Perturbation results on the zero-locus of a polynomial *

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Abstract

Let f and g be complex multivariate polynomials of the same degree. Extending Beauzamy's results which hold in the univariate case, we bound the Euclidean distance of points belonging to the zero-loci of f and g in terms of the Bombieri norm of the difference $g - f$. We also discuss real perturbations of real polynomials.

Introduction

In this work we address the problem of evaluating how much the zero-locus of a polynomial varies, if some perturbations on the polynomial coefficients are permitted. We start with a simple illustrative example where we explicitly show that, locally, it is possible to bound the Euclidean distance of points belonging to two different algebraic plane curves of equations $f = 0$ and $g = 0$ in terms of the difference between f and g measured using a suitable norm in the space of polynomials.

Example 0.1 (Descartes Folium) In the affine plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$ consider the cubic curve $\mathcal{C}$ of equation $f(x, y) = 0$, where $f(x, y) = x^3 + y^3 - xy$ (see Figure 1). Further, consider two other cubic curves $\mathcal{C}_1 : g_1(x, y) = 0$ and $\mathcal{C}_2 : g_2(x, y) = 0$, where $g_1(x, y) = x^3 + y^3 - (1 + \frac{\sqrt{6}}{100})xy$ and $g_2(x, y) = x^3 + y^3 - xy - \frac{1}{100}$ (see Figure 2).

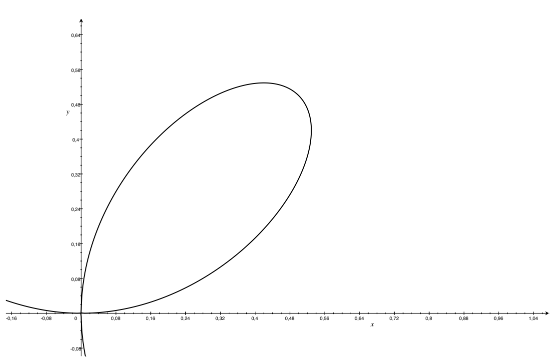

Figure 1: The curve $f = 0$.

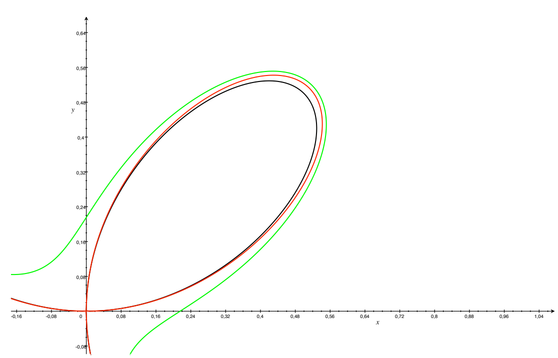

Figure 2: The three curves $f = 0$ (black), $g_1 = 0$ (red), and $g_2 = 0$ (green).

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Note that the coefficients of the polynomials $g_1(x, y)$ and $g_2(x, y)$ "slightly" differ from the corresponding coefficients of the polynomial $f(x, y)$, and this leads to consider the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ as "small" perturbations of the curve $\mathcal{C}$. In order to quantify the size of each perturbation, we need to have a measure of the difference between the "original" curve $\mathcal{C}$ and each "perturbed" curve $\mathcal{C}_1$ or $\mathcal{C}_2$. This is feasible, for instance, by computing the Bombieri norm (introduced in Definition 2.1) of the polynomials differences, that is, $g_1 - f$ or $g_2 - f$. In this case, the norm is always equal to 0.01, although the geometry of the real parts of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ is completely different (see again Figure 2).

Here, the matter is to see whether or not the size of the perturbation can be used to give local information on the Euclidean distance between the curves $\mathcal{C}_1$ (or $\mathcal{C}_2$) and $\mathcal{C}$. The answer depends on the point p we fix on $\mathcal{C}$. Indeed, take first $p = (0, 0) \in \mathcal{C}$, so that $p \in \mathcal{C}_1 : g_1 = 0$, whereas $p \notin \mathcal{C}_2 : g_2 = 0$ (see Figure 3). Then, taking $p = (0.5, 0.5) \in \mathcal{C}$, one has $p \notin \mathcal{C}_1$ and $p \notin \mathcal{C}_2$ (see Figure 4).

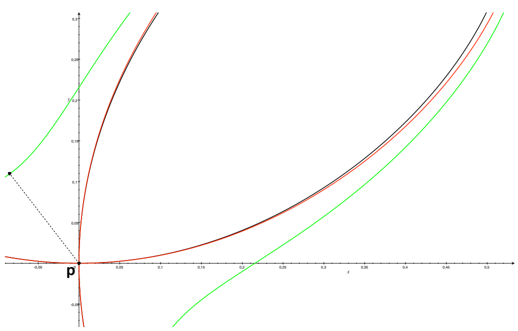

Figure 3: $p = (0, 0) \in \mathcal{C}$, $p \in \mathcal{C}_1$, $p \notin \mathcal{C}_2$.

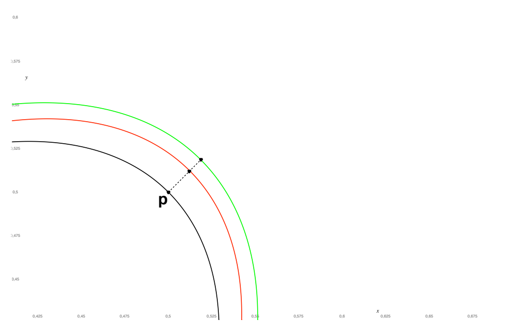

Figure 4: $p = (0.5, 0.5) \in \mathcal{C}$, $p \notin \mathcal{C}_1$, $p \notin \mathcal{C}_2$.

Inspired by the previous example, we formulate the addressed problem asking whether and how much a small perturbation of a polynomial f in the Bombieri norm is still close to f.

In Section 1 we recall the Walsh's Contraction Principle, a result on the zeros of multivariate complex symmetric polynomials which are linear with respect to each variable, on which our main result, Theorem 3.4, is funded. In Section 2, we recall the notion of the Bombieri–Weyl norm on the space of polynomials, and some of its properties we use throughout the paper.

Section 3 is devoted to state and prove our main result. Generalizing the situation highlighted in the above examples, let f be a polynomial of $\mathbb{C}[x_1, \dots, x_n]$ of degree d and let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a point of $f = 0$ of multiplicity $s \geq 1$. Let ε be a positive real number and let g be any degree d polynomial satisfying the conditions that $\frac{\partial^s g}{\partial x_i^s}(p) \neq 0$ for some index i and the Bombieri norm of the difference $g - f$ is less than ε (that is, g is a perturbation of f). Then, for all such polynomials g, we find a constant $k = k(f, p, \varepsilon)$, depending only on f, p, ε, and a zero q of $g = 0$ such that the Euclidean distance $\|q - p\|_2$ satisfies $\|q - p\|_2 \leq k(f, p, \varepsilon)$.

Since the Bombieri norms are explicitly defined through the coefficients of a polynomial, all estimates are effective and involve explicit constants. This generalizes results of [3], where the univariate case is considered. We also provide some examples to illustrate that the bounds we give are near to be sharp.

In Section 4, we specialize the results of Section 3 to the non-singular case, discussing a connection with a result of Dégot [9], which is based on a first-order analysis perspective and generalizes a result of Shub and Smale [16].

As we said, the above results are funded on Walsh's Contraction Principle which is false over the reals. Having in mind some applications (see [18]), it seems natural asking for some perturbation results over the reals. In Section 5, we propose a perturbation result for real polynomials based on Rouché's theorem, a classical result giving a useful criterion to locating regions of the complex plane in which an analytic function has zeros. We end up in Section 6 offering an interpretation of Walsh's Contraction Theorem as a statement in several complex variables.

We refer to [17] and [10] for somehow related results, though approached using different techniques.

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1 Preliminaries

We let $x_1, \dots, x_n$ be indeterminates. The multivariate polynomial ring with complex coefficients $\mathbb{C}[x_1, \dots, x_n]$ is denoted by P. We choose a new indeterminate x_0, also called *homogenizing indeterminate*, and denote the polynomial ring $\mathbb{C}[x_0, x_1, \dots, x_n]$ by $\overline{P}$.

Let f be a polynomial of P and let $f = f_d + \cdots + f_0$ be its decomposition into homogeneous components, where each $f_i \in P$ is homogeneous of degree i. The *homogenization of f with respect to x_0* is the polynomial $f^{\mathrm{hom}} = f_d + x_0 f_{d-1} + \cdots + x_0^d f_0 \in \overline{P}$. For the zero polynomial, we set $0^{\mathrm{hom}} = 0$. Let F be a homogeneous polynomial of $\overline{P}$. The *dehomogenization of F with respect to x_0* is the polynomial $F^{\mathrm{deh}} = F(1, x_1, \dots, x_n) \in P$ (see [15, Section 4.3]). The homogenization and dehomogenization of polynomials obey the following rules (see [15], Proposition 4.3.2).

Proposition 1.1 *Let f, g be polynomials of P, and let F, G be homogeneous polynomials of $\overline{P}$. The following equalities hold true:*

1. $(f^{\mathrm{hom}})^{\mathrm{deh}} = f$.
2. $(fg)^{\mathrm{hom}} = f^{\mathrm{hom}} g^{\mathrm{hom}}$.
3. $(F + G)^{\mathrm{deh}} = F^{\mathrm{deh}} + G^{\mathrm{deh}}$.
4. $(FG)^{\mathrm{deh}} = F^{\mathrm{deh}} G^{\mathrm{deh}}$.

Given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, and letting $\boldsymbol{x} = (x_1, \dots, x_n)$, we denote by $|\boldsymbol{\alpha}|$ the number $\alpha_1 + \cdots + \alpha_n$, by $\boldsymbol{x}^{\boldsymbol{\alpha}}$ the power product $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and by $\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ the $\boldsymbol{\alpha}$-partial derivative of a polynomial $f = f(\boldsymbol{x}) \in P$. Moreover, following the standard notation, we denote by $\mathrm{Jac}_f(\boldsymbol{x}) := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ the *Jacobian* (or *gradient*) of f.

Let us recall the Walsh's Contraction Principle (see [19], [20] and [4]), a result on the zeros of multivariate symmetric polynomials which are linear with respect to each variable. We recall that, in Walsh's terminology, a (*closed*) *circular region* of the complex plane $\mathbb{C}$

is either a (closed) disk, or a (closed) half-plane, or the (closed) exterior of a disk. Let us also mention the interesting complementary result [7, Theorem 2].

Theorem 1.2 (Walsh's Contraction Principle) *Let $f \in P$ be a polynomial with the following properties:*

1. *f is linear w.r.t. each variable x_i, $i = 1, \dots, n$.*
2. *f is symmetric w.r.t. the variables x_i, that is, $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$ for each order n permutation σ (equivalently, f is invariant under permutations of the variables x_i).*

Let $D \subset \mathbb{C}$ be a circular region of the complex plane, and assume that there are $z_1, \dots, z_n \in D$ such that $f(z_1, \dots, z_n) = 0$. Then there exists a point $z \in D$ such that $f(z, \dots, z) = 0$.

Now, we recall the definition of some vectorial norms we need. Given a positive integer n, let $\boldsymbol{v} = (v_1, \dots, v_n)$ be a vector in $\mathbb{C}^n$, and let r be a positive integer. The *r-norm* $\|\boldsymbol{v}\|_r$ of $\boldsymbol{v}$ is defined by the formula

$$\|\boldsymbol{v}\|_r := \left(\sum_{i=1}^{n} |v_i|^r \right)^{\frac{1}{r}},$$

where "$|\cdot|$" denotes the module of complex numbers. In particular, if $r = 1$, we get the expression $\|\boldsymbol{v}\|_1 = \sum_{i=1}^n |v_i|$. If $r = 2$ we get the well-known *Euclidean norm* $\|\boldsymbol{v}\|_2 = \left(\sum_{i=1}^n |v_i|^2\right)^{1/2}$. While, if $r \to \infty$, the r-norm approaches the *∞-norm* defined by $\|\boldsymbol{v}\|_\infty := \max_{i=1,\dots,n} \{|v_i|\}$.

2 The Bombieri–Weyl norm

In this section we recall the notion of Bombieri–Weyl's norm on the space of polynomials and some of its main properties in both the homogeneous and the affine case (see [5]).

Definition 2.1 Let $F = \sum_{|\boldsymbol{\alpha}|=d} c_{\boldsymbol{\alpha}} x_0^{\alpha_0} \dots x_n^{\alpha_n}$ and $G = \sum_{|\boldsymbol{\alpha}|=d} c'_{\boldsymbol{\alpha}} x_0^{\alpha_0} \dots x_n^{\alpha_n}$ be two homogeneous polynomials of $\overline{P}$ of degree d. Then the *Bombieri scalar product* of F and G is defined as

$$(F, G)_{(d)} = \sum_{|\boldsymbol{\alpha}|=d} \frac{\alpha_0! \dots \alpha_n!}{d!} c_{\boldsymbol{\alpha}} \overline{c'_{\boldsymbol{\alpha}}},$$

where "$\overline{\cdot}$" denotes the conjugate of complex numbers.

Such a scalar product induces an inner product on the linear space of all the degree d polynomials of $\overline{P}$. We can then consider the canonically associated *Bombieri's norm*, defined as

$$\|F\|_{(d)} = \left(\sum_{|\boldsymbol{\alpha}|=d} \frac{\alpha_0! \dots \alpha_n!}{d!} |c_{\boldsymbol{\alpha}}|^2 \right)^{1/2}.$$

Moreover, for degree d polynomials f and g of P, the *Bombieri scalar product* of f and g is defined as Bombieri's scalar product of the homogenization of f and g, that is,

$$(f, g)_{(d)} = (f^{\text{hom}}, g^{\text{hom}})_{(d)}.$$

And the *Bombieri norm* of f is defined as the Bombieri norm of the homogenization of f, that is,

$$\|f\|_{(d)} = \|f^{\text{hom}}\|_{(d)}.$$

Let us stress the fact that in Bombieri's scalar product the polynomials must be of the same degree d (or considered so) and that the scalar product depends on d.

Every inner product satisfies the Cauchy–Schwarz inequality (e.g., see [14, Chapter 14]) in terms of the associated norm, that is,

$$|(f,g)_{(d)}| = |(f^{\text{hom}}, g^{\text{hom}})_{(d)}| \leq \|f^{\text{hom}}\|_{(d)}\|g^{\text{hom}}\|_{(d)} = \|f\|_{(d)}\|g\|_{(d)}. \tag{1}$$

We will use over and over such an inequality, without explicitly referring to it.

In the example below we point out a property we use in the sequel.

Example 2.2 We consider a polynomial $(c_1x_1 + \cdots + c_nx_n)^d \in P$, where $c_1, \ldots, c_n$ are complex numbers. We explicitly show that $\|(c_1x_1 + \cdots + c_nx_n)^d\|_{(d)} = \|(c_1, \ldots, c_n)\|_2^d$. To this end we rewrite the polynomial in the form

$$(c_1x_1 + \cdots + c_nx_n)^d = \sum_{|\boldsymbol{\alpha}|=d} \frac{d!}{\alpha_1! \ldots \alpha_n!} c_1^{\alpha_1} \ldots c_n^{\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Thus we find

$$\begin{aligned}
\|(c_1x_1 + \cdots + c_nx_n)^d\|_{(d)} &= \left(\sum_{|\boldsymbol{\alpha}|=d} \frac{\alpha_1! \ldots \alpha_n!}{d!} \frac{d!^2}{\alpha_1!^2 \ldots \alpha_n!^2} |c_1|^{2\alpha_1} \ldots |c_n|^{2\alpha_n} \right)^{1/2} \\
&= \left(\sum_{|\boldsymbol{\alpha}|=d} \frac{d!}{\alpha_1! \ldots \alpha_n!} |c_1|^{2\alpha_1} \ldots |c_n|^{2\alpha_n} \right)^{1/2} \\
&= \left(|c_1|^2 + \cdots + |c_n|^2 \right)^{d/2} = \|(c_1, \ldots, c_n)\|_2^d,
\end{aligned}$$

as we want.

We need the following Bombieri's scalar product properties (for a proof, see [6] Lemma 9, Corollary 10, and Proposition 2).

Lemma 2.3 *Let F and G be homogeneous polynomials of $\overline{P}$ of degree $d-1$ and d respectively. Then*

$$(x_iF, G)_{(d)} = \frac{1}{d}\left(F, \frac{\partial G}{\partial x_i}\right)_{(d-1)}, \quad i = 0, \ldots, n.$$

Lemma 2.4 *Let $F \in \overline{P}$ be a homogeneous polynomial of degree d, and let $p = (p_0, \ldots, p_n) \in \mathbb{A}_{\mathbb{C}}^{n+1}$. The evaluation of F at p can be expressed in terms of Bombieri's scalar product as*

$$F(p) = \left(F, (\overline{p_0}x_0 + \cdots + \overline{p_n}x_n)^d\right)_{(d)}.$$

Extensions to the affine case are generally easy, fixing the first variable to 1. Lemma 2.4 rewrites as

Lemma 2.5 *Let $f \in P$ be a polynomial of degree d, and let $p = (p_1, \dots, p_n) \in \mathbb{A}^n_{\mathbb{C}}$. The evaluation of f at p can be expressed in terms of Bombieri's scalar product as*

$$f(p) = \left(f, (1 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^d\right)_{(d)}.$$

Proof. By definition of Bombieri's scalar product and norm in the affine case, Lemma 2.4 and Proposition 1.1 give

$$f(p) = f^{\mathrm{hom}}(1, p) = \left(f^{\mathrm{hom}}, (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^d\right)_{(d)} = \left(f, (1 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^d\right)_{(d)},$$

which concludes the proof. Q.E.D.

We need the following submultiplicativity-type result (see also [2, Formula (6)] and [1, Proposition 5]).

Lemma 2.6 *Let d, s be positive integers, and consider in P a degree d polynomial f Then, for each $i = 1, \dots, n$,*

$$\|x_i^s f\|_{(d+s)} \leq \|f\|_{(d)}.$$

Proof. Let $f = \sum_{\boldsymbol{\alpha} \leq d} b_{\boldsymbol{\alpha}} x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Passing to the homogenization, write $f^{\mathrm{hom}} = \sum_{\alpha_0 + |\boldsymbol{\alpha}| = d} b_{\boldsymbol{\alpha}} x_1^{\alpha_1} \dots x_n^{\alpha_n} x_0^{\alpha_0}$. Then

$$\begin{aligned} x_i^s f^{\mathrm{hom}} &= \sum_{\alpha_0 + |\boldsymbol{\alpha}| = d+s} b_{\boldsymbol{\alpha}} x_0^{\alpha_0} x_1^{\alpha_1} \dots x_i^{\alpha_i + s} \dots x_n^{\alpha_n} \\ &= \sum_{\alpha_0' + |\boldsymbol{\alpha}'| = d+s} b_{\boldsymbol{\alpha}'} x_0^{\alpha_0'} x_1^{\alpha_1'} \dots x_i^{\alpha_i'} \dots x_n^{\alpha_n'}, \end{aligned}$$

where $\alpha_j' = \alpha_j$, $j \neq i$, $\alpha_i' = \alpha_i + s$. Thus,

$$\begin{aligned} \|x_i^s f^{\mathrm{hom}}\|_{(d+s)} &= \left(\sum_{\alpha_0 + |\boldsymbol{\alpha}'| = d+s} \frac{\alpha_0'! \dots \alpha_i'! \dots \alpha_n'!}{(d+s)!} |b_{\boldsymbol{\alpha}'}|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{\alpha_0 + |\boldsymbol{\alpha}| = d+s} \frac{\alpha_0! \dots (\alpha_i + s)! \dots \alpha_n!}{(d+s)!} |b_{\boldsymbol{\alpha}}|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\alpha_0 + |\boldsymbol{\alpha}| = d+s} \frac{\alpha_0! \dots \alpha_i! \dots \alpha_n!}{d!} |b_{\boldsymbol{\alpha}}|^2 \right)^{\frac{1}{2}} = \|f^{\mathrm{hom}}\|_{(d)}, \end{aligned}$$

where the inequality immediately follows by noting that $\frac{(\alpha_i+s)!}{(d+s)!} \leq \frac{\alpha_i!}{d!}$ since $\alpha_i \leq d$, $i = 1, \dots, n$. Coming back to affine coordinates gives

$$\|x_i^s f\|_{(d+s)} = \|x_i^s f^{\mathrm{hom}}\|_{(d+s)} \leq \|f^{\mathrm{hom}}\|_{(d)} = \|f\|_{(d)}.$$

Q.E.D.

3 Perturbation bounds for multivariate polynomials

In this section, we bound the Euclidean distance of points belonging to the zero-loci of complex multivariate degree d polynomials f and g in terms of a given bound of Bombieri's norm $\|g-f\|_{(d')}$, with $d' = \deg(g-f) \leq d$. Specifically, we propose a generalization of [3, theorems 1 and 4], which gives back Beauzamy's results in the univariate case.

As in Beauzamy [3], we use the following fact, a restatement of the Fundamental Theorem of Algebra.

Lemma 3.1 *Let $f(x) = a(x-z_1)\ldots(x-z_k)$ be a univariate polynomial in $\mathbb{C}[x]$, with $a \neq 0$, and let ε be a positive real number. If $|f(z)| \leq \varepsilon$ for some $z \in \mathbb{A}^1_{\mathbb{C}}$, then one of the $f(x)$'s roots, say z_1, satisfies $|z - z_1| \leq \left(\frac{\varepsilon}{|a|}\right)^{1/k}$.*

Proof. If $|z-z_1|\ldots|z-z_k| \leq \frac{\varepsilon}{|a|}$, then one of the factors $|z - z_j|$'s, $j \in \{1,\ldots,k\}$, say z_1, satisfies $|z - z_1| \leq \left(\frac{\varepsilon}{|a|}\right)^{1/k}$. Q.E.D.

We also need the following technical fact. Recall that the support, $\mathrm{Supp}(f)$, of a degree d polynomial $f = \sum_{|\boldsymbol{\alpha}|\leq d} c_{\boldsymbol{\alpha}} x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ of P, $\alpha := \alpha_1 + \cdots + \alpha_n$, is defined as the set of monomials $\{x_1^{\alpha_1}\ldots x_n^{\alpha_n} \mid c_{\boldsymbol{\alpha}} \neq 0\}$.

Lemma 3.2 *Let f be a degree d polynomial of P and let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a point such that $\frac{\partial^s f}{\partial x_i^s}(p) \neq 0$ for some integer $s \geq 1$ and $i \in \{1,\ldots,n\}$. Then ($d \geq s$ and) x_i^s belongs to* $\mathrm{Supp}(f)$.

Proof. Since f is a differentiable function from $\mathbb{C}^n$ to $\mathbb{C}$, it can be expressed using the classical Taylor formula, which in our notations reads

$$f(x_1,\ldots,x_n) = \sum_{m=0}^{d} \frac{1}{m!}\left(\sum_{|\boldsymbol{\alpha}|=m} \binom{m}{\boldsymbol{\alpha}} \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}(p)(x_1-p_1)^{\alpha_1}\ldots(x_n-p_n)^{\alpha_n}\right).$$

Let $\boldsymbol{\alpha}_s := s\, e_i$, where e_i is the i-th elementary (unit) vector of $\mathbb{C}^n$. Then, the term of the expansion of f corresponding to $\boldsymbol{\alpha}_s$ gives the contribution

$$\frac{\partial^s f}{s!\partial x_i^s}(p)(x_i-p_i)^s = \frac{\partial^s f}{s!\partial x_i^s}(p)x_i^s + r(x_i),$$

where $r(x_i)$ is a univariate polynomial in x_i of degree $\leq s-1$. Since by hypothesis $\frac{\partial^s f}{\partial x_i^s}(p) \neq 0$ and the term x_i^s cannot occur anywhere else in the Taylor formula, we conclude that x_i^s is an element of $\mathrm{Supp}(f)$. Q.E.D.

To prove our main result, Theorem 3.4, we make use of the following application of Walsh's Contraction Theorem which provides an upper bound for the Euclidean distance of points of $f = 0$ and $g = 0$, where f and g are polynomials of P satisfying given constraints.

Lemma 3.3 *Let f and g be polynomials of P of degree d and let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a point of $f = 0$ of multiplicity $s \geq 1$ such that $\frac{\partial^s g}{\partial x_i^s}(p) \neq 0$ for some index i. Let ε be a positive real* *number, and suppose that* $\|g-f\|_{(d')} \le \varepsilon$*, where* $d' = \deg(g-f) \le d$*. Then there exists a point* $q \in \mathbb{A}^n_{\mathbb{C}}$ *belonging to* $g=0$ *such that*

$$\|q-p\|_2 \le \left(\frac{d!}{(d-s)!} \frac{(1+\|p\|_2^2)^{d/2}}{\left\| \left(\frac{\partial^s g}{\partial x_1^s}(p), \ldots, \frac{\partial^s g}{\partial x_n^s}(p) \right) \right\|_1} \right)^{1/s} \varepsilon^{1/s}.$$

Proof. Let $p = (p_1, \ldots, p_n)$. Since $f(p) = 0$ we can evaluate g at p as $g(p) = g(p) - f(p) = (g-f)(p)$. Thus, by combining Lemma 2.5 with the Cauchy–Schwarz inequality (1) and recalling the assumption $\|g-f\|_{(d')} \le \varepsilon$, the same computation as in Example 2.2 yields

$$\begin{aligned} |g(p)| &= \left| \left(g-f, (1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^{d'} \right)_{(d')} \right| \\ &\le \|g-f\|_{(d')} \, \|(1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^{d'}\|_{(d')} \\ &= \|g-f\|_{(d')} \, \|(1, \overline{p_1}, \ldots, \overline{p_n})\|_2^{d'} \\ &= \|g-f\|_{(d')} \, (1+\|p\|_2^2)^{d'/2} \le \varepsilon (1+\|p\|_2^2)^{d/2}. \end{aligned} \tag{2}$$

Now, define

$$I := \left\{ i \in \{1, \ldots, n\} \mid \frac{\partial^s g}{\partial x_i^s}(p) \ne 0 \right\}.$$

Note that $I \ne \emptyset$ by the assumption made. For $i \in I$, we consider the univariate functions $h_i : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{C}$ defined by

$$\begin{aligned} &h_i(\xi) \\ &:= \left(g, (1+\overline{p_1}x_1 + \cdots + \overline{p_i}x_i + \cdots + \overline{p_n}x_n)^{d-s} (1+\overline{p_1}x_1 + \cdots + \overline{\xi}x_i + \cdots + \overline{p_n}x_n)^s \right)_{(d)}. \end{aligned}$$

On the other hand, by using Lemma 2.5 again, we can express $g(p)$ as

$$g(p) = \left(g, (1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^d \right)_{(d)}.$$

Thus, $h_i(p_i) = g(p)$, so that relation (2) reads

$$|h_i(p_i)| \le \varepsilon (1+\|p\|_2^2)^{d/2}, \quad i \in I. \tag{3}$$

We claim that each $h_i(\xi)$ is a polynomial in ξ of degree s. For each $i \in I$, rewrite

$$(1+\overline{p_1}x_1 + \cdots + \overline{\xi}x_i + \cdots + \overline{p_n}x_n)^s = x_i^s \overline{\xi}^s + \sum_{j=0}^{s-1} a_j^i \overline{\xi}^j,$$

where $a_j^i := a_j^i(p_1, \ldots, p_n; x_1, \ldots, x_n)$ polynomially depends on $p_1, \ldots, p_n$ and $x_1, \ldots, x_n$, and $\deg(a_j^i) = s$. Therefore, by elementary properties of complex vector spaces equipped with an inner product (see [14, Proposition 14.1.5]), we then obtain

$$\begin{aligned} h_i(\xi) &= \left(g, (1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^{d-s} \left(x_i^s \overline{\xi}^s + \sum_{j=0}^{s-1} a_j^i \overline{\xi}^j \right) \right)_{(d)} \\ &= \left(g, (1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^{d-s} x_i^s \right)_{(d)} \xi^s \\ &+ \sum_{j=0}^{s-1} \left(g, (1+\overline{p_1}x_1 + \cdots + \overline{p_n}x_n)^{d-s} a_j^i \right)_{(d)} \xi^j. \end{aligned} \tag{4}$$

Let us compute the first summand of the right-hand side of expression (4). By iteratively using Lemma 2.3, we get

$$
\begin{aligned}
\big(g, x_i^s(1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}\big)_{(d)} &= \big(g^{\mathrm{hom}}, x_i^s(x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}\big)_{(d)} \\
&= \overline{\big(x_i^s(x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}, g^{\mathrm{hom}}\big)_{(d)}} \\
&= \overline{\big(x_i x_i^{s-1}(x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}, g^{\mathrm{hom}}\big)_{(d)}} \\
&= \frac{\overline{\left(x_i^{s-1}(x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}, \frac{\partial g^{\mathrm{hom}}}{\partial x_i}\right)_{(d-1)}}}{d} \\
&\vdots \\
&= \frac{\overline{\left(x_i^{s-\ell}(x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}, \frac{\partial^\ell g^{\mathrm{hom}}}{\partial x_i^\ell}\right)_{(d-\ell)}}}{d(d-1)\dots(d-\ell+1)} \\
&\vdots \\
&= \frac{\overline{\left((x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}, \frac{\partial^s g^{\mathrm{hom}}}{\partial x_i^s}\right)_{(d-s)}}}{d(d-1)\dots(d-s+1)},
\end{aligned}
$$

where $2 \le \ell \le s-1$ and at each step the total degree of $\frac{\partial^\ell g^{\mathrm{hom}}}{\partial x_i^\ell}$, $\ell \le s$, equals $d-\ell \ge s-\ell$, since by Lemma 3.2 applied to g we have that x_i^s is an element of $\mathrm{Supp}(g)$, and so of $\mathrm{Supp}(g^{\mathrm{hom}})$, $i \in I$. Therefore,

$$
\begin{aligned}
\big(g, x_i^s(1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}\big)_{(d)} &= \frac{\left(\frac{\partial^s g^{\mathrm{hom}}}{\partial x_i^s}, (x_0+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}\right)_{(d-s)}}{d(d-1)\dots(d-s+1)} \\
&= \frac{\left(\frac{\partial^s g}{\partial x_i^s}, (1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}\right)_{(d-s)}}{d(d-1)\dots(d-s+1)} \\
&= \frac{1}{d(d-1)\dots(d-s+1)}\frac{\partial^s g}{\partial x_i^s}(p),
\end{aligned}
$$

the last equality being a consequence of Lemma 2.5. Thus, relation (4) becomes

$$
h_i(\xi) = \frac{(d-s)!}{d!}\frac{\partial^s g}{\partial x_i^s}(p)\xi^s + \sum_{j=0}^{s-1}\big(g, (1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d-s}a_j^i\big)_{(d)}\xi^j,
$$

that we rewrite in the form

$$
h_i(\xi) = k_i r_i(\xi),
$$

where $k_i := \frac{(d-s)!}{d!}\frac{\partial^s g}{\partial x_i^s}(p)$ is by assumption a non-zero constant and $r_i(\xi) \in \mathbb{C}[\xi]$. From Lemma 3.1 applied to the univariate polynomial $h_i(\xi)$, it thus follows that there exists a value $p_i' \in \mathbb{A}^1_{\mathbb{C}}$ such that $h_i(p_i') = 0$, and satisfying the condition

$$
|p_i' - p_i| \le \left(\frac{(1+\|p\|_2^2)^{d/2}}{k_i}\varepsilon\right)^{1/s} = \left(\frac{d!}{(d-s)!}\frac{(1+\|p\|_2^2)^{d/2}}{\left|\frac{\partial^s g}{\partial x_i^s}(p)\right|}\varepsilon\right)^{1/s}. \tag{5}
$$

Let now $u_1, \ldots, u_d$ be new variables. For each $i \in I$, let's consider the functions $\varphi_i = \varphi_i(u_1, \ldots, u_d) : \mathbb{A}^d_{\mathbb{C}} \to \mathbb{C}$ defined by Bombieri's scalar product, that is, φ_i maps $(u_1, \ldots, u_d)$ to

$$\big(g, (1 + \overline{p_1}x_1 + \cdots + \overline{u_1}x_i + \cdots + \overline{p_n}x_n) \cdots (1 + \overline{p_1}x_1 + \cdots + \overline{u_d}x_i + \cdots + \overline{p_n}x_n)\big)_{(d)}.$$

Each function $\varphi_i(u_1, \ldots, u_d)$ is linear with respect to the d variables $u_1, \ldots, u_d$ and invariant under permutation of them. Furthermore, from the definition of $h_i(\xi)$ and the equality $h_i(p'_i) = 0$, it follows that

$$\varphi_i(\underbrace{p_i, \ldots, p_i}_{d-s}, \underbrace{p'_i, \ldots, p'_i}_{s}) = h_i(p'_i) = 0. \tag{6}$$

Let D_i be the closed disk of the complex plane $\mathbb{C}$ centered at the point p_i and with radius $|p'_i - p_i|$. Applying Walsh's Contraction Principle to each polynomial $\varphi_i(u_1, \ldots, u_d) \in \mathbb{C}[u_1, \ldots, u_d]$, we conclude by (6) that there exists a point $z_i \in D_i$ such that $\varphi_i(z_i, \ldots, z_i) = 0$. For each index $i \in I$, take the point $q_i = (p_1, \ldots, p_{i-1}, z_i, p_{i+1}, \ldots, p_n) \in \mathbb{A}^n_{\mathbb{C}}$. Thus, from the previous vanishing condition, Lemma 2.5, and the definition of $\varphi_i(u_1, \ldots, u_d)$, we have

$$g(q_i) = \big(g, (1 + \overline{p_1}x_1 + \cdots + \overline{p_{i-1}}x_{i-1} + \overline{z_i}x_i + \overline{p_{i+1}}x_{i+1} + \cdots + \overline{p_n}x_n)^d\big)_{(d)} = \varphi_i(z_i, \ldots, z_i) = 0.$$

Therefore the points q_i, $i \in I$, belong to $g = 0$. Furthermore, since $z_i \in D_i$, inequality (5) gives

$$\|q_i - p\|_2 = |z_i - p_i| \leq |p'_i - p_i| \leq \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left|\frac{\partial^s g}{\partial x_i^s}(p)\right|} \varepsilon \right)^{1/s}.$$

Let $q \in \mathbb{A}^n_{\mathbb{C}}$ be the point such that $\|q - p\|_2 = \min_{i \in I} \|q_i - p\|_2$. The above bound then yields

$$\begin{aligned} \|q - p\|_2 \leq \min_{i \in I} \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left|\frac{\partial^s g}{\partial x_i^s}(p)\right|} \varepsilon \right)^{1/s} &= \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\max_{i \in I} \left|\frac{\partial^s g}{\partial x_i^s}(p)\right|} \varepsilon \right)^{1/s} \\ &= \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\max_{i=1,\ldots,n} \left|\frac{\partial^s g}{\partial x_i^s}(p)\right|} \varepsilon \right)^{1/s}. \end{aligned}$$

Thus, the result follows. Q.E.D.

The following is our main result: it provides an evaluation of how much the zero-locus of a polynomial locally varies if some (small enough) perturbations of its coefficients are permitted.

Theorem 3.4 *Let f and g be polynomials of P of degree d and let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a point of $f = 0$ of multiplicity $s \geq 1$ such that $\frac{\partial^s g}{\partial x_i^s}(p) \neq 0$ for some index i. Let ε be a positive real number, and suppose that $\|g - f\|_{(d')} \leq \varepsilon$, where $d' = \deg(g - f) \leq d$. Further assume that*

$$\varepsilon \leq \frac{(d-s)!}{2d!} \frac{\left\| \left(\frac{\partial^s f}{\partial x_1^s}(p), \ldots, \frac{\partial^s f}{\partial x_n^s}(p) \right) \right\|_1}{(1 + \|p\|_2^2)^{\frac{d-s}{2}}}.$$

Then there exists a point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ such that

$$\|q - p\|_2 \le \left(\frac{2d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left\| \left(\frac{\partial^s f}{\partial x_1^s}(p), \ldots, \frac{\partial^s f}{\partial x_n^s}(p) \right) \right\|_1} \right)^{1/s} \varepsilon^{1/s}.$$

Proof. Set $\mathcal{D}^s f := \left(\frac{\partial^s f}{\partial x_1^s}, \ldots, \frac{\partial^s f}{\partial x_n^s} \right)$ and $\mathcal{D}^s g := \left(\frac{\partial^s g}{\partial x_1^s}, \ldots, \frac{\partial^s g}{\partial x_n^s} \right)$, and use the estimate

$$\begin{aligned} \big| \|\mathcal{D}^s g(p)\|_1 - \|\mathcal{D}^s f(p)\|_1 \big| &\le \|\mathcal{D}^s g(p) - \mathcal{D}^s f(p)\|_1 \\ &= \|\mathcal{D}^s (g-f)(p)\|_1 = \max_{i=1,\ldots,n} \left| \frac{\partial^s (g-f)}{\partial x_i^s}(p) \right|. \end{aligned} \tag{7}$$

If the maximum in (7) is zero, we would have $\|\mathcal{D}^s g(p)\|_1 = \|\mathcal{D}^s f(p)\|_1$, so that by simply using Lemma 3.3 we would obtain

$$\|q - p\|_2 \le \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left\| \left(\frac{\partial^s f}{\partial x_1^s}(p), \ldots, \frac{\partial^s f}{\partial x_n^s}(p) \right) \right\|_1} \right)^{1/s} \varepsilon^{1/s},$$

from which the statement clearly follows.

Thus, we can assume that $\frac{\partial^s (g-f)}{\partial x_i^s}(p) \neq 0$ for some index $i \in \{1, \ldots, n\}$ (whence, in particular, $d' = \deg(g - f) \ge s$), and we confine to consider such indices i's for the rest of the proof. Then the same argument as above, by iteratively using Lemma 2.3, gives

$$\begin{aligned} \left(g - f, x_i^s (1 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}\right)_{(d')} &= \left((g-f)^{\mathrm{hom}}, x_i^s (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}\right)_{(d')} \\ &= \overline{\left(x_i^s (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}, (g-f)^{\mathrm{hom}}\right)_{(d')}} \\ &= \overline{\left(x_i x_i^{s-1} (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}, (g-f)^{\mathrm{hom}}\right)_{(d')}} \\ &= \frac{\overline{\left(x_i^{s-1} (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}, \frac{\partial (g-f)^{\mathrm{hom}}}{\partial x_i}\right)_{(d'-1)}}}{d'} \\ &\;\;\vdots \\ &= \frac{\overline{\left(x_i^{s-\ell} (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}, \frac{\partial^\ell (g-f)^{\mathrm{hom}}}{\partial^\ell x_i^\ell}\right)_{(d'-\ell)}}}{d'(d'-1)\ldots(d'-\ell+1)} \\ &\;\;\vdots \\ &= \frac{\overline{\left((x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}, \frac{\partial^s (g-f)^{\mathrm{hom}}}{\partial x_i^s}\right)_{(d'-s)}}}{d'(d'-1)\ldots(d'-s+1)}, \end{aligned}$$

where $2 \le \ell \le s - 1$ and at each step the total degree of $\frac{\partial^\ell (g-f)^{\mathrm{hom}}}{\partial x_i^\ell}$, $\ell \le s$, equals $d' - \ell \ge s' - \ell$, since by Lemma 3.2 applied to the polynomial $g - f$, we have that x_i^s is an element of $\mathrm{Supp}(g - f)$, and so of $\mathrm{Supp}\big((g - f)^{\mathrm{hom}}\big)$. Therefore,

$$\begin{aligned} \left(g - f, x_i^s (1 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}\right)_{(d')} &= \frac{\left(\frac{\partial^s (g-f)^{\mathrm{hom}}}{\partial x_i^s}, (x_0 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}\right)_{(d'-s)}}{d'(d'-1)\ldots(d'-s+1)} \\ &= \frac{\left(\frac{\partial^s (g-f)}{\partial x_i^s}, (1 + \overline{p_1} x_1 + \cdots + \overline{p_n} x_n)^{d'-s}\right)_{(d'-s)}}{d'(d'-1)\ldots(d'-s+1)} \\ &= \frac{1}{d'(d'-1)\ldots(d'-s+1)} \frac{\partial^s (g-f)}{\partial x_i^s}(p), \end{aligned}$$

the last equality being a consequence of Lemma 2.5. Thus,

$$\begin{aligned}
\left|\frac{\partial^s(g-f)}{\partial x_i^s}(p)\right| &= d'(d'-1)\dots(d'-s+1)\left|\left(g-f, x_i^s(1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d'-s}\right)_{(d')}\right| \\
&\le d'(d'-1)\dots(d'-s+1)\|g-f\|_{(d')}\ \|x_i^s(1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d'-s}\|_{(d')} \\
&\le d'(d'-1)\dots(d'-s+1)\|g-f\|_{(d')}\ \|(1+\overline{p_1}x_1+\cdots+\overline{p_n}x_n)^{d'-s}\|_{(d'-s)} \\
&\le d'(d'-1)\dots(d'-s+1)(1+\|p\|_2^2)^{\frac{d'-s}{2}}\varepsilon \\
&\le d(d-1)\dots(d-s+1)(1+\|p\|_2^2)^{\frac{d-s}{2}}\varepsilon \\
&= \frac{d!}{(d-s)!}(1+\|p\|_2^2)^{\frac{d-s}{2}}\varepsilon, \qquad (8)
\end{aligned}$$

where the first, the second, and the third inequalities follow from the Cauchy–Schwarz inequality (1), Lemma 2.6, and inequality (2), respectively.

From estimate (7), relation (8) and the assumption on ε, we find

$$\|\mathcal{D}^s g(p)\|_1 \ge \|\mathcal{D}^s f(p)\|_1 - \frac{d!}{(d-s)!}(1+\|p\|_2^2)^{\frac{d-s}{2}}\varepsilon \ge \|\mathcal{D}^s f(p)\|_1 - \frac{1}{2}\|\mathcal{D}^s f(p)\|_1 = \frac{1}{2}\|\mathcal{D}^s f(p)\|_1.$$

By combining the previous inequality with Lemma 3.3, the stated bound is proved. Q.E.D.

Let's note that, as in [3], both the estimates of Lemma 3.3 and Theorem 3.4 above are invariant under scalar multiplication. In fact, by multiplying the polynomials f and g by the same non-zero constant λ, we observe that the quantities $\left\|\left(\frac{\partial^s g}{\partial x_1^s}(p),\dots,\frac{\partial^s g}{\partial x_n^s}(p)\right)\right\|_1$ and $\left\|\left(\frac{\partial^s f}{\partial x_1^s}(p),\dots,\frac{\partial^s f}{\partial x_n^s}(p)\right)\right\|_1$ result in a multiplication by λ, so both the estimates are not modified.

Example 3.5 (Non-singular case) We aim to show how much sharp are the bounds given in Lemma 3.3 and Theorem 3.4. For simplicity, we consider the case of multiplicity $s=1$. Let $f = x_1^d+\cdots+x_{n-1}^d-1$ and $g = x_1^d+\cdots+x_{n-1}^d+\varepsilon x_n^d+\varepsilon\sqrt{\binom{d}{d/2}}x_1^{d/2}-1$, with even degree $d=2k$, and let $p=(1,0,\dots,0)$ be a (non-singular) point of $f=0$. It is easy to check that $\|g-f\|_{(d)}=\sqrt{2}\varepsilon$ and that the point

$$q = \left(\left(\sqrt{1+\frac{\varepsilon^2}{4}\binom{d}{d/2}}-\frac{\varepsilon}{2}\sqrt{\binom{d}{d/2}}\right)^{2/d},0,\dots,0\right)$$

is a real zero of $g=0$. Given a real function $\omega:\mathbb{R}^n\to\mathbb{R}$, we write $\omega(x_1,\dots,x_n)=\mathrm{O}(\varepsilon^r)$, $r\in\mathbb{N}$, to mean that $\frac{\omega(x_1,\dots,x_n)}{\varepsilon^r}$ is bounded near the origin. We then find

$$\begin{aligned}
\|q-p\|_2 &= \left|\left(\sqrt{1+\frac{\varepsilon^2}{4}\binom{d}{d/2}}-\frac{\varepsilon}{2}\sqrt{\binom{d}{d/2}}\right)^{2/d}-1\right| \\
&= \left|\left(1+\frac{\varepsilon^2}{8}\binom{d}{d/2}+\mathrm{O}(\varepsilon^4)-\frac{\varepsilon}{2}\sqrt{\binom{d}{d/2}}\right)^{2/d}-1\right| \\
&= \left|\left(1+\mathrm{O}(\varepsilon^2)-\frac{\varepsilon}{2}\sqrt{\binom{d}{d/2}}\right)^{2/d}-1\right| \\
&= \left|1-\frac{1}{d}\sqrt{\binom{d}{d/2}}\varepsilon+\mathrm{O}(\varepsilon^2)-1\right| = \frac{1}{d}\sqrt{\binom{d}{d/2}}\varepsilon+\mathrm{O}(\varepsilon^2) = \frac{1}{d}\frac{\sqrt{d!}}{(d/2)!}\varepsilon+\mathrm{O}(\varepsilon^2),
\end{aligned}$$

where the second and fourth equalities follow by Taylor series expansion of the quantities $\sqrt{1+\frac{\varepsilon^2}{4}\binom{d}{d/2}}$ and $\left(1+\mathrm{O}(\varepsilon^2)-\frac{\varepsilon}{2}\sqrt{\binom{d}{d/2}}\right)^{2/d}$, respectively. Thus, in a first order error analysis and by using Stirling's formula[1], we get

$$\begin{aligned}\|q-p\|_2 &= \frac{1}{d}\left(\left(\frac{d}{e}\right)^d\sqrt{2\pi d}\right)^{1/2}\left(\frac{e}{d/2}\right)^{d/2}\frac{1}{\sqrt{2\pi d/2}}\varepsilon \\ &= \frac{1}{d}\left(\frac{d}{e}\right)^{d/2}(2\pi d)^{1/4}\left(\frac{2e}{d}\right)^{d/2}\frac{1}{(\pi d)^{1/2}}\varepsilon \\ &= \frac{1}{d}2^{d/2}\left(\frac{2}{\pi d}\right)^{1/4}\varepsilon. \qquad (9)\end{aligned}$$

Then Theorem 3.4 yields the bound $2^{(d+1)/2}\varepsilon$, whose order of magnitude is comparable with the one given in (9).

Example 3.6 (Singular case) Let $f = f(x_1,\ldots,x_n) \in P$ be a degree d polynomial, with the origin $p = (0,\ldots,0)$ a point of multiplicity d of the hypersurface $f = 0$. Thus, f is a homogeneous form of degree d, that can be written as

$$f = c_1x_1^d + c_2x_2^d + \cdots + c_nx_n^d + h_d(x_1,\ldots,x_n),$$

where $h_d(x_1,\ldots,x_n)$ only contains mixed degree d terms in the x_i's.

For a positive real number ε, let $g = g(x_1,\ldots,x_n) = f - \varepsilon$. For each $i \in \{1,\ldots,n\}$ such that $c_i \neq 0$, consider

$$g(0,\ldots,0,x_i,0,\ldots,0) = f(0,\ldots,0,x_i,0,\ldots,0) - \varepsilon = c_ix_i^d - \varepsilon.$$

For such indices i's, letting $p_i := \left(0,\ldots,0,\left(\frac{\varepsilon}{c_i}\right)^{\frac{1}{d}},0,\ldots,0\right)$, we then have $g(p_i) = 0$. We note that $\|p_i - p\|_2 = \|p_i\|_2 = \left(\frac{\varepsilon}{|c_i|}\right)^{\frac{1}{d}}$. Therefore, among the points p_i's, that one which has minimum distance from p, say $i = 1$,

$$\|p_1 - p\|_2 = \left(\frac{\varepsilon}{|c_1|}\right)^{\frac{1}{d}} = \min_{i=1,\ldots,n}\|p_i - p\|_2, \qquad (10)$$

satisfies the condition $|c_1| = \max_{i=1,\ldots,n}|c_i|$. On the other hand, $\frac{\partial^d g}{\partial x_i^d}(\boldsymbol{x}) = d!c_i$, so that

$$\left\|\left(\frac{\partial^d g}{\partial x_1^d}(p),\ldots,\frac{\partial^d g}{\partial x_n^d}(p)\right)\right\|_1 = d!\max_{i=1,\ldots,n}|c_i| = d!|c_1|.$$

It then follows that our upper bound in Lemma 3.3, with $s = d$, becomes

$$\left(\frac{d!\ \varepsilon}{d!\ |c_1|}\right)^{\frac{1}{d}} = \left(\frac{\varepsilon}{|c_1|}\right)^{\frac{1}{d}}, \qquad (11)$$

showing that the bound is "almost" sharp (this, in view of (10) and noting that, as it is clear, $\|p_1\|_2 = \|p_1 - p\|_2 \geq \min_{i=1,\ldots,n}\{\|q - p\|_2 \mid g(q) = 0\}$).

To have an explicit example, consider the quartic curve of equation

$$f = f(x,y) = \left(\frac{1}{2}x + y\right)\left(\frac{1}{2}x - y\right)(2x + y)(2x - y) = x^4 + y^4 + h_4(x,y).$$

[1]The correct formulation is $\lim_{m\to+\infty}\frac{1}{m!}\sqrt{2\pi m}(\frac{m}{e})^m = 1$, which is often written as $m! \approx \sqrt{2\pi m}(\frac{m}{e})^m$, giving a good approximation of $m!$ for $m \gg 0$.

Take $\varepsilon = \frac{1}{10}$ and let $g = g(x,y) = f - \frac{1}{10}$. Since $c_1 = c_2 = 1$, we can restrict to either one of the x, y axis. E.g., consider

$$g(x,0) = f(x,0) - \frac{1}{10} = x^4 - \frac{1}{10}.$$

Then $g(p) = 0$, where $p = \big((\frac{1}{10})^{1/4}, 0\big)$, and the bound given by (11) is now $(\frac{1}{10})^{1/4}$. We observe that in this case the points $\big((\frac{1}{10})^{1/4}, 0\big)$, $\big(0, (\frac{1}{10})^{1/4}\big)$ minimize the distance of the perturbed curve $g(x,y) = 0$ from the origin.

Let us add the following numerical example, related to Example 0.1.

Example 3.7 In the affine plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$ consider the cubic curve $\mathcal{C}$ of equation $f(x,y) = (x-y)^3 + (x+y)^3 - x^2 + y^2 = 0$, the same Descartes Folium as in Example 0.1, up to a rotation of axes. Then consider the perturbed curve $\mathcal{C}'$ of equation $g(x,y) = f(x,y) - \frac{1}{100}$. The cubic $\mathcal{C}$ has a double point at the origin $p = (0,0)$.

We have $\|g - f\|_{(0)} = \frac{1}{100}$ and $\big\|\big(\frac{\partial^2 g}{\partial x^2}(p), \frac{\partial^2 g}{\partial y^2}(p)\big)\big\|_1 = 2$. Then apply Lemma 3.3 to conclude that there exists a point $q \in \mathbb{A}^2_{\mathbb{C}}$ such that $g(q) = 0$ and satisfying the condition

$$\|q - p\|_2 \leq \frac{\sqrt{3}}{10} \approx 0.179.$$

On the other hand, a standard tied minimum computation shows that the minimum distance when the point q varies on the curve $\mathcal{C}' : g = 0$ is $\|q - p\|_2 \approx 0.136$. For reader's convenience we include some detail. We consider the polynomial system, obtained via Lagrange multipliers,

$$\begin{cases} 2x + \mu(3(x-y)^2 + 3(x+y)^2 - 2x) = 0 \\ 2y + \mu(-3(x-y)^2 + 3(x+y)^2 + 2y) = 0 \\ (x-y)^3 + (x+y)^3 - x^2 + y^2 - \frac{1}{100} = 0 \end{cases}$$

By using the software CoCoA5 [8], we then solve it by computing the reduced Gröbner basis, with respect to the lexicographical order, of the ideal associated to the system.

4 A comparison with a first-order analysis perspective

We keep the setting and the notation as in Section 3. We specialize here to the non-singular case, in connection with a result of Dégot [9]. In such a case, Lemma 3.3 and Theorem 3.4 rewrite in the following form.

Theorem 4.1 *Let f and g be polynomials of P of degree d, and let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a non-singular point of $f = 0$ such that $\mathrm{Jac}_g(p)$ is non-zero. Suppose that $\|g - f\|_{(d')} \leq \varepsilon$ for some positive real number ε, where $d' = \deg(g - f) \leq d$. Then there exists a point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ such that:*

1. $\|q - p\|_2 \leq \dfrac{d(1 + \|p\|_2^2)^{d/2}}{\|\mathrm{Jac}_g(p)\|_1}\varepsilon.$

2. *If ε is small enough, namely $\varepsilon \leq \dfrac{1}{2}\dfrac{\|\mathrm{Jac}_f(p)\|_1}{d(1 + \|p\|_2^2)^{\frac{d-1}{2}}}$, then*

$$\|q - p\|_2 \leq \frac{2d(1 + \|p\|_2^2)^{d/2}}{\|\mathrm{Jac}_f(p)\|_1}\varepsilon.$$

Clearly, Theorem 4.1(2) yields an upper bound on the minimum of $\|q - p\|_2$ when the point q varies on the hypersurface $g = 0$. Precisely,

$$\min_{q \in \mathbb{A}^n_{\mathbb{C}}}\{\|q - p\|_2 \;\big|\; g(q) = 0\} \leq \frac{2d(1 + \|p\|_2^2)^{d/2}}{\|\mathrm{Jac}_f(p)\|_1}\varepsilon. \tag{12}$$

Now, consider Taylor's expansion of $g(\boldsymbol{x})$ at p, that is,

$$g(\boldsymbol{x}) = g(p) + \operatorname{Jac}_g(p)(\boldsymbol{x}-p)^t + \mathrm{O}(\|\boldsymbol{x}-p\|_2^2).$$

We want to find $\boldsymbol{x}$ such that $g(\boldsymbol{x}) = 0$, up to a first-order analysis, that is, disregarding second-order contributions. To this aim, by using the above expression and recalling that $f(p) = 0$, we find

$$(g-f)(p) + \operatorname{Jac}_g(p)(\boldsymbol{x}-p)^t = 0. \tag{13}$$

The solution $\boldsymbol{x} := q^*$, expressed by

$$(q^* - p)^t = -\frac{\operatorname{Jac}_g(p)^t}{\|\operatorname{Jac}_g(p)\|_2^2}(g-f)(p),$$

satisfies the minimality distance condition

$$\|q^* - p\|_2 = \min_{\boldsymbol{x}\in\mathbb{A}^n_{\mathbb{C}}}\{\|\boldsymbol{x}-p\|_2 \bigm| (g-f)(p) + \operatorname{Jac}_g(p)(\boldsymbol{x}-p)^t = 0\}.$$

To see this, compute the minimal distance from p to the hyperplane of equation (13). Consider the line through p with normal direction $\operatorname{Jac}_g(p)$, that is, the line ℓ of parametric equation $\boldsymbol{x} = p + u\operatorname{Jac}_g(p)$, $u \in \mathbb{R}$, or $(\boldsymbol{x}-p)^t = u\operatorname{Jac}_g(p)^t$. Intersecting the hyperplane with ℓ, we find

$$u = -\frac{(g-f)(p)}{\|\operatorname{Jac}_g(p)\|_2^2}.$$

Thus, $(\boldsymbol{x}-p)^t = -\frac{\operatorname{Jac}_g(p)^t}{\|\operatorname{Jac}_g(p)\|_2^2}(g-f)(p)$, which gives the solution q^* we are searching for.

Letting $\Delta p := q^* - p$, we summarize the linear approximation process as described above, by writing

$$\Delta p \approx \min_{q\in\mathbb{A}^n_{\mathbb{C}}}\{\|q-p\|_2 \bigm| g(q) = 0\},$$

and we say that Δp is the *first-order perturbation of p, corresponding to the infinitesimal perturbation $g-f$ of f* (in the sense that $\|g-f\|_{(d')} \leq \varepsilon$).

In [9, Section 2], extending a result of Shub and Smale [16] to the more general case of polynomial systems with less equations than unknowns, Dégot provides a bound, which, in the previous setting, rewrites as

$$\|\Delta p\|_2 \leq \frac{\sqrt{d}(1+\|p\|_2^2)^{d/2}}{\|\operatorname{Jac}_{f^{\mathrm{hom}}}(1,p)\|_2}\varepsilon. \tag{14}$$

Since clearly $\|\operatorname{Jac}_{f^{\mathrm{hom}}}(1,p)\|_2 \geq \|\operatorname{Jac}_f(p)\|_2 \geq \|\operatorname{Jac}_f(p)\|_1$, the above bound is smaller than the bound provided in Theorem 4.1(2). Nevertheless, the two bounds give estimates for two different quantities; namely, the minimum of $\|q-p\|_2$ when the point q varies either on the hypersurface $g(\boldsymbol{x}) = 0$ or on the hyperplane of equation (13).

Thus, to compare the first-order analysis perspective as in [9] with our approach based on Walsh's Contraction Principle, it would be meaningful to exhibit examples satisfying the condition

$$\|\Delta p\|_2 \leq \frac{\sqrt{d}(1+\|p\|_2^2)^{d/2}}{\|\operatorname{Jac}_{f^{\mathrm{hom}}}(1,p)\|_2}\varepsilon \leq \|q-p\|_2.$$

In the univariate case the Wilkinson polynomial shows how the location of the roots may be very sensitive to even small perturbations of polynomial's coefficients.

Example 4.2 We consider the degree $d = 20$ polynomial $f(x) = \prod_{i=1}^{20}(x-i) \in \mathbb{C}[x]$ and a small perturbation $g(x)$ of $f(x)$, that is,

$$g(x) = f(x) + \varepsilon\sqrt{\binom{20}{19}}x^{19}, \quad \varepsilon \in \mathbb{R}.$$

In order to compare with Dégot's bound, let's pass to homogenization. We then find $\|g^{\text{hom}} - f^{\text{hom}}\|_{(20)} = |\varepsilon|$. We choose $\varepsilon = 2^{-32}$. Take the root $p = 20$ of $f = 0$, and denote by q the root of $g = 0$ nearest to p. According to (14), we find

$$\frac{\sqrt{d}(1 + \|p\|_2^2)^{d/2}}{\|\text{Jac}_{f^{\text{hom}}}(1, p)\|_2}\varepsilon \approx 0.045,$$

which does not upper bound the roots difference $|q - p| \approx 0.05$.

5 Real perturbations of real polynomials

The results discussed in Section 3 are based on Walsh's Contraction Theorem. Though the Walsh Theorem is false over the reals (see the Example 5.1 below), it is natural to ask for perturbation results over the real numbers analogous to those over the complex numbers. In this section we present a perturbation result for real polynomials based on Rouché's theorem, a classical result used to locate regions of the complex plane in which an analytic function has zeros (see [11, Theorem III.7.7]). We also refer to [12] for related results.

Example 5.1 Consider the polynomial $f(x, y) = xy + 1$, linear and symmetric with respect to the variables x, y. Then $f(-1, 1) = 0$. Assuming Walsh's Contraction Principle valid over the reals, it would exist a (closed) circular region D of $\mathbb{R}$ (in the sense of Theorem 1.2) and a real point $x \in D$ such that $f(x, x) = 0$, leading to the contradiction $x^2 = -1$.

Theorem 5.2 (Rouché) *Let f, h be analytic functions defined on a simply-connected open set $D \subset \mathbb{C}$ and let $\mathcal{C}$ be a closed non-singular curve in D. Assume that*

$$|f(\zeta)| > |h(\zeta)| \ \text{ for each point } \zeta \in \mathcal{C}.$$

Then the functions f, $f + h$ have no zeros on $\mathcal{C}$ and f and $f + h$ have the same number of zeros, counting multiplicities in the interior of the open set bounded by $\mathcal{C}$.

Having in mind some specific applications in the real context (see [18, Section 4]), it looks in fact quite appropriate to ask how much the real zero loci of real polynomials vary up to small real perturbations of their coefficients (instead of considering Bombieri's norm as done before).

In the univariate case, Theorem 5.3 and Remark 5.4 summarize our result.

Theorem 5.3 *Let f and g be monic polynomials in $\mathbb{R}[x]$ of the same degree d. Let $p \in \mathbb{A}^1_{\mathbb{C}}$ be a real solution of $f = 0$ of odd multiplicity $s \geq 1$. Let ε be a positive real number, and let $g - f = \sum_{i=0}^{d} \varepsilon_i x^i$ with $|\varepsilon_i| \leq \varepsilon$. Then, for ε small enough, there exists a real $0 < r < 1$ such that the polynomial g has a real zero in the disk $\Delta_r \subset \mathbb{C}$ of radius r centered at p.*

Proof. Let's first consider the case of multiplicity $s = 1$. We can clearly assume that p is the origin of the coordinates of the complex plane. Let $f = \sum_{i=1}^{d} a_i x^i$. Assume $d \geq 2$. Then, for $z \in \mathbb{C}$, we can write

$$\begin{aligned} |f(z)| &= |z||a_1 + a_2 z + a_3 z^2 + \cdots + a_d z^{d-1}| \\ &\geq |z|\big(|a_1| - |z|(|a_2| + |a_3||z| + \cdots + |a_d||z^{d-2}|)\big). \end{aligned} \tag{15}$$

Note that there is some small $r > 0$ with the right-hand side of inequality (15) strictly positive for $0 < |z| \leq r$.

Since f, g are monic polynomials, then $h := g - f = \sum_{i=0}^{d-1} \varepsilon_i x^i$ is a polynomial of degree $\leq d - 1$. By assumption, we have

$$\begin{aligned} |h(z)| &\leq \varepsilon + \varepsilon|z| + \cdots + \varepsilon|z|^{d-1} \\ &= \varepsilon\frac{1 - |z|^d}{1 - |z|} \leq \varepsilon\frac{1}{1 - |z|}, \end{aligned} \tag{16}$$

for $|z| < 1$.

Assume also that

$$r < \frac{|a_1|}{2\big(|a_2| + |a_3| + \cdots + |a_d|\big)},$$

and let

$$c := r\big(|a_1| - r(|a_2| + |a_3|r + \cdots + |a_d|r^{d-2})\big).$$

By the assumptions on r it then follows that

$$\begin{aligned} c &> r\big(|a_1| - r(|a_2| + |a_3| + \cdots + |a_d|)\big) > \\ &> r\left(|a_1| - \frac{|a_1|}{2\big(|a_2| + |a_3| + \cdots + |a_d|\big)}(|a_2| + |a_3| + \cdots + |a_d|)\right) \\ &= \frac{r|a_1|}{2}. \end{aligned}$$

Thus, as soon as $r > \frac{2\varepsilon}{|a_1|}$, one has $1 - \frac{\varepsilon}{c} > 0$, so that we can choose

$$\left(\frac{2\varepsilon}{|a_1|} <\right) r < 1 - \frac{\varepsilon}{c}. \tag{17}$$

Hence $c(1-r) > \varepsilon$, which, by definition of c, is the same as

$$r\big(|a_1| - r(|a_2| + |a_3|r + \cdots + |a_d|r^{d-2})\big) > \frac{\varepsilon}{1-r}. \tag{18}$$

By combining relations (15), (16), and (18), we finally have

$$|f(z)| > \frac{\varepsilon}{1-r} \geq |h(z)|,$$

whence $|f(z)| > |h(z)|$ for each point $z \in \partial\Delta_r$. Thus, applying Theorem 5.2 with $\mathcal{C} := \partial\Delta_r$, we conclude that the polynomials f and $f + h = g$ have in the interior of $\mathcal{C}$ the same number of zeros, counting multiplicities. Hence g has a zero, say q, in the interior of the disk. If q is not real, then the conjugate point $\overline{q}$ (which belongs to the interior of the disk) would be as well a zero of g. This contradicts Rouché's theorem since f has only p as zero in the disk.

In the case $d = 1$, the key inequality $|f(z)| > |h(z)|$ holds true as soon as we assume $r < 1$ and $r|a_1| > \frac{\varepsilon}{1-r}$ (somehow, a particular case of (18)), without the need of using the quantity c.

Up to minor changes, the same argument as above extends to the case of odd multiplicity $s \geq 3$.

Q.E.D.

Let us make more explicit how ε and r depend on the polynomial f.

Remark 5.4 (Computing the quantities ε and r) Notation as in the proof of Theorem 5.3. With no loss of generality, we can assume p to be the origin of the complex plane. Moreover, we can as well suppose $d \geq 2$. First, assume that p is a non-singular point. Recall that $\frac{2\varepsilon}{|a_1|} = \frac{2\varepsilon}{|f'(p)|} < r < 1$. Further assume

$$r < \frac{\delta|a_1|}{|a_2| + |a_3| + \cdots + |a_d|}, \tag{19}$$

for some $0 < \delta < 1$. Then, since $c > r\big(|a_1| - r(|a_2| + |a_3| + \cdots + |a_d|)\big)$,

$$\begin{aligned} c(1-r) &> r\big(|a_1| - r(|a_2| + |a_3| + \cdots + |a_d|)\big)(1-r) \\ &> r(|a_1| - \delta|a_1|)(1-r) = r|a_1|(1-\delta)(1-r). \end{aligned} \tag{20}$$

We see that the inequality $r|a_1|(1-\delta)(1-r) > \varepsilon$, or $r^2 - r + \frac{\varepsilon}{|a_1|(1-\delta)} < 0$, holds true as soon as

$$r_1 := \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{(1-\delta)|a_1|}} < r < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{(1-\delta)|a_1|}} =: r_2, \tag{21}$$

and

$$\varepsilon < \frac{(1-\delta)|a_1|}{4} \tag{22}$$

Note also that both the left-hand and the right-hand terms r_j in (21) always satisfy the conditions $0 < r_j < 1$, $j = 1, 2$.

Thus, we conclude that the key inequality (18) in the proof of Theorem 5.3 can be explicitly expressed by mean of inequalities (21) and (22) involving r and ε. This also shows that both these quantities only depend (up to the choice of the constant δ) on the derivative $|f'(p)| = |a_1|$.

It is just the case to note that, if p is a singular point of multiplicity $s > 1$, then conditions (19), (21) and (22) become,

$$r < \frac{\delta|a_s|}{|a_{s+1}| + |a_{s+2}| + \cdots + |a_d|},$$

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{(1-\delta)|a_s|}} < r < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{(1-\delta)|a_s|}},$$

and

$$\varepsilon < \frac{(1-\delta)|a_s|}{4}.$$

For smooth points, Theorem 5.3 extends to the multivariate case as follows. In the sequel, we set $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$, $n > 1$.

Theorem 5.5 *Let f and g be polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, of the same degree d. Further assume that f and g have the same leading form. Let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a real point of $f = 0$ such that $\mathrm{Jac}_f(p)$ is non-zero. Let ε be a positive real number, and let $g - f = \sum_{|\boldsymbol{\alpha}| \leq d} \varepsilon_{\boldsymbol{\alpha}} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $|\varepsilon_{\boldsymbol{\alpha}}| \leq \dfrac{\varepsilon}{n^{d/2}}$. Then, for ε small enough, there exists a real $0 < r < 1$ such that the polynomial g has a real zero in the ball $\mathbf{B}_r(p) \subset \mathbb{C}^n$ of radius r centered at p.*

Proof. We can assume $p = (0, \ldots, 0)$ to be the origin in $\mathbb{A}^n_{\mathbb{C}}$. Let $f = \sum_{|\boldsymbol{\alpha}| \leq d} c_{\boldsymbol{\alpha}} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and consider the line ℓ through the origin of equations $\ell : x_i = u_i t$, $i = 1, \ldots, n$, for a complex parameter $t \in \mathbb{C}$, where

$$u_i := \frac{\partial f}{\partial x_i}(p) \frac{1}{\sqrt{\sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(p)\right|^2}}, \quad i = 1, \ldots, n.$$

That is, $u := (u_1, \ldots, u_n)$ is the unit vector along the normal direction to the hypersurface $f = 0$ at p. Restricting $f = 0$ to ℓ, we find the polynomial, of degree $d_\ell \leq d$,

$$f_\ell(t) := f(u_1 t, \ldots, u_n t) = \sum_{|\boldsymbol{\alpha}| \leq d_\ell} c_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} t^{|\boldsymbol{\alpha}|} = \sum_{k=1}^{d_\ell} \Big(\sum_{|\boldsymbol{\alpha}| = k} c_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} \Big) t^k =: \sum_{k=1}^{d_\ell} a_i t^k,$$

where $u^{\boldsymbol{\alpha}} = u_1^{\alpha_1} \ldots u_n^{\alpha_n}$. Note also that, denoting by the prime symbol the derivative with respect to t, we have $f'_\ell(t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}(t)$, whence

$$f'_\ell(0) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}(p) = 1 \ (\neq 0),$$

so that $t = 0$ is a real solution of multiplicity 1 of the polynomial $f_\ell(t) \in \mathbb{C}[t]$. Then, for $z \in \mathbb{C}$, we can write as in the proof of Theorem 5.3,

$$\begin{array}{lcll} |f_\ell(z)| & \geq & |z|\big(|a_1| - |z|(|a_2| + |a_3||z| + \cdots + |a_d||z^{d_\ell - 2}|)\big) & \text{if} \quad d_\ell \geq 2 \\ |f_\ell(z)| & = & |z||a_1| & \text{if} \quad d_\ell = 1, \end{array}$$

where the right-hand side of the above inequality is > 0 for $|z|$ small.

The assumption that f, g have the same leading form allows us to conclude that, when restricting to ℓ, the polynomial $h_\ell := g_\ell - f_\ell$ has degree $\leq d_\ell$. Then

$$h_\ell(t) := \sum_{|\boldsymbol{\alpha}| \leq d_\ell} \varepsilon_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} t^{|\boldsymbol{\alpha}|} = \sum_{k=0}^{d_\ell} \Big(\sum_{|\boldsymbol{\alpha}|=k} \varepsilon_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} \Big) t^k.$$

To upper bound $|h_\ell(z)|$ for $z \in \mathbb{C}$, let's compute

$$\Big| \sum_{|\boldsymbol{\alpha}|=k} \varepsilon_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} \Big| \leq \sum_{|\boldsymbol{\alpha}|=k} |\varepsilon_{\boldsymbol{\alpha}}||u^{\boldsymbol{\alpha}}| \leq \frac{\varepsilon}{n^{d/2}} \sum_{|\boldsymbol{\alpha}|=k} |u^{\boldsymbol{\alpha}}|. \tag{23}$$

Moreover,

$$\sum_{|\boldsymbol{\alpha}|=k} |u^{\boldsymbol{\alpha}}| \leq \sum_{|\boldsymbol{\alpha}|=k} \binom{k}{\boldsymbol{\alpha}} |u^{\boldsymbol{\alpha}}| = \Big(\sum_{i=1}^n |u_i| \Big)^k \leq n^{k/2}, \tag{24}$$

where $\binom{k}{\boldsymbol{\alpha}} = \frac{k!}{\alpha_1! \ldots \alpha_n!}$, the equality follows from the multinomial theorem, and the right-hand inequality is a consequence of

$$\sum_{i=1}^n |u_i| \leq \sqrt{\sum_{i=1}^n |u_i|^2} \sqrt{n} = \sqrt{n}. \tag{25}$$

Thus, we get

$$\Big| \sum_{|\boldsymbol{\alpha}|=k} \varepsilon_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} \Big| \leq \frac{\varepsilon}{n^{d/2}} n^{k/2} \leq \varepsilon, \tag{26}$$

which allows us to obtain

$$|h_\ell(z)| \leq \varepsilon \sum_{k=0}^{d_\ell} |z|^k = \varepsilon \frac{1 - |z|^{d_\ell + 1}}{1 - |z|} \leq \varepsilon \frac{1}{1 - |z|}, \tag{27}$$

as soon as we take $r := |z| < 1$. Now, exactly the same argument as in the proof of Theorem 5.3 applies to conclude that

$$|f_\ell(z)| > |h_\ell(z)| \ \text{ for each point } \ z \in \partial \Delta_r,$$

where $\Delta_r \subset \mathbb{R}^2$ is the disk of radius r centered at the origin. And again, by using Theorem 5.2, we conclude that g has a real zero, say $\overline{q}$, when restricted to the line ℓ. Let $\overline{t}$ be the corresponding value of the parameter t. Thus, the hypersurface $g = 0$ vanishes at the real point $(u_1\overline{t}, \ldots, u_n\overline{t})$, belonging to the ball $\mathbf{B}_r(p) \subset \mathbb{C}^n$ of radius r centered at p.

Remark 5.6 With the notation as above, and assuming that p is the origin of the complex plane, let $f_\ell(t) = \sum_{k=1}^{d_\ell} \Big(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} u^{\boldsymbol{\alpha}} \Big) t^k =: \sum_{k=1}^{d_\ell} a_k t^k$. As far as the the computation of the quantities ε and r is concerned, note that conditions (19), (21) and (22) as in Remark 5.4 hold still true, up to restricting to $f_\ell(t)$ and writing d_ℓ instead of d.

The following example suggests that the bounds in (the proof of) Theorem 5.5 should be best possible.

Example 5.7 Let $f(x,y) = 4(x+y) - 2\sqrt{2}(x-y)^2 + 2(x-y)^3$, and let $p = (0,0)$ be a point of $f = 0$. One has $\mathrm{Jac}_f(p) = (4,4)$. In this case, $d = 3$, $n = 2$, so that $n^{d/2} = 2\sqrt{2}$. Let $g(x,y) := f(x,y) + \frac{1}{2}$. Then f and g have the same leading form, and $g - f = \frac{1}{2} = \varepsilon_{\boldsymbol{\alpha}} = \frac{\varepsilon}{n^{d/2}} = \frac{1}{2}$, with $\varepsilon = \sqrt{2}$. Moreover,

$$(u_1, u_2) = \frac{\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right)}{\sqrt{\left|\frac{\partial f}{\partial x}(p)\right|^2 + \left|\frac{\partial f}{\partial y}(p)\right|^2}} = \frac{(4,4)}{\sqrt{32}} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

Computing, we find,

$$f_\ell(t) = f(u_1 t, u_2 t) = 4\sqrt{2}t,$$

and

$$h_\ell(t) = (g_\ell - f_\ell)(t) = g(u_1 t, u_2 t) - f(u_1 t, u_2 t) = \frac{1}{2}.$$

In particular, $\deg(f_\ell(t)) = 1$, $\deg(h_\ell(t)) = 0$, leading to equalities in formulas (23), (24), (25). While, in (26), one has equality on the left-hand side with strict inequality on the right-hand side.

6 A final comment

Coming back to Walsh's Contraction Theorem, we would like to end by interpreting it as a statement in several complex variables, and posing some questions raised by that interpretation. Let S_N denote the symmetric group on $\{1, \ldots, N\}$ and let $\sigma_1, \ldots, \sigma_N$ denote the N symmetric functions

$$\sigma_j(z_1, \ldots, z_N) := \sum_{\tau \in S_N} z_{\tau(1)} \cdots z_{\tau(j)},$$

in the variables $z_1, \ldots, z_N$. Let $\sigma : \mathbb{C}^N \to \mathbb{C}^N$ be the map given by sending $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ to $(\sigma_1(z), \ldots, \sigma_N(z)) \in \mathbb{C}^N$. Let Δ^N denote the open polydisk where $\Delta \subset \mathbb{C}$ is a disk of some fixed radius; let D denote the diagonal of Δ^N; let $P = \sigma(\Delta^N)$; and let $\mathcal{D} = \sigma(D)$.

Theorem 6.1 (Walsh's Contraction Theorem) *Let $L(z) = a_0 + a_1 z_1 + \cdots + a_N z_N$ be any linear function. It follows that $L(P) = L(\mathcal{D})$.*

The question this raises is:

> For what other pairs of a bounded domain B in $\mathbb{C}^N$ and a curve $\mathcal{C} \subset B$ does Theorem 6.1 hold with $(B, \mathcal{C})$ in place of $(P, \mathcal{D})$?

In particular, if we replace the σ_j with the power functions

$$\pi_j(z) := z_1^j + \cdots + z_N^j$$

and let π denote the map sending $z \in \mathbb{C}^N$ to $(\pi_1(z), \ldots, \pi_N(z))$, does Theorem 6.1 hold with $(\pi(\Delta^N), \pi(D))$ in place of $(P, \mathcal{D})$?

The question is most interesting with the restriction that $\deg(\mathcal{C}) \leq N$.

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