

Proceedings of WANCSA 2017  
Normandie University, Le Havre, France - July 4-5, 2017

# EXPLORING CHAOTIC DYNAMICS BY PARTITION OF BIFURCATION DIAGRAM

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**Abstract.** Chaotic dynamical systems have been recently successfully used to replace uniform probability functions in several algorithms in optimization and machine learning. In this work, we propose a study on the use of bifurcation diagrams and first return map in the Rössler system for producing chaotic dynamics. Then, we plan to use these chaotic dynamic for optimization problem. With a bifurcation diagram we can also distinguish the periodic solutions apart from the chaotic solutions. By studying the chaotic solutions, we can then achieve a first return map which is a signature of the dynamical system and thoroughly study the complexity of the latter with a certain set of parameters. As a result, the partition in the bifurcation diagram is provided. From the first return maps, we are able to confirm the complexity of the dynamics in those partitions along with the transitions between them.

**Keywords.** Bifurcation Diagram, Chaotic Dynamics, First Return Map, Rössler System, Uniform Random Function

## 1 Related works

Rössler system was first introduced as a system for studying chemical reactions [4]. The system is defined by three differential equations:

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases} \quad (1)$$

The solutions that are retrieved from the system can be an attractor (chaotic solution) or a periodic solution. The first derivative  $\dot{x}$ ,  $\dot{y}$  and  $\dot{z}$  is the definition of the problem where the  $x$ ,  $y$  and  $z$  are the variables of the system; all variables at  $t = 0$  are called initial conditions. A series of variables  $x$ ,  $y$  and  $z$  from  $t = 0 \rightarrow \infty$  is the solution of the system. Lastly,  $a$ ,  $b$  and  $c$  are parameters to alter the behavior of the system. Regarding the parameters, Sprott and Li [5] proposed a method to generalize these parameters ( $a$ ,  $b$  and  $c$ ) modification with  $\alpha$  as seen in:

$$\begin{cases} a = 0.2 + 0.09\alpha \\ b = 0.2 - 0.06\alpha \\ c = 5.7 - 1.18\alpha \end{cases} \quad (2)$$

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From this parametrization, it has been shown that the system can only generate attractors with the same dynamics, i.e., with no more than three branches Fig. 1 (readers are encouraged to consult [3] for further details).

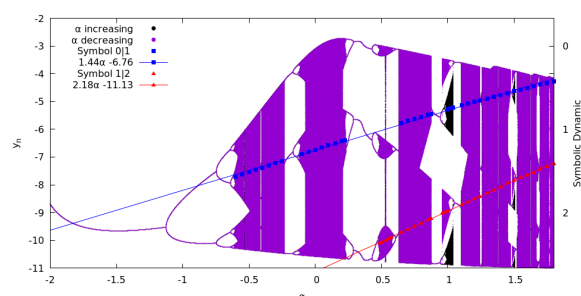


Figure 1: The bifurcation diagram where  $\alpha$  is varied in Eq. (2) for the Rössler system Eq. (1).

The purpose of our work is to obtain more complex chaotic dynamics from this bifurcation diagram by using the banded chaos parts. We expect to improve algorithm performances by introducing these type of chaotic behavior.

## 2 Bifurcation Diagram Partition

A bifurcation diagram can be used as a tool to illustrate the behavior of each state in a chaotic dynamical system when a parameter is varied. This bifurcation diagram can only be obtained after the system discretization. We used the Poincaré section to complete this task.

The bifurcation diagram of the Rössler system with  $\alpha = [-0.5, 1.2]$  is presented in Fig. 2. It can be clearly observed that there are three kinds of areas in the diagram. The first kind is a very sparse area, e.g. in the period of  $\alpha = [0.3, 0.4]$ . These values of  $\alpha$  give periodic solutions. The second kind is a dense area, e.g. the period of  $\alpha = [-0.1, 0.2]$ . These  $\alpha$  yield chaotic solutions. The third kind is a dense area with a gap, e.g. the period of  $\alpha = [1.1, 1.15]$ . The attractor yielded from these  $\alpha$  is called “Banded Chaos”. The characteristic of the banded chaos is more complex than previously mentioned chaotic attractors. Readers who would like to pursue further details should refer to [3].

We compute our first return maps with this equation:

$$\rho_n = (y_n - UB)/(LB - UB) \quad (3)$$

where  $LB$  stands for “Lower Bound” and  $UB$  is for “Upper Bound”. These upper bound and lower bound are based on the bifurcation diagram. As shown in Fig. 2, the bifurcation diagram does not have a linear shape. Therefore, the diagram has been divided into several parts to ensure that we extract the values of  $\rho_n$  in  $[0, 1]$  and retrieve non-equivalent dynamics from the bifurcation diagram. This is because the Rössler dynamics can be considered similar as a whole using templates and subtemplates according to [3].  $LB$  and  $UB$  in Eq. (3) can be found in Eq. (4) and Eq. (5) respectively. In this work, we provide the partitions of  $\alpha = [-0.5, 1.2]$  since the period of  $\alpha < -0.5$  are mostly periodic and not entirely our focus.

$$LB = 0.110626 \times \alpha^2 - 10.585 + 0.4141 \times \alpha \quad (4)$$

$$UB = \begin{cases} 6.04162\alpha - 2.8 & \text{if } \alpha = [-0.5, -0.22] \\ -7.16446\alpha^2 + 3.31662\alpha - 3.05252 & \text{if } \alpha = (-0.22, 0.192] \\ -0.99127\alpha - 2.49 & \text{if } \alpha = (0.192, 0.790] \\ -4.31521\alpha - 4.01651 & \text{if } \alpha = [0.790, 0.876] \\ -0.99127\alpha - 2.49 & \text{if } \alpha = (0.876, 1.102] \\ 23.673\alpha - 34.2435 & \text{if } \alpha = [1.102, 1.155] \\ -0.99127\alpha - 2.49 & \text{if } \alpha = (1.155, 1.2] \end{cases} \quad (5)$$

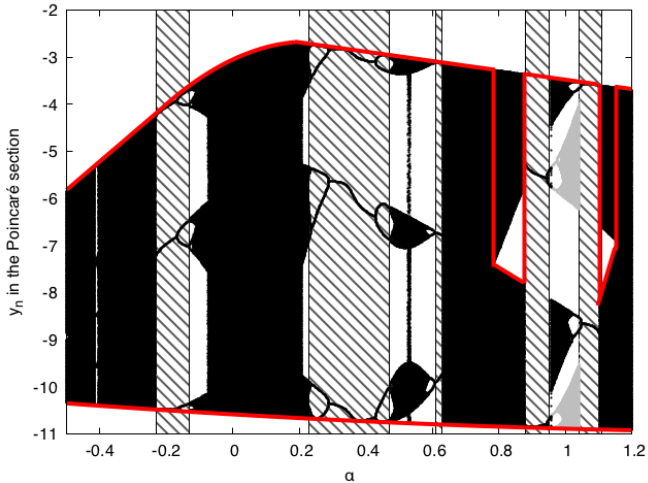


Figure 2: The bifurcation diagram with the upper and lower bounds.

A first return map is a dynamical signature of a system under an influence of a set of parameters. In the case of our work, the parameter is  $\alpha$ . The map is drawn from a plot of  $\rho_n$  against  $\rho_{n+1}$  obtained from the partitions. The first return map does not only help in discerning whether the state of system is periodic or chaotic, but also aids in studying the complexity of the system behavior through the concept of branch in the first return map as described in [3]. The difference can be seen in Fig. 3 and Fig. 4, where our partition is not applied in the former, but the latter for the same value of  $\alpha$ . Thus, a first return map with four branches (more complex) is obtained.

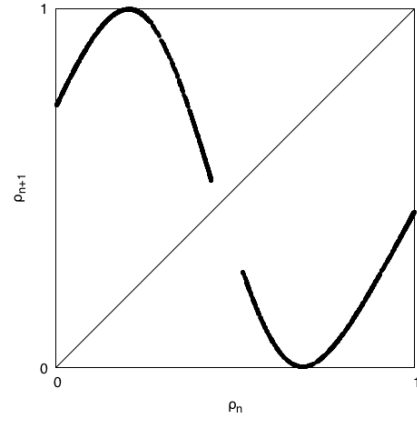


Figure 3: First Return Map of  $\alpha = 1.135$  when the whole bands of “Banded Chaos” is considered.

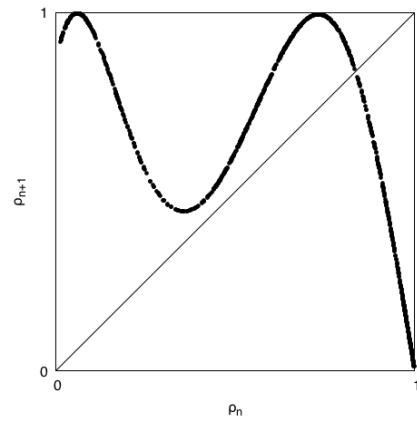


Figure 4: First Return Map of  $\alpha = 1.135$  when only a lower band of “Banded Chaos” is considered.

### 3 Conclusion

In this paper, we provide a novel partition method that can extract high complexity behaviors from a banded chaotic attractor through a bifurcation diagram in the Rössler system. As future work, we aim to deploy the Rössler system in optimization algorithms such those that appeared in [1, 2] and graph traversal algorithm.

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