REAL REPRESENTATIONS OF PREFERENCES AND APPLICATIONS

Dottorato di ricerca in "Assicurazione e Finanza:Matematica e Gestione", Universit`a di Trieste,

ANNO ACCADEMICO 2014-2015,
XXVIII ciclo

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Chapter 1

General introduction

In this thesis we first review some results concerning the real representation of preferences. Different kind of preferences are considered and a particular attention is devoted to the case of intransitivity of the associated indifference relation. Therefore, interval orders and semiorders appear as relevant types of binary relations for which indifference is not transitive. It is particulary interesting to guarantee the existence of continuous or at least upper semicontinuous representations. Homogeneity of the representations is studied in connection with the aforementioned continuity when the space of preferences is endowed with an algebraic structure.

As relevant applications of the above results, we are then concerned with the space L^1_+ (L^2_+) of integrable (respectively, square integrable) random variables which very frequently comes into consideration in the literature. In the case of a total preorder, we deal with the concept of a certainty equivalent and some results are presented involving both positive homogeneity and translation invariance. We then prove some results concerning the existence of a certainty equivalent for interval orders. As a further relevant application, we study catastrophic risks and present various impossibility theorems excluding the existence of (upper semi)continuous representations in connection with a property of a preorder called safety-first principle. Such a property, which has been recently investigated in the literature, seems natural when dealing with catastrophic risks, but actually it is extremely restrictive.

The thesis is structured as follows.

- Chapter 2 presents the classical axioms concerning binary relations which are defined on a set. In general, such a set is not endowed neither with a topological nor with an algebraic stucture. The concept of an order-preserving function on a preordered set is presented. We also define the concept of an interval order together with its real representation by means of a pair of real-valued functions. The semiorder case appears as a very interesting particular case of an interval order. Typically, the possibility of a threshold representation (u, δ) is investigated.

- Chapter 3 recalls the main topological notions concerning a metric space and more generally a topological space. The notion of upper semicontinuity and continuity of a binary relation on a topological space is here presented.

- Chapter 4 presents the concepts of semicontinuous and continuous functions. Continuity of interval orders on a topological space is considered.

- Chapter 5 concerns binary relations among random variables. Homogeneous representations are studied for both total preorders and interval orders.

- Chapter 6 is devoted to certainty equivalence. The relevant properties of translation invariant and subadditivity of a certainty equivalence functional are considered. We introduce some possibilities for defining a certainty equivalence functional for interval orders.

- Chapter 7 concerns catastrophic risks. The classical concepts of stochastic dominance are recalled. The safety-first principle excludes the possibility of various types of semicontinuous representations. Therefore we present some new impossibility results excluding various semicontinuous representations.

Chapter 2

Binary relations and their representation

2.1 Introduction

In this chapter we present the basic concepts concerning the binary relations and their real representations. These are the basic models that are adopted in order to describe individual preferences in economics and social sciences.

We are concerned not only with (*partial*) *preorders*, but also with *interval orders* and *semiorders*. Indeed, these two latter models of preferences are known to be of particular interest since not only the intransitivity of the *indifference* is allowed, but also because, under not very restrictive assumptions, preferences ofthis kind can be fully characterized by means of two real-valued functions. In particular, after the section where we present the general definitions,we introduce in particular the interval orders in Section 2.3, and then the semiorders in Section 2.4.

Since the seminal work of Fishburn [30], it is of particular interest to consider the so called *traces* associated to an intervabrder or a semiorder.

These are total preorders naturally associated to the originalinterval order (or semiorder). Needless to say, the consideration of the traces is particularly important not only for theoretical, say, reasons but also in order to guarantee more easily the existence of a real representation.

The adoption of interval orders is motivated by introducing examples. We further present the *set theoretical characterizations* of interval orders (i.e., the characterizations of interval orders based on convenient assumptions concerning the set of all the *lower sections*). Indeed, it is shown that a preference is an interval order if and only if the set of all the lower sections is linearly (totally) ordered by set inclusion.

2.2 Definitions and preliminaries

In the following definition we summarize the basic assumption that may concern a binary relation on a set.

Definition 2.2.1 [**axioms concerning binary relations**]. A binary relation R on a nonempty set X (i.e. a subset of the Cartesian product $X \times X$) is said to be [1]

(i) *reflexive,* if $xRx \quad \forall x \in X,$

(ii) *irreflexive,* if $xRy \Rightarrow \neg(yRx) \quad \forall x, y \in X,$

(iii) *transitive,* if $(xRy) \wedge (yRz) \Rightarrow xRz \quad \forall x, y, z \in X,$

(iv) *negatively transitive,* if $\neg(xRy) \wedge \neg(yRz) \Rightarrow \neg(xRz) \quad \forall x, y, z \in X,$

[1] In what follows, given a binary relation R on a set X, for any two elements $x, y \in X$ we shall write xRy instead of $(x, y) \in R$.

(v) *symmetric,* *if* $xRy \Rightarrow yRx \quad \forall x, y \in X$,

(vi) *asymmetric,* *if* $xRy \Rightarrow \neg(yRx) \quad \forall x, y \in X$,

(vii) *antisymmetric,* *if* $(xRy) \wedge (yRx) \Rightarrow x = y \quad \forall x, y \in X$,

($viii$) *acyclic,* *if* $(x_0 R x_1) \wedge (x_2 R x_3) \wedge \ldots \wedge (x_{n-1} R x_n) \Rightarrow \neg(x_n R x_0) \forall n \geq 1, \forall x_0, \ldots, x_n \in X$,

(ix) *total,* *if* $(xRy) \vee (yRx) \quad \forall x, y \in X$,

(x) *complete,* *if* $(xRy) \vee (yRx) \quad \forall x, y \in X$ *such that* $x \neq y$.

The pair (X, R) will be referred to as a related set.

Let us observe that, given any related set (X, R),

(i) if R is total, then it is reflexive;
(ii) if R is asymmetric, then it is irreflexive;
(iii) if R is irreflexive and transitive, then it is acyclic;
(iv) if R is acyclic, then it is asymmetric and not necessarily transitive.

Definition 2.2.2 [preorders] A *preorder* - on a nonempty set X is a binary relation on X which is reflexive and transitive. If in addition - is antisymmetric, then we shall refer to - as an *order*. If X is a nonempty set, and - is a preorder (order) on X, then the related set $(X, -)$ will be referred to as a *preordered set* (respectively, an *ordered set*).

Definition 2.2.3 [total preorder]. A preordered set (X, -) is said to be *totally preordered* if the binary relation - on X is *total*. (i.e. if $(x - y) \vee (y - x)$ $\forall x, y \in X$).

Definition 2.2.4 [strict part]. The strict part $\prec$ of a preorder - is defined as follows for all $x, y \in X$: $x \prec y \Leftrightarrow (x - y)$ *and* *not* $(y - x)$.

Definition 2.2.5 [non trivial]. A preorder - is said to be *non trivial* if there exist $x, y \in X$ such that $x \prec y$.

Definition 2.2.6 [increasing function]. Given a preordered set $(X, -)$, a function $f : (X, -) \rightarrow (\mathbf{R}, \leq)$ is said to be *a real-valued increasing function* on $(X, -)$ if

$$(x, y \in X) \wedge (x - y) \Rightarrow f(x) \leq f(y)].$$

The existence of a real-valued increasing function f on a preordered set $(X, -)$ does not give enough information on the preorder -. Indeed, given any constant real-valued function f on an arbitrary set X, for every preorder - on X we have that f is increasing on the preordered set $(X, -)$. Nevertheless, given any preorder - on an arbitrary set X, if there exists a countable family $\{f_n\}_{n \in \mathbf{N}^+}$ of real-valued increasing functions on $(X, -)$ such that for every $x, y \in X$ there exists $n \in \mathbf{N}^+$ with $f_n(x) < f_n(y)$, then the real-valued function f on $(X, -)$ defined by $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ is order-preserving. So there is a strict connection between the existence of an order preserving function for a given preorder - and the existence of a countable family of increasing functions which separate points in the graph of the asymmetric part $\prec$ of -.

Definition 2.2.7 [order-preserving function]. Given a preordered set $(X, -)$, a function $f : (X, -) \rightarrow (\mathbf{R}, \leq)$ is said *order-preserving* function on $(X, -)$ if it is increasing on $(X, -)$ and

$$(x, y \in X) \wedge (x \prec y) \Rightarrow f(x) < f(y).$$

In this case, f is also said to be a *utility function* on $(X, -)$.

If f is a real-valued function on a totally preordered set $(X, -)$, then, for every $x, y \in X$,

$$x - y \Leftrightarrow f(x) \leq f(y).$$

This is the classical definition of a utility function.

Given an order-preserving function f on a preordered set $(X, -)$, it is clear that the composition $f' = \phi \circ f$ of f with any strictly increasing real-valued function in f(X) (i.e., a function $\phi : (f(X), \leq) \rightarrow (\mathbf{R}, \leq)$) is also an order-preserving function on $(X, -)$.

2.3 Interval orders

In order to allow non-transitivity of the indifference relation the concept of interval order was introduced (see Fishburn[30]).

Definition 2.3.1 [interval order]. A binary relation - on a set X is said to be an *interval order* if - is reflexive and the following condition holds for all $x, y, z, w \in X$:

$$(x - y) \wedge (z - w) \Rightarrow (x - w) \vee (z - y) \; \forall x, y, z, w \in X.$$

Interval orders are particular interesting since they are not transitive in general. It is clear that a total preorder is in particular an interval order.

Nevertheless we observe that the strict part $\prec$ of any interval order - is transitive. Moreover, an interval order - on a set X may be fully described by means of a pair of real-valued functions. This fact will be illustrated in the sequel.

Example 2.3.1 [An intuitive idea of what an interval order means].

Suppose that two new models of cars (say A and B) appear in the market. The price of any of them could vary from a car-seller to another depending of bargaining, so that the car A could be sold at prices that vary in an interval $[m_A, M_A]$,where m_A is the minimum price of the car A among the car-sellers, and M_A is its maximum price. Obviously the same thing will happen to the car B, that could be sold at prices varying in an interval $[m_B, M_B]$. It is plain that one will consider that the car B is (undoubtably) more expensive than the car A if $M_A < m_B$. Also, if the intervals $[m_A, M_A]$ and $[m_B, M_B]$ meet, there is at least a price at which both cars could be sold, so that it could be said that those cars are, in some sense, indifferent. The indifference is not transitive, in general.

A more classical example which is usually referred to when justifying the adoption of a model of preferences with intransitive indifference refers to the cup of sugar. It is due to Armstrong [4].

Considers a man that prefers a cup of coffee with a whole portion of sugar, to a cup of coffee with no sugar at all. If such man is forced to declare his preference between a cup with no sugar at all and a cup with only one molecule of sugar, he will declare them indifferent. The same will occur if he compares a cup with n molecules and a cup with $n + 1$ *molecules of sugar.*

However, after a very large number of intermediate comparisons we would finally confront him with a cup that has a whole portion of sugar that he is able to discriminate from the cup with no sugar at all. Here, we observe a clear intransitivity of indifferences.

Definition 2.3.2 [pair of functions representing an interval order]. An interval order $\precsim$ on a set X is represented by a pair (u, v) of real-valued functions on X if for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

Notice that we must have $u(x) \leq v(x)$ for all $x \in X$ due to the fact that $\precsim$ is reflexive. We are particularly interested in the existence of a pair (u, v) of functions that satisfy algebraic and/or topological properties when the set X is endowed with some algebraic and topological structure.

If $\precsim$ is any binary relation on X such that there exists a pair (u, v) of real-valued functions on X with $x \precsim y \Leftrightarrow u(x) \leq v(y)$ for all $x, y \in X$, and $u(x) \leq v(x)$ for all $x \in X$, then $\precsim$ is necessarily an interval order. Indeed, it is clear that $\precsim$ is reflexive. Further, since for all $x, y, z, w \in X$, $(x \precsim y) \wedge (z \precsim w) \wedge (x \not\precsim w)$ is equivalent to $(u(x) \leq v(y)) \wedge (u(z) \leq v(w)) \wedge (v(w) < u(x))$, which in turn implies that $u(z) \leq v(y)$. Therefore, $\precsim$ is an interval order according to Definition 2.3.1.

It is immediate to check that an interval order $\precsim$ on a set X is pseudo-transitive, in the sense that, for every $x, x', y, y' \in X$,

$$x \prec x' \precsim y' \prec y \Rightarrow x \prec y.$$

The interpretation of the existence of a pair of real-valued functions (u, v) representing an interval order $\precsim$ on a set X is the following: it is possible to associate to each element $x \in X$ a closed interval $[u(x), v(x)]$ in such a way that, for all $x, y \in X$, we have that $x \prec y$ if and only if the interval $[u(x), v(x)]$ associated to x is completely to the left with respect to the interval $[u(y), v(y)]$ associated to y.

In general a binary relation $\precsim$ on X is said to be *total* if for two elements $x, y \in X$, either $x \precsim y$ or $y \precsim x$. It is clear that an interval order is not necessarily transitive, and that a *total preorder* is also an interval order. Further, an interval order is total (see Oloriz, Candeal and Indur´ain [37]).

Proposition 2.3.1 [an interval order is total]. Let $\precsim$ be an interval order on a set X. Then $\precsim$ is total.

Proof. Let $\precsim$ be an interval order on a set X and assume that $\precsim$ is not total. Then there exist $x, y \in X$ such that $x \not\precsim y$ and $y \not\precsim x$. Hence either $x \not\precsim x$ or $y \not\precsim y$, and this is contradictiory since $\precsim$ is reflexive. This consideration completes the proof.

Definition 2.3.3 [associated binary relations]. Let $\precsim$ be an interval order on a set X. Then define two binary relations (*traces*) $\precsim^*$ and $\precsim^{**}$ on X as follows:

$$x \precsim^* y \Leftrightarrow (z \precsim x \Rightarrow z \precsim y, \forall z \in X) \ (x, y \in X),$$

$$x \precsim^{**} y \Leftrightarrow (y \precsim z \Rightarrow x \precsim z, \forall z \in X) \ (x, y \in X).$$

Definition 2.3.4 [partial orders]. A partial order $\prec$ on a nonempty set X is a binary relation on X which is irreflexive and transitive. In this case, the related set $(X, \prec)$ will be referred to as a partially ordered set.

Definition 2.3.5 [incomparability relation]. Given a partial order $\prec$ on a set X, define, for every $x, y \in X$,

(2.1) $x \precsim y \Leftrightarrow \neg(y \prec x)$,

(2.2) $x \sim y \Leftrightarrow \neg(x \prec y) \wedge \neg(y \prec x)$.

The binary relations $\precsim$ and $\sim$ defined above will be called the *preference-incomparability relation* and the *incomparability relation* associated to the partial order $\prec$.

Definition 2.3.6 [weak orders]. A weak order $\prec$ on a nonempty set X is a binary relation on X which is asymmetric and negatively transitive. In this case, the pair $(X, \prec)$ will be referred to as a weakly ordered set.

Proposition 2.3.2 [weak orders and total preorders]. Let $(X, \precsim)$ be a totally preordered set. Then the asymmetric part $\prec$ of $\precsim$ is a *weak order* on X. Conversely, if $(X, \prec)$ is a weakly ordered set, then the preference-indifference relation $\precsim$ associated to $\prec$ is a total preorder on X.

Proof. Let $(X, \precsim)$ be a totally preordered set, and consider the binary relation $\prec$. Since it is clear that $\prec$ is asymmetric, let us show that $\prec$ is negatively transitive. Consider $x, y, z \in X$ such that $\neg(x \prec y) \wedge \neg(y \prec z)$. Then, using the fact that $\precsim$ is total, we obtain $(y \precsim x) \wedge (z \precsim y)$, which in turn implies $z \precsim x$ since $\precsim$ is transitive. Therefore $x \prec z$ is contradictory. Conversely, let $(X, \prec)$ be a weakly ordered set, and consider the preference-indifference relation $\precsim$. Since $\prec$ is irreflexive, it is clear that $\precsim$ is reflexive. Observe that transitivity of $\precsim$ is equivalent to negative transitivity of $\prec$. Finally, let us show that $\precsim$ is total. Assume that there exist two elements

$x, y \in X$ such that $\neg(x \precsim y) \wedge \neg(y \precsim x)$. Then we have $(y \prec x) \wedge (x \prec y)$, and this is contradictory since $\prec$ is transitive and asymmetric.

Proposition 2.3.3 [associated total preorders]. Let $\precsim$ be a reflexive binary relation on a set X. Then $\precsim$ is an interval order if and only if the associated binary relations $\precsim^*$ and $\precsim^{**}$ in Definition 2.3.3 are both total preorders.

Proof. Let $\precsim$ be any reflexive binary relation on a set X. First assume that $\precsim$ is an interval order. Let us prove that the associated binary relation $\precsim^*$ in Definition 2.3.3 is a total preorder. In order to show that $\precsim^*$ is transitive, consider $x, y, z \in X$ with $x \precsim^* y \precsim^* z$. Then for all $w \in X$, $w \precsim x$ entails $w \precsim y$, which in turn entails $w \precsim z$, and therefore we have $x \precsim^* z$ from the definition of the binary relation $\precsim^*$. In order to show that $\precsim^*$ is total, assume by contraposition that there exist two elements $x, y \in X$ such that neither $x \precsim^* y$ nor $y \precsim^* x$. Then from the definition of $\precsim^*$ there exist two elements $z, w \in X$ with $z \precsim x$, $z \not\precsim y$, $w \precsim y$, $w \not\precsim x$, and this contradicts the fact that $\precsim$ is an interval order. Hence $\precsim^*$ is a total preorder. Analogously it can be proven that $\precsim^{**}$ is a total preorder. Conversely, assume that the binary relations $\precsim^*$ and $\precsim^{**}$ are both total preorders. In order to show that $\precsim$ is an interval order, consider $x, y, z, w \in X$ such that $x \precsim y$ and $z \precsim w$. If neither $x \precsim w$ nor $z \precsim y$, then the asymmetric property would be violated since $w \prec x \precsim y \Rightarrow w \prec^* y$ and $y \prec z \precsim w \Rightarrow y \prec^* w$, we have that both $w \prec^* y$ and $y \prec^* w$, and this is contradictory since $\prec^*$ is a weak order by Proposition 2.3.2. Hence $\precsim$ must be an interval order and the proof is complete.

It easy to prove that, in addition, an interval order $\precsim$ is transitive (that is $\precsim$ is a total preorder) if and only if $\precsim$, $\precsim^*$, and $\precsim^{**}$ coincide.

Definition 2.3.7 [i.o.-separability and strong i.o.-separability]. We say that an interval order - on X is *(strongly) i.o.-separable* if there exists a countable set $D \subseteq X$ such that, for every $x, y \in X$ with $x \prec y$, there exists $d \in D$ with $x \prec d$ - ** y (respectively, there exists a countable set $D \subseteq X$ such that, for every $x, y \in X$ with $x \prec y$, there exists $d \in D$ with $x \prec d \prec^{**} y$). D is said to be a (strongly) i.o.-dense subset of X (see Oloriz, Candeal and Indur´ain [37]).

It is clear that the assumption of strong i.o.-separability is weaker that the assumption of strong separability introduced by Chateauneuf [23], according to which there exists a countable set $D \subseteq X$ such that, for every $x, y \in X$ with $x \prec y$, there exist $d_1, d_2 \in D$ with $x \prec d_1$ - $d_2 \prec y$.

Set theoretical characterizations

Definition 2.3.8 [lower and upper sections]. If (X, R) is a related set, then define, for every $x \in X$,

(2.3) $L_R(x) = \{z \in X : zRx\}, \qquad U_R(x) = \{z \in X : xRz\}.$

$L_R(x)$ and $U_R(x)$ are said to be the lower section and respectively the upper section of the element $x \in X$ according to the binary relation R.When there is no ambiguity about the binary relation involved, the subscript R will be omitted, and we shall simply write $L(x)$ and $U(x)$. Moreover define

(2.4) $L_R = \{L_R(x) : x \in X\}, \qquad U_R = \{U_R(x) : x \in X\}.$

As usual, reflexive and irreflexive binary relations will be denoted by -, and respectively by $\prec$.

Let us define $L(X)_{\precsim}$ as the collection of all the weak lower sections of an interval order (in particular a total preorder) $\precsim$ on a set X, that is

$$L(X)_{\precsim} = \{L_{\precsim}(x) : x \in X\}.$$

Proposition 2.3.4 [lower sections in a totally preordered set]. Let $(X, \precsim)$ be a preordered set. Then the preorder $\precsim$ on X is a total if and only if $L(X)_{\precsim}$ is totally ordered by set inclusion.

Proof. Consider any preordered set $(X, \precsim)$. If $\precsim$ is total, then for two elements $x, y \in X$ either $x \precsim y$ or $y \precsim x$. Hence, by transitivity of $\precsim$, either $L_{\precsim}(x) \subseteq L_{\precsim}(y)$ or $L_{\precsim}(y) \subseteq L_{\precsim}(x)$, and therefore $L(X)_{\precsim}$ is totally ordered by set inclusion. In order to show that if $L(X)_{\precsim}$ is totally ordered by set inclusion then the preorder $\precsim$ on X is total, assume by contraposition that $\precsim$ is not total. Then there exist two elements $x, y \in X$ such that neither $x \precsim y$ nor $y \precsim x$. Hence we have $x \notin L_{\precsim}(y)$ and $y \notin L_{\precsim}(x)$. Since it is clear that $x \in L_{\precsim}(x)$ and $y \in L_{\precsim}(y)$ by reflexivity of $\precsim$, $L(X)_{\precsim}$ is not totally ordered by set inclusion, since neither $L_{\precsim}(x) \subseteq L_{\precsim}(y)$ nor $L_{\precsim}(y) \subseteq L_{\precsim}(x)$. This consideration completes the proof.

The following proposition is well known and it is due to Rabinovitch [41]

Proposition 2.3.5 [lower sections of an interval order]. Let $\precsim$ be a reflexive binary relation on a set X. Then $\precsim$ is an interval order if and only if $L(X)_{\precsim}$ is totally ordered by set inclusion.

Proof. Let $\precsim$ be a reflexive binary relation on a set X. First assume that $\precsim$ is an interval order and consider any two elements $x, y \in X$. If $L_{\precsim}(x) \not\subseteq L_{\precsim}(y)$ then there exists an element $z \in L(x) \setminus L_{\precsim}(y)$. Since $z \precsim x$, $z \not\precsim y$, we have that $w \in L_{\precsim}(y)$ entails $w \in L_{\precsim}(x)$. Indeed, $z \precsim x$ and $w \precsim y$ entails either $z \precsim y$ or $w \precsim x$, and $z \precsim y$ is not true. Hence, $L(X)_{\precsim}$ is totally ordered by set inclusion. Conversely, assume that $L(X)_{\precsim}$ is totally ordered by set inclusion, and consider four elements $x, y, z, w \in X$ such that $x \precsim y$ and $z \precsim w$. If $x \quad w$, then we have $x \in L_{\precsim}(y)$ and $x \notin L_{\precsim}(w)$, and therefore $L_{\precsim}(y) \not\subseteq L_{\precsim}(w)$. Since it must be $L_{\precsim}(w) \subseteq L_{\precsim}(y)$, $z \precsim w$ entails $z \precsim y$. So the proof is complete.

Proposition 2.3.6 [sections of associated preorders]. Let $\precsim$ be an interval order on a set X. Then the following assertions hold for every $x, y \in X$:

(i) $x \prec y \Rightarrow (L_{\prec^*}(x) \subset L_{\prec}(y)) \wedge (U_{\prec^{**}}(y) \subset U_{\prec}(x))$;
(ii) $x \prec^* y \Rightarrow U_{\prec}(y) \subseteq U_{\prec}(x)$;
(ii) $x \prec^{**} y \Rightarrow L_{\prec}(x) \subseteq L_{\prec}(y)$.

Proof. Let $\precsim$ be an interval order on a set X.
(i). Consider any two points $x, y \in X$ such that $x \prec y$. If $z \in X$ is any point such that $z \prec^* x$, then from the definition of the total prorder $\precsim^*$ there exists $\xi \in X$ such that $z \prec \xi \precsim x \prec y$, and therefore it must be $z \prec y$ since $\precsim$ is an interval order. Further, $x \prec y$ but clearly $x \not\prec^* x$. Hence we have shown that $L_{\prec^*}(x) \subset L_{\prec}(y)$ whenever $x \prec y$. Analogously it can be proven that $U_{\prec^{**}}(y) \subset U_{\prec}(x)$ whenever $x \prec y$.
(ii). If $x, y \in X$ are such that $x \prec^* y$, then there exists $\xi \in X$ with $x \prec \xi \precsim y$, and therefore $y \prec z$ entails $x \prec z$ for every $z \in X$. Hence $U_{\prec}(y) \subseteq U_{\prec}(x)$ whenever $x \prec^* y$.
(iii) The proof is perfectly analogous to the proof of the statement (ii).

2.4 Semiorders

Let us now consider the relevant case of a semiorder (see e.g. Pirlot and Vincke [40]).

In many fields connected with decision-aid (economy, operations research, acturial sciences,finance), the potencial decisions (projects,candidates,...) are evaluated on quantitative criteria, so that comparing decisions is equivalent to comparing numbers. The classical model underlying all these fields is the following:

if A is the set of potencial decisions and g the function which associates a value g(a) to every element a of A, then, decision a "is at least as good as" decision b (a % *b) iff g(a)≥g(b).*

Making the distinction between the relation "is strictly better than" and the relation "is as good as" (i.e. the asymmetric and symmetric parts of relation %) one obtains,

$$\begin{cases} a \succ b \ iff \ g(a) > g(b) \\ a \sim b \ iff \ g(a) = g(b) \end{cases}.$$

However, reflection suggests that it is not very reasonable to consider that a decision a is strictly better than b as soon as the value of a is higher than the value of b; the unavoidable imprecisions on the evaluations of the decisions often force to consider as equal, values which are very close to each other. This leads to the introduction of a positive threshold q (indifference, sensitivity or tolerance threshold) such that, $\forall\ a, b \in A,$

$$\begin{cases} a \succ b \ iff \ g(a) > g(b) + q \\ a \sim b \ iff \ | \ g(a) = g(b) \ | \leq q \end{cases}$$

$$a \succsim b \ \textit{iff} \ \ g(a) \geq g(b) - q.$$

Such a relation $\succsim$ will be called a semiorder and its asymmetric part $\succ$ will be called a strict semiorder.

It is important to note that $\succsim$ is reflexive and total so that the knowledge of $\succ$ ("strictly better" relation) implies that of $\succsim$ ("at least as good " relation); this is due to the fact that the "as good as" relation is given by

$$a \sim b \ \textit{iff} \ \ \neg(a \succ b) \textit{and} \neg(b \succ a).$$

More generally, the threshold q may vary along the numerical scale of the values of g. This will be the case, for example, if a and b are considered as indifferent when the difference between their values is smaller than a percentage of the smallest of themIt can be proved that if, $\forall a, b \in A,$

$$g(a) > g(b) \Rightarrow g(a) + q_a \geq g(b) + q_b,$$

(where q_a *and* q_b are thresholds respectively associated to $g(a)$ and $g(b)$), then relation $\succsim$ is still a semiorder (which means that $\succsim$ has the same mathematical properties as when the threshold is constant). The concept of semiorder is also encountered when the evaluation of each decision is an interval between a minimal (pessimistic) value and a maximal (optimistic) value. A possible attitude consists, in such a situation, to declare that decision a is strictly better than b if the interval associated to a lies entirely to the right of the interval associated to bBoth decisions are then considered as indifferent when their intervals have a non-empty intersectionIf no interval is strictly included in another, we obtain again a semiorder; indeed, denoting g(a), the left end point of the interval associated to a and $g(a) + q$ its right end point leads to the proceding situation.

Definition 2.4.1 [semiorder]. A binary relation - on a set X is said to be a *semiorder* if - is an interval order and the following condition holds for all $x, y, z, w \in X$:

$$(x - y) \wedge (y - z) \Rightarrow (x - w) \vee (w - z) \; \forall x, y, z, w \in X.$$

Definition 2.4.2 [threshold representation]. A semiorder - on a set X is represented by a real-valued function u on X and a *threshold* $\delta > 0$ (that is, it admits a representation (u, δ) for short) if for all $x, y \in X$,

$$x - y \Leftrightarrow u(x) \leq u(y) + \delta.$$

The concept of a semiorder was apparently first introduced by Luce [35] to deal with inaccuracies in measurements where a nonnegative threshold of discrimination is considered. The original idea of Luce was to present a mathematical model of preferences enable to capture situations of intransitive indifference with a threshold of discrimination:

Suppose, for instance, that a man is not able to declare different two quantities of a same thing when such two quantities do not differ more than a threshold of discrimination or perception, α. This threshold is a non-negative real number, and it is supposed to be the same for every individual. That is, if $a \prec b$ *means here a man is able to realize that the quantity a is smaller than b, then we have* $a \prec b \Leftrightarrow a + \alpha < b$.

We recall that the concept of semiorder first appears in Wiener [45] with a different name. It was Luce [35] who developed this important concept.

Then Fishburn [30] presented a deep study about preferences with intransitive indifferences and their real representations. The threshold representations were first studied By Scott and Suppes [43].The famous *Scott-Suppes Theorem* shows that every semiorder - on a finite set X admits a threshold representation $(u, 1)$.

Remark 2.4.1. It should be noted that the definition of a threshold representation could be formulated in terms of the strict part $\prec$ of -. Indeed if a semiorder - admits a threshold representation (u, δ) then, for all $x, y \in X$, $x \prec y \Leftrightarrow u(x) + \delta < u(y)$.

Let us now prove the following simple proposition which motivates the introduction of a threshold representation (u, δ) in connection with semiorders.

Proposition 2.4.1 [threshold implies semiorder]. Let - be any binary relation satisfying the following condition for some real-valued function u on X and a threshold $\delta > 0$;

$$x - y \Leftrightarrow u(x) \leq u(y) + \delta \qquad \text{for all } x, y \in X.$$

Then - is a semiorder.

Proof. It is clear that - is reflexive due to the fact that $u(x) < u(x) + \delta$ for all $x \in X$. In addition, the pair $(u, u + \delta)$ represents - as interval order. Finally, if for some $x, y, z, w \in X$ it occurs that $w \prec x - y - z \prec w$, then we have that $u(w)+\delta < u(x) \leq u(y)+\delta \leq u(z)+2\delta < u(w)+\delta$, a contradiction.

Definition 2.4.3 [weak order associated to a semiorder]. If - is a preference relation on a set X, then define, for $x, y \in X$,

$$x \prec^0 y \Leftrightarrow (x \prec^* y) \; or \; (x \prec^{**} y).$$

Fishburn (1970) proved that if $\prec$ is a semiorder, then $\prec^0$ is a weak order, and therefore the associated preference-indifference relation $-^0$ is a total preorder. It is straightforward to prove that this result may be strengthened as follows.

Proposition 2.4.2 Let $\prec$ be an interval order on a set X. Then $\prec^0$ is asymmetric if and only if $\prec$ is a semiorder, in which case $-^0$ is a total preorder.

Proof . Let $\prec$ be an interval order on a set X. Then both $\prec^*$ and $\prec^{**}$ are asymmetric. If $\prec$ is a semiorder, then $x \not\prec^* y$ entails $x -^{**} y$, and $x \prec^{**} y$ entails $x -^* y$, as well. So it is easily seen that $\prec^0$ is asymmetric. Conversely, assume that $\prec$ is an interval order, and that $\prec^0$ is asymmetric. Suppose that there exist x, ξ, η, y with $x - \eta \prec y \prec \xi - x$. Hence $x \prec^{**} y$ and $y \prec^* x$, and this is impossible because $\prec^0$ is asymmetric. This contradiction completes the proof.

It is clear that if - is an interval order for which $\prec^* = \prec^{**}$, then - is semiorder. Indeed, in this case $\prec^0 = \prec^* = \prec^{**}$ is asymmetric by Proposition 2.3.3.

From Proposition 2.4.2, we may refer to $-^0$ as the total preorder intimately connected with the semiorder $\prec$. Is is easily seen that if $\prec$ is a semiorder, then

$$x -^0 y \Leftrightarrow (U_\prec(y) \subseteq U_\prec(x)) \; and \; (L_\prec(x) \subseteq L_\prec(y)).$$

The following corollary relates the utility functions representing the total preorder $-^*$ and $-^{**}$, respectively, to the utility function representing $-^0$.

Corollary 2.4.1 Let $\prec$ be a semiorder on a set X. If u and v are utility functions for $\precsim^{**}$ and $\precsim^{*}$, respectively, then $u + v$ is a utility function for $\precsim^{0}$.

Proof. Let $\prec$ be a semiorder on a set X, and assume that u and v are utility functions for $\precsim^{**}$ and $\precsim^{*}$, respectively. From Proposition 2.4.2, $\precsim^{0}$ is a total preorder. First consider x, y with $x \precsim^{0} y$. Then both $x \precsim^{**} y$ and $x \precsim^{*} y$. So $x \precsim^{0} y$ entails $u(x) + v(x) \leq u(y) + v(y)$. Now consider x, y with $y \prec^{0} x$, that is $not(x \precsim^{0} y)$. Then either $y \prec^{*} x$ or $y \prec^{**} x$. If $y \prec^{*} x$, then $v(y) < v(x)$, and this entalis $u(y) \leq u(x)$. So $y \prec^{0} x$ entails $u(y) + v(y) < u(x) + v(x)$. In a perfectly analogous way we proceed in case that $y \prec^{**} x$. Now the proof is complete (see Bosi [14, Corollary 1]).

Numerical representations of interval orders

We recall that a pair (u, v) of real-valued functions on X is said to represent an interval order $\precsim$ on X if, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

It should be noted that a strict interval order $\prec$ on a set X is then represented by a pair (u, v) of real-valued functions on X if, for all $x, y \in X$,

$$[x \prec y \Leftrightarrow v(x) < u(y)].$$

Definition 2.4.4 [weak utility]. A real-valued function u on X is said to be a weak utility for an interval order $\prec$ if for all $x, y \in X,$

$$x \prec y \Rightarrow u(x) < u(y).$$

It is immediate to check that if (u, v) is a representation of an interval order - on X, then u is a weak utility for the weak order $\prec^{**}$. More generally, the following proposition holds.

Proposition 2.4.3 Let - be an interval order relation on a nonempty set X that is represented by a pair of real-valued functions (u, v). Then we have that

- u is a weak utility for $\prec^{**}$.
- v is a weak utility for $\prec^{*}$.

Proof . $x \prec^{**} y \Longleftrightarrow \exists \xi \in X:\quad x$ - $\xi \prec y \Rightarrow u(x) \leq v(\xi) < u(y) \Rightarrow u(x) < u(y)$.

Proof. $x \prec^{*} y \Longleftrightarrow \exists \eta \in X:\quad x \prec \eta$ - $y \Rightarrow v(x) < u(\eta) \leq v(y) \Rightarrow v(x) < v(y)$.

Chapter 3

Topological and vector related spaces

3.1 Introduction

In this chapter we recall the basic definitions concerning *metric spaces* (more generally *topological spaces*) and *vector spaces*. Indeed, we shall then be concerned with the real representation of preferences on such structures. Typically, algebraic and/or continuity properties concerning the representing functions are required with a view to the applications to economics. After reviewing the basic concepts concerning metric and topological spaces in Section 3.2, we are then concerned in Section 3.3 with *topological related spaces*. In particular the concept of *continuity* of a binary relation is presented. Such a concept is particularly interesting in connection with the existence of *continuous* or at least *upper semicontinuous* representations of binary relations.

3.2 Basic definitions

Definition 3.2.1 [metric space]. A *metric space* (X, d) consists of a non-empty set X and a function $d : X \times X \to [0, \infty)$ such that the following conditions are verified:

(i) *Positivity*. For all $x, y \in X$, $d(x, y) \geq 0$ with equality if and only if $x = y$.

(ii) *Symmetry*. For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iii) *Triangle Inequality*. For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A function d satisfying conditions (i)-(iii), is called a *metric* on X.

Definition 3.2.2 [vector space]. A *vector space* is a set V on which two operations + and · are defined, called *vector addition* and *scalar multiplication*:

The operation + (vector addition) must satisfy the following conditions:

(1) *Closure*: If u and v are any vectors in V, then the sum $u+v$ belongs to V.

(2) *Commutative law* : For all vectors u and v in V, $u + v = v + u$

(3) *Associative law* : For all vectors u, v, w in V, $u+(v+w) = (u+v)+w$

(4) *Additive identity* : The set V contains an additive identity element,

denoted by 0, such that for any vector v in V , $0 + v = v$ and $v + 0 = v$.

(5) *Additive inverses*: For each vector v in V , the equations $v + x = 0$ and $x + v = 0$ have a solution x in V , called an additive inverse of v, and denoted by - v.

The operation · (scalar multiplication) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

(6) *Closure*: If v is any vector in V , and c is any real number, then the product $c \cdot v$ belongs to V .

(7) *Distributive law* : For all real numbers c and all vectors u, v in V , $c \cdot (u + v) = c \cdot u + c \cdot v$

(8) *Distributive law* : For all real numbers c, d and all vectors v in V , $(c + d) \cdot v = c \cdot v + d \cdot v$

(9) *Associative law* : For all real numbers c, d and all vectors v in V, $c \cdot (d \cdot v) = (cd) \cdot v$

(10) *Unitary law* : For all vectors v in V, $1 \cdot v = v$

Definition 3.2.3 [norm of a metric].

If V is a *(real) vector space*, a function $| \cdot | : V \rightarrow$ R is called a *norm* if the following conditions are satisfied:

(i) For all $x \in V$, $|x| \geq 0$ with equality if and only if $x = 0$.

(ii) $|\alpha x| = |\alpha||x|$ for all $\alpha \in$ R and all $x \in V$.

(iii) $|x + y| \leq |x| + |y|$ for all $x, y \in V$.

Convergence and continuity.

We begin the study of metric spaces by defining convergence of sequences. A sequence $\{x_n\}$ in a metric space X is just a collection $\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ of elements in X enumerated by the natural numbers.

Definition 3.2.4 [Convergence]. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if for every $\varepsilon > 0$ there exists an $N \in$ N such that $d(x_n, a) < \varepsilon$ for all n $\geq$ N. We write $\lim_{n\to\infty} x_n = a$ or $x_n \to a$.

Lemma 3.2.1. A sequence $\{x_n\}$ in a metric space (X, d) converges to a if and only if $\lim_{n\to\infty} d(x_n, a) = 0$.

Proof. The distances $\{d(x_n, a)\}$ form a sequence of non negative numbers. The sequence converges to 0 if and only if there for every $\varepsilon > 0$ exists an $N \in$ N such that $d(x_n, a) < \varepsilon$ when n $\geq$ N. But this is exactly what the definition says.

Proposition 3.2.1. A sequence in a metric space cannot converge to more than one point.

Proof. Assume that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$. We must show that this is only possible if $a = b$. According to the triangle inequality $d(a, b) \leq d(a, x_n) + d(x_n, b)$. Taking limits, we get :

$$d(a, b) \leq \lim_{n\to\infty} d(a, x_n) + \lim_{n\to\infty} d(x_n, b) = 0 + 0 = 0$$

Consequently, $d(a, b) = 0$, and according to point (i) (positivity) in the definition of metric spaces, $a = b$.

Open and closed sets.

Definition 3.2.5 [Open ball] Let a be a point in a metric space (X, d), and assume that r is a positive, real number. The *(open) ball centered at a with radius r* is the set :

$$B(a; r) = \{x \in X | d(x, a) < r\}$$

Definition 3.2.6 [Closed ball] Let a be a point in a metric space (X, d), and assume that r is a positive, real number. The *closed ball centered at a with radius r* is the set :

$$\overline{B}(a; r) = \{x \in X | d(x, a) \leq r\}$$

If A is a subset of X and x is a point in X, there are three possibilities:

(i) There is a ball $B(x; r)$ around x which is contained in A. In this case x is called an *interior point* of A.

(ii) There is a ball $B(x; r)$ around x which is contained in the complement A^c. In this case x is called an *exterior point* of A.

(iii) All balls $B(x; r)$ around x contains points in A as well as points in the complement A^c. In this case x is a *boundary point* of A.

Note that an interior point always belongs to A, while an exterior point never belongs to A. A boundary point will some times belong to A, and some times to A^c.

Proposition 3.2.2. A subset A of a metric space is open if it does not contain any of its boundary points, and it is closed if it contains all its boundary points.

Most sets contain some, but not all of their boundary points, and are hence neither open nor closed. The empty set $\emptyset$ and the entire space X are both open and closed as they do not have any boundary points. Here is an obvious, but useful reformulation of the definition of an open set.

Proposition 3.2.3. A subset A of a metric space X is *open* if and only if it only consists of interior points, i.e. for all $a \in A$, there is a ball $B(a; r)$ around a which is contained in A.

Observe that a set A and its complement A^c have exactly the same boundary points. This leads to the following useful result.

Proposition 3.2.4. A subset A of a metric space X is open if and only if its complement A^c is closed.

Proof. If A is open, it does not contain any of the (common) boundary points. Hence they all belong to A^c, and A^c must be closed. Conversely, if A^c is closed, it contains all boundary points, and hence A can not have any. This means that A is open.

We can also phrase the notion of convergence in more geometric terms. If a is an element of a metric space X, and r is a positive number, the (open) *ball centered at a with radius r* is the set:

$$B(a; r) = \{x \in X | d(x, a) < r\}$$

as the terminology suggests, we think of $B(a; r)$ as *a ball around a with radius r*. Note that $x \in B(a; r)$ means exactly the same as $d(x, a) \leq r$. The definition of convergence can now rephrased by saying that $\{x_n\}$ converges to a if the terms of the sequence $\{x_n\}$ eventually end up inside any ball B(a; ε) around a. So this proof is complete.

Definition 3.2.7 [Continuity in metric spaces]. Assume that (X, d_X), (Y, d_Y) are two metric spaces. A function $f : X \to Y$ is continuous at a point $a \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$.

This definition says exactly the same as the usual definitions of continuity for functions of one or several variables; we can get the distance between $f(x)$ and $f(a)$ smaller than ϵ by choosing x such that the distance between x and a is smaller than δ. The only difference is that we are now using the metrics d_X and d_Y to measure the distances. A more geometric formulation of the definition is to say that for any open ball $B(f(a); \epsilon)$ around $f(a)$, there is an open ball $B(a, \delta)$ around a such that $f(B(a; \delta)) \subset B(f(a); \epsilon)$.

There is a close connection between continuity and convergence which reflects our intuitive feeling that f is continuous at a point a if $f(x)$ approaches $f(a)$ whenever x approaches a.

Definition 3.2.8 [continuity in metric spaces]. A function $f: X \to Y$ between two metrics spaces is called continuous if is continuous at all points x in X.

3.3 Topological related spaces

Definition 3.3.1 [topological space]. A family τ of subsets of a nonempty set X (called the family of all the open sets) is a *topology* on X if the following conditions are verified:

(i) $X, \emptyset \in \tau$;

(ii) the union of an arbitrary family of sets all belonging to τ belongs to τ ;

(iii) the intersection of a finite family of sets all belonging to τ belongs to τ .

The pair (X, τ) is said to be a *topological space*.

Definition 3.3.2 [closure]. Given a topological space (X, τ), the (topological) closure $\overline{U}$ of any subset U of X is the intersection of all the closed subsets of X containing U .

Definition 3.3.3 [continuous function]. A function f of a topological space (X, τ) into a topological space (Y, θ) is continuous if $f^{-1}(V) = \{x \in X : f(x) \in V\} \in \tau$ for every $V \in \theta$.

Definition 3.3.4 [directed sets and nets]. A directed set $(I, <)$ is any nonempty set endowed with a binary relation $<$ such that (i) $<$ is transitive; (ii) for every $a, b \in I$ there exists $c \in I$ such that $(a < c) \wedge (b < c)$. A net in a topological space (X, τ) is any mapping ϕ of a directed set $(I, <)$ into (X, τ). We shall denote a net by $\{x_\alpha\}_{\alpha \in I}$.

An example of a net in a topological space (X, τ) is provided by a sequence $\{x_n\}_{n \in \mathbf{N}}$ of points in X. Indeed, the set **N** of the natural numbers endowed with the natural order $<$ is obviously a directed set.

Definition 3.3.5 [convergence of a net]. A net $\{x_\alpha\}_{\alpha \in I}$ in a topological space (X, τ) converges to a point $x \in X$ if for every neighborhood U of x there exists $\alpha \in I$ such that $x_\beta \in U$ for every $\beta \in I$ such that $\alpha < \beta$.

Definition 3.3.6 [topological related space]. A triplet (X, τ, R) is said to be a topological related space if (X, R) is a related set and (X, τ) is a topological space.

Definition 3.3.7 [continuity of a transitive binary relation]. Given a topological related space (X, τ, R), the binary relation R is said to be (τ-)continuous if either R is reflexive and transitive and $L_R(x)$ and $U_R(x)$ are closed sets for every $x \in X$, or R is irreflexive and transitive and $L_R(x)$ and $U_R(x)$ are open sets for every $x \in X$.

Remark 3.3.1. We recall that a topology τ on a set X is said to be

principal if τ is closed under arbitrary intersections. If a topological space (X, τ) is principal, then every point $x \in X$ has a minimal neighborhood with respect to the partial order of strict set inclusion. It is clear that any topology on a finite set X is principal.
A topology τ on a set X is said to be T_0 if, given $x, y \in X$ with $x \neq y$, there exists $U \in \tau$ such that either $x \in U, y \notin U$ or $y \in U, x \notin U$.

Example 3.3.1 The usual topology (the natural topology) τ_{nat} on the real line is not a principal topology. Indeed consider for every $n \in \mathbb{N}^+$ the open interval $I_n =]-\frac{1}{n}, \frac{1}{n}[$. We have that $\bigcap_{n=1}^{\infty} I_n = \{0\}$ which is not open (actually it is closed).

Theorem 3.3.1 Let X be a nonempty set. There is a bijection ϕ from the set of all the T_0-principal topologies on X into the set of all the partial orders on X.

Proof. Consider a T_0-principal topology τ on X, and, for every $x \in X$, denote by U_x the open set which is the intersection of all the open sets containing x. Observe that U_x is open since τ is principal. Define a binary relation $\prec$ on X as follows:

$$x \prec y \Leftrightarrow y \in U_x \setminus \{x\}.$$

It is clear that $\prec$ is irreflexive. In order to show that $\prec$ is transitive, first observe that, since τ is principal, $y \in U_x \setminus \{x\}$ entails $U_y \subseteq U_x$ (indeed U_y is the minimal open set containing y and $y \in U_x$). Further, τ is T_0 and therefore $y \in U_x \setminus \{x\}$ entails $x \notin U_y \setminus \{y\}$. Hence $(x \prec y) \wedge (y \prec z) \Leftrightarrow (y \in U_x \setminus \{x\}) \wedge (z \in U_y \setminus \{y\}) \Rightarrow z \in U_x \setminus \{x\} \Leftrightarrow x \prec z$.
So $\prec$ is a partial order on X. Define an application ϕ from the set of all the T_0-principal topologies on X into the set of all the partial orders on X

by letting $\phi(\tau) = \prec$. In order to show that ϕ is injective, we just observe that, if τ_1 and τ_2 are two different T_0-principal topologies on X, then there exists $x \in X$ such that the minimal τ_1 neighborhood of x is different from the minimal τ_2 neighborhood of x, so that $\phi(\tau_1) \neq \phi(\tau_2)$. To show that ϕ is surjective, let $\prec$ be any partial order on X, and consider the T_0-principal topology τ on X such that $U_\prec(x) \cup \{x\}$ is the minimal neighborhood of x. It is clear that $\phi(\tau) = \prec$. So the proof is complete.

Chapter 4

Semicontinuous utility representations

4.1 Introduction

In Section 4.2, we first recall the definition of a real-valued upper (lower) semicontinuous function on a topological space. We then present the concepts of continuity of a total preorder and a proof of the classical *Rader's theorem*, according to which there exists an upper semicontinuous utility representation for every upper semicontinuous total preorder on a *second countable topological space* (in particular, the reader may recall that this is the case of a *separable metric space*). In Section 4.3, we present a slightly modified version of an already existing characterization of the existence of a pair of upper semicontinuous functions representing an interval order on a topological space. Finally, in Section 4.4 we deal with the semiorder case, and we present a necessary condition for the existence of an upper semicontinuous thereshold representation. We further present an example, illustrating the fact that the existence of an upper semicontinuous threshold representation doesn't imply that the traces are upper semicontinuous.

4.2 Basic definitions and preliminary results

Definition 4.2.1 [upper semicontinuous real-valued function]. A real-valued function f on an arbitrary topological space (X, τ) is said to be *upper semicontinuous* (*lower semicontinuous*) if $f^{-1}(]-\infty, \alpha[) = \{x \in X : f(x) < \alpha\} \in \tau$ for every $\alpha \in \mathbf{R}$ (respectively, $f^{-1}(]\alpha, +\infty[) = \{x \in X : \alpha < f(x)\} \in \tau$ for every $\alpha \in \mathbf{R}$). Further, f is said to be *continuous* if and only if it is both upper and lower semicontinuous.

Alternative characterization of real-valued semicontinuous function on a metric space.

It is easy to show that a real-valued function on a topological space (X, τ) is upper semicontinuous if and only if the following condition is satisfied (see e.g. Herden and Mehta [34]):

For every point $x \in X$ and every net $\{x_\alpha\}_{\alpha \in I}$ in (X, τ) converging to x the equation $f(x) = \lim_{\alpha \in I} \sup f(x_\alpha)$ holds.

In the case when the topology is induced by a distance (i.e.,(X, τ) is a metric space) the following easier definition holds true, which actually applies to every real-valued function on a *first countable space*[1]:

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[1] A topological space (X, τ) is said to be *first countable* if for every $x \in X$ there exists a countable family $B^x = \{B_n^x : n \in \mathrm{N}\} \subseteq \tau$ such that for every nonempty open subset O of X containing x there exists $n \in \mathrm{N}$ with $x \in B_n^x \subset O$.

For every real number α, every sequence $\{x_n : n \in \mathbb{N}\} \subset X$ and $\bar{x} \in X$ with $x_n \to \bar{x}$, $f(x_n) \geq \alpha$ for all $n \in \mathbb{N}$ entails $f(\bar{x}) \geq \alpha$.

Definition 4.2.2 [semicontinuity of a preorder]. A preorder $\precsim$ on a topological space (X, τ) is said to be *upper semicontinuous* (*lower semicontinuous*) if $U_{\precsim}(x)$ ($L_{\precsim}(x)$) is a closed set for every $x \in X$. Further, $\precsim$ is said to be continuous if it is both upper and lower semicontinuous.

Remark 4.2.1 [upper semicontinuous preorder on a metric space]. It should be noted that, in the particular case when we consider a preorder $\precsim$ on a metric space (X, τ), we have that $\precsim$ is upper semicontinuous if and only if the following condition holds:

For all points $x, \bar{x} \in X$ and every sequence $\{x_n : n \in \mathbb{N}\} \subset X$ with $x_n \to \bar{x}$ and $x \precsim x_n$ for all $n \in \mathbb{N}$, we have that $x \precsim \bar{x}$. Therefore, we can say that an individual whose preferences are described by a semicontinuous preorder $\precsim$ on a topological space (X, τ) is *consistent in the small*.

Separation and continuity for a total preorder.

Proposition 4.2.1 [condition for continuity of a total preorder]. Let $(X, \tau, \precsim)$ be a totally preordered topological space. Then $\precsim$ is continuous if the following condition is verified:

(i) for every $x, y \in X$ such that $x \prec y$ there exists a real-valued continuous increasing function $f_{x,y}$ on $(X, \tau, \precsim)$ with values in $[0, 1]_{\mathbb{R}}$ such that $f_{x,y}(x) = 0$ and $f_{x,y}(y) = 1$.

Proof. Let $(X, \tau, \precsim)$ be any totally preordered topological space, and assume that condition (i) is verified. Since $\precsim$ is total, it is continuous if and only if $L_{\prec}(x) = \{z \in X : z \prec x\}$ and $U_{\prec}(x) = \{z \in X : x \prec z\}$ are open sets for every $x \in X$. Consider any point $x \in X$, and let $z \in L_{\prec}(x)$. By condition (i), there exists a real-valued continuous increasing function $f_{z,x}$ on $(X, \tau, \precsim)$ with values in $[0, 1]_{\mathbf{R}}$ such that $f_{z,x}(z) = 0$ and $f_{z,x}(x) = 1$. Then $f_{z,x}^{-1}([0, f_{z,x}(x)[)$ is an open subset[2] of $L_{\prec}(x)$ containing z, and therefore $L_{\prec}(x)$ is an open set. Analogously it can be shown that $U_{\prec}(x)$ is an open set for every $x \in X$. So the proof is complete.

Definition 4.2.3 [weakly order-separable]. A preorder $\precsim$ is *weakly order-separable* if there exists a countable set $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exist $d_1, d_2 \in D$ such that $x \precsim d_1 \prec d_2 \precsim y$.

Theorem 4.2.1 Let $(X, \tau, \precsim)$ be a topological totally preordered space. Then the following conditions are equivalent:

(i) There exists a real-valued continuous order-preserving function f on $(X, \tau, \precsim)$ with values in $[0, 1]_{\mathbf{R}}$;

(ii) The total preorder $\precsim$ on X is weakly order-separable and continuous.

We recall that a topological space (X, τ) is said to be second countable if there exists a countable base[3] $\mathcal{B} = \{B_n : n \in \mathbf{N}\}$ for (X, τ).

---

[2]Consider that $f_{z,x}^{-1}([0, f_{z,x}(x)[) \not\subset L_{\prec}(x)$ would imply the existence of $w \in X$ such that $f_{z,x}(w) < f_{z,x}(x)$ and $w \not\prec x$, but this is impossible since $w \not\prec x$ is equivalent to $x \precsim w$ due to the fact that $\precsim$ is total, and $f_{z,x}$ is increasing.

[3]A family $\mathcal{B} \subseteq \tau$ is said to be a *base* for a topological space (X, τ) if every nonempty open subset of X can be represented as the union of a subfamily of $\mathcal{B}$.

We furnish a proof of Rader's classic theorem [42].

Theorem 4.2.2 [Rader's theorem]. Let $\precsim$ be an upper semicontinuous total preorder on a second countable topological space (X, τ). Then there exists an upper semicontinuous utility function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$.

Proof. Let $\precsim$ be an upper semicontinuous total preorder on L^1_+. Then denote by τ_L the topology generated by the family $L = \{L_{\prec}(x)\}_{x \in X}$. Since τ_L is a linearly ordered subtopology of τ and τ is second countable, we have that also τ_L is second countable (see Bosi and Herden [12]). Let $\{O_n\}_{n \in N}$ be a countable base for the topology τ_L on X consisting of (open) decreasing subsets of X. Since the preorder $\precsim$ on (X, τ) is upper semicontinuous, we have that for all $x, y \in X$ such that $x \prec y$ there exists $n \in N$ such that $x \in O_n \subset L_{\prec}(y), y \notin L_{\prec}(y)$.

Now consider, for every $n \in N$, the upper semicontinuous increasing function with values in $[0, 1]$ defined as follows:

$$u_n(x) = \begin{cases} 0 & \text{if } x \in O_n \\ 1 & \text{if } x \notin O_n \end{cases}.$$

It is now almost immediate to check that the function

$$u = \sum_{n \in N} 2^{-n} u_n$$

is an upper semicontinuous utility function for $\precsim$ on (X, τ). Indeed, it is clear that u is upper semicontinuous since u_n is upper semicontinuous for all $n \in N$. Further, u is a utility function for the preorder $\precsim$ on X since u is increasing and for all $x, y \in X$ such that $x \prec y$ there exists some $n \in N$ with

$u_n(x) = 0$ and $u_n(y) = 1$ (clearly, if $x \prec y$ then $u_n(x) \leq u_n(y)$ for all n). So the proof is complete.

4.3 Upper semicontinuous representations of interval orders

Definition 4.3.1 [upper semicontinuous interval orders]. An interval order - on a metric space $(X; d)$ is said to be upper semicontinuous if $L_{\prec}(x)$ is an open subset of X for every $x \in X$. Clearly, upper semicontinuity of the traces $\prec^{*}$ and $\prec^{**}$ is defined in a perfectly analogous way.

In order to illustrate the fact that the existence of a pair of upper semicontinuous real-valued functions (u, v) representing an interval order - on a metric space (X, d) does not imply that the associated weak order $\prec^{**}$ is upper semicontinuous (i.e. $L_{\prec^{**}}(x) = \{y \in X : y \prec^{**} x\}$ is an open subset of X for all $x \in X$) let us consider the following example.

Example 4.3.1 Let X be the set [1,3] ∪[9,10] endowed with subspace topology and consider the interval order - on X defined as follows for all $x, y \in X$: $x - y \Leftrightarrow x \leq y^{2}$.

Then it is clear that (u, v) is (upper semi) continuous representation of - as soon as we define $u(x) = x$ and $v(x) = x^{2}$ for every $x \in X$. We have that the associated weak order $\precsim^{**}$ is not upper semicontinuous. Indeed, consider for example that $L_{\prec^{**}}(10) = [1,3] \cup \{9\}$ is not open set. Notice that $x^{2} < 10$ for all $x \in [1, 3]$, $9 \prec^{**} 10$ since $9 \leq 3^{2} < 10$ but for no $9 < x < 10$ we have that $x \prec^{**} 10$ because this would imply the existence of $\eta \in X$ such that $9 < x \leq \eta^2 < 10$. Notice that in this case the topology on X fails to be connected.

Lemma 4.3.1[characterization of an upper semicontinuous representation of an interval order in terms of sets].

Let - be an interval order on a topological space (X, τ). Then the following conditions are equivalent:

(i) There exists a pair (u, v) of upper semicontinuous real-valued functions on (X, τ) representing the interval order -;

(ii) There exist a countable family $\{(A_n, B_n)\}_{n \in \mathbb{N}^+}$ of pairs of open subsets of X satisfying the following conditions:

(a) $x \prec y$ and $y \in B_n$ imply $x \in A_n$ for all $x, y \in X$ and for all $n \in \mathbb{N}^+$;

(b) x - y and $y \in A_n$ imply $x \in B_n$ for all $x, y \in X$ and for all $n \in \mathbb{N}^+$;

(c) for all $x, y \in X$ such that $x \prec y$ there exist $n \in \mathbb{N}^+$ such that $x \in A_n$, $y \notin B_n$.

Proof. (i) ⇒ (ii). If (u, v) is a continuous representation of the interval order - and u and v are both upper semicontinuous, then just define, for all $q \in \mathbb{Q}$, $A_q = \{x \in X : v(x) < q\}$, $B_q = \{x \in X : u(x) < q\}$ in order to immediately verify that the set of pairs $\{(A_q, B_q)\}_{q \in \mathbb{Q}}$ satisfies the above conditions (a), (b) and (c).

(ii) ⇒ (i). Assume that there is a countable family $\{(A_n, B_n)\}_{n \in \mathbb{N}^+}$ of pairs of open subsets of X satisfying the above conditions (a) through (c). Define, for all $n \in \mathbb{N}^+$, the following upper semicontinuous functions u_n and v_n:

$$u_n(x) = \begin{cases} 0 & \text{if } x \in B_n \\ 1 & \text{if } x \notin B_n \end{cases}, \quad v_n(x) = \begin{cases} 0 & \text{if } x \in A_n \\ 1 & \text{if } x \notin A_n \end{cases}.$$

Then define two functions $u, v : X \to [0, 1]$ by

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \quad v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x).$$

It is clear that (u, v) is a pair of upper semicontinuous functions on (X, τ). We claim that the pair (u, v) represents the interval order $\precsim$. In order to prove this fact, first consider any two elements $x, y \in X$ such that $x \precsim y$, and observe that, for every $n \in \mathbb{N}^+$, if $y \in A_n$ then it must be that $x \in B_n$ by the above condition (b). Hence, it must be $u(x) \leq v(y)$ from the definition of u and v. Now consider any two elements $x, y \in X$ such that $x \prec y$. Then we have that $v_n(x) \leq u_n(y)$ for every $n \in \mathbb{N}^+$ by condition (a). Further, by condition (c), there exists $n \in \mathbb{N}^+$ such that $x \in A_n$, $y \notin B_n$. Hence, we have that $v(x) < u(y)$. This consideration completes the proof.

Theorem 4.3.1 [characterization of an upper semicontinuous representation of an interval order]. Let $\precsim$ be an interval order on a topological space (X, τ). Then the following conditions are equivalent:

(i) There exists a pair (u, v) of upper semicontinuous real-valued functions on (X, τ) representing the interval order $\precsim$;

(ii) The following conditions are verified:

(a) The interval order $\precsim$ on X is i.o.-separable;

(b) - is upper semicontinuous;

(c) There exists an upper semicontinuous weak utility u for $\prec^{**}$,

Proof. (i) ⇒ (ii). Assume that there exists a representation (u, v) of the interval order - on (X, τ) with u and v upper semicontinuous. Since the interval order - on X is representable by a pair of real-valued functions on X, then - is i.o.-separable (see Bosi et al. [6]) and therefore condition (a) is verified. It is clear that - is upper semicontinuous (condition (b)) and u is an upper semicontinuous weak utility for the weak order $\precsim^{**}$ by Proposition 2.4.3 (condition (c)).

(ii) ⇒ (i). Let - be an i.o.-separable and upper semicontinuous interval order on a topological space (X, τ) which in addition satisfies the above condition (c). Without loss of generality, from very well known topological properties of the real line we can assume that the i.o.-order dense set D is such that, for all $z \in X$ and $n \in \mathrm{N}^+$, if $z \sim^{**} d_n \in D$ then there exists $d_m \in D$ such that $d_m \sim^{**} d_n$ and $u(d_m) \leq u(z)$ (u is an upper semicontinuous weak utility for $\prec^{**}$). Indeed, let us start from an i.o.-order dense set $D' \subset X$. Consider, for every $n \in \mathrm{N}^+$, the set of numbers $C_n = \{u(z) : z \sim^{**} d_n\}$. If there exists $\min C_n = \alpha_n$, then just consider any element $x_n \in u^{-1}(\{\alpha_n\})$. If $\inf C_n = \alpha_n$ does not belong to C_n, then consider any sequence $\{c_{nk}\}_{k \in \mathrm{N}^+}$ converging to α_n, and for every c_{nk} choose an element $x_{nk} \in u^{-1}(\{c_{nk}\})$. Then consider a new i.o.-dense set D resulting from the union of D' and the elements x_n and the elements x_{nk}, in order to realize that D is a countable i.o.-dense subset of X such that, for all $z \in X$ and $n \in \mathrm{N}^+$, if $z \sim^{**} d_n \in D$ then there exists $d_m \in D$ such that $d_m \sim^{**} d_n$ and $u(d_m) \leq u(z)$. Denote this observation and property by (*).

Define, for all $n \in \mathrm{N}^+$,

$$A_n = L_{\prec}(d_n),$$

$$B_n = u^{-1}(]-\infty, u(d_n)[)$$

We claim that $\{(A_n, B_n)\}_{n \in \mathbb{N}^+}$ is a countable family of pairs of open subsets of X satisfying conditions (a) through (c) of Lemma 4.3.1. Please observe that each set B_n is open as a consequence of the fact that u is upper semicontinuous.

In order to first show that condition (a) of Lemma 4.3.1 is satisfied, consider $x, y \in X$, and $n \in \mathbb{N}^+$ such that $x \prec y$, $u(y) < u(d_n)$. Then we must have $x \prec d_n$ because otherwise $d_n \precsim x \prec y$ implies that $d_n \prec^{**} y$ and therefore we arrive at the contradiction $u(d_n) < u(y)$.

In order to show that condition (b) is valid, consider $x, y \in X$, and $n \in \mathbb{N}^+$ such that $x \precsim y \prec d_n$. Then $x \prec^{**} d_n$, and we must have that $u(x) < u(d_n)$, or equivalently $x \in B_n$.

Finally, in order to show that also condition (c) of Lemma 4.3.1 is verified, consider $x, y \in X$ such that $x \prec y$. Since we considered an i.o.-order dense subset of X satisfying property (*) above, there exists $d_n \in D$ such that $x \prec d_n \precsim^{**} y$ and $u(d_n) \leq u(y)$. Hence $x \in A_n = L_{\prec}(d_n)$ and $y \notin B_n = u^{-1}(]-\infty, u(d_n)[)$. This consideration completes the proof.

Remark 4.3.1 [upper semicontinuous representations on finite sets]. It is immediate to check that any topology τ on a finite set X is principal. Indeed, since there are finitely many subsets of X, and therefore finitely many open subsets of X, the intersection of any family of open sets is necessarily the intersection of finitely many open sets, which is necessarily open. Therefore, if for all $x \in X$ we denote by U_x the minimal neighbourhood of x, and $\precsim$ is an interval order on X, we have that $\precsim$ is upper semicontinuous if and only if $U_z \subset L_{\prec}(x)$ for all $z \in X$ such that $z \prec x$. If this condition is verified and in addition there exists an upper semicontinuous weak utility u' for $\prec^{**}$, we have that there exists a pair (u, v) of upper semicontinuous real-valued functions on (X, τ) representing the interval order $\precsim$ by Proposition

4.2. Indeed, it is clear that - is i.o.-separable due to the fact that X is finite (X itself is an i.o.-dense of itself).

4.4 Upper semicontinuous threshold representations of semiorders

Let us prove the following proposition which makes explicit the natural necessary conditions for the existence of a semicontinuous threshold representations of a semiorder.

Proposition 4.4.1 [condition for a semiorder]. Let $\prec$ be an interval order on a set X. If there is a real-valued function u on X that is a weak utility for both $\prec^{*}$ and $\prec^{**}$, then $\prec$ is semiorder.

Proof. Consider $x, y, z, w \in X$ with $w - x \prec y \prec z - w$, then $w \prec^{**} y \prec^{*} w$. Hence a function u cannot be a weak utility for both $\prec^{*}$ and $\prec^{**}$, since otherwise we have that $u(w) < u(y) < u(w)$. Then the proof is complete.

Proposition 4.4.2 [sufficient condition for semicontinuous threshold representations]. Let $\prec$ be a semiorder on a topological space (X, τ). If $\prec$ is upper semicontinuous and $\prec^{*} = \prec^{**}$, then there exists a pair (u, v) of upper semicontinuous real-valued functions on (X, τ) representing $\prec$ provided that there exists a pair (u', v') of real-valued functions on X representing $\prec$.

Proof. The proposition is a consequence of Theorem 4.3.1. Indeed, since $\prec$ is upper semicontinuous and $\prec^{*} = \prec^{**}$. we have that the trace $\prec^{**}$ is also upper semicontinuous. Since it is not difficult to realize that there exists an upper semicontinuous weak utility u' for $\prec^{**}$ due to i.o. separability of $\prec$, the thesis follows from Theorem 4.3.1.

Proposition 4.4.3 [necessary conditions for semicontinuous threshold representations] Let $\prec$ be a semiorder on a topological space (X, τ) which admits a threshold representation (u, δ) with u upper semicontinuous. Then the following conditions are verified:

(i) $\prec$ is upper semicontinuous;

(ii) u is an upper semicontinuous weak utility for both $\prec^*$ and $\prec^{**}$ (therefore for $\prec^0$).

Proof. Since u is upper semicontinuous, it is clear that also $u + \delta$ is upper semicontinuous. Therefore, $L_\prec(x) = \{z : u(z) + \delta < u(x)\} = (u + \delta)^{-1}(]-\infty, u(x)[)$ is an open subset of X for all $x \in X$, or equivalently $\prec$ is upper semicontinuous. Hence, (i) is proved. In order to show that also condition (ii) is satisfied, we just observe that this fact is an immediate consequence of Proposition 2.3 (for all $x, y \in X$, $x \prec^* y$ implies $u(x) + \delta < u(y) + \delta \Leftrightarrow u(x) < u(y)$). The proof is now complete.

In order to support the assertion that in a threshold representation (u, δ) not necessarily u is a (two-way) utility for the traces, we present the following example.

Example 4.4.1 [u is not a utility for the traces] Let $X = [1, 2] \cup [\sqrt{5}, \sqrt{6}]$ be the union of two real intervals on the real line R endowed with the usual natural induced topology, and consider the semiorder $\prec$ on X defined as follows for all $x, y \in X$

$$x \prec y \Leftrightarrow x^2 + 1 < y^2.$$

Then it is clear that $(u, 1)$ is an (upper semi)continuous representation of $\prec$

when we define $u(x) = x^2$ for all $x \in X$. We have that the associated weak order $\prec^{**}$ is not upper semicontinuous. Indeed, consider for example that $L_{\prec^{**}}(\sqrt{6}) = [1, 2] \cup \{\sqrt{5}\}$ is not an open set. Notice that $x \prec \sqrt{6}$ for all $x \in [1, 2]$, $\sqrt{5} \prec^{**} \sqrt{6}$ since $\sqrt{5}$ - 2 $\prec \sqrt{6}$ but for no $\sqrt{5} < x < \sqrt{6}$ we have that $x \prec^{**} \sqrt{6}$ because this would imply the existence of $\eta \in X$ such that $5 < x^2 \leq \eta^2 + 1 < 6$.

Chapter 5

Preferences among random variables and their representations

5.1 Introduction

We are now concerned with preferences defined on a normed space of random variables on a common probability space.

In Section 5.2 we refresh the concept of L_+^p space and we define *homotheticity* of a preference with to the aim of then presenting results guaranteeing the existence of an *homogeneous representation*. The concept of *Choquet integral* is viewed as a relevant example of a positively homogeneous functional. Then the classical concepts of *stochastic dominance* are presented as fundamental examples of preorders.

In section 5.3 we restate the main concepts of continuity in our context and we present a proposition which furnishes a characterization of continuity of a positively homogeneous functional which is in addition a utility functional for some total preorder.

Section 5.4 is dedicated to the existence of positively homogeneous utility representation of total preorders, while Section 5.5 is devoted to positively homogeneous representability of interval orders.

5.2 Basic concepts

Denote by $\mathbb{R}$ ($\mathbb{R}_+$) the set of all real numbers (respectively, the set of all nonnegative real numbers). Further $\mathbb{R}_{++}$ stands for the set of all positive real numbers and $\mathbb{Q}_{++}$ stands for the set of the positive rational numbers.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and denote by χ_F the indicator function of any subset F of Ω, that is

$$\chi_F(x) = \begin{cases} 1 & if\, x \in F \\ 0 & if\, x \notin F \end{cases}.$$

Let L_+ be a vector space of nonnegative real random variables on $(\Omega, \mathcal{F}, P)$. In particular, L_+ could be specialized as the space L^1_+ (L^2_+) of integrable (respectively, square integrable) nonnegative random variables on $(\Omega, \mathcal{F}, P)$.

In the sequel we shall be concerned with an interval order (or in particular a total preorder) - on L^1_+ (L^2_+) and we shall consider its real representation which may satisfy appropriate conditions.

Homogeneous representation

We recall the well known general definition of L^p-space ($1 \leq p < \infty$).

Definition 5.2.1 [L^p-space]. The space $L^p(\Omega)$ (the short version of $L^p(\Omega, \mathcal{F}, P)$) consists of all measurable functions $X : \Omega \longrightarrow R$ such that

$$\int |X|^p \, dP = E|X|^p < \infty.$$

The L^p-norm of $X \in L^p(\Omega)$ is defined by

$$\| X \|_{L^p} = \left(\int | X |^p \, dP \right)^{\frac{1}{p}} .$$

For the sake of convenience, in the sequel we shall write L^p_+ instead of $L^p_+(\Omega)$.

Example 5.1 A classical example of a L^2-norm continuous utility functional is

$$U(X) = E[X] - \alpha Var[X] \ (\alpha > 0).$$

If for two $X, Y \in L^p_+$ we have that $X(\omega) \preceq Y(\omega)$ for $\omega \in \Omega$, P-almost surely, then we shall simply write $X \leq Y$. We have that $\preceq$ is a preorder on L^p_+.

Definition 5.2.2 [homothetic preorder]. A preorder $\preceq$ on L^1_+ is said to be homothetic if, for every $X, Y \in L^1_+$ and $t \in \mathbf{R}_{++}$,

$$X \preceq Y \Leftrightarrow tX \preceq tY.$$

Definition 5.2.3 [first order stochastic dominance]. The first-order stochastic dominance relation $\preceq_{FSD}$ on L^1_+ is defined as follows:

$$X \preceq_{FSD} Y \Leftrightarrow P\{\omega \in \Omega : X(\omega) \leq k\} \geq P\{\omega \in \Omega : Y(\omega) \leq k\} \ \forall \, k \in \mathbf{R}.$$

It is immediate to check that $\preceq_{FSD}$ is homothetic.

Example 5.2.2 [Choquet integral]. Consider a space L^1_+ of nonnegative real random variables on a common probability space (Ω, F, P). Let

$\gamma : F \to [0, \infty)$ be a monotone finite set function (i.e. $\gamma(\emptyset) = 0$, $\gamma(\Omega) = 1$, $A, B \in F$, $A \subseteq B \Rightarrow \gamma(A) \le \gamma(B)$). The Choquet integral of $X \in L^1_+$ is defined as follows:

$$\int_\Omega X\, d\gamma = \int_0^\infty \gamma\{\omega \in \Omega : X(\omega) > u\} du.$$

Let $\precsim$ be the total preorder on L^1_+ defined by

$$X \precsim Y \Leftrightarrow \int_\Omega X\, d\gamma \le \int_\Omega Y\, d\gamma.$$

Then $\precsim$ is homothetic.

Definition 5.2.4 [Positively homogeneous function]. A real-valued function f on a space L^1_+ (L^2_+) is said to be *positively homogeneous* of degree one (for brief, positively homogeneous) if, for every $t \in \mathbf{R}_{++}$ and $X \in L^1_+ (L^2_+)$,

$$f(tX) = tf(X).$$

Example 5.2.3 [Positively homogeneous function]. A classical example of a L_2-norm continuous and positively homogeneous function is :

$$U(X) = E[X] - \alpha \sqrt{Var[X]} \quad (\alpha > 0).$$

We prove this fact.

Proof. We have to prove that $U(tX) = tU(X)$ for all $t > 0$.

In fact: $\forall t > 0$. $U(tX) = E(tX) - \alpha\sqrt{Var(tX)} \Rightarrow t(EX) - \alpha\sqrt{t^2 Var(X)} \Rightarrow \underbrace{t[E(X) - \alpha\sqrt{Var(X)}]}_{} = tU(X)$.

Stochastic Dominance

Introduction

Stochastic Dominance (SD) is a fundamental concept in decision theory with uncertainty. It describes when a particular random prospect, say a lottery, is better than another random prospect based on preferences regarding outcomes (which may be expressed in terms of monetary values or utility values). Essentially the question boils down to in what sense(s) can we say $X \leq Y$, where X and Y are two random variables. The simplest example of SD is state-by-state dominance: $X(w) \leq Y(w), \quad \forall (w) \in \Omega$ or slightly more weakly, absolute or almost sure dominance: we say that $X \leq Y$ *almost surely* if $P(X \leq Y) = 1$. In other words, a random variable or lottery Y is said to be (almost surely) state-by-state dominant over lottery X when Y provides a better outcome than X for each possible state of nature, except possibly for a set of states with probability 0. For example, if one pound is added to one or more prizes in a lottery, the new lottery is state-by-state dominant the old one. In this chapter, we focus on a more probabilistic sense in which $X \leq Y$, namely that Y has more chance of being bigger than X. We shall mainly be concerned with two major types of SD, namely, the first-order stochastic dominance (FSD) and the second-order stochastic dominance (SSD).

Absolute and First-Order Stochastic Dominance

Definition 5.2.5 [(Absolute dominance/almost-sure dominance)].
Y is absolutely dominant over X ($X -_{AD} Y$) if $P(X \leq Y) = 1$.

It is clear that Y is absolutely dominant over X under state-by-state dominance of Y over X.

The following definition is much more popular and frequently encountered in the literature.

Definition 5.2.6 [(First-order stochastic dominance)]. Y is *first-order stochastically dominant* over X if $F_Y(y) \leq F_X(y)$ for all y (equivalently $\overline{F}_X(y) \leq \overline{F}_Y(y)$ for all y). In addition, it is natural to say that Y is *strictly first-order stochastically dominant* over X if Y is first-order stochastically dominant over X and in addition there is at least one y such that $F_Y(y) < F_X(y)$ (equivalently $\overline{F}_X(y) < \overline{F}_Y(y)$). In other words, Y has more chance than X of being bigger than any given value y.

If Y is first-order stochastically dominant over X, we write $Y \geq_{sd} X$. It is clear that $\geq_{sd}$ is a not necessarily total preorder. This concept is profitably used in connection with suitable monotonicity assumption of an original pre-order -. Indeed, we may introduce the following definition, that is very common in the literature.

Definition 5.2.7 [(Monotonicity of a preorder with respect to First-order stochastic dominance)]. A preorder - is said to be monotone with respect to first-order stochastic dominance if, for all random variables X, Y, $X \leq_{sd} Y$ implies that X - Y. In addition a preorder - is said to be *strictly-monotone* with respect to first-order stochastic dominance if it is monotone with respect to first order stochastic dominance and in addition, for all random variables X, Y if, $F_Y(y) \leq F_X(y)$ for all y and there is at least one y such that $F_Y(y) < F_X(y)$, then $X \prec Y$.

We shall meet again this concept when introducing suitable properties of a *certainty equivalence functional*.

Definition 5.2.8 [(Monotonicity of a functional with respect to First-order stochastic dominance)]. A functional C is said to be *monotone with respect to first-order stochastic dominance* if, for all random variables X, Y, $X \leq_{sd} Y$ implies that $C(X) \leq C(Y)$. A functional C is said to be *strictly monotone with respect to first-order stochastic dominance* if it is monotone with respect to first order stochastic dominance and in addition, for all random variables X, Y, if, $F_Y(y) \leq F_X(y)$ for all y and there is at least one y such that $F_Y(y) < F_X(y)$, then $C(X) < C(Y)$.

Second-Order Stochastic Dominance

Definition 5.2.9 [(Second-order stochastic dominance)]. Y is second-order stochastically dominant over X if

$$\int_{-\infty}^{x} F_Y(y)y \leq \int_{-\infty}^{x} F_X(y)y$$

for all x, and there is at least one x for which the above inequality is strict.

It should be noted that FSD is stronger than SSD (i.e. FSD $\Rightarrow$ SSD).

5.3 Representation of Preferences on L_+^1, L_+^2

Definition 5.3.1 [order separable preordered set]. From Bridges and Mehta [18], a preordered set $(X, \precsim)$ is said to be *order separable* if there exists a countable set $A(\subseteq X)$ such that

$$(x \prec y) \text{ and } (x, y \in X) \Rightarrow \exists a \in A : x \prec a \prec y.$$

A is termed a countable *order dense* subset of X.

Definition 5.3.2 [upper semicontinuous function]. A real-valued function f on L_+^1 (L_+^2) is said to be *upper semicontinuous* if

$$f^{-1}(]-\infty, \alpha[) = \{Z \in L_+^1 (L_+^2) : f(Z) < \alpha\}$$

is an open set for every $\alpha \in$ R (i.e., if for every sequence $\{Z_n\} \in L_+^1$ (L_+^2) and $Y \in L_+^1$, (L_+^2) such that $Z_n \to Y$, $f(Z_n) \geq \alpha$ $\forall n \in$ N entails that $f(Y) \geq \alpha$).

We also present the concept of a lower semicontinuous function, since it is widely used in the theory of *premium functionals* in *Insurance Mathematics*.

Definition 5.3.3 [lower semicontinuous function]. A real-valued function f on L_+^1 (L_+^2) is said to be *lower semicontinuous* if

$$f^{-1}(]\alpha, +\infty[) = \{Z \in L_+^1 (L_+^2) : \alpha < f(Z)\}$$

is an open set for every $\alpha \in$ R (i.e., if for every sequence $\{Z_n\} \in L_+^1$ (L_+^2) and $Y \in L_+^1$, (L_+^2) such that $Z_n \to Y$, $f(Z_n) \leq \alpha$ $\forall n \in$ N entails that $f(Y) \leq \alpha$).

Definition 5.3.4 [Closed sections]. Let $\precsim$ a reflexive binary relation on L_+^1 (L_+^2). Then for every $X \in L_+^1$ we define:

$$d_{\precsim}(X) = \{Z \in L_+^1 : Z \precsim X\}$$

$i_{-}(X) = \{Z \in L^1_+ : X - Z\}$

Definition 5.3.5 [Upper semicontinuity]. Let - an interval order on L^1_+, (L^2_+). Then - is *upper semicontinuous* if $i_{-}(X)$ is a closed set for every $X \in L^1_+$, namely if for every sequence $\{Z_n\} \subseteq L^1_+$, (L^2_+) and $Y \in L^1_+$, (L^2_+) such that $Z_n \to Y \Longleftrightarrow lim_{n\to\infty} E(|Y - Z_n|) \to 0$ ($lim_{n\to\infty} E(|Y - Z_n|^2) \to 0$).

$X - Z_n$ for every $n \in$ N entails $X - Y \Longleftrightarrow Y \in i_{-}(X)$.

Definition 5.3.6 [Lower semicontinuity]. Let - an interval order on L^1_+, (L^2_+). Then - is *lower semicontinuous* if $d_{-}(X)$ is a closed set for every $X \in L^1_+$, namely if for every sequence $\{Z_n\} \subseteq L^1_+$, (L^2_+) and $Y \in L^1_+$, (L^2_+) such that $Z_n \to Y \Longleftrightarrow lim_{n\to\infty} E(|Y - Z_n|) \to 0$ ($lim_{n\to\infty} E(|Y - Z_n|^2) \to 0$).

X % Z_n for every $n \in$ N entails X % $Y \Longleftrightarrow Y \in d_{-}(X)$.

Definition 5.3.7 [Continuity of intervalorder]. Let - an interval order on L^1_+, (L^2_+), then - is continuous if $i_{-}(X)$ and $d_{-}(X)$ are closed sets for every $X \in L^1_+$, (L^2_+).

Remark 5.3.1 [Continuity of total preorder]. Since a total preorder is an interval order it is clear that the same definition holds for a total preorder too. (i.e., a total preorder is continuous if $i_{-}(X)$ and $d_{-}(X)$ are closed sets for every $X \in L^1_+$, (L^2_+)).

In the sequel we shall denote by **0** the constant random variable equal to 0.

Proposition 5.3.1 Let $\precsim$ be a nontrivial total preorder on L^1_+, such that $\mathbf{0} \precsim X$ for every $X \in L^1_+$. Assume that there exists a homogeneous of degree one utility function u for $\precsim$. Then u is continuous if and only if $\precsim$ is continuous.

Proof. The only if part of the lemma is obvious, since if a total preorder $\precsim$ admits a continuous utility representation u then it is continuous. Indeed in this case we have that $i_{\precsim}(X) = u^{-1}([u(X), +\infty[)$ and $d_{\precsim}(X) = u^{-1}(]-\infty, u(X)])$, which are closed sets. So assume that $\precsim$ is a nontrivial, total and continuous preorder on a L^1_+, such that $\mathbf{0} \precsim X$ for every $X \in L^1_+$. Let u be a homogeneous of degree one utility function for $\precsim$, which is assumed to be continuous. Then it must be $u(\mathbf{0}) = 0$, so that u is nonnegative. Let $X_0 \in L^1_+$ be such that $\mathbf{0} \prec X_0$, and therefore $0 < u(X_0)$. In order to prove that u is upper semicontinuous, consider $X \in L^1_+$ and $\alpha \in \Re$, such that $u(X) < \alpha$. Observe that there exists $t \in \mathbb{R}_{++}$ such that $u(X) < tu(X_0) < \alpha$. Since u is homogeneous of degree one, the previous inequalities can be equivalently written as $u(X) < u(tX_0) < \alpha$. Further $\precsim$ is continuous, we have that $]\leftarrow, tX_0[= \{Z : Z \prec tX_0\}$ is an open set containing X, such that $u(Z) < \alpha$ for every $Z \in]\leftarrow, tX_0[$. Similarly it may be proven that u is lower semicontinuous. So the proof is complete.

Now we are able to present necessary and sufficient axioms for the existence of a homogeneous of degree one and continuous utility fuction u for a total preorder $\precsim$ on L^1_+.

5.4 Positively homogeneous representations of total preorders

Some authors were concerned with the existence of a positively homogeneous and (semi)continuous real-valued functional u representing a total preorder $\precsim$ on a space L^1_+ (L^2_+) (see, e.g., Dow [25] and Werlang , Bosi [5] , and Bosi, Candeal, and Indur´ain [9]).

We now present a caracterization of the existence of a positively homogeneous and continuous utility function for a total preorder.

To this aim, we introduce a new condition that substitutes another one that appears in Bosi [5] and consists of a separability assumption. Before presenting the aforementioned result, we need the following lemma which guarantees the order separability of a total preorder admitting a positively homogeneous representation.

Lemma 5.4.1 If there exists a positively homogeneous utility function u' for non trivial total preorder $\precsim$ and ($\mathbf{0} \precsim X$) for every $X \in L^1_+$ then (L^1_+, $\precsim$) is order separable and $A = \{qX_0 : q \in \mathbb{Q}_{++}\}$ is a countable order dense subset of L^1_+, (L^2_+) for every $X_0 \in L^1_+$, (L^2_+) such that ($\mathbf{0} \prec X_0$).

Proof. Consider $X_0 \in L^1_+$ such that ($\mathbf{0} \prec X_0$). Then $u'(0) = 0$ since u is positively homogeneous and $0 < u'(X_0)$, since u' is a utility function for $\precsim$. Let $A = \{qX_0 : q \in \mathbb{Q}_{++}\}$. If $X \prec Y$, $X, Y \in L^1_+$, then $\exists\, q \in \mathbb{Q}_{++}$: $u'(X) < qu'(X_0) < u'(Y) \Longleftrightarrow u'(X) < u'(qX_0) < u'(Y) \Leftrightarrow X < qX_0 < Y$. So the proof is complete.

We are now ready to furnish a characterization of the existence of a positively homogeneous representation of a total preorder.

Theorem 5.4.1 [positively homogeneous utility]. Let ≾ be a non *trivial* total preorder on L^1_+ and assume that ($\mathbf{0} \precsim X$) for every $X \in L^1_+$. There exists a nonnegative-positively homogeneous and continuous utility function u for ≾ if and only if the following conditions are verified.

(i) ≾ is homothetic,

(ii) ≾ is continuous,

(iii) There exists a positively homogeneous utility function u' for ≾ on L^1_+.

Proof. Let ≾ be a nontrivial total preorder on a L^1_+, and assume that $\mathbf{0} \precsim X$ for every $X \in L^1_+$. It is easily seen that conditions (i), (ii) and (iii) are necessary for the existence of a nonnegative, homogeneous of degree one and continuous utility function for ≾. Indeed condition (i) is verified since, for all $X \precsim Y$ $X, Y \in L^1_+$) and $t > 0 (t \in \mathbb{R}_{++})$, we have that:

$$X \precsim Y \Leftrightarrow u(X) \leq u(Y) \Leftrightarrow tu(X) \leq tu(Y) \Leftrightarrow u(tX) \leq u(tY) \Leftrightarrow tX \precsim tY.$$

Further, it is clear that condition (ii) is verified since a total preorder is continuous as soon at it admits a continuous utility function. Finally, it is trivial to observe that also condition (iii) holds.

So assume that axioms (i), (ii) and (iii) are verified. By nontriviality of ≾, there exists $X_0 \in L^1_+$ such that $\mathbf{0} \prec X_0$. By condition (iii), there exists a positively homogeneous utility function u' for ≾. Since ≾ is homothetic and continuous, we have that $X \prec tX$ for every $X \in L^1_+$ such that $\mathbf{0} \prec X$, and for every real number $t > 1$. Otherwise, by homotheticity of ≾, there exist $X \in L^1_+$ with $\mathbf{0} \prec X$, and $t' \in (0, 1)$ such that $X \precsim (t')^n X$ for every $n \geq 1$. Since the operation of scalar multiplication is continuous, $(t')^n X \to \mathbf{0}$ as

$n \to \infty$, and therefore X - $\mathbf{0}$ by upper semicontinuity of - (this part of the proof of theorem 1.7 in Dow and Werlang [25] works under our assumptions). So there are no maximal elements relative to -, and the range of u' is actually $[0, +\infty)$. Moreover, it is $u'(\mathbf{0}) = 0$ since u' is positively homogeneous. Hence, it must be $u'(X_0) > 0$. Define, for every $X \in L^1_+$,

$$u(X) = \inf\{qu'(X_0) : X \prec qX_0, \; q \in \mathbb{Q}_{++}\}.$$

By axiom (i), u is well defined since there are no maximal elements relative to -. Clearly u is nonnegative. We first prove that u is a utility function for -. Consider $X, Y \in L^1_+$ such that X - Y. By axiom (i), using the fact that there is not a maximal element relative to -, there exists $\bar{q} \in \mathbb{Q}_{++}$ such that $Y \prec \bar{q}X_0$, $u(Y) < \bar{q}u'(X_0)$. Moreover, $Y \prec qX_0$ entails $X \prec qX_0$, and therefore $u(X) \leq u(Y)$ from the definition of u. If $Y \prec X$, then by Lemma 5.4.1 there exists $\bar{q} \in \mathbb{Q}_{++}$ such that $Y \prec \bar{q}X_0 \prec X$, $u(Y) < \bar{q}u'(X_0)$, and therefore it must be $u(Y) < u(X)$.

Now let us prove that u is homogeneous of degree one. First assume that there exist $t \in \mathbb{R}_{++}$ and $X \in L^1_+$ such that $u(tX) < tu(X)$. Hence, from the definition of u, there exists $q' \in \mathbb{Q}_{++}$ such that $u(tX) < q'u'(X_0) < tu(X)$, $tX \prec q'X_0$.

So, by homotheticity of -, it is $X \prec \frac{q'}{t}X_0$, and therefore $u(X) < \frac{q'}{t}u'(X_0)$ from the definition of u. This is contradictory, since $q'u'(X_0) < tu(X)$. Now assume that there exist $t \in \mathbb{R}_{++}$ and $X \in L^1_+$ such that $tu(X) < u(tX)$. Hence, from the definition of u, there exists $q' \in \mathbb{Q}_{++}$ such that $u(X) < q'u'(X_0) < \frac{1}{t}u(tX)$, $X \prec q'X_0$. Since it is also $tq'u'(X_0) < u(tX)$, it must be $tq'X_0$ - tX from the definition of u. Hence, by homotheticity of -, it is $q'X_0$ - X, and this is contradictory. So u is homogeneous of degree one.

Finally, since u is a homogeneous of degree one utility function for -, and - is continuous, then u is continuous by Proposition 5.3.1. So the proof is complete.

5.5 Positively homogeneous representation of interval orders

In the previous paragraph a characterization has been presented of the existence of a positively homogeneous and continuous representation for a total preorder. However, the restrictivity of such a representation has been already underlined in the literature and the consideration of interval orders instead has been justified. Indeed, one can argue that from a decision-theoretic viewpoint the transitivity assumption concerning the binary relation - is too restrictive. In order to explain such an assertion, consider the following situation.

Given a (reflexive) binary relation - (to be interpreted as a preference-indifference relation) on L^1_+, assume that - is homothetic (i.e., $[X - Y \Leftrightarrow tX - tY]$ for every $X, Y \in L^1_+$, and $t \in \mathbb{R}_{++}$) and upper semicontinuous (i.e., $\{Z \in L^1_+ : X - Z\}$ is a closed subset of L^1_+). If - is a total preorder (i.e., a transitive and total binary relation) on L^1_+, it seems that a too accurate assessment of the preferences is required. Indeed, it is easily seen that, for every strictly positive real number ε, and for every real random variable $X \in L^1_+$ such that $\mathbf{0} \prec X$, it must be $X \prec (1 + \varepsilon)X$ (otherwise, we have that $X - (1 + \varepsilon)^{-n} X$ for every integer $n \geq 1$ by homoteticity and transitivity of -, so that, by continuity of scalar multiplication and upper semicontinuity of -, we arrive at the contradiction $X - \mathbf{0}$).

Therefore, a more general model for the preference-indifference - allowing nontransitivity could be thought of as more realistic. It is well known that the simplest model of this kind is represented by an interval order - (i.e., a reflexive binary relation - on L^1_+ such that, for every $X, Y, Z, W \in L^1_+$, $[(X - Z) \wedge (Y - W) \Rightarrow (X - W) \vee (Y - Z)]$.

We now discuss the existence of a pair $\langle u, v \rangle$ of real-valued functionals representing a given interval order $\precsim$ on L_+^1 (in the sense that, for every $X, Y \in L_+^1$, $[X \precsim Y \Leftrightarrow u(X) \leq v(Y)]$), such that u is lower semicontinuous, v is upper semicontinuous, and u and v are both positively homogeneous.

Definition 5.5.1 [Homothetic interval order]. An interval order $\precsim$ on L_+^1 is said to be *homothetic* if for all $X, Y \in L_+^1$ and $t > 0$ we have that $X \precsim Y \Leftrightarrow tX \precsim tY$.

In the following theorem we provide a characterization of the existence of a pair of semicontinuous positively homogeneous real-valued functionals representing an interval order on a L_+^1. This theorem generalizes the previous theorem on the continuous and homogeneous representation of total preorders. A lemma is needed in order to facilitate the proof.

Lemma 5.5.1. If there exists (u', v') representation of an interval order $\precsim$ on L_+^1 with (u', v') positively homogeneous, then

$$\mathbb{Q}_{++}(X_0) = \{qX_0 : q \in \mathbb{Q}_{++}\}$$

is a strongly i.o. dense subset of $(\mathcal{L}, \precsim)$ for every $X_0 \in L_+^1$ such that $\mathbf{0} \prec X_0$.

Proof. Consider any $\mathbf{0} \prec X_0$ and let $X \prec Y$. Then $v'(X) < u'(Y)$ and $v(\mathbf{0}) = 0 < u'(X_0) \leq v'(X_0)$. Therefore there are $q_1, q_2 \in Q_{++}$ such that $v'(X) < q_1 u'(X_0) < q_2 v'(X_0) < u'(Y)$. Hence $X \prec q_1 X_0 \precsim q_2 X_0 \prec Y$, imply that $X \prec q_1 X_0 \prec^{**} Y$. So the proof is complete.

Lemma 5.5.2. If an interval order $\precsim$ on L_+^1 is homothetic, then the associated total preorders $\precsim^*$ and $\precsim^{**}$ are both homothetic.

Proof. Just consider that, for all $X, Y \in L_+^1$,

$$X \precsim^* Y \Leftrightarrow (t(Z) \precsim t(X) \Rightarrow t(Z) \precsim t(Y)), \forall X, Y, Z \in X), t \in \mathbb{R}_{++}$$

$$X \precsim^{**} Y \Leftrightarrow (t(Y) \precsim t(Z) \Rightarrow t(X) \precsim t(Z), \forall X, Y, Z \in \mathcal{X}), t \in \mathbb{R}_{++}$$

Theorem 5.5.1 Let $\precsim$ be an interval order on a L^1_+, and assume that $\mathbf{0} \precsim^{*} X$ for every $X \in L^1_+$. There exists a pair $\langle u, v \rangle$ of nonnegative positively homogeneous real-valued functionals on L^1_+ representing $\precsim$, such that u is lower semicontinuous and v is upper semicontinuous, if and only if the following conditions are verified:

(i) $\precsim$ is homothetic;

(ii) $\precsim$ is continuous;

(iii) There exists a pair (u', v') of positively homogeneous functions representing $\precsim$.

Proof. It is clear that conditions (i), (ii) and (iii) are necessary for the existence of a representation $\langle u, v \rangle$ with the indicated properties. So, assume that conditions (i), (ii) and (iii) hold. From Lemma 5.5.2, homotheticity of the interval order $\precsim$ implies homotheticity of the associated total preorder $\precsim^{**}$. Further, it is clear that, since it must be $\mathbf{0} \precsim X$ for every $X \in L^1_+$, we have that $\mathbf{0} \precsim^{**} X$ for every $X \in L^1_+$. From condition (iii) and Lemma 5.5.1 we have that, for every $X_0 \in L^1_+$ such that $\mathbf{0} \prec X_0$, given $X, Y \in L^1_+$ with $X \prec^{**} Y$, there exists $q \in \mathbb{Q}_{++}$ such that $X \prec^{**} qX_0 \prec^{**} Y$ (i.e., the totally preordered set $(L^1_+, \precsim^{**})$ is order separable, and $\{qX_0 : q \in \mathbb{Q}_{++}\}$ is an order dense subset of $(L^1_+, \precsim^{**})$ for every $X_0 \in L^1_+$ such that $\mathbf{0} \prec X_0$). Condition (ii) implies lower semicontinuity of $\precsim^{**}$, since, for every $X \in L^1_+$, $\{Z \in L^1_+ : X \prec^{**} Z\} = \bigcup_{\{Z' \in L^1_+ : X \precsim Z'\}} \{Z \in L^1_+ : Z' \prec Z\}$. Then, it can be shown that there exists a nonnegative, positively homogeneous and lower semicontinuous real-valued utility functional u for $\precsim^{**}$ (see the proof of Theorem 5.4.1). Given any element $X_0 \in L^1_+$ such that $\mathbf{0} \prec X_0$, define a nonnegative real-valued functional v on L^1_+ as follows:

$$v(X) = \begin{cases} \inf\{tu(X_0) : X \prec tX_0,\ t \in \mathbb{R}_{++}\} & \text{if } \mathbf{0} \precsim Z \prec X \text{ for some } Z \in L^1_+ \\ 0 & \text{otherwise} \end{cases}$$

We claim that the pair $\langle u, v\rangle$ represents the interval order $\precsim$, and that v is positively homogeneous and upper semicontinuous. It is clear that $tv(X) \geq 0$ for all $X \in L^1_+$.

Let us first show that $\langle u, v\rangle$ is a representation of $\precsim$. First consider any two elements $X, Y \in L^1_+$ such that $X \prec Y$. Then, from condition (iii) and Lemma 5.5.1, there exists $q \in \mathbb{Q}_{++}$ such that $X \prec qX_0 \prec^{**} Y$, and therefore, from the definition of v and since u is a positively homogeneous utility functional for $\precsim^{**}$, it is $v(X) < qu(X_0) < u(Y)$, which obviously implies $v(X) < u(Y)$. Now consider any two elements $X, Y \in L^1_+$ such that $X \precsim Y$. If $X \sim^{**} \mathbf{0}$, then it is $u(X) = 0$, and it is clear that $u(X) \leq v(Y)$, since v is nonnegative. So assume that $\mathbf{0} \prec^{**} X$. Then there exists $Z \in L^1_+$ such that $\mathbf{0} \precsim Z \prec X \precsim Y$. Since $Y \prec tX_0$ entails $X \prec^{**} tX_0$, it is clear that $u(X) \leq v(Y)$.

Now let us prove that v is positively homogeneous. By contradiction, assume that there exist $X \in L^1_+$, and $t \in \mathbb{R}_{++}$, such that $v(tX) < tv(X)$. Then there is $t' \in \mathbb{R}_{++}$ such that $v(tX) < t'u(X_0) < tv(X)$, $tX \prec t'X_0$. Then, since $\precsim$ is homothetic, it is also $X \prec t^{-1}t'X_0$, and therefore, using the fact that $\langle u, v\rangle$ is a representation of $\precsim$, and u is positively homogeneous, we arrive at the contradiction $v(X) < t^{-1}t'u(X_0)$. Analogously, it can be shown that for no $X \in L^1_+$, and $t \in \mathbb{R}_{++}$, it is $tv(X) < v(tX)$.

Finally, let us show that v is upper semicontinuous. Consider any $X \in L^1_+$, and $\alpha \in \mathbb{R}_{++}$, such that $v(X) < \alpha$. Since u is positively homogeneous, there exists $t \in \mathbb{R}_{++}$ such that $v(X) < u(tX_0) < \alpha$, $X \prec tX_0$. By upper semicontinuity of $\precsim$, $\{Z \in L^1_+ : Z \prec tX_0\}$ is an open set containing X such that $v(Z') < \alpha$ for every $Z' \in \{Z \in L^1_+ : Z \prec tX_0\}$. This consideration completes the proof.

We also have the following corollaries.

Corollary 5.5.1. Let $\precsim$ be an interval order on a L^1_+. Assume that $\mathbf{0} \precsim^* X$ for every $X \in L^1_+$, and $\precsim^{**}$ is upper semicontinuous. There exists a pair $\langle u, v\rangle$ of nonnegative positively homogeneous real-valued functionals on L^1_+ representing $\precsim$, such that u is continuous and v is upper semicontinuous, if and only if $\precsim$ is homothetic and continuous.

Proof. It is clear that, if there exists a representation $\langle u, v\rangle$ of the interval order $\precsim$, such that u is continuous, v is upper semicontinuous and u and v are both positively homogeneous, then $\precsim$ is homothetic and continuous. So assume that $\precsim$ is homothetic and continuous. Since the total preorder $\precsim^{**}$ is upper semicontinuous, from considerations in the proof of the previous theorem we have that $\precsim^{**}$ is actually continuous, and therefore, from the corollary in Bosi, Candeal and Indur´ain[9] , there exists a nonnegative, positively homogeneous and continuous utility functional u for $\precsim^{**}$. Hence, it suffices to show that, under our assumptions, condition (iii) of the previous theorem is verified. Consider $X_0, X, Y \in L^1_+$ such that $\mathbf{0} \prec X_0, X \prec Y$. Define $\alpha = sup\{t \in \mathbb{R}_+ : tX_0 \precsim X\}$, $\beta = inf\{t \in \mathbb{R}_{++} : Y \precsim^{**} tX_0\}$. Observe that β is well defined, since, if $tX_0 \precsim^{**} Y$ for every $t \in \mathbb{R}_{++}$, then $X_0 \precsim^{**} \mathbf{0}$ by upper semicontinuity of $\precsim^{**}$, and this is contradictory. From continuity of $\precsim$, and upper semicontinuity of $\precsim^{**}$, it is $\alpha X_0 \precsim X$, $Y \precsim^{**} \beta X_0$ ($\alpha < \beta$). Then, for every $\bar{q} \in \mathbb{Q}_{++}$ such that $\alpha < \bar{q} < \beta$, it must be $X \prec \bar{q}X_0 \prec^{**} Y$, and this means that $\mathbb{Q}_{++}(X_0)$ is a strongly i.o. dense subset of $(L^1_+, \precsim)$ for every $X_0 \in L^1_+$ such that $\mathbf{0} \prec X_0$. This consideration completes the proof.

Corollary 5.5.2. Let - be an interval order on a L^1_+. Assume that **0** - * X for every $X \in L^1_+$, - ** is upper semicontinuous and - * is lower semicontinuous. There exists a pair $\langle u, v \rangle$ of nonnegative, positively homogeneous and continuous real-valued functionals on L^1_+ representing - if and only if - is homothetic and continuous.

Proof. It is clear that, if there exists a representation $\langle u, v \rangle$ of the interval order -, such that u and v are both positively homogeneous and continuous, then - is homothetic and continuous. So assume that - is homothetic and continuous. From Corollary 5.5.1, there exists a pair $\langle u, v \rangle$ of nonnegative positively homogeneous real-valued functionals on L^1_+ representing -, such that u is continuous and v is upper semicontinuous. Observe that, under our assumptions, the total preorder -* is continuous, since continuity of the interval order - implies upper semicontinuity of - *. Indeed, we have that, for every $X \in L^1_+$, $\{Z \in L^1_+ : Z \prec^* X\} = \bigcup_{\{Z' \in L^1_+ : Z' - X\}} \{Z \in L^1_+ : Z \prec Z'\}$.
Let us show that the real-valued functional v defined in the proof of the theorem is a utility functional for - *. First consider $X, Y \in L^1_+$ such that X - * Y. If $v(X) = 0$, then it is clear that $v(X) \leq v(Y)$, since v is nonnegative. If $v(X) > 0$, then there exists $Z \in L^1_+$ such that **0** - $Z \prec^* X$, which implies **0** - $Z \prec^* Y$. Hence, $Y \prec tX_0$ implies $X \prec tX_0$, and therefore it is $v(X) \leq v(Y)$ from the definition of v. Now consider $X, Y \in L^1_+$ such that $X \prec^* Y$. Then there exist $Z \in L^1_+$ such that $X \prec Z$ - Y. Since $\langle u, v \rangle$ is a representation of -, it is $v(X) < u(Z) \leq v(Y)$, which obviously implies $v(X) < v(Y)$.
From the proof of theorem 5.5.1, v is positively homogeneous utility functional for the continuous total preorder - *, and therefore v must be continuous ([13] Lemma 1). So the proof is complete.

Remark 5.5.1. The real-valued functional u in the representation $\langle u, v\rangle$ whose existence is guaranteed in the theorem is actually a utility functional for the total preorder $\precsim^{**}$. Further, from the proof of Corollary 5.5.2, the real-valued functional v defined in the proof of the theorem is a utility functional for $\precsim^{*}$. In the particular case when the interval order $\precsim$ in Corollary 5.5.2 is a total preorder, we have that $\precsim = \precsim^{*} = \precsim^{**}$. Then, it must be $u = v$ (this is not always the case, as Bridges[18] observed). Indeed, if there exists $X \in L^1_+$ such that $u(X) < v(X)$, then from positive homogeneity of v there exists $t \in \mathbb{R}_{++}$ with $u(X) < v(tX) < v(X)$, and therefore we arrive at the contradiction $X \precsim tX \prec X$. Therefore, Corollary 5.5.2 provides a generalization of the Corollary in Bosi, Candeal and Indur´ain [9] in the case when $\mathbf{0} \precsim X$ for every $X \in L^1_+$, since we have that there exists a nonnegative, positively homogeneous and continuous utility functional u for a total preorder $\precsim$ on a L^1_+, such that $\mathbf{0} \precsim X$ for every $X \in L^1_+$, if and only if $\precsim$ is homothetic and continuous.

Example 5.5.1. Given any positive real number α, define the real-valued functional u_α on L^2_+ by

$$u_\alpha(X) = E(X) + \alpha\sqrt{Var(X)}.$$

If $\precsim$ is the binary relation on L^1_+ defined by $X \precsim Y \Leftrightarrow u_\alpha(X) \le u_\beta(Y)$ $(\alpha \le \beta)$, then $\precsim$ is an interval order on L^1_+ which is represented by the pair $\langle u_\alpha, u_\beta\rangle$ of nonnegative, positively homogeneous and L_2-(pseudo)norm continuous real-valued functionals.

Example 5.5.2. Let L^1_+ be a real space of nonnegative real random variables in $L_p(\Omega, A, P)$, endowed with the L_p-(pseudo)norm topology (p is any positive real number). Given any increasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$, $g(1) = 1$, the Choquet integral of $X \in L^1_+$ with respect to the distorted probability $g \circ P$ is defined as follows:

$$\int_\Omega X dg \circ P = \int_0^\infty g \circ P\{\omega \in \Omega : X(\omega) > t\} dt.$$

Given two increasing, concave and continuous functions $g_u, g_v : [0, 1] \rightarrow [0, 1]$ such that $g_u(0) = g_v(0) = 0$, $g_u(1) = g_v(1) = 1$, $g_u(t) \leq g_v(t)$ for every $t \in [0, 1]$, let $u(X) = \int_\Omega X dg_u \circ P$, $v(X) = \int_\Omega X dg_v \circ P$ $(X \in L^1_+)$, and consider the binary relation $\precsim$ on L^1_+ defined by $[X \precsim Y \Leftrightarrow u(X) \leq v(Y)]$. Then $\precsim$ is an interval order on L^1_+, which is represented by the pair $\langle u, v \rangle$ of nonnegative, positively homogeneous and L^1-(pseudo)norm continuous real-valued functionals (see Denneberg [24], Proposition 9.4).

Chapter 6

Certainty equivalence

6.1 Introduction

It is well known that the problem of associating certainty equivalents to preferences over stochastic situations, arises in a number of different fields, like, for example, the theory of risk attitudes or the analysis of stochastic cooperative games (see e.g., Luce [36] and Suijs and Borm [44]). The possibility of endowing such preferences with certainty equivalence functionals that satisfy relevant requirements (such as positive homogeneity, translation invariance, monotonicity with respect to first-order stochastic dominance and subadditivity) has been already investigated by Alcantud and Bosi [3].

We recall that a certainty equivalent is a particular utility functional which associates to every random variable a value that is indifferent to the random variable itself.

In this chapter we review the existing results in the literature concerning continuous certainty equivalents for total preorders and we present some new concepts and proposals in the case of interval orders. In particular, in Section 6.2 we discuss the existence of a continuous certainty equivalent for a total preorder, which is in addition translation invariant and subadditive.

In Section 6.3 we present a possible concept of certainty equivalence functional for interval orders and we prove some sufficient conditions in this direction.

6.2 Existence of a continuous certainty equivalent

In this section we shall discuss the existence ofa continuous certainty equivalence functional on L^1_+ endowed with relevant propertiesLike before, we shall denote by (Ω, F, P) a probability space.

Let L^1_+(R) be the space of all the real-valued random variables with finite expectation on a common probability space (Ω, F, P), interpreted as the space of stochastic payoffs.

In what follows, it is assumed that L^1_+(R) is endowed with the L^1_+-*pseudonorm topology* corresponding to the *pseudonorm* $|| \cdot ||_1$ (i.e., $|| X ||_1 = E[\,|X|\,] = \int_\Omega |X|\, dP$).

Denote by C(R) (C(Q)) the set of all the constant real-valued (rational-valued) random variables.

For any given real number d, the constant random variable equal to d will also be denoted by X_d. Observe that, given a total preorder - on L^1_+(R), and $d_1, d_2 \in$ R, $X_{d_1} \prec X_{d_2}$ ($d_1, d_2 \in C$(R)) means that the deterministic payoff equal to d_2 with certainty is strictly preferred to the deterministic payoff equal to d_1 with certainty.

In the sequel, the term continuous referred to either a total preorder or a real-valued functional on L^1_+(R) means continuous in the L^1_+(R)-pseudonorm topology.

As usual a total preorder $\precsim$ on $L^1_+(\mathbb{R})$ is said to be continuous if

$$\{Y \in L^1_+(\mathbb{R}) : Y \precsim X\}, \qquad \{Y \in L^1_+(\mathbb{R}) : X \precsim Y\}$$

are closed sets for every $X \in L^1_+(\mathbb{R})$.

Definition 6.2.1 [certainty equivalence functional]. Given a total preorder $\precsim$ on $L^1_+(\mathbb{R})$, we say that a functional $C : L^1_+(\mathbb{R}) \to \mathbb{R}$ is a *certainty equivalence functional* for $\precsim$ if C satisfies the following conditions:

(M 1) for every $X, Y \in L^1_+(\mathbb{R})$: $X \precsim Y$ if and only if $C(X) \leq C(Y)$) (i.e., C is a utility functional for $\precsim$);

(M 2) for every $d \in \mathbb{R}$: $C(X_d) = d$ (i.e., the value that C associates to each deterministic payoff X_d is precisely equal to d).

If in addition the following property is verified:

(M 3) for every $X \in L^1_+(\mathbb{R})$, and for every $d \in \mathbb{R}$: $C(d + X) = d + C(X)$ (i.e., C is linearly separable in the deterministic amount of money d for every $d \in IR$),

then C is said to be *translation invariant*.

For an interpretation of conditions (M 1), (M 2) and (M 3), see paragraph 3 in Suijs and Borm [44]. We just recall that, if C satisfies conditions (M 1) and (M 2) then the following condition holds:

($M2'$) for every $X \in L^1_+(\mathbb{R})$: $X \sim X_{C(X)}$ (i.e., the deterministic payoff

$C(X)$ is indifferent to the stochastic payoff X for every $X \in L^1_+(\mathbb{R})$). This consideration motivates the name of *Certainty equivalence*.

In order to show that property (M2') holds, consider that $C(X) = C(X_{C(X)})$ (a property that is directly implied by condition (M2)) is equivalent to $X \sim X_{C(X)}$ by property (M1).

Definition 6.2.2 [traslation invariant total preorder]. We say that a total preorder $\precsim$ on L^1_+ is *translation invariant* if the following condition is verified:

(T) for every $d \in IR$, and for every $X, Y \in L^1_+$: $X \precsim Y$ if and only if $X + X_d \precsim Y + X_d$.

Remark 6.2.1. It is easily seen that translation invariance of a total preorder $\precsim$ on L^1_+ is a necessary condition for the existence of a real-valued functional C on L^1_+ satisfying conditions (M1), (M2) and (M3). Indeed $X \precsim Y \Leftrightarrow C(X) \overset{M1}{\leq} C(Y) \Leftrightarrow C(X) + d \leq C(Y) + d \overset{M2}{\Leftrightarrow} C(X) + C(X_d) \leq C(Y) + C(X_d) \overset{M3}{\Leftrightarrow} C(X + X_d) \leq C(Y + X_d) \overset{M1}{\Leftrightarrow} X + X_d \precsim Y + X_d$

We recall that a total preorder $\precsim$ on $L^1_+(\mathbb{R})$ is said to be *order-separable* if the following condition holds:

(S) there exists a countable subset Z of L^1_+ such that, for every $X, Y \in L^1_+$, if $X \prec Y$ then there exists $Z \in Z$ such that $X \prec Z \prec Y$.

In the previous definition, Z is said to be an order-dense subset of L.

In this section, we characterize the existence of a unique continuous certainty equivalence functional C for a total preorder - on L^1_+(R) satisfying relevant properties.

We first present necessary and sufficient conditions for the existence of a unique continuous real-valued functional C satisfying conditions (M 1) and (M 2). The following proposition has been already proved by Alcantud and Bosi (2003, Theorem 3.3) in a slightly different context.

Proposition 6.2.1. Let - be a total preorder on L^1_+(R). Then the following conditions are equivalent:

(i) There exists a unique continuous real-valued functional C on L^1_+(R) satisfying conditions (M 1) and (M 2);

(ii) The following conditions are verified:

(a) - is order-separable;

(b) - is continuous;

(c) for every $d_1, d_2 \in IR$: $d_1 < d_2 \Rightarrow X_{d_1} \prec X_{d_2}$.

In the following lemma, we clarify the strict connection between continuity of a real-valued functional C on L^1_+(R) satisfying conditions (M 1) and (M 2) and continuity of the total preorder - on L^1_+(R). The immediate proof is omitted since property (M 1) simply says that a certainty equivalence function is in particular a utility functional.

In the sequel, for the sake of convenience we shall use indifferently the notation X_α and $\overline{\alpha}$ for every non negative real number α.

Lemma 6.2.1. Let - be a total preorder on $L^1_+(\mathbb{R})$. A real-valued functional C on $L^1_+(\mathbb{R})$ satisfying conditions ($M1$) is continuous if and only if - is continuous.

We are ready to present a characterization of the existence of a continuous and traslation invariant certainty equivalence functional. Just before we present a very simple example,that is well know in actuarial mathematics since it is related to a very popular premium principle, of a functional that is traslation invariant and continuous but not positively homogeneous.

Example 6.2.1 Define on $L^2_+(\mathbb{R})$ the following functional C:

$$C(X) = E(X) + \alpha Var(X).$$

Then C is traslation invariant and continuous but not positively homogeneous.

Theorem 6.2.1. Let - be a total preorder on L^1_+. There exists a certainty equivalence functional C for - satisfying $C(X_0) = 0$ such that

$$\begin{cases} \underline{A1}. & C \text{ is nonnegative,} \\ \underline{A2}. & C \text{ is translation invariant,} \\ \underline{A3}. & C \text{ is continuous,} \end{cases}$$

if and only if the following conditions are verified:

$$\begin{cases} \underline{B1}. & X_0 \precsim X \textit{ for every } X \in L^1_+, \\ \underline{B2}. & \precsim \textit{ is translation invariant,} \\ \underline{B3}. & X \prec X + \bar{\lambda} \textit{ for every } X \in L^1_+, \lambda \in \mathbb{R}_{++}, \\ \underline{B4}. & \precsim \textit{ is continuous,} \\ \underline{B5}. & \textit{There is a utility functional } U \textit{ for } \precsim. \end{cases}$$

Proof. It is clear that conditions $A1$ through $A3$ togheter with the condition $C(X_0) = 0$ imply conditions $B1$ through $B5$. So assume that conditions $\underline{B1}$ through $\underline{B5}$ hold. By conditions $\underline{B4}$ and $\underline{B5}$, there exists a continuous utility functional U for the total preorder $\precsim$ (see e.g. Bridges and Mehta [6, Theorem 3.2.9]). Observe that condition $\underline{B3}$ imply the following condition:
$(^*)$ $X_{\lambda_1} \prec X_{\lambda_2}$ for every $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $\lambda_1 < \lambda_2$.
Then it can be shown that there exists a continuous certainty equivalence functional C for $\precsim$ (by Proposition 6.2.1). Since it is necessarily $C(\bar{0}) = 0$, we have that C is nonnegative by condition $\underline{B1}$. Define a real functional C on L^1_+ by

$$C(X) = \inf\{C(Z) + \lambda : X \prec Z + \bar{\lambda},\ \lambda \in \mathbb{R}_{++},\ Z \in L^1_+\} \qquad (X \in L^1_+).$$

Since $X < X + X_\lambda$ for every $\lambda \in \mathbb{R}_{++}$ by the definition of C and condition $\underline{B3}$, we have that $C(X) \leq C(Z) + \lambda$ for every $\lambda \in \mathbb{R}_{++}$ that implies $C(X) \leq C(X) + \lambda$ for every $X \in L^1_+$. We claim that C is a certainty equivalence functional for $\precsim$ which satisfies conditions $\underline{A1}$ through $\underline{A3}$.
In order to prove that C is a certainty equivalence functional for $\precsim$, we first show that C is a utility functional for $\precsim$. Consider $X, Y \in L^1_+$ such that $X \precsim Y$. Since $Y \prec Z + \bar{\lambda}$ entails $X \prec Z + \bar{\lambda}$, it is $C(X) \leq C(Y)$ from the definition of C. Now consider $X, Y \in L^1_+$ such that $Y \prec X$. Then by conditions $\underline{B3}$ and $\underline{B4}$ there exists $\lambda \in \mathbb{R}_{++}$ such that $Y \prec Y + \bar{\lambda} \prec X$.

Indeed, if $X \succsim Y + \overline{\lambda}$ for every $\lambda \in \mathbb{R}_{++}$, then $X \succsim Y$ by continuity of $\succsim$ and continuity of the vector operation +. Since $C(Y) < C(Y) + \lambda$ from the definition of C, in order to prove that $C(Y) < C(X)$ it suffices to show that it must be $C(Y)+\lambda \leq C(X)$. By contradiction, assume that $C(X) < C(Y)+\lambda$. Then, from the definition of C, there exist $Z \in L^1_+$, and $\mu \in \mathbb{R}_{++}$ such that $X \prec Z + \overline{\mu}$, $C(X) < C(Z) + \mu < C(Y) + \lambda$. Then, since $\succsim$ is translation invariant, C is a certainty equivalence functional for $\succsim$, and property (*) above holds, we have that $X \prec Z + \overline{\mu} \sim \overline{C(Z) + \mu} \prec \overline{C(Y) + \lambda}$, and therefore it is $X \prec \overline{C(Y) + \lambda}$, which implies $X \prec Y + \overline{\lambda}$ (a contradiction).

In order to prove that $C(\overline{\lambda}) = \lambda$ for every $\lambda \in \mathbb{R}_{++}$, first observe that $C(\overline{\lambda}) \leq \lambda = C(\overline{\lambda})$. If there exists $\lambda \in \mathbb{R}_{++}$ such that $C(\overline{\lambda}) < \lambda$, then perfectly analogous considerations leads to the existence of $Z \in L^1_+$, and $\mu \in \mathbb{R}_{++}$ such that $\overline{\lambda} \prec Z + \overline{\mu} \prec \overline{\lambda}$, and this is contradictory.

Let us show that C is translation invariant (i.e., condition $\underline{A2}$ holds). By contradiction, assume that there exist $X \in L^1_+$, and $\lambda \in \mathbb{R}_{++}$ such that $C(X + \overline{\lambda}) < C(X) + \lambda$. Then, from the definition of C, there exist $Z \in L^1_+$, and $\mu \in \mathbb{R}_{++}$ such that $C(X + \overline{\lambda}) < C(Z)+\mu < C(X)+\lambda$, $X + \overline{\lambda} \prec Z + \overline{\mu}$. If $\mu \geq \lambda$, then $X \prec Z + \overline{\mu - \lambda}$ by translation invariance of $\succsim$ (condition $B2$), and therefore it is $C(X) < C(Z)+\mu-\lambda$ (a contradiction). If $\mu < \lambda$, then $X + \overline{\lambda - \mu} \prec Z$ by translation invariance of $\succsim$. Since C is a certainty equivalence functional for $\succsim$, we have that $X + \overline{\lambda - \mu} \prec \overline{C(Z)}$. If $C(Z) \leq \lambda - \mu$, then we have $X + \overline{\lambda - \mu - C(Z)} \prec \overline{0}$ from translation invariance of $\succsim$, and this contradicts condition $\underline{B1}$. So it must be $C(Z) > \lambda - \mu$. Then by translation invariance of $\succsim$, we have $X \prec \overline{C(Z) - \lambda + \mu}$. Hence, using the fact that C is a certainty equivalence functional for $\succsim$, it is $C(X) < C(Z) - \lambda + \mu$, which is a contradiction. Analogously, it can be shown that for no $X \in L^1_+$, and $\lambda \in \mathbb{R}_{++}$, it is $C(X) + \lambda < C(X + \overline{\lambda})$.

It remains to show that C is continuous. In order to prove that C is upper semicontinuous, consider $X \in L^1_+$, and $\alpha \in \mathbb{R}_{++}$ such that $C(X) < \alpha$.

Then, from translation invariance of C, there exists $\lambda \in \mathbb{R}_{++}$ such that $C(X) < C(X + \bar{\lambda}) < \alpha$. Hence, from continuity of $\precsim$, and using the fact that C is a utility functional for $\precsim$, we have that $\mathbf{C}(X + \bar{\lambda}) = \{Z \in L^1_+ : Z \prec X + \bar{\lambda}\}$ is an open subset of L^1_+ containing X, such that $C(Z) < \alpha$ for every $Z \in L^1_+(X + \bar{\lambda})$. Finally, in order to show that C is lower semicontinuous, consider $X \in L^1_+$, and $\alpha \in \mathbb{R}_+$, such that $0 \leq \alpha < C(X)$. Consider any $\lambda \in \mathbb{R}_{++}$ with $\alpha < \lambda < C(X)$. Since C is a certainty equivalence functional for $\precsim$, and $\precsim$ is continuous, we have that $\mathbf{C}(\bar{\lambda}) = \{Z \in L^1_+ : \bar{\lambda} \prec Z\}$ is an open subset of L^1_+ containing X, such that $C(Z) > \alpha$ for every $Z \in \mathbf{C}(\bar{\lambda})$. This consideration completes the proof.

Now let us characterize the existence of a nonnegative, positively homogeneous, translation invariant, subadditive and continuous certainty equivalence functional C for a total preorder $\precsim$ on L^1_+.

Definition 6.2.3.[subadditive functional]. A real functional C on L^1_+ is said to be subadditive if, for every $X, Y \in L^1_+$,

$$C(X + Y) \leq C(X) + C(Y).$$

Theorem 6.2.2. Let $\precsim$ be a total preorder on L^1_+. There exists a certainty equivalence functional C for $\precsim$ such that

$$\begin{cases} \underline{C1}. & C \textit{ is nonnegative,} \\ \underline{C2}. & C \textit{ is positively homogeneous,} \\ \underline{C3}. & C \textit{ is translation invariant,} \\ \underline{C4}. & C \textit{ is continuous,} \end{cases}$$

if and only if the following conditions are verified:

$$\begin{cases} \underline{D1}. & \bar{0} - X \textit{ for every } X \in L^1_+, \\ \underline{D2}. & - \textit{ satisfies constant relative risk aversion,} \\ \underline{D3}. & - \textit{ is translation invariant,} \\ \underline{D4}. & X \prec X + \bar{\lambda} \textit{ for every } X \in L^1_+, \lambda \in \mathbb{R}_{++}, \\ \underline{D5}. & - \textit{ is continuous.} \end{cases}$$

The functional C above is also subadditive if and only if in addition the following condition holds:

$$\underline{D6}.\ (X - \bar{\lambda}) \text{ and } (Y - \bar{\mu}) \Rightarrow X + Y - \overline{\lambda + \mu} \quad \textit{for every } X, Y \in L^1_+, \textit{ and } \lambda, \mu \in \mathbb{R}_{++}.$$

Proof. It is clear that conditions $\underline{C1}$ through $\underline{C4}$ imply conditions $\underline{D1}$ through $\underline{D5}$. So assume that conditions $\underline{D1}$ through $\underline{D5}$ hold. By conditions $\underline{D1}$, $\underline{D2}$ and $\underline{D5}$, there exists a nonnegative,positively homogeneous and continuous utility functional U for - (see the corollary in Bosi, Candeal and Indur´ain [9]). Since by condition $\underline{D4}$ it must be $\bar{0} \prec \bar{1}$, and therefore $U(\bar{1}) > 0$, it is immediate to check that the functional C on L^1_+ defined by $C(\cdot) = (U(\bar{1}))^{-1} U(\cdot)$ is a nonnegative, positively homogeneous and continuous certainty equivalence functional for -. From the proof of Theorem 6.2.1, we have that the functional C defined there is a nonnegative, translation invariant and continuous certainty equivalence functional for -. We have just to show that such a functional is positively homogeneous, as a consequence of the fact that C' is a positively homogeneous certainty equivalence functional for -. By contradiction , assume that there exists $X \in L^1_+$, and $\lambda \in \mathbb{R}_{++}$ such that $C(\lambda X) < \lambda C(X)$. From the definition of C, there exists $\mu \in \mathbb{R}_{++}$, and $Z \in L^1_+$, with $C(\lambda X) < C(Z) + \mu < \lambda C(X)$, $\lambda X \prec Z + \bar{\mu}$. By condition $\underline{D2}$, it is $X \prec \lambda^{-1} Z + \overline{\lambda^{-1}\mu}$, and therefore, using the fact that C is positively homogeneous,we arrive at the contradiction $C(X) < \lambda^{-1} C(Z) + \lambda^{-1}\mu$ as a

consequence of the definition of C. Analogously, it can be shown that for no $X \in L^1_+$, and $\lambda \in \mathbb{R}_{++}$, it is $\lambda C(X) < C(\lambda X)$.
Finally, if C is any subadditive certainty equivalence functional for $\precsim$, then it is clear that condition $D6$ is satisfied. Conversely, assume that condition $D6$ holds in addition to conditions $D1$ through $D5$. Let us prove that the certainty equivalence functional C defined in the proof of Theorem 6.2.1 is subadditive. If there exist $X, Y \in L^1_+$ such that $C(X) + C(Y) < C(X + Y)$, then from the definition of C there exist $Z_1, Z_2 \in L^1_+$, and $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$ with $C(X) + C(Y) < C(Z_1) + \lambda_1 + C(Z_2) + \lambda_2 < C(X + Y)$, $X \prec Z_1 + \lambda_1$, $Y \prec Z_2 + \lambda_2$. Define $\lambda = C(Z_1) + \lambda_1$, $\mu = C(Z_2) + \lambda_2$. Using the fact that C is also a certainty equivalence functional for $\precsim$, and $\precsim$ is translation invariant, we have that $X \prec \bar{\lambda}$ and $Y \prec \bar{\mu}$, and therefore $X + Y \prec \overline{\lambda + \mu}$ by condition $D6$. Hence $\lambda + \mu < C(X + Y)$ is contradictory, since C is a certainty equivalence funtional for $\precsim$. So the proof is complete.

A number of interesting examples of preferences which admit a certainty equivalence functional are found in Suijs and Borm [44]. Let us only present a further example concerning the existence of a certainty equivalence functional which is also continuous.

Example 6.2.2 Let $g : [0, 1]_{\mathbb{R}} \to [0, 1]_{\mathbb{R}}$ be a increasing and concave continuous function such that $g(0) = 0$, $g(1) = 1$. The Choquet integral of $X \in L^1_+(\mathbb{R})$ with respect to the distorted probability $\mu = g \circ P$, denoted by $\int_\Omega X d\mu$, is

$$\int_0^\infty \mu\{\omega \in \Omega : X(\omega) \geq u\} du + \int_{-\infty}^0 [\mu\{\omega \in \Omega : X(\omega) \geq u\} - 1]\, du.$$

Consider the real-valued functional C on $L^1_+(\mathbb{R})$ defined by $C(X) = \int_\Omega X d\mu$ ($X \in L^1_+(\mathbb{R})$), and let $\precsim$ be the total preorder on $L^1_+(\mathbb{R})$ defined by $X \precsim Y$ if and only if $C(X) \leq C(Y)$ ($X, Y \in L^1(\mathbb{R})$). From standard properties of the

Choquet integral, we have that C is a certainty equivalence functional for -. Further, C is continuous from Proposition 9.4 in Denneberg [24].

Remark 6.2.2. The fact that we have considered preferences on the space L^1_+ (R) of real-valued random variables with finite expectation. Such an assumption is made only for the ease of exposition, and because such a space is traditionally used in the literature concerning stochastic game theory. All the previous results are still valid if preferences are defined on any pseudometric space of real-valued random variables on a common probability space (in particular, L^p(R) with p any positive real number).

6.3 Certainty equivalence with interval orders

How to define a certainty equivalence functional in the case of an interval order? We can take advantage of the considerations contained in the previous section together with the results concerning the traces of an interval order. Let us begin from the following proposition.

Proposition 6.3.1 If an interval order - on L^1_+ is represented by a pair (u, v) of real-valued functions with $u(X_C) = C$ and $v(X_C) = C, \quad \forall C \in$ R₊ then the following conditions are equivalent for every random variable $X \in L^1_+$, and $C \in$ R⁺ :

(i) X_C is such that $X \sim X_C$

(ii) $u(X) \leq C \leq v(X)$.

Proof. Consider that X - $X_C \Leftrightarrow u(X) \leq v(X_C)$ = C and X_C - $X \Leftrightarrow$

$u(X_C) = C \leq v(X)$. Therefore, $X \sim X_C$ is equivalent to both $X \precsim X_C$ and $X_C \precsim X$, and it is in turn equivalent to $u(X) \leq C \leq v(X)$. The converse is immediate. This consideration completes the proof.

Hence, it is clear that, in the particular case when, in the above proposition, we have that $u = v$, then $\precsim$ turns out to be a total preorder with a certainty equivalence functional u (the situation that we have just studied).

In general, it is clear that the certain random variable that is indifferent to any random variable X is not uniquely determined when we consider the more general case of an interval order.

Our considerations suggest the following definition of a certainty equivalence functional for an interval order.

Definition 6.3.1 Let $\precsim$ be an interval order on L^1_+ that is represented by a pair (u, v) of real-valued functions such that $u(X_C) = C$, and $v(X_C) = C$, $\forall C \in \mathbb{R}_{++}$. Then we say that a functional $\phi : L^1_+ \rightarrow \mathbb{R}_+$ is a *certainty equivalence functional* for $\precsim$ if the following condition holds for all $X \in L^1_+$:

$$\phi(X) \in [u(X), v(X)].$$

It should be noted that the definition of a certainty equivalent for an interval order does not require the existence of a certainty equivalence functional neither for $\precsim^{**}$ (the case when u is a utility function for $\precsim^{**}$) nor for $\precsim^{*}$ (the case when v is a utility function for $\precsim^{*}$ in the above representation (u, v)).

The following proposition is an immediate consequence of the definition above.

Proposition 6.3.2 Let ≾ be an interval order on L_+^1 that is represented by a pair (u, v) of real-valued functions such that $u(X_C) = C,$ and $v(X_C) = C,$ $\forall C \in \mathbb{R}_{++}$. Then both u and v are certainty equivalence functionals for ≾.

As an immediate corollary, we get the following result.

Corollary 6.3.1 Let ≾ be an interval order on L_+^1 that is represented by a pair (u, v) of positively homogeneous real-valued functions such that $u(X_1) = 1,$ and $v(X_1) = 1,$. Then both u and v are certainty equivalence functionals for ≾.

Proof. Consider that if u is a positively homogeneous real-valued function such that $u(X_1) = 1$, then $u(X_C) = u(CX_1) = Cu(X_1) = C$ for all $C \in \mathbb{R}^+$. Then Proposition 6.3.2 applies.

Proposition 6.3.3 Let ≾ be an interval order on L_+^1, If there exists a certainty equivalence functional ϕ for either ≾* or ≾**, then ϕ is a certainty equivalence functional for ≾.

Proof. Consider that, for all $X, Y \in L_+^1$,

$$(X \sim^* Y) \text{ or } (X \sim^{**} Y) \Rightarrow X \sim Y.$$

Hence, if for example there exists a certainty equivalence functional ϕ for ≾*, we have that, for all $X \in L_+^1$,

$$X \sim^{**} X_{\phi(X)} \Rightarrow X \sim X_{\phi(X)}.$$

The proof is now complete.

Proposition 6.3.4 Let - be an upper semicontinuous order-separable interval order on L^1_+. Then there exists a continuous certainty equivalence functional for - provided that:

(i) - * is lower semicontinuous;

(ii) $d_1 < d_2 \Rightarrow X_{d_1} \prec^* X_{d_2} (d_1, d_2 \in$ R$_+$).

Proof. We first show that the trace -* satisfies conditions (a), (b) and (c) of Proposition 6.2.1.First of all, it is clear that - * is order separableIndeed, assume that there exists a countable set D such that if $X \prec Y$ then there exist $D \in D$ such that $X \prec D \prec Y$. Then $X \prec^* Y \Longleftrightarrow X \prec Z$ - Y implies that there exist $D_1, D_2 \in D$ with $X \prec D_1 \prec D_2 \prec Z$ - $Y \Rightarrow X \prec^* D_2 \prec^* Y$. Therefore, - * is order-separable.

We now prove that - is continuous. Clearly, due to our assumptions, it sufficies to show that -* is upper semicontinuous.This is clear, since for all $X \in L^1_+$:

$$L_{\prec^*}(X) = \{Z \in L^1_+ : Z \prec^* X\} = \bigcup_{Z' - X} L_{\prec}(Z')$$

and therefore $L_{\prec^*}(X)$ is an open set since - is upper semicontinuous. Therefore, since condition (ii) holds, there exists a certainty equivalence functional C for - * by Proposition 6.2.1. Finally, C is a certainty equivalence function for - by Proposition 6.3.3.

Chapter 7

Risk adverse decision making under catastrophic risk

7.1 Introduction

Catastrophic events such as hurricane and earthquakes are the dominant source of risk for many property casualty insurersPrimary insurers usually limit the scale and geographic scope of their operations in order to focus on core competencies such as marketing,underwriting and loss control. But his often leaves them without sufficient geographic spread to diversify catastrophe risk. The traditional hedge for the primary insurer is reinsurance. Specialist reinsurers achieve a specialspread of risk and can therefore bear catastrophe risk that is undiversifiable to the primary. But the transaction costs associated with reinsurance,and therefore premiums,are high. High premiums, coupled with the fact that catastrophe losses exhibit little correlation with capital market indices, has attracted considerable activity in

Wall Street in searching for new instruments that securitize catastrophe risk. Indeed many players are now talking of catastrophe risk being a new “asset class” and new instruments such as catastrophe options and catastrophe bonds are starting to appear. The rationale for these new instruments is usually developed as follows. Recent catastrophe events such as Hurricane Andrew and the Northridge earthquake have imposed costs on the insurance industry of an order of magnitude not thought possible only a decade ago. More sophisticated modeling now presents potential losses to the industry of $50 billion or more. Examples would be Andrew hitting Miami, a major quake on the New Madrid Fault and a repeat of the 1906 San Francisco earthquake. These events could wipe out 25% or more of the entire industry’s net worth which currently is in the order of $200 billion. Two such events, or one such event combined with continued mass tort claims (e.g. successful plaintiff claims in tobacco litigation) could cripple the whole industry. However, losses of this size would hardly cause a ripple in capital markets. The U.S. capital market currently consists of securities representing some $13 trillion of investor wealth and the loss scenarios cited above amount to less than one standard deviation of daily trading volume. Presentations by merchant bankers, reinsurance brokers and others have echoed this potential for diversifying catastrophe risk within the capital market. The high transactions costs of reinsurance offers potential for hedging instruments to be offered to primary insurers that are both competitive with current reinsurance and which offer investors high rates of return. Moreover, since catastrophe risk is uncorrelated with market indices, the benchmark for such investments is just the risk free rate. Pricing new instruments requires that the expected loss be estimated with some. Until recently, insurers and reinsurers had a comparative advantage in information on catastrophic events. But in the past decade a number of modeling firms have developed models that combine seismic and meteorological information with data on the construction, siting, and value of individual buildings. These models can be used to simulate the economic effects of many thousands of storms and earthquakes. Although such models are used by the insurance firms and reinsurers, mainly for loss estimation and re-balancing their exposure, the same models are now available to other companies and investors. The arrival of the modelers and their models is eroding the comparative information advantage of insurers and reinsurers and opening the door to new players. Insurers will retain their comparative advantage over, say, merchant banks in related insurance services such as marketing, underwriting and loss settlement facilities. But the stage has been set for an unbundling of insurance products with insurers retaining marketing underwriting and settlement services and risk bearing by-passing the reinsurance industry and being provided more directly from the capital market. But the combination of high transaction costs for reinsurance and the vast capacity of the capital market for diversification, is not sufficient to ensure the success of these new instruments. The costs associated with reinsurance do not necessarily reflect monopoly rent. Relationships between primary insurers and reinsurers involve moral hazard; the relationship relaxes the incentive for the insurer to underwrite carefully or to settle claims efficiently. Consequently, the reinsurer will monitor the primary. Moreover, long term relationships are often formed to counter such expropriation. The apparently high transaction costs of reinsurance may simply reflect the resolution of moral hazard. If new instruments such as catastrophe options and bonds are to compete successfully with reinsurance, they must be able resolve incentive conflicts between the primary insurer and the ultimate risk bearer. Indeed, if moral hazard is not resolved, using past insurance loss data to estimate the potential returns for purchasers of catastrophe bonds, etc, is spurious.

7.2 Risk adverse decision making and the safety principle

In the existing theories of choice, uncertain outcomes are usually modeled by finite-valued random variables, i.emeasurable functions from a probability space (Ω, F, P) to the real line R (where F is a sigma algebra of sets in Ω, and P is a probability measure on (Ω, F), which will be referred to as a *standard probabilistic setting*). Agent's preference relation is then represented by weak relation - on arbitrary set A of r.v.'s. (i.e. random variables),where Y - X means that either a random variable X is preferred over a random variable Y ($Y \prec X$) or X and Y are equally preferable $X \sim Y$. Usually, the attention is restricted to r.v.'s from $L^1(\Omega)$ (= $L^1(\Omega, F, P)$), i.e. such that $E|X| < \infty$, where $E|\cdot|$ denotes the expected value.

Let $F_X(x) = P\,|X \leq x|$ be a cumulative distribution function (CDF) of X. If $F_X(x) \equiv F_Y(y)$ for two r.v.'s X and Y, we say that X and Y have the same distribution and write $X \overset{d}{\sim} Y$.

Let F be the set of possible CDFs, i.e. the set of non-decreasing right-continuous functions F with $lim_{k\to-\infty} F(x) = 0$ and $lim_{k\to+\infty} F(x) = 1$. The probability space Ω is assumed to be *atom-less*, i.e. there exists a random variable with a continuous CDF. For any r.v.'s X and Y and every $\lambda \in [0, 1]$, an r.v. Z with the CDF $F_Z(z) = \lambda F_X(z) + (1\text{-}\lambda)F_Y(z)$ is called λ- lottery of X and Y and is denoted by $Z = \lambda X + (1\text{-}\lambda)Y$. Also any constant C corresponds to a constant r.v.X_C such that $P[X_C = C] = 1$.

A preference relation - is called *rational* if it satisfies a certain set of principles (axioms) of rational behavior that traditionally includes the axioms of *completeness*, *monotonicity*, and *continuity*.

Often, - is also assumed to be *risk adverse*, i.e. a sure outcome C is preferred to any lottery with the expected payoff of C.

We recall now the context and terminology that we are going to refer to in this chapter.

(i) *Completeness*: - defines a *total order* on A, namely, - is *antisymmetric*, *transitive* (Y -X and Z - Y imply that Z -X) and *total* (Y -X or X- Y for every X and Y).

(ii) *Monotonicity* : Y -X when $P[X \geq Y] = 1$. If, in addition, $Y \prec X$ when $P[X > Y] > 0$, then is called strictly monotone.

(iii) *Continuity* : the sets $\{Y \in A : X - Y\}$ and $\{Y \in A : Y - X\}$ are closed (Equivalently, $X \succ$Y implies $X_n \succ Y_n$ for large enough n when $X_n \to X$ and $Y_n \to Y$ as n$\to \infty$), where "closedness" is defined in a topology specified for each aplication. If $A = L^1(\Omega)$, then the continuity with respect to L^1 norm is a natural choice. The continuity axiom guarantees that infinitely small variations in an r.v. can't drastically change preference relations.

Definition 7.2.1 [risk adverse preference relation] Let X and Y be r.v.'s such that $E[Z/X=x] = 0$ for all x. If $Y \overset{d}{\sim} X + Z$, we say that Y can be obtained from X by *mean-preserving spread*. Then - is *risk adverse* if Y - X when Y is obtainable from X by mean-preserving spread. By the definition above, $X \overset{d}{\sim}$ Y implies $X \sim Y$, i.e. - depends only on the CDFs' of X and Y . In fact, such - is called *law invariant*. Also, it follows from definition that Y - $E[Y]$ for all Y (*risk aversion*). A monotone and law-invariant

- is also called *consistent with the first-order stochastic dominance* (FSD), while monotone and risk averse - is called *consistent with the second-order stochastic dominance* (SSD).

Definition 7.2.2 [safety first principle]. Let X and Y be r.v.'s, and let $\alpha(X)$ and $\alpha(Y)$ be the probabilities of catastrophic events associated with X and Y, respectively.A preorder % is *consistent with the safety-first principle* if $X \succ Y$ when $\alpha(X) \leq \alpha(Y)$.

In particular, according to the definition proposed from Grechuk[33], that appears a little bit inaccurate, we shall adopt the following definitionIn the paper by Grechuk it is, al least implicity, assumed that total preorders must be taken into consideration. Actually this assumption is not needed and therefore we can simply deal with not necessary total preoders.

Definition 7.2.3. A preference relation % is said to be *consistent with the safety-first principle* if $X \succ Y$ when the following property is verified:

$$\exists c \in \mathbb{R} : F_X(c') \leq F_Y(c') \quad \forall c' < c$$

(*) and

$$F_X(c) < F_Y(c)$$

It is well-known, however,that this definition of the safety-first principle is inconsistent with some basic axioms on a preference relation,namely with continuity and risk aversion. (see Proposition 1 in Grechuk[33]).

In our standard probabilistic setting we have two generic random variables X, Y and a third Z that is a *Mixture* of the others two random variables obtained through the disintegration formula on the partition of certainly event. We have that $Z = \lambda X + (1-\lambda)Y$ (or we can write $Z = X\lambda Y$) where the probability of Z, $P(Z)$ is equal to $\lambda \in [0, 1]$. We will obtain that the Cumulative distribution function of Z is: $F_Z(z) = F_X(z)\lambda + F_Y(z)(1-\lambda)$.

Proposition 7.2.1 [safety-first principle in the standard probabilistic setting]. If - is consistent with the safety-first principle in the standard probabilistic setting, then:

(i) - is not continuous (in any topology in which $X_{c+1/n} \to X_c$ as $n \to \infty$).

(ii) - is not risk adverse.

Proof (i). To show that in our standard probabilistic setting the safety-first principle implies that % is not continuous we take the set $\{Y \in A : X^* \text{-} Y\}$ with $X^* = 1/2X_c + 1/2X_{c+1}$ in order to show that such a set is not closed. Indeed, we know that a sequence of constants $\{X_{c+1/n}\}_{n=1}^{\infty}$ converges to X_c and further $X_{c+1/n} \succ X^*$ for any $n \in$ N, by the safety-first principle. On the other hand, by the safety first-principle again, it must be $X^* \succ X_c \iff X_c \notin \{Y \in A : X^* \text{-} Y\}$. Recall that the postulates of the axiom of continuity require that any two sets $\{Y \in A : X \text{-} Y\}$, $\{Y \in A : Y \text{-} X\}$ are closed, that means that $X \succ Y$ implies $X_n \succ Y_n$ for large enough n when $X_n \to X$ and $Y_n \to Y$ with $n \to \infty$ (i.e. that occurs up to the limit).

From the definition of the Safety-first principle we see that when $a(X) < a(Y) \Rightarrow X \succ Y$. So we have that $\alpha(X^*) = P(X^* \leq c) = F_{X^*}(c) = [F_{X_c}(c) + F_{X_{c+1}}(c)]$. So we take $\alpha(X_{c+1/n}) = P(X_{c+1/n} \leq c) = F(X_{c+1/n})$. Now we make the comparison to negate the continuity in this case. We have to prove that $\alpha(X_{c+1/n}) < \alpha(X^*)$. In our particular case if we replace the values obtained from the analysis of the cumulative distribution function, we see that:

$F_{X_{c+1/n}}(c) < 1/2 F_{X_c}(c) + 1/2 F_{X_{c+1}}(c) \Rightarrow 0 < 1/2,$
$F_{X_{c+1/n}}(c) = P(X_{c+1/n} \leq c) = 0,$
$F_{X_c}(c) = P(X_c \leq c) = 1,$
$F_{X_{c+1}}(c) = P(X_{c+1} \leq c) = 0.$

Considering then the analysis of the Cumulative Distribution Function we can then say that under the safety-first principle the preorder is not continuous. So the proof is complete.

Proof (ii). To show that - is not risk adverse we find a case that denies the relation of risk adverse only in our standard probabilistic setting. We take $Y^* \succ X_{E[Y^*]}$ for $Y^* = \frac{1}{2}X_{c-1} + \frac{1}{2}X_{c+1}$. We remember that our decision maker is risk adverse if Y - $X \Leftarrow Y \overset{d}{\sim} X + Z$ (i.e. it means that he prefers X to Y because X is less dispersed).Even in this case $X \overset{d}{\sim}$ Y implies that the preference - depends only on the Cumulative Distribution Function of X and Y. This property is referred to as law invariance; we can say that our decision maker is *Risk adverse* if prefers a sure outcome $E[Y]$ to a random value Y. To deny the axiom we have to verify that $\alpha(Y^*) > \alpha(X_{E[Y^*]})$. Now we replace the Cumulative Distribution Function: $P(Y^* \leq c) \leq P(X_{E[Y^*]} \leq c)$ and replacing to $E[Y^*]$ the value of Y^* we have that $E[Y^*] = E[X_{c-1} \; \frac{1}{2} \; X_{c+1}] = \frac{1}{2}(c-1) + \frac{1}{2}(c+1) = \frac{1}{2}(c-1+c+1) = c$. Now we have that $P(X_{E[Y^*]} \leq c) = 1$, while $P(Y^* \leq c) = \frac{1}{2}$ implies that $P(Y^* \leq c) \leq P(X_{E[Y^*]} \leq c)$ and so we can say that the Safety-first Principle implies that % is not risk adverse (otherwise we would have to find that Y - X. So the proof is complete.

Remark 7.2.1 It should be noted that the preorder - in the statement of the previous theorem is not required to be total. Therefore, as far as we know, since the existence of an upper semicontinuous order preserving function u for - doesn't imply that - is upper semicontinuous, we have that a preorder - can at the same time satisfy the safety-first principle and admit an upper semicontinuous order-preserving function.

Remark 7.2.2. Proposition 7.2.1 actually ensures that a total preorder that is consistent with the safety-first principle is not upper semi-continuous and therefore it cannot admits an upper semi-continuous utility function.

Remark 7.2.3. If a total preorder - is consistent with the safety-first principle in the standard probabilistic setting, then it doesn't admit a continuous utility function (because otherwise it is continuous).

7.3 Negative results in the presence of the safety principle

We now enrich the previous considerations by introducing interval orders. We first need the following definition.

Definition 7.3.1 [safety first principle for interval orders]. An interval order % is consistent with safety first principle if $X \succ Y$ when condition (*) in definition 7.2.3 holds.

Corollary 7.3.1. If an interval order - is consistent with the safety-first principle in the standard probabilistic setting and it admits an (upper semi)continuous utility representation (u, v) than neither u is a utility function for - ** nor v is a utility function for - *.

Proof . We recall that if - is an interval order then, for all X, Y we have that

$$X \prec Y \Rightarrow (X \prec^{*} Y) \; and \; (X \prec^{**} Y)$$

Therefore it is clear that if an interval order - satisfies the safety first principle then both the associated total preorders -* and -** satisfy the safety first principle. Indeed if for any two random variables X, Y condition (*) of definition 7.2.3 is verified then we have that $X \succ Y$ that in turn implies that $X \succ^{*} Y$ and also $X \succ^{**} Y$. Then the traces cannot admit any continuous utility function by Remark 7.2.3.

Definition 7.3.2 A preorder . is said to be an *extension* of a preorder - if for all $X, Y \in L^{1}_{+}$,

$$X \text{ - } Y \Rightarrow X \text{ . } Y,$$

$$X \prec Y \Rightarrow X < Y.$$

The following Corollary to Proposition 7.2.1 may be viewed as interesting.

Corollary 7.3.2 If a preorder - is consistent with the safety-first principle in the standard probabilistic setting, then it cannot admit an upper semicontinuous order preserving function u (or,more generally,there is no upper semicontinuous weak utility u for its strict part $\prec$).

Proof. Consider a preorder - which is consistent with the safety-first principle in the standard probabilistic setting, and assume by contraposition that there exists an upper semicontinuous weak utility u for its strict part

$\prec$. Define, for all $X, Y \in L^1_+$.

$$X \,.\, Y \Leftrightarrow u(X) \leq u(Y).$$

Then - is an upper semicontinuous total preorder on L^1_+, with an upper semicontinuous utility u. Therefore, it is clear that - is upper semicontinuous (i.e., $i_.(X) = \{Z \in L^1_+ : X \,.\, Z\} = u^{-1}([u(Z), +\infty[)$ is a closed subset of L^1_+ for all $X \in L^1_+$. This is contradictory by Proposition 7.2.1, since, for all $X, Y \in L^1_+$ if there exists a constant c such that $F_X(c) < F_Y(c)$ then $X \succ Y$ which implies $u(X) > u(Y)$ and therefore $X > Y$. This means that - is an upper semicontinuous preorder which is consistent with the safety-first principle in the standard probabilistic setting (impossible). This consideration completes the proof.

Corollary 7.3.3 If an interval order - is consistent with the safety-first principle in the standard probabilistic setting, and it admits a representation (u, v), then neither u is upper (or lower) semicontinuous nor v is upper (or lower) semicontinuous.

Proof. Since $-^{**}$ is also consistent with the safety-first principle due to the fact that, for all $X, Y \in L^1_+$, $F_X(c) < F_Y(c)$ implies that $X \succ Y$ and in turn $X \succ^{**} Y$, and since u is a weak utility for $\succ^{**}$, we have that u cannot be upper semicontinuous by Corollary 7.3.2 Analogous considerations concern $-^{*}$ in connection with v. This consideration completes the proof.

As a particular case, we get the following corollary.

Corollary 7.3.4 If a semiorder . admits a representation (u, δ) (in the sense that $X \,.\, Y \Leftrightarrow u(X) \leq u(Y) + \delta$ for all $X, Y \in L^1_+$), then u is neither upper nor lower semicontinuous.

Proof. The corollary is just a particular case of Corollary 7.3.1 with $u = u$ and $v = u + \delta$.

The following definition is essentially due to Evren and Ok [29]

Definition 7.3.3 [multi-utility representation]. A preorder $\precsim$ on L^1_+ is said to have a *multi-utility representation* if there exists a family F of increasing (isotone) real-valued functions on L such that, for every pair of elements $(X, Y) \in L^1_+ \times L^1_+$,

$$X \precsim Y \Leftrightarrow f(X) \leq f(Y) \qquad \forall f \in F.$$

In decision theory multi-utility representations have been mainly studied in connection with possible generalizations of expected utility representations of incomplete preferences (see e.g. Evren[28]). A (continuous) multi-utility representation is important since it characterizes numerically a not necessarily total preorder.

Definition 7.3.4 [upper semicontinuous multi-utility representation]. A preorder $\precsim$ on L^1_+ is said to have an *upper semicontinuous multi-utility representation* if F is a multi-utility representation of $\precsim$ and every function $f \in F$ is upper semicontinuous.

Before stating an interesting impossibility result that is associated to the concept of safety-first principle in connection with the notion of un upper semicontinuous multi- utility representation we need the following lemma.

Lemma 7.3.1. If a preorder $\precsim$ on L^1_+ admits an upper semicontinuous multi-utility representation, then it is upper semicontinuous.

Proof. Let - a preorder on L^1_+ and assume that - admits un upper semicontinuous multi-utility representation functions. In order to show that $i_{-}(X)$ is closed for all $X \in L^1_+$, consider $Z \notin i_{-}(X) \Leftrightarrow$ not (X - Z). Then there exists $f \in F$ such that $f(Z) < f(X)$. Then since F only consists of upper semicontinuous functions, we have that $f^{-1}(]-\infty, f(X)[$ is an open subset of L^1_+ that contains Z and is disjoint from $i_{-}(X)$. Therefore $i(X)$ is closed (since its complement is open). So the proof is complete.

Proposition 7.3.1 If a preorder - on L^1_+ satisfies the safety-first principle then it cannot admit an upper semicontinuous multi-utility representation.

Proof. By Lemma 7.3.1, if a preorder - on L^1_+ admits an upper semicontinuous multi-utility representation, then it is upper semicontinuous. But this is in contrast with the safety first-principle by proposition 7.2.1. So the proof is complete.

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