Modal extensions of Ł$_n$-valued logics, coalgebraically

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We show how previous work on modal extensions of Ł$_n$-valued logics fits naturally into the coalgebraic framework and indicate some of the ensuing generalisations.

Modal extensions of Ł$_n$-valued logics. We study logics with a modal operator $\Box$ and built from a countable set of propositional variables Prop using the connectors $\neg, \to, \Box, 1$ in the usual way. To interpret formulas on structures, we use a (crisp) many-valued generalization of the KRIPKE models. We fix a positive integer n and we denote by Ł$_n$ the subalgebra Ł$_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ of the standard MV-algebra $\langle [0,1], \neg, \to, 1\rangle$. A *frame* is a couple $\langle W, R\rangle$ where W is a nonempty set and R is an binary relation. We denote by FR the class of frames.

Definition 0.1 ([2, 4, 5, 9])**.** An *Ł$_n$-valued model*, or a *model* for short, is a couple $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val}\rangle$ where $\mathfrak{F} = \langle W, R\rangle$ is a frame and $\mathrm{Val}\colon W \times \mathsf{Prop} \to$ Ł$_n$. The valuation map Val is extended inductively to $W \times \mathsf{Form}$ using ŁUKASIEWICZ' interpretation of the connectors 0, $\neg$ and $\to$ in $[0,1]$ and the rule

$$\mathrm{Val}(u, \Box\phi) = \min\{\mathrm{Val}(w, \phi) \mid w \in Ru\}. \tag{1}$$

A formula ϕ is *true* in an Ł$_n$-valued model $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val}\rangle$, in notation $\mathcal{M} \models \phi$, if $\mathrm{Val}(u, \phi) = 1$ for every world u of $\mathfrak{F}$. If Φ is a set of formulas that are true in every Ł$_n$-valued model based on an frame $\mathfrak{F}$, we write

$$\mathfrak{F} \models_n \Phi$$

and say that Φ is Ł$_n$*-valid* in $\mathfrak{F}$.

Apart from the signature of frames, there is another first-order signature that can be used to interpret formulas. We denote by $\preceq$ the dual order of divisibility on $\mathbb{N}$, that is, for every $\ell, k \in \mathbb{N}$ we write $\ell \preceq k$ if ℓ is a divisor of k, and $\ell \prec k$ if ℓ is a proper divisor of k.

Definition 0.2 (n-frames, [5, 9])**.** An *n-frame* is a tuple $\langle W, (r_m)_{m\preceq n}, R\rangle$ where $\langle W, R\rangle$ is a frame, $r_m \subseteq W$ for every $m \preceq n$, and

1. $r_n = W$ and $r_m \cap r_q = r_{\gcd(m,q)}$ for any $m, q \preceq n$,
2. $Ru \subseteq r_m$ for any $m \preceq n$ and $u \in r_m$.

FR^n is the class of n-frames. For $\mathfrak{F} \in \mathsf{FR}^n$, a model $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val}\rangle$ is based on $\mathfrak{F}$ if $\mathrm{Val}(u, \mathsf{Prop}) \subseteq$ Ł$_m$ for every $m \preceq n$ and $u \in r_m$. We write

$$\mathfrak{F} \models \Phi$$

if Φ holds in all models based on $\mathfrak{F}$.

It is apparent from [4, 8, 5, 9] that $\models$ is better behaved then $\models_n$ because there is a nice duality between n-frames and modal $\mathcal{MV}_n$-algebras, very much analogous to the classical duality between Kripke frames and Boolean algebras with operators. For example, the Goldblatt-Thomason theorem for modal Ł$_n$-valued logic in [9] is first proved for n-frames and $\models$. The Goldblatt-Thomason theorem for frames and $\models_n$ then appears as a corollary. Morevoer, the canonical extension of a modal $\mathcal{MV}_n$-algebra $\mathbf{A}$ can be obtained as

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the complex algebra of a canonical n-frame associated with $\mathbf{A}$. This construction leads to completeness-through-canonicity results [5] with regards to classes of n-frames.

Modal extensions of $Ł_n$-valued logics, coalgebraically. We account for $\models_n$ by following well-established coalgebraic methodology, summarised in

$$T \circlearrowright \mathsf{Set} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{MV_n} \circlearrowleft L \tag{2}$$

where $T = \mathcal{P}$ is the powerset functor and LA is the free $\mathcal{MV}_n$ algebra generated by $\{\Box a \mid a \in A\}$ modulo the axioms of modal $\mathcal{MV}_n$-algebras. P and S are the contravariant functors given by homming into $Ł_n$. (1) allows us to extend P to a functor $\widetilde{P}$ from T-coalgebras to L-algebras, assigning to a T-coalgebra its 'complex algebra'. Similarly, the functor S can be extended to a functor $\widetilde{S}$ from L-algebras to T-coalgebras assigning to an L-algebra its 'canonical structure'.

A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is exactly a T-coalgebra (for $T = \mathcal{P}$). The Lindenbaum algebras (over a set of atomic propositions) are free L-algebras. We have $\mathfrak{F} \models \phi$ iff all morphisms from the free L-algebra (over the atomic propositions of ϕ) to $\widetilde{P}\mathfrak{F}$ map ϕ to W.

To account for $\models$, we replace, in (2), Set by the category $\mathsf{Set}_{\mathcal{V}_n}$ defined as follows. Let $\mathcal{V}_n = \{1, \ldots, n\}$ be the lattice of all divisors of n ordered by $n \leq m$ if m divides n (so that n is bottom and 1 is top). Then $\mathsf{Set}_{\mathcal{V}_n}$ has as objects pairs (X, v) with $v : X \to \mathcal{V}$ and arrows are maps $f : (X, v) \to (X', v')$ such that $v' f x \geq v x$. Note that this definition makes sense for any complete lattice $\mathcal{V}$ and that $\mathsf{Set}_{\mathcal{V}}$ coincides with Goguen's category of fuzzy sets [3].[1]

In order to extend functors $T : \mathsf{Set} \to \mathsf{Set}$ as in (2) to functors $\mathsf{Set}_{\mathcal{V}_n} \to \mathsf{Set}_{\mathcal{V}_n}$ we notice that $\mathsf{Set}_{\mathcal{V}}$ can be described equivalently as a category of 'continuous presheaves'. A continuous presheaf is a collection of sets $(X, (X_i)_{i \in \mathcal{V}})$ such that (i) $i \leq j$ only if $X_j \subseteq X_i$ (ii) $X_{\bigvee I} = \bigcap_{i \in I} X_i$ (iii) $X_0 = X$. Under mild conditions, this allows us to extend T pointwise by mapping $(X, (X_i)_{i \in \mathcal{V}})$ to $(TX, (TX_i)_{i \in \mathcal{V}})$.

In case of $\mathcal{V} = \mathcal{V}_n$ and $T = \mathcal{P}$, a T-coalgebra is precisely an n-frame, and capture the situation for $\models$:

$$T \circlearrowright \mathsf{Set}_{\mathcal{V}_n} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{MV_n} \circlearrowleft L \tag{3}$$

The adjunction (3) has better properties than (2). In particular, (3) restricts to a dual equivalence on finite structures. This shows that (3) falls into the framework of [7] and allows us to obtain the Goldblatt-Thomason theorems of [9] from the coalgebraic Goldblatt-Thomason theorem of [6]. In particular, this generalises the theorems of [9] to other functors T.

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