$, also the Virasoro algebra $\mathcal{V}$ becomes a $\mathbb{Z}$-graded Lie algebra.
There are three popular realizations of the Witt algebra. First of all, the Witt algebra has an algebraic description in terms of the Lie algebra of derivations of the infinite-dimensional associative $\mathbb{K}$-algebra of Laurent polynomials $\mathbb{K}\left[z^{-1}, z\right]$.
Secondly, considering $\mathbb{K} = \mathbb{C}$, a geometrical realization of the Witt algebra is given by the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 and ∞. In this realization, the generators of the Witt algebra are given by:

$$e_n = z^{n+1}\frac{d}{dz}\,,$$

where z is the quasi-global complex coordinate.
Finally, another geometrical realization of the Witt algebra can be obtained by complexifying $Vect_{pol}(S^1)$, the Lie algebra of polynomial vector fields on the circle S^1, which gives the Witt algebra over $\mathbb{C}$. In this realization, the generators of the Witt algebra are given by:

$$e_n = e^{in\varphi}\frac{d}{d\varphi}\,,$$

where φ is the angle coordinate along S^1.

3. The cohomology of Lie algebras

3.1. **The Chevalley-Eilenberg cohomology.** In this section, we will introduce the Chevalley-Eilenberg cohomology [2], which is the cohomology of Lie algebras and the counterpart to the Hochschild cohomology [21] of associative algebras.
Let $\mathcal{L}$ be a Lie algebra and M an $\mathcal{L}$-module. Moreover, let $C^q(\mathcal{L}, M)$ be the vector space of q-multilinear alternating maps with values in M,

$$C^q(\mathcal{L}, M) := \{\psi : \mathcal{L} \times \cdots \times \mathcal{L} \to M \mid \psi \text{ is } q\text{-multilinear and alternating}\}\,.$$

The elements of $C^q(\mathcal{L}, M)$ are called *q-cochains*. By convention, we set $C^0(\mathcal{L}, M) := M$. Next, we introduce the family of coboundary operators δ_q defined by:

$$\forall q \in \mathbb{N}, \qquad \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q\psi\,,$$

$$\begin{aligned}(\delta_q\psi)(x_1, \dots x_{q+1}) : \; &= \textstyle\sum_{1\le i<j\le q+1}(-1)^{i+j+1}\;\psi([x_i, x_j]\,, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \\ &\quad + \textstyle\sum_{i=1}^{q+1}(-1)^i\; x_i \cdot \psi(x_1, \dots, \hat{x}_i, \dots, x_{q+1})\,.\end{aligned} \tag{2}$$

with $x_1, \dots, x_{q+1} \in \mathcal{L}$, $\hat{x}_i$ means that the entry x_i is omitted and the dot $\cdot$ stands for the module structure. In our case, we consider the adjoint module, i.e. the module M corresponds to the Lie algebra $\mathcal{L}$ itself and the module structure corresponds to the Lie algebra structure, i.e. $\cdot = [\cdot, \cdot]$. Since the coboundary operators fulfill $\delta_{q+1} \circ \delta_q = 0\ \forall\ q \in \mathbb{N}$, we obtain the following complex of vector spaces:

$$\begin{aligned}\{0\} \xrightarrow{\delta_{-1}} M \xrightarrow{\delta_0} C^1(\mathcal{L}, M) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{q-2}} C^{q-1}(\mathcal{L}, M) \\ \xrightarrow{\delta_{q-1}} C^q(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} C^{q+1}(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} \dots \longrightarrow \dots\end{aligned}$$

[1] The Kronecker delta δ_n^m is defined as being equal to 1 if $n = m$, and zero else.

where $\delta_{-1} := 0$. The vector space of *q-cocycles* is given by $Z^q(\mathcal{L}, M) := \ker \delta_q$, and the *q-coboundaries* are in $B^q(\mathcal{L}, M) := \operatorname{im} \delta_{q-1}$. The q^{th} Lie algebra cohomology space of $\mathcal{L}$ with values in M is given by:

$$H^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)\,.$$

The *Chevalley-Eilenberg cohomology* [2] associated to the Lie algebra $\mathcal{L}$ with values in M is given by the total cohomology space:

$$H^*(\mathcal{L}, M) := \bigoplus_{q=0}^{\infty} H^q(\mathcal{L}, M)\,.$$

In case of graded Lie algebras such as the Witt algebra, the notion of degree for cochains can be introduced as a helpful tool. This is presented in the next section.

3.2. **Degree of homogeneous cochains.** As already mentioned, this article focuses on cohomology with values in the adjoint module. Thus, we will concentrate on the main case $M = \mathcal{L}$.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$. A q-cochain ψ is *homogeneous of degree* d if there exists a $d \in \mathbb{Z}$ such that for all q-tuple $x_1, \ldots, x_q$ of homogeneous elements $x_i \in \mathcal{L}_{deg(x_i)}$, we have:

$$\psi(x_1, \ldots, x_q) \in \mathcal{L}_n \text{ with } n = \sum_{i=1}^{q} deg(x_i) + d\,.$$

The subspace of homogeneous q-cochains of degree d is denoted by $C^q_{(d)}(\mathcal{L}, \mathcal{L})$, and every q-cochain can be expressed as a formal infinite sum,

$$\psi = \sum_{d\in\mathbb{Z}} \psi_{(d)}, \qquad \psi_{(d)} \in C^q_{(d)}(\mathcal{L}, \mathcal{L})\,.$$

Applied on a fixed q-tuple of elements, only a finite number of terms will be non-zero.

The coboundary operators δ_q are of degree zero. If $\psi = \sum_d \psi_{(d)}$ is a q-cocycle, then $\delta_q\psi = \sum_d \delta_q\psi_{(d)} = 0$. The sum cancels only if all the individual $\psi_{(d)}$ are q-cocycles. In fact, the terms cannot cancel each other as the sum goes over cochains of different degrees and the coboundary operator is of degree zero. Moreover, if ψ is a q-coboundary of degree d, i.e. $\psi = \delta_{q-1}\phi$ for ϕ some $(q-1)$-cochain ϕ, we can always preform a cohomological change such that $\psi = \delta_{q-1}\phi'$ where ϕ' is a $(q-1)$-cochain of degree d. Hence, every cohomology class $[\psi] \in H^q(\mathcal{L}, \mathcal{L})$ can be decomposed as a formal sum:

$$[\psi] = \sum_{d\in\mathbb{Z}} \left[\psi_{(d)}\right], \qquad \left[\psi_{(d)}\right] \in H^q_{(d)}(\mathcal{L}, \mathcal{L})\,,$$

where $H^q_{(d)}(\mathcal{L}, \mathcal{L})$ is the subspace consisting of classes of q-cocycles of degree d modulo q-coboundaries of degree d. Therefore, we have:

$$H^q(\mathcal{L}, \mathcal{L}) = \bigoplus_{d\in\mathbb{Z}} H^q_{(d)}(\mathcal{L}, \mathcal{L})\,,$$

and the inspection of the cohomology group $H^q(\mathcal{L}, \mathcal{L})$ can be performed by analyzing each of its components $H^q_{(d)}(\mathcal{L}, \mathcal{L})$ separately.

An import result states that in the case of internally $\mathbb{Z}$-graded Lie algebras, nonzero cohomology groups can only exist for degree zero, see Fuks [14]:

$$\mathrm{H}^q_{(d)}(\mathcal{L},\mathcal{L}) = \{0\} \quad \text{for} \quad d \neq 0 \;, \tag{3a}$$

$$\mathrm{H}^q(\mathcal{L},\mathcal{L}) = \mathrm{H}^q_{(0)}(\mathcal{L},\mathcal{L}) \;. \tag{3b}$$

Nevertheless, we prove this statement again explicitly for $q = 1$ and $q = 3$ in the case of the Witt algebra, since the corresponding proofs presented in this article are short and very simple. The aim is to render this article as self-contained as possible.
The discussion above focused on the adjoint module, but it holds true in the generic case of a graded $\mathcal{L}$-module M which is internally graded with respect to the same grading element e_0 as the Lie algebra $\mathcal{L}$, i.e.

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \,,$$
$$\text{with } M_n = \{x \in M \mid e_0 \cdot x = n\, x\} \,.$$

Aside from the adjoint module, another example of such a module is given by the trivial module $\mathbb{K}$, which can be decomposed as $\mathbb{K} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}_n$ with trivial grading $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_n = \{0\}$ for $n \neq 0$. In particular, the result (3) of Fuks is true accordingly.

3.3. **Interpretation of the first three cohomology groups.** The cohomology of algebras has an interpretation in terms of deformations, as was shown by Gerstenhaber [16–19]. The proofs in the present article are only concerned with the first and the third cohomology groups, since the second group has already been analyzed in [8, 29, 30]. In order to be complete though, we shall briefly present all three cohomology groups below. More precisely, we shall provide an interpretation of the first three cohomology groups of a Lie algebra with values in the adjoint module.

3.3.1. *Outer derivations.* In order to determine $H^1(\mathcal{L},\mathcal{L})$, the kernel of δ_1 has to be computed. Putting the expression in (2) for $q = 1$ equal to zero yields:

$$(\delta_1\psi)(x_1,x_2) = \psi\left([x_1,x_2]\right) - [x_1,\psi(x_2)] - [\psi(x_1),x_2] = 0$$
$$\Leftrightarrow\ \psi\left([x_1,x_2]\right) = [x_1,\psi(x_2)] + [\psi(x_1),x_2] \;.$$

From the Leibniz rule appearing in the last line in the expression above, we see that a 1-cochain ψ is a 1-cocycle if and only if it is a derivation for $\mathcal{L}$. Next, the image of δ_0 has to be determined. Setting $q = 0$ in the expression in (2), one obtains:

$$(\delta_0\phi)(x) = -\,[x,\phi] = ad_\phi(x)\;,$$

where $\phi \in C^0(\mathcal{L},\mathcal{L}) = \mathcal{L}$. The map ad_ϕ is an inner derivation for all $\phi \in \mathcal{L}$, implying that the 1-coboundaries $(\delta_0\phi)\ \forall\ \phi \in \mathcal{L}$ correspond exactly to the inner derivations of $\mathcal{L}$. The first cohomology group $H^1(\mathcal{L},\mathcal{L})$ is given by the following quotient:

$$H^1(\mathcal{L},\mathcal{L}) = \frac{\ker(\delta_1 : C^1(\mathcal{L},\mathcal{L}) \to C^2(\mathcal{L},\mathcal{L}))}{\mathrm{im}(\delta_0 : C^0(\mathcal{L},\mathcal{L}) \to C^1(\mathcal{L},\mathcal{L}))} = \frac{\{\text{derivations}\}}{\{\text{inner derivations}\}} = \{\text{outer derivations}\}\,.$$

The elements of the first cohomology group $H^1(\mathcal{L},\mathcal{L})$ thus correspond to outer derivations. Note that another interpretation of $H^1(\mathcal{L},\mathcal{L})$ is given by one-dimensional "right extensions" of $\mathcal{L}$ (see [14]), but we will not introduce these here.

3.3.2. *Infinitesimal deformations.* A Lie algebra $\mathcal{L}$ over $\mathbb{K}$ and its Lie algebra structure $[\cdot,\cdot]$ can be expressed in terms of an anti-symmetric bilinear map ψ_0:

$$\psi_0 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \qquad (x_1, x_2) \mapsto \psi_0(x_1, x_2) = [x_1, x_2] \ .$$

The map ψ_0 has to satisfy the Jacobi identity in order for $\mathcal{L}$ to be a Lie algebra. Next, consider the following family of Lie algebra structures defined on the same vector space $\mathcal{L}$ is defined on:

$$\mu_t = \psi_0 + \psi_1 \ t + \psi_2 \ t^2 + \dots \ , \tag{4}$$

The maps ψ_i are bilinear anti-symmetric maps $\psi_i : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that the Lie algebra structure μ_t fulfills the Jacobi identity. In that case, the family $\mathcal{L}_t := (\mathcal{L}, \mu_t)$ is a Lie algebra and the original Lie algebra $\mathcal{L}_0 = (\mathcal{L}, \psi_0)$ is retrieved by setting $t = 0$. The family $\{\mathcal{L}_t\}$ is a *deformation* of $\mathcal{L}_0$.

Regarding the deformation parameter t, different situations can be considered:

(1) The parameter t is a variable which allows to plug in numbers $\alpha \in \mathbb{K}$. In this case, $\mathcal{L}_\alpha$ is a Lie algebra for every α for which the expression (4) is defined. Unfortunately, this situation is in general very hard to tackle.
(2) The parameter t can be taken as a formal variable, in which case the family $\mathcal{L}_t$ is considered over the ring of formal power series $\mathbb{K}[[t]]$. We might try to plug in numbers of our base field $\mathbb{K}$ for the formal variable t. If we obtain a convergent series, we have a deformation in the previous sense. However, it might also happen that the series converges only when we plug in zero for t. A deformation with the formal power series as parameter space is called a *formal deformation*.
(3) If a deformation $\mathcal{L}_t$ is taken over the quotient $\mathbb{K}[[X]]/(X^{n+1})$, it is called a *$n$-deformation*. In that case, the sum in (4) contains maximally $n + 1$ terms. The particular case corresponding to $n = 1$ is called *infinitesimal deformation*, i.e. the parameter t is considered as an infinitesimal variable with $t^2 = 0$.

More complicated parameter spaces can be considered, which yield surprising properties in the case of infinite-dimensional Lie algebras, see the work by Fialowski and Schlichenmaier [10–12, 27, 28].

In general, two deformations μ_t and μ'_t of ψ_0 are called *equivalent* if there exists a linear automorphism ψ_t,

$$\psi_t = id + \alpha_1 \ t + \alpha_2 \ t^2 + \dots \ ,$$

with linear maps $\alpha_i : \mathcal{L} \to \mathcal{L}$ such that:

$$\mu'_t(x_1, x_2) = \psi_t^{-1}(\mu_t(\psi_t(x_1), \psi_t(x_2)))\,.$$

Let us come back to the family μ_t in (4). The condition that μ_t has to fulfill the Jacobi identity must be valid up to all orders, i.e.

$$\begin{aligned} &\mu_t(\mu_t(x_1, x_2), x_3) + \text{ cyclic permutations of } (x_1, x_2, x_3) = 0 \\ \Leftrightarrow &\sum_{i,j \geq 0} \psi_i(\psi_j(x_1, x_2), x_3) \ t^{i+j} + \text{ cyclic permutations of } (x_1, x_2, x_3) = 0 \end{aligned} \tag{5}$$

has to be fulfilled for all t. In particular, infinitesimal deformations give rise to two conditions. The condition for order zero t^0 corresponds to the original Jacobi identity of ψ_0 on $\mathcal{L}$. The condition for order one t^1 corresponds to:

$$\psi_1([x_1, x_2], x_3) + \text{ cycl. perm. } + [\psi_1(x_1, x_2), x_3] + \text{ cycl. perm. } = 0\,. \tag{6}$$

As we have $t^2 = 0$ for an infinitesimal deformation, no terms of higher order have to be verified.
Next, let us compare these results to the elements of the second cohomology group $H^2(\mathcal{L},\mathcal{L})$. An alternating bilinear cochain ψ is in $Z^2(\mathcal{L},\mathcal{L})$ if it lies in the kernel of δ_2. Putting the expression (2) with $q = 2$ equal to zero, we obtain the 2-cocycle condition:

$$\begin{aligned}&\psi\left([x_1,x_2],x_3\right)+\psi\left([x_2,x_3],x_1\right)+\psi\left([x_3,x_1],x_2\right)\\&-[x_1,\psi(x_2,x_3)]+[x_2,\psi(x_1,x_3)]-[x_3,\psi(x_1,x_2)]=0\,.\end{aligned}$$

The equation above corresponds exactly to (6). This means that the family $\mu_t = \psi_0 + \psi_1\ t$ is an infinitesimal deformation if and only if ψ_1 is a Lie algebra 2-cocycle with values in the adjoint module, i.e. $\psi_1 \in Z^2(\mathcal{L},\mathcal{L})$. In general, a necessary condition for a generic family μ_t to be a deformation is that the first non-vanishing coefficient ψ_i has to be a 2-cocycle.
Furthermore, if two deformations μ_t and μ'_t are equivalent, then the corresponding ψ_i and ψ'_i are cohomologous in $H^2(\mathcal{L},\mathcal{L})$. In particular, two infinitesimal deformations $\mu_t = \psi_0 + \psi_1\ t$ and $\mu'_t = \psi_0 + \psi'_1\ t$ are equivalent if and only if ψ_1 and ψ'_1 are cohomologous. As a conclusion, we obtain that the elements of $H^2(\mathcal{L},\mathcal{L})$ correspond to infinitesimal deformations up to equivalence.
Note that if $H^2(\mathcal{L},\mathcal{L}) = \{0\}$, then $\mathcal{L}$ is infinitesimally and formally *rigid*, meaning every deformation of such type is equivalent to the trivial deformation, see Fialowski and Fuchs [9], Fialowski [6, 7], Gerstenhaber [17–19], and Nijenhuis and Richardson [25]. However, contrary to finite-dimensional Lie algebras [17–19, 24], the vanishing of $H^2(\mathcal{L},\mathcal{L})$ does not imply rigidity with respect to other parameter spaces in the case of infinite-dimensional Lie algebras, see [10–12, 27, 28].

3.3.3. *Obstructions.* In general, given an infinitesimal deformation it will not be possible to plug in for t elements of our base field $\mathbb{K}$ to obtain a "honest" deformation. In particular, a deformation $\mu_t = \psi_0 + \psi_1 t$ with $\psi_1 \in C^2(\mathcal{L},\mathcal{L})$ will not be a Lie algebra in general, because in the Jacobi identity higher order terms for t will appear.
Hence, not every infinitesimal deformation will yield a honest deformation. Usually, it is not an easy task to determine which infinitesimal deformations can be lifted to a genuine deformation. Slightly more accessible is the question which infinitesimal deformations allow a lift at least on the formal level, i.e. which ones can be given by a formal power series to all orders of t. This problem boils down to a step n to a step $n+1$ lifting property. Generally, obstructions to this lifting will appear, which will live in $H^3(\mathcal{L},\mathcal{L})$.
Let us consider a n-deformation, i.e. the sum in (4) contains maximally $n+1$ terms. In view of our lifting problem, we ask the question whether a given n-deformation $\mu_t = \sum_{i=0}^{n} \psi_i\ t^i$ can be extended to an $n+1$-deformation $\mu'_t = \sum_{i=0}^{n+1} \psi_i\ t^i$. The sum in (5) is zero if the coefficients of t^k are zero for all k, i.e. the following conditions hold:

$$\sum_{i+j=k,i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \qquad 0 \leq k \leq n+1\,.$$

Since the n-deformation is given, the equations above are satisfied for $0 \leq k \leq n$. The n-deformation can be extended to a $n+1$-deformation if the last equation for $k = n+1$ is fulfilled:

$$\begin{aligned}
&\sum_{i+j=n+1,i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \\
\Leftrightarrow\ & [(\psi_0(\psi_{n+1}(x_1,x_2),x_3) + \psi_{n+1}(\psi_0(x_1,x_2),x_3)) + \text{ cycl. perm.}] \\
&+ \left[\sum_{i+j=n+1,i,j>0} (\psi_i(\psi_j(x_1,x_2),x_3)) + \text{ cycl. perm.}\right] = 0 \\
\Leftrightarrow\ & (\delta_2\psi_{n+1})(x_1,x_2,x_3) + \left[\sum_{i+j=n+1,i,j>0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.}\right] = 0\,.
\end{aligned}$$

Hence, the condition for extending the n-deformation μ_t to the $n+1$-deformation μ'_t boils down to a condition on a 3-coboundary term for ψ_{n+1} plus an extra term which is called *obstruction*, given by:

$$\Psi_{n+1} := \sum_{i+j=n+1,i,j>0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.}\,.$$

It can be shown that $\delta_3\Psi_{n+1} = 0$, i.e. $\Psi_{n+1} \in Z^3(\mathcal{L},\mathcal{L})$ is a 3-cocycle [17–19]. The equivalence class $[\Psi_{n+1}]$ in $H^3(\mathcal{L},\mathcal{L})$ can be seen as an obstruction to a deformation as follows: A n-deformation can be extended to an $n+1$-deformation if and only if $[\Psi_{n+1}] = 0$ in $H^3(\mathcal{L},\mathcal{L})$, as in that case there exists a ψ_{n+1} such that $\delta_2\psi_{n+1} = -\Psi_{n+1}$.
In particular, in case $H^3(\mathcal{L},\mathcal{L}) = \{0\}$, all obstructions will vanish at all levels, hence each infinitesimal deformation can be extended to a formal deformation. For more details, see e.g. [3].

4. Analysis of $\mathrm{H}^1(\mathcal{W},\mathcal{W})$

In this section, we analyze the first cohomology group of the Witt algebra with values in the adjoint module. It is already known that all derivations of the Witt algebra are inner derivations, see Zhu and Meng [37], i.e. $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = 0$ in the language of cohomological algebra. See also e.g. Jiang [22], Shen and Jiang [31] and Fu and Gao [13] for related work on other infinite-dimensional Lie algebras, as well as Yang and Yu and Yao [36], and Fa and Han and Yue [5]. However, we shall prove this result again by using our algebraic techniques, in order to introduce our notation and to provide a warm-up example of the used procedures. The main aim of this section is to proof the following theorem:

Theorem 4.1. *The first cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)$= 0$ and values in the adjoint module vanishes, i.e.*

$$\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \{0\}\,.$$

The proof follows in two steps, the first step concentrating on the non-zero degree part of the Witt algebra, the second step focusing on the degree zero part.
Recall that the coboundary condition for a 1-cocycle $\psi \in \mathrm{H}^1(\mathcal{W},\mathcal{W})$ is given by:

$$\psi(x) = (\delta_0\phi)(x) = -x\cdot\phi = [\phi, x]\ ,$$

with $x \in \mathcal{W}$ and $\phi \in \mathrm{C}^0(\mathcal{W},\mathcal{W}) = \mathcal{W}$. As the values are taken in the adjoint module, $\cdot = [.,.]$.

The cocycle condition for a 1-cocycle ψ is given by:

$$\delta_1\psi(x_1,x_2) = 0 = \psi\left([x_1,x_2]\right) - [x_1,\psi(x_2)] - [\psi(x_1),x_2] \ ,$$

with $x_1,x_2 \in \mathcal{W}$.

4.1. **The non-zero degree part of the Witt algebra.** The proposition proved in this section shall be the following:

Proposition 4.1.1. *The following hold:*

$$\begin{aligned} &\mathrm{H}^1_{(d)}(\mathcal{W},\mathcal{W}) = \{0\} \quad \textit{for} \quad d \neq 0 \ , \\ &\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \mathrm{H}^1_{(0)}(\mathcal{W},\mathcal{W}) \ . \end{aligned}$$

Proof. Let $\psi \in \mathrm{H}^1_{(d\neq 0)}(\mathcal{W},\mathcal{W})$.
Let us perform a cohomological change $\psi' = \psi - \delta_0\phi$ with the following 0-cochain ϕ:

$$\phi = -\frac{1}{d}\psi(e_0) \in \mathcal{W} \Rightarrow (\delta_0\phi)(x) = \frac{1}{d}\left[x,\psi(e_0)\right] \ ,$$

which gives us:

$$\begin{aligned} \psi'(x) &= \psi(x) - (\delta_0\phi)(x) = \psi(x) + \frac{1}{d}\left[\psi(e_0),x\right] \\ \Rightarrow \psi'(e_0) &= \psi(e_0) + \frac{1}{d}\left[\psi(e_0),e_0\right] \\ &= \psi(e_0) - \frac{1}{d}\ \deg(\psi(e_0))\ \psi(e_0) \\ &= \psi(e_0) - \frac{1}{d}\ d\ \psi(e_0) = 0 \ . \end{aligned}$$

We thus have $\boxed{\psi'(e_0) = 0}$.
Next, let us write down the cocycle condition for ψ' on the doublet (x,e_0):

$$\begin{aligned} 0 &= \psi'\left([x,e_0]\right) - \underbrace{[x,\psi'(e_0)]}_{=0} - [\psi'(x),e_0] \\ \Leftrightarrow 0 &= \psi'(-\deg(x)x) + \deg(\psi'(x))\psi'(x) \\ \Leftrightarrow 0 &= -\deg(x)\psi'(x) + (\deg(x)+d)\psi'(x) \\ \Leftrightarrow 0 &= d\ \psi'(x) \Leftrightarrow\ 0 = \psi'(x) \text{ as } d \neq 0 \ . \end{aligned}$$

We conclude that the first cohomology of the Witt algebra reduces to the degree zero part. □

4.2. **The degree zero part for the Witt algebra.** The proposition we shall prove in this section is the following:

Proposition 4.2.1. *The following holds:*

$$\mathrm{H}^1_{(0)}(\mathcal{W},\mathcal{W}) = \{0\} \ .$$

Proof. Let ψ be a degree zero 1-cocycle, i.e. we can write it as $\psi(e_i) = \psi_i e_i$ with suitable coefficients $\psi_i \in \mathbb{K}$. Consider the following 0-cochain $\phi = \psi_1 e_0$. The coboundary condition for ϕ gives:

$$(\delta_0\phi)(e_i) = [\phi,e_i] = i\psi_1 e_i \, .$$

The cohomological change $\psi' = \psi - \delta_0\phi$ leads to $\psi'_1 = 0$. In the following, we will work with a 1-cocycle normalized to $\psi'_1 = 0$, although we will drop the apostrophe in order to augment readability.
The 1-cocycle condition for ψ on the doublet (e_i, e_j) becomes:

$$\begin{aligned} 0 &= \psi\left([e_i, e_j]\right) - [e_i, \psi(e_j)] - [\psi(e_i), e_j] \\ \Leftrightarrow 0 &= (j-i)\left(\psi_{i+j} - \psi_j - \psi_i\right) . \end{aligned}$$

For $j = 1$ and $i = 0$, we obtain from the 1-cocycle condition: $\psi_0 = 0$.
For $j = 1$ and $i < 0$ decreasing, we obtain from the 1-cocycle condition: $\psi_i = \psi_{i+1} = 0$.
For $j = 1$ and $i > 1$ increasing, we obtain from the 1-cocycle condition: $\psi_{i+1} = \psi_i = \psi_2$, where the value of ψ_2 is unknown for the moment.
Next, taking $j = 2$ and for example $i = 3$, we obtain:

$$\begin{aligned} &\psi_5 - \psi_2 - \psi_3 = 0 \\ \Leftrightarrow\ &\psi_2 - \psi_2 - \psi_2 = 0 \text{ as we have } \psi_i = \psi_2\ \forall i > 1 \\ \Leftrightarrow\ &\psi_2 = 0 . \end{aligned}$$

All in all, we conclude $\boxed{\psi_i = 0\ \forall i \in \mathbb{Z}}$ □

This concludes the proof of Theorem 4.1.

5. Analysis of $\mathrm{H}^3(\mathcal{W}, \mathcal{W})$

In this section, we analyze the third cohomology group of the Witt algebra with values in the adjoint module. The main aim of this section is to proof the following theorem:

Theorem 5.1. *The third cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)$= 0$ and values in the adjoint module vanishes, i.e.*

$$\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\} .$$

The proof follows again in two steps, the first step concentrating on the non-zero degree part of the Witt algebra, the second step focusing on the degree zero part.
The coboundary condition for a 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{W}, \mathcal{W})$ is given by:

$$\begin{aligned} \psi(x_1, x_2, x_3) = (\delta_2\phi)(x_1, x_2, x_3) =& \phi\left([x_1, x_2], x_3\right) + \phi\left([x_2, x_3], x_1\right) + \phi\left([x_3, x_1], x_2\right) \\ & - [x_1, \phi(x_2, x_3)] + [x_2, \phi(x_1, x_3)] - [x_3, \phi(x_1, x_2)] , \end{aligned}$$

where $x_1, x_2, x_3 \in \mathcal{W}$ and $\phi \in \mathrm{C}^2(\mathcal{W}, \mathcal{W})$.
The cocycle condition for a 3-cocycle ψ is given by:

$$\begin{aligned} &(\delta_3\psi)(x_1, x_2, x_3, x_4) \\ =&\psi\left([x_1, x_2], x_3, x_4\right) - \psi\left([x_1, x_3], x_2, x_4\right) + \psi\left([x_1, x_4], x_2, x_3\right) \\ &+ \psi\left([x_2, x_3], x_1, x_4\right) - \psi\left([x_2, x_4], x_1, x_3\right) + \psi\left([x_3, x_4], x_1, x_2\right) \\ &- [x_1, \psi(x_2, x_3, x_4)] + [x_2, \psi(x_1, x_3, x_4)] - [x_3, \psi(x_1, x_2, x_4)] + [x_4, \psi(x_1, x_2, x_3)] = 0 . \end{aligned}$$

with $x_1, x_2, x_3, x_4 \in \mathcal{W}$

5.1. **The non-zero degree part for the Witt algebra.** The proposition proved in this section shall be the following:

Proposition 5.1.1. *The following hold:*

$$\mathrm{H}^3_{(d)}(\mathcal{W},\mathcal{W}) = \{0\} \quad \textit{for} \quad d \neq 0 \ ,$$
$$\mathrm{H}^3(\mathcal{W},\mathcal{W}) = \mathrm{H}^3_{(0)}(\mathcal{W},\mathcal{W}) \ .$$

Proof. Let $\psi \in \mathrm{H}^3_{(d\neq 0)}(\mathcal{W},\mathcal{W})$.
Let us perform a cohomological change $\psi' = \psi - \delta_2\phi$ with the following 2-cochain ϕ:

$$\phi(x_1,x_2) = -\frac{1}{d}\psi(x_1,x_2,e_0)\,,$$

which gives us, taking into account that $\phi(e_0,\cdot) = \phi(\cdot,e_0) = 0$:

$$\begin{aligned}
&\psi'(x_1,x_2,e_0) = \psi(x_1,x_2,e_0) - (\delta_2\phi)(x_1,x_2,e_0)\\
&= \psi(x_1,x_2,e_0) - \underbrace{\phi\left([x_1,x_2],e_0\right)}_{=0} - \phi\left([x_2,e_0],x_1\right) - \phi\left([e_0,x_1],x_2\right)\\
&+ [x_1,\underbrace{\phi(x_2,e_0)}_{=0}] - [x_2,\underbrace{\phi(x_1,e_0)}_{=0}] + [e_0,\phi(x_1,x_2)]\\
&= \psi(x_1,x_2,e_0) + \deg(x_2)\underbrace{\phi(x_2,x_1)}_{=-\phi(x_1,x_2)} - \deg(x_1)\phi(x_1,x_2) + \left(\deg(x_1)+\deg(x_2)+d\right)\phi(x_1,x_2)\\
&= -d\ \phi(x_1,x_2) + d\ \phi(x_1,x_2) = 0\,.
\end{aligned}$$

We thus have $\boxed{\psi'(x_1,x_2,e_0) = 0}$.
Next, let us write down the cocycle condition for ψ' on the quadruplet (x_1,x_2,x_3,e_0):

$$\begin{aligned}
&(\delta_3\psi')(x_1,x_2,x_3,e_0) = 0\\
\Leftrightarrow\ &\underbrace{\psi'\left([x_1,x_2],x_3,e_0\right)}_{=0} - \underbrace{\psi'\left([x_1,x_3],x_2,e_0\right)}_{=0} + \psi'\left([x_1,e_0],x_2,x_3\right)\\
&+ \underbrace{\psi'\left([x_2,x_3],x_1,e_0\right)}_{=0} - \psi'\left([x_2,e_0],x_1,x_3\right) + \psi'\left([x_3,e_0],x_1,x_2\right)\\
&- [x_1,\underbrace{\psi'(x_2,x_3,e_0)}_{=0}] + [x_2,\underbrace{\psi'(x_1,x_3,e_0)}_{=0}] - [x_3,\underbrace{\psi'(x_1,x_2,e_0)}_{=0}] + [e_0,\psi'(x_1,x_2,x_3)] = 0\\
\Leftrightarrow\ &-\deg(x_1)\psi(x_1,x_2,x_3) + \deg(x_2)\underbrace{\psi(x_2,x_1,x_3)}_{=-\psi(x_1,x_2,x_3)} - \deg(x_3)\underbrace{\psi(x_3,x_1,x_2)}_{=\psi(x_1,x_2,x_3)}\\
&+ \left(\deg(x_1)+\deg(x_2)+\deg(x_3)+d\right)\psi(x_1,x_2,x_3) = 0\\
\Leftrightarrow\ &d\ \psi(x_1,x_2,x_3) = 0 \Leftrightarrow\ \psi(x_1,x_2,x_3) = 0 \text{ as } d \neq 0\,.
\end{aligned}$$

We conclude that the third cohomology of the Witt algebra reduces to the degree zero part. □

5.2. **The degree zero part for the Witt algebra.** The proposition we shall prove in this section is the following:

Proposition 5.2.1. *The following holds:*

$$\mathrm{H}^3_{(0)}(\mathcal{W},\mathcal{W}) = \{0\} \ .$$

Clearly, Proposition 5.2.1 together with Proposition 5.1.1 shows Theorem 5.1. The proof of Proposition 5.2.1 is accomplished in six steps and is similar to the proof performed for $\mathrm{H}^2_{(0)}(\mathcal{W},\mathcal{W})$ in [29, 30].

Let ψ be a degree zero 3-cocycle, i.e. we can write it as $\psi(e_i, e_j, e_k) = \psi_{i,j,k} e_{i+j+k}$ with suitable coefficients $\psi_{i,j,k} \in \mathbb{K}$. We say that $\psi_{\cdot,\cdot,\cdot}$ is of *level* $l \in \mathbb{Z}$ if one of its indices is equal to l, i.e. $\psi_{\cdot,\cdot,\cdot} = \psi_{\cdot,\cdot,l}$ or some permutation thereof.

Consequently, five steps of the proof correspond to the analysis of the levels plus one, minus one, zero, plus two and minus two. The final step consists in the analysis of generic levels, which is obtained by induction. In each step, there are always three cases to consider depending on the signs of the indices. One of the three indices corresponds to the level and is fixed. In that case, the three cases to consider correspond to both remaining indices being negative, both being positive, or one being negative and one being positive. It does not matter which of the indices are chosen to be positive or negative, nor does it matter which one of the three indices is chosen to be fixed, because of the alternating property of the cochains. In the following, we provide a brief and superficial summary of the proof:

- **Level plus one / minus one:** There is a cohomological change $\psi' = \psi - \delta_2\phi$, $\phi \in C^2(\mathcal{W},\mathcal{W})$ which allows to normalize to zero either the coefficients of level plus one or the coefficients of level minus one, depending on the signs of the two remaining indices. More precisely, we normalize ψ' to $\psi'_{i,j,-1} = 0$ if i and j are both positive and $\psi'_{i,j,1} = 0$ else.
  The aim is to use the coboundary condition to produce recurrence relations which provide a consistent definition of ϕ, i.e. of all the $\phi_{i,j}$ $\forall\ i,j \in \mathbb{Z}$. Each degree of freedom given by some $\phi_{\cdot,\cdot}$ should be used to cancel some coefficient of the form $\psi_{\cdot,\cdot,1}$ or $\psi_{\cdot,\cdot,-1}$. In the case where both indices of $\phi_{i,j}$ have the same sign, the definition of the $\phi_{i,j}$'s can be obtained in a straightforward manner from the recurrence relations. In the case where the two indices are of opposite sign, poles occur in the recurrence relations, and the definition of the $\phi_{i,j}$'s has to be obtained in a somewhat roundabout manner.
- **Level zero:** For a cocycle ψ normalized as described in the previous bullet point, the cocycle conditions imply $\psi_{i,j,0} = 0$ $\forall\ i,j \in \mathbb{Z}$.
  The cocycle conditions provide recurrence relations which allow to deduce the result immediately for i and j of the same sign. For i and j of different sign, the proof is an (almost) straightforward generalization of the proof of $H^2(\mathcal{W},\mathcal{W})_{(0)} = \{0\}$ given in [29, 30].
- **Level minus one / plus one:** The cocycle conditions imply $\psi_{i,j,1} = 0$ if i and j are both positive and $\psi_{i,j,-1} = 0$ else. Together with the result of the first bullet point, we have $\psi_{i,j,1} = \psi_{i,j,-1} = 0$ $\forall\ i,j \in \mathbb{Z}$.
  This step is the simplest one of the entire proof. The cocycle conditions provide again recurrence relations which allow to deduce the results directly.
- **Levels plus two and minus two / Generic Level k:** The cocycle conditions imply $\psi_{i,j,-2} = 0$ and $\psi_{i,j,2} = 0$ $\forall\ i,j \in \mathbb{Z}$. Induction on k subsequently implies $\psi_{i,j,k} = 0$ $\forall\ i,j,k \in \mathbb{Z}$.

For both indices i and j negative, the first step consists in proving that level minus two is zero, i.e. $\psi_{i,j,-2} = 0$. Induction on the third index allows to conclude that the coefficients $\psi_{i,j,k}$ are zero for all negative indices $i, j, k \leq 0$. These results can be obtained directly from the recurrence relations given by the cocycle conditions.
In the case of one positive and one negative index, the first step consists in proving that both levels plus two and minus two are zero, $\psi_{i,j,2} = \psi_{i,j,-2} = 0$. This has to be done by using induction on either i or j depending on the level under consideration. Note that in the proof of $H^2(\mathcal{W}, \mathcal{W})_{(0)} = \{0\}$ in [29, 30], the vanishing of the levels plus two and minus two could be proved directly without using induction. Obviously, the number of times induction has to be used increases with the number of indices. Due to poles and zeros in the recurrence relations, the proof again follows a somewhat roundabout way. The second and final step consists in using induction on the third index in order to prove $\psi_{i,j,k} = 0$ for mixed indices, i.e. two indices positive and one index negative or two indices negative and one index positive.
The final case with both indices i and j positive starts with the proof that level plus two is zero, i.e. $\psi_{i,j,2} = 0$. Induction on the third index allows to conclude that the coefficients $\psi_{i,j,k}$ are zero for all positive indices $i, j, k \geq 0$. These results follow directly from the recurrence relations.

We now come to the detailed proof. Let us write down the coboundary and cocycle conditions for later use. If ϕ is a degree zero 2-cochain, i.e. $\phi(e_i, e_j) = \phi_{i,j} e_{i+j}$, the coboundary condition for ψ on the triplet (e_i, e_j, e_k) becomes:

$$\begin{aligned}\psi_{i,j,k} = (\delta_2\phi)_{i,j,k} =& (j-i)\phi_{i+j,k} + (k-j)\phi_{k+j,i} + (i-k)\phi_{i+k,j} \\ &- (j+k-i)\phi_{j,k} + (i+k-j)\phi_{i,k} - (i+j-k)\phi_{i,j}\,.\end{aligned}$$

The cocycle condition for ψ on the quadruplet (e_i, e_j, e_k, e_l) becomes:

$$\begin{aligned}(\delta_3\psi)_{i,j,k,l} =& (j-i)\psi_{i+j,k,l} - (k-i)\psi_{i+k,j,l} + (l-i)\psi_{i+l,j,k} \\ &+ (k-j)\psi_{k+j,i,l} - (l-j)\psi_{l+j,i,k} + (l-k)\psi_{l+k,i,j} \\ &- (j+k+l-i)\psi_{j,k,l} + (i+k+l-j)\psi_{i,k,l} \\ &- (i+j+l-k)\psi_{i,j,l} + (i+j+k-l)\psi_{i,j,k} = 0\,.\end{aligned}$$

The first step of the proof is achieved with a cohomological change:

Lemma 5.2.1. *Every 3-cocycle ψ of degree zero is cohomologous to a degree zero 3-cocycle ψ' with:*

$$\begin{array}{llll} & \psi'_{i,j,1} = 0 & \forall\ i \leq 0,\ \forall\ j \in \mathbb{Z}\,, & \\ \textit{and} & \psi'_{i,j,-1} = 0 & \forall\ i, j > 0\,, & \\ \textit{and} & \psi'_{i,-1,2} = 0 & \forall\ i \in \mathbb{Z}\,, & \\ \textit{and} & \psi'_{-4,2,-2} = 0\,. & & \end{array} \tag{7}$$

Proof. If ϕ is a 2-cochain, it can always be normalized to $\phi_{i,1} = 0\ \forall\ i \in \mathbb{Z}$ and $\phi_{-1,2} = 0$ with a cohomological change. The proof can be found in [29, 30] where it was performed in the context of ϕ being a 2-cocycle. However, as the cocycle condition is not used in the proof, the result is valid for any 2-cochain ϕ. Hence, we will perform a cohomological change $\psi' = \psi - \delta_2\phi$ with ϕ normalized to $\boxed{\phi_{i,1} = 0\ \forall\ i \in \mathbb{Z}}$ and $\boxed{\phi_{-1,2} = 0}$. This simplifies the notations considerably.

To increase the readability of the proof, we will separate the analysis depending on the signs of the indices i, j. Let us start with the case i and j both being negative.

$\boxed{\textbf{Case 1: } i, j \leq 0}$

Our aim is to show that we can find coefficients $\phi_{i,j}$ such that $\psi'_{i,j,1} = 0$. Writing down the coboundary condition for $(i, j, 1)$ and dropping the terms of the form $\phi_{\cdot,1}$, we need:

$$\psi_{i,j,1} = -(i + j - 1)\phi_{i,j} + (i - 1)\phi_{i+1,j} - (j - 1)\phi_{j+1,i} \,.$$

This is the case if we define ϕ:

$$\phi_{i,j} := \frac{i-1}{i+j-1}\phi_{i+1,j} - \frac{j-1}{i+j-1}\phi_{j+1,i} - \frac{\psi_{i,j,1}}{i+j-1} \,.$$

Starting with $i = 0$, $j = -1$, j decreasing and using $\phi_{\cdot,1} = 0$, this recurrence relation defines in a first step $\phi_{0,j}$ for $j \leq -1$. In a second step, $\phi_{-1,j}$ with $j \leq -2$ can be obtained, and so on for all $i \leq -2$ with $j < i$. It is sufficient to consider $j < i$ due to the alternating character of the cochains. Thus, this recurrence relation defines $\phi_{i,j}$ for $i, j \leq 0$. It follows that we can perform a cohomological change such that $\psi'_{i,j,1} = 0 \ \forall\ i, j \leq 0$.

$\boxed{\textbf{Case 2: } i \leq 0 \text{ and } j > 0}$

We will start by proving that we can obtain $\psi'_{i,2,-1} = 0 \ \forall\ i \leq 0$ for a suitable choice of the coefficients $\phi_{i,j}$.

Let us consider the coboundary condition for $(-3, 2, -1)$. Taking into account the normalization $\phi_{2,-1} = 0$ we obtain:

$$-2\phi_{-4,2} - 6\phi_{-3,-1} = \psi_{-3,2,-1} \,.$$

The quantity $\phi_{-3,-1}$ has been defined in the previous case $i, j \leq 0$. Thus, we obtain a definition for $\phi_{-4,2}$. From there, we can obtain $\phi_{i,2}$ $i \leq -5$ by using the coboundary condition for $(i, 2, -1)$ and $\phi_{2,-1} = 0$, which gives us:

$$\phi_{i-1,2} = \frac{3+i}{i+1}\phi_{i,2} + \frac{\psi_{i,2,-1}}{i+1} - \frac{i-3}{i+1}\phi_{i,-1} + \frac{i-2}{i+1}\phi_{i+2,-1} \,. \tag{8}$$

The last two terms have been defined in the previous case $i, j \leq 0$. Thus, this defines $\phi_{i,2}$ $i \leq -4$ such that we have $\psi'_{i,2,-1} = 0 \ \forall\ i \leq -3$. Next, let us consider the coboundary condition for $(-4, 2, -2)$:

$$\psi_{-4,2,-2} = -2\phi_{-6,2} - 8\phi_{-4,-2} - 4\phi_{0,-4} - 4\phi_{2,-2} \,.$$

The coefficients $\phi_{-4,-2}$ and $\phi_{0,-4}$ have been defined in the previous case $i, j \leq 0$. The coefficient $\phi_{-6,2}$ has been defined in (8) for $i \leq -4$. Therefore, we obtain a definition for $\phi_{2,-2}$, which annihilates $\psi'_{-4,2,-2}$, $\boxed{\psi'_{-4,2,-2} = 0}$.

As $\phi_{2,-2}$ is now defined, we can come back to Equation (8), insert $i = -2$ and obtain a definition for $\phi_{-3,2}$, annihilating $\psi'_{-2,2,-1}$. Since $\phi_{-1,2} = 0$ due to our normalization, the only remaining $\phi_{i,2}$ $i \leq 0$ to define is $\phi_{0,2}$.

Let us write down the coboundary condition for $(0, 2, -1)$:

$$\begin{aligned} &-(3\phi_{0,2} + 3\phi_{0,-1}) = \psi_{0,2,-1} \\ \Leftrightarrow\ &\phi_{0,2} = -\phi_{0,-1} - \frac{1}{3}\psi_{0,2,-1} \,. \end{aligned}$$

This defines $\phi_{0,2}$ and consequently, $\psi'_{0,2,-1} = 0$. Since $\psi'_{-1,2,-1} = 0$ due to the alternating property, we obtain all in all that $\psi'_{i,2,-1} = 0 \ \forall\ i \leq 0$.
Next, let us prove that we can obtain $\psi'_{i,j,1} = 0 \ \forall\ i \leq 0 \ \forall\ j > 0$. It suffices to write down the coboundary condition for $(i, j, 1)$ in the following way:

$$\phi_{i,j+1} := \frac{i+j-1}{j-1}\phi_{i,j} - \frac{i-1}{j-1}\phi_{i+1,j} + \frac{\psi_{i,j,1}}{j-1}\,.$$

Fixing $i = 0$, and starting with $j = 2$ (recall that $\phi_{i,1} = 0$ and that we have just defined all $\phi_{i,2}\ i \leq 0$), j increasing, we obtain $\phi_{0,j}\ \forall\ j > 2$ and $\psi'_{0,j,1} = 0\ \forall\ j \geq 2$. Similarly, fixing $i = -1$, and starting with $j = 2$, j increasing, we obtain $\phi_{-1,j}\ \forall\ j > 2$ and $\psi'_{-1,j,1} = 0\ \forall\ j \geq 2$. Continuing along the same lines, we obtain $\phi_{i,j}\ \forall\ i \leq 0,\ j > 0$ and $\psi'_{i,j,1} = 0\ \forall\ i \leq 0,\ j > 0$. Together with the result $\psi'_{i,j,1} = 0\ \forall\ i,j \leq 0$ obtained from the previous case with $i, j \leq 0$, we get $\boxed{\psi'_{i,j,1} = 0\ \forall\ i \leq 0,\ \forall\ j \in \mathbb{Z}}$

$\boxed{\textbf{Case 3: } i > 0 \text{ and } j > 0}$

Let us write down the coboundary condition for $(i, j, -1)$:

$$\begin{aligned}\psi_{i,j,-1} =& (i+1)\phi_{i-1,j} + (i-j-1)\phi_{i,-1} - (1+i+j)\phi_{i,j} \\ &+ (j+1)\phi_{i,j-1} + (1+i-j)\phi_{j,-1} + (j-i)\phi_{i+j,-1}\,.\end{aligned}$$

From there, we can define ϕ via recurrence as follows:

$$\begin{aligned}\phi_{i,j} =& \frac{(i+1)}{(1+i+j)}\phi_{i-1,j} + \frac{(j+1)}{(1+i+j)}\phi_{i,j-1} - \frac{\psi_{i,j,-1}}{(1+i+j)} \\ &+ \frac{(i-j-1)}{(1+i+j)}\phi_{i,-1} + \frac{(1+i-j)}{(1+i+j)}\phi_{j,-1} + \frac{(j-i)}{(1+i+j)}\phi_{i+j,-1}\,.\end{aligned}$$

Note that $\phi_{\cdot,-1}$ have been defined in the previous case for $i \leq 0,\ j > 0$. Starting with $i = 2$, $j = 3$ and j increasing, we obtain in a first step $\phi_{2,j},\ \forall\ j \geq 3$ and $\psi'_{2,j,-1} = 0\ \forall\ j \geq 3$. Next, fixing $i = 3$, starting with $j = 4$ and j increasing, we obtain in a second step $\phi_{3,j},\ \forall\ j \geq 4$ and $\psi'_{3,j,-1} = 0\ \forall\ j \geq 4$. Continuing similarly with i increasing, we finally obtain all $\phi_{i,j},\ \forall\ i,j > 0$, and $\boxed{\psi'_{i,j,-1} = 0\ \forall\ i,j > 0}$. Note that we already have $\psi'_{1,j,-1} = 0\ \forall\ j > 0$ due to the previous case, which yielded $\psi'_{i,j,1} = 0\ \forall\ i \leq 0,\ \forall\ j \in \mathbb{Z}$. Combining the result $\psi'_{i,j,-1} = 0\ \forall\ i,j > 0$ with the result $\psi'_{i,2,-1} = 0\ \forall\ i \leq 0$ from the previous case $i \leq 0,\ j > 0$, we also obtain $\boxed{\psi'_{i,2,-1} = 0\ \forall\ i \in \mathbb{Z}}$. □

Lemma 5.2.2. *Let ψ be a degree zero 3-cocycle such that:*

$$\begin{aligned} & \psi_{i,j,1} = 0 \quad && \forall\ i \leq 0,\ \forall\ j \in \mathbb{Z}\,, \\ \textit{and} \quad & \psi_{i,j,-1} = 0 && \forall\ i,j > 0\,, \\ \textit{and} \quad & \psi_{i,-1,2} = 0 && \forall\ i \in \mathbb{Z}\,, \end{aligned}$$

then

$$\psi_{i,j,0} = 0 \quad \forall\ i,j \in \mathbb{Z}\,. \tag{9}$$

Proof. Again, we split the proof into the three cases depending on the signs of i and j.

$\boxed{\textbf{Case 1: } i,j \leq 0}$

Let us write down the cocycle condition for $(i, j, 0, 1)$, neglecting the terms of the form $\psi_{i,j,1}$ $i, j \leq 0$:

$$(i + j - 1)\psi_{i,j,0} - (i - 1)\psi_{i+1,j,0} + (j - 1)\psi_{j+1,i,0} = 0\,.$$

We can define the following recurrence relation for i and j decreasing:

$$\psi_{i,j,0} = \frac{(i-1)}{(i+j-1)}\psi_{i+1,j,0} - \frac{(j-1)}{(i+j-1)}\psi_{j+1,i,0}\,.$$

Fixing $i = -1$, starting with $j = -2$ and j decreasing, we obtain $\psi_{-1,j,0} = 0 \;\forall\; j \leq -2$. Repeating the same procedure with decreasing values for i and $j < i$, we obtain $\psi_{i,j,0} = 0 \;\forall\; i, j \leq 0$.

$\boxed{\textbf{Case 2: } i \leq 0,\ j > 0}$

Let us write down the cocycle condition for $(i, 2, 0, -1)$:

$$\begin{aligned}&-\cancel{\psi_{-1,i,2}} + 3\psi_{1,i,0} + (-1+i)\psi_{2,0,-1} - 2\cancel{\psi_{2,i,-1}} - (1+i)\psi_{-1+i,2,0}\\ &+ (-3+i)\psi_{i,0,-1} - \cancel{\psi_{i,2,-1}} + (3+i)\psi_{i,2,0} - (-2+i)\psi_{2+i,0,-1} = 0\,.\end{aligned}$$

The slashed terms cancel each other, although they are zero anyway as we have $\psi_{i,2,-1} = 0 \;\forall\; i \in \mathbb{Z}$. The term $\psi_{1,i,0}$ is zero as we have $\psi_{i,j,1} = 0 \;\forall\; i, j \leq 0$. The term $\psi_{2,0,-1}$ is zero due to $\psi_{i,2,-1} = 0 \;\forall\; i \in \mathbb{Z}$. The terms $\psi_{i,0,-1}$ and $\psi_{2+i,0,-1}$ (for $i \leq -2$) are zero because of the previous case, $\psi_{i,j,0} = 0 \;\forall\; i, j \leq 0$. Therefore, we are left with:

$$\psi_{i-1,2,0} = \frac{i+3}{i+1}\psi_{i,2,0}\,. \tag{10}$$

Putting $i = -3$ in the equation above, this recurrence relation implies $\psi_{-4,2,0} = 0$ and by recursion $\psi_{i,2,0} = 0 \;\forall i \leq -4$. Next, consider the cocycle condition for $(i, 2, -2, 0)$:

$$\begin{aligned}&2\cancel{\psi_{-2,i,2}} + i\psi_{2,-2,0} + 2\cancel{\psi_{2,i,-2}} + (2+i)\psi_{-2+i,2,0}\\ &+ (-4+i)\psi_{i,-2,0} - (4+i)\psi_{i,2,0} - (-2+i)\psi_{2+i,-2,0} = 0\,.\end{aligned}$$

The slashed terms cancel each other, the terms $\psi_{i,-2,0}$ and $\psi_{2+i,-2,0}$ (for $i \leq -2$) are zero because of $\psi_{i,j,0} = 0 \;\forall\; i, j \leq 0$. As we have $\psi_{i,2,0} = 0 \;\forall i \leq -4$, we can put for example $i = -4$ in the equation above and obtain $\psi_{2,-2,0} = 0$. Inserting this value in Equation (10) with $i = -2$, we obtain $\psi_{-3,2,0} = 0$. Recall that we also have $\psi_{-1,2,0} = 0$ due to $\psi_{i,-1,2} = 0 \;\forall\; i \in \mathbb{Z}$. All in all, we have $\psi_{i,2,0} \;\forall\; i \leq 0$.

This result is needed to write down a well-defined recurrence relation. Writing down the cocycle condition for $(i, j, 0, 1)$ and neglecting the terms of the form $\psi_{i,j,1}$ with $i, j \leq 0$ and $i \leq 0$, $j > 0$, we obtain the following recurrence relation:

$$\psi_{i,j+1,0} = \frac{i+j-1}{j-1}\psi_{i,j,0} - \frac{i-1}{j-1}\psi_{i+1,j,0}\,.$$

Fixing $i = -1$, one starts with $j = 2$ (since we already have $\psi_{i,j,1} = 0$ $i, j \leq 0$ and $\psi_{i,2,0} = 0$ $i \leq 0$), which gives, with increasing j, $\psi_{-1,j,0} = 0$ $j \geq 3$. Continuing with fixing $i = -2$, starting again with $j = 2$ and increasing j, we obtain $\psi_{-2,j,0} = 0$ $j \geq 3$. Doing this for all $i \leq 0$, we finally obtain $\psi_{i,j,0} = 0 \;\forall\; i \leq 0,\ j > 0$.

$\boxed{\textbf{Case 3: } i, j > 0}$

Writing down the cocycle condition for $(i,j,0,-1)$, we obtain:

$$\begin{aligned}&-\cancel{\psi_{-1,i,j}}-(1+i)\psi_{-1+i,j,0}+(-1+i-j)\psi_{i,0,-1}+i\cancel{\psi_{i,j,-1}}-\cancel{(-1+i+j)\psi_{i,j,-1}}\\&+(1+i+j)\psi_{i,j,0}+(1+j)\psi_{-1+j,i,0}+(1+i-j)\psi_{j,0,-1}-j\cancel{\psi_{j,i,-1}}+(-i+j)\psi_{i+j,0,-1}=0\,.\end{aligned}$$

The slashed terms cancel each other, though they are zero anyway due to $\psi_{i,j,-1}=0\ i,j>0$. The terms $\psi_{i,0,-1}$, $\psi_{j,0,-1}$ and $\psi_{i+j,0,-1}$ are zero due to the previous case, $\psi_{i,j,0}=0\ i\leq 0,\ j>0$. Thus, we obtain the following recurrence relation:

$$\psi_{i,j,0}=\frac{(1+i)}{(1+i+j)}\psi_{-1+i,j,0}-\frac{(1+j)}{(1+i+j)}\psi_{-1+j,i,0}\,.$$

Fixing $i=1$, starting with $j=2$, j increasing, we obtain $\psi_{1,j,0}=0\ j\geq 2$. Fixing $i=2$, starting with $j=3$, we get $\psi_{2,j,0}=0\ j\geq 3$. Continuing with increasing i and keeping $j>i$ due to skew-symmetry, we finally obtain $\psi_{i,j,0}=0\ \forall\ i,j>0$.
Taking all three cases together, we obtain the announced result, $\psi_{i,j,0}=0\ \forall\ i,j\in\mathbb{Z}$. □

Lemma 5.2.3. *Let ψ be a degree zero 3-cocycle such that:*

$$\begin{aligned}&\psi_{i,j,1}=0 &&\forall\ i\leq 0,\ \forall\ j\in\mathbb{Z}\,,\\ \textit{and}\quad&\psi_{i,j,-1}=0 &&\forall\ i,j>0\,,\\ \textit{and}\quad&\psi_{i,-1,2}=0 &&\forall\ i\in\mathbb{Z}\,,\\ \textit{and}\quad&\psi_{i,j,0}=0 &&\forall\ i,j\in\mathbb{Z}\,,\end{aligned}$$

then

$$\psi_{i,j,1}=\psi_{i,j,-1}=0\quad\forall\ i,j\in\mathbb{Z}\,.$$

Proof. Again, the proof is split into the three cases depending on the signs of i,j.
$\boxed{\textbf{Case 1: } i,j\leq 0}$
Writing down the cocycle condition for $(i,j,1,-1)$ and neglecting $\psi_{i,j,1}\ i,j\leq 0$ as well as $\psi_{i,j,0}\ i,j\leq 0$, we obtain:

$$\begin{aligned}&-(-2+i+j)\psi_{i,j,-1}+(-1+i)\psi_{1+i,j,-1}-(-1+j)\psi_{1+j,i,-1}=0\\ \Leftrightarrow\ &\psi_{i,j,-1}=\frac{(-1+i)}{(-2+i+j)}\psi_{1+i,j,-1}-\frac{(-1+j)}{(-2+i+j)}\psi_{1+j,i,-1}\,.\end{aligned}$$

Fixing $i=-2$ (since level zero $\psi_{0,j,-1}\ j\leq 0$ is already done and $\psi_{-1,j,-1}=0$), starting with $j=-3$ and j decreasing, we obtain $\psi_{-2,j,-1}=0\ j\leq -3$. Fixing $i=-3$, starting with $j=-4$ and j decreasing, we get $\psi_{-3,j,-1}=0\ j\leq -4$. Continuing along the same lines, we obtain $\psi_{i,j,-1}=0\ i,j\leq 0$.
$\boxed{\textbf{Case 2: } i\leq 0,\ j>0}$
Writing down the cocycle condition for $(i,j,1,-1)$ and neglecting $\psi_{i,j,1}\ i\leq 0,\ j>0$, $\psi_{i,j,1}\ i,j\leq 0$ as well as $\psi_{i,j,0}\ i\leq 0,\ j>0$, we obtain:

$$\begin{aligned}&-(-2+i+j)\psi_{i,j,-1}+(-1+i)\psi_{1+i,j,-1}-(-1+j)\psi_{1+j,i,-1}=0\\ \Leftrightarrow\ &\psi_{i,1+j,-1}=\frac{(-2+i+j)}{(-1+j)}\psi_{i,j,-1}-\frac{(-1+i)}{(-1+j)}\psi_{1+i,j,-1}\,.\end{aligned}$$

Fixing $i=-2$ (since level zero $\psi_{0,j,-1}\ j>0$ is already done and $\psi_{-1,j,-1}=0$) and starting with $j=2$ (since $\psi_{i,2,-1}=0\ i\leq 0$), increasing j, we obtain $\psi_{-2,j,-1}=0\ j\geq 3$. Fixing $i=-3$, starting again with $j=2$, j increasing, we get $\psi_{-3,j,-1}=0\ j\geq 3$. Continuing with

i decreasing, we get $\psi_{i,j,-1} = 0\ i \leq 0,\ j > 0$.

$\boxed{\textbf{Case 3: } i, j > 0}$

Writing again down the cocycle condition for $(i, j, 1, -1)$, this time neglecting the terms $\psi_{i,j,-1}\ i, j > 0$ and $\psi_{i,j,0}\ i, j > 0$, we obtain:

$$\begin{aligned} &-(1+i)\psi_{-1+i,j,1} + (2+i+j)\psi_{i,j,1} + (1+j)\psi_{-1+j,i,1} = 0 \\ \Leftrightarrow\ &\psi_{i,j,1} = \frac{(1+i)}{(2+i+j)}\psi_{-1+i,j,1} - \frac{(1+j)}{(2+i+j)}\psi_{-1+j,i,1}\,. \end{aligned}$$

Fixing $i = 2$ (since $\psi_{1,j,1} = 0$) and starting with $j = 3$, increasing j, we obtain $\psi_{2,j,1} = 0\ j \geq 3$. Increasing i and keeping $j > i$ we finally obtain $\psi_{i,j,1} = 0\ \forall\ i, j > 0$.
Taking all three cases together, we have proven that $\psi_{i,j,1} = 0\ \forall\ i, j \in \mathbb{Z}$ and $\psi_{i,j,-1} = 0\ \forall\ i, j \in \mathbb{Z}$. □

Lemma 5.2.4. *Let ψ be a degree zero 3-cocycle such that:*

$$\psi_{i,j,1} = \psi_{i,j,-1} = \psi_{i,j,0} = 0 \quad \forall\ i, j \in \mathbb{Z} \quad \textit{and} \quad \psi_{-4,2,-2} = 0\,,$$

then

$$\psi_{i,j,k} = 0 \quad \forall\ i, j, k \in \mathbb{Z}\,.$$

Proof. Again, the proof is split in the three cases depending on the signs of i and j.

$\boxed{\textbf{Case 1: } i, j \leq 0}$

In a first step, we shall prove the following statement: $\boxed{\psi_{i,j,-2} = 0\ \forall\ i, j \leq 0}$.
Writing down the cocycle condition for $(i, j, -2, 1)$ and neglecting the terms of level one $\psi_{i,j,1}$ and level minus one $\psi_{i,j,-1}$, we obtain the following:

$$\begin{aligned} &(-3+i+j)\psi_{i,j,-2} - (-1+i)\psi_{1+i,j,-2} + (-1+j)\psi_{1+j,i,-2} = 0 \\ \Leftrightarrow\ &\psi_{i,j,-2} = \frac{(-1+i)}{(-3+i+j)}\psi_{1+i,j,-2} - \frac{(-1+j)}{(-3+i+j)}\psi_{1+j,i,-2}\,. \end{aligned}$$

Fixing $i = -3$ (since the levels zero $\psi_{i,j,0}$ and minus one $\psi_{i,j,-1}$ are already done and $\psi_{-2,j,-2} = 0$), starting with $j = -4$ and decreasing j, we obtain $\psi_{-3,j,-2} = 0\ j \leq -4$. Continuing along the same lines with decreasing i and keeping $j < i$, we obtain $\psi_{i,j,-2} = 0\ \forall\ i, j \leq 0$ as a first step.
In a second step, we shall prove $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j, k \leq 0}$. This can be done by induction. We know the result is true for $k = 0, -1, -2$. Hence, we will assume it is true for some $k \leq -2$ and check whether it remains true for $k - 1$. The cocycle condition for $(i, j, k, -1)$ is given by, after omitting terms of level minus one $\psi_{i,j,-1}$:

$$\begin{aligned} &-(1+i)\psi_{-1+i,j,k} + (1+i+j+k)\psi_{i,j,k} + (1+j)\psi_{-1+j,i,k} - (1+k)\psi_{-1+k,i,j} = 0 \\ &\Leftrightarrow\ -(1+k)\psi_{-1+k,i,j} = 0 \Leftrightarrow\ \psi_{-1+k,i,j} = 0 \text{ as } k \leq -2\,. \end{aligned}$$

The terms $\psi_{-1+i,j,k}$, $\psi_{i,j,k}$ and $\psi_{-1+j,i,k}$ are zero since they are of level k and thus zero by induction hypothesis. It follows $\psi_{i,j,k} = 0\ \forall\ i, j, k \leq 0$.

$\boxed{\textbf{Case 2: } i \leq 0,\ j > 0}$

In a first step, we shall prove the following statements: $\boxed{\psi_{i,j,-2} = 0\ \forall\ i \leq 0,\ j > 0}$ and $\boxed{\psi_{i,j,2} = 0\ \forall\ i \leq 0,\ j > 0}$.

The cocycle condition for $(-3,2,-2,-1)$ reads, after dropping the terms of level one, minus one and zero:

$$2\psi_{-4,2,-2} - 2\psi_{-3,2,-2} = 0\,.$$

Since we have $\psi_{-4,2,-2} = 0$, the equation above implies $\psi_{-3,2,-2} = 0$. Next, let us write down the cocycle condition for $(-3,j,-2,1)$, which gives after dropping terms of level one and of level minus one:

$$\begin{aligned}&(-6+j)\psi_{-3,j,-2} + (-1+j)\psi_{1+j,-3,-2} = 0\\ \Leftrightarrow\;\; &\psi_{1+j,-3,-2} = \frac{(-6+j)}{(-1+j)}\psi_{j,-3,-2}\,.\end{aligned}$$

Starting with $j=2$, we obtain $\psi_{j,-3,-2} = 0\ \forall\ j \geq 3$ since the starting point is zero: $\psi_{2,-3,-2} = 0$. Adding the level one, we obtain $\psi_{j,-3,-2} = 0\ \forall\ j > 0$.
Next, let us write down the cocycle condition for $(i,3,2,-1)$ after dropping terms of level one and level minus one:

$$\begin{aligned}&-(1+i)\psi_{-1+i,3,2} + (6+i)\psi_{i,3,2} = 0\\ \Leftrightarrow\;\; &\psi_{-1+i,3,2} = \frac{(6+i)}{(1+i)}\psi_{i,3,2}\,.\end{aligned}\tag{11}$$

This gives us for $i=-2$: $\psi_{-3,3,2} = -4\psi_{-2,3,2}$
For $i=-3$: $\psi_{-4,3,2} = \frac{3}{-2}\psi_{-3,3,2} = 6\psi_{-2,3,2}$
For $i=-4$: $\psi_{-5,3,2} = \frac{2}{-3}\psi_{-4,3,2} = -4\psi_{-2,3,2}$
Now, let us write down the cocycle condition for $(-3,3,2,-2)$ and drop the terms of level one, level minus one and level zero, as well as the terms of the form $\psi_{j,-3,-2}\ j > 0$:

$$\begin{aligned}&\psi_{-5,3,2} + 4\psi_{-3,3,2} - 6\psi_{3,2,-2} = 0\\ \Leftrightarrow\;\; &-4\psi_{-2,3,2} - 16\psi_{-2,3,2} - 6\psi_{-2,3,2} = 0\\ \Leftrightarrow\;\; &\psi_{-2,3,2} = 0\,.\end{aligned}$$

To obtain the second line, the first two terms were simply replaced by their expressions computed above. Putting $i=-2$ and $\psi_{-2,3,2} = 0$ in the recurrence relation (11), we obtain $\psi_{i,3,2} = 0\ \forall\ i \leq -3$. Together with the levels minus one and zero, we obtain $\psi_{i,3,2} = 0\ \forall\ i \leq 0$.
To prove $\psi_{i,j,-2} = 0\ \forall\ i \leq 0,\ j > 0$, we will use induction on i. Indeed, we have proven that $\psi_{i,j,-2} = 0\ \forall\ j > 0$, for $i = 0,-1,-2,-3$ (recall that we have $\psi_{j,-3,-2} = 0\ j > 0$). Suppose the statement holds true down to $i+1$, $i \leq -4$, and let us see what happens for i. The cocycle condition for $(i,j,-2,1)$ gives, after dropping terms of level one and level minus one:

$$\begin{aligned}&(-3+i+j)\psi_{i,j,-2} - (-1+i)\underbrace{\psi_{1+i,j,-2}}_{=0} + (-1+j)\psi_{1+j,i,-2} = 0\\ \Leftrightarrow\;\; &\psi_{i,j+1,-2} = \frac{(-3+i+j)}{(-1+j)}\psi_{i,j,-2}\,.\end{aligned}\tag{12}$$

The term in the middle is zero due to the induction hypothesis.
This gives, for $j=2$: $\psi_{i,3,-2} = (-1+i)\psi_{i,2,-2}$.
For $j=3$: $\psi_{i,4,-2} = \frac{i}{2}\psi_{i,3,-2} = \frac{(-1+i)(i)}{2}\psi_{i,2,-2}$.
For $j=4$: $\psi_{i,5,-2} = \frac{(1+i)}{3}\psi_{i,4,-2} = \frac{(-1+i)(i)(1+i)}{6}\psi_{i,2,-2}$.

Next, we will insert these values into the cocycle condition for $(i, 3, -2, 2)$, after dropping terms of level zero and level one:

$$
\begin{aligned}
&(-3+i)\psi_{3,-2,2} + \psi_{5,i,-2} + (2+i)\psi_{-2+i,3,2} + (-3+i)\psi_{i,-2,2}\\
&+ (-1+i)\psi_{i,3,-2} - (7+i)\psi_{i,3,2} - (-2+i)\psi_{2+i,3,-2} - (-3+i)\psi_{3+i,-2,2} = 0\\
&\Leftrightarrow\ -\frac{(-1+i)(i)(1+i)}{6}\psi_{i,2,-2} - (-3+i)\psi_{i,2,-2} + (-1+i)(-1+i)\psi_{i,2,-2} = 0\\
&\Leftrightarrow\ (i-3)(i^2-3i+8)\psi_{i,2,-2} = 0\,.
\end{aligned}
$$

The terms $\psi_{3,-2,2}$, $\psi_{-2+i,3,2}$ and $\psi_{i,3,2}$ are zero due to what was proved before, $\psi_{i,3,2} = 0\ \forall\ i \leq 0$. The terms $\psi_{2+i,3,-2}$ and $\psi_{3+i,-2,2}$ are zero as a consequence of the induction hypothesis. In the last line, we have $(i-3) \neq 0$, since $i \leq -4$, and also $(i^2 - 3i + 8) \neq 0$ since its discriminant is negative. It follows $\psi_{i,2,-2} = 0$. Reinserting this into (12) and taking into account that level one is zero, we obtain that the induction holds true for i, and thus: $\psi_{i,j,-2} = 0\ \forall\ i \leq 0, j > 0$.

Next, we proceed similarly, but with induction on j, to prove $\psi_{i,j,2} = 0\ \forall\ i \leq 0, j > 0$. We already know that the statement holds true for $j = 1, 2, 3$. Let us suppose it is true up to $j-1$, $j \geq 4$, and show that it remains true for j. Let us write down the cocycle condition for $(i, j, 2, -1)$, after dropping terms of level one and level minus one:

$$
\begin{aligned}
&-(1+i)\psi_{-1+i,j,2} + (3+i+j)\psi_{i,j,2} + (1+j)\underbrace{\psi_{-1+j,i,2}}_{=0} = 0\\
&\Leftrightarrow\ \psi_{-1+i,j,2} = \frac{(3+i+j)}{(1+i)}\psi_{i,j,2}\,.
\end{aligned}
\tag{13}
$$

The third term is zero due to the induction hypothesis. From the recurrence relation above, we obtain for $i = -2$: $\psi_{-3,j,2} = -(1+j)\psi_{-2,j,2}$.

For $i = -3$: $\psi_{-4,j,2} = \frac{j}{(-2)}\psi_{-3,j,2} = \frac{j(1+j)}{2}\psi_{-2,j,2}$

For $i = -4$: $\psi_{-5,j,2} = \frac{(-1+j)}{(-3)}\psi_{-4,j,2} = -\frac{(-1+j)j(1+j)}{6}\psi_{-2,j,2}$.

Next, we insert these values into the cocycle condition for $(-3, j, 2, -2)$ after dropping terms of level zero and level minus one:

$$
\begin{aligned}
&\psi_{-5,j,2} - (3+j)\psi_{-3,2,-2} - (-7+j)\psi_{-3,j,-2} + (1+j)\psi_{-3,j,2}\\
&+ (3+j)\psi_{-3+j,2,-2} + (2+j)\psi_{-2+j,-3,2} - (3+j)\psi_{j,2,-2} - (-2+j)\psi_{2+j,-3,-2} = 0\\
&\Leftrightarrow\ -\frac{(-1+j)j(1+j)}{6}\psi_{-2,j,2} - (1+j)(1+j)\psi_{-2,j,2} - (3+j)\psi_{-2,j,2} = 0\\
&\Leftrightarrow\ (j+3)(8+3j+j^2)\psi_{-2,j,2} = 0\,.
\end{aligned}
$$

The terms $\psi_{-3,2,-2}$, $\psi_{-3,j,-2}$ and $\psi_{2+j,-3,-2}$ are zero due to what was shown before in this proof, $\psi_{j,-3,-2} = 0\,\forall\, j > 0$. The terms $\psi_{-3+j,2,-2}$ and $\psi_{-2+j,-3,2}$ are zero due to the induction hypothesis. In the last line, we have $(j+3) \neq 0$, since $j \geq 4$, and also $(j^2 + 3j + 8) \neq 0$ since its discriminant is negative. It follows $\psi_{-2,j,2} = 0$. Reinserting this into (13) and taking into account that level minus one is zero, we obtain that the induction holds true for j, and thus: $\psi_{i,j,2} = 0\ \forall\ i \leq 0, j > 0$.

Now that the terms of level two and of level minus two are zero for the case $i \leq 0, j > 0$, we can use induction on k to first prove $\boxed{\psi_{i,j,k} = 0,\ \forall\ i \leq 0, j > 0, k \geq 0}$ and then

$\boxed{\psi_{i,j,k} = 0,\ \forall\ i \leq 0, j > 0, k \leq 0}$.
The result is true for $k = 0, 1, 2$. Let us assume the result is true for k, $k \geq 2$ and show that it remains true for $k+1$. The cocycle condition for $(i, j, k, 1)$ gives, after dropping terms of level one,

$$
\begin{aligned}
&(-1+i+j+k)\psi_{i,j,k} - (-1+i)\psi_{1+i,j,k} + (-1+j)\psi_{1+j,i,k} - (-1+k)\psi_{1+k,i,j} = 0 \\
&\Leftrightarrow\ (-1+k)\psi_{1+k,i,j} = 0 \Leftrightarrow\ \psi_{1+k,i,j} = 0 \text{ as } k \geq 2\,.
\end{aligned}
$$

The terms $\psi_{i,j,k}$, $\psi_{1+i,j,k}$ and $\psi_{1+j,i,k}$ are zero because of the induction hypothesis (if $1+i = 1$, the term $\psi_{1+i,j,k}$ is still zero because the level plus one is zero for all $j, k \in \mathbb{Z}$). It follows that the result holds true for $k+1$.
All the same, the result is true for $k = 0, -1, -2$. Let us assume it is true for k, $k \leq -2$, and show that it holds true for $k-1$. The cocycle condition for $(i, j, k, -1)$ yields, after dropping terms of level minus one,

$$
\begin{aligned}
&-(1+i)\psi_{-1+i,j,k} + (1+i+j+k)\psi_{i,j,k} + (1+j)\psi_{-1+j,i,k} - (1+k)\psi_{-1+k,i,j} = 0 \\
&\Leftrightarrow\ (1+k)\psi_{-1+k,i,j} = 0 \Leftrightarrow\ \psi_{-1+k,i,j} = 0 \text{ as } k \leq -2\,.
\end{aligned}
$$

The terms $\psi_{-1+i,j,k}$, $\psi_{i,j,k}$ and $\psi_{-1+j,i,k}$ are zero because of the induction hypothesis (if $-1+j = 0$, the term $\psi_{-1+j,i,k}$ is still zero because the level zero vanishes for all $i, k \in \mathbb{Z}$). It follows that the result holds true for $k-1$. Thus, we have obtained the desired result for the case $i \leq 0, j > 0$.

$\boxed{\textbf{Case 3: } i > 0, j > 0}$

In a first step, we shall prove the following statement: $\boxed{\psi_{i,j,2} = 0\ \forall\ i, j > 0}$. The cocycle condition for $(i, j, 2, -1)$ yields, after dropping the terms of level one and of level minus one:

$$
\begin{aligned}
&-(1+i)\psi_{-1+i,j,2} + (3+i+j)\psi_{i,j,2} + (1+j)\psi_{-1+j,i,2} = 0 \\
&\Leftrightarrow\ \psi_{i,j,2} = \frac{(1+i)}{(3+i+j)}\psi_{-1+i,j,2} + \frac{(1+j)}{(3+i+j)}\psi_{i,-1+j,2}\,.
\end{aligned}
$$

Fixing $i = 3$ and starting with $j = 4$, j ascending, we obtain $\psi_{3,j,2} = 0\ \forall j \geq 4$. Fixing $i = 4$ and starting with $j = 5$, j ascending, we get $\psi_{4,j,2} = 0\ \forall j \geq 5$. Continuing with ascending i, keeping $j > i$, we finally obtain $\psi_{i,j,2} = 0\ \forall\ i, j > 0$.
Finally, we want to prove $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j > 0, k \geq 0}$. This can be done with induction on k. Indeed, the result is true for level zero, level one and level two, i.e. $k = 0, 1, 2$. Thus, let us assume the result is true for k, $k \geq 2$ and show that it holds true for $k+1$. The cocycle condition for $(i, j, k, 1)$ gives, after dropping the terms of level one:

$$
\begin{aligned}
&(-1+i+j+k)\psi_{i,j,k} - (-1+i)\psi_{1+i,j,k} + (-1+j)\psi_{1+j,i,k} - (-1+k)\psi_{1+k,i,j} = 0 \\
&\Leftrightarrow\ (-1+k)\psi_{1+k,i,j} = 0 \Leftrightarrow\ \psi_{1+k,i,j} = 0 \text{ as } k \geq 2\,.
\end{aligned}
$$

The terms $\psi_{i,j,k}$, $\psi_{1+i,j,k}$ and $\psi_{1+j,i,k}$ are zero because of the induction hypothesis. It follows that the statement holds true for $k+1$.
Taking all three cases together, we find the announced result $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j, k \in \mathbb{Z}}$ □

Proof of the Proposition 5.2.1

Proof. Let us collect the statements of the four lemmata. Let ψ be a degree-zero 3-cocycle of $\mathcal{W}$ with values in $\mathcal{W}$. By Lemma 5.2.1 we can perform a cohomological change such that we obtain a cohomologous degree-zero 3-cocycle with coefficients fulfilling (7). Hence, the assumptions of Lemma 5.2.2 are satisfied and we obtain (9). Together with Lemma 5.2.3, the assumptions of Lemma 5.2.4 are fulfilled and Lemma 5.2.4 shows $\boxed{\psi_{i,j,k} = 0 \ \forall \ i,j,k \in \mathbb{Z}}$, which proves the Proposition 5.2.1. □

6. Results for the Virasoro algebra

In this section, we present a series of statements concerning the first and the third cohomology groups involving the Virasoro algebra[2]. More precisely, we prove the vanishing of the first cohomology group with values in the adjoint module of the Virasoro algebra. The proof uses the result obtained for the first cohomology group of the Witt algebra and is similar to the one given in [30] to prove the vanishing of the second cohomology group of the Virasoro algebra. Moreover, we will announce results concerning the third cohomology group of the Virasoro algebra without providing a proof here. In fact, numerical evidence corroborates the statements but we do not have an algebraic proof yet. Details will be left for future work [4].
We will start by proving $H^1(\mathcal{V},\mathbb{K}) = \{0\}$, since this result will be used in the proof of $H^1(\mathcal{V},\mathcal{V}) = \{0\}$.

Proposition 6.1. *The first cohomology groups of the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module vanish, i.e.*

$$H^1(\mathcal{W},\mathbb{K}) = H^1(\mathcal{V},\mathbb{K}) = \{0\}\,.$$

Proof. Recall that for a Lie algebra $\mathcal{L}$ the following holds (c.f. e.g. [35]):

$$H^1(\mathcal{L},\mathbb{K}) = \left(\frac{\mathcal{L}}{[\mathcal{L},\mathcal{L}]}\right)^* \,,$$

where the star * stands for the dual space. The conclusion of the proof follows from the fact that both the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ are perfect Lie algebras. □

In the next step, we show $H^1(\mathcal{V},\mathcal{V}) = \{0\}$ by using long exact sequences.

Theorem 6.1. *The first cohomology group of the Virasoro algebra with values in the adjoint module is zero, i.e.:*

$$H^1(\mathcal{V},\mathcal{V}) = \{0\}\,.$$

Proof. The following short exact sequence of Lie algebras,

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{V} \xrightarrow{\nu} \mathcal{W} \longrightarrow 0\,,$$

is an exact sequence of Lie modules over $\mathcal{V}$. Such sequences give rise to long exact sequences in cohomology. For the first cohomology, the sequence looks as follows:

$$\ldots \longrightarrow H^1(\mathcal{V},\mathbb{K}) \longrightarrow H^1(\mathcal{V},\mathcal{V}) \xrightarrow{\nu_*} H^1(\mathcal{V},\mathcal{W}) \longrightarrow \ldots\,.$$

Since we already have $H^1(\mathcal{V},\mathbb{K}) = \{0\}$ and also $H^1(\mathcal{W},\mathcal{W}) = \{0\}$, it suffices to prove $H^1(\mathcal{V},\mathcal{W}) \cong H^1(\mathcal{W},\mathcal{W})$ in order to conclude. Moreover, this time we will focus only on the

[2]For the second cohomology group, see [30]

degree zero part accordingly to the result (3) by Fuks. The proof consists of two steps. First, we will compare the cocycles of $Z^1(\mathcal{V},\mathcal{W})$ to the cocycles of $Z^1(\mathcal{W},\mathcal{W})$. In the second step, we will compare the coboundaries of $B^1(\mathcal{V},\mathcal{W})$ to the ones of $B^1(\mathcal{W},\mathcal{W})$.
Let $\hat{\psi}:\mathcal{V}\to\mathcal{W}$ be a cocycle of $Z^1(\mathcal{V},\mathcal{W})$. Our aim is to show that the restriction of this cocycle to $\mathcal{W}$, i.e. $\psi:=\hat{\psi}|_{\mathcal{W}}:\mathcal{W}\to\mathcal{W}$, is a cocycle of $Z^1(\mathcal{W},\mathcal{W})$. Let $x_1,x_2\in\mathcal{V}$. Writing the Virasoro product $[\cdot,\cdot]_{\mathcal{V}}$ in terms of the Witt product $[\cdot,\cdot]_{\mathcal{W}}$ and the 2-cocycle $\alpha(\cdot,\cdot)$ giving the central extension, i.e. $[\cdot,\cdot]_{\mathcal{V}}=[\cdot,\cdot]_{\mathcal{W}}+\alpha(\cdot,\cdot)\cdot t$, the cocycle condition for $\hat{\psi}$ becomes:

$$\begin{aligned}
&0=(\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2)=\hat{\psi}\left([x_1,x_2]^{\mathcal{V}}\right)-x_1\cdot\hat{\psi}(x_2)+x_2\cdot\hat{\psi}(x_1)\\
&\Leftrightarrow 0=(\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2)=\hat{\psi}\left([x_1,x_2]^{\mathcal{W}}\right)+\alpha(x_1,x_2)\ \hat{\psi}(t)-[x_1,\hat{\psi}(x_2)]^{\mathcal{W}}+[x_2,\hat{\psi}(x_1)]^{\mathcal{W}}\\
&\Leftrightarrow 0=(\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2)=(\delta_1^{\mathcal{W}}\hat{\psi})(x_1,x_2)+\alpha(x_1,x_2)\ \hat{\psi}(t)\,. \qquad (14)
\end{aligned}$$

Since we are considering degree-zero cocycles, the cocycle $\hat{\psi}$ evaluated on the central element reads as follows:

$$\hat{\psi}(t)=c\ e_0\,,$$

for suitable $c\in\mathbb{K}$. Next, let us insert this expression into the cocycle condition for (e_1,t), which yields:

$$\begin{aligned}
&(\delta_1^{\mathcal{V}}\hat{\psi})(e_1,t)=\hat{\psi}\left(\cancel{[e_1,t]^{\mathcal{V}}}\right)-e_1\cdot\hat{\psi}(t)+\cancel{t\cdot\hat{\psi}(e_1)}=0\\
&\Leftrightarrow -[e_1,\hat{\psi}(t)]^{\mathcal{W}}=-c\ [e_1,e_0]^{\mathcal{W}}=c\ e_1=0\\
&\Leftrightarrow c=0\,.
\end{aligned}$$

Inserting $\hat{\psi}(t)=0$ into (14), we obtain

$$0=(\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2)=(\delta_1^{\mathcal{W}}\psi)(x_1,x_2)\,. \qquad (15)$$

This means that a cocycle $\hat{\psi}\in Z^1(\mathcal{V},\mathcal{W})$ corresponds to a cocycle $\psi\in Z^1(\mathcal{W},\mathcal{W})$ when projected to $\mathcal{W}$. Moreover, a cocycle ψ of $Z^1(\mathcal{W},\mathcal{W})$ can also be lifted to a cocycle $\hat{\psi}:=\psi\circ\nu$ in $Z^1(\mathcal{V},\mathcal{W})$. By definition, we thus have $\hat{\psi}(t)=0$ and the relation (15) holds true. Hence, a cocycle $\psi\in Z^1(\mathcal{W},\mathcal{W})$ yields a cocycle $\hat{\psi}\in Z^1(\mathcal{V},\mathcal{W})$ and we have $Z^1(\mathcal{V},\mathcal{W})\cong Z^1(\mathcal{W},\mathcal{W})$ in a canonical way.
The second step of the proof consists in comparing the coboundaries of $B^1(\mathcal{V},\mathcal{W})$ and those of $B^1(\mathcal{W},\mathcal{W})$. However, this is trivial. In fact, the coboundary condition applied on a 0-cochain $\phi\in C^0(\mathcal{V},\mathcal{W})$ is the same as the one applied on a 0-cochain $\phi\in C^0(\mathcal{W},\mathcal{W})$, yielding in both cases:

$$(\delta_0\phi)(x)=-x\cdot\phi \text{ with } \phi\in\mathcal{W}\,.$$

Since the central element of $\mathcal{V}$ acts trivially on $\mathcal{W}$, we have $B^1(\mathcal{V},\mathcal{W})\cong B^1(\mathcal{W},\mathcal{W})$. All in all, we conclude $H^1(\mathcal{V},\mathcal{W})\cong H^1(\mathcal{W},\mathcal{W})$ in a canonical way[3]. □

We also have an algebraic proof of a result concerning the third cohomology groups of the Witt and the Virasoro algebra with values in the trivial module:

[3]The result $H^1(\mathcal{V},\mathcal{W})\cong H^1(\mathcal{W},\mathcal{W})$ can also be obtained via Hochschild-Serre spectral sequences. However, we decided to keep the proof elementary here.

Theorem 6.2. *The third cohomology groups of the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module have the same dimension, i.e.*

$$dim(H^3(\mathcal{W}, \mathbb{K})) = dim(H^3(\mathcal{V}, \mathbb{K}))\,.$$

Moreover, the dimension of these spaces is at most one.

Proof. The proof uses the same techniques as the one provided to show $H^3(\mathcal{W}, \mathcal{W}) = \{0\}$. Details can be found in [4]. □

Concerning the exact dimension of $H^3(\mathcal{W}, \mathbb{K})$ and $H^3(\mathcal{V}, \mathbb{K})$, numerical evidence strongly suggests that it is equal to one. In general, it is hard to provide an algebraic proof for the determination of the dimension of a cohomological group when it is different from zero. This is due to the fact that the resolution of certain recurrence relations can lead to such complicated expressions that it is very awkward to prove algebraically that they are solutions of the cocycle condition. Therefore, we will just announce the results here and leave the details for future work [4].

Conjecture 6.1. *The third cohomology groups of the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module are one-dimensional, i.e.*

$$dim(H^3(\mathcal{W}, \mathbb{K})) = dim(H^3(\mathcal{V}, \mathbb{K})) = 1\,.$$

In addition to the numerical evidence, our conjecture is corroborated by the fact that $dim(H^3(Vect(S^1), \mathbb{R})) = 1$ in the case of continuous cohomology [14, 15].

Theorem 6.3. *Under the assumption that Conjecture 6.1 is true, it follows that the third cohomology group of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the adjoint module is one-dimensional, i.e.*

$$dim(H^3(\mathcal{V}, \mathcal{V})) = 1\,.$$

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