ON SPECTRAL SYNTHESIS IN VARIETIES CONTAINING THE SOLUTIONS OF INHOMOGENEOUS LINEAR FUNCTIONAL EQUATIONS

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Abstract. As a continuation of our previous work [20] the aim of the recent paper is to investigate the solutions of special inhomogeneous linear functional equations by using spectral synthesis in translation invariant closed linear subspaces of additive/multiadditive functions containing the restrictions of the solutions to finitely generated fields. The idea is based on the fundamental work of [3]. Using spectral analysis in some related varieties we can prove the existence of special solutions (automorphisms) of the functional equation but the spectral synthesis allows us to describe the entire space of solutions on a large class of finitely generated fields. It is spanned by the so-called exponential monomials which can be given in terms of automorphisms of $\mathbb{C}$ and differential operators. We apply the general theory to some inhomogeneous problems motivated by quadrature rules of approximate integration [6], see also [7] and [8].

1. Introduction and preliminaries

Let $\mathbb{C}$ denote the field of complex numbers. We are going to investigate the family of functional equations of type

$$\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y) = c_p \cdot \sum_{l=0}^{p} x^l y^{p-l} \qquad (x, y \in \mathbb{C}), \tag{1}$$

where α_i and β_i $(i = 1, \ldots, n)$ are given real or complex parameters, $p = 1, \ldots, 2n-1$ and $c_p \in \mathbb{C}$ is a constant depending on p. For some

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technical reasons we pay a special attention to the case of $p = 1$, i.e.

$$\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y) = c \cdot (x + y) \qquad (x, y \in \mathbb{C}), \tag{2}$$

where c_1 is rewritten as c for the sake of simplicity. The problem of solving the family of equations (1) as p runs through its possible values $1, \dots, 2n - 1$ is equivalent to the solution of equation

$$F(y) - F(x) = (y - x) \sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y), \tag{3}$$

where $x, y \in \mathbb{C}$ and $f, F : \mathbb{C} \to \mathbb{C}$ are unknown functions. It is motivated by quadrature rules of approximate integration [6], see also [7] and [8].

Remark 1.1. In order to substitute $x = 0$ or $y = 0$ into (1) we agree that $0^0 := 1$. Such a special choice reproduces the pair of equations

$$\sum_{i=1}^{n} a_i f(\alpha_i x) = c_p \cdot x^p \quad \text{and} \quad \sum_{i=1}^{n} a_i f(\beta_i y) = c_p \cdot y^p \qquad (x, y \in \mathbb{C}) \tag{4}$$

that are consequences of (3) for monomial solutions of degree p. For a more detailed survey of the preliminary results see [20].

Let $(G, *)$ be an Abelian group; $\mathbb{C}^G$ denotes the set of complex valued functions defined on G. A function $f : G \to \mathbb{C}$ is a *generalized polynomial*, if there is a non-negative integer p such that

$$\Delta_{g_1} \dots \Delta_{g_{p+1}} f = 0 \tag{5}$$

for any $g_1, \dots, g_{p+1} \in G$. Here Δ_g is the difference operator defined by $\Delta_g f(x) = f(g * x) - f(x)$ $(x \in G)$, where $f \in \mathbb{C}^G$ and $g \in G$. The smallest p for which (5) holds for any $g_1, \dots, g_{p+1} \in G$ is the *degree* of the generalized polynomial f. A function $F : G^p \to \mathbb{C}$ is *p-additive*, if it is additive in each of its variables. A function $f \in \mathbb{C}^G$ is called a *generalized monomial of degree p*, if there is a symmetric p-additive function F such that $f(x) = F(x, \dots, x)$ for any $x \in G$. It is known that any generalized polynomial function can be written as the sum of generalized monomials [13]. By a general result of M. Sablik [12] any solution of (3) is a generalized polynomial of degree at most $2n-1$ under some mild conditions for the parameters in the functional equation:

(1) $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$ or $\mathbb{C}$,
(2) $\alpha_i + \beta_i \neq 0$,

(3)

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{vmatrix} \neq 0, \quad i \neq j, \ i, j \in \{1, \dots, n\}; \tag{6}$$

see also Lemma 2 in [7]. The generalized polynomial solutions of (3) are constituted by the sum of the diagonalizations of p-additive functions satisfying equations of type (1).

In what follows we are going to use spectral synthesis in translation invariant closed linear subspaces of additive functions on some finitely generated fields containing the restrictions of the solutions of functional equation (2). Note that the translation invariance is taken with respect to the multiplicative group structure. We will use spectral synthesis in some related varieties of equation (1) for any $p > 1$ too. The idea is based on [3]. To describe the space of the solutions of the inhomogeneous equation we need a non-zero particular solution in the first step. In some special cases it is enough to use spectral analysis to find such a solution [20]. Otherwise the spectral analysis proves the existence of special solutions (automorphisms) of the functional equations in the homogeneous case; see e.g. [1], [4], [16] and [20]. This means that we have only some necessary conditions for the existence of a nonzero solution of the inhomogeneous problem and we need the application of spectral synthesis in the varieties to give the description of the solution space on a large class of finitely generated fields. It is spanned by the so-called exponential monomials which can be given in terms of automorphisms of $\mathbb{C}$ and differential operators. Unfortunately, the description of all exponential monomials spanning the entire space of the solutions seems to be beyond hope in general; see Example 2 in subsection 4.2. Our results give an explicit and unified technic to solve the problem of finding solutions at all: it is based on the spectral analysis in the first part [20] of the investigations and the present paper completes the solution of the problem by the application of spectral synthesis.

1.1. **Varieties generated by non-trivial solutions of linear functional equations.**

The varieties we are going to investigate have been constructed in our previous work [20] for the application of the so-called spectral analysis. In what follows we summarize the basic steps of the constructions. Let G be an Abelian group. By a *variety* we mean a translation invariant closed linear subspace of $\mathbb{C}^G$.

1.1.1. *Varieties of additive solutions.* Let a finitely generated subfield $K \subset \mathbb{C}$ containing the parameters α_i, β_i $(i = 1, \dots, n)$ be fixed. If V_1 is the set of additive functions on K then it is a closed linear subspace in $\mathbb{C}^K$; for the proof see [3].

Definition 1.2. *Let S_1 be the subset of V_1, where $\tilde{f} \in S_1$ if and only if there exists $\tilde{c} \in \mathbb{C}$ such that*

$$\sum_{i=1}^{n} a_i \tilde{f}(\alpha_i x + \beta_i y) = \tilde{c} \cdot (x + y) \qquad (x, y \in K). \tag{7}$$

By Lemma 2.3 in [20], S_1 is a closed linear subspace of V_1. Let $K^* = \{x \in K : x \neq 0\}$ be the Abelian group with respect to the multiplication in K. We also put $V_1^* = \{f|_{K^*} : f \in V_1\}$ and $S_1^* = \{\tilde{f}|_{K^*} : \tilde{f} \in S_1\}$. By Lemma 2.5 in [20], V_1^* and S_1^* are varieties in $\mathbb{C}^{K^*}$. Recall that the translation invariance is taken with respect to the multiplicative group structure, i.e. if $\tilde{f} \in S_1^*$, then the map $\tau_a \tilde{f} : x \in K^* \mapsto \tilde{f}(ax)$ also belongs to S_1^* for every $a \in K^*$.

Definition 1.3. *S_1^0 is the subspace of S_1 belonging to the homogeneous case $\tilde{c} = 0$.*

1.1.2. *Varieties generated by higher order monomial solutions.* Let a finitely generated subfield $K \subset \mathbb{C}$ containing the parameters α_i, β_i $(i = 1, \dots, n)$ be fixed. If V_p is the set of p-additive functions on K then it is a closed linear subspace in $\mathbb{C}^G$, where $G = K \times \ldots \times K$ is the Cartesian product of K with itself (p-times); for the proof see [3]. For any p-additive function F_p let us define F_p^σ as

$$F_p^\sigma(w_1, \dots, w_p) := F_p(w_{\sigma(1)}, \dots, w_{\sigma(p)}),$$

where σ is a permutation of the elements $1, \dots, p$.

Definition 1.4. *Let S_p be the subset of V_p, where $\tilde{F}_p \in S_p$ if and only if there exists $\tilde{c}_p \in \mathbb{C}$ such that*

$$\begin{aligned}
&\sum_{i=1}^{n} a_i \tilde{F}_p^\sigma(\alpha_i x_1, \dots, \alpha_i x_p) = \tilde{c}_p \cdot x_1 \cdot \ldots \cdot x_p, \\
&\sum_{i=1}^{n} a_i \tilde{F}_p^\sigma(\beta_i y_1, \dots, \beta_i y_p) = \tilde{c}_p \cdot y_1 \cdot \ldots \cdot y_p, \\
&\sum_{i=1}^{n} a_i \binom{p}{l} \tilde{F}_p^\sigma(\alpha_i x_1, \dots, \alpha_i x_l, \beta_i y_1, \dots, \beta_i y_{p-l}) = \tilde{c}_p \cdot x_1 \cdot \ldots \cdot x_l \cdot y_1 \cdot \ldots \cdot y_{p-l}
\end{aligned} \tag{8}$$

$(l = 1, \dots, p-1)$ for any permutation σ of the elements $1, \dots, p$.

S_p is a closed linear subspace of V_p. If the sets V_p^* and S_p^* consist of the restrictions of the elements in V_p and S_p to $G^* := K^* \times \ldots \times K^*$ ($p-$times), respectively then they are varieties in $\mathbb{C}^{G^*}$. Lemma 2.8 in [20] shows that S_p^* is the variety in $\mathbb{C}^{G^*}$ generated by the restriction of symmetric p-additive functions to G^* provided that the diagonalizations are the solutions of functional equation

$$\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y) = \tilde{c}_p \cdot \sum_{l=0}^{p} x^l y^{p-l} \qquad (x, y \in K). \tag{9}$$

for some $\tilde{c}_p \in \mathbb{C}$.

Definition 1.5. *S_p^0 is the subspace of S_p belonging to the homogeneous case $\tilde{c}_p = 0$.*

1.2. **Applications of spectral analysis.** Let $(G, *)$ be an Abelian group. A function $m\colon G \to \mathbb{C}$ is called *exponential* if it is multiplicative: $m(x * y) = m(x)m(y)$ for any $x, y \in G$. If a variety contains an exponential function then we say that *spectral analysis holds in this variety*. If spectral analysis holds in each variety on G, then *spectral analysis holds on G*. The main results of [20] (Theorem 3.3 and Theorem 4.1) are based on the application of spectral analysis in the variety S_p^* $(p = 1, \ldots, 2n-1)$. They can be summarized as follows[1].

Theorem 1.6. *The existence of a nonzero additive solution of* (2) *implies that there exist a finitely generated subfield $K \subset \mathbb{C}$ containing α_i and β_i $(i = 1, \ldots, n)$ and an automorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ as the extension of an exponential element in S_1^* such that*

$$\sum_{i=1}^{n} a_i \phi(\alpha_i x + \beta_i y) = \tilde{c} \cdot (x + y) \quad (x, y \in K) \tag{10}$$

for some $\tilde{c} \in \mathbb{C}$. Especially,

$$\sum_{i=1}^{n} a_i \phi(\alpha_i) = \sum_{i=1}^{n} a_i \phi(\beta_i) = \tilde{c}. \tag{11}$$

If $\tilde{c} = 0$ then

$$\sum_{i=1}^{n} a_i \phi(\alpha_i x + \beta_i y) = 0 \quad (x, y \in \mathbb{C}), \tag{12}$$

[1] Since the variety S_1^* contains the restrictions of additive functions, the exponential property results in an automorphism of $\mathbb{C}$ by an extension process.

i.e. ϕ is the solution of the homogeneous equation on $\mathbb{C}$. If $\tilde{c} \neq 0$ then $\phi(x) = x$ $(x \in K)$ and

$$\sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} a_i \beta_i = \tilde{c} \neq 0. \tag{13}$$

Conversely, if (13) holds then $f := (c/\tilde{c}) \cdot x$ is a nonzero particular additive solution of (2) on $\mathbb{C}$.

The result says that if there are no automorphisms satisfying (11) with $\tilde{c} = 0$, i.e. S_1^0 is trivial[2] for any finitely generated subfield $K \subset \mathbb{C}$ containing the parameters α_i and β_i $(i = 1, \ldots, n)$ then the only nonzero additive solution of (2) must be the proportional of the identity function provided that (13) holds. In what follows we are interested in another possible case: if $\tilde{c} = 0$ for any exponential function in S_1^* then the exponentials give only translation parts in the solution of the inhomogeneous equation on K and we need to apply spectral synthesis in the variety S_1^* to decide the existence of a nonzero particular solution of the inhomogeneous equation on finitely generated fields containing the parameters α_i and β_i $(i = 1, \ldots, n)$. The higher order analogue of Theorem 1.6 formulates the consequences of the application of spectral analysis in S_p^*: the existence of a nonzero monomial solution of degree $p > 1$ of (1) implies that there exist a finitely generated subfield $K \subset \mathbb{C}$ containing α_i and β_i $(i = 1, \ldots, n)$ and some automorphisms $\phi_i \colon \mathbb{C} \to \mathbb{C}$ $(i = 1, \ldots, p)$ such that

$$\sum_{i=1}^{n} a_i \operatorname{diag} \phi(\alpha_i x + \beta_i y) = \tilde{c}_p \cdot \sum_{l=0}^{p} x^l y^{p-l} \qquad (x, y \in K) \tag{14}$$

for some $\tilde{c}_p \in \mathbb{C}$, where the product $\phi = \phi_1 \cdot \ldots \cdot \phi_p$ is an exponential function in S_p^* and diag means the diagonalization of the mappings. Especially

(15)

$$\sum_{i=1}^{n} a_i \phi_1(\alpha_i) \cdot \ldots \cdot \phi_p(\alpha_i) = \sum_{i=1}^{n} a_i \phi_1(\beta_i) \cdot \ldots \cdot \phi_p(\beta_i) =$$
$$\sum_{i=1}^{n} a_i \binom{p}{l} \phi_{\sigma(1)}(\alpha_i) \cdot \ldots \cdot \phi_{\sigma(l)}(\alpha_i) \cdot \phi_{\sigma(l+1)}(\beta_i) \cdot \ldots \cdot \phi_{\sigma(p)}(\beta_i) = \tilde{c}_p,$$

[2]The so-called characteristic polynomial method helps us to investigate such an existence problem in terms of polynomials whose coefficients depend algebraically on the parameters α_i and β_i $(i = 1, \ldots, n)$; [17], see also [15], [18] and [19].

where $l = 1, \dots, p-1$ and σ is an arbitrary permutation of the indices. If $\tilde{c}_p = 0$ then

$$\sum_{i=1}^{n} a_i \operatorname{diag}\phi(\alpha_i x + \beta_i y) = 0 \qquad (x, y \in \mathbb{C}), \tag{16}$$

i.e. $\operatorname{diag}\phi$ is the solution of the homogeneous equation on $\mathbb{C}$. If $\tilde{c}_p \neq 0$ then $\phi_1(x) = \ldots = \phi_p(x) = x$ $(x \in K)$ and

(17)
$$\sum_{i=1}^{n} a_i \alpha_i^p = \sum_{i=1}^{n} a_i \beta_i^p = \sum_{i=1}^{n} a_i \binom{p}{l} \alpha_i^l \beta_i^{p-l} = \tilde{c}_p \neq 0 \quad (l = 1, \dots, p-1).$$

Conversely, if (17) holds then $f(x) := (c/\tilde{c}_p) \cdot x^p$ is a nonzero particular monomial solution of degree p of (1) on $\mathbb{C}$; see Theorem 4.1 in [20]. In what follows we are interested in the application of spectral synthesis in the variety S_p^*. It is spanned by the so-called exponential monomials which can be given in terms of automorphisms of $\mathbb{C}$ and differential operators. The conditions for the exponential monomial solutions are formulated on a finitely generated field $K \subset \mathbb{C}$ containing the parameters α_i and β_i $(i = 1, \dots, n)$. Therefore both the exponential function and the differential operator solutions provide solutions of the functional equation on $\mathbb{C}$ by some (typically transfinite) extension processes. Note that the direct application of the discrete spectral theory (analysis and synthesis) to $\mathbb{C}$ is not possible (see [10, 11]) but any solution on $\mathbb{C}$ can be naturally embedded in the varieties by a simple restriction. Therefore we also restrict our investigations to finitely generated fields to provide the area for the application of spectral theory (analysis and synthesis).

2. Spectral synthesis in the variety containing additive functions

An *exponential monomial* is the product of a generalized polynomial and an exponential function. If a variety $V \subset \mathbb{C}^G$ is spanned by exponential monomials belonging to V then we say that *spectral synthesis holds in the variety* V. If spectral synthesis holds in each variety on G, then *spectral synthesis holds on* G. If spectral synthesis holds in a variety V then spectral analysis holds in V, as well. Lemma 2.2 in [14] states the explicit result as follows.

Lemma 2.1. *Let p be a nonzero generalized polynomial, m is an exponential function on the Abelian group G; if the exponential monomial $p \cdot m$ belongs to the variety $V \subset \mathbb{C}^G$ then $\Delta_h p \cdot m$ also belongs to V;*

especially V contains the exponential function m after finitely many steps of applying the difference operator to the polynomial term.

The following result was proved in [3].

Theorem 2.2. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on K^* consisting of additive functions with respect to addition.*

To give a more precise description of the solutions of functional equation (2) we also need the notion of differential operators.

2.1. **Differential operators on a finitely generated field K.** Suppose that the complex numbers $t_1, \dots, t_n$ are algebraically independent over $\mathbb{Q}$. The elements of the field $\mathbb{Q}(t_1, \dots, t_n)$ are the rational functions of $t_1, \dots, t_n$ with rational coefficients. By a *differential operator on* $\mathbb{Q}(t_1, \dots, t_n)$ we mean an operator of the form

$$D = \sum c_{i_1 \dots i_n} \cdot \frac{\partial^{i_1+\dots+i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}, \tag{18}$$

where $\partial/\partial t_i$ is the usual partial derivative, the sum is finite, the coefficient is a complex number in each term and the exponents $i_1, \dots, i_n$ are nonnegative integers. If $i_1 = \dots = i_n = 0$, then $\partial^{i_1+\dots+i_n}/\partial t_1^{i_1} \cdots \partial t_n^{i_n}$ means the identity operator on $\mathbb{Q}(t_1, \dots, t_n)$. The degree of the differential operator D is the maximum of the numbers $i_1 + \dots + i_n$ such that $c_{i_1 \dots i_n} \neq 0$. It is clear that $\partial/\partial t_i$ is a derivation on $\mathbb{Q}(t_1, \dots, t_n)$ for every $i = 1, \dots, n$, i.e. it is an additive function satisfying the Leibnitz rule. Therefore, any differential operator on $\mathbb{Q}(t_1, \dots, t_n)$ is the complex linear combination of finitely many maps of the form $d_1 \circ \dots \circ d_k$, where $d_1, \dots, d_k$ are derivations on $\mathbb{Q}(t_1, \dots, t_n)$. This observation motivates the following definition.

Definition 2.3. Let K be a subfield of $\mathbb{C}$. We say that the map $D : K \to \mathbb{C}$ is a *differential operator* on K, if D is the complex linear combination of finitely many maps of the form $d_1 \circ \dots \circ d_k$, where $d_1, \dots, d_k$ are derivations on K. If $k = 0$ then $d_1 \circ \dots \circ d_k$ means the identity function on K.

Differential operators in the sense of formula (18) and Definition 2.3 mean the same objects on $K = \mathbb{Q}(t_1, \dots, t_n)$ as the following Proposition shows; see [3].

Proposition 2.4. *Let K be a subfield of $\mathbb{C}$ and suppose that the elements $t_1, \dots, t_n \in K$ are algebraically independent over $\mathbb{Q}$. If D is a* *differential operator on K then the restriction of D to $\mathbb{Q}(t_1, \dots, t_n)$ is of the form* (18).

Definition 2.5. *The action of a field automorphism $\phi \colon \mathbb{C} \to \mathbb{C}$ on the differential operator*

$$D = \sum c_{i_1 \dots i_n} \cdot \frac{\partial^{i_1 + \dots + i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}} \tag{19}$$

is defined as

$$D^\phi = \sum c'_{i_1 \dots i_n} \cdot \frac{\partial^{i_1 + \dots + i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}, \tag{20}$$

where $c'_{i_1 \dots i_n} := \phi(c_{i_1 \dots i_n})$ for any coefficient of D.

Remark 2.6. Note that if $L \subset K \subset \mathbb{C}$ are fields and D is a differential operator on L, then D can be extended to K as a differential operator. This is clear from the fact that every derivation can be extended from L to K. If K is the algebraic extension of L, then the extension of the differential operator is uniquely determined.

3. Applications of spectral synthesis for additive solutions of linear functional equations

Now we are going to apply spectral synthesis to the functional equation

$$\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y) = c \cdot (x + y) \qquad (x, y \in K) \tag{21}$$

and the related variety S_1^*. Recall that $K \subset \mathbb{C}$ is a finitely generated subfield containing the parameters α_i, β_i $(i = 1, \dots, n)$, V_1 is the set of additive functions on K, $S_1 \subset V_1$, where $\tilde{f} \in S_1$ if and only if there exists $\tilde{c} \in \mathbb{C}$ such that

$$\sum_{i=1}^{n} a_i \tilde{f}(\alpha_i x + \beta_i y) = \tilde{c} \cdot (x + y) \qquad (x, y \in K) \tag{22}$$

and S_1^* is the variety containing the restrictions of the elements in S_1 to the multiplicative subgroup K^* of K. Since K is a finitely generated field we can apply Theorem 2.2 to conclude that spectral synthesis holds in S_1^*. Therefore it is spanned by exponential monomials. By Theorem 4.2 in [3] we have the following basic theorem.

Theorem 3.1. *The function $\tilde{f} \in S_1^*$ is an exponential monomial on K^* if and only if $\tilde{f} = \phi \circ D|_{K^*}$, where $\phi \colon \mathbb{C} \to \mathbb{C}$ is the extension of* *an exponential function in* S_1^* *to an automorphism of* $\mathbb{C}$ *and* D *is a differential operator on* K.

In what follows we are going to test the most simple generating elements: if there is a solution of the form $\phi \circ D$ then it is also a polynomial exponential function of the form $p \cdot m$ by Lemma 4.2 in [3]. As Lemma 2.2 in [14] shows, if the generalized polynomial p is not constant, then $\Delta_h p \cdot m$ is also a solution. In the most simple case p is additive and the polynomial exponential function $p \cdot m$ can be written into the form $\phi \circ d$ where $d \colon K \to K$ is a derivation; the details can be found in [5].

Proposition 3.2. *Suppose that*

$$\sum_{i=1}^{n} a_i \tilde{f}(\alpha_i x + \beta_i y) = \tilde{c} \cdot (x+y) \quad (x, y \in K) \tag{23}$$

has a nonzero exponential monomial solution of the form $\phi \circ d$, *where* $\phi \colon \mathbb{C} \to \mathbb{C}$ *is the extension of an exponential function in* S_1^* *to an automorphism of* $\mathbb{C}$ *and* $d : K \to K$ *is a derivation. If* $\tilde{c} \neq 0$ *then* $\phi(x) = x$ $(x \in K)$,

$$\sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} a_i \beta_i = 0 \tag{24}$$

and

$$\sum_{i=1}^{n} a_i d(\alpha_i) = \sum_{i=1}^{n} a_i d(\beta_i) = \tilde{c} \neq 0. \tag{25}$$

Conversely, if $\tilde{c} \neq 0$ *then* (24) *and* (25) *imply that* $\tilde{f} = (c/\tilde{c}) \cdot d$ *is a nonzero particular additive solution of* (21).

Proof. If $y = 0$ then

$$\sum_{i=1}^{n} a_i \tilde{f}(\alpha_i x) = \tilde{c} \cdot x, \tag{26}$$

where $\tilde{f}$ is of the form $\phi \circ d$ and we have that

$$\sum_{i=1}^{n} a_i \phi(d(\alpha_i x)) = \tilde{c} \cdot x, \quad \text{i.e.} \quad \sum_{i=1}^{n} a_i \phi(d(\alpha_i) x + \alpha_i d(x)) = \tilde{c} \cdot x$$

$$\left(\sum_{i=1}^{n} a_i \phi(d(\alpha_i)) \right) \phi(x) + \left(\sum_{i=1}^{n} a_i \phi(\alpha_i) \right) \phi(dx) = \tilde{c} \cdot x. \tag{27}$$

If $x = 1$ then

$$\sum_{i=1}^{n} a_i\phi(d(\alpha_i)) = \tilde{c}, \quad \text{i.e.} \quad \tilde{c}\cdot\phi(x) + \left(\sum_{i=1}^{n} a_i\phi(\alpha_i)\right)\phi(dx) = \tilde{c}\cdot x \tag{28}$$

which means that

$$\left(\sum_{i=1}^{n} a_i\phi(\alpha_i)\right)\phi(dx) = \tilde{c}\cdot(x - \phi(x)). \tag{29}$$

In a similar way,

$$\begin{aligned} \left(\sum_{i=1}^{n} a_i\phi(\alpha_i)\right)\phi(dy) &= \tilde{c}\cdot(y - \phi(y)), \\ \left(\sum_{i=1}^{n} a_i\phi(\alpha_i)\right)\phi(d(xy)) &= \tilde{c}\cdot(xy - \phi(xy)). \end{aligned} \tag{30}$$

Expanding both sides of the second equation in (30), equations (28) - (30) imply that $\phi(x) = x$ in case of $\tilde{c} \neq 0$. On the other hand

$$\sum_{i=1}^{n} a_i\alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} a_i d(\alpha_i) = \tilde{c}$$

in the sense of (28). The corresponding relations for β_i's can be derived in a similar way by substitution $x = 0$. The converse of the statement is trivial. □

Remark 3.3. The proof of the previous theorem shows that the result and its consequences can be formulated by separating the terms containing x and y, respectively.

Corollary 3.4. *If $\tilde{c} \neq 0$ for an exponential monomial $\phi \circ d$ in S_p^* then the space of the additive solutions of equation* (21) *is*

$$\frac{c}{\tilde{c}}\cdot d + S_1^0,$$

where $S_1^0 \subset S_1$ is the subspace belonging to the homogeneous case $\tilde{c} = 0$.

Remark 3.5. If $\tilde{c} \neq 0$ then equation (24) shows that any exponential function in S_1^* solves the homogeneous equation because

$$\sum_{i=1}^{n} a_i\phi(\alpha_i) = \sum_{i=1}^{n} a_i\phi(\beta_i) \neq 0$$

gives a contradiction in the sense of Theorem 1.6. In case of $\tilde{c}=0$

$$
\begin{aligned}
&\sum_{i=1}^{n} a_i\phi(\alpha_i)=\sum_{i=1}^{n} a_i\phi(\beta_i)=0,\\
&\sum_{i=1}^{n} a_i\phi(d(\alpha_i))=\sum_{i=1}^{n} a_i\phi(d(\beta_i))=0,
\end{aligned} \tag{31}
$$

i.e. both the exponential function ϕ and the exponential monomial $\phi\circ d$ are the solutions of the homogeneous equation. It is useful to apply the inverse automorphism on both sides to unify (31) for the application of the characteristic polynomial method:

$$
\begin{aligned}
&\sum_{i=1}^{n} \phi^{-1}(a_i)\alpha_i=\sum_{i=1}^{n} \phi^{-1}(a_i)\beta_i=0,\\
&\sum_{i=1}^{n} \phi^{-1}(a_i)d(\alpha_i)=\sum_{i=1}^{n} \phi^{-1}(a_i)d(\beta_i)=0;
\end{aligned} \tag{32}
$$

[17], see also [15], [18], [19].

Corollary 3.6. *A derivation $d\colon K\to K$ is a solution of* (23) *on K with $\tilde{c}\neq 0$ if and only if*

$$\sum_{i=1}^{n} a_i\alpha_i=\sum_{i=1}^{n} a_i\beta_i=0 \tag{33}$$

and

$$\sum_{i=1}^{n} a_i d(\alpha_i)=\sum_{i=1}^{n} a_i d(\beta_i)=\tilde{c}\neq 0. \tag{34}$$

3.1. **An observation on differential operators and field isomorphisms.** Before testing the generating element $\phi\circ D$ of S_1^* we need the following key lemma. Let $t_1,\dots,t_k$ be an algebraically independent system. For the sake of simplicity let us introduce the following abbreviations:

$$\partial_1=\frac{\partial}{\partial t_1},\dots,\partial_k=\frac{\partial}{\partial t_k}.$$

In the sense of Proposition 2.4 any difference operator on $\mathbb{Q}(t_1,\dots,t_k)$ is of the form

$$D:=\sum_{j_1=0}^{J_1}\dots\sum_{j_k=0}^{J_k} c_{j_1\dots j_k}\partial_1^{j_1}\dots\partial_k^{j_k}$$

and it has a unique extension to the algebraic closure of $\mathbb{Q}(t_1,\dots,t_k)$.

Lemma 3.7. *Suppose that*

$$D = \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} c_{j_1 \dots j_k} \partial_1^{j_1} \dots \partial_k^{j_k}$$

is a differential operator on $\mathbb{Q}(t_1, \dots, t_k)$ with the uniquely determined extension to the field K, where $\mathbb{Q}(t_1, \dots, t_k) \subset K$ and K is contained in the algebraic closure of $\mathbb{Q}(t_1, \dots, t_k)$. If

$$D(x) = c \cdot \phi(x)$$

for any $x \in K$, where c is a nonzero complex number, $\phi \colon \mathbb{C} \to \mathbb{C}$ is an automorphism then $c_{j_1 \dots j_k} = 0$ if $j_1 + \dots + j_m \geq 1$, $c_{0 \dots 0} = c$ and $\phi(x) = x$ $(x \in K)$.

Proof. First we show the statement for $k = 1$. For the sake of simplicity let $t_1 = t$ and $\partial_1 = \partial$. We have that

$$\sum_{j=1}^{J} c_j \partial^j (x) + c_0 x = c \cdot \phi(x). \tag{35}$$

If $x = 1$ then $c_0 = c$. Substituting $x = t^j$, $j = M, \dots, N$:

$$\begin{aligned} \frac{M!}{(M-J)!} \cdot c_J \cdot t^{M-J} + \dots + M \cdot c_1 \cdot t^{M-1} + c \cdot t^M &= c \cdot v^M, \\ \dots &= \dots \\ \frac{N!}{(N-J)!} \cdot c_J \cdot t^{N-J} + \dots + N \cdot c_1 \cdot t^{N-1} + c \cdot t^N &= c \cdot v^N, \end{aligned}$$

where $v = \phi(t)$. Let us divide the equations by $t^M, \dots, t^N$, respectively:

$$\begin{aligned} \frac{M!}{(M-J)!} \cdot c_J \cdot t^{-J} + \dots + M \cdot c_1 \cdot t^{-1} + c &= c \cdot \left(\frac{v}{t}\right)^M, \\ \dots &= \dots \\ \frac{N!}{(N-L)!} \cdot c_J \cdot t^{-J} + \dots + N \cdot c_1 \cdot t^{-1} + c &= c \cdot \left(\frac{v}{t}\right)^N \end{aligned}$$

Using the notation $c_j \cdot t^{-j} = \mu_j$ we can write

$$\begin{aligned} \frac{M!}{(M-J)!} \cdot \mu_J + \dots + M \cdot \mu_1 + c &= c \cdot \left(\frac{v}{t}\right)^M, \\ \dots &= \dots \\ \frac{N!}{(N-J)!} \cdot \mu_J + \dots + N \cdot \mu_1 + c &= c \cdot \left(\frac{v}{t}\right)^N. \end{aligned}$$

The left hand sides of the equations are polynomial expressions in M, ..., N, respectively (the degree of the polynomial is the degree J of the differential operator). The right hand sides of these equations are exponential expressions in M, ..., N, repectively. It follows that c must be zero or $v = t$ and $v/t = 1$. In both cases we get that

$$\begin{bmatrix} \frac{M!}{(M-J)!} & \cdots & M \\ \cdots & \cdots & \cdots \\ \frac{N!}{(N-J)!} & \cdots & N \end{bmatrix} \times \begin{bmatrix} \mu_J \\ \cdots \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}.$$

If $N - M = J$ then it is a quadratic matrix. Using that

$$M(M-1)+M = M^2, \ \ M(M-1)(M-2)+3M(M-1)+M = M^3, \ \ldots$$

we have that its determinant is the same as that of the usual Vandermonde-matrix

$$\begin{bmatrix} M^J & \cdots & M \\ \cdots & \cdots & \cdots \\ N^J & \cdots & N \end{bmatrix}.$$

The nonzero determinant implies that $\mu_j = 0$ for any $j = 1, \ldots, J$, i.e. $c_1 = \ldots = c_J = 0$ and $c_0 = c$ as we have seen above. Using that the extension of D to K is uniquely determined it follows that $D(x) = c \cdot x$ $(x \in K)$ and, consequently, $\phi(x) = x$ $(x \in K)$. We sketch the inductive step for $k = 2$. Let D be a differential operator of the variables t_1 and t_2. If one of the variables, say t_2, is keeping constant then we can repeat the previous procedure by the substitution of $t_1^M \cdot t_2^{J_2}, \ldots, t_1^N \cdot t_2^{J_2}$, where $N - M = J_1$. We have that for any $j_1 = 1, \ldots, J_1$

$$\mu_{j_1} := \sum_{j_2=0}^{J_2} c_{j_1 j_2} \frac{J_2!}{(J_2 - j_2)!} t_2^{J_2 - j_2} \cdot t_1^{-j_1} = 0,$$

i.e.

$$\sum_{j_2=0}^{J_2} c_{j_1 j_2} \frac{J_2!}{(J_2 - j_2)!} t_2^{J_2 - j_2} = 0,$$

where the left hand side is a polynomial expression of the variable t_2. This means that $c_{j_1 j_2} = 0$, where $j_1 = 1, \ldots, J_1$ and $j_2 = 0, \ldots, J_2$. Changing the role of the variables: $c_{j_1 j_2} = 0$, where $j_1 = 0, \ldots, J_1$ and $j_2 = 1, \ldots, J_2$. Therefore $D(x) = c \cdot x$ for any $x \in \mathbb{Q}(t_1, t_2)$. Using that the extension of D to K is uniquely determined it follows that $D(x) = c \cdot x$ $(x \in K)$ and, consequently, $\phi(x) = x$ $(x \in K)$. $\square$

Remark 3.8. Lemma 3.7 is of the same type as the statement in [2] about the linear independency of the iterates of any nonzero real derivation.

In what follows we are going to test the generating elements of the form $\phi \circ D$ by the transcendence degree of the parameters $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ over the rationals.

3.2. **The case of transcendence degree** 0 **(algebraic parameters).** Suppose that the parameters $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ are algebraic numbers over the rationals.Since there is no nontrivial derivation on the field

$$K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$$

, any element $\phi \circ D \in S_1$ reduces to ϕ; see e.g. [5] and [9]. By subsection 1.2 this means that

$$\sum_{i=1}^{n} a_i \phi(\alpha_i) = \sum_{i=1}^{n} a_i \phi(\beta_i) = \tilde{c}$$

and we have two possibilities: if $\tilde{c} \neq 0$ then $\phi(x) = x$ $(x \in K)$ and any solution of (21) must be of the form

$$\frac{c}{\tilde{c}} \cdot x + \sum_j c_j \cdot \phi_j(x) \quad (x \in K) \tag{36}$$

provided that $\sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} a_i \beta_i = \tilde{c} \neq 0$ (Theorem 1.6). The second term in (36) contains the linear combination of automorphisms satisfying

$$\sum_{i=1}^{n} a_i \phi_j(\alpha_i) = \sum_{i=1}^{n} a_i \phi_j(\beta_i) = 0. \tag{37}$$

According to the characteristic polynomial method [17], see also [15], [18], [19] there is only finitely many different ϕ_j's, i.e. the space S_1^0 is finitely generated. If

$$\sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} a_i \beta_i = 0 \tag{38}$$

then both the exponential functions and the differential operators solve the homogeneous equation[3]. Therefore we have no solutions of (21) with $c \neq 0$.

Remark 3.9. Suppose that K is a finitely generated field containing the algebraic parameters $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that K contains at least one transcendental number to provide the existence of

[3]This means that they belong to S_1^0.

nontrivial differential operators. Then we can formulate the following result too: any solution of (21) must be of the form

$$\frac{c}{\tilde{c}} \cdot x + \sum c_j \phi_j \circ D_j(x)$$

provided that $\sum_{i=1}^n a_i\alpha_i = \sum_{i=1}^n a_i\beta_i = \tilde{c} \neq 0$, the automorphism ϕ_j satisfies (37) and D_j is an arbitrary differential operator on K for any $j = 1, \dots, n$. Observe that if the automorphism part ϕ solves the homogeneous equation then $\phi \circ D$ is also a solution of the homogeneous equation for any differential operator D. The case of algebraic parameters is closely related to the theory of spectral analysis because of the trivial action of the differential operators on algebraic elements. The transcendence degree of the embedding (finitely generated) field has no any influence in this sense. In case of (38) we have no solutions of (21) with $c \neq 0$.

3.3. **The case of higher transcendence degree.** Suppose that the transcendence degree of the parameters $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ is k, i.e. we have a field extension $L = \mathbb{Q}(t_1, \dots, t_k)$ such that $t_1, \dots, t_k$ are algebraically independent over the rationals and the algebraic closure of L contains the parameters α_i's and β_i's. Let us introduce the abbreviations

$$\partial_1 = \frac{\partial}{\partial t_1}, \dots, \partial_k = \frac{\partial}{\partial t_k}.$$

By a simple induction we have that

$$\partial^n(xy) = \sum_{k=0}^{n} \binom{n}{k} \partial^k(x)\partial^{n-k}(y) \quad (x, y \in K), \tag{39}$$

where K is a finitely generated field containing the parameters α_i and β_i $(i = 1, \dots, n)$ such that $L \subset K$ and K is contained in the algebraic closure of L. Using formula (39)

$$\partial_1^{j_1} \dots \partial_k^{j_k}(xy) = \partial_1^{j_1} \dots \partial_{k-1}^{j_{k-1}} \left(\sum_{l_k=0}^{j_k} \binom{j_k}{l_k} \partial_k^{j_k-l_k}(x)\partial_k^{l_k}(y) \right) =$$

$$\sum_{l_k=0}^{j_k} \dots \sum_{l_1=0}^{j_1} \binom{j_k}{l_k} \cdot \ldots \cdot \binom{j_1}{l_1} \partial_1^{j_1-l_1} \dots \partial_k^{j_k-l_k}(x)\partial_1^{l_1} \dots \partial_k^{l_k}(y) \quad (x, y \in K).$$

Recall that the action of an automorphism ϕ on a differential operator

$$D := \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} c_{j_1 \dots j_k} \partial_1^{j_1} \dots \partial_k^{j_k}$$

is defined as

$$D^\phi=\sum_{j_1=0}^{J_1}\dots\sum_{j_k=0}^{J_k}c'_{j_1\dots j_k}\partial_1^{j_1}\dots\partial_k^{j_k},$$

where $c'_{i_1\dots i_n}:=\phi(c_{i_1\dots i_n})$ for any coefficient of D; see formula (20). In what follows we frequently need the following family of differential operators generated by D: if $0\le m_1\le J_1$, …, $0\le m_k\le J_k$ then

$$D_{m_1\dots m_k}:=\sum_{j_1=m_1}^{J_1}\dots\sum_{j_k=m_k}^{J_k}c_{j_1\dots j_k}\binom{j_k}{m_k}\cdot\ldots\cdot\binom{j_1}{m_1}\partial_1^{j_1-m_1}\dots\partial_k^{j_k-m_k};$$

especially $D_{0\dots0}=D$. Therefore

$$D^\phi_{m_1\dots m_k}:=\sum_{j_1=m_1}^{J_1}\dots\sum_{j_k=m_k}^{J_k}c'_{j_1\dots j_k}\binom{j_k}{m_k}\cdot\ldots\cdot\binom{j_1}{m_1}\partial_1^{j_1-m_1}\dots\partial_k^{j_k-m_k}.$$

Proposition 3.10. *Suppose that*

$$\sum_{i=1}^n a_i\tilde f(\alpha_i x+\beta_i y)=\tilde c\cdot(x+y)\quad(x,y\in K)\tag{40}$$

has a nonzero exponential monomial solution of the form $\phi\circ D$, where $\phi\colon\mathbb{C}\to\mathbb{C}$ is the extension of an exponential function in S_1^* to an automorphism of $\mathbb{C}$ and

$$D:=\sum_{j_1=0}^{J_1}\dots\sum_{j_k=0}^{J_k}c_{j_1\dots j_k}\partial_1^{j_1}\dots\partial_k^{j_k}\tag{41}$$

is a differential operator on K by its uniquely determined extension to the algebraic closure of L. If $\tilde c\ne0$ then $\phi(x)=x$ $(x\in K)$,

$$\sum_{i=1}^n a_iD^\phi_{m_1\dots m_k}(\alpha_i)=\sum_{i=1}^n a_iD^\phi_{m_1\dots m_k}(\beta_i)=0\tag{42}$$

if $m_1+\ldots+m_k\ge1$ and

$$\sum_{i=1}^n a_iD^\phi(\alpha_i)=\sum_{i=1}^n a_iD^\phi(\beta_i)=\tilde c.\tag{43}$$

Conversely, if $\tilde c\ne0$ then (42) and (43) imply that $f:=(c/\tilde c)\cdot D^\phi$ is a nonzero particular additive solution of (21).

Proof. If $y=0$ then equation (40) reduces to

$$\sum_{i=1}^n a_i\tilde f(\alpha_i x)=\tilde c\cdot x,\tag{44}$$

where, by our assumption, $\tilde{f}$ is of the form $\phi \circ D$,

$$D := \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} c_{j_1\dots j_k} \partial_1^{j_1} \dots \partial_k^{j_k}.$$

Substituting in (44)

$$\sum_{i=1}^{n} a_i \phi(D(\alpha_i x)) = \tilde{c} \cdot x,$$

$$\sum_{i=1}^{n} a_i \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} \phi(c_{j_1\dots j_k}) \phi(\partial_1^{j_1} \dots \partial_k^{j_k}(\alpha_i x)) = \tilde{c} \cdot x$$

$$\sum_{i=1}^{n} a_i \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} \phi(c_{j_1\dots j_k}) \sum_{l_k=0}^{j_k} \dots \sum_{l_1=0}^{j_1} \binom{j_k}{l_k} \cdot \ldots \cdot \binom{j_1}{l_1}$$
$$\phi(\partial_1^{j_1-l_1} \dots \partial_k^{j_k-l_k}(\alpha_i)) \phi(\partial_1^{l_1} \dots \partial_k^{l_k}(x)) = \tilde{c} \cdot x.$$

Applying the inverse automorphism ϕ^{-1} to both sides

$$\sum_{i=1}^{n} \phi^{-1}(a_i) \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} c_{j_1\dots j_k} \sum_{l_k=0}^{j_k} \dots \sum_{l_1=0}^{j_1} \binom{j_k}{l_k} \cdot \ldots \cdot \binom{j_1}{l_1}$$
$$\partial_1^{j_1-l_1} \dots \partial_k^{j_k-l_k}(\alpha_i) \partial_1^{l_1} \dots \partial_k^{l_k}(x) = \phi^{-1}(\tilde{c}) \phi^{-1}(x).$$

If $\tilde{c} \neq 0$ then, by Lemma 3.7,

$$\lambda_{m_1\dots m_k} := \sum_{i=1}^{n} \phi^{-1}(a_i) \sum_{j_1=m_1}^{J_1} \dots \sum_{j_k=m_k}^{J_k} c_{j_1\dots j_k} \binom{j_k}{m_k} \cdot \ldots \cdot \binom{j_1}{m_1}$$
$$\partial_1^{j_1-m_1} \dots \partial_k^{j_k-m_k}(\alpha_i) = 0$$

if $m_1 + \ldots + m_k \geq 1$,

$$\lambda_{0\dots 0} := \sum_{i=1}^{n} \phi^{-1}(a_i) \sum_{j_1=0}^{J_1} \dots \sum_{j_k=0}^{J_k} c_{j_1\dots j_k} \partial_1^{j_1} \dots \partial_k^{j_k}(\alpha_i) = \phi^{-1}(\tilde{c})$$

if $m_1 = \ldots = m_k = 0$ and $\phi^{-1}(x) = x$ $(x \in K)$. Taking the action of ϕ on both sides of the equations it follows that

$$\sum_{i=1}^{n} a_i D^{\phi}_{m_1\dots m_k}(\alpha_i) = 0 \quad (m_1 + \ldots + m_k \geq 1) \quad \text{and} \quad \sum_{i=1}^{n} a_i D^{\phi}(\alpha_i) = \tilde{c}$$

because of $D_{0\dots 0} = D$ and $\phi(x) = x$ for any $x \in K$. Note that the terms of the form $\partial_1^{j_1-m_1} \dots \partial_k^{j_k-m_k}(\alpha_i)$ also belongs to K (see Remark 3.16). The corresponding system of equations for the parameters β_i's can be derived in a similar way by substitution $x = 0$. The converse of the statement is trivial. $\square$

Corollary 3.11. *If $\tilde{c} \neq 0$ for an exponential monomial $\phi \circ D$ in S_p^* then the space of the additive solutions of equation (21) is*

$$\frac{c}{\tilde{c}} \cdot D^{\phi} + S_1^0,$$

where $S_1^0 \subset S_1$ is the subspace belonging to the homogeneous case $\tilde{c} = 0$.

Remark 3.12. Equations (42) and (43) can be considered as linear systems of equations for the unknown quantities $c'_{0\ldots0}$, $\ldots$, $c'_{j_1\ldots j_k}$, $\ldots$, $c'_{J_1\ldots J_k}$ in the expression of the solution D^{Φ}. Following the lexicographic ordering we have upper triangle matrices from the definition of $D_{m_1\ldots m_k}$. The coefficients of $c'_{m_1\ldots m_k}$ in equations (42) and (43) are $\displaystyle\sum_{i=1}^{n} a_i\alpha_i$ and $\displaystyle\sum_{i=1}^{n} a_i\beta_i$, respectively. Suppose (for example) that the matrix containing α_i's is regular, i.e. $\displaystyle\sum_{i=1}^{n} a_i\alpha_i \neq 0$. For an upper triangle fundamental matrix Cramer's rule says that

$$c'_{0\ldots0} = \tilde{c} \cdot \frac{\left(\sum_{i=1}^{n} a_i\alpha_i\right)^{N-1}}{\left(\sum_{i=1}^{n} a_i\alpha_i\right)^{N}} = \frac{\tilde{c}}{\sum_{i=1}^{n} a_i\alpha_i},$$

where $N = J_1 \cdot \ldots \cdot J_k$ and $c_{j_1\ldots j_k} = 0$ if $j_1 + \ldots + j_k > 0$. Therefore we have two possible cases:

$$\sum_{i=1}^{n} a_i\alpha_i = \sum_{i=1}^{n} a_i\beta_i \neq 0$$

and D^{ϕ} reduces to the proportional of the identity:

$$D^{\phi}(x) = c' \cdot x \quad (x \in K), \quad \text{where} \quad c' = \frac{\tilde{c}}{\sum_{i=1}^{n} a_i\alpha_i} = \frac{\tilde{c}}{\sum_{i=1}^{n} a_i\beta_i}.$$

Otherwise

$$\sum_{i=1}^{n} a_i\alpha_i = \sum_{i=1}^{n} a_i\beta_i = 0$$

and we can reduce the order of the upper triangle matrices of systems (42) and (43): $c'_{0\ldots0}$ can be arbitrarily choosen and we can delete the first column and the last row to give a reduced system of linear equations for the unknown quantities $c'_{0\ldots 01}$, ..., $c'_{j_1\ldots j_k}$, ..., $c'_{J_1\ldots J_k}$. This means that if D is an at least first order differential operator then

$$\sum_{i=1}^{n} a_i\alpha_i = \sum_{i=1}^{n} a_i\beta_i = 0$$

implies that any exponential function ϕ must be in S_1^0, i.e. ϕ is the solution of the homogeneous equation on K. By contraposition, if there is an exponential function in S_1^* with $\tilde{c} \neq 0$ then it must be the proportional of the identity function on K together with the exponential monomial $\phi \circ D$, i.e. D is a differential operator of degree zero.

According to the technical difficulties of the discussion of the matrix forms of (42) and (43) we omit the details in general. The case of transcendence degree 1 can be entirely solved by the technic of linear systems of equations as we shall see in subsection 3.3.1; see also subsection 5.2.1. In case of the higher transcendence degree we omit the further theoretical computations but we present how the method is working in explicit cases; section 4.

Remark 3.13. If $\tilde{c} = 0$ then

$$\sum_{i=1}^{n} \phi^{-1}(a_i) D_{m_1\ldots m_k}(\alpha_i) = \sum_{i=1}^{n} \phi^{-1}(a_i) D_{m_1\ldots m_k}(\beta_i) = 0 \tag{45}$$

for any $0 \leq m_i \leq J_i$ $(i = 1, \ldots, k)$. These equations can be also considered as linear systems of equations for the unknown quantities $c_{0\ldots 0}$, ..., $c_{j_1\ldots j_k}$, ..., $c_{J_1\ldots J_k}$; cf. Remark 3.12. The diagonal elements of the fundamental matrices are $\sum_{i=1}^{n} \phi^{-1}(a_i)\alpha_i$ and $\sum_{i=1}^{n} \phi^{-1}(a_i)\beta_i$, respectively. They have zero determinant because of $\sum_{i=1}^{n} a_i\phi(\alpha_i) = \sum_{i=1}^{n} a_i\phi(\beta_i) = 0$, where ϕ is the extension of an exponential function in S_1^{0*} to an automorphism of $\mathbb{C}$; cf. equation (32).

Unfortunately the action of ϕ is hard to compute in general. We need to apply the characteristic polynomial method to (45). This results in a system of polynomial equations. The transcendence degree l of a_i's over the rational gives the number of the variables of the polynomials. Their coefficients depend on

- the particular actions
$$\partial_1^{j_1-m_1} \ldots \partial_k^{j_k-m_k}(\alpha_i) \quad \text{or} \quad \partial_1^{j_1-m_1} \ldots \partial_k^{j_k-m_k}(\beta_i)$$

of the differential operator (see Remark 3.16),

- the coefficients of the defining polynomial of an algebraic elements u over $\mathbb{Q}(a_1, \ldots, a_l)$, where $a_1, \ldots, a_l$ form a maximal algebraically independent system and u is choosen such that the missing outer parameters belong to the simple algebraic extension $\mathbb{Q}(a_1, \ldots, a_l)(u)$,
- the (rational) coefficients of the polynomials p_j and q_j, where
$$a_j = p_j(a_1, \ldots, a_k)/q_j(a_1, \ldots, a_k) \quad j = l+1, \ldots, n.$$

For the details we can refer to [17], see also [15], [18], [19]. Practically we simultaneously determine the actions $w_1 := \phi^{-1}(a_1)$, ..., $w_l := \phi^{-1}(a_l)$ and the quantities $c_{0\ldots0}$, ..., $c_{j_1\ldots j_k}$, ..., $c_{J_1\ldots J_k}$ as the solutions of a system of (multivariate) polynomial equations. They constitute an exponential monomial in S_1^0 by the composition $\phi \circ D$.

Remark 3.14. The proof of the previous theorem shows that the result and its consequences can be formulated by separating the terms containing x and y, respectively.

Corollary 3.15. *A differential operator*
$$D = \sum_{j_1=0}^{J_1} \ldots \sum_{j_k=0}^{J_k} c_{j_1\ldots j_k} \partial_1^{j_1} \ldots \partial_k^{j_k}$$
is a solution of (40) *on* K *with* $\tilde{c} \neq 0$ *if and only if*
$$\sum_{i=1}^{n} a_i D_{m_1\ldots m_k}(\alpha_i) = \sum_{i=1}^{n} a_i D_{m_1\ldots m_k}(\beta_i) = 0 \tag{46}$$
if $m_1 + \ldots + m_k \geq 1$ *and*
$$\sum_{i=1}^{n} a_i D(\alpha_i) = \sum_{i=1}^{n} a_i D(\beta_i) = \tilde{c} \neq 0. \tag{47}$$

Remark 3.16. To compute the particular action of a differential operator on an algebraic element u over $L = \mathbb{Q}(t_1, \ldots, t_k)$ we need its defining polynomial
$$u^m + r_{m-1}u^{m-1} + \ldots r_1 u + r_0 = 0.$$
Then
$$(mu^{m-1} + (m-1)r_{m-1}u^{m-2} + ... + r_1)\partial_i(u) + \sum_{j=0}^{m-1} (\partial_i r_j)u^j = 0,$$
where $r_j = p_j(t_1, \ldots, t_k)/q_j(t_1, \ldots, t_k)$ is a rational fraction and $\partial_i r_j$ means the usual partial differentiation. Observe that $\partial_i(u)$ is also an

algebraic number over $L=\mathbb{Q}(t_1,\dots,t_k)$ because the algebraic numbers form a field and $m=\deg u$ implies that

$$mu^{m-1}+(m-1)r_{m-1}u^{m-2}+...+r_1\neq 0.$$

Moreover $\partial_i(u)\in\mathbb{Q}(t_1,\dots,t_k)(u)$ and the process can be repeated to compute the action $\partial_1^{j_1}\dots\partial_k^{j_k}(u)$ of the higher order term of the differential operator.

3.3.1. *The case of transcendence degree* 1. This special case gives the best result from the viewpoint of the application of the general theory. It is due to the relatively simple matrix form of the equations and the uniquely determined main terms in the differential operators. Suppose that the transcendence degree of the parameters α_1, ..., α_n, β_1, ..., β_n is 1, i.e. we have a field extension $L=\mathbb{Q}(t)$ for some transcendental number over the rationals such that its algebraic closure contains the parameters α_i's and β_i's. In case of $k=1$ systems (42) and (43) can be written in a more detailed form (cf. Remark 3.12). For example

(48)
$$c_0'\sum_{i=1}^n a_i\alpha_i+c_1'\sum_{i=1}^n a_i\partial(\alpha_i)+c_2'\sum_{i=1}^n a_i\partial^2(\alpha_i)+\ldots+c_J'\sum_{i=1}^n a_i\partial^J(\alpha_i)=\tilde c$$
$$c_1'\sum_{i=1}^n a_i\alpha_i+\binom{2}{1}c_2'\sum_{i=1}^n a_i\partial(\alpha_i)+\ldots+c_J'\binom{J}{1}\sum_{i=1}^n a_i\partial^{J-1}(\alpha_i)=0$$
$$\dots$$
$$c_{J-1}'\sum_{i=1}^n a_i\alpha_i+\binom{J}{J-1}c_J'\sum_{i=1}^n a_i\partial(\alpha_i)=0$$
$$c_J'\sum_{i=1}^n a_i\alpha_i=0.$$

The corresponding system of equations containing the parameters β_i's is of the same form. Since we have upper triangle matrices the following result can be easily concluded[4].

Proposition 3.17. *Suppose that*

$$\sum_{i=1}^n a_i\tilde f(\alpha_i x+\beta_i y)=\tilde c\cdot(x+y)\quad (x,y\in K) \tag{49}$$

[4]Recall that K is a finitely generated field containing the parameters α_i's and β_i's such that $L\subset K$ and K is contained in the algebraic closure of L.

has a nonzero exponential monomial solution of the form $\phi \circ D$ *on* K*, where* $\phi\colon \mathbb{C} \to \mathbb{C}$ *is the extension of an exponential function in* S_1^* *to an automorphism of* $\mathbb{C}$ *and*

$$D = \sum_{j=0}^{J} c_j \partial^j, \quad c_J \neq 0 \tag{50}$$

is a differential operator on K *by its uniquely determined extension to the algebraic closure of* L*. If* $\tilde{c} \neq 0$ *then* $\phi(x) = x$ $(x \in K)$*,*

$$\begin{aligned} &\sum_{i=1}^{n} a_i \alpha_i = 0,\ \sum_{i=1}^{n} a_i \partial(\alpha_i) = 0,\ \sum_{i=1}^{n} a_i \partial^2(\alpha_i) = 0,\ \dots,\ \sum_{i=1}^{n} a_i \partial^{J-1}(\alpha_i) = 0,\\ &\sum_{i=1}^{n} a_i \beta_i = 0,\ \sum_{i=1}^{n} a_i \partial(\beta_i) = 0,\ \sum_{i=1}^{n} a_i \partial^2(\beta_i) = 0,\ \dots,\ \sum_{i=1}^{n} a_i \partial^{J-1}(\beta_i) = 0, \end{aligned} \tag{51}$$

i.e. the coefficients $c_0, \dots, c_{J-1}$ *can be arbitrarily choosen and*

$$\sum_{i=1}^{n} a_i \partial^J(\alpha_i) = \sum_{i=1}^{n} a_i \partial^J(\beta_i) \neq 0, \quad c'_J = \frac{\tilde{c}}{\sum_{i=1}^{n} a_i \partial^J(\alpha_i)} = \frac{\tilde{c}}{\sum_{i=1}^{n} a_i \partial^J(\beta_i)}, \tag{52}$$

where $c'_J = \phi(c_J)$*. Conversely, if* $\tilde{c} \neq 0$ *then* (51) *and* (52) *imply that* $f := (c/\tilde{c}) \cdot D^\phi$ *is a nonzero particular additive solution of* (21)*, where*

$$D^\phi = \sum_{j=0}^{J} c'_j \partial^j, \quad c'_J \neq 0,$$

the coefficients $c'_0, \dots, c'_{J-1}$ *can be arbitrarily choosen and* c'_J *is determined by* (52)*.*

Remark 3.18. If $\tilde{c} = 0$ then

$$\sum_{i=1}^{n} \phi^{-1}(a_i) D_m(\alpha_i) = \sum_{i=1}^{n} \phi^{-1}(a_i) D_m(\beta_i) = 0 \tag{53}$$

for any $0 \leq m \leq J$. The analogue result of (32) up to order J of the differential operator can be easily concluded by using (48).

Corollary 3.19. *A differential operator*

$$D = \sum_{j=0}^{J} c_j \partial^j, \quad c_J \neq 0$$

is a solution of (49) *with* $\tilde{c} \neq 0$ *if and only if* (51) *and* (52) *are satisfied with* $c_J = c'_J$*.*

Remark 3.20. Note that the existence of a uniquely determined main term of the differential operator is an essential difference relative to the case of higher transcendence degree; see e.g. subsection 4.2.

4. Examples

The following examples illustrate how the results are working in explicit cases. For the sake of simplicity we use their separated version in the sense of Remark 3.3 and Remark 3.14. In case of transcendence degree 1 we can describe the entire space of solutions (Example 1). Otherwise the situation becomes more difficult because of the missing main term of the differential operators (Example 2).

4.1. **Example 1.** Let $c \neq 0$ be a complex number and consider functional equation

$$t^3 f(tx) - t^2 f(t^2 x) - t f(t^3 x) + f(t^4 x) = c \cdot x, \tag{54}$$

i.e. $n = 4$ and $a_1 = t^3$, $a_2 = -t^2$, $a_3 = -t$, $a_4 = 1$, $\alpha_1 = t$, $\alpha_2 = t^2$, $\alpha_3 = t^3$, $\alpha_4 = t^4$, where t is a transcendental number over the rationals, $L = \mathbb{Q}(t)$ and $L \subset K$ such that K is a finitely generated field and it is contained in the algebraic closure of L. Using that

$$\begin{aligned} &\sum_{i=1}^{4} a_i \alpha_i = t^3 \cdot t - t^2 \cdot t^2 - t \cdot t^3 + t^4 = 0 \\ &\sum_{i=1}^{4} a_i \partial(\alpha_i) = t^3 - t^2 \cdot (2t) - t \cdot (3t^2) + 4t^3 = 0 \end{aligned} \tag{55}$$

but

$$\sum_{i=1}^{4} a_i \partial^2(\alpha_i) = -2t^2 - 6t^2 + 12t^2 = 4t^2 \neq 0 \tag{56}$$

we have, by Corollary 3.19, that the differential operator

$$D = c_0 + c_1 \partial + \frac{c}{4t^2} \partial^2 \quad (c_0, c_1 \in \mathbb{C})$$

is a non-zero particular solution of equation (54) and the space of the solutions on K is $D + S_1^0$. Equation $\sum_{i=1}^{4} a_i \alpha_i = 0$ implies that any exponential function in S_1 belongs to S_1^0, i.e.

$$t^3 \phi(t) - t^2 \phi(t^2) - t\phi(t^3) + \phi(t^4) = 0.$$

Substituting $s = \phi(t)$ we have

$$t^3 \cdot s - t^2 \cdot s^2 - t \cdot s^3 + s^4 = 0 \quad \Rightarrow \quad s \cdot (s-t) \cdot (s^2 - t^2) = 0.$$

Therefore $\phi(t) = t$ or $\phi(t) = -t$. To find the generating elements of the form $\phi \circ D$ in S_1^0 we need to use system (53):

- If $\phi(t) = t$ then $\phi^{-1}(a_i) = a_i$ $(i = 1, \ldots, 4)$ and we have

$$\begin{aligned}
&c_0 \sum_{i=1}^4 a_i \alpha_i + c_1 \sum_{i=1}^4 a_i \partial(\alpha_i) + c_2 \sum_{i=1}^4 a_i \partial^2(\alpha_i) + \ldots + c_J \sum_{i=1}^4 a_i \partial^J(\alpha_i) = 0 \\
&c_1 \sum_{i=1}^4 a_i \alpha_i + \binom{2}{1} c_2 \sum_{i=1}^4 a_i \partial(\alpha_i) + \ldots + c_J \binom{J}{1} \sum_{i=1}^4 a_i \partial^{J-1}(\alpha_i) = 0 \\
&\ldots \\
&c_{J-1} \sum_{i=1}^4 a_i \alpha_i + \binom{J}{J-1} c_J \sum_{i=1}^4 a_i \partial(\alpha_i) = 0 \\
&c_J \sum_{i=1}^4 a_i \alpha_i = 0.
\end{aligned} \tag{57}$$

By equations (55) and (56) it follows that $c_2 = \ldots = c_J = 0$, i.e. $D = c_0 + c_1 \partial$ $(c_0, c_1 \in \mathbb{C})$.

- If $\phi(t) = -t$ then $\phi^{-1}(a_1) = -a_1$, $\phi^{-1}(a_2) = a_2$, $\phi^{-1}(a_3) = -a_3$ and $\phi^{-1}(a_4) = a_4$. System (53) has vanishing diagonal elements because of

$$\sum_{i=1}^4 \phi^{-1}(a_i)\alpha_i = -t^3 \cdot t - t^2 \cdot t^2 + t \cdot t^3 + t^4 = 0$$

but

$$\sum_{i=1}^4 \phi^{-1}(a_i)\partial(\alpha_i) = -t^3 - t^2 \cdot (2t) + t \cdot (3t^2) + 4t^3 = 4t^3 \neq 0.$$

Therefore $c_1 = \ldots = c_J = 0$ and D reduces to the proportional of the identity function on K.

This means that S_1^0 is spanned by the extensions of ϕ_1, ϕ_2 and $\phi_1 \circ D$, where $\phi_1(t) = t$, $\phi_2(t) = -t$ and $D(x) = c_0 x + c_1 \partial(x)$ $(c_0, c_1 \in \mathbb{C}$ and $x \in L)$.

4.2. **Example 2.** Let $c \neq 0$ be a complex number and consider functional equation

$$(t_1^3 + t_2^3) f(x) - (t_1^2 + t_2^2) f((t_1 + t_2)x) + t_2 f(t_1^2 x) + t_1 f(t_2^2 x) = c \cdot x, \tag{58}$$

i.e. $n = 4$ and $a_1 = t_1^3 + t_2^3$, $a_2 = -(t_1^2 + t_2^2)$, $a_3 = t_2$, $a_4 = t_1$, $\alpha_1 = 1$, $\alpha_2 = t_1 + t_2$, $\alpha_3 = t_1^2$, $\alpha_4 = t_2^2$, where t_1 and t_2 are algebraically independent numbers over the rationals, $L = \mathbb{Q}(t_1, t_2)$ and $L \subset K$ such that K is finitely generated and it is contained in the algebraic closure of L. First of all compute the possible coefficients of the corresponding linear system of equations:

$$\begin{aligned}
&\sum_{i=1}^{4} a_i\alpha_i = 0,\\
&\sum_{i=1}^{4} a_i\partial_1(\alpha_i) = \sum_{i=1}^{4} a_i\partial_2(\alpha_i) = -(t_1 - t_2)^2, \qquad (59)\\
&\sum_{i=1}^{4} a_i\partial_1\partial_2(\alpha_i) = 0, \quad \sum_{i=1}^{4} a_i\partial_1^2(\alpha_i) = 2t_2, \quad \sum_{i=1}^{4} a_i\partial_2^2(\alpha_i) = 2t_1;
\end{aligned}$$

note that any higher order derivative must be zero because α_i's are at most second order polynomials of the variables t_1 and t_2. We are going to use Corollary 3.15 to compute differential operator solutions

$$D = \sum_{j_1=0}^{J_1}\sum_{j_2=0}^{J_2} c_{j_1 j_2}\partial_1^{j_1}\partial_2^{j_2}$$

of degree one, two and three.

- If $J_1 = J_2 = 1$ then we have four equations for the coefficients of D.

Using (59)

$$\begin{aligned}
&\sum_{i=1}^{4} a_i D_{11}(\alpha_i) = 0, \qquad \sum_{i=1}^{4} a_i D_{01}(\alpha_i) = 0,\\
&\sum_{i=1}^{4} a_i D_{10}(\alpha_i) = -c_{11}(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{11} = 0,\\
&\sum_{i=1}^{4} a_i D_{00}(\alpha_i) = -(c_{10} + c_{01})(t_1 - t_2)^2 \neq 0
\end{aligned}$$

and a non-zero particular solution is $(c/\tilde{c}) \cdot D$, where

$$D = c_{00} + c_{10}\partial_1 + c_{01}\partial_2,$$

c_{00}, c_{10} and $c_{01} \in \mathbb{C}$ such that $\tilde{c} := -(c_{10} + c_{01})(t_1 - t_2)^2 \neq 0$.

- If $J_1 = J_2 = 2$ then we have nine equations for the coefficients of D. Using (59)

$$\sum_{i=1}^{4} a_i D_{22}(\alpha_i) = 0,$$
$$\sum_{i=1}^{4} a_i D_{21}(\alpha_i) = -2c_{22}(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{22} = 0$$
$$\sum_{i=1}^{4} a_i D_{20}(\alpha_i) = -c_{21}(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{21} = 0,$$
$$\sum_{i=1}^{4} a_i D_{12}(\alpha_i) = 0,$$
$$\sum_{i=1}^{4} a_i D_{11}(\alpha_i) = -2c_{12}(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{12} = 0,$$
$$\sum_{i=1}^{4} a_i D_{10}(\alpha_i) = -\left(2c_{20} + c_{11}\right)(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{11} = -2c_{20},$$
$$\sum_{i=1}^{4} a_i D_{02}(\alpha_i) = 0,$$
$$\sum_{i=1}^{4} a_i D_{01}(\alpha_i) = -\left(2c_{02} + c_{11}\right)(t_1 - t_2)^2 = 0 \quad \Rightarrow \quad c_{11} = -2c_{02},$$
$$\sum_{i=1}^{4} a_i D_{00}(\alpha_i) = -(c_{10} + c_{01})(t_1 - t_2)^2 - c_{11}(t_1 + t_2) \neq 0$$

and a non-zero particular solution is $(c/\tilde{c}) \cdot D$, where

$$D = c_{00} + c_{10}\partial_1 + c_{01}\partial_2 - \frac{c_{11}}{2}\left(\partial_1^2 - 2\partial_1\partial_2 + \partial_2^2\right),$$

where c_{00}, c_{10}, c_{01} and $c_{11} \in \mathbb{C}$ such that

$$\tilde{c} := -(c_{10} + c_{01})(t_1 - t_2)^2 - c_{11}(t_1 + t_2) \neq 0.$$

- If $J_1 = J_2 = 3$ then we have sixteen equations for the coefficients of D.

Using (59)

$$
\begin{aligned}
&\sum_{i=1}^{4} a_i D_{33}(\alpha_i) = 0,\\
&\sum_{i=1}^{4} a_i D_{32}(\alpha_i) = -3c_{33}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{33} = 0\\
&\sum_{i=1}^{4} a_i D_{31}(\alpha_i) = -2c_{32}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{32} = 0,\\
&\sum_{i=1}^{4} a_i D_{30}(\alpha_i) = -c_{31}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{31} = 0,\\
&\sum_{i=1}^{4} a_i D_{23}(\alpha_i) = 0,\\
&\sum_{i=1}^{4} a_i D_{22}(\alpha_i) = -3c_{23}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{23} = 0,\\
&\sum_{i=1}^{4} a_i D_{21}(\alpha_i) = -2c_{22}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{22} = 0,\\
&\sum_{i=1}^{4} a_i D_{20}(\alpha_i) = -(c_{21}+3c_{30}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{21} = -3c_{30},\\
&\sum_{i=1}^{4} a_i D_{13}(\alpha_i) = 0,\\
&\sum_{i=1}^{4} a_i D_{12}(\alpha_i) = -3c_{13}(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{13} = 0\\
&\sum_{i=1}^{4} a_i D_{11}(\alpha_i) = -2(c_{12}+c_{21})(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{12} = -c_{21},\\
&\sum_{i=1}^{4} a_i D_{10}(\alpha_i) = -(c_{11}+2c_{20})(t_1-t_2)^2 + 2t_1c_{12} + 6t_2c_{30} = 0 \quad \Rightarrow\\
&(c_{11}+2c_{20})(t_1-t_2)^2 = 2t_1c_{12} + 6t_2c_{30},
\end{aligned}
$$

$$\sum_{i=1}^{4} a_i D_{03}(\alpha_i) = 0,$$

$$\sum_{i=1}^{4} a_i D_{02}(\alpha_i) = -(c_{12}+3c_{03})(t_1-t_2)^2 = 0 \quad \Rightarrow \quad c_{12} = -3c_{03},$$

$$\sum_{i=1}^{4} a_i D_{01}(\alpha_i) = -(c_{11}+2c_{02})(t_1-t_2)^2 + 2t_2c_{21} + 6t_1c_{03} = 0 \quad \Rightarrow$$

$$(c_{11}+2c_{02})(t_1-t_2)^2 = 2t_2c_{21} + 6t_1c_{03},$$

$$\sum_{i=1}^{4} a_i D_{00}(\alpha_i) = -(c_{01}+c_{10})(t_1-t_2)^2 + 2t_1c_{02} + 2t_2c_{20} \neq 0$$

and a non-zero particular solution is $(c/\tilde{c}) \cdot D$, where

$$D = c_{00}+c_{10}\partial_1+c_{01}\partial_2-\frac{c_{11}}{2}\left(\partial_1^2 - 2\partial_1\partial_2 + \partial_2^2\right)+\frac{t_1+t_2}{(t_1-t_2)^2}c_{12}\left(\partial_1^2-\partial_2^2\right)+ \frac{c_{12}}{3}\left(\partial_1^3+3\partial_1\partial_2^2-3\partial_2\partial_1^2-\partial_2^3\right),$$

$c_{00}, c_{10}, c_{01}, c_{11}$ and $c_{12} \in \mathbb{C}$ such that

$$\tilde{c} := -(c_{10}+c_{01})(t_1-t_2)^2 - c_{11}(t_1+t_2) - 2\frac{t_1+t_2}{t_1-t_2}c_{12} \neq 0.$$

The space of the solutions on K is $D + S_1^0$. Equation $\sum_{i=1}^{4} a_i\alpha_i = 0$ implies that any exponential function in S_1 belongs to S_1^0, i.e.

$$(t_1^3+t_2^3) - (t_1^2+t_2^2)(s_1+s_2) + t_2 \cdot s_1^2 + t_1 \cdot s_2^2 = 0, \tag{60}$$

where $\phi(t_1) = s_1$ and $\phi(t_2) = s_2$ are algebraically independent over the rationals. Since the coefficient t_2/t_1 of the normalized characteristic polynomial

$$p(x,y) = \frac{t_1^3+t_2^3}{t_1} - \frac{t_1^2+t_2^2}{t_1}(x+y) + \frac{t_2}{t_1} \cdot x^2 + y^2$$

is transcendent it has algebraically independent roots; see [18] and [19]. Especially $\phi(t_1) = t_1$ and $\phi(t_2) = t_2$ are solutions of equation (60). To find the generating elements of the form $\phi \circ D$ in S_1^0 we need to use (45):

$$\sum_{i=1}^{4} \phi^{-1}(a_i) D_{m_1m_2}(\alpha_i) = 0 \quad (0 \le m_1 \le J_1, 0 \le m_2 \le J_2).$$

If $\phi(t_1) = t_1$ and $\phi(t_2) = t_2$ then $\phi^{-1}(a_i) = a_i$ $(i = 1, \dots, 4)$. Therefore $\phi \circ D \in S_1^0$, where D is one of the differential operators above provided that the surviving coefficients is choosen such that $\tilde{c} = 0$. For example, a differential operator of degree one in S_1^0 is

$$D = c_{00} + c_{10}\partial_1 + c_{01}\partial_2, \quad \text{where} \quad c_{10} + c_{01} = 0.$$

5. Spectral synthesis in the variety containing monomial solutions of higher degree

In what follows we are going to give a survey of the higher order version of section 3. By Theorem 6.3 in [3] the space S_p is spanned by the functions of the form

$$(x_1, \ldots, x_p) \mapsto \phi_1 \circ D_1(x_1) \cdot \ldots \cdot \phi_p \circ D_p(x_p),$$

where $\phi := \phi_1 \cdot \ldots \cdot \phi_p$ is an exponential element in S_p and $D_1, \ldots, D_p$ are differential operators on K, where K is a finitely generated field containing the parameters α_i and β_i $(i = 1, \ldots, n)$.

5.1. **The case of transcendence degree 0 (algebraic parameters).** If the parameters $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are algebraic numbers then we have no non-trivial differential operators on $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$; cf. subsection 3.2. The analogue results can be easily formulated for $p > 1$ by using the higher order version of Theorem 1.6; see also section 4 in [20].

5.2. **The case of higher transcendence degree.** Suppose that the transcendence degree of the parameters $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ is k, i.e. we have a field extension $L = \mathbb{Q}(t_1, \ldots, t_k)$ such that $t_1, \ldots, t_k$ are algebraically independent over the rationals, $L \subset K$ and K is contained in the algebraic closure of L; especially it contains the parameters α_i's and β_i's.

Lemma 5.1. *Suppose that $\phi(x) = x$ $(x \in K)$; then*

$$\phi \circ D(\alpha_i x) = \sum_{m_1=0}^{J_1} \ldots \sum_{m_k=0}^{J_k} D^{\phi}_{m_1 \ldots m_k}(\alpha_i)\partial_1^{m_1} \ldots \partial_k^{m_k}(x)$$

for any $x \in K$.

The proof is a straightforward calculation by formula (39). The key step of the application of the higher order spectral synthesis is to formulate the necessary and sufficient conditions for

$$\tilde{F}_p(x_1, \ldots, x_p) := \phi_1 \circ D_1(x_1) \cdot \ldots \cdot \phi_p \circ D_p(x_p)$$

to satisfy system (8). In what follows we discuss only the case of $p = 2$. The case of $p > 2$ can be investigated in a similar way because of the inductive argument as follows. The first equation of system (8) implies that

$$\sum_{i=1}^{n} a_i\phi_1 \circ D_1(\alpha_i x_1) \cdot \phi_2 \circ D_2(\alpha_i x_2) = \tilde{c} \cdot x_1 \cdot x_2 \tag{61}$$

In order to use the results in the previous sections let the variables x_2 be considered as a non-zero given constant in K; the most simple choice is $x_2 = 1$. Then

$$\sum_{i=1}^{n} \underbrace{a_i\phi_2 \circ D_2(\alpha_i x_2)}_{\text{new coefficients}} \cdot \phi_1 \circ D_1(\alpha_i x_1) = \tilde{c} \cdot x_1 \cdot x_2, \tag{62}$$

where the new constant is $\tilde{c} \cdot x_2$, i.e. $\phi_1 \circ D_1$ satisfies one of the conditions to be the additive solution of an inhomogeneous equation with some new coefficients. If $\tilde{c} \neq 0$ then we have, by Proposition 3.10, that $\phi_1(x) = x$ $(x \in K)$ and

(63)
$$\sum_{i=1}^{n} a_i\phi_2 \circ D_2(\alpha_i x_2) \cdot \left(D_1^{\phi_1}\right)_{m_{11}\dots m_{1k}}(\alpha_i) = 0 \quad \text{if} \quad m_{11} + \ldots + m_{12} \geq 1$$
$$\sum_{i=1}^{n} a_i\phi_2 \circ D_2(\alpha_i x_2) \cdot D_1^{\phi_1}(\alpha_i) = \tilde{c} \cdot x_2.$$

In a similar way, $\phi_2(x) = x$ $(x \in K)$ and, by using Lemma 5.1, the left hand side of the first equation is the action of a differential operator at x_2 for any given $m_{11}, \ldots, m_{1k}$:

$$\sum_{m_{21}=0}^{J_{21}} \cdots \sum_{m_{2k}=0}^{J_{2k}} \sum_{i=1}^{n} a_i \left(D_2^{\phi_2}\right)_{m_{21}\dots m_{2k}}(\alpha_i) \cdot \left(D_1^{\phi_1}\right)_{m_{11}\dots m_{1k}}(\alpha_i) \cdot \partial_1^{m_{21}} \ldots \partial_k^{m_{2k}}(x) = 0.$$

Therefore

(64)
$$\sum_{i=1}^{n} a_i \left(D_1^{\phi_1}\right)_{m_{11}\dots m_{1k}}(\alpha_i) \cdot \left(D_2^{\phi_2}\right)_{m_{21}\dots m_{2k}}(\alpha_i) = 0$$
$$\text{if} \quad m_{11} + \ldots + m_{1k} \geq 1 \quad \text{and} \quad 0 \leq m_{21} \leq J_{21}, \ldots, 0 \leq m_{2k} \leq J_{2k}.$$

On the other hand the second equation of (63) shows that $\phi_2 \circ D_2$ satisfies one of the conditions to be the additive solution of an inhomogeneous equation with some new coefficients. By Proposition 3.10

$$\begin{aligned}
&\sum_{i=1}^{n} a_i \left(D_2^{\phi_2}\right)_{m_{21}\dots m_{2k}} (\alpha_i)\cdot D_1^{\phi_1}(\alpha_i) = 0 \;\text{ if }\; m_{21}+\ldots+m_{2k}\geq 1 \\
&\sum_{i=1}^{n} a_i D_2^{\phi_2}(\alpha_i)\cdot D_1^{\phi_1}(\alpha_i) = \tilde{c}.
\end{aligned} \tag{65}$$

Equations (64) and (65) give that

$$\begin{aligned}
&\sum_{i=1}^{n} a_i \left(D_1^{\phi_1}\right)_{m_{11}\dots m_{1k}} (\alpha_i)\cdot \left(D_2^{\phi_2}\right)_{m_{21}\dots m_{2k}} (\alpha_i) = 0 \\
&\text{if }\; m_{11}+\ldots+m_{1k}+m_{21}+\ldots+m_{2k}\geq 1 \\
&\sum_{i=1}^{n} a_i D_1^{\phi_1}(\alpha_i)\cdot D_2^{\phi_2}(\alpha_i) = \tilde{c}.
\end{aligned} \tag{66}$$

System (8) implies similar equations containing the parameters $\beta_1, \ldots, \beta_n$ (pure case) or $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ (mixed cases). According to the definition of $D_{m_1\dots m_k}$ the result is a non-linear (polynomial; especially quadratic because of $p=2$) system (66) for the quantities $c'_{j_{11}\dots j_{1k}}$'s and $c'_{j_{21}\dots j_{2k}}$'s, where

$$D_1^{\phi_1} = \sum_{j_{11}=0}^{J_{11}} \cdots \sum_{j_{1k}=0}^{J_{1k}} c'_{j_{11}\dots j_{1k}} \partial_1^{j_{11}} \ldots \partial_k^{j_{1k}},$$

$$D_2^{\phi_2} = \sum_{j_{21}=0}^{J_{21}} \cdots \sum_{j_{2k}=0}^{J_{2k}} c'_{j_{21}\dots j_{2k}} \partial_1^{j_{21}} \ldots \partial_k^{j_{2k}},$$

$$c'_{j_{11}\dots j_{1k}} = \phi_1(c_{j_{11}\dots j_{1k}}), \quad c'_{j_{21}\dots j_{2k}} = \phi_2(c_{j_{21}\dots j_{2k}}).$$

5.2.1. *The case of transcendence degree* 1. For the sake of simplicity consider the case of $k=1$ (transcendence degree one), $p=2$ and the separated version of the functional equations as in section 4 (see also Remark 3.3 and Remark 3.14). For any fixed index m_{21} system (66) results in a diagonal form like (48) with some new coefficients:

$$(67)\qquad \begin{aligned}
&\sum_{i=1}^{n} \underbrace{a_i \cdot \left(D_2^{\phi_2}\right)_{m_{21}} (\alpha_i)}_{\text{new coefficients}} \cdot \left(D_1^{\phi_1}\right)_{m_{11}} (\alpha_i) = 0 \ \text{ if } \ m_{11} + m_{21} \geq 1 \\
&\sum_{i=1}^{n} \underbrace{a_i \cdot D_2^{\phi_2}(\alpha_i)}_{\text{new coefficients}} \cdot D_1^{\phi_1}(\alpha_i) = \tilde{c}.
\end{aligned}$$

Therefore we can conclude that

$$(68)\qquad \begin{aligned}
&\sum_{i=1}^{n} a_i \alpha_i \left(D_2^{\phi_2}\right)_{m_{21}} (\alpha_i) = 0 \\
&\sum_{i=1}^{n} a_i \partial(\alpha_i) \left(D_2^{\phi_2}\right)_{m_{21}} (\alpha_i) = 0 \\
&\cdots \\
&\sum_{i=1}^{n} a_i \partial^{J_{11}-1}(\alpha_i) \left(D_2^{\phi_2}\right)_{m_{21}} (\alpha_i) = 0 \\
&\sum_{i=1}^{n} a_i \partial^{J_{11}}(\alpha_i) \left(D_2^{\phi_2}\right)_{m_{21}} (\alpha_i) = \begin{cases} 0 \ \text{ if } \ m_{21} \geq 1 \\ \tilde{c} \ \text{ if } \ m_{21} = 0 \end{cases}
\end{aligned}$$

Taking the terms of the form $a_i \partial^{j_{11}}(\alpha_i)$ $(i = 1, \ldots, n)$ as new coefficients in the j_{11}th equation we can give similar conclusions by (48) in case of each equation above:

$$(69)\qquad \begin{aligned}
&\sum_{i=1}^{n} a_i \alpha_i^2 = 0 \quad (j_{11} + j_{21} = 0), \\
&\cdots \\
&\sum_{i=1}^{n} a_i \partial^{j_{11}}(\alpha_i) \partial^{j_{21}}(\alpha_i) = 0 \quad (j_{11} + j_{21} < J_{11} + J_{21}), \\
&c'_{J_{11}} c'_{J_{21}} \sum_{i=1}^{n} a_i \partial^{J_{11}}(\alpha_i) \partial^{J_{21}}(\alpha_i) = \tilde{c} \quad (j_{11} + j_{21} = J_{11} + J_{21}).
\end{aligned}$$

These are necessary and sufficient conditions in case of transcendence degree 1 for the generating elements of the space S_2. The coefficients can be arbitrarily chosen except the greatest one in both D_1 and D_2 provided that system (69) holds.

5.2.2. *An explicite example.* Let $c \neq 0$ be a complex number and consider functional equation

$$t^6 f(tx) - t^4 f(t^2x) - t^2 f(t^3x) + f(t^4x) = c \cdot x^2 \tag{70}$$

i.e. $n = 4$ and $a_1 = t^6$, $a_2 = -t^4$, $a_3 = -t^2$, $a_4 = 1$, $\alpha_1 = t$, $\alpha_2 = t^2$, $\alpha_3 = t^3$, $\alpha_4 = t^4$, where t is a transcendental number over the rationals, $L = \mathbb{Q}(t)$ and $L \subset K$ such that K is a finitely generated field and it is contained in the algebraic closure of L. Using that

$$\begin{aligned} &\sum_{i=1}^{4} a_i \alpha_i^2 = t^6 \cdot t^2 - t^4 \cdot t^4 - t^2 \cdot t^6 + t^8 = 0 \\ &\sum_{i=1}^{4} a_i \alpha_i \partial(\alpha_i) = t^7 - t^4 \cdot t^2 \cdot (2t) - t^2 \cdot t^3 \cdot (3t^2) + t^4 \cdot (4t^3) = 0 \end{aligned} \tag{71}$$

but

$$\sum_{i=1}^{4} a_i \partial(a_i) \partial(\alpha_i) = t^6 - t^4 \cdot (2t)^2 - t^2 \cdot (3t^2)^2 + (4t^3)^2 = 4t^6 \neq 0 \tag{72}$$

it follows that the generating elements of S_2 are the products of first order exponential monomials $D_1^{\phi_1}(x) = c'_{10}x + c'_{11}\partial(x)$ and $D_2^{\phi_2}(x) = c'_{20}x + c'_{21}\partial(x)$ such that $c'_{11} \cdot c'_{21} = \frac{c}{4t^6}$, where $c'_{11} = \phi_1(c_{11})$, $c'_{21} = \phi_2(c_{21})$ and the differential operators are of the form $D_1(x) = c_{10}x + c_{11}\partial(x)$ and $D_2(x) = c_{20}x + c_{21}\partial(x)$.

6. Concluding remarks

The spectral synthesis in S_p^* allows us to describe the entire space of solutions of an inhomogeneous linear functional equation on a large class of finitely generated fields, at least theoretically. In general we need to solve inhomogeneous linear systems of equations to find the solutions. The conclusions for the homogeneous case has been also formulated step by step in terms of some remarks. The discussion of the case of algebraic parameters (the transcendence degree is zero) is relatively simple because of the trivial action of any differential operator on algebraic numbers. The case of transcendence degree one gives the best results independently of the degree of the monomial solutions. In case of higher degree of transcendence the problem becomes much more difficult because of the missing main terms of the differential operators and the increasing number of the equations. Some explicit examples are also presented to illustrate both the effectivity and the difficulties of the method in practice. Computer assisted methods can be successful for the formulation and the solution of large systems of linear equations belonging to the case of higher order monomial solutions and/or higher degree of transcendence.

References

[1] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equations, Aequationes Math. September 2014, Volume 88, Issue 1, pp. 151-162.

[2] E. Gselmann and Zs. Páles, Additive solvability and linear independence of the solutions of a system of functional equations, *Acta Sci. Math. Szeged* **82** (1-2), (2016), 101-110 34/2014

[3] G. Kiss and M. Laczkovich, Linear functional equations, differential operators and spectral synthesis, *Aequationes Math.* April 2015, Volume **89**, Issue 2, 301-328.

[4] G. Kiss, A. Varga and Cs. Vincze, *Algebraic methods for the solution of linear functional equations*, Acta Math. Hungar. June 2015, Vol. 146 (1), pp. 128-141.

[5] G. Kiss, Linear functional equations with algebraic parameters, *Publ. Math. Debrecen* **85**/1-2 (2014), 145-160.

[6] B. Koclega-Kulpa, T. Szostok, On a functional equation connected to Gauss quadrature rule, *Ann. Math. Sylesianae* **22** (2008), 27-40.

[7] B. Koclega-Kulpa, T. Szostok, On a class of eqautions stemming form various quadrature rules, *Acta. Math. Hungar.* **130 (4)** (2011), 340-348.

[8] B. Koclega-Kulpa, T. Szostok and S. Wasowicz, On functional equations connected with quadrature rules, *Georgian Math. J.* **16** (2009), 725-736.

[9] M. Kuczma, An introduction to the Theory of Functional Equations and Inequalities, Polish Scientific Publishers, Warsaw (1985).

[10] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete abelien groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

[11] M. Laczkovich, Local spectral synthesis on Abelian groups, *Acta Math. Hung.* **143**(2014), no. 2, 313-329.

[12] M. Sablik, Taylor's theorem and functional equations, *Aequationes Math.* **60** (2000), 258-267.

[13] L. Székelyhidi, On a class of linear functional equations, *Publ. Math.* (Debrecen) **29** (1982), 19-28.

[14] L. Székelyhidi, Discrete spectral synthesis, *Annales Mathematicae and Informaticae* **32** (2005), 141-152.

[15] Cs. Vincze and A. Varga, On Daróczy's problem for additive functions *Publ. Math. Debrecen* 75 (1-2) (2009), 299-310.

[16] Cs. Vincze and A. Varga, Nontrivial solutions of linear functional equations: methods and examples *Opuscula Math.* 35, no. 6 (2015), 957-972

[17] Cs. Vincze and A. Varga, On the characteristic polynomials of linear functional equations, *Period. Math. Hungar.* December 2015, Volume 71, Issue 2, 250-260.

[18] Cs. Vincze, Algebraic dependency of roots of multivariate polynomials and its applications to linear functional equations, accepted for publication in *Period. Math. Hungar.*

[19] Cs. Vincze and A. Varga, On a sufficient and necessary condition for a multivariate polynomial to have algebraic dependent roots - an elementary proof, accepted for publication in *Acta Math. Acad. Paedagog. Nyházi.*

[20] G. Kiss and Cs. Vincze, On spectral analysis in varieties containing the solutions of inhomogeneous linear functional equations, submitted to *Aequat. Math.*

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