UNIVERSITÀ DEGLI STUDI DI TRIESTE

XXVIII CICLO DEL DOTTORATO DI RICERCA IN

FISICA

QUANTUM FLUCTUATIONS AND ENTANGLEMENT IN MESOSCOPIC SYSTEMS

Settore scientifico-disciplinare: FIS/02 Fisica Teorica, Modelli e Metodi Matematici

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QUANTUM FLUCTUATIONS AND ENTANGLEMENT IN MESOSCOPIC SYSTEMS

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Chapter 1
Introduction

One of the most intriguing properties of quantum mechanics is the possibility of establishing correlations between physical systems that have no classical counterpart, the most peculiar ones being known under the name of entanglement. At first considered as a mere curiosity [1, 2], entanglement has nowadays become a physical resource allowing the realization of protocols and tasks in quantum information and quantum technology not permitted by classical physics [3–5].
However, entanglement is a very fragile resource, and it can be rapidly spoiled by the presence of an external environment. In general, any quantum system can hardly be considered to be completely isolated: coupling to its surroundings is unavoidable and, usually, this leads to decoherence effects and to the emergence of a classical behaviour [6–13]. Indeed, although the dynamics of the total system, system plus environment, is reversible, the time-evolution of the system alone, obtained by averaging over the infinitely many, uncontrollable degrees of freedom of the environment, turns out to be irreversible, rather complicated and hardly amenable to an analytic treatment. Nevertheless, in many physical instances, the coupling between system and environment can be considered to be weak; further, the system evolution usually occurs on time scales much larger than the decaying time of correlations in the environment. In such situations, very well met in many physical applications, the reduced dynamics of the system alone can be given in terms of a quantum dynamical semigroup: this is a collection (a one-parameter (=time) family of) linear maps on the set of system's states, composing only forward in time (semigroup property), with the additional characteristic of being completely positive, property that guarantees consistency in all physical situations. This framework, the so-called *open quantum system* paradigm, is very general, and has been successfully used to describe environment induced decoherence effects in atomic and molecular physics, quantum optics and condensed matter models.
However, an environment not always degrades quantum coherence and entanglement: in some situations, it may happen that it can enhance them through purely mixing mechanisms. Indeed, it has been realized that, in certain circumstances, two independent, non-interacting systems can become entangled by the action of a common bath in which they are immersed [14–20]. In general, the obvious way of entangling two

quantum systems is through a direct Hamiltonian coupling accounting for interaction among them. A different possibility is, indeed, to put them in contact with a same external environment; commonly, in these situations entanglement is still created by an Hamiltonian coupling between the systems generated by the action of the bath. Remarkably, it is possible to show that, even excluding any possible direct Hamiltonian interaction, the presence of the bath induces a mixing-enhancing mechanism able to actually generate quantum correlations among systems immersed within it.

So far, such an interesting possibility has been explored and ascertained in microscopic systems, made of few qubits or oscillators [20–22]. In view of the recent developments in optomechanics, spintronics and in the preparation and manipulation of trapped ultracold gases, one may ask whether a similar mechanism may work also for "large", many-body systems.

At first sight this possibility seems rather remote: the larger a system becomes, the less one is expecting quantum properties to be shown. This clearly holds for many-body quantum systems, *i.e.* quantum systems composed by a large number N of elementary constituents. In such systems, the study of single particle properties is impractical and the only sensible information, that can usually be gathered, concerns the behaviour of collective observables, *i.e.* observables involving all system degrees of freedom.

In general, such collective observables represent extensive properties of the system, growing indefinitely with N. Collective observables need therefore to be normalized by suitable powers of $1/N$. Provided the system density N/V is kept fixed, V being the system volume, these normalized observables become independent from the number N, allowing one to work in the so-called thermodynamic, large N limit.

Typical examples of collective observables are *average* observables, *i.e.* suitable means of single particle quantities computed over all constituents, an example of which is the mean magnetization in spin systems. Although single particle observables possess a quantum character, average observables show in general a classical behaviour as the number N of constituents increases, thus becoming so-called *macroscopic* observables. The well-established theory of these average observables [23, 24] precisely describes many-body systems at this macroscopic level.

Nevertheless, recently there have been studies reporting the observation of some sort of quantum behaviour also in systems made of a large number of particles; typically, these systems either involve Bose-Einstein condensates [25–27], namely thousands of ultracold atoms trapped in optical lattices [28, 29], or optomechanical systems [30] made of micro-oscillators (cantilevers).

Clearly, macroscopic observables, being averages quantities, scaling as $1/N$ for large N, can not be used to explain such a behaviour. Indeed, it turns out that the scaling of these averages is too strong for them to retain quantum properties when N is large. However, other kinds of collective observables have been introduced and studied in many-body systems; in analogy with classical probability theory, they are called *fluctuations* [31]. They still involve all the degrees of freedom of the system, but they account for quantum deviations around the macroscopic average behaviour. They scale as $1/\sqrt{N}$ and exhibit some quantum properties even in the large N limit. Being half-way between the microscopic observables, namely those describing the behaviour of single particles in the system, and the macroscopic averages, they are called *mesoscopic* observables.

The set of fluctuation observables forms an algebra, that, irrespective of the nature of the microscopic constituents, turns out to be non-classical, *i.e.* non-commutative, and

always of bosonic character: it is at the elements of this algebra that one should look in order to properly describe quantum features of large systems.
Although the properties and the time-evolution of the fluctuation operators algebra have been studied in various physical models [31–33], very little is known of its behaviour in open many-body systems [34], *i.e.* in large systems in contact with an external environment. As already mentioned, this is the most common situation encountered in actual experiments, where these systems can never be thought of as completely isolated from their thermal surroundings. Aim of this thesis is precisely to give first a comprehensive analysis of the dissipative dynamics of many-body fluctuation operators and then to study whether such open system time-evolutions are able to generate quantum correlations through a purely mixing-enhancing mechanism. We shall see that quite in general two non-interacting many-body systems, made of a collection of spin variables, and immersed in a common bath, can indeed become entangled at the level of mesoscopic fluctuations solely because of the presence of mixing effects. Even more strikingly, in certain situations, the created entanglement can persist for asymptotically long times.
In more detail, the thesis is organized as follows:

- In Chapter 2, the basic mathematical tools for the description of many-body quantum systems are briefly reviewed. They are based on the algebraic approach to quantum mechanics, which represents the most general formulation of the theory, valid for both finite and infinite dimensional systems. Furthermore, such an approach is necessary for the definition and the study of collective observables.

- Chapter 3 focuses on the properties of collective many-body observables and in particular on the algebraic structure generated by fluctuation operators. In many-body systems characterized by short-range correlations, the large N limit leads to fluctuation operators that are bosonic quantum degrees of freedom with Gaussian characteristic function. Such a limiting behaviour can be shown by means of an extension to the quantum setting of the classical central limit theorem [35].

- Chapter 4 is dedicated to the description of the dissipative dynamics of fluctuation operators. A brief presentation of the theory of open quantum systems is first given, assuming the coupling between system and environment to be weak. In such situations, as already mentioned, physically motivated approximations lead to reduced microscopic dynamics of the system that can be very well described by a Markovian, *i.e.* memoryless, time evolution, generated by a master equation in Kossakowski-Lindblad form. The dynamics is chosen in such a way to leave the microscopic reference state of the system invariant and to map into itself the linear span of relevant single-site observables. Under this condition, we show that the emergent, large N mesoscopic dynamics for the bosonic fluctuations results in a quantum dynamical semigroup of quasi-free type, thus preserving the initial Gaussian character of the fluctuation algebra.

- In Chapter 5, this general result is applied to the study of the behaviour of a many-body system composed by two, independent spin-1/2 chains, immersed in a common thermal bath. The two chains are initially prepared in a microscopic Gibbs state, with a separable, tensor product structure that excludes long-range

correlations. The attention is then focused on a suitable set of single-site operators giving rise to quantum fluctuations that, in the large N limit, identify collective bosonic degrees of freedom exclusively referring to either one or the other of the two chains. Despite the lack of direct microscopic interactions among the spins either in a same or in different chains, the dissipative dynamics due to the presence of the bath is able to create mesoscopic collective entanglement between the two many-body systems at the level of their fluctuation operators through a purely noisy mechanism. Remarkably, in certain situations the created entanglement can persist for asymptotic long times.
The behaviour of the created collective quantum correlations is then studied in detail as a function of the bath temperature and other properties of the dissipative dynamics. One then discovers that a sort of entanglement phase transition is at work: a critical temperature can always be identified, above which entanglement between mesoscopic observables can not be created.

The Appendices contain technical calculations and proofs that is not appropriate to include in the main text.

Finally, we would like to point out that the obtained results are quite general and independent from specific models. As such, they can find direct applications in all instances where mesoscopic, coherent quantum behaviours are expected to emerge, *e.g.* in experiments involving spin-like and optomechanical systems, or ultra-cold gases trapped in optical lattices [25–27,36–38]: the possibility of entangling these many-body systems through a purely mixing mechanism may reinforce their use for the actual realization of quantum information and communication protocols [39–41].

Chapter 2

Mathematical Description of Infinite Systems

Physics deals with reproducible phenomena that can be tested and verified through experiments. Any experiment consists in two main procedures: the preparation of the system under study in an initial state and the measurement of some of its properties or observables. The statistical interpretation of the measurement stems from the possibility of preparing many times the system in the same state.
On the other hand, an observable of a system is a physical quantity identified by the apparatus used for its measurement [42].
From experiments one obtains outcomes of measurement processes, consisting of real finite numbers associated to the correspondent observables; it is the aim of physics to provide a mathematical setting where these outcomes can be interpreted and predicted.

2.1 Algebraic Approach to Quantum Mechanics

Any physical system is characterized by a set of independent measurable quantities; to each of these quantities one associates an element in an operator set. Its spectrum consists of the possible outcomes of a measurement; thus, the operator has to be self-adjoint in order to possess real eigenvalues. Using these abstract elements one can construct, by means of products and linear combinations, an algebra $\mathcal{A}$, whose self-adjoint elements correspond to all possible measurable quantities.
The algebra $\mathcal{A}$ turns out to be a C^*-algebra; this means that it is a linear, associative algebra (with unity) over the set of complex numbers $\mathbb{C}$. Further, $\mathcal{A}$ is endowed with an anti-linear involution $* : \mathcal{A} \to \mathcal{A}$, such that $\left(a^\dagger\right)^\dagger = a$, $\forall a \in \mathcal{A}$. In addition, a norm $\|\cdot\|$ is defined on $\mathcal{A}$, satisfying $\|ab\| \leq \|a\|\,\|b\|$, $\forall\, a, b \in \mathcal{A}$, such that $\left\|a^\dagger a\right\| = \|a\|^2$; $\mathcal{A}$ is closed under this norm, *i.e.* $\mathcal{A}$ is a complete space with respect to the topology induced by the norm.
The main difference between quantum and classical mechanics lies in the character of this algebra: observables of a quantum system do not commute, meaning that during

experiments the ordering of the different measures is relevant.
To connect the abstract elements of the algebra $\mathcal{A}$ to actual experiments, a mathematical representation of the physical state of the system is needed. Since the complete information about the condition of a system lies in the knowledge of all moments and correlation functions of its observables, it is natural to identify its physical state with a linear functional on the algebra $\mathcal{A}$. Of course, in order for these functionals to be interpreted as states, they must obey some physical properties.

Definition 2.1. *A quantum state is a functional $\omega : \mathcal{A} \to \mathbb{C}$, such that:*

$$\begin{aligned}
&\textit{1)} \quad \omega\,(a + \lambda b) = \omega(a) + \lambda\omega(b), \quad \forall\, a, b \in \mathcal{A},\ \lambda \in \mathbb{C}\,,\\
&\textit{2)} \quad \omega\left(a^{\dagger} a\right) \geq 0,\ \forall\, a \in \mathcal{A}\,,\\
&\textit{3)} \quad \omega(\mathbf{1}) = 1\,.
\end{aligned}$$

The first condition represents linearity; the second one embodies the positivity requirement: the expectation of any positive observable must be positive. The last condition is the normalization requirement, which is needed to conform with the statistical interpretation of quantum mechanics.

Remark 2.1. *The above definition of quantum states as positive linear normalized functionals, reduces to the familiar one in terms of density operators, in the case of finite dimensional quantum systems. In such cases, the set of all density matrices forms the convex space of positive, unit trace operators on the system Hilbert space $\mathcal{H}$:*

$$\mathcal{S}(\mathcal{H}) = \{\rho : \mathcal{H} \to \mathcal{H}\,|\,\mathrm{Tr}(\rho) = 1,\ \rho \geq 0\}\,.$$

Being hermitian the generic element of such space can always be written as follows:

$$\begin{aligned}
&\mathcal{S}(\mathcal{H}) \ni \rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|,\\
&\langle\psi_i|\psi_j\rangle = \delta_{ij}, \quad 0 \leq \lambda_i \leq 1,\ \forall i, \quad \sum_i \lambda_i = 1\,.
\end{aligned}$$

Pure states are projectors, obtained when just one λ_i is equal to 1 all others being zero. Within this formalism, expectation values over states are computed by means of the trace operation:

$$\langle o\rangle = \mathrm{Tr}(\rho\, o)\,, \qquad \forall\, o \in \mathcal{A},\ \rho \in \mathcal{S}(\mathcal{H})\,.$$

Therefore, the state ω of a finite-dimensional system can always be written as

$$\omega(o) = \mathrm{Tr}(\rho\, o)\,,$$

and one can check that it obeys the rules of Definition 2.1.

The description of a quantum system as a C^*-algebra containing the observables and a functional ω on this algebra, serving as state, is the minimal one, sufficient for the description of any quantum system, finite or infinite dimensional. It further allows a Hilbert space interpretation through the so-called GNS-construction.

2.1.1 The GNS-construction

An important mathematical result, with wide application in many-body systems, goes under the name of GNS-construction (Gelfand-Naimark-Segal) [43, 44]. It shows that, given any C^*-algebra $\mathcal{A}$ together with a state ω, one can find a state dependent triple $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, π_ω a representation of the algebra into linear bounded operators and Ψ_ω is a cyclic vector for the Hilbert space $\mathcal{H}_\omega$, such that

$$\omega(a) = \langle \Psi_\omega | \pi_\omega(a) | \Psi_\omega \rangle\,, \quad \forall\, a \in \mathcal{A}\,.$$

This construction allows to pass from an algebra and a state, to the more familiar description by means of Hilbert spaces and bounded operators [23, 45–48].

Theorem 2.1. (GNS Construction) *Let ω be a state over the C^*-algebra $\mathcal{A}$. It follows that there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ of $\mathcal{A}$ such that:*

$$\omega(a) = \langle \Psi_\omega | \pi_\omega(a) | \Psi_\omega \rangle\,, \quad \forall\, a \in \mathcal{A}\,,$$

consequently, $\|\Psi_\omega\|^2 = \langle \Psi_\omega | \Psi_\omega \rangle = 1$. Moreover, the representation is unique up to unitary equivalence.

The so constructed triple $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ provides the suited tools for the standard description of quantum systems and makes it apparent that the notion of Hilbert space associated to a quantum system is not a primary concept, but an emergent one.

Example 2.1. *Consider a two-level system: it is described by the spin operator algebra $\mathcal{A}_s = M_2(\mathbb{C})$, the algebra of 2×2 matrices; a basis of the algebra is provided by the operator $\{s_\mu\}_{\mu=0}^3$, obeying the following commutation relations $[s_\mu, s_\nu] = i\epsilon_{\mu\nu\eta} s_\eta$, $\mu, \nu, \eta = 1, 2, 3$, while $[s_0, s_\mu] = 0$, and $\mathrm{Tr}(s_\mu^2) = \frac{1}{2}$, $\forall\, \mu$. Further, on this algebra let us consider the functional such that:*

$$\omega_\beta(s_1) = \omega_\beta(s_2) = 0, \qquad \omega_\beta(s_3) = -\frac{1}{2} \tanh\left(\frac{\beta}{2}\right)\,.$$

It is straightforward to show that one can represent this functional by means of a density matrix, interpretable as a thermal state at inverse temperature β:

$$\omega_\beta(\cdot) = \mathrm{Tr}(\rho_\beta \cdot)\,, \tag{2.1}$$

$$\rho_\beta = \frac{e^{-\beta s_3}}{2\cosh(\frac{\beta}{2})}\,. \tag{2.2}$$

By purification [49] of the density operator[1]*, one finds the following unit GNS-vector $|\Psi_{\omega_\beta}\rangle \in \mathbb{C}^4$:*

$$|\Psi_{\omega_\beta}\rangle = \lambda_+(\beta)|+\rangle \otimes |+\rangle + \lambda_-(\beta)|-\rangle \otimes |-\rangle$$

[1]Given a d-dimensional quantum system, a generic density matrix $\rho \in \mathcal{S}\left(\mathbb{C}^d\right)$ can be written as (see Remark 2.1) $\rho = \sum_{i=1}^d r_i |i\rangle\langle i|$, with $\{|i\rangle\}_{i=1}^d$, an orthonormal basis of $\mathbb{C}^d$. The purified vector $|\Psi\rangle$ related to ρ, in the enlarged Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, is of the form $|\Psi\rangle = \sum_{i=1}^d \sqrt{r_i} |i\rangle \otimes |i\rangle$, and it is such that

$$\mathrm{Tr}_{II}\left(|\Psi\rangle\langle\Psi|\right) = \rho\,,$$

being $\mathrm{Tr}_{II}\left(\cdot\right)$ the partial trace over the second copy of the Hilbert space $\mathbb{C}^d$.

with $|\pm\rangle$ *eigenstates of* s_3*, such that* $s_3|\pm\rangle = \pm\frac{1}{2}|\pm\rangle$*, and*

$$\lambda_+(\beta) = \sqrt{\frac{e^{-\frac{\beta}{2}}}{2\cosh(\frac{\beta}{2})}}\,, \qquad \lambda_-(\beta) = \sqrt{\frac{e^{\frac{\beta}{2}}}{2\cosh(\frac{\beta}{2})}}\ .$$

Further, representing the algebra on $\mathbb{C}^4$ *as* $\pi_{\omega_\beta}(\mathcal{A}_s) = M_2(\mathbb{C}) \otimes \mathbf{1}$*, one can verify that:*

$$\omega_\beta(s_\mu) = \langle\Psi_{\omega_\beta}|\frac{1}{2}\sigma_\mu \otimes \mathbf{1}|\Psi_{\omega_\beta}\rangle\,, \qquad \forall\,\mu \in \{0,1,2,3\},$$

with $(\mathbf{1}, \sigma_1, \sigma_2, \sigma_3)$ *the Pauli matrices. The GNS-Hilbert space* $\mathcal{H}_{\omega_\beta}$*, is obtained completing the complex linear span formed by the vectors* $\{|\Psi_\mu\rangle\}_{\mu=1}^4$*, with* $|\Psi_\mu\rangle = \sigma_\mu \otimes \mathbf{1}|\Psi_{\omega_\beta}\rangle$*, with respect to the vector norm* $\||\Psi_\mu\rangle\|^2 = \langle\Psi_\mu|\Psi_\mu\rangle$*.*
This example is also useful to deduce some properties of representations; the commutant *of a representation* $\pi(\mathcal{A}) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *(where* $\mathcal{B}(\mathcal{H})$ *is the algebra of bounded linear operators on* $\mathcal{H}$*) is defined as the set [50]:*

$$\pi'(\mathcal{A}) = \left\{ b \in \mathcal{B}(\mathcal{H}) \middle| [a,b] = 0,\, \forall\, a \in \pi(\mathcal{A}) \right\}\ . \tag{2.3}$$

In the example what one can see is that the commutant is isomorphic to the representation itself, indeed $\pi'_{\omega_\beta}(\mathcal{A}_s) = \mathbf{1} \otimes M_2(\mathbb{C})$*; an algebra with non-trivial commutant is called reducible.*
Nevertheless, one can check that the center *[50], defined as the intersection of the representation with its commutant*

$$\mathcal{Z} = \pi_{\omega_\beta}(\mathcal{A}_s) \cap \pi'_{\omega_\beta}(\mathcal{A}_s) = \left\{ \alpha\mathbf{1} \otimes \mathbf{1} \middle| \alpha \in \mathbb{C} \right\}\ ,$$

consists of multiples of the identity. Such a representation is a reducible factor representation.
It is interesting to consider the limiting case of zero temperature, so that $\beta \to \infty$*; what happens is that the GNS-vector becomes*

$$\lim_{\beta\to\infty} |\Psi_{\omega_\beta}\rangle = |-\rangle \otimes |-\rangle\ ,$$

and there is no need of enlarging the Hilbert space up to $\mathbb{C}^4$*. In this situation, with the vector* $|\Psi_{\omega_\infty}\rangle = |-\rangle$ *and the representation* $\pi_{\omega_\infty}(\mathcal{A}_s) = M_2(\mathbb{C})$*, one is able to recover all expectations*

$$\omega_\infty(s_\mu) = \langle\Psi_{\omega_\infty}|\frac{1}{2}\sigma_\mu|\Psi_{\omega_\infty}\rangle\ .$$

The state ω_∞ *is a pure state and the arising GNS representation* π_{ω_∞} *has a trivial commutant, hence irreducible.*

2.2 Disjoint Phases and Inequivalent Representations

We have seen that, given an algebra $\mathcal{A}$ and a state ω on it, one can construct a cyclic representation of $\mathcal{A}$, allowing for a fully physical description of the quantum

system. Varying the state, one obtains several representations; it turns out that in finite dimensions all these representations are unitarily equivalent, in the sense of Theorem 2.1.
On the contrary, when the system is made of an infinite number of particles, this is in general no longer true. Such a feature is not just a mathematical artefact, it reflects specific physical conditions. Indeed, one has that the same infinite system can be found in different inequivalent configurations corresponding to different physical phases. In general, the physical configurations of a same system differ just for finite number of operations that could be performed on the system, while others require an infinite number of operations to be obtained one from the other, namely an infinite amount of energy; experimentally, infinite amounts of energy are not accessible. The following examples [50], clarify these points.

Example 2.2. *Let us start with a system made by a number N of spin-$\frac{1}{2}$ particles; the algebra of each of these particles is the same finite dimensional algebra $\mathcal{A}_s$ considered before in Example 2.1, and to each particle is associated a Hilbert space $\mathbb{C}^2$.*
The Hilbert space of the total system is constructed by means of the tensor product structure

$$\mathcal{H}_N = \bigotimes_{k=1}^{N} \left(\mathbb{C}^2\right)^{(k)} , \tag{2.4}$$

where k labels different spins. Taking two vectors, representing the situation in which all particles are in the same state $|\psi\rangle$ or $|\varphi\rangle$,

$$\begin{aligned} |\psi_N\rangle &= \bigotimes_{k=1}^{N} |\psi^{(k)}\rangle , \\ |\varphi_N\rangle &= \bigotimes_{k=1}^{N} |\varphi^{(k)}\rangle, \qquad |\psi\rangle, |\varphi\rangle \in \mathbb{C}^2,\, 0 < |\langle\varphi|\psi\rangle| < 1 , \end{aligned}$$

the scalar product is defined multiplicatively:

$$\langle\varphi_N|\psi_N\rangle = \prod_{k=1}^{N} \langle\varphi^{(k)}|\psi^{(k)}\rangle = \langle\varphi|\psi\rangle^N . \tag{2.5}$$

As N increases, the scalar product becomes smaller and smaller, and it eventually goes to zero in the limit of infinite number of particles, even if $\langle\varphi|\psi\rangle \neq 0$.
The scalar product on this infinite tensor product Hilbert space defines equivalence classes of vectors belonging to the same phase of the system. Thus, the latter two vectors become orthogonal and belong to different separable Hilbert spaces, describing disjoint phases of the system.
In general, different equivalence classes cannot be unitarily equivalent. For instance, on the single particle Hilbert space, one can always identify a one parameter rotation U_α, $U_0 = \mathbf{1}$, such that there exists an $\bar{\alpha}$, giving:

$$U_{\bar{\alpha}}|\varphi\rangle = |\psi\rangle ;$$

thus, the operator $U_{\bar{\alpha}}^N = \bigotimes_{k=1}^N U_{\bar{\alpha}}^{(k)}$ rotates $|\varphi_N\rangle$ into $|\psi_N\rangle$; nevertheless, the formal writing $U_{\bar{\alpha}}^\infty$ is meaningless, since such an operator would not even be a weakly contin-

uous[2] *unitary operator in* α*; indeed, it is easy to check that*

$$\langle\varphi_\infty|U_\alpha^\infty|\varphi_\infty\rangle = \begin{cases} 1 & \textit{if} \quad \alpha = 0 \\ 0 & \textit{whenever} \quad \alpha \neq 0 \end{cases} .$$

By Stone theorem [50], the existence of a generator for such a rotation is equivalent to the strong continuity (see Footnote 2) of U_α^∞*, but such an operator is not even weakly continuous in* α*, thus there is no generator (notice that strong continuity would imply the weak one). Physically speaking, the experimentalist should be able to provide enough energy to rotate all particles of the system from* $|\varphi\rangle$ *to* $|\psi\rangle$*, but this operation would cost infinite energy; the two states are thus meaningfully interpreted as different phases of the system.*

Example 2.3. *Let us consider again spin-*$\frac{1}{2}$ *particles; we show how in the infinite limit inequivalent representations emerge.*
We focus on the class of factorized, translation invariant states $|\bar{n}\rangle$ *with spins directed along the direction* $\bar{n}$ *for all, infinitely many particles; thus*

$$|\bar{n}\rangle = \bigotimes_{k=1}^{\infty} |\bar{n}^{(k)}\rangle, \qquad \langle\bar{n}|s_\mu^{(k)}|\bar{n}\rangle = n_\mu .$$

Considering the C^**-algebra* $\mathcal{A}$ *containing all the observables of the systems (for a precise definition, see below Section 2.3.1), by means of the GNS-construction, one can find the triples* $(\mathcal{H}_{\bar{n}}, \pi_{\bar{n}}, \Psi_{\bar{n}})$*, based on the states* $|\bar{n}\rangle$*.*
It is clear from equation (2.5), that $\langle\bar{n}|\bar{n}'\rangle = 0$*, whenever* $\bar{n} \neq \bar{n}'$*; furthermore, no local action can change the convergence of the latter scalar product; this means that,* $\forall\, a \in \mathcal{A}$*,*

$$\langle\bar{n}|\pi_{\bar{n}}(a)|\bar{n}'\rangle = 0 .$$

With the help of a particular class of observables, which will be thoroughly investigated in Section 3.1, we shall show that there exists no unitary operator relating the two representations $\pi_{\bar{n}}, \pi_{\bar{n}'}$*. Let us define the average magnetization along the* α*-axis*

$$\bar{S}_\alpha^N = \frac{1}{N}\sum_{k=1}^{N} \pi_{\bar{n}}\left(s_\alpha^{(k)}\right) ; \tag{2.6}$$

in the limit $N \to \infty$*, this observable converges in the strong topology of the GNS-representation to a multiple of the identity, proportional to the mean-value [23, 31, 50]. Such a type of convergence means that*

$$\lim_{N\to\infty} \langle\bar{n}|\pi_{\bar{n}}\left(a^\dagger\right)\left(\bar{S}_\alpha^N - n_\alpha \mathbf{1}\right)^2 \pi_{\bar{n}}\left(a\right)|\bar{n}\rangle = 0, \quad \forall\, a \in \mathcal{A}\,, \tag{2.7}$$

and is denoted as

$$s - \lim_{N\to\infty} \bar{S}_\alpha^N = n_\alpha \mathbf{1} .$$

[2] Given a Hilbert space $\mathcal{H}$, and an operator A_t depending on $t \in \mathbb{R}$, acting on such Hilbert space, the operator A_t is said to be *weakly continuous* if $\lim_{t\to t_0}\langle\phi|\left(A_t - A_{t_0}\right)|\chi\rangle = 0$, $\forall\chi, \phi \in \mathcal{H}$. In other words, an operator A_t is weakly continuous if all of its matrix elements are continuous functions of t. Similarly, the operator A_t is said to be *strongly continuous* if $\lim_{t\to t_0}\langle\phi|\left(A_t - A_{t_0}\right)^\dagger\left(A_t - A_{t_0}\right)|\phi\rangle = 0$, $\forall\phi \in \mathcal{H}$.

If a unitary operator U mapping the two representations one into the other existed, once applied to the averages, it would imply that

$$U n_\alpha U^{-1} = n'_\alpha \,.$$

This is impossible since different scalars can not be mapped one into the other by a unitary operator.
Thus, the two representations $\pi_{\bar{n}}, \pi_{\bar{n}'}$ are inequivalent.

2.3 Infinite Number of Distinguishable Particles

In this thesis, we shall deal with many-body systems composed by an infinite number of distinguishable particles, such that one is able to address each one of them independently (*e.g.* lattice systems). We first discuss the proper definition of the algebra $\mathcal{A}$ containing all the observables of the system and then its relevant states.

2.3.1 The *quasi-local* Algebra

It is assumed that each particle can be described by means of a same d-dimensional C^*-algebra $\mathcal{A}_d$, containing all relevant single-particle observables. Since one is able to distinguish between different particles, the latter will be labelled by an integer number $k \in \mathbb{Z}$; in particular $\mathcal{A}_d^{(k)}$ will denote the algebra relative to particle k. Referring to different degrees of freedom, we have

$$\left[\mathcal{A}_d^{(k)}, \mathcal{A}_d^{(h)}\right] = 0, \qquad \forall\, k \neq h \in \mathbb{Z}\,; \tag{2.8}$$

by means of the tensor product structure one can construct local algebras, including just a finite number of particles. For instance, the algebra

$$\mathcal{A}_{[q,p]} = \bigotimes_{k=q}^{p} \mathcal{A}_d^{(k)}, \quad q, p \in \mathbb{Z},\, q \leq p \;, \tag{2.9}$$

contains all operators of the set of particles from q to p. The family of local algebras $\left\{\mathcal{A}_{[q,p]}\right\}_{q\leq p}$ possesses the following properties [23]:

$$\left[\mathcal{A}_{[q_1,p_1]}, \mathcal{A}_{[q_2,p_2]}\right] = 0 \qquad \text{if} \qquad [q_1,p_1] \cap [q_2,p_2] = \emptyset \tag{2.10}$$

$$\mathcal{A}_{[q_1,p_1]} \subseteq \mathcal{A}_{[q_2,p_2]} \qquad \text{if} \qquad [q_1,p_1] \subseteq [q_2,p_2] \tag{2.11}$$

The union of these algebras over all possible finite sets of particles, contains all the observables of the system.

Definition 2.2. *Microscopic observables are all the Hermitian elements of the C^*-algebra $\mathcal{A}$, defined as:*

$$\mathcal{A} = \overline{\bigcup_{\forall\, q \leq p \in \mathbb{Z}} \mathcal{A}_{[q,p]}}^{\|\cdot\|}$$

where the notation means completion with respect to the norm topology. The algebra $\mathcal{A}$ is called the quasi-local algebra.
Within this algebra one can write the operator a of the particle k in the following way:

$$a^{(k)} = \mathbf{1}_{k-1]} \otimes a \otimes \mathbf{1}_{[k+1}\,,$$

where $\mathbf{1}_{k-1]}$ is the tensor product of identities from $-\infty$ to the particle $k-1$, and $\mathbf{1}_{[k+1}$ is the product of identities from the particle $k+1$ to $+\infty$. Clearly, $a^{(k)}, \forall\, a \in \mathcal{A}_d$ acts non trivially only on the k-th particle.
Such an algebra, is naturally embedded with the translation automorphism $\tau : \mathcal{A} \to \mathcal{A}$, acting on the operator $a^{(k)}$ of the particle k in the following way:

$$\tau(a^{(k)}) = a^{(k+1)}, \qquad \forall\, a \in \mathcal{A}_d, \forall\, k \in \mathbb{Z}\,.$$

Clearly, some operators in this quasi-local algebra $\mathcal{A}$ act non-trivially only on a finite set of particles and can be named *local*; to each of these local operators one can associate a set, called *support*, providing a measure of the spreading of its action on the infinite system.

Definition 2.3. *An operator $O \in \mathcal{A}$ is said to be local, if there exists a set of intervals of lattice sites $[k, h]$, with $k \leq h \in \mathbb{Z}$, such that $\left[O, x^{(j)}\right] = 0$, $\forall\, x \in \mathcal{A}_d$, and $\forall\, j \notin [k, h]$. The length of such intervals $[k, h]$ is given by $\ell\,([k, h]) = h - k + 1$, and the smallest among all possible ones, according to such a length, is called the support of the operator O.*

Remark 2.2. *The fact that, the quasi-local algebra $\mathcal{A}$ is the norm closure of the union of all possible local algebras, has an important consequence in the approximation of any given element $a \in \mathcal{A}$. Indeed, it means that, $\forall \epsilon > 0$, there exists a strictly local operator a_ϵ such that in the norm topology*

$$\|a_\epsilon - a\| < \epsilon\,;$$

in other terms, the error considering a local observable a_ϵ instead of $a \in \mathcal{A}$ can be made arbitrarily small. This means that the set of strictly local operators is dense in $\mathcal{A}$.

Such an algebra possesses an interesting property that has also a relevant physical meaning; given the very large number of particles, it is physically expected that measurements of observables far away from each other become compatible.
It turns out that indeed such an algebra, possesses the strongest form of such a property, usually called asymptotic abelianess [23, 46].

Proposition 2.1. *The quasi-local algebra $\mathcal{A}$ is norm asymptotic abelian, in the sense that:*

$$\lim_{|k|\to\infty} \left\|\left[\tau^k(a), b\right]\right\| = 0, \quad \forall a, b \in \mathcal{A}\,.$$

Proof. Let us consider two operators $a, b \in \mathcal{A}$; because of Remark 2.2 above, $\forall \epsilon > 0$ one has

$$\|a - a_\epsilon\| < \epsilon\,,$$
$$\|b - b_\epsilon\| < \epsilon\,,$$

with a_ϵ, b_ϵ strictly local. Therefore

$$\left\| \left[\tau^k(a), b\right] \right\| < 2\epsilon \left(\|a_\epsilon\| + \|b\|\right) + \left\| \left[\tau^k(a_\epsilon), b_\epsilon\right] \right\| ,$$

and for k large enough, the support of $\tau^k(a_\epsilon)$ does not intersect the one of b_ϵ, therefore the two operators belong to disjoint local algebras, and the commutator, in the above equation, becomes zero for the locality property (2.10). This means that the norm of the commutator can be made arbitrarily small for k large.

□

2.3.2 Physically Relevant Representations

After having introduced the main properties of the algebra of the microscopic observables, one has to specify a reasonable class of physical states that can only be dictated by actual experimental conditions.
Given a many-body system, it is then reasonable to require that observables related to regions that are far away from each other show no correlations. By means of the translation automorphism τ, this property can be expressed in the following way:

$$\lim_{|k|\to\infty} \left[\omega\left(\tau^k(a)b\right) - \omega\left(\tau^k(a)\right)\omega\left(b\right)\right] = 0, \quad \forall\, a, b \in \mathcal{A}\,. \tag{2.12}$$

Therefore, the requirement (2.12), together with the asymptotic abelianess of the algebra, gives the usual cluster property of states.

Definition 2.4. *A state ω is said to be clustering if $\forall\, a, b, c \in \mathcal{A}$, one has:*

$$\lim_{|k|\to\infty} \omega\left(a\tau^k(c)b\right) = \omega(ab) \lim_{|k|\to\infty} \omega\left(\tau^k(c)\right) .$$

Another assumption that is usually made on states of infinite systems is translation invariance. We give now the definition and then explain the physical reasons why this mathematical requirement does not restrict the class of physical operations that can be performed during an experiment.

Definition 2.5. *A state ω is called translation invariant if $\forall\, a \in \mathcal{A}$, one has:*

$$\omega\left(\tau^k(a)\right) = \omega\left(\tau^h(a)\right) = \omega(a), \qquad \forall\, h, k \in \mathbb{Z}\,.$$

Any physical operation necessarily consists of a finite amount of energy transferred to the system; this means that in an experiment only a finite number of particles can be simultaneously addressed. Therefore, the phase of the system, as outlined in Section 2.2, is determined by the boundary conditions, roughly speaking by what happens to particles at infinity. The most simple reference state for the system is therefore [46] a translation invariant one, all other states being obtained by means of local (*quasi-local*) perturbations of this reference state. Thus, considering a translation invariant state ω and a physical operation described by the operator $b \in \mathcal{A}$, a state ω_b can be obtained from the invariant one as follows:

$$\omega_b(\cdot) = \frac{\omega\left(b^\dagger \cdot b\right)}{\omega\left(b^\dagger b\right)}\,. \tag{2.13}$$

Therefore, assuming a translation invariant reference state ω, one can benefit from relevant mathematical simplifications without limiting the class of states that can be considered.

Definition 2.6. *A state $\omega : \mathcal{A} \to \mathbb{C}$ is said to be a translation invariant, clustering state if:*

$$\begin{aligned} &1) \quad \omega\left(\tau^k(a)\right) = \omega\left(\tau^h(a)\right) &&= \omega(a), \qquad \forall\, h, k \in \mathbb{Z} &(2.14)\\ &2) \quad \lim_{|k|\to\infty} \omega\left(a\,\tau^k(c)\,b\right) &&= \omega(a\,b)\omega\left(c\right), \quad \forall\, a, b, c \in \mathcal{A}\,. &(2.15)\end{aligned}$$

In the following we shall always consider states ω satisfying these two general properties. With any of these states, and the algebra $\mathcal{A}$, by means of the GNS construction one can identify the triple $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$, where the Hilbert space $\mathcal{H}_\omega$ contains all possible states that can be obtained by quasi-local manipulations of the unique translation invariant vector Ψ_ω.

Example 2.4. *Let us consider again an infinite number of spin-$\frac{1}{2}$ particles, all with the same thermal state as in Example 2.1. Clearly, the corresponding state ω_β^∞ is translation invariant. Notice that this state can not be obtained by extending the finite N density operator*

$$\rho_\beta^N = \frac{e^{-\beta\sum_{k=-N}^N s_3^{(k)}}}{\mathrm{Tr}\left(e^{-\beta\sum_{k=-N}^N s_3^{(k)}}\right)}\,,$$

to an infinite number of particles. In fact, when $N \to \infty$ this operator is ill defined as it converges in norm to zero [50]:

$$\lim_{N\to\infty} \left\|\rho_\beta^N\right\| \leq \lim_{N\to\infty} \left(\frac{e^{\frac{\beta}{2}}}{e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}}\right)^{2N+1} = 0\,, \tag{2.16}$$

although it is a unit trace operator $\forall\, N$. In the infinite setting, states are indeed not, in general, represented by density matrices; instead, the functional ω_β^∞

$$\omega_\beta^\infty\left(\cdot\right) = \bigotimes_{k=-\infty}^{\infty} \omega_\beta^{(k)}\left(\cdot\right), \qquad \omega_\beta^{(k)}(\cdot) = \mathrm{Tr}\left(\rho_\beta \cdot\right)\,, \tag{2.17}$$

with ρ_β as in (2.2) *has a precise mathematical meaning.*
Starting from this initial translation invariant clustering state, one can obtain all other states in the GNS representation; for instance, the state in which the k-th spin is directed down in the third direction can be obtained as follows

$$\tilde{\omega}\left(\cdot\right) = \frac{\omega_\beta^\infty\left(s_+^{(k)} \cdot s_-^{(k)}\right)}{\omega_\beta^\infty\left(s_+^{(k)} s_-^{(k)}\right)}\,, \tag{2.18}$$

where $s_\pm^{(k)}$ are the ladder operators defined as $s_\pm^{(k)} = s_1^{(k)} \pm i s_2^{(k)}$.

As a matter of fact, usual states in quantum statistical mechanics are either ground states of a given Hamiltonian or generalized thermal states (KMS-states [51–53]); since the GNS representations arising from these two kind of states possess nice properties, it is useful to review some of them explicitly.

Ground state

Given a Hamiltonian, the eigenprojections onto its one-dimensional energy eigenvectors are pure quantum states. Within the algebraic formalism, one can rephrase the notion of purity by means of the following definition [46].

Definition 2.7. *A state ω over a C^*-algebra is said to be pure if it can not be written as a convex sum*

$$\omega = \lambda\,\omega_1 + (1-\lambda)\omega_2\,, \quad 0 < \lambda < 1\,,$$

of other two states ω_1, ω_2.

As already seen in Example 2.1, representations arising from pure states are irreducible, meaning that the commutant of such representations contains only multiples of the identity. This result is a consequence of the following theorem [23].

Theorem 2.2. *Let ω be a pure state over the C^*-algebra $\mathcal{A}$, and $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ the associated GNS representation, then $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible.*

KMS-states

Other relevant states are the so-called Kubo-Martin-Schwinger (KMS) states [23, 46, 50]; for finite dimensional systems, these are just the Gibbs states with respect to a given Hamiltonian at inverse temperature β. As discussed in Example 2.4, the density operator formalism becomes ill-defined in infinite dimensions, and the proper generalization of Gibbs states is as follows:

Definition 2.8. *A state ω is a KMS-state with respect to an automorphism α_t and the inverse temperature β, if:*

$$\omega\left(\alpha_t(a)b\right) = \omega\left(b\,\alpha_{t+i\beta}(a)\right)\ , \quad \forall\, a, b \in \mathcal{A}\,,$$

which is the so-called Kubo-Martin-Schwinger condition.
The state ω is an extremal KMS-state if it can not be written as a convex sum

$$\omega = \lambda\,\omega_1 + (1-\lambda)\omega_2\,, \quad 0 < \lambda < 1\,,$$

of two other KMS-state ω_1, ω_2.

For extremal KMS-states one has the following Theorem [50].

Theorem 2.3. *If a KMS-state ω is extremal, then its GNS-representation π_ω is a factor,* i.e. *the center $\mathcal{Z} = \pi_\omega(\mathcal{A}) \cap \pi'_\omega(\mathcal{A})$ consists of multiples of the identity.*

In this Chapter, we have provided the basic tools for the description of the microscopic degrees of freedom of an infinite quantum system. As mentioned in the Introduction, these degrees of freedom are essentially local and, as such, unable to provide collective descriptions of many-body systems. In the next Chapter two relevant classes of collective observables will be introduced and their properties studied.

Chapter 3

Collective Observables of Many-body Quantum Systems

In the study of many-body systems, made of a large number of microscopic constituents, the relevant observables are collective ones, those involving all the degrees of freedom; indeed, microscopic observables referring to single particles are usually experimentally inaccessible. Collective observables are suitably scaled sums of microscopic operators, as for instance macroscopic averages, but also fluctuations around these averages.
Given the large amount of particles considered in these collective observables, one expects their quantum features to fade away as the number of constituents increases; this is indeed what happens to macroscopic averages. This behaviour is not surprising: averages are believed to describe the macroscopic world, where quantum features should be replaced by classical behaviours. However, taking inspiration from mesoscopic classical physics, *e.g.* Brownian motion, one can identify another class of collective observables retaining some quantum features: fluctuations around macroscopic averages. As we shall see, this second class of collective observables will be very useful in the study of the quantum behaviour of many-body systems at the mesoscopic scale, half-way between the microscopic description of single-particle observables and the classical one essentially based on macroscopic averages.

3.1 Macroscopic Observables

The first, most natural class of collective observables one can construct are averages; considering the set of particles Λ_N from $-N$ to N, and the single-particle observable $x \in \mathcal{A}_d$, the average of x over Λ_N is defined as

$$M_N(x) := \frac{1}{N_T} \sum_{k=-N}^{N} x^{(k)} \,, \tag{3.1}$$

$N_T = 2N + 1$, the collective nature of the large N operator being then transparent. For finite N, such an operator inherits the properties of its microscopic building blocks and thus behaves in a quantum way. For instance, one might consider the commutator of this average with a local operator o, whose support is contained in the set Λ_{N_o}: we have $(N > N_o)$,

$$[M_N(x), o] = \frac{1}{N_T} \sum_{k=-N_o}^{N_o} \left[x^{(k)}, o\right] \; ; \tag{3.2}$$

notice that the sum runs inside a finite set Λ_{N_o}, so that the commutator vanishes as N_T becomes large.
This behaviour holds, even if one considers quasi-local operators $o \in \mathcal{A}$ and not just strictly local operators (see Remark 2.2). From these results one can deduce that the limiting point of the sequence (3.1) belongs to the commutant of the GNS representation $\pi'_\omega(\mathcal{A})$, being ω any state of the system; in other terms, one finds that, in norm,

$$\left[\lim_{N\to\infty} M_N(x),\, \mathcal{A}\right] = 0\,, \quad \forall x \in \mathcal{A}_d\,. \tag{3.3}$$

Remark 3.1. *In order to understand what kind of operator is the limiting point of the sequence* (3.1) *as N increases, one needs to find a proper operator topology, where a good convergence is reached. For instance, such limiting operator does not belong to the quasi-local algebra $\mathcal{A}$. If it were so, since the C^*-algebra $\mathcal{A}$ is closed in the norm topology, then the sequence $\{M_N(x)\}_N$ should be a Cauchy sequence*[1] *with respect to such topology, but this is not true.*
To show it, let us consider the quantity:

$$I_{NK} = \|M_N(x) - M_K(x)\| \; ;$$

without loss of generality, we assume $N > K$, thus collecting the equal terms of the summation

$$I_{NK} = \left\| \left(\frac{1}{N_T} - \frac{1}{K_T}\right) \sum_{h=-K}^{K} x^{(h)} + \frac{1}{N_T} \sum_{h=\Lambda_N \setminus \Lambda_K} x^{(h)} \right\| \,.$$

Since the norm is greater than the expectation over any possible state, we consider:

$$|\psi_{NK}\rangle = \bigotimes_{h=-N}^{-K-1} |x_{max}\rangle^{(h)} \otimes \bigotimes_{h=-K}^{K} |x_{min}\rangle^{(h)} \otimes \bigotimes_{h=K+1}^{N} |x_{max}\rangle^{(h)}$$

where $|x_{min}\rangle, |x_{max}\rangle$ are the eigenstates corresponding to the minimum eigenvalue x_{min}, respectively the maximum x_{max}; therefore

$$I_{NK} \geq |\langle\psi_{NK}|M_N(x) - M_K(x)|\psi_{NK}\rangle| = \left| \left(\frac{K_T}{N_T} - 1\right) x_{min} + \left(1 - \frac{K_T}{N_T}\right) x_{max} \right| \,,$$

that can be written as

$$I_{NK} \geq \left(1 - \frac{K_T}{N_T}\right)(x_{max} - x_{min}) \,, \tag{3.4}$$

showing that $\{M_N(x)\}_N$ is not a Cauchy sequence in the norm topology.

[1] A sequence of operators $\{A_n\}_n$ is a Cauchy sequence in the norm topology, if $\forall\, \epsilon > 0$, there exists an integer n_0, such that

$$\|A_n - A_m\| < \epsilon\,, \quad \forall\, n, m > n_0\,.$$

Nevertheless, with respect to translation invariant clustering states (see Definition 2.6, equations (2.14),(2.15)), the limit of the sequence (3.1) can properly be defined.

Theorem 3.1. *Given the quasi-local algebra $\mathcal{A}$ and a translation-invariant clustering state ω, in the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ one has:*

$$\pi_\omega\left(s - \lim_{N\to\infty} M_N(x)\right) = \omega(x)\mathbf{1}, \quad \forall\, x \in \mathcal{A}_d.$$

with $M_N(x)$ as in (3.1)*.*

Proof. We can consider $x = x^\dagger$, but the proof can easily be extended to non-Hermitian operators.
Recalling Theorem 2.1, and equation (2.7), in order to prove convergence in the strong operator topology of the GNS representation π_ω, one has to show that

$$\lim_{N\to\infty} \omega\left(a^\dagger \left(M_N(x) - \omega(x)\right)^2 a\right) = 0\,, \quad \forall\, a \in \mathcal{A}\,.$$

It has been already shown that in the limit $M_N(x)$ commutes with the whole algebra $\mathcal{A}$, thus we have:

$$\lim_{N\to\infty} \omega\left(a^\dagger \left(M_N(x) - \omega(x)\right)^2 a\right) = \lim_{N\to\infty} \omega\left(a^\dagger a \left(M_N(x) - \omega(x)\right)^2\right)\,;$$

using the Cauchy-Schwarz inequality and the norm bound for x and a, it follows that

$$\omega\left(a^\dagger \left(M_N(x) - \omega(x)\right)^2 a\right) \le 2\|a\|^2\|x\|\sqrt{\omega\left(\left(M_N(x) - \omega(x)\right)^2\right)}\,,$$

showing that the convergence of this sequence does not depend on a. The relevant contribution is given by the argument of the square root; expanding the square and using the translation invariance of the state, one writes

$$\lim_{N\to\infty} \omega\left(\left(M_N(x) - \omega(x)\right)^2\right) = \lim_{N\to\infty}\left(\omega\left(M_N(x)^2\right) - \omega(x)^2\right).$$

Focusing on the argument of the limit,

$$\omega\left(M_N(x)^2\right) - \omega(x)^2 = \frac{1}{N_T^2}\sum_{k,h=-N}^{N}\left(\omega\left(x^{(k)}x^{(h)}\right) - \omega(x)^2\right),$$

by translation invariance of the state, it can be written as follows:

$$\omega\left(M_N(x)^2\right) - \omega(x)^2 = \frac{1}{N_T^2}\sum_{k=-N}^{N}\sum_{m=-k-N}^{N-k}\left(\omega\left(x^{(0)}x^{(m)}\right) - \omega(x)^2\right);$$

taking the modulus, and bounding it by means of the sum of the moduli, and extending the second summation, we finally find:

$$\begin{aligned}\left|\omega\left(M_N(x)^2\right) - \omega(x)^2\right| &\le \frac{1}{N_T^2}\sum_{k=-N}^{N}\sum_{m=-N_T}^{N_T}\left|\omega\left(x^{(0)}x^{(m)}\right) - \omega(x)^2\right| = \\ &= \frac{1}{N_T}\sum_{m=-N_T}^{N_T}\left|\omega\left(x^{(0)}x^{(m)}\right) - \omega(x)^2\right|.\end{aligned} \tag{3.5}$$

Because of the clustering condition,

$$\lim_{m\to\pm\infty}\left|\omega\left(x^{(0)}x^{(m)}\right)-\omega(x)^2\right|=0\,,$$

by Cesàro mean[2] [54] one gets:

$$\lim_{N\to\infty}\left|\omega\left(M_N(x)^2\right)-\omega(x)^2\right|=0\,.$$

□

3.1.1 The Classical Algebra of Macroscopic Observables

As a result of the previous theorem, one can conclude that no quantum description is possible by means of macroscopic observables. The proper mathematical description of the set of these classical observables can be given as follows.
Let us consider an orthonormal Hermitian basis of the single-particle algebra $\mathcal{A}_d$, $\{v_\mu\}_{\mu=1}^{d^2}$; this means that:

$$\mathrm{Tr}(v_\mu v_\nu)=\delta_{\mu\nu}\,,\qquad \|v_\mu\|\leq 1,\,\forall\,\mu\,.$$

With these operators we construct the vector:

$$\vec{m}_N=\left(M_N(v_1),M_N(v_2),\ldots,M_N(v_{d^2})\right)^{tr}\,,$$

where ^{tr} means transposition, and considering a translation-invariant clustering state ω, also the vector of expectations:

$$\vec{\omega}=\left(\omega(v_1),\omega(v_2),\ldots,\omega(v_{d^2})\right)^{tr}\,.$$

The space $\mathcal{C}$ of all possible configurations of $\vec{\omega}$ is constrained by the positivity of the functional ω; indeed, such a functional defines a single-particle density matrix ρ, acting on the single-particle Hilbert space, given by

$$\rho=\sum_{\alpha=1}^{d^2}\omega(v_\alpha)v_\alpha,\quad \rho\geq 0,\quad \mathrm{Tr}(\rho)=1\,.$$

Therefore, the only physical $\vec{\omega}$ are those $\vec{\omega}\in\mathcal{C}$, where

$$\mathcal{C}=\left\{\vec{x}:\sum_{\alpha=1}^{d^2}x_\alpha v_\alpha\geq 0,\,\sum_{\alpha=1}^{d^2}x_\alpha\mathrm{Tr}(v_\alpha)=1\right\}\,.$$

Assuming $\vec{x}\in\mathcal{C}$, given any continuous function $f(\vec{x})$, from Theorem 3.1, it follows that:

$$\lim_{N\to\infty}\omega\left(f(\vec{m}_N)\right)=f(\vec{\omega})\,. \tag{3.6}$$

[2] Let $\{A_n\}$ be a sequence of numbers, if $\lim_{n\to\infty}A_n=A$, then also

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}A_k=A\,.$$

This allows us to consider the commutative algebra $\mathcal{L}$ of continuous functions $f(\vec{x})$ on the space of configurations $\mathcal{C}$, together with a state $\delta_{\vec{\omega}}(\cdot)$ of the following form:

$$\delta_{\vec{\omega}}(g) = \int_{\mathcal{C}} d\vec{y}\, \delta(\vec{y} - \vec{\omega})\, g(\vec{y}) = g(\vec{\omega}) \tag{3.7}$$

and Theorem 3.1 and relation (3.6) show

$$\lim_{N\to\infty} \omega(f(\vec{m}_N)) = \delta_{\vec{\omega}}(f) = f(\vec{\omega})\,. \tag{3.8}$$

Therefore, the algebra of macroscopic observables is, in the infinite particles limit, the algebra of continuous functions $\mathcal{L}(\mathcal{C})$, where the actual information about the quasi-local state ω is encoded in the functional $\delta_{\vec{\omega}} : \mathcal{L}(\mathcal{C}) \to \mathbb{C}$.

3.2 Quantum Fluctuation Operators

In order to get inspiration on how to construct collective operators that retain some quantum behaviour, it is useful to recall some results in classical probability.

3.2.1 Classical Fluctuations and the Central Limit Theorem

For the sake of simplicity, let us consider a discrete family $\bar{X}_N = (X_1, X_2, \ldots, X_N)$ of independent stochastic variables with the same distribution $p(X)$. Thus, calling $\mathbb{E}[g]$ the expectation of the function $g(\bar{X})$, we have

$$\begin{aligned} &\mathbb{E}[X_i] = \mu,\ \forall\, i; \\ &\mathbb{E}[(X_i - \mu)(X_j - \mu)] = \sigma^2 \delta_{ij}\,; \\ &\mathbb{E}[g(X_i)h(X_j)] = \mathbb{E}[g(X_i)]\,\mathbb{E}[h(X_j)]\,, \qquad \forall\, i \neq j\,. \end{aligned} \tag{3.9}$$

In classical probability theory, one can construct two quantities, both with well behaved limit as N becomes large. The first one is the average and is governed by the law of large numbers. Considering the stochastic variable $\Sigma_N = \sum_{k=1}^{N} X_k$ one has that

$$\lim_{N\to\infty} \frac{1}{N}\Sigma_N = \mu\,,$$

where the convergence has to be understood *in distribution*; this means that, when N increases, the average of the considered process gets closer and closer to the deterministic value μ, which is the single stochastic variable expectation. These averages are obviously the classical analogue of the macroscopic observables introduced in Section 3.1.
The second relevant collective stochastic variables are the fluctuations around mean-values, $F(\bar{X}_N)$, defined as follows:

$$\begin{gathered} F(\bar{X}_N) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - \mu) = \frac{1}{\sqrt{N}}\Sigma_N - \sqrt{N}\mu\,, \\ \mathbb{E}\left[F(\bar{X}_N)\right] = 0\,. \end{gathered} \tag{3.10}$$

Their convergence in the large N limit can be studied by means of their characteristic function

$$\lim_{N\to\infty} \mathbb{E}\left[e^{i\alpha F(\bar{X}_N)}\right] = \chi^{\alpha}_{F(\bar{X})}\,. \tag{3.11}$$

Expanding the exponential function, and using the third of the properties (3.9), one has:

$$\chi^{\alpha}_{F(\bar{X})} = \lim_{N\to\infty} \prod_{k=1}^{N} \left(1 - \frac{\alpha^2}{2N}\mathbb{E}\left[(X_i-\mu)^2\right] + o\left(\frac{1}{N}\right)\right) = \lim_{N\to\infty}\left(1-\frac{\alpha^2}{2N}\sigma^2\right)^N\,, \tag{3.12}$$

showing that the stochastic variable $F(\bar{X}_N)$ tends to a Gaussian variable with zero mean and covariance given by σ^2:

$$\chi^{\alpha}_{F(\bar{X})} = \lim_{N\to\infty}\left(1-\frac{\alpha^2}{2N}\sigma^2\right)^N = e^{-\alpha^2\frac{\sigma^2}{2}}\,.$$

What one would like to investigate is whether a quantum analogue of these fluctuations can be defined having good convergence properties. Such a problem has been investigated in [31, 32, 35], and it was found that the quantum fluctuations can be defined and that they tend, in the large N limit, to bosonic operators, obeying a Weyl algebra. Before introducing the definition of quantum fluctuation operators, it is then convenient to recall the main properties of Weyl algebras.

3.2.2 Weyl Algebras and quasi-free states

Let H be the real vector space spanned by the sequence of the linearly independent elements $\{x_\alpha\}_{\alpha=1}^{n}$; a generic element of this space can then be expressed as

$$q_r = (r,x) = \sum_{\mu=1}^{n} r_\mu x_\mu, \qquad \text{with } r_\mu \in \mathbb{R}\,. \tag{3.13}$$

The space H is equipped with a bilinear anti-symmetric form σ[3], defined as follows:

$$\begin{aligned} &\sigma : H\times H \to \mathbb{R}, && q_r \times q_s \to \sigma(q_r,q_s)\,,\\ &\sigma(q_r,q_s) = -\sigma(q_s,q_r), && \sigma(\alpha q_r, \beta q_s) = \alpha\beta\sigma(q_r,q_s)\,. \end{aligned} \tag{3.14}$$

To the space (H,σ), one can associate the Weyl algebra $\mathcal{W}(H,\sigma)$, generated by the Weyl operators $\{W(r)|(r,x)\in H\}$, defined by the following relations:

$$\begin{aligned} W(r)W(s) =& W(r+s)e^{-\frac{i}{2}\sigma(r,s)}\,,\\ W(r)^\dagger =& W(-r)\,. \end{aligned} \tag{3.15}$$

As for any algebra, a state Ω for $\mathcal{W}(H,\sigma)$ is a positive, normalized, linear functional $\Omega : \mathcal{W}(H,\sigma) \to \mathbb{C}$; with it, we can construct the GNS triple $(\mathcal{H}_\Omega, \pi_\Omega, \Psi_\Omega)$.

[3]In some of the original papers about fluctuations, the Authors refer to such a form as "a possibly degenerate symplectic form". Here, the form will be generally called bilinear anti-symmetric, while it will be named symplectic only if it can be shown that the form is also non-degenerate.

Quasi-free states on $\mathcal{W}(H,\sigma)$ An important class of states on Weyl algebras is the set of the so-called quasi-free states. They are identified by the Gaussian character of their expectations on Weyl operators[4]:

$$\Omega\left(W(r)\right) = e^{-\frac{1}{2}(r,\Sigma\, r)}\,, \tag{3.16}$$

where Σ is a positive symmetric matrix. It turns out that the corresponding GNS representation is regular, so that the Weyl operator can be expressed as follows,

$$\pi_\Omega\left(W(r)\right) = e^{i(r,B)}\,, \tag{3.17}$$

in terms of a vector $B = (B_1, B_2, \dots B_n)^{tr}$ of (unbounded) Bose-field operators. This allows us to give an explicit form to the matrix Σ, whose elements are ($\{a,b\} := ab+ba$):

$$\Sigma_{\mu\nu} = \frac{1}{2}\Omega\left(\{B_\mu, B_\nu\}\right)\,.$$

Furthermore, using relations (3.16),(3.17) one can write

$$\Omega\left(W(r)\right) = \langle\Psi_\Omega|e^{i(r,B)}|\Psi_\Omega\rangle = e^{-\frac{1}{2}(r,\Sigma\, r)}\,, \tag{3.18}$$

which shows that B_i's are quantum observables with zero average and covariance matrix given by Σ.
From the first of relations (3.15), together with the Baker-Campbell-Hausdorff formula, one derives the commutation relations of the Bose-field operators, that can be collected in the anti-symmetric matrix σ:

$$\sigma_{\mu\nu} := -i\left[B_\mu, B_\nu\right] = \sigma\left(x_\mu, x_\nu\right)\,. \tag{3.19}$$

It is important to recall that in order for Ω to be a positive functional on the algebra, the following condition has to be verified [55, 56]:

$$\Sigma + \frac{i}{2}\sigma \geq 0\,. \tag{3.20}$$

Example 3.1. *The simplest example of Weyl algebra is the one with real vector space H of dimension two. Thus, the independent elements of the vector space are x_1, x_2; on this we consider the bilinear anti-symmetric form*

$$\sigma(x_1, x_2) = 1\,,$$

giving rise to the anti-symmetric matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

Considering the regular state Ω such that:

$$\Omega\left(W(r)\right) = e^{-\frac{1}{4}\|r\|^2}$$

that is positive since $\frac{\mathbf{1}+i\sigma}{2} \geq 0$, we can represent Weyl operators in the following way

$$\pi_\Omega\left(W(r)\right) = e^{i(r,B)}\,,$$

[4] We consider here Gaussian states with zero average.

with $B = (B_1, B_2)^{tr}$ *obeying* $[B_1, B_2] = i$ *(*cf. (3.19)*). In this case,* B_1 *and* B_2 *are position and momentum like operators and the Weyl operators can be represented as*

$$\pi_\Omega\left(W(r)\right) = e^{i(r_1\, x + r_2\, p)}$$

i.e. *through displacement operators as introduced in quantum optics.*
By means of the following operators

$$a = \frac{1}{\sqrt{2}}\left(x + ip\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - ip\right)\,,$$

one can check that the chosen state Ω *corresponds to the expectation*

$$\Omega\left(\cdot\right) = \langle 0|\cdot|0\rangle\,,$$

where $|0\rangle$ *is the vacuum state annihilated by* a*,* $a|0\rangle = 0$*; indeed, by explicit computation one finds*

$$\Omega\left(W(r)\right) = e^{-\frac{1}{4}\|r\|^2} = \langle 0|e^{i(r_1\, x + r_2\, p)}|0\rangle\,.$$

3.2.3 Algebra of Fluctuation Operators

Inspired by the classical results summarized in Section 3.2.1, given a translation invariant state ω, one is led to define quantum fluctuation operators for the single-particle operators $x = x^\dagger \in \mathcal{A}_d$ as

$$F_N(x) := \frac{1}{\sqrt{N_T}} \sum_{k=-N}^{N} \left(x^{(k)} - \omega(x)\right)\,. \tag{3.21}$$

First, let us show that these operators retain some quantum features as N gets large. We choose two observables $x, y \in \mathcal{A}_d$, $\left[x^{(k)}, y^{(h)}\right] = iz^{(k)}\delta_{kh}$, construct the relative fluctuations $F_N(x), F_N(y)$ and study the commutator $[F_N(x), F_N(y)]$. We have

$$[F_N(x), F_N(y)] = \frac{i}{N_T} \sum_{k=-N}^{N} z^{(k)} = iM_N(z)\,, \tag{3.22}$$

showing that the commutator of two fluctuations is proportional to an average macroscopic observable, converging to a multiple of the identity. This suggests that fluctuations, might give rise in the large N limit to Bose-field operators.
We would like to give a meaning to the limit of $F_N(x)$ in (3.21), as N gets large. These operators are more difficult to study than macroscopic observables, since their norm is not bounded

$$\|F_N(x)\| = \sqrt{N}\,\|x - \omega(x)\|\,.$$

Nevertheless, the subtraction of the mean value, suggests the possibility of some good convergence properties in some state induced topology.
Indeed, the expectation over the state gives a zero average, for any N

$$\omega\left(F_N(x)\right) = 0\,,$$

while $\omega(F_N(x)^2)$ can be made finite for some states ω. One finds

$$\omega\left(F_N(x)^2\right) = \frac{1}{N_T}\sum_{k,h=-N}^{N}\left(\omega\left(x^{(k)}x^{(h)}\right) - \omega(x)^2\right) \tag{3.23}$$

so that:

$$\lim_{N\to\infty}\omega\left(F_N(x)^2\right) \leq \sum_{m=-\infty}^{\infty}\left|\omega\left(x^{(0)}x^{(m)}\right) - \omega(x)^2\right| ; \tag{3.24}$$

the convergence of the series on the right hand side of the above inequality is a sufficient condition for the existence of the variance of the fluctuation operator in the limit of infinite particles. This condition represents an L^1-clustering constraint on the state ω relative to the operator x, that needs to be enforced in order to have a well-defined variance for $F_N(x)$.
In order to give a meaning to the convergence of these observables in the large N limit, one can first try to adopt the same procedure used for average operators. In this case, the sequence $\{F_N(x)\}_N$ should turn out to be a Cauchy sequence in the strong operator topology induced by the GNS-construction. As a consequence, defining

$$\tilde{I}_{NM} = \omega\left((F_N(x) - F_M(x))^2\right) \tag{3.25}$$

this should be arbitrarily small for N, M greater than a certain N_0. In order to show that this is not possible, let us consider the case of a tensor product state $\omega\left(x^{(k)}x^{(h)}\right) = \omega(x)\omega(x)$, $\forall\, k \neq h$. By direct computation, one has:

$$\tilde{I}_{NM} = 2\left(\omega(x^2) - \omega(x)^2\right) - \frac{1}{\sqrt{N_T M_T}}\sum_{k=-N}^{N}\sum_{h=-M}^{M}\omega\left(\left\{(x-\omega(x))^{(k)}, (x-\omega(x))^{(h)}\right\}\right), \tag{3.26}$$

and without loss of generality considering $N > M$,

$$\tilde{I}_{NM} = 2\left(\omega(x^2) - \omega(x)^2\right)\left(1 - \frac{\sqrt{M_T}}{\sqrt{N_T}}\right), \tag{3.27}$$

that is not a Cauchy sequence. It turns out that, as in the case of the classical central limit theorem, the proper convergence is a convergence in distribution [35].

The Characteristic Function of Fluctuations

Given a physical system, one can in general select a set $\{x_1, x_2, \ldots, x_n\}$, $x_i \in \mathcal{A}_d$ of Hermitian single-particle observables physically relevant, in particular from the mesoscopic collective point of view of their fluctuations.

Definition 3.1. *Given the set $\{x_\alpha\}_{\alpha=1}^n$, of independent single-site observables relevant for the description of a quantum system, one can construct the real linear span K, whose elements are of the form*

$$q_r = (r, x) := \sum_{\mu=1}^{n} r_\mu\, x_\mu,\; r \in \mathbb{R}^n .$$

One can then construct the corresponding fluctuation operators.

Definition 3.2. *Given a generic element (r,x) of the real linear space K, and a translation invariant state ω, we associate to it the fluctuation $F_N(q_r)$, defined by*

$$(r,F_N) = F_N(q_r) := \frac{1}{\sqrt{N_T}} \sum_{k=-N}^{N} \left(q_r^{(k)} - \omega(q_r)\right) , \tag{3.28}$$

where

$$q_r^{(k)} = (r, x^{(k)}) := \sum_{\mu=1}^{n} r_\mu \, x_\mu^{(k)} .$$

The exponential of such an operator, referred in the following as a pre-Weyl operator, is explicitly given by

$$W_N(r) = e^{i(r,F_N)} := \sum_{n=0}^{\infty} \frac{i^n \, (r,F_N)^n}{n!} . \tag{3.29}$$

With the previous definition at hand, one has that the characteristic function of the quantum observables $F_N(x_1), F_N(x_2), \dots F_N(x_n)$ is given by

$$\chi_N\left(r\right) = \omega\left(W_N(r)\right) . \tag{3.30}$$

As discussed before, for suitable choices of the state ω, this characteristic function converges in the large N limit to a Gaussian one. The set of states for which this holds is specified by the following [31]:

Definition 3.3. *Given the Hermitian subspace $K \subset \mathcal{A}_d$, the system (K, ω), with ω a translation invariant clustering state, is said to have* normal quantum fluctuations, *if:*

$$\textit{1)} \sum_{k=-\infty}^{\infty} \left|\omega\left(x^{(0)} y^{(k)}\right) - \omega(x)\omega(y)\right| < \infty, \quad \forall\, x, y \in K , \tag{3.31}$$

$$\textit{2)} \lim_{N\to\infty} \omega\left(e^{i\alpha F_N(x)}\right) = e^{-\frac{\alpha^2}{2} s_\omega(x,x)}, \quad \forall\, x \in K,\, \alpha \in \mathbb{R} , \tag{3.32}$$

where $s_\omega(x,y) := \lim_{N\to\infty} \frac{1}{2}\omega\left(\{F_N(x), F_N(y)\}\right)$ for all $x, y \in K$.

We shall now show how studying the behaviour of these operators $W_N(r)$, Weyl-like relations (3.15) naturally arise.

Weyl-like Relations for Exponentials of Fluctuations

Let us consider two exponentials of fluctuations $e^{iF_N(x)}$, $e^{iF_N(y)}$; by Baker-Campbell-Hausdorff formula, we have

$$\begin{aligned} e^{iF_N(x)} e^{iF_N(y)} = \exp\Bigg\{ & i\big(F_N(x) + F_N(y)\big) - \frac{1}{2}\left[F_N(x), F_N(y)\right] + \\ & -\frac{i}{12}\left(\left[F_N(x), \left[F_N(x), F_N(y)\right]\right] - \left[F_N(y), \left[F_N(x), F_N(y)\right]\right]\right) + \dots \Bigg\} , \end{aligned} \tag{3.33}$$

where the omitted terms involve higher order commutations. The first commutator, as already seen, is proportional to an average macroscopic observable, so, with respect to clustering states, it is expected to converge to a multiple of the identity. All other terms in the above formula starting from the double commutators go to zero in norm in the limit $N \to \infty$; indeed, as an example, we have

$$\lim_{N\to\infty} \|[F_N(x),[F_N(x),F_N(y)]]\| = \\ = \lim_{N\to\infty} \frac{1}{\sqrt{N_T}^3} \left\| \sum_{k=-N}^{N} \left[x^{(k)},\left[x^{(k)},y^{(k)}\right]\right] \right\| \le \lim_{N\to\infty} \frac{1}{\sqrt{N_T}} 4\|x\|^2\|y\| = 0\,. \tag{3.34}$$

These results suggest that, in the large N limit, equation (3.33) reduces to

$$e^{iF_N(x)}e^{iF_N(y)} \sim e^{i\left(F_N(x)+F_N(y)\right)}e^{-\frac{1}{2}[F_N(x),F_N(y)]}\,.$$

In order to better formalize this result, let us first introduce a proper anti-symmetric matrix:

Definition 3.4. *Given the ordered set $\{x_\alpha\}_{\alpha=1}^n$ of generators of the real linear space K, we define the operator valued matrix T^N to be the matrix whose entries are*

$$T^N_{\mu\nu} := -i\left[F_N(x_\mu),F_N(x_\nu)\right]\,, \tag{3.35}$$

and, by means of a translation invariant state ω, the matrix of its expectations,

$$\sigma^{(\omega)}_{\mu\nu} := \omega\left(T^N_{\mu\nu}\right) = -i\omega\left([x_\mu,x_\nu]\right)\,. \tag{3.36}$$

Both matrices are anti-symmetric and we have:

$$\lim_{N\to\infty} \omega\left([F_N(q_r),F_N(q_s)]\right) = i \lim_{N\to\infty} \omega\left(\left(r,T^N\,s\right)\right)\,,$$

where

$$\left(r,T^N\,s\right) = \sum_{\mu,\nu=1}^{n} r_\mu s_\nu T^N_{\mu\nu}\,,$$

so that:

$$\lim_{N\to\infty} \omega\left([F_N(q_r),F_N(q_s)]\right) = i\left(r,\sigma^{(\omega)}s\right)\,.$$

Lemma 3.1. *Given a normal fluctuations system $(K,\,\omega)$, the following equality holds*

$$\lim_{N\to\infty} \omega\left(W_N(r)W_N(s_1)W_N(s_2)W_N(t)\right) = \\ = \lim_{N\to\infty} \omega\left(W_N(r)W_N(s_1+s_2)W_N(t)\right)e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}, \quad \forall\, t,s_1,s_2,r \in \mathbb{R}^n\,,$$

where W_N's are as in (3.29)*, showing that in the large N limit, under state expectation, $W_N(s_1)W_N(s_2)$ behaves as $W_N(s_1+s_2)e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}$.*

Proof. We divide the proof in two steps: we first show that

$$\lim_{N\to\infty} \left\| W_N(s_1)W_N(s_2) - W_N(s_1+s_2)e^{-\frac{i}{2}\left(s_1,T^N s_2\right)} \right\| = 0\,,$$

then, the proof follows thanks to the strong convergence of macroscopic observables (Theorem 3.1).
We start with an algebraic identity:

$$e^{iF_N(q_{s_1})}e^{iF_N(q_{s_2})}-e^{i\left(F_N(q_{s_1})+F_N(q_{s_2})\right)}e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}=$$
$$\int_0^1 d\alpha\,\frac{d}{d\alpha}\left(e^{i\alpha F_N(q_{s_1})}e^{i\alpha F_N(q_{s_2})}e^{i(1-\alpha)\left(F_N(q_{s_1})+F_N(q_{s_2})\right)}e^{-\frac{i(1-\alpha^2)}{2}\left(s_1,T^N s_2\right)}\right);$$

computing explicitly the derivative, we get

$$\frac{d}{d\alpha}\left(e^{i\alpha F_N(q_{s_1})}e^{i\alpha F_N(q_{s_2})}e^{i(1-\alpha)\left(F_N(q_{s_1})+F_N(q_{s_2})\right)}e^{-\frac{i(1-\alpha^2)}{2}\left(s_1,T^N s_2\right)}\right)=$$
$$=ie^{i\alpha F_N(q_{s_1})}\left[F_N(q_{s_1}),e^{i\alpha F_N(q_{s_2})}\right]e^{i(1-\alpha)\left(F_N(q_{s_1})+F_N(q_{s_2})\right)}e^{-\frac{i(1-\alpha^2)}{2}\left(s_1,T^N s_2\right)}+$$
$$+e^{i\alpha F_N(q_{s_1})}e^{i\alpha F_N(q_{s_2})}e^{i(1-\alpha)\left(F_N(q_{s_1})+F_N(q_{s_2})\right)}i\alpha\left(s_1,T^N s_2\right)e^{-\frac{i(1-\alpha^2)}{2}\left(s_1,T^N s_2\right)}.$$

Using the result of Appendix A, and recalling the definition of T_N in (3.35) we have

$$\lim_{N\to\infty}\left\|\left[F_N(q_{s_1}),e^{i\alpha F_N(q_{s_2})}\right]+\alpha\left(s_1,T^N\,s_2\right)e^{i\alpha F_N(q_{s_2})}\right\|=0\,.$$

This result together with the observation that macroscopic observables commute with fluctuations in the large N limit (see Appendix A), allows one to conclude that the limit of the norm of the derivative goes to zero as N becomes large. Then,

$$\lim_{N\to\infty}\omega\left(W_N(r)W_N(s_1)W_N(s_2)W_N(t)\right)=$$
$$\lim_{N\to\infty}\omega\left(W_N(r)\left[W_N(s_1+s_2)e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}\right]W_N(t)\right)\,.$$

From this, using that $e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}$ commutes with $W_N(t)$ in the large N limit (see Appendix A), and by adding and subtracting $e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}$, one can recast the previous result in the following form:

$$\lim_{N\to\infty}\omega\left(W_N(r)W_N(s_1+s_2)W_N(t)e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}\right)=$$
$$\lim_{N\to\infty}\omega\left(W_N(r)W_N(s_1+s_2)W_N(t)\left(e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}-e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}\right)\right)+$$
$$+\lim_{N\to\infty}\omega\left(W_N(r)W_N(s_1+s_2)W_N(t)\right)e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}\,.$$

Finally, the limit in the second line of the above equation vanishes; indeed, using the Cauchy-Schwarz inequality, and observing that W_N's are unitary, this term is bounded by

$$\lim_{N\to\infty}\left|\omega\left(W_N(r)W_N(s_1+s_2)W_N(t)\left(e^{-\frac{i}{2}\left(s_1,T^N s_2\right)}-e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}\right)\right)\right|\leq$$
$$\leq\lim_{N\to\infty}\sqrt{\omega\left(\left(e^{-\frac{i}{2}(s_1,T^N s_2)}-e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}\right)^{\dagger}\left(e^{-\frac{i}{2}(s_1,T^N s_2)}-e^{-\frac{i}{2}\left(s_1,\sigma^{(\omega)}s_2\right)}\right)\right)}\,,$$

which goes to zero thanks to the result of Theorem 3.1 and equations (3.35),(3.36).
□

The algebraic structure of fluctuations

The results just proven allow a complete characterization of the algebra obeyed by fluctuation operators. It should be already clear that, given the system of normal quantum fluctuations $(K,\,\omega)$, the operators $W_N(r)$ as in Definition 3.2 behave, under the state expectation, as Weyl operators in the large N limit; namely

$$\lim_{N\to\infty} W_N(r) \to W(r)\,,$$

where $W(r)$ belongs to the Weyl algebra $\mathcal{W}\left(K,\sigma^{(\omega)}\right)$, with the anti-symmetric form given in (3.36). Furthermore, the condition (3.32) suggests the existence of a quasi-free state Ω on this algebra $\mathcal{W}\left(K,\sigma^{(\omega)}\right)$ such that:

$$\Omega\left(W(r)\right) = e^{-\frac{1}{2}s_\omega(q_r,q_r)}\,.$$

The regularity of this state allows to represent Weyl operators as exponential of Bose-field operators[5]

$$\pi_\Omega\left(W(r)\right) = e^{i(r,F)}\,,$$

where F is the vector $F = (F_1, F_2, \ldots, F_n)^{tr}$ corresponding to the large N limit of the vector F_N.
This makes clear the convergence of these fluctuation operators to Bose-field operators

$$\lim_{N\to\infty} F_N(x_\alpha) \to F_\alpha, \quad \alpha = 1, 2, \ldots, n\,,$$

where the convergence has to be understood in the quantum central limit sense [35]. This is the content of the following Theorem 3.2 [31].

Theorem 3.2. *If the system $(K,\,\omega)$ has normal quantum fluctuations (Definition 3.3), then for $q_r, q_s \in K$, there exists a quasi-free state Ω on the Weyl algebra $\mathcal{W}\left(K,\sigma^{(\omega)}\right)$ of the quantum fluctuations such that:*

$$\begin{aligned}&\lim_{N\to\infty}\omega\left(e^{iF_N(q_r)}e^{iF_N(q_s)}\right) = \\ &\qquad = \exp\left(-\frac{1}{2}s_\omega\left(q_r+q_s, q_r+q_s\right) - \frac{i}{2}\left(r, \sigma^{(\omega)}\, s\right)\right) = \Omega\left(W(r)W(s)\right)\,,\end{aligned} \tag{3.37}$$

From the previous Theorem, and the Weyl-like commutation relations of Lemma 3.1, it further follows that

$$\lim_{N\to\infty}\omega\left(W_N(r_1)W_N(r_2)\ldots W_N(r_m)\right) = \Omega\left(W(r_1)W(r_2)\ldots W(r_m)\right),\, \forall r_i \in \mathbb{R}^n\,.$$

This result can be reinterpreted through the introduction of what can be called a *mesoscopic limit.*

Definition 3.5. *Given a sequence $\{X_N\}_N$ of operators made by polynomial or exponentials of fluctuation operators, it converges in the mesoscopic topology to the operator X, if*

$$\lim_{N\to\infty}\omega\left(W_N(r_1)X_N W_N(r_2)\right) = \Omega\left(W(r_1)XW(r_2)\right)\,, \quad \forall\, r_{1,2} \in \mathbb{R}^n\,, \tag{3.38}$$

[5] This representation holds in the GNS-construction induced by the regular state Ω.

where $W_N(r_{1,2}) = e^{i(r_{1,2},F_N)}$ *are as in Definition 3.2,* $W(r_{1,2}) \in \mathcal{W}(K, \sigma^{(\omega)})$ *and* Ω *a quasi-free state as defined in* (3.37).
In other words, one can say that X *is the mesoscopic limit of* X_N*, denoting it as:*

$$m - \lim_{N\to\infty} X_N = X\,.$$

Notice that for any element X in the Weyl algebra $\mathcal{W}\left(K, \sigma^{(\omega)}\right)$, the quantities $\Omega\left(W(r_1)XW(r_2)\right)$ represent all possible matrix elements of X, and therefore they define uniquely the operator X.
What is remarkable is that, despite being collective, fluctuations retain some of the quantum nature of the microscopic underlying system, providing a useful setting where to look for the description of quantum effects in many-body mesoscopic systems [57–60]. Their quasi-free bosonic nature allows for a description of the state in terms of covariance matrices. Indeed, defining the matrix with entries

$$\begin{aligned}\Sigma^{(\omega)}_{\mu\nu} &= \frac{1}{2}\lim_{N\to\infty} \omega\left(\{F_N(x_\mu), F_N(x_\nu)\}\right)\,,\\ (r, \Sigma^{(\omega)} s) &= \sum_{\mu,\nu=1}^{n} r_\mu s_\nu \Sigma^{(\omega)}_{\mu\nu}\,,\end{aligned} \tag{3.39}$$

one has that

$$\Omega\left(W(r)\right) = e^{-\frac{1}{2}\left(r,\Sigma^{(\omega)} r\right)}\,,$$

and because of the Gaussian characteristic function, all possible moments of the quantum variables $F_1, F_2, \dots F_n$ are encoded in $\Sigma^{(\omega)}$.

Example 3.2. *We consider again an infinite spin-*$\frac{1}{2}$ *chain as in Example 2.4 with the same thermal state,* ω^∞_β*. The following fluctuations can be constructed*

$$F_N(s_\mu) = \frac{1}{\sqrt{N_T}} \sum_{k=-N}^{N} \left(s^{(k)}_\mu - \omega^\infty_\beta\left(s_\mu\right)\right)\,,$$

and also the anti-symmetric form

$$\begin{aligned}\sigma^{(\omega^\infty_\beta)}_{\mu\nu} &= -i\omega^\infty_\beta\left([s_\mu, s_\nu]\right)\,,\\ \sigma^{(\omega^\infty_\beta)} &= \begin{pmatrix} 0 & \eta & 0\\ -\eta & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\,, \quad \eta = \omega^\infty_\beta(s_3) = -\frac{1}{2}\tanh\left(\frac{\beta}{2}\right)\,.\end{aligned}$$

From this matrix, it is clear that the asymptotic fluctuation $\lim_{N\to\infty} F_N(s_3) \to F_3$ *represents a classical degree of freedom, since it commutes with the rest of the bosonic algebra.*
Out of the two remaining fluctuations $F_N\left(s_{1,2}\right)$*, we can construct the following two operators*

$$x_N = \frac{F_N(s_2)}{\sqrt{|\eta|}}\,, \quad p_N = \frac{F_N(s_1)}{\sqrt{|\eta|}}\,,$$

such that their limiting points respectively x, p*, in the mesoscopic limit reviewed in this Chapter, obey* $[x, p] = i$*. The quasi-free state* Ω_β *on the generic asymptotic Weyl operators* $e^{i(\alpha x + \gamma p)}$ *is thermal, indeed*

$$\lim_{N\to\infty} \omega\left(e^{i(\alpha x_N + \gamma p_N)}\right) = \exp\left(-\frac{1}{2}\frac{(\alpha^2+\gamma^2)\coth\left(\frac{\beta}{2}\right)}{2}\right) = \Omega_\beta\left(e^{i(\alpha x + \gamma p)}\right)$$

so that

$$\Omega_\beta(\cdot) = \frac{\mathrm{Tr}\left(e^{-\beta H}\cdot\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}\,, \quad H = \frac{x^2}{2} + \frac{p^2}{2}\,.$$

The operators x, p *are position and momentum like operators, but their physical meaning is not transparent; this can be better appreciated if one passes from* x, p *to creation and annihilation operators* $a, a^\dagger$*, with* $a = \frac{1}{\sqrt{2}}(x + ip)$*. Indeed, in this description,* $a^\dagger$ *represents the creator operator of an excitation of fluctuations, as it can be deduced by looking at the following expectation*

$$\Omega_\beta\left(a^\dagger a\right) = \frac{1}{2|\eta|}\lim_{N\to\infty}\omega_\beta^\infty\left(F_N^2(s_1) + F_N^2(s_2)\right) - \frac{1}{2}\,,$$

where the right-hand side of the equality, is proportional to total amount of fluctuations in the system.

Chapter 4

Dissipative Short-Range Evolution of Fluctuations

In the previous Chapter, the kinematics of collective degrees of freedom was introduced; here, instead we shall look at their dynamical properties.
A lot of work has been done in the study of unitary time-evolutions on quantum fluctuations [31–33, 60], while very little is known about dissipative ones [34]. The latter account for situations in which the many-body system is in interaction with an environment; this is the most common situation encountered in actual experiments, typically involving cold atoms, optomechanical or spin-like systems [30, 61], that can never be thought of as completely isolated from their surroundings.
The behaviour of the environment is usually not of interest; one just focuses on system observables, tracing out the environmental degrees of freedom. This leads to an irreversible dynamics showing dissipative and noisy effects. In many physical situations one can consider the interaction system-environment to be weak, and neglect memory effects; in such a case the dynamics of the system can be described by effective, reduced dynamics involving only the system degrees of freedom, that satisfies the forward-in-time semigroup composition law.
After reviewing the theory of such open quantum systems we shall apply it to the study of the open dynamics of fluctuations.

4.1 Effective Open Quantum Dynamics

For sake of completeness, we shall first review the standard approach to open quantum systems, *i.e.* systems weakly coupled to external environments. The literature on this topic is large, we refer to [8, 9] for details on the derivation of the master equation, and to [6, 7], for a detailed analysis of the properties of quantum dynamical evolutions and semi-groups.

4.1.1 Open Quantum Systems

Let us consider a system S, whose observables belong to the C^*-algebra $\mathcal{A}_S$, interacting with an environment E, whose degrees of freedom are collected in the algebra $\mathcal{A}_E$. The compound system, described by the total algebra $\mathcal{A}_T = \mathcal{A}_S \otimes \mathcal{A}_E$, is considered to be isolated from the rest and, as such, evolves unitarily, according to the Heisenberg equations of motion induced by the total Hamiltonian H_T,

$$\frac{\mathrm{d}}{\mathrm{d}t} O_t = i\,[H_T, O_t]\,, \qquad \forall O \in \mathcal{A}_T\,. \tag{4.1}$$

Such a Hamiltonian can always be decomposed in the following way

$$H_T = H_S \otimes \mathbf{1} + \mathbf{1} \otimes H_E + \lambda H_I \tag{4.2}$$

where H_S, respectively H_E represent the Hamiltonian generating the free evolution of the system S, respectively the environment E, while H_I is the interaction term made by operators of both system and environment, λ being a dimensionless coupling constant. The solution to the operatorial equation (4.1), is given in terms of a family of one-parameter unitary automorphisms $\mathbb{U}_t^{S+E} : \mathcal{A}_T \to \mathcal{A}_T$, obeying the composition law

$$\mathbb{U}_t^{S+E} \circ \mathbb{U}_s^{S+E} = \mathbb{U}_{t+s}^{S+E}\,, \qquad t, s \in \mathbb{R}\,,$$

and such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{U}_t^{S+E}\,[\cdot] = \mathbb{H} \circ \mathbb{U}_t^{S+E}[\cdot], \quad \mathbb{H}[\cdot] = i\,[H_T, \cdot]\ .$$

In usual situations though, one is interested in the evolution of the system S, only. This is obtained by tracing over the environment degrees of freedom. Furthermore, in many physical cases, the system can be prepared independently from the environment, so, one can consider the state of the compound system $S + E$ to be a product state $\omega_T = \omega_S \otimes \omega_E$. Given such a functional over the total algebra $\mathcal{A}_T$, one implements the partial trace over the environment degrees of freedom by focusing upon the expectations with respect to the state ω_E on the total evolution of the system S observables of the form $O \otimes \mathbf{1} \in \mathcal{A}_T$,

$$O_t^S = \omega_E\left(\mathbb{U}_t^{S+E}\,[O \otimes \mathbf{1}]\right)\ .$$

The resulting operators belong to the algebra of the system, $O_t^S \in \mathcal{A}_S$, $\forall\, O \in \mathcal{A}_S$ and the partial expectation defines a family of maps $\Lambda_t : \mathcal{A}_S \to \mathcal{A}_S$,

$$\Lambda_t\,[O] := \omega_E\left(\mathbb{U}_t^{S+E}\,[O \otimes \mathbf{1}]\right)\ . \tag{4.3}$$

In general these maps are very complicated, do not obey the group composition law, $\Lambda_t \circ \Lambda_s \neq \Lambda_{t+s}$, and, because of the partial expectation over the environment, with whom the system interacts, they also embody the irreversible character of the reduced dynamics in that $\Lambda_t^{-1} \neq \Lambda_{-t}$, and one then ought to consider positive times, only: $t \geq 0$.
Nevertheless, these maps are linear

$$\begin{aligned}\Lambda_t\,[A + \delta B] = \omega_E\left(\mathbb{U}_t^{S+E}\,[(A + \delta B) \otimes \mathbf{1}]\right) = \\ \omega_E\left(\mathbb{U}_t^{S+E}\,[A \otimes \mathbf{1}] + \delta\,\mathbb{U}_t^{S+E}\,[B \otimes \mathbf{1}]\right) = \Lambda_t\,[A] + \delta\,\Lambda_t\,[B]\ ,\end{aligned} \tag{4.4}$$

and also completely positive.

Definition 4.1. *A linear map $\Lambda : \mathcal{A} \to \mathcal{A}$ is completely positive, if and only if $\Lambda \otimes \mathbf{1}_m$ is positive, on $\mathcal{A} \otimes \mathcal{A}_m$, $\forall\, m \in \mathbb{N}$, being $\mathcal{A}_m$ the algebra of $m \times m$ complex matrices.*

Indeed, one can check the complete positivity of the maps Λ_t based on the following argument. For any $m \in \mathbb{N}$ a generic operator $O \in \mathcal{A}_S \otimes \mathcal{A}_m$ can be written as

$$O = \sum_{\alpha,\beta} c_{\alpha\beta}\, X_\alpha^S \otimes X_\beta^m, \quad \text{where } X_\alpha^S \in \mathcal{A}_S, \quad X_\beta^m \in \mathcal{A}_m\,,$$

with $c_{\alpha\beta}$ complex coefficients; from the linearity of the map (4.4) one has

$$\Lambda_t \otimes \mathbf{1}_m \left[O^\dagger O\right] = \sum_{\alpha,\beta,\gamma,\delta} \bar{c}_{\alpha\beta} c_{\gamma\delta} \Lambda_t \left[X_\alpha^{S\dagger} X_\gamma^S\right] \otimes X_\beta^{m\dagger} X_\delta^m\,. \tag{4.5}$$

Being $\mathbb{U}_t^{S+E}$ automorphisms, one can write:

$$\Lambda_t \left[X_\alpha^{S\dagger} X_\gamma^S\right] = \omega_E \left(\mathbb{U}_t^{S+E}\left[X_\alpha^{S\dagger} X_\gamma^S \otimes \mathbf{1}\right]\right) = \omega_E \left(\mathbb{U}_t^{S+E}\left[X_\alpha^{S\dagger} \otimes \mathbf{1}\right] \mathbb{U}_t^{S+E}\left[X_\gamma^S \otimes \mathbf{1}\right]\right)\,,$$

and introducing the new operator $\tilde{O} \in \mathcal{A}_S \otimes \mathcal{A}_E \otimes \mathcal{A}_m$,

$$\tilde{O} = \sum_{\alpha,\beta} c_{\alpha\beta}\, \mathbb{U}_t^{S+E}\left[X_\alpha^S \otimes \mathbf{1}\right] \otimes X_\beta^m\,,$$

one recasts (4.5) as follows

$$\Lambda_t \otimes \mathbf{1}_m \left[O^\dagger O\right] = \omega_E \left(\tilde{O}^\dagger \tilde{O}\right)\,.$$

Since $\tilde{O}$ is an operator $\tilde{O} \in \mathcal{A}_S \otimes \mathcal{A}_E \otimes \mathcal{A}_m$, it can also be decomposed as follows

$$\tilde{O} = \sum_{\bar{\alpha}} d_{\bar{\alpha}}(t)\, Y_{\alpha_1}^S \otimes Y_{\alpha_2}^E \otimes Y_{\alpha_3}^m\,,$$

with $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a collection of indices, $d_{\bar{\alpha}}(t)$ suitable time-dependent coefficients and $Y_{\alpha_1}^S$, $Y_{\alpha_2}^E$, $Y_{\alpha_3}^m$ proper operators of the system, bath and of the m-level system, respectively. Therefore, one has:

$$\mathcal{A}_S \otimes \mathcal{A}_m \ni \omega_E \left(\tilde{O}^\dagger \tilde{O}\right) = \sum_{\bar{\alpha},\bar{\beta}} \bar{d}_{\bar{\alpha}}(t) d_{\bar{\beta}}(t)\, \Theta_{\alpha_2\beta_2}\, Y_{\alpha_1}^{S\dagger} Y_{\beta_1}^S \otimes Y_{\alpha_3}^{m\dagger} Y_{\beta_3}^m\,,$$

with $\Theta_{\alpha_2\beta_2} = \omega_E\left(Y_{\alpha_2}^{E\dagger} Y_{\beta_2}^E\right)$, entries of a positive semi-definite matrix Θ. Recasting the previous relation, one writes

$$\omega_E \left(\tilde{O}^\dagger \tilde{O}\right) = \sum_{\alpha_2,\beta_2} \Theta_{\alpha_2\beta_2} \left(\sum_{\alpha_1,\alpha_3} d_{\bar{\alpha}}(t)\, Y_{\alpha_1}^S \otimes Y_{\alpha_3}^m\right)^\dagger \left(\sum_{\beta_1,\beta_3} d_{\bar{\beta}}(t)\, Y_{\beta_1}^S \otimes Y_{\beta_3}^m\right)\,;$$

the positivity of the matrix Θ, together with the structure of the operatorial part in the right-hand side of the above equation, ensures that

$$\Lambda_t \otimes \mathbf{1}_m \left[O^\dagger O\right] = \omega_E \left(\tilde{O}^\dagger \tilde{O}\right) \geq 0\,,$$

implying that the maps Λ_t are indeed completely positive.

Remark 4.1. *Complete positivity is essential for the physical consistency of quantum dynamical maps [8, 9]. Its necessity is strictly related to the possibility of having the system initially entangled with an inert ancilla.*
In these cases, positivity of the dynamical maps Λ_t *does not guarantee that the lifted ones* $\Lambda_t \otimes \mathbf{1}_m$*, implementing the evolution of system plus ancilla, are mapping positive operators into positive ones. In absence of complete positivity, one could always find a time* t*, an initial positive operator, and an entangled system-ancilla initial state for which the expectation at the given time* t *provides a negative value. This contradicts any possible experimental result.*

4.1.2 Quantum Dynamical Semi-groups

Maps defined as in equation (4.3), are usually rather complicated and not amenable to analytical treatment. Nevertheless, in many physical situations interaction between system and environment can be considered to be weak $\lambda \ll 1$. Moreover, when the typical bath relaxation time-scales are much smaller than the characteristic time-scale of the system, memory effects can be neglected. The resulting dynamics can then be described by completely positive unital semigroups.
In this section, we briefly review how, by means of the weak coupling limit, one can retrieve this kind of dynamics. We will follow the procedure presented in [8], revisited according to our definition of state.

Weak-coupling limit

As already said, our aim is to find an effective description for the evolution of expectations of system's observables, namely,

$$\langle O \rangle_t := \omega_T\Big(\mathbb{U}_t^{S+E}\left[O \otimes \mathbf{1}\right]\Big) .$$

By means of the automorphism implemented solely by the free Hamiltonians $\mathbb{U}_t^0$, we define the interaction picture functional ω_t^I in the following way:

$$\omega_t^I := \omega_T \circ \mathbb{U}_t^{S+E} \circ \left(\mathbb{U}_t^0\right)^{-1} ;$$

this is the fully evolved initial state, rotated back by means of the free evolution.
The equation of motion for the functional becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^I = \omega_t^I \circ \mathbb{H}_t^I, \qquad \mathbb{H}_t^I[\cdot] = i\lambda\left[H_I(t), \cdot\right], \quad H_I(t) = \mathbb{U}_t^0\left[H_I\right] . \tag{4.6}$$

Formally integrating the equation, one obtains

$$\omega_t^I = \omega_T + \int_0^t ds\, \omega_s^I \circ \mathbb{H}_s^I ,$$

and inserting it in (4.6), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^I = \omega_T \circ \mathbb{H}_t^I + \int_0^t ds\, \omega_s^I \circ \mathbb{H}_s^I \circ \mathbb{H}_t^I . \tag{4.7}$$

Since we are interested in the system S observables, only, we construct a reduced functional on the algebra $\mathcal{A}_S$ in the following way:

$$\omega_t^S\left(O\right) := \omega_t^I\left(O \otimes \mathbf{1}\right) \,.$$

As it will be clear below, one can always recast the Hamiltonian in such a way that $\omega_T\left(\mathbb{H}_t^I\left[O \otimes \mathbf{1}\right]\right) = 0$, thus we can rewrite the above equation in the following way

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^I = \int_0^t ds\, \omega_s^I \circ \mathbb{H}_s^I \circ \mathbb{H}_t^I \,.$$

Now the first approximation comes into play [8]: because of the weak system-environment interaction, the influence of the system on the large environment is negligible, so that the state of the total system at any time t can be approximately described as the following tensor product (Born approximation)

$$\omega_t^I = \omega_t^S \otimes \omega_E \,.$$

Therefore, we have, on operators of the form $O \otimes \mathbf{1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^S = \int_0^t ds\, \omega_s^S \otimes \omega_E \circ \mathbb{H}_s^I \circ \mathbb{H}_t^I \,.$$

Further, when memory effects can be neglected, one can perform the so-called Markov approximation; this is done in two steps. At first ω_s^S is replaced by ω_t^S, in order to get a time-local differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^S = \omega_t^S\left(\int_0^t ds\, \omega_E \circ \mathbb{H}_s^I \circ \mathbb{H}_t^I\right) \,;$$

secondly, after operating the substitution $s \to t - s$, one lets the upper limit of the integral go to infinity; this is permitted when the time-scales of the evolution of the system are much larger than the bath relaxation time. With the two approximations one gets

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\omega_t^S = \omega_t^S \circ \mathbb{K}_t \\ \mathbb{K}_t : \mathcal{A}_S \to \mathcal{A}_S, \qquad &\mathbb{K}_t\left[O\right] = \omega_E\left(\int_0^\infty ds\, \mathbb{H}_{t-s}^I \circ \mathbb{H}_t^I\left[O \otimes \mathbf{1}\right]\right) \,. \end{aligned} \tag{4.8}$$

These equations do not in general lead to completely positive dynamical semigroup [62, 63]; this is achieved from (4.8), averaging over the rapidly oscillating terms in the map $\mathbb{K}_t$ (secular approximation).
One first observes that in general the interaction Hamiltonian can be taken of the following form

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha \,,$$

with $A_\alpha^\dagger = A_\alpha$, $B_\alpha^\dagger = B_\alpha$. Assuming H_S to posses a discrete spectrum, with eigenvalues ϵ relative to spectral projectors $\Pi(\epsilon)$, one can define the operators

$$A_\alpha(\delta) := \sum_{\epsilon' - \epsilon = \delta} \Pi(\epsilon) A_\alpha \Pi(\epsilon') \,.$$

As a consequence, one has

$$\begin{aligned}
&[H_S, A_\alpha(\delta)] = -\delta\, A_\alpha(\delta)\,,\\
&\left[H_S, A^\dagger_\alpha(\delta)\right] = \delta\, A_\alpha(\delta)\,,\\
&\left[H_S, A^\dagger_\alpha(\delta) A_\beta(\delta)\right] = 0, \quad A^\dagger_\alpha(\delta) = A_\alpha(-\delta)\,;
\end{aligned}$$

this enables us to recast the interaction Hamiltonian in the following form

$$H_I = \sum_{\alpha,\delta} A_\alpha(\delta) \otimes B_\alpha = \sum_{\alpha,\delta} A^\dagger_\alpha(\delta) \otimes B_\alpha\,.$$

Therefore, in the interaction picture

$$H_I(t) = \sum_{\alpha,\delta} e^{-i\delta t} A_\alpha(\delta) \otimes B_\alpha(t) = \sum_{\alpha,\delta} e^{i\delta t} A^\dagger_\alpha(\delta) \otimes B_\alpha(t), \quad B_\alpha(t) = \mathbb{U}^0_t\left[B_\alpha\right]\,,$$

and the condition $\omega_E\left([H_I(t), O \otimes \mathbf{1}]\right) = 0$, $\forall\, O$, becomes $\omega_E\left(B_\alpha(t)\right) = 0$.
Substituting the interaction Hamiltonian into $\mathbb{K}_t$ one gets

$$\begin{aligned}
\mathbb{K}_t\left[O\right] = \lambda^2 \sum_{\alpha,\beta,\delta,\delta'} & e^{i(\delta-\delta')t} \left[A^\dagger_\alpha(\delta), O\right] A_\beta\left(\delta'\right) \Gamma_{\alpha\beta}\left(\delta'\right) + \\
& + \lambda^2 \sum_{\alpha,\beta,\delta,\delta'} e^{i(\delta'-\delta)t} A^\dagger_\beta(\delta') \left[O, A_\alpha(\delta)\right] \bar{\Gamma}_{\alpha\beta}\left(\delta'\right)\,,
\end{aligned} \tag{4.9}$$

where, we have defined the quantities

$$\Gamma_{\alpha\beta}\left(\delta'\right) := \int_0^\infty ds\; e^{is\delta'}\, \omega_E\left(B_\alpha(t) B_\beta(t-s)\right)\,,$$

depending on the environment correlation functions. If we also assume the state ω_E to be stationary with respect to the free environment evolution, one has that $\Gamma_{\alpha\beta}(\delta')$ does not depend on t, since

$$\omega_E\left(B_\alpha(t) B_\beta(t-s)\right) = \omega_E\left(B_\alpha(s) B_\beta(0)\right)\,.$$

Typical time-scales of the system are represented by $|\delta - \delta'|^{-1}$, $\delta' \neq \delta$; if these time-scales are large compared to the relaxation time τ_R of the open system, the non-secular terms in (4.9), the ones with $\delta \neq \delta'$ may be neglected, since they oscillate very rapidly during the time τ_R over which the system state varies appreciably. Thus, the generator becomes

$$\begin{aligned}
\mathbb{K}\left[O\right] = \lambda^2 \sum_{\alpha,\beta,\delta} & \left[A^\dagger_\alpha(\delta), O\right] A_\beta\left(\delta\right) \Gamma_{\alpha\beta}\left(\delta\right) + \\
& + \lambda^2 \sum_{\alpha,\beta,\delta} A^\dagger_\alpha(\delta) \left[O, A_\beta(\delta)\right] \bar{\Gamma}_{\beta\alpha}(\delta)\,.
\end{aligned} \tag{4.10}$$

The half-Fourier transforms $\Gamma_{\alpha\beta}$ can be decomposed in the following way:

$$\begin{aligned}
\Gamma_{\alpha\beta}(\delta) &= \frac{1}{2}\gamma_{\alpha\beta}(\delta) + i S_{\alpha\beta}(\delta)\,,\\
\gamma_{\alpha\beta}(\delta) &= \Gamma_{\alpha\beta}(\delta) + \bar{\Gamma}_{\beta\alpha}(\delta) = \int_{-\infty}^{+\infty} ds\, e^{i\delta s} \omega_E\left(B_\alpha(s) B_\beta(0)\right)\,,\\
S_{\alpha\beta}(\delta) &= \frac{1}{2i}\left(\Gamma_{\alpha\beta}(\delta) - \bar{\Gamma}_{\beta\alpha}(\delta)\right)\,,
\end{aligned}$$

therefore, the following relations hold

$$\Gamma_{\alpha\beta}(\delta) = \frac{1}{2}\gamma_{\alpha\beta}(\delta) + iS_{\alpha\beta}(\delta)\,,$$
$$\bar{\Gamma}_{\beta\alpha}(\delta) = \frac{1}{2}\gamma_{\alpha\beta}(\delta) - iS_{\alpha\beta}(\delta)\,,$$

and one can recast the generator in the usual Lindblad form [64, 65]

$$\mathbb{K}\,[O] = \lambda^2 \sum_{\delta}\sum_{\alpha\beta} \gamma_{\alpha\beta}(\delta)\Big(\,\left[A^\dagger_\alpha(\delta), O\right] A_\beta(\delta) + A^\dagger_\alpha(\delta)\left[A_\beta(\delta), O\right]\Big) + \tag{4.11}$$

$$+ i\lambda^2 \sum_{\delta}\sum_{\alpha\beta} S_{\alpha\beta}(\delta)\left[A^\dagger_\alpha(\delta) A_\beta(\delta), O\right]\,, \tag{4.12}$$

since the matrices $\gamma_{\alpha\beta}(\delta)$ are positive and the generator in (4.12) is Hermitian.
Within these approximations, we found that the evolution of the functional in the interaction picture is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega^S_t = \omega^S_t \circ \mathbb{K}\,,$$

whose solution is given by

$$\omega^S_t = \omega_S \circ e^{t\mathbb{K}} := \omega_S \circ \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\mathbb{K} \circ \mathbb{K} \circ \cdots \circ \mathbb{K}}_{n-\text{times}}\,.$$

The second part of the generator (4.12), represents a contribution from a Lamb shift Hamiltonian, which leads to a renormalization of the energy levels induced by the coupling system-environment; the first term (4.11), called dissipator, is instead the part of the generator taking into account the non-coherent aspect of the time-evolution which is due to the interaction between the system and the environment.
The reduced dynamics acting on the algebra of the system $e^{t\mathbb{K}} : \mathcal{A}_S \to \mathcal{A}_S$, is such that it obeys the semi-group composition law, indeed $e^{t\mathbb{K}} \circ e^{s\mathbb{K}} = e^{(t+s)\mathbb{K}}$, $t, s \geq 0$, and provides us with an effective description of the evolution of observables. Indeed, the above derivation shows that, under the performed approximations, we have, $\forall\, O \in \mathcal{A}_S$

$$\omega_E\left(\mathbb{U}^{S+E}_t \circ \left(\mathbb{U}^0_t\right)^{-1}[O \otimes \mathbf{1}]\right) \sim e^{t\mathbb{K}}\,[O]\,.$$

When the open system S is an $m-$ *level* system, the results sketched above can be summarized as follows [64, 65].

Theorem 4.1. *Let $\mathcal{A}_S$ be the algebra of an m-level system and $\Lambda_t : \mathcal{A}_S \to \mathcal{A}_S$ form a time-continuous semi-group of unital, completely positive, hermiticity-preserving linear maps. Then, the semi-group has the form $\Lambda_t = e^{t\mathbb{L}}$ with generator consisting of*

$$\begin{aligned}
&\mathbb{L}[O] = \mathbb{H}[O] + \mathbb{D}[O]\,,\\
&\mathbb{H}[O] = i\,[H, O]\,, \quad H = H^\dagger\,,\\
&\mathbb{D}[O] = \sum_{\mu,\nu=1}^{m^2-1} \frac{C_{\mu\nu}}{2}\left([V_\mu, O]\, V^\dagger_\nu + V_\mu\left[O, V^\dagger_\nu\right]\right)\,,
\end{aligned} \tag{4.13}$$

where the matrix of coefficients $C_{\mu\nu}$ is the so-called Kossakowski matrix, and the operators V_μ are such that

$$V_{m^2} = \frac{\mathbf{1}}{\sqrt{m}}, \qquad \mathrm{Tr}\left(V_\mu^\dagger V_\nu\right) = \delta_{\mu\nu}, \quad 0 \le \mu, \nu \le m^2 \, .$$

The matrix C must be positive semi-definite in order to ensure the complete positivity of the maps Λ_t.

4.2 The Dissipative Generator

We shall now apply these results to the description of the open dynamics of fluctuation operators.
As discussed in the previous Chapter, we shall study a system formed by a bi-infinite lattice, where each site supports a same d-level algebra $\mathcal{A}_d$.
The algebraic description of such a system is in term of a quasi-local algebra $\mathcal{A}$.
One then selects an ordered set of single-site independent operators $\{x_i\}_{i=1}^n$, $x_i = x_i^\dagger \in \mathcal{A}_d$, whose fluctuations are supposed of physical interest and constructs the real linear span K (see Definition 3.1). For these operators, the corresponding fluctuations are defined as in (3.28).
Given a translation-invariant clustering state ω (see Definition 2.6, (2.14)-(2.15)) on this quasi-local algebra $\mathcal{A}$, we will assume the system (K, ω) to have normal quantum fluctuations as in Definition 3.3, equations (3.31),(3.32).
The aim is to study the dynamics of fluctuations inherited from a microscopic irreversible dynamics such that, in the Heisenberg picture,

$$\partial_t X_t \;=\; \mathbb{L}_N[X_t] \;, \tag{4.14}$$

for all local operators $X \in \mathcal{A}_{[-N,N]}$, where $\mathbb{L}_N$ is a generator of Lindblad form (see (4.13)).

$$\mathbb{L}_N\left[X\right] = \mathbb{H}_N\left[X\right] + \mathbb{D}_N\left[X\right] \;. \tag{4.15}$$

The Hamiltonian contribution $\mathbb{H}_N\left[X\right] = i\left[H_N, X\right]$ will be sought with H_N of the form

$$H_N = \sum_{k=-N}^{N} h^{(k)} \;, \tag{4.16}$$

namely it is the sum of single-site Hamiltonians $h^{(k)} = \left(h^{(k)}\right)^\dagger$, while noise and dissipation, accounted for by the dissipator $\mathbb{D}_N[X]$, will be given by

$$\mathbb{D}_N[X] = \sum_{k,\ell=-N}^{N} J_{k\ell} \sum_{\mu,\nu=1}^{p} \frac{D_{\mu\nu}}{2}\left(\left[v_\mu^{(k)}\,,\, X\right](v_\nu^\dagger)^{(\ell)} \,+\, v_\mu^{(k)}\left[X\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right) \,. \tag{4.17}$$

In the above expression the operators $v_\nu^{(k)}$ are single site operators while the coefficients $J_{k\ell}$ and $D_{\mu\nu}$ form the Kossakowski matrix $J \otimes D$. We shall assume $J \ge 0$, $D \ge 0$ so that $J \otimes D$ results positive semi-definite guaranteeing the complete positivity of the dynamics. The matrix D contains information about the noisy and mixing effects due to environment on couples of sites, while the matrix J accounts for the strength of these effects as a function of the distance between sites. The dissipative contribution is made translation invariant by setting

$$J_{k\ell} = J(k-\ell) \ , \quad J_{kk} = J_0 > 0 \, , \qquad \forall \, k, \ell \in \mathbb{Z} \ . \tag{4.18}$$

We also assume that

$$\sum_{\ell=-\infty}^{\infty} |J_{k\ell}| = \sum_{r=-\infty}^{\infty} |J(r)| < \infty \, , \qquad \forall \, k \in \mathbb{Z} \ , \tag{4.19}$$

condition that establishes a fast decay of the strength of the statistical coupling of far separated chain sites, thus describing short-range mixing effects due to environment. A dissipative contribution of the form (4.17) is very general and accounts for a generic lattice-environment weak interaction. As explained in the previous Section, it can be obtained by means of a weak-coupling limit, assuming an interaction Hamiltonian $H_I = H_I^\dagger$ of the following form

$$H_I = \sum_\alpha f_\alpha \sum_{k,h} g_{kh} \, v_\alpha^{(k)} \otimes B_h \ ,$$

with $v_\alpha^{(k)}$'s single-particle operators of the lattice system, B_h's operators of the environment, and f_α, g_{kh} suitable coefficients.

4.2.1 Locality Conditions

The main purpose of this Chapter is to find the structure of the dynamics of fluctuations, in particular of Weyl operators $W(r)$, inherited from a microscopic dissipative spin chain dynamics of the type (4.15),(4.16),(4.17).
In general, the action of the local Lindblad generator $\mathbb{L}_N$ on $(r, F_N) = \sum_{j=1}^n r_j F_N(x_j)$ maps it into fluctuations of a single-site operator that is not in the linear span K, or it might even generate fluctuations of operators that act non-trivially on more than just one site. For the mesoscopic dynamics to be a map from the Weyl algebra $\mathcal{W}(K, \sigma^{(\omega)})$ into itself, one has to assume

$$\mathbb{L}_N[x_i^{(k)}] = \sum_{j=1}^n \mathcal{L}_{ij} x_j^{(k)} \ , \quad \mathcal{L} = \mathcal{H} + \mathcal{D} \ , \tag{4.20}$$

for all $x_i \in K$, with $k \in [-N, N]$, where $\mathcal{L}$ is the $n \times n$ matrix with entries $\mathcal{L}_{ij}$ and $\mathcal{H}$, $\mathcal{D}$ are the $n \times n$ matrices with entries defined by

$$i \left[H_N, x_i^{(k)} \right] = \sum_{j=1}^n \mathcal{H}_{ij} x_j^{(k)} \ , \quad \mathbb{D}_N \left[x_i^{(k)} \right] = \sum_{j=1}^n \mathcal{D}_{ij} x_j^{(k)} \ . \tag{4.21}$$

In other terms, the above condition consists in asking the linear span K to be mapped into itself by the generator $\mathbb{L}_N$.
Such a constraint on the dynamics has also implications on the action of the generator on products of operators from different sites.

Lemma 4.1. *Given a single-site operator basis* $\{o_\alpha\}_{\alpha=1}^{d^2}$ *in* $\mathcal{A}_d$ *and a generator* $\mathbb{L}_N$ *satisfying*

$$\mathbb{L}_N\left[o_\alpha^{(k)}\right] = \sum_{\beta=1}^{d^2} c_{\alpha\beta}^k\, o_\beta^{(k)} \ ,$$

then:

$$\mathbb{L}_N\left[o_{\alpha_1}^{(k_1)} x_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}\right] = \sum_{\bar\beta} c_{\bar\alpha\,\bar\beta}^{\bar k}\, o_{\beta_1}^{(k_1)} o_{\beta_2}^{(k_2)} \dots x_{\beta_m}^{(k_m)} \ ,$$

with multi-indices $\bar\beta = (\beta_1, \beta_2, \dots, \beta_m)$ *and suitable coefficients* $c_{\bar\alpha,\bar\beta}^{\bar k}$.

We give here the explicit proof of this result, since it shows the general methodology that needs to be adopted in order to show many of the results discussed in the following sections.

Proof. Due to the properties of commutators, the lemma is certainly true for the Hamiltonian term $\mathbb{H}_N$ of the Lindblad generator. For the dissipative term $\mathbb{D}_N$ we proceed by induction: the statement is true in the case $m = 1$ thus, assuming this to hold also for the product of m single-site operators, we want to show that it is valid also for products of $m + 1$ operators.
Using the algebraic relation

$$b\Big(a\,[d\,,\,c] + [a\,,\,d]\,c\Big) + \Big(a\,[b\,,\,c] + [a\,,\,b]\,c\Big)\,d - a\,[bd\,,\,c] - [a\,,\,bd]\,c \;=\; -2\,[a\,,\,b]\,[d\,,\,c]\ ,$$

one derives that

$$\mathbb{D}_N[ab] = \mathbb{D}_N[a]\,b \,+\, a\,\mathbb{D}_N[b] \,+\, 2\sum_{\mu,\nu=1}^{p} \frac{D_{\mu\nu}}{2} \sum_{k,\ell=-N}^{N} J_{k\ell}\left[v_\mu^{(k)}\,,\,a\right]\,\left[b\,,\,v_\nu^{(\ell)}\right] \ ,$$

for all operators $a, b \in \mathcal{A}_{[-N,N]}$. Let $a = o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}$ and $b = o_{\alpha_{m+1}}^{(k_{m+1})}$; then,

$$\begin{aligned}\mathbb{D}_N\left[o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)} o_{\alpha_{m+1}}^{(k_{m+1})}\right] &= o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}\,\mathbb{D}_N\left[o_{\alpha_{m+1}}^{(k_{m+1})}\right] \,+ \\ &+\mathbb{D}_N\left[o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}\right]\,o_{\alpha_{m+1}}^{(k_{m+1})} \,+ \\ &+2\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}\sum_{k,\ell=-N}^{N} J_{k\ell}\,\left[v_\mu^{(k)}\,,\,o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}\right]\left[o_{\alpha_{m+1}}^{(k_{m+1})}, v_\nu^{\dagger(\ell)}\right]\ .\end{aligned}$$

Due to the assumptions and the induction hypothesis, the first two contributions can again be expressed as linear combinations of products of single-site basis operators at sites $k_1, k_2, \dots k_{m+1}$. As for the last term, it amounts to

$$\begin{aligned}&\sum_{k,\ell=-N}^{N} J_{k\ell}\,\left[v_\mu^{(k)}\,,\,o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_m}^{(k_m)}\right]\,\left[o_{\alpha_{m+1}}^{(k_{m+1})}\,,\,v_\nu^{(\dagger\ell)}\right] = \\ &= \sum_{q=k_1,k_2,\dots k_m} J_{qk_{m+1}}\, o_{\alpha_1}^{(k_1)} o_{\alpha_2}^{(k_2)} \dots o_{\alpha_{q-1}}^{(k_{q-1})}\left[v_\mu^{(q)}, o_{\alpha_q}^{(q)}\right] o_{\alpha_{q+1}}^{(k_{q+1})} \dots o_{\alpha_m}^{(k_m)}\,\left[o_{\alpha_{m+1}}^{(k_{m+1})}, v_\nu^{\dagger(k_{m+1})}\right]\ .\end{aligned}$$

Therefore, by expanding the various commutators with respect to the single-site matrix basis $\{o_\alpha\}_{\alpha=1}^{d^2}$, it can also be written as a linear combination of products of basis operators at sites $k_1, k_2, \dots k_{m+1}$.

□

4.3 Mesoscopic Dissipative Dynamics

In this section we shall show that, under the conditions (4.19),(4.20) on the microscopic Lindblad generator preserving the microscopic state ω, the mesoscopic dynamics that emerges in the limit $N \to +\infty$ is described by a semi-group $\{\Phi_t\}_{t\geq 0}$ of completely positive, unital maps on the quantum fluctuation algebra.
The assumption that the generator preserves the microscopic reference state ω means that we are considering dynamics such that

$$\omega\Big(\Phi_t^N(X)\Big) = \omega(X) \;\Leftrightarrow\; \omega\Big(\mathbb{L}_N[X]\Big) = 0 \;.$$

This is what happens in many physical situations: asking $\omega \circ \Phi_t^N = \omega$ is tantamount to considering a dynamics that does not change the initial phase of the many-body system, as discussed in Section 2.2. For instance, with this assumption one can describe all those experimental settings where the focus is on the dynamical properties of states prepared perturbing, by means of local manipulations, an equilibrium time-invariant state ω (see equation (2.13) and the discussion above it)[1].
Recalling the definition of the mesoscopic limit in (3.38), one naturally defines the action of Φ_t in the following way:

Definition 4.2. *The microscopic dissipative dynamical maps Φ_t^N on the local algebras $\mathcal{A}_{[-N,N]}$ define the corresponding mesoscopic dynamical maps Φ_t on the Weyl algebra $\mathcal{W}(K,\sigma^{(\omega)})$ of quantum fluctuations if the following mesoscopic limit*

$$\lim_{N\to+\infty} \omega\left(W_N(s_1)\Phi_t^N[W_N(r)]\,W_N(s_2)\right) = \Omega\Big(W(s_1)\,\Phi_t(W(r))\,W(s_2)\Big) \;, \tag{4.22}$$

is well-defined for all Weyl-like operators $W_N(s_1)$, $W_N(s_2)$, $W_N(r)$, with $W(s_1)$, $W(s_2)$ and $W(r)$ the corresponding limiting Weyl operators, with ω the state on the quasi-local algebra $\mathcal{A}$, and Ω the mesoscopic state on the Weyl algebra defined in (3.37) *by Theorem 3.2.*

We will look for dynamical maps Φ_t of quasi-free type, namely mapping Weyl operators into Weyl operators:

$$\Phi_t(W(r)) = \mathrm{e}^{f_r(t)}\,W(r_t)\,, \qquad \forall\, r \in \mathbb{R}^n \;, \tag{4.23}$$

where both the time-dependent function $f_r(t)$ and vector $r_t \in \mathbb{R}^n$ are unknowns to be determined. The maps are unital, $\Phi_t[1] = 1$, and must be completely positive. As such they must obey the Schwartz positivity inequality

$$\Phi_t(X^\dagger X) \;\geq\; \Phi_t(X^\dagger)\,\Phi_t(X)\,, \qquad \forall X \in \mathcal{W}(K,\sigma^{(\omega)}) \;. \tag{4.24}$$

Then, since Weyl operators $W(r)$ are unitary, $f_r(t)$ must satisfy

$$\|\Phi_t(W(r))\| = \left|\mathrm{e}^{f_r(t)}\right| \leq \|\Phi_t[1]\| = 1 \;. \tag{4.25}$$

[1]Such an assumption can be relaxed, allowing the many-body system to dynamically change its phase. This leads to two main technical complications: on one hand, the definition of fluctuations as in (3.21) must be changed, subtracting a time-dependent mean-value; on the other hand, also the commutation relations of the large N fluctuation operators would evolve in time, according to the evolution of averages. Such an interesting situation has been studied in [66, 67].

The proof of equations (4.22),(4.23) will be based on a family of local microscopic maps Ψ_t^N on the quantum lattice system interpolating between the microscopic, Φ_t^N, and the mesoscopic dissipative time-evolution, Φ_t, defined by:

$$\begin{aligned} \Psi_t^N\left[W_N(r)\right] &= \mathrm{e}^{f_r(t)}\, W_N(r_t) \;=\; \mathrm{e}^{f_r(t)}\, \mathrm{e}^{i(r_t,F_N)} && (4.26)\\ r_t &= \mathcal{X}_t^{tr}\, r\ , \quad \mathcal{X}_t = \mathrm{e}^{t\,(\mathcal{D}+\mathcal{H})} && (4.27)\\ f_r(t) &= -\left(r,\mathcal{Y}_t r\right)\ , \quad \mathcal{Y}_t = \frac{1}{2}\left(\Sigma^{(\omega)} \,-\, \mathcal{X}_t\,\Sigma^{(\omega)}\,\mathcal{X}_t^{tr}\right)\ , && (4.28) \end{aligned}$$

where $\mathcal{X}^{tr}$ denotes the transposition of the $n\times n$ matrix $\mathcal{X}$ and $\Sigma^{(\omega)}$ is the fluctuation covariance matrix. Because of Lemma 3.1 and Theorem 3.2, we know that

$$\lim_{N\to+\infty}\omega\left(W_N(s_1)\,W_N(r_t)\,W_N(s_2)\right) = \Omega\Big(W(s_1)\,W(r_t)\,W(s_2)\Big)\ .$$

We are going to show that the mesoscopic dynamical maps Φ_t in (4.22) are of the form (4.23), $\Phi_t\left[W(r)\right] = \mathrm{e}^{f_r(t)}\,W(r_t)$, where $f_r(t)$ and r_t are given by (4.27) and (4.28). The maps Φ_t compose as a semigroup; indeed, for all $s,t\geq 0$,

$$\begin{aligned} \Phi_s\circ\Phi_t\left[W(r)\right] &= \mathrm{e}^{-(r,\mathcal{Y}_t\,r)-(r_t,\mathcal{Y}_s\,r_t)}\,W((r_t)_s)\\ &= \mathrm{e}^{-(r,\mathcal{Y}_t\,r)-(r,\mathcal{X}_t\mathcal{Y}_s\mathcal{X}_t^{tr}\,r)}\,W(r_{t+s}) \qquad (4.29)\\ &= \mathrm{e}^{-(r,\mathcal{Y}_{t+s}\,r)}\,W(r_{t+s}) = \Phi_{t+s}\left[W(r)\right]\ . \end{aligned}$$

Furthermore, as required by complete positivity and unitality, and proved by the following lemma, the function $f_r(t)$ defined by (4.28) is such that $\exp(f_r(t))\leq 1$.

Lemma 4.2. *The invariance of the microscopic state ω with respect to the microscopic dissipative dynamics Φ_t^N implies $f_r(t)\leq 0$.*

Proof. We shall show that $\omega\circ\Phi_t^N=\omega$, $t\geq 0$, makes negative semi-definite, $\mathcal{Y}_t\leq 0$, the matrix defined by (4.28), for all $t\geq 0$. Let $\lambda\in\mathbb{C}^n$ be a generic complex vector and set $q_\lambda=\sum_{j=1}^n\lambda_j\,x_j$; then, using Schwartz positivity, and the time-invariance of the state that allows one to use equations (C.22) in Appendix C, one estimates

$$\begin{aligned} \frac{1}{2}\sum_{i,j=1}^n\lambda_i^*\lambda_j\,\omega\Big(\Big\{F_N(x_i)\,,\,F_N(x_j)\Big\}\Big) &= \frac{1}{2}\sum_{i,j=1}^n\lambda_i^*\lambda_j\,\omega\Big(\Phi_t^N\Big[\Big\{F_N(x_i)\,,\,F_N(x_j)\Big]\Big\}\Big)\\ &\geq\frac{1}{2}\omega\Big(\Phi_t^N[F_N(q_\lambda^\dagger)]\,\Phi_t^N\left[F_N(q_\lambda)\right]\Big)\,+\,\frac{1}{2}\omega\Big(\Phi_t^N\left[F_N(q_\lambda)\right]\,\Phi_t^N[F_N(q_\lambda^\dagger)]\Big)\\ &=\frac{1}{2}\omega\Big(\Big(\lambda,\mathcal{X}_t\,F_N\Big)\Big(\lambda^*,\mathcal{X}_t\,F_N\Big)\Big)\,+\,\frac{1}{2}\omega\Big(\Big(\lambda^*,\mathcal{X}_t\,F_N\Big)\Big(\lambda,\mathcal{X}_t\,F_N\Big)\Big)\\ &=\frac{1}{2}\sum_{i,j;r,s=1}^n\lambda_i^*\lambda_r\mathcal{X}_t^{ij}\,\mathcal{X}_t^{rs}\omega\Big(\Big\{F_N(x_j)\,,\,F_N(x_s)\Big\}\Big)\ . \end{aligned}$$

In the large N limit one thus obtain, for all $\lambda\in\mathbb{C}^n$,

$$\Big(\lambda,\Sigma^{(\omega)}\,\lambda\Big)\geq\sum_{i,j;r,s=1}^n\lambda_i^*\lambda_r\mathcal{X}_t^{ij}\,\mathcal{X}_t^{rs}\,\Sigma_{js}^{(\omega)}=\Big(\lambda,\mathcal{X}_t\Sigma^{(\omega)}\mathcal{X}_t^{tr}\,\lambda\Big)\ .$$

□

In conclusion, in order to prove (4.22) with (4.23), we need to show that

$$\lim_{N\to+\infty} \omega\left(W_N(s_1)\Big(\Psi_t^N\,[W_N(r)] \,-\, \Phi_t^N\,[W_N(r)]\,\Big)W_N(s_2)\right) = 0, \quad \forall\, s_1, s_2 \in \mathbb{R}^n\,.$$

Actually, like all positive, normalised linear functionals on the Weyl algebra, ω satisfies the Cauchy-Schwartz inequality $|\omega(a^\dagger\, b)|^2 \leq \omega(a^\dagger a)\,\omega(b^\dagger b)$, whence the unitarity of the Weyl-like operators $W_N(r)$ yields

$$\Big|\omega\left(W_N(s_1)\,\Delta_N(t,r)\,W_N(s_2)\right)\Big|^2 \;\leq\; \omega\Big(W_N(s_1)\Delta_N(t,r)\Delta_N^\dagger(t,r)\,W_N^\dagger(s_1)\Big)\,, \tag{4.30}$$

$$\Delta_N(t,r) \;=\; \Psi_t^N[W_N(r)] \,-\, \Phi_t^N\,[W_N(r)]\;. \tag{4.31}$$

In order to show that the right hand side of the above inequality vanishes with $N \to +\infty$, we need relate the interpolating map Ψ_t^N to the local microscopic dissipative dynamics $\Phi_t^N = \mathrm{e}^{t\,\mathbb{L}_N}$; namely, we need study the time-derivative of Ψ_t^N and its relations with the generator $\mathbb{L}_N$. The structure of the time derivative can be derived by means of the following lemma whose proof can be found in Appendix B.

Lemma 4.3. *Let M_t be a time-dependent Hermitian matrix and $N_t = \mathrm{e}^{iM_t}$. Then,*

$$\dot{N}_t := \frac{\mathrm{d}N_t}{\mathrm{d}t} = O_t\,N_t\;, \quad O_t := \sum_{k=1}^{\infty}\frac{i^k}{k!}\mathbb{K}_{M_t}^{k-1}[\dot{M}_t]\;, \tag{4.32}$$

where $\mathbb{K}_{M_t}^n[\dot{M}_t] = \Big[M_t\,,\,\mathbb{K}_{M_t}^{n-1}[\dot{M}_t]\Big]$ and $\mathbb{K}_{M_t}^0[\dot{M}_t] = \dot{M}_t$.

Equipped with this result, we can show that, for large N, all terms in the series expansion of $\dfrac{\mathrm{d}}{\mathrm{d}t}\Psi_t^N[W_N(r)]$ of order larger than 2 vanish in norm.

Proposition 4.1. *For large N, the behaviour of $\dfrac{\mathrm{d}}{\mathrm{d}t}\Psi_t^N[W_N(r)]$ can be approximated by*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^N[W_N(r)] \;\simeq\; &\Bigg(i\,(r_t,(\mathcal{H}\,+\,\mathcal{D})F_N) \,-\, \frac{1}{2}\,[(r_t,F_N)\,,\,(r_t,(\mathcal{H}\,+\,\mathcal{D})F_N)] \,+\\ &+\;\left(r_t,(\mathcal{H}\,+\,\mathcal{D})\,\Sigma^{(\omega)}\,r_t\right)\Bigg)\,\Psi_t^N\,[W_N(r)]\;,\end{aligned} \tag{4.33}$$

the error vanishing in norm.

Proof. The time-derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^N\,[W_N(r)] = \frac{\mathrm{d}f_r(t)}{\mathrm{d}t}\,\Psi_t^N\,[W_N(r)] \,+\, \mathrm{e}^{f_r(t)}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{i(r_t,F_N)}$$

consists of two terms: from (4.28), by direct computation, the first one contains

$$\dot{f}_r(t) = \frac{1}{2}\left(r_t\,,\,\left((\mathcal{H}+\mathcal{D})\Sigma^{(\omega)} \,+\, \Sigma^{(\omega)}(\mathcal{H}+\mathcal{D})^{tr}\right)r_t\right) = \left(r_t\,,\,(\mathcal{H}+\mathcal{D})\Sigma^{(\omega)}\,r_t\right)\;, \tag{4.34}$$

where in the last equality use has been made of the reality of the vector r_t and of the fact that the covariance matrix is real symmetric, namely $\Sigma^{(\omega)} = \left(\Sigma^{(\omega)}\right)^{tr}$. Using Lemma 4.3 and the notation of Definition 3.2, the second term contains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{i(r_t,F_N)} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{i\,F_N(q_{r_t})} = \Big(i\,F_N\left(\dot{q}_{r_t}\right) - \frac{1}{2}\Big[F_N\left(q_{r_t}\right)\,,\,F_N\left(\dot{q}_{r_t}\right)\Big]\Big)\,\mathrm{e}^{i(r_t,F_N)}\\ &+ \sum_{h=3}^{+\infty}\frac{i^h}{h!}\,\mathbb{K}^{h-1}_{F_N(q_{r_t})}\left[F_N(\dot{q}_{r_t})\right]\,\mathrm{e}^{i(r_t,F_N)}\ ,\end{aligned}$$

where $\dot{q}_{r_t} = (\dot{r}_t, F_N) = \Big(r_t, (\mathcal{H}{+}\mathcal{D})F_N\Big)$, and $\mathbb{K}^h_{F_N(q_{r_t})}[F_N(\dot{q}_{r_t})]$, is the multi-commutator defined by

$$\mathbb{K}^h_x\left[z\right] = \left[x, \mathbb{K}^{h-1}_x[z]\right]\ ,\quad \mathbb{K}^0_x[z] = z\ . \tag{4.35}$$

Then, since operators at different sites commute, one estimates

$$\begin{aligned}\left\|\sum_{h=3}^{+\infty}\frac{i^h}{h!}\,\mathbb{K}^{h-1}_{F_N(q_{r_t})}\left[F_N(\dot{q}_{r_t})\right]\,\mathrm{e}^{i(r_t,F_N)}\right\| &\leq \left\|\sum_{h=3}^{+\infty}\frac{1}{h!}\sum_{k=-N}^{N}\frac{1}{N_T^{h/2}}\,\mathbb{K}^{h-1}_{q_{r_t}}\left[\dot{q}^{(k)}_t\right]\right\|\\ &\leq \frac{1}{\sqrt{N_T}}\sum_{h=3}^{+\infty}\frac{(2\|q_{r_t}\|)^{h-1}}{h!}\,\|\dot{q}_{r_t}\| \leq \frac{\mathrm{e}^{2\|q_{r_t}\|}}{\sqrt{N_T}}\,\|\dot{q}_{r_t}\|\ .\end{aligned}$$

The result thus follows as q_{r_t} and $\dot{q}_{r_t}$ are bounded single-site operators for all $t \geq 0$ belonging to finite intervals of time.

□

According to the previous discussion, the convergence of the microscopic dissipative dynamics $\Phi^N_t = \mathrm{e}^{t\mathbb{L}_N}$ to the mesoscopic dissipative dynamics Φ_t in (4.23) amounts to the validity of the following result.

Theorem 4.2. *Given a quantum chain with normal quantum fluctuations* (K,ω)*,* K *a linear set of single-site observables and a local Lindblad generator satisfying assumptions* (4.19),(4.20) *and preserving the microscopic state* ω*, then*

$$\lim_{N\to+\infty}\omega\left(W_N(s_1)\,\Delta_N(t,r)\,\Delta^\dagger_N(t,r)\,W^\dagger_N(s_1)\right) = 0\ , \tag{4.36}$$

where $\Delta_N(t,r) = \Psi^N_t\left[W_N(r)\right]\,-\,\Phi^N_t\left[W_N(r)\right]$ *and* Ψ^N_t *is defined as in* (4.26)–(4.28).

Proof. The first step in the proof is the analysis of the action of the local Lindblad generator (4.20) on the pre-Weyl operator given in (3.29). This requires various technical steps that can be found in Appendix C.
Then, coming to the proof of the Theorem, notice that

$$\begin{aligned}\Psi^N_t\left[W_N(r)\right] - \Phi^N_t\left[W_N(r)\right] &= \int_0^t dy\frac{d}{dy}e^{(t-y)\mathbb{L}_N}\left[\Psi^N_y\left[W_N(r)\right]\right] =\\ &= \int_0^t dy\ \mathrm{e}^{(t-y)\mathbb{L}_N}\left[\frac{d}{dy}\Psi^N_y\left[W_N(r)\right]\,-\,\mathbb{L}_N\left[\Psi^N_y\left[W_N(r)\right]\right]\right]\ .\end{aligned}$$

From Lemma C.1 and using (C.22) (Appendix C) because of the time-invariance of the reference state ω, for large N, one approximates

$$\mathbb{L}_N\left[W_N(r_y)\right] \simeq \left(i\left(r_y, (\mathcal{H}+\mathcal{D})\, F_N\right) + \right.$$

$$\left. -\frac{1}{2}\left[\left(r_y, F_N\right), \left(r_y, (\mathcal{H}+\mathcal{D})F_N\right)\right] + S(r_y;N)\right) W_N(r_y) ,$$

with $S(r_y;N)$ as defined in (C.3), in Appendix C. On the other hand, Lemma 4.3 asserts that the time derivative can be approximated as follows

$$\frac{d}{dy}\Psi_t^N\left[W_N(r_y)\right] \simeq \left(i\left(r_y, (\mathcal{H}+\mathcal{D})F_N\right) - \frac{1}{2}\left[\left(r_y, F_N\right), \left(r_y, (\mathcal{H}+\mathcal{D})F_N\right)\right] + \right.$$

$$\left. + \left(r_y, (\mathcal{H}+\mathcal{D})\,\Sigma^{(\omega)}\, r_y\right)\right) \Psi_t^N\left[W_N(r_y)\right] .$$

Since the errors in these approximations vanish in norm for all finite $t \geq 0$ and Φ_t^N is a contracting map, $\|\Phi_t^N[a^\dagger a]\| \leq \|a\|^2$, what remains to be studied is the quantity

$$\begin{aligned}
&\omega\left(W_N(s_1)\Delta_N(t,r)\,\Delta_N^\dagger(t,r)\,W_N^\dagger(s_1)\right) = \\
&\quad = \int_0^t \mathrm{d}y \int_0^t \mathrm{d}z\; \omega\left(W_N(s_1)\Phi_{t-y}^N\left[D(r_y;N)\right]\,\Phi_{t-z}^N\left[D^\dagger(r_z;N)\right]\,W_N^\dagger(s_1)\right) \\
&\quad D(r_y;N) := Z(r_y;N)\Psi_y^N\left[e^{i(r,F_N)}\right] \\
&\quad Z(r_y;N) := \left(r_y, (\mathcal{H}+\mathcal{D})\,\Sigma^{(\omega)}\, r_y\right) - S(r_y;N) .
\end{aligned}$$

Using the Cauchy-Schwarz inequality (4.30) and then twice the Schwartz positivity inequality (4.24), once for Φ_t^N and the other for Ψ_t^N, the proof of the theorem reduces to bounding

$$\begin{aligned}
&\omega\left(W_N(s_1)\Phi_{t-y}^N\left[D(r_y;N)\,D^\dagger(r_y;N)\right]\right)\,W_N^\dagger(s_1)) \leq \\
&\qquad \leq \mathrm{e}^{2f_r(y)}\,\omega\left(W_N(s_1)\,\Phi_{t-y}^N\left[Z^2(r_y;N)\right]\,W_N^\dagger(s_1)\right) \leq \\
&\qquad \leq \omega\left(W_N(s_1)\,\Phi_{t-y}^N\left[Z^2(r_y;N)\right]\,W_N^\dagger(s_1)\right) .
\end{aligned}$$

Indeed, $Z_N(r_y;N)$ is Hermitian and $\mathrm{e}^{2f_r(y)} \leq 1$ (see Lemma 4.2). Consider now

$$Z^2(r_y;N) = \left(S(r_y;N) - \dot{f}_r(y)\right)^2 , \tag{4.37}$$

where use has been made of (4.34). The operator $S(r_y;N)$ can be written as

$$S(r_y;N) = \frac{1}{N_T}\sum_{k,\ell=-N}^{N} J_{k\ell} \sum_{\alpha,\beta=1}^{d^2} o_\alpha^{(k)} o_\beta^{(\ell)} ,$$

with o_α, o_β suitable single-particle operators, showing that it is a finite summation of terms of the form (C.24). Lemma 4.1 implies that the support of the operators $o_\alpha^{(k)} o_\beta^{(\ell)}$

is not altered by the local dissipative dynamics Φ_t^N so that

$$\Phi_{t-y}^N\left[S(r_y;N)\right] = \frac{1}{N_T}\sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\alpha,\beta=1}^{d^2} c_{k\ell}^{\alpha\beta}(t-y)\, o_\alpha^{(k)} o_\beta^{(\ell)} \ ,$$

where $c_{k\ell}^{\alpha\beta}(t)$ are suitable coefficients, bounded for all finite $t \geq 0$, and $\{o_\alpha\}_{\alpha=1}^{d^2}$ is a single-site operator basis in the algebra $\mathcal{A}_d$. Analogously,

$$\Phi_{t-y}^N\left[S^2(r_y;N)\right] = \frac{1}{N_T^2}\sum_{k,\ell;p,q=-N}^{N}\ \sum_{\alpha\beta;\mu\nu=1}^{d^2} J_{k\ell}\, J_{pq}\, d_{k\ell,pq}^{\alpha\beta,\mu\nu}(t-y)\, o_\alpha^{(k)} o_\beta^{(\ell)}\, o_\mu^{(p)} o_\nu^{(q)} \ ,$$

with $d_{k\ell,p,q}^{\alpha\beta,\mu\nu}(t)$ bounded coefficients for all finite positive times $t \geq 0$.

Lemma C.3 asserts that the sums over k, ℓ and p, q commute with the Weyl-like operators $W_N(r)$ when $N \to +\infty$; then, using the time-invariance under Φ_t^N of the microscopic state ω, we get:

$$\begin{aligned}\lim_{N\to+\infty}\omega\left(W_N(s_1)\Phi_t^N\left[Z^2(r_y;N)\right]W_N^\dagger(s_1)\right) &= \lim_{N\to+\infty}\omega\left(\Phi_t^N\left[Z^2(r_y;N)\right]W_N(s_1)W_N^\dagger(s_1)\right)\\ &= \lim_{N\to+\infty}\omega\left(\Phi_t^N\left[Z^2(r_y;N)\right]\right) = \lim_{N\to+\infty}\omega\left(Z^2(r_y;N)\right) \ .\end{aligned}$$

The proof is thus completed by means of (C.23) and of Proposition C.3 which imply

$$\lim_{N\to+\infty}\omega\left(S^2(r;N)\right) = \left(\lim_{N\to+\infty}\omega\left(S(r;N)\right)\right)^2 = \dot{f}_r^2(t) \ .$$

□

Remark 4.2. *As the dynamical maps Φ_t transform Weyl operators into Weyl operators, their dual maps that act on the states $\hat{\Omega}$ on the Weyl algebra sending them into $\hat{\Omega}_t = \hat{\Omega}\circ\Phi_t$, transform Gaussian states into Gaussian states. For instance, as expected for it emerges from a microscopic time invariant state ω, the state Ω in Theorem 3.2 is left invariant by Φ_t:*

$$\begin{aligned}\Omega\left(\Phi_t\left[W(r)\right]\right) &= \mathrm{e}^{f_r(t)}\,\Omega\left(W(r_t)\right) = \mathrm{e}^{-\frac{1}{2}(r,\mathcal{Y}_t r)-\frac{1}{2}\left(r_t,\Sigma^{(\omega)}\, r_t\right)}\\ &= \mathrm{e}^{-\frac{1}{2}(r,\mathcal{Y}_t r)-\frac{1}{2}\left(r,\mathcal{X}_t\Sigma^{(\omega)}\,\mathcal{X}_t^{tr} r\right)} = \mathrm{e}^{-\frac{1}{2}\left(r,\Sigma^{(\omega)}\, r\right)} = \Omega\left(W(r)\right) \ .\end{aligned}$$

*As they inherit the semigroup property from the Φ_t, the dual maps have a generator and this generator must then be at most quadratic in the mesoscopic operators $F(x_i)$ arising from the local fluctuation operators $F_N(x_i)$. When the anti-symmetric matrix $\sigma^{(\omega)}$ is invertible (*i.e. *$\sigma^{(\omega)}$ is symplectic), the explicit form of the generator is derived by duality and by explicitly computing the time-derivative of $\Phi_t\left[W(r)\right]$, using the Weyl algebraic relations to reconstruct it by means of the action $\mathbb{L}\left[W(r)\right]$: it turns out that the resulting Kossakowski matrix is positive semi-definite so that the maps Φ_t on $\mathcal{W}(K,\sigma^{(\omega)})$ are completely positive.*
In the cases where $\sigma^{(\omega)}$ can not be inverted, thanks to the quasi-free character of the maps Φ_t, complete positivity can nevertheless be proved applying the same strategy adopted in [68], as shown in Appendix D.

Example 4.1. *The following model is useful to understand the main result of this Chapter. Consider a bi-infinite one-dimensional lattice of spin-1 systems. At each site, one has the angular momentum operators* J_1, J_2, J_3*, such that:*

$$[J_1, J_2] = iJ_3\,,$$

and cyclic permutations. A convenient basis, is the one made by the eigenstate of J_3*,*

$$J_3|\mu\rangle = \mu|\mu\rangle, \quad \mu = -1, 0, 1\,;$$

using also the ladder operators $J_\pm = J_1 \pm iJ_2$*, whose action on the previous basis is described by*

$$J_+|1\rangle = J_-|-1\rangle = 0, \quad J_\pm|0\rangle = \sqrt{2}|\pm 1\rangle\,,$$

one obtains the action of all operators

$$J_1|\pm 1\rangle = \tfrac{1}{\sqrt{2}}|0\rangle, \qquad J_1|0\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}\,;$$
$$J_2|\pm 1\rangle = \pm\tfrac{i}{\sqrt{2}}|0\rangle, \qquad J_2|0\rangle = \frac{|1\rangle - |-1\rangle}{\sqrt{2}i}\,.$$

On the quasi-local algebra of such spin-1 chain, we consider the following translation-invariant factorized state ω_β*,*

$$\begin{aligned}\omega_\beta(J_1) &= \omega_\beta(J_2) = 0\\ \omega_\beta(J_3) &= -2\frac{\sinh(\beta\varepsilon)}{1 + 2\cosh(\beta\varepsilon)}\,,\end{aligned} \tag{4.38}$$

corresponding to a tensor product of Gibbs states at inverse temperature β *for each site, with respect to the single-site Hamiltonian* εJ_3*.*
We consider K *as the linear span formed by the elements* J_1, J_2*, and therefore construct the main local fluctuations*

$$F_N(J_{1,2}) = \frac{1}{\sqrt{2N+1}} \sum_{k=-N}^{N} J_{1,2}^{(k)}\,.$$

As reviewed in Chapter 3, these converge in the large N *limit to Bose field operators* $F(J_1), F(J_2)$*, such that*

$$[F(J_1), F(J_2)] = i\omega_\beta(J_3) = i\eta\,,$$

and whose covariance matrix is

$$\Sigma^{(\omega_\beta)} = \frac{1 + \cosh(\beta\varepsilon)}{1 + 2\cosh(\beta\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, introducing the exponentials

$$W_N(r) = \exp\left(i\left(r_1 F_N(J_1) + r_2 F_N(J_2)\right)\right)$$

in the large N *limit they tend to Weyl operators* $W(r)$ *such that*

$$\lim_{N\to\infty} \omega_\beta(W_N(r)) = e^{-\frac{1}{2}\left(r, \Sigma^{(\omega_\beta)} r\right)} = \Omega_\beta(W(r))\,,$$

where Ω_β is the quasi-free state on the Weyl algebra generated by $W(r)$, induced by ω_β. Let us now consider the following dynamical generator

$$\begin{aligned}\mathbb{L}_N\left[X\right] &= i\varepsilon\left[\sum_{k=-N}^{N} J_3^{(k)}, X\right] + \lambda\mathbb{D}_N\left[X\right], \quad \lambda > 0, \\ \mathbb{D}_N\left[X\right] &= \sum_{k=-N}^{N}\left(J_3^{(k)} X J_3^{(k)} - \frac{1}{2}\left\{\left(J_3^{(k)}\right)^2, X\right\}\right);\end{aligned} \tag{4.39}$$

since both state and dynamics are factorized, one can check the time-invariance of the state on single-site operators. The generic operator can be written in term of the basis formed by the eigenvectors $|\mu\rangle$ of J_3. Thus, one has

$$\omega_\beta \circ \mathbb{L}_N\left(|\mu\rangle\langle\nu|^{(k)}\right) = \left(i\varepsilon\left(\mu-\nu\right) + \lambda\left(\mu\nu - \frac{\mu^2+\nu^2}{2}\right)\right)\omega_\beta\left(|\mu\rangle\langle\nu|\right)$$

and $\omega \circ \mathbb{L}_N = 0$ follows from $\omega_\beta\left(|\mu\rangle\langle\nu|\right) = \delta_{\mu\nu}$. This dynamical system obeys the main assumptions under which Theorem 4.2 holds, and computing

$$\mathbb{L}_N\left[r_1 F_N(J_1) + r_2 F_N(J_2)\right] = (r_1, r_2)\begin{pmatrix} -\frac{\lambda}{2} & -\varepsilon \\ \varepsilon & -\frac{\lambda}{2}\end{pmatrix}\begin{pmatrix} F_N(J_1) \\ F_N(J_2)\end{pmatrix},$$

according to equations (C.22), *one has*

$$e^{t\mathbb{L}_N}\left[(r,\, F_N)\right] = (r,\, \mathcal{X}_t\, F_N)\ ,$$

with

$$\mathcal{X}_t = e^{-\frac{\lambda}{2}t}\begin{pmatrix}\cos\left(\varepsilon\, t\right) & -\sin\left(\varepsilon\, t\right) \\ \sin\left(\varepsilon\, t\right) & \cos\left(\varepsilon\, t\right)\end{pmatrix}.$$

Therefore, the generic Weyl operator $W(r)$, evolves according to

$$\Phi_t\left[W(r)\right] = W\left(\mathcal{X}_t^{tr} r\right)\exp\left(-\frac{1-e^{\lambda t}}{2}\left(r, \Sigma^{(\omega_\beta)}\, r\right)\right). \tag{4.40}$$

Notice that

$$\lim_{t\to\infty}\Phi_t\left[W(r)\right] = \mathbf{1}e^{-\frac{1}{2}\left(r,\Sigma^{(\omega_\beta)}\, r\right)};$$

this means that whatever initial state $\tilde{\Omega}$ acting on the Weyl algebra generated by $W(r)$ is chosen, it will asymptotically converge to the thermal one Ω_β; indeed,

$$\lim_{t\to\infty}\tilde{\Omega}\left(\Phi_t\left[W(r)\right]\right) = e^{-\frac{1}{2}\left(r,\Sigma^{(\omega_\beta)}\, r\right)} = \Omega_\beta\left(W(r)\right)$$

which is the characteristic function of the thermal state.

Remark 4.3. *In the unitary case [31, 33, 60], there are situations where the large N dynamical maps on fluctuations are dependent on the reference state of the system ω. Nevertheless, it is known [33] that a factorized Hamiltonian, such as the one in* (4.16), *acting separately on different particles without any interaction among them, determines a dynamics for the fluctuations that depends only on the Hamiltonian itself, with no trace of the microscopic reference state ω.*

This can also be appreciated in the previous Example (4.39)*: indeed, consider the generator* (4.39)*, and set* $\lambda = 0$*. The resulting map (equation* (4.40)*) is*

$$\begin{aligned}\Phi_t\left[W(r)\right] &= e^{ir_1(t)F(J_1)+ir_2(t)F(J_2)}\,,\\ r_1(t) &= r_1\cos(\varepsilon t) + r_2\sin(\varepsilon t)\,,\\ r_2(t) &= r_2\cos(\varepsilon t) - r_1\sin(\varepsilon t)\,.\end{aligned}$$

In these equations, there is no dependence on the state ω_β *of* (4.38)*.*
On the contrary, let us consider the purely dissipative map obtained setting $\varepsilon = 0$*,* $\lambda > 0$ *in the generator* (4.39)*; the dynamics is of the following form*

$$\Phi_t\left[W(r)\right] = W\left(e^{-\frac{\lambda}{2}t}r\right)\exp\left(-\frac{1-e^{\lambda t}}{2}\left(r, \Sigma^{(\omega_\beta)}\, r\right)\right)\,,$$

where the dependence on the reference state ω_β *is embodied by the dependence on the temperature parameter* β *in the exponential factor.*
Therefore, even in the case of a strictly local dissipative evolution, with no mixing effects among different sites, microscopic Lindblad evolutions give rise to dynamical maps on fluctuation operators that are dependent on the reference state ω*.*

Another interesting comment concerns the asymptotic time-behaviour of these evolutions Φ_t (4.23)*, with* $\mathcal{X}_t$*,* $f_r(t)$ *as in* (4.27)*,* (4.28)*, respectively. If the microscopic Lindblad dynamics, generated by* (4.15)*, is such that the matrix* $\mathcal{X}_t$ *of equation* (4.27)*, converges, for large times, in the norm topology, to the null matrix,*

$$\lim_{t\to\infty}\|\mathcal{X}_t\| = 0\,,$$

then, whatever initial state $\tilde{\Omega}$ *on the Weyl algebra* $\mathcal{W}\left(K, \sigma^{(\omega)}\right)$ *is considered, in the large time limit, as a result of theorem 4.2, the characteristic function of fluctuations will converge to*

$$\lim_{t\to\infty}\tilde{\Omega}\left(\Phi_t\left[W(r)\right]\right) = e^{-\frac{1}{2}(r,\Sigma^{(\omega)}\, r)}\,.$$

The right-hand side of the above equation represents the characteristic function of the mesoscopic state Ω *defined by the reference state* ω*, in the sense of Theorem 3.2. This means that, when* $\lim_{t\to\infty}\|\mathcal{X}_t\| = 0$*, any initial mesoscopic state converges, asymptotically in time, to a unique mesoscopic state* Ω *on fluctuation operators, defined by the reference state* ω*.*

In this Chapter we have shown that, in general, given a lattice system with normal quantum fluctuations, in weak interaction with a heat bath in the sense of Section 4.1, the dynamics of fluctuations, provided that the locality condition (4.20) and short-range mixing effects (4.19) are guaranteed, consists of a semigroup of unital, completely positive maps that transform mesoscopic Gaussian states into states of the same kind. In the next Chapter, such a result will be used in order to prove the possibility of entangling two many-body systems through the presence of a common heat bath.

Chapter 5

Environment Induced Entanglement in Many-body Systems

The presence of an external environment typically affects quantum systems in weak interaction with it via loss of quantum correlations due to decohering and mixing-enhancing effects [6–8, 11]. Nevertheless, it has also been established that suitable environments are capable of creating and enhancing quantum entanglement among quantum open sub-systems immersed in them instead of destroying it [14–20]. It is remarkable that entanglement can be generated solely by the mixing structure of the irreversible dynamics, without any environment induced, direct interaction between the quantum sub-systems. This mechanism of environment induced entanglement generation has been studied for systems made of few qubits or oscillator modes [20–22] and specific protocols have been proposed to prepare predefined entangled states via the action of suitably engineered environments [69]. Instead, here, we want to study the possibility that entanglement be created through a purely noisy mechanism in many-body systems.[1] As already outlined in Chapter 3, in a quantum system made of a large number N of constituents, typical accessible observables are collective ones, *i.e.* those involving the degrees of freedom of all its elementary parts. There, it was pointed out that fluctuations retain quantum properties in the infinite number of particles limit, providing a suitable framework where to look for truly quantum behaviours in such systems. The repeated claim of having detected "macroscopic" entanglement in several experiments [36, 37, 74] poses a serious challenge in trying to interpret theoretically those results: fluctuations may indeed play a relevant role [60].
In the following, we shall show that quantum behaviour can indeed be present at the mesoscopic level in open many-body systems and, even more strikingly, that entanglement can be generated in mesoscopic systems by the presence of an external bath. A bipartite many-body system will be considered, with no direct Hamiltonian interaction

[1]For different approaches to entanglement in many-body systems, see [29, 70–73] and references therein.

between the two subsystems, that are nevertheless immersed in a common heat bath. It will be shown that emergent dissipative quantum time-evolutions on fluctuations, as the ones studied in Chapter 4, are capable of entangling fluctuations of the two infinite sub-systems; moreover, such dissipative generated entanglement presents interesting features when studied as a function of the temperature of the heat bath.

5.1 The Model and its Fluctuation Algebra

The model we shall study consists of two infinite quantum spin-$\frac{1}{2}$ chains (namely two one-dimensional bi-infinite lattice systems, supporting on each site a spin-$\frac{1}{2}$ system), whose spins do not directly interact, but are immersed into a same environment and therefore behave as an open quantum system; in the regime of weak coupling, the two chains undergo a microscopic dissipative quantum dynamics described by a semi-group with a generator in Kossakowski-Lindblad form. As discussed in the previous Chapter, this induces a dissipative dynamics of the level of fluctuations; we shall show that, solely because of its statistical mixing properties, this dynamics may induce mesoscopic entanglement between the two spin chains.
Let us describe the system in more detail. At each site of both chains we attach the algebra $M_2(\mathbb{C})$ of complex 2×2 matrices generated by the identity and the Pauli matrices $\sigma_{1,2,3}$ satisfying the algebraic rules

$$[\sigma_i\,,\,\sigma_j] = 2i\epsilon_{ijk}\,\sigma_k\ .$$

We shall pair sites from the two chains so that the single site algebra is $\mathcal{A}_4^{(k)}$, represented by the matrix algebra $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ supported by the k-th sites of the double chain. The *quasi-local* algebra $\mathcal{A}$ describing the double chain will then be the tensor product of the quasi-local algebras of the single chains, with $a \otimes 1$ and $1 \otimes a$ denoting operators pertaining to the first, respectively the second chain. We consider on $\mathcal{A}$ the microscopic thermal tensor product state at inverse temperature β, such that the only non-vanishing single-site expectations are

$$\omega_\beta\Big(\sigma_3^{(j)} \otimes 1\Big) = \omega_\beta\Big(1 \otimes \sigma_3^{(j)}\Big) = -\epsilon\,, \tag{5.1}$$

$$\omega_\beta(\sigma_3^{(j)} \otimes \sigma_3^{(k)}) = \epsilon^2\ . \tag{5.2}$$

where $\epsilon = \tanh(\beta\eta/2)$, with η a positive constant. Namely, each site of the two chains is equipped with a Gibbs state at inverse temperature β defined by the Hamiltonian

$$H = \frac{\eta}{2}\big(\sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3\big)\ . \tag{5.3}$$

Such a state does not support correlations between the two spin chains and manifestly obeys the clustering condition (2.15) of Definition 2.6.

In the following, we shall focus on observables in the real linear span K generated by the set $\{x_j\}_{j=1}^8$ consisting of the following 4×4 Hermitian matrices

$$x_1 = \sigma_1 \otimes \mathbf{1}\ ,\ x_2 = \sigma_2 \otimes \mathbf{1}\ ,\ x_3 = \mathbf{1} \otimes \sigma_1\ ,\ x_4 = \mathbf{1} \otimes \sigma_2 \tag{5.4}$$

$$x_5 = \sigma_1 \otimes \sigma_3\ ,\ x_6 = \sigma_2 \otimes \sigma_3\ ,\ x_7 = \sigma_3 \otimes \sigma_1\ ,\ x_8 = \sigma_3 \otimes \sigma_2\ . \tag{5.5}$$

One easily sees that $\omega_\beta(x_j) = 0$ for all $j = 1, \ldots, 8$, and that the condition (3.31) in Definition 3.3 is satisfied; indeed,

$$\sum_{k=-\infty}^{\infty} \left|\omega_\beta(x_i^{(0)} x_j^{(k)}) - \omega_\beta(x_i)\,\omega_\beta(x_j)\right| = \left|\omega_\beta(x_i x_j)\right| . \tag{5.6}$$

Using the notation of the previous Chapters, in the case of N-site chains the local fluctuation operators corresponding to the observables x_j are given by

$$F_N(x_j) = \frac{1}{\sqrt{N_T}} \sum_{k=-N}^{N} \Big(x_j^{(k)} - \omega(x_j)\Big) = \frac{1}{\sqrt{N_T}} \sum_{k=-N}^{N} x_j^{(k)} . \tag{5.7}$$

Notice that the matrices $x_{1,2}$ and $x_{3,4}$ refer to single sites belonging to different spin chains: their fluctuations will provide collective degrees of freedom associated to the first, the second chain, respectively.[2] The matrix $C^{(\omega_\beta)}$ that in the large N limit gives the correlations of fluctuations corresponding to the operators $\{x_j\}_{j=1}^8$ is the matrix with entries

$$C_{ij}^{(\omega_\beta)} = \lim_{N\to\infty} \omega_\beta\Big(F_N(x_i) F_N(x_j)\Big) . \tag{5.8}$$

Such an 8×8 matrix can be expressed as a three-fold tensor products of 2×2 matrices:

$$C^{(\omega_\beta)} = (\mathbf{1} - \epsilon\,\sigma_1) \otimes \mathbf{1} \otimes (\mathbf{1} + \epsilon\,\sigma_2) . \tag{5.9}$$

In computing tensor products, we adopt the convention in which the entries of a matrix are multiplied by the matrix to its right. According to Chapter 3, the algebraic relations among the emerging mesoscopic bosonic operators $F(x_j)$, to which $F_N(x_j)$ tend in the mesoscopic limit, are described by the symplectic matrix with entries

$$\begin{aligned} \sigma_{ij}^{(\omega_\beta)} &= -i\omega_\beta\big([x_i\,,\,x_j]\big)\,, \\ \sigma^{(\omega_\beta)} &= -2i\epsilon(\mathbf{1} - \epsilon\sigma_1) \otimes \mathbf{1} \otimes \sigma_2\,, \end{aligned} \tag{5.10}$$

while the covariance matrix of these bosonic degrees of freedom is

$$\begin{aligned} \Sigma_{ij}^{(\omega_\beta)} &= \frac{1}{2}\,\omega_\beta\big(\,\{x_i\,,\,x_j\}\big)\,, \\ \Sigma^{(\omega_\beta)} &= \frac{1}{2}\left(C^{(\omega_\beta)} + (C^{(\omega_\beta)})^{tr}\right) = (\mathbf{1} - \epsilon\sigma_1) \otimes \mathbf{1} \otimes \mathbf{1} . \end{aligned} \tag{5.11}$$

The inverse of the symplectic matrix $\sigma^{(\omega_\beta)}$ can be computed and one explicitly finds:

$$(\sigma^{(\omega_\beta)})^{-1} = \frac{1}{2c^2\epsilon}\,(\mathbf{1} + \epsilon\sigma_1) \otimes \mathbf{1} \otimes\, i\sigma_2\,, \qquad c = \sqrt{1-\epsilon^2} . \tag{5.12}$$

As discussed in Chapter 3, it is useful to introduce pre-Weyl operators (3.29),

$$W_N(r) = \mathrm{e}^{iF_N(q_r)} = \mathrm{e}^{i(r\,,\,F_N)} \tag{5.13}$$

$$(r\,,\,F_N) = \sum_{j=1}^{8} r_j\,F_N(x_j) = F_N(q_r)\,, \tag{5.14}$$

[2] There are 16 single site independent observables of the form $\sigma_\mu \otimes \sigma_\nu$, $\mu, \nu = 0, 1, 2, 3$, $\sigma_0 = \mathbf{1}$. It turns out that fluctuations corresponding to the chosen subset $\{x_j\}_{j=1}^8$ give rise to a set of mesoscopic bosonic operators, whose Weyl algebra commutes with the one generated by the remaining eight elements.

where $F_N = \{F_N(x_j)\}_{j=1}^8$ is the vector of local fluctuations and the vector r is now 8-dimensional, $r \in \mathbb{R}^8$. In the mesoscopic, large N limit, they define the Weyl operators $W(r)$, forming the fluctuation algebra $\mathcal{W}(K, \sigma^{(\omega_\beta)})$. Thanks to the regularity of the quasi-free state (see Theorem 3.2), one has the representation

$$\pi\left(W(r)\right) = \mathrm{e}^{iF(q_r)} = \mathrm{e}^{i\sum_{j=1}^{8} r_j\, F(x_j)} = \mathrm{e}^{i(r\,,F)}\ , \qquad q_r = \sum_{j=1}^{8} r_j\, x_j\ , \tag{5.15}$$

where F is the eight-dimensional operator valued vector with components $F(x_j)$, $1 \leq j \leq 8$. From (3.15) and (5.10), one also finds:

$$\pi\left(W(r)\right)\, F(x_i)\, \pi\left(W^\dagger(r)\right) = F(x_i) + i\big[(r, F)\,,\, F(x_i)\big] = F(x_i) + \sum_{j=1}^{8} \sigma_{ij}^{(\omega_\beta)}\, r_j\ . \tag{5.16}$$

The Weyl algebraic structure associated with the chosen set K of local observables and the thermal state ω_β allow for the mesoscopic description to be formulated in terms of four-mode bosonic annihilation and creation operators $a_i^\# \equiv (a_i,\, a_i^\dagger)$, $1 \leq i \leq 4$, satisfying the canonical commutation relations

$$[a_i\,,\, a_j^\dagger] = \delta_{ij}\ , \quad [a_i\,,\, a_j] = [a_i^\dagger\,,\, a_j^\dagger] = 0\ . \tag{5.17}$$

Indeed, introducing the following four-dimensional complex vectors $f_i \in \mathbb{C}^4$,

$$f_1 = \sqrt{\epsilon}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\ , \qquad f_3 \;=\; \sqrt{\epsilon}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\ , \tag{5.18}$$

$$f_5 = \sqrt{\epsilon}\begin{pmatrix}-\epsilon\\ \sqrt{1-\epsilon^2}\\ 0\\ 0\end{pmatrix}\ , \qquad f_7 \;=\; \sqrt{\epsilon}\begin{pmatrix}0\\ 0\\ -\epsilon\\ \sqrt{1-\epsilon^2}\,,\end{pmatrix} \tag{5.19}$$

and $f_{2j} = -if_{2j-1}$, $j = 1, 2, 3, 4$, one can obtain the commutation relations of the bosonic degrees of freedom as

$$\big[F(x_i)\,,\, F(x_j)\big] = 2\,i\,\mathcal{I}m\left((f_i, f_j)\right)\ , \quad (f_i, f_j) = \epsilon\, \Sigma_{ij}^{(\omega_\beta)} + \frac{i}{2}\sigma_{ij}^{(\omega_\beta)}\ , \tag{5.20}$$

thus showing that these can be written in terms of the four modes $a_i^\#$, $1 \leq i \leq 4$,

$$F(x_i) = a(f_i) + a^\dagger(f_i)\ , \quad a^\dagger(f_i) = \sum_{j=1}^{4} [f_i]_j\, a_j^\dagger\ ,\ 1 \leq i \leq 8\ . \tag{5.21}$$

Setting

$$B_i = (a_i, a_i^\dagger)^{tr}\ , \quad A = (B_1, B_2, B_3, B_4)^{tr}\ , \tag{5.22}$$

one has

$$F = \mathcal{M}\, A\,, \tag{5.23}$$

where

$$\mathcal{M} = \sqrt{\epsilon} \begin{pmatrix} \mathcal{M}_1 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M}_1 & 0 \\ -\epsilon\mathcal{M}_1 & c\mathcal{M}_1 & 0 & 0 \\ 0 & 0 & -\epsilon\mathcal{M}_1 & c\mathcal{M}_1 \end{pmatrix} , \tag{5.24}$$

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} .$$

The 8×8 matrix $\mathcal{M}$ can be inverted and used to write $A = \mathcal{M}^{-1}F$:

$$\mathcal{M}^{-1} = \begin{pmatrix} \frac{1}{2\sqrt{\epsilon}}\mathcal{M}_1^\dagger & 0 & 0 & 0 \\ \frac{\sqrt{\epsilon}}{2c}\mathcal{M}_1^\dagger & 0 & \frac{1}{2c\sqrt{\epsilon}}\mathcal{M}_1^\dagger & 0 \\ 0 & \frac{1}{2\sqrt{\epsilon}}\mathcal{M}_1^\dagger & 0 & 0 \\ 0 & \frac{\sqrt{\epsilon}}{2c}\mathcal{M}_1^\dagger & 0 & \frac{1}{2c\sqrt{\epsilon}}\mathcal{M}_1^\dagger \end{pmatrix} . \tag{5.25}$$

From the structure of $\mathcal{M}^{-1}$, one notices that the creation and annihilation operators $a_1^\#$, respectively $a_3^\#$ come from single site operators $x_{1,2}$, respectively $x_{3,4}$, pertaining to the first, respectively the second chain. Then, $a_1^\#$ and $a_3^\#$ describe two independent mesoscopic degrees of freedom emerging from different chains. Instead, $a_2^\#$ and $a_4^\#$ result from combinations of spin operators involving both chains at the same time.

Remark 5.1. *If the temperature vanishes, i.e.* $\epsilon = 1$, $c = 0$, *the matrix* $\mathcal{M}$ *becomes singular, thus showing that the representation in terms of bosonic creation and annihilation operators can be reduced. Indeed, in such a degenerate case, only the following two bosonic modes can be accommodated:*

$$a_1^\dagger = \frac{F(x_1) \,+\, i\,F(x_2)}{2} \ , \quad a_2^\dagger = \frac{F(x_3) \,+\, i\,F(x_4)}{2} \ . \tag{5.26}$$

This degeneracy is due to a so-called coarse graining effect [31] that prevents distinguishing the mesoscopic limits of distinct fluctuation operators. In other terms, it may happen that

$$\lim_{N\to\infty} \omega\Big(\big[F_N(q_{r_1}) - F_N(q_{r_2})\big]^2 \Big) = 0 \ ,$$

even when $q_{r_1} \neq q_{r_2}$.

In the creation and annihilation operator formalism, the Weyl operators become displacement operators $D(z)$ labelled by complex vectors $z \in \mathbb{C}^4$.
Let $\mathcal{Z}_i = (z_i, z_i^*)^{tr}$, and $Z = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)^{tr} \in \mathbb{C}^8$ and Σ_3 denote the diagonal 8×8 matrix $\mathrm{diag}(-1, 1, -1, 1, -1, 1, -1, 1)$; then,

$$D(z) := \mathrm{e}^{(Z, \Sigma_3\, A)} = \exp\left(\sum_{j=1}^4 \Big(z_j\, a_j^\dagger - z_j^*\, a_j \Big) \right) . \tag{5.27}$$

Lemma 5.1. *Given the creation and annihilation operators* $a_i^\#$, $1 \leq i \leq 4$, *Weyl and displacement operators are related by*

$$W(r) \;=\; \mathrm{e}^{i(r,F)} = D(z_r) \ , \qquad Z_r = \begin{pmatrix} z_r \\ z_r^* \end{pmatrix} = -i\Sigma_3\, \mathcal{M}^\dagger\, r \tag{5.28}$$

$$D(z) \;=\; W(r_z) \ , \quad r_z = i(\mathcal{M}^\dagger)^{-1}\Sigma_3\, Z \ . \tag{5.29}$$

According to Theorem 3.2, the mesoscopic algebra $\mathcal{W}(K,\,\sigma^{(\omega_\beta)})$ inherits a regular quasi-free state from the microscopic state ω_β.

Proposition 5.1. *The quasi-free state Ω_β on the Weyl algebra of quantum fluctuations $\mathcal{W}(K,\sigma^{(\omega_\beta)})$ is such that*

$$\Omega_\beta(W(r)) = \exp\Big(-\frac{1}{2}(r\,,\Sigma^{(\omega_\beta)}\,r)\Big)\ , \tag{5.30}$$

with covariance matrix $\Sigma^{(\omega_\beta)}$ given by (5.11)*. In the creation and annihilation operator formalism, it amounts to the expectation functional $\Omega_\beta(W(r)) = \mathrm{Tr}(R_\beta\,W(r))$, where*

$$R_\beta = \frac{\mathrm{e}^{-\beta\,\hat{H}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\,\hat{H}}\right)}\ ,\qquad \hat{H} = \eta\sum_{j=1}^{4} a_j^\dagger a_j\ , \tag{5.31}$$

namely to a Gibbs state at inverse temperature β with respect to the quadratic Hamiltonian $\hat{H}$.

Proof. The tensor product structure and translation-invariance of ω_β yield

$$\begin{aligned}\omega_\beta\left(W_N(r)\right) &= \left(\omega_\beta\left(\mathrm{e}^{i/\sqrt{N}\sum_{j=1}^{8} r_j\,x_j}\right)\right)^N\\ &= \left(1-\frac{1}{2N}\sum_{i,j=1}^{8} r_i r_j \omega_\beta(x_i\,x_j)\,+\,o\left(\frac{1}{N}\right)\right)^N\ ,\end{aligned}$$

whence, since $r\in\mathbb{R}^8$,

$$\lim_{N\to\infty}\omega_\beta\big(W_N(r)\big) = \lim_{N\to\infty}\omega_\beta\left(\mathrm{e}^{i(r\,,F_N)}\right) = \exp\Big(-\frac{1}{2}(r\,,\Sigma^{(\omega_\beta)}\,r)\Big)\ .$$

On the other hand, writing $W(r)$ as a displacement operator $D(z_r)$, from (5.28), its expectation with respect to the state Ω_β reads

$$\Omega_\beta(W(r)) = \exp\Big(-\frac{\|Z_r\|^2}{4\epsilon}\Big) = \exp\Big(-\frac{\sum_{i,j=1}^{8} r_i r_j\,(f_i,f_j)}{2\epsilon}\Big)\ .$$

Then, the result follows from $(f_i,f_j) = \epsilon\,\Sigma_{ij}^{(\omega_\beta)} + \frac{i}{2}\sigma_{ij}^{(\omega_\beta)}$ of (5.20), and noticing that r is a real-valued vector and $\sigma^{(\omega_\beta)}$ an anti-symmetric matrix.

□

5.2 Dissipative Dynamics of Gaussian Fluctuations and Entanglement

We shall focus upon dissipative dynamics of the type studied in the previous Chapter; therefore, given our set $\{x_i\}_{i=1}^8$, we assume the dynamics to be generated by a Lindblad map of the type (4.15),(4.16),(4.17) obeying assumptions (4.19),(4.20). The previous Theorem 4.2 shows that, when the linear space K of selected single-site operators is stable under the action of the local Lindblad generator, then the emergent mesoscopic irreversible dynamics maps Weyl operators into themselves: it corresponds to a semigroup of unital, completely positive maps on the Weyl algebra $\mathcal{W}(K,\,\sigma^{(\omega_\beta)})$. Since in the present case the matrix $\sigma^{(\omega_\beta)}$ is invertible, one is able to provide a general form of the bosonic Lindblad generator, at most quadratic in the fluctuation operators $F(x_i)$, that implements such dynamics.

Corollary 5.1. *The maps* $\mathcal{W}(K,\,\sigma^{(\omega_\beta)}) \ni W(r) \mapsto \Phi_t[W(r)] = W_t(r) = \mathrm{e}^{f_r(t)}\, W(r_t)$ *with* $r_t \in \mathbb{R}^8$ *and* $f_r(t)$ *given by* (4.27)*, respectively* (4.28)*, satisfy the time-evolution equation* $\partial_t W_t(r) = \mathbb{L}[W_t(r)]$*, where the generator* $\mathbb{L}$ *is given by*

$$\mathbb{L}[W_t(r)] = \frac{i}{2}\sum_{i,j=1}^{8} H^{(1)}_{ij}\big[F(x_i)F(x_j)\,,\,W_t(r)\big] \tag{5.32}$$

$$+\sum_{i,j=1}^{8} D^{(1)}_{ij}\left(F(x_i)\,W_t(r)\,F(x_j) \,-\, \frac{1}{2}\big\{F(x_i)F(x_j)\,,\,W_t(r)\big\}\right), \tag{5.33}$$

with $H^{(1)}$ *a Hermitian* 8×8 *matrix and* $D^{(1)}$ *a positive semi-definite* 8×8 *Hermitian matrix, given by*

$$H^{(1)} = -i(\sigma^{(\omega_\beta)})^{-1}\left(\mathcal{L}\,C^{(\omega_\beta)} \,-\, C^{(\omega_\beta)}\,\mathcal{L}^{tr}\right)(\sigma^{(\omega_\beta)})^{-1}\,, \tag{5.34}$$

$$D^{(1)} = (\sigma^{(\omega_\beta)})^{-1}\left(\mathcal{L}\,C^{(\omega_\beta)} \,+\, C^{(\omega_\beta)}\mathcal{L}^{tr}\right)(\sigma^{(\omega_\beta)})^{-1}\,. \tag{5.35}$$

In the creation and annihilation operator formalism, using the notation introduced in (5.22)*, the generator reads*

$$\mathbb{L}[D_t(z)] = \frac{i}{2}\sum_{i,j=1}^{8} H^{(2)}_{ij}\left[A_i^\dagger\,A_j\,,\,D_t(z)\right] \tag{5.36}$$

$$+\sum_{i,j=1}^{8} D^{(2)}_{ij}\left(A_i^\dagger\,D_t(z)\,A_j \,-\, \frac{1}{2}\left\{A_i^\dagger A^\dagger\,,\,D_t(z)\right\}\right), \tag{5.37}$$

where $D_t(z)$ *is the time-evolved displacement operator* (5.29) *corresponding to the time-evolved Weyl operator* $W_t(r)$ *and* $H^{(2)}$ *and* $D^{(2)}$ *are* 8×8 *matrices, given by*

$$H^{(2)} = \mathcal{M}^\dagger\,H^{(1)}\,\mathcal{M}\,,\qquad D^{(2)} = \mathcal{M}^\dagger\,D^{(1)}\,\mathcal{M}\,, \tag{5.38}$$

where $\mathcal{M}$ *is the matrix in* (5.24)*.*

Proof. Using Lemma 4.3, the explicit expressions for $\dot r_t$, $f_r(t)$ and the relation

$$C^{(\omega_\beta)} = \Sigma^{(\omega_\beta)} + \frac{i}{2}\sigma^{(\omega_\beta)}\,,$$

one computes

$$\begin{aligned}
\partial_t W_t(r) &= \left(\dot f_r(t) \,+\, i(\dot r_t, F) \,-\, \frac{1}{2}\big[(r_t, F)\,,\,(\dot r_t, F)\big]\right) W_t(r)\\
&= \left(i(r_t, \mathcal{L}\,F) \,+\, (r_t, \mathcal{L}\,\Sigma^{(\omega_\beta)} r_t) \,+\, \frac{i}{2}(r_t, \mathcal{L}\,\sigma^{(\omega_\beta)} r_t)\right) W_t(r)\\
&= \left(i(r_t, \mathcal{L}\,F) \,+\, (r_t, \mathcal{L}\,C^{(\omega_\beta)} r_t)\right) W_t(r)\,.
\end{aligned}$$

In order to show how to match this time-derivative with the action on $W_t(r)$ of a linear map as in the statement of the Corollary, it is useful to recall (5.16), which gives

$$W_t(r)\,F(x_i) = \left(F(x_i)\,+\,\sum_{j=1}^{8}\sigma^{(\omega_\beta)}_{ij}\,r_{t_j}\right)\,W_t(r)\ .$$

It is then straightforward to derive that

$$\begin{aligned}\mathbb{L}[W_t(r)] &= \frac{i}{2}\Big(\,\big(r_t,\sigma^{(\omega_\beta)}\big(H^{(1)}+(H^{(1)})^{tr}\big)F\big)\,+\,\big(r_t,\sigma^{(\omega_\beta)}\,H^{(1)}\,\sigma^{(\omega_\beta)}r_t\big)\,\Big)\,W_t(r)\\ &+\ \frac{1}{2}\Big(\,\big(r_t,\sigma^{(\omega_\beta)}\big(D^{(1)}-(D^{(1)})^{tr}\big)F\big)\,+\,\big(r_t,\sigma^{(\omega_\beta)}\,D^{(1)}\,\sigma^{(\omega_\beta)}r_t\big)\,\Big)\,W_t(r)\ .\end{aligned}$$

By equating the operatorial, respectively the scalar contributions, from the time-derivative and the generator action, one obtains

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sigma^{(\omega_\beta)}\left(H^{(1)}\,+\,(H^{(1)})^{tr}\right)\,-\,\frac{i}{2}\sigma^{(\omega_\beta)}\left(D^{(1)}-(D^{(1)})^{tr}\right)\\ \mathcal{L}\,C^{(\omega_\beta)} &= \sigma^{(\omega_\beta)}\,\frac{i\,H^{(1)}\,+\,D^{(1)}}{2}\,\sigma^{(\omega_\beta)}\ ,\end{aligned}$$

whence, by the invertibility of $\sigma^{(\omega_\beta)}$ (see (5.12)), the hermiticity of $C^{(\omega_\beta)}$ and the the fact that $\mathcal{L}^\dagger = \mathcal{L}^{tr}$ (since it is a real matrix), the result follows from

$$\mathcal{L}\,C^{(\omega_\beta)}\pm C^{(\omega_\beta)}\mathcal{L}^{tr} = \sigma^{(\omega_\beta)}\left(\frac{i\,H^{(1)}\,+\,D^{(1)}}{2}\,\mp\,\frac{i\,H^{(1)}\,-\,D^{(1)}}{2}\right)\,\sigma^{(\omega_\beta)}\ .$$

The second part of the corollary follows from using (5.23) and inserting it into (5.32) and (5.33)

$$F(x_i) = F^\dagger(x_i) = \sum_{k=1}^{8}\mathcal{M}^*_{ik}\,A^\dagger_k\ ,\quad F(x_j) = \sum_{\ell=1}^{8}\mathcal{M}_{i\ell}\,A_\ell\ .$$

□

5.2.1 Quasi-Free States

The mesoscopic dissipative dynamics Φ_t obtained in the previous Chapter is quasi-free as it maps Weyl operators into Weyl operators. The dual maps Φ^*_t acts on the states ρ on the Weyl algebra $\mathcal{W}(K,\,\sigma^{(\omega_\beta)})$, sending them into $\rho_t = \Phi^*_t[\rho]$ according to the duality relation

$$\rho_t(W(r)) = \rho\big(\Phi_t[W(r)]\big)\ ,\qquad \forall\,W(r)\in\mathcal{W}(K,\,\sigma^{(\omega_\beta)})\ . \tag{5.39}$$

Particularly useful states on $\mathcal{W}(K,\,\sigma^{(\omega_\beta)})$ are the Gaussian ones ρ_G which are identified by their characteristic functions being Gaussian, *i.e.* by the following expectations of Weyl operators[3]

$$\rho_G\big(W(r)\big) = \rho_G\left(\mathrm{e}^{i(r,F)}\right) = \exp\left(-\frac{1}{2}(r,G\,r)\right)\ ,\qquad \forall r\in\mathbb{R}^8\ , \tag{5.40}$$

$$G = [G_{ij}]\ ,\quad G_{ij} = \frac{1}{2}\rho_G\left(\Big\{F(x_i)\,,\,F(x_j)\Big\}\right)\,,\ i,j = 1,2\ldots,8\ . \tag{5.41}$$

[3]For simplicity we limit the discussion to Gaussian states with zero averages.

These states are completely identified by their covariance matrix G; in particular, as already observed in Chapter 3, positivity of ρ_G is equivalent to the following condition on G [56, 75]:

$$G + \frac{i}{2}\sigma^{(\omega_\beta)} \geq 0 \ , \tag{5.42}$$

where $\sigma^{(\omega_\beta)}$ is the symplectic matrix in (5.10). Clearly, the maps Φ_t^* transform Gaussian states into Gaussian states:

$$\begin{aligned} \Phi_t^*[\rho_G](W(r)) &= \rho_G\Big(\Phi_t[W(r)]\Big) = \mathrm{e}^{f_r(t)}\,\rho_G\Big(W(r_t)\Big) \\ &= \exp\left(f_r(t) - \frac{1}{2}(r_t, G\, r_t)\right) = \rho_{G_t}\Big(W(r)\Big) \ , \end{aligned} \tag{5.43}$$

with the time-dependent covariance matrix G_t obtained recalling (4.36) of Theorem 4.2, together with (4.26)–(4.28):

$$G_t = \Sigma^{(\omega_\beta)} \ - \ \mathrm{e}^{t\mathcal{L}}\,\Sigma^{(\omega_\beta)}\,\mathrm{e}^{t\mathcal{L}^{tr}} + \mathrm{e}^{t\mathcal{L}}\,G\,\mathrm{e}^{t\mathcal{L}^{tr}} \ . \tag{5.44}$$

It follows that the mesoscopic state Ω_β in (5.30) is Gaussian with covariance matrix $G = \Sigma^{(\omega_\beta)}$ and thus, as the microscopic state ω_β is invariant under the local dissipative dynamics Φ_t^N, Ω_β is invariant under the mesoscopic dissipative dynamics Φ_t^*, *i.e.* $G_t = \Sigma^{(\omega_\beta)}$.

A useful equivalent expression for the covariance matrix can be obtained by passing to the language of creation and annihilation operators. The expectation of the displacement operator $D(z)$ with respect to a Gaussian state ρ_G reads

$$\rho_G\left(D(z)\right) = \exp\left(-\frac{1}{2}(Z, \tilde{G}, Z)\right) \ , \tag{5.45}$$

with the new covariance matrix $\tilde{G}$ explicitly given by

$$\tilde{G} = \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} & \tilde{G}_{13} & \tilde{G}_{14} \\ \tilde{G}_{21} & \tilde{G}_{22} & \tilde{G}_{23} & \tilde{G}_{24} \\ \tilde{G}_{31} & \tilde{G}_{32} & \tilde{G}_{33} & \tilde{G}_{34} \\ \tilde{G}_{41} & \tilde{G}_{42} & \tilde{G}_{43} & \tilde{G}_{44} \end{pmatrix} \ , \tag{5.46}$$

where

$$\tilde{G}_{ij} = \frac{1}{2}\begin{pmatrix} \rho_G\left(\{a_i, a_j^\dagger\}\right) & -\rho_G\left(\{a_i, a_j\}\right) \\ -\rho_G\left(\{a_i^\dagger, a_j^\dagger\}\right) & \rho_G\left(\{a_i^\dagger, a_j\}\right) \end{pmatrix} , \qquad i, j = 1, 2, 3, 4 \, . \tag{5.47}$$

The 2×2 matrices along the diagonal represent single-mode covariance matrices, while the off-diagonal ones account for correlations among the various modes.

5.2.2 Entanglement Measure in Bipartite Gaussian States

Using the previous results, and in particular the quasi-free property of the maps Φ_t, we want now to study: 1) whether it is possible to generate mesosocopic entanglement between different chains entirely by means of the dissipative microscopic dynamics and further 2) investigate the fate of the generated entanglement in the course of time and of its dependence on the strength of the coupling with the environment and on the temperature of the given microscopic invariant state.

By *mesoscopic entanglement* we mean the existence of mesoscopic states carrying non-local, quantum correlations among the fluctuation operators pertaining to different chains. More precisely, we shall focus on the creation and annihilation operators $a_1^\#$ and $a_3^\#$ that, as observed before, are collective degrees of freedom attached to the first, second chain, respectively. We shall then study the time-evolution of two-mode Gaussian states $\rho^{(13)}$, obtained by tracing a full four-mode Gaussian state over $a_2^\#$ and $a_4^\#$. In the case of two-mode Gaussian states, the presence of entanglement can be ascertained using the partial transposition criterion, *i.e.* by looking at their behaviour when a_1 and $a_1^\dagger$ are exchanged while keeping $a_1^\dagger a_1$ and $a_1 a_1^\dagger$ unchanged and without touching a_3 and $a_3^\dagger$. If under this substitution, $\rho^{(13)}$ does not remain a positive functional, then it carries quantum correlations between the modes 1 and 3 and thus results entangled. Vice versa, a Gaussian state with respect to these two modes that remains positive under the above substitution is for sure separable. This is the content of the so-called Simon entanglement criterion [76]. Notice that the state Ω_β in (5.30), besides being time-invariant, is separable with respect to all its four modes; indeed, its density matrix representation R_β in (5.31) can be written as a product of four independent density matrices one for each of the modes. The corresponding covariance matrix $\widetilde{\Sigma}^{(\beta)}$ results diagonal when expressed in the representation (5.45), (5.46), thus showing neither quantum nor classical correlations between the different modes. In order to obtain a non-trivial mesoscopic dynamics, we shall consider initial states that are obtained from R_β by the action of suitable squeezing operators in the modes 1 and 3, *i.e.* Gaussian states of the form

$$\rho^{(\beta)}_{r_1 r_3} = S_1(r_1) S_3(r_3)\, R_\beta\, S_3^\dagger(r_3) S_1^\dagger(r_1)\ , \tag{5.48}$$

where $S_j(r_j)$, $r_j \in \mathbb{R}$, are single-mode squeezing operators such that

$$S_j^\dagger(r_j)\, a_j^\dagger\, S_j(r_j) = \cosh(r_j)\, a_j^\dagger \,-\, \sinh(r_j)\, a_j\ , \quad j = 1, 3\ .$$

The squeezing operators map displacement operators $D(z)$ in (5.27) into displacement operators

$$D(z') = S_3^\dagger(r_3) S_1^\dagger(r_1)\, D(z)\, S_1(r_1) S_3(r_3)\ ,$$

where $z' = (z_1', z_2, z_3', z_4)$ with $z_{1,3}' = \cosh(r_{1,3}) z_{1,3} - \sinh(r_{1,3}) \bar{z}_{1,3}$. Further, the modes are not mixed by the squeezing so that $\rho^{(\beta)}_{r_1 r_3}$ is also a separable Gaussian state relatively to all four modes. In particular, after squeezing, the 8×8 covariance matrix $\widetilde{\Sigma}^{(\beta)}$ of the thermal state R_β is mapped into the following one:

$$\widetilde{\Sigma}^{(\beta)}_{r_1,r_3} = \frac{1}{2\epsilon}\begin{pmatrix} \mathcal{S}(r_1) & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathcal{S}(r_3)\end{pmatrix}\ , \qquad \mathcal{S}(r) = \begin{pmatrix} \cosh(2r) & -\sinh(2r) & 0 & 0 \\ -\sinh(2r) & \cosh(2r) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{5.49}$$

where $\mathbf{0}_4$ is the null matrix in four dimensions; the block diagonal form, again shows that also these squeezed states do not carry correlations between the modes.

Moreover, a state $\rho^{(13)}$ on the Bose algebra generated by $a^{\#}_{1,3}$ can be obtained from $\rho^{(\beta)}_{r_1 r_3}$ by restricting its action on displacement operators of the form $D(z_{13})$ with $z_{13} = (z_1, 0, z_3, 0)$. Namely, $\rho^{(13)}$ is completely defined by the expectations

$$\rho^{(13)}\big(D(z_{13})\big) = \mathrm{Tr}\left(\rho^{(\beta)}_{r_1 r_3}\, D(z_{13})\right) = \mathrm{Tr}\big(R_\beta\, D(z'_{13})\big) \ , \tag{5.50}$$

and then inherits the Gaussian character of R_β as these expectations are Gaussian functions of $z_{1,3}$. Finally, the same argument shows that the mesoscopic, dissipative time-evolution Φ_t transforms it in a Gaussian state at all times $t \geq 0$:

$$\rho^{(13)}_t\left(D(z_{13})\right) = \mathrm{Tr}\Big(\rho^{(\beta)}_{r_1 r_3}\, \Phi_t\big[D(z_{13})\big]\Big) \ . \tag{5.51}$$

In practice, the covariance matrix of interest, that involves only the modes $1, 3$, can be retrieved from the total matrix in the form (5.46) by discarding the blocks relative to modes $2, 4$. Explicitly,

$$\begin{aligned} \widetilde{G}_{red}(t) &= \begin{pmatrix} \rho^{(13)}_t(a_1^\dagger a_1) + \frac{1}{2} & -\rho^{(13)}_t(a_1^2) & \rho^{(13)}_t(a_1 a_3^\dagger) & -\rho^{(13)}_t(a_1 a_3) \\ -\rho^{(13)}_t(a_1^{\dagger 2}) & \rho^{(13)}_t(a_1^\dagger a_1) + \frac{1}{2} & -\rho^{(13)}_t(a_1^\dagger a_3^\dagger) & \rho^{(13)}_t(a_1^\dagger a_3) \\ \rho^{(13)}_t(a_1^\dagger a_3) & -\rho^{(13)}_t(a_1 a_3) & \rho^{(13)}_t(a_3^\dagger a_3) + \frac{1}{2} & -\rho^{(13)}_t(a_3^2) \\ -\rho^{(13)}_t(a_1^\dagger a_3^\dagger) & \rho^{(13)}_t(a_1 a_3^\dagger) & -\rho^{(13)}_t(a_3^{\dagger 2}) & \rho^{(13)}_t(a_3^\dagger a_3) + \frac{1}{2} \end{pmatrix} \\ \widetilde{G}_{red}(t) &\equiv \begin{pmatrix} \Sigma_1 & \Sigma_c \\ \Sigma_c^\dagger & \Sigma_2 \end{pmatrix} \ . \end{aligned} \tag{5.52}$$

For two mode-Gaussian states, the already mentioned Simon's criterion not only provides an exhaustive entanglement witness, but it also offers a means to quantify it [76]. It is nevertheless convenient to formulate the criterion in terms of the previous covariance matrix [77]. Consider the block structure of $\widetilde{G}_{red}(t)$ and define:

$$\begin{gathered} I_1 = \det(\Sigma_1) \ , \qquad I_2 = \det(\Sigma_2) \qquad I_3 = \det(\Sigma_c) \ , \\ I_4 = \mathrm{Tr}\Big(\Sigma_1 \sigma_3 \Sigma_c \sigma_3 \Sigma_2 \sigma_3 \Sigma_c^\dagger \sigma_3\Big) \ . \end{gathered} \tag{5.53}$$

Then, the necessary and sufficient condition for a state to be separable is:

$$S \equiv I_1 I_2 + \Big(\frac{1}{4} - |I_3|\Big)^2 - I_4 - \frac{(I_1 + I_2)}{4} \geq 0 \ . \tag{5.54}$$

Further, the amount of entanglement in two-mode Gaussian states can be measured through the so-called logarithmic negativity of the state:

$$E = \max\left\{0, -\frac{1}{2}\log_2\left(4\mathcal{I}\right)\right\} \ , \tag{5.55}$$

where

$$\mathcal{I} = \frac{I_1 + I_2}{2} - I_3 - \left(\left[\frac{I_1 + I_2}{2} - I_3\right]^2 - \left(I_1 I_2 + I_3^2 - I_4\right)\right)^{1/2} \ . \tag{5.56}$$

5.3 Witnessing Environment Induced Mesoscopic Entanglement

In the following we shall apply the theoretical tools developed so far to the study of the dissipative generation of mesoscopic entanglement in two different models: in the first one, the microscopic Lindblad generator contains contributions involving single-site operators from both chains, while in the second one all terms contain single-site operators from one chain only.

Model 1 We shall consider a Lindblad generator of the form (4.15), with Hamiltonian term

$$\mathbb{H}_N[X] = i\big[H_N,\, X\big] \ , \qquad H_N = \frac{\eta}{2}\sum_{k=-N}^{N} h^{(k)} \ , \quad h^{(k)} = \sigma_3^{(k)} \otimes \mathbf{1}^{(k)} + \mathbf{1}^{(k)} \otimes \sigma_3^{(k)} \ , \tag{5.57}$$

and dissipative contribution of the generic form (4.17),

$$\mathbb{D}_N[X] = \frac{1}{2}\sum_{k,\ell=-N}^{N} J_{k\ell} \sum_{\mu,\nu=1}^{4} D_{\mu\nu}\Big(v_\mu^{(k)}\ \big[X\,,\,(v_\nu^\dagger)^{(\ell)}\big] \ + \ \big[v_\mu^{(k)}\,,\,X\big]\ (v_\nu^\dagger)^{(\ell)}\Big) \ , \tag{5.58}$$

with the following single-site Kraus operators

$$v_1 = \sigma_+ \otimes \sigma_- \ , \quad v_2 = \sigma_- \otimes \sigma_+ \, , \quad v_3 = \frac{1}{2}\big(\sigma_3 \otimes \mathbf{1}\big) \, , \quad \sigma_4 = \frac{1}{2}\big(\mathbf{1} \otimes \sigma_3\big) \ , \tag{5.59}$$

where $\sigma_\pm = (\sigma_1 \pm i\,\sigma_2)/2$, while the 4×4 matrix D is given by

$$D = \begin{pmatrix} \delta & 0 & \gamma & \gamma \\ 0 & \delta & \gamma & \gamma \\ \gamma & \gamma & \delta & 0 \\ \gamma & \gamma & 0 & \delta \end{pmatrix} \ ; \tag{5.60}$$

by choosing $|\gamma| \leq \delta/2$, D results positive semi-definite. In this case, one can recast $\mathbb{D}_N$ in a double commutator form:

$$\mathbb{D}_N[X] = \frac{1}{2}\sum_{k,\ell=-N}^{N} J_{k\ell} \sum_{\mu,\nu=1}^{4} D_{\mu\nu}\Big[\ \big[v_\mu^{(k)}\,,\,X\big]\ ,\,(v_\nu^\dagger)^{(\ell)}\Big] \ . \tag{5.61}$$

In the following we shall study the emergent mesoscopic dynamics corresponding to the microscopic dissipative dynamics locally generated by $\mathbb{L}_N[X] = \mathbb{H}_N[X] + \mathbb{D}_N[X]$ as given above.
The thermal state ω_β reduces on the local algebra $\mathcal{A}_{[-N,N]}$ to

$$\rho_N^{(\beta)} = \bigotimes_{j=-N}^{N} \frac{1}{4\cosh^2(\eta\beta/2)}\, \mathrm{e}^{-\beta\eta h^{(k)}/2} \ , \tag{5.62}$$

that evolves according to the master equation involving the dual generator $\mathbb{L}_N^\star$:

$$\partial_t \rho_N(t) = \mathbb{L}_N^\star[\rho_N(t)] = -i\big[H_N,\, \rho_N(t)\big] + \mathbb{D}_N[\rho_N(t)] \ . \tag{5.63}$$

The microscopic thermal state $\rho_N^{(\beta)}$ is left invariant by the dissipative dynamics; indeed, $\mathbb{L}_N^\star[\rho_N^{(\beta)}] = 0$, as it follows from

$$[\sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3 \, , \, v_\mu] = 0 \qquad \forall \mu = 1, 2, 3, 4 \; .$$

Further, since spin operators at different sites commute, given the Lindblad generator $\mathbb{L}_N$, its action on the self adjoint element $x_i^{(k)}$ from the set $\{x_j\}_{j=1}^8$ at site k is given by:

$$\begin{aligned} \mathbb{L}_N \left[x_i^{(k)}\right] &= i\frac{\eta}{2}\left[\sigma_3^{(k)} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3^{(k)} \, , \, x_i^{(k)}\right] \\ &+ J_0 \sum_{\mu,\nu=1}^{4} \frac{D_{\mu\nu}}{2}\left[\left[v_\mu^{(k)} \, , \, x_i^{(k)}\right] \, , \, (v_\nu^\dagger)^{(k)}\right] \; . \end{aligned}$$

This action maps the linear span K in itself; indeed, $\mathbb{L}_N\left[x_i^{(k)}\right] = \sum_{j=1}^8 \mathcal{L}_{ij}\, x_j^{(k)}$, with the 8×8 matrix $\mathcal{L} = \mathcal{H} + \mathcal{D}$, where

$$\mathcal{H} = \eta \begin{pmatrix} \mathcal{S} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathcal{S} \end{pmatrix} \; , \qquad \mathcal{D} = \begin{pmatrix} -\delta \mathbf{1}_4 & \Gamma \\ \Gamma & -\delta \mathbf{1}_4 \end{pmatrix} \tag{5.64}$$

$$\mathcal{S} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; , \quad \Gamma = \gamma \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \; .$$

Then, the generator of the mesoscopic dissipative dynamics as given in *Corollary 5.1* is completely determined by the 8×8 matrices $H^{(1)}$ and $D^{(1)}$ in (5.34), (5.35) or $H^{(2)}$ and $D^{(2)}$ in (5.38). Here, we give the form of the generator with respect to creation and annihilation operators.

Proposition 5.2. *In terms of annihilation and creation operators* $a_i^\#$, $i = 1, 2, 3, 4$, *the mesoscopic Lindblad generator acts on displacement operators* $D(z)$ *as* $\mathbb{L} = \mathbb{H} + \mathbb{D}$, *with* $\mathbb{H}$ *and* $\mathbb{D}$ *given by*

$$\mathbb{H}[D(z)] = i\eta\Big[\sum_{j=1}^{4} a_j^\dagger a_j \, , \, D(z)\Big] \tag{5.65}$$

$$\mathbb{D}[D(z)] = \sum_{i,j=1}^{8} K_{ij}^{(\beta)} \left(A_i^\dagger \, D(z)\, A_j - \frac{1}{2}\left\{A_i^\dagger \, A_j \, , \, D(z)\right\}\right) \; , \tag{5.66}$$

where $A = (a_1, a_1^\dagger, a_2, a_2^\dagger, a_3, a_3^\dagger, a_4, a_4^\dagger)^{tr}$ *and Kossakowski matrix*

$$K^{(\beta)} = \frac{J_0}{\epsilon}\begin{pmatrix} A_\beta & B_\beta \\ B_\beta & A_\beta \end{pmatrix} \; , \quad A_\beta = \delta \begin{pmatrix} 1+\epsilon & 0 & 0 & 0 \\ 0 & 1-\epsilon & 0 & 0 \\ 0 & 0 & 1+\epsilon & 0 \\ 0 & 0 & 0 & 1-\epsilon \end{pmatrix} \tag{5.67}$$

$$B_\beta = \gamma \begin{pmatrix} \epsilon(1+\epsilon) & 0 & -(1+\epsilon)c & 0 \\ 0 & \epsilon(1-\epsilon) & 0 & -(1-\epsilon)c \\ -(1+\epsilon)c & 0 & -\epsilon(1+\epsilon) & 0 \\ 0 & -(1-\epsilon)c & 0 & -\epsilon(1-\epsilon) \end{pmatrix} \; , \tag{5.68}$$

where $\epsilon = \tanh(\eta\beta/2)$ *and* $c = \sqrt{1-\epsilon^2}$ *as before.*

Proof. The expressions of the 8×8 matrices $H^{(1)}$ and $D^{(1)}$ in (5.34) and (5.35) that define the action of the mesoscopic dissipative generator in (5.32)-(5.33) can be readily computed by means of the following quantities

$$\mathcal{L}\, C^{(\beta)} \;-\; C^{(\beta)}\, \mathcal{L}^{tr} = -2i\eta\, (\mathbf{1} - \epsilon\sigma_1) \otimes \mathbf{1} \otimes (\epsilon + \sigma_2)$$
$$\mathcal{L}\, C^{(\beta)} \;+\; C^{(\beta)}\, \mathcal{L}^{tr} = -2J_0\Big(\delta\, (\mathbf{1} - \epsilon\sigma_1) \otimes \mathbf{1} \;-\; \gamma\, (\sigma_1 - \epsilon) \otimes \sigma_1\Big) \otimes (\mathbf{1} + \epsilon\sigma_2)\ .$$

From (5.34), *i.e.*

$$H^{(1)} = -i(\sigma^{(\beta)})^{-1} \left(\mathcal{L}C^{(\beta)} \;-\; C^{(\beta)}\mathcal{L}^{tr}\right) (\sigma^{(\beta)})^{-1}\ ,$$

one derives that the Hamiltonian coupling among the $F(x_i)$ is given by

$$H^{(1)} = \frac{\eta}{2c^2\epsilon^2}\, (\mathbf{1} + \epsilon\sigma_1) \otimes \mathbf{1} \otimes (\epsilon + \sigma_2) = \frac{\eta}{2c^2\epsilon^2} \begin{pmatrix} \mathcal{E} & \epsilon\, \mathcal{E} \\ \epsilon\, \mathcal{E} & \mathcal{E} \end{pmatrix}\ , \tag{5.69}$$

with

$$\mathcal{E} = \begin{pmatrix} \epsilon & -i & 0 & 0 \\ i & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & -i \\ 0 & 0 & i & \epsilon \end{pmatrix}\ .$$

Similarly, the Hamiltonian contribution expressed in terms of creation and annihilation operators in (5.38) gives rise to the matrix $H^{(2)} = \mathcal{M}^\dagger\, H^{(1)}\, \mathcal{M}$, explicitly given by

$$H^{(2)} = \frac{1}{\epsilon} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}\ . \tag{5.70}$$

Inserting such a diagonal matrix in the expression (5.36) and using the commutation relations $[a_i, a_i^\dagger] = 1$ one recovers the Hamiltonian generator in (5.65).
For what concerns the dissipative part, from (5.35), *i.e.*

$$D^{(1)} = (\sigma^{(\beta)})^{-1} \left(\mathcal{L}C^{(\beta)} \;+\; C^{(\beta)}\mathcal{L}^{tr}\right) (\sigma^{(\beta)})^{-1}\ ,$$

one derives the explicit expression of the Kossakowski matrix:

$$\begin{aligned} D^{(1)} &= \frac{J_0}{2c^2\epsilon^2}\Big(\delta\, (\mathbf{1} + \epsilon\sigma_1) \otimes \mathbf{1} - \gamma\, (\epsilon + \sigma_1) \otimes \sigma_1\Big) \otimes (\mathbf{1} + \epsilon\sigma_2) \\ &= \frac{J_0}{2c^2\epsilon^2} \begin{pmatrix} D_1 & \epsilon D_2 & \epsilon D_1 & D_2 \\ \epsilon D_2 & D_1 & D_2 & \epsilon D_1 \\ \epsilon D_1 & D_2 & D_1 & \epsilon D_2 \\ D_2 & \epsilon D_1 & \epsilon D_2 & D_1 \end{pmatrix}\ , \\ D_1 &= \delta \begin{pmatrix} 1 & -i\epsilon \\ i\epsilon & 1 \end{pmatrix}\ , \qquad D_2 = -\gamma \begin{pmatrix} 1 & -i\epsilon \\ i\epsilon & 1 \end{pmatrix}\ . \end{aligned}$$

Using the transformation (5.38), one can rewrite it in the language of creation and annihilation operators, thus obtaining the expression (5.67),(5.68).

□

Remark 5.2. *From the above expression of the Lindblad generator there emerge two main features of the mesoscopic dissipative dynamics: 1) the unitary contribution* $\mathbb{H}$ *to the collective dynamics of the Boson degrees of freedom shows no interactions among them. The mesoscopic Hamiltonian is proportional to the number operator and as such it does commute with the dissipative contribution:* $\mathbb{D}\circ\mathbb{H} = \mathbb{H}\circ\mathbb{D}$. *In fact,* $\mathbb{D}$ *is gauge-invariant, it does not change by sending* a_i *into* $\mathrm{e}^{i\phi}a_i$ *and* $a_i^\dagger$ *into* $\mathrm{e}^{-i\phi}a_i^\dagger$, $i = 1,2,3,4$. *Furthermore, 2) were it not for the off-diagonal blocks* B_β *in* (5.68) *in the Kossakowski matrix, the dissipative dynamics would correspond to decaying process affecting independently the various bosonic degrees of freedom. For instance, in absence of off-diagonal terms in the Kossakowski matrix, one would have*

$$\mathbb{L}[a_i] = -\,(i\omega\,+\,J_0\delta)\;a_i\ .$$

Instead, the presence of $B_\beta \neq 0$ *statistically couples the collective operators,* $a^\#_{1,3}$, $a^\#_{2,4}$ *referring to different chains.*

Model 2 While the Lindblad operators v's of the first model involve contributions from both chains (*c.f.* (5.59)) and different sites are statistically coupled by the coefficients $J_{k\ell}$, in this second case we shall consider a Lindblad generator with the same Hamiltonian term as in (5.57), and a diagonal dissipative contribution of the form:

$$\mathbb{D}_N[X] = \sum_{k=-N}^{N}\mathbb{D}^{(k)}[X]\ ,\quad \mathbb{D}_N^{(k)}[X] = \sum_{\mu,\nu=1}^{6} D_{\mu\nu}\left(v_\mu^{(k)}\,X\,v_\nu^{(k)} - \frac{1}{2}\left\{v_\mu^{(k)}\,v_\nu^{(k)}\,,\,X\right\}\right)\ , \tag{5.71}$$

with self-adjoint Lindblad operators,

$$v_{1,2,3} = \sigma_{1,2,3}\otimes\mathbf{1}\ ,\quad v_{4,5,6} = \mathbf{1}\otimes\sigma_{1,2,3}\ , \tag{5.72}$$

and 6×6 Kossakowski matrix D given by

$$D = \begin{pmatrix} M & M\\ M & M\end{pmatrix}\ ,\quad M = \begin{pmatrix}1 & -i\epsilon & 0\\ i\epsilon & 1 & 0\\ 0 & 0 & \xi\end{pmatrix}\ , \tag{5.73}$$

where the conditions $\xi \geq 0$ and $\epsilon = \tanh(\eta\beta/2) \leq 1$ guarantee $D \geq 0$. Because of the symmetry of the Kossakowski matrix, each single site contribution to the Lindblad generator can be recast in the simpler form:

$$\mathbb{D}_N^{(k)}[X] \;=\; \sum_{\mu,\nu=1}^{3} M_{\mu\nu}\left(w_\mu^{(k)}\,X\,w_\nu^{(k)} - \frac{1}{2}\left\{w_\mu^{(k)}\,w_\nu^{(k)}\,,\,X\right\}\right) \tag{5.74}$$

$$= \frac{1}{2}\Big(\left[w_1^{(k)}\,,\left[X\,,\,w_1^{(k)}\right]\right] + \left[w_2^{(k)}\,,\left[X\,,\,w_2^{(k)}\right]\right] + \gamma\left[w_3^{(k)}\,,\left[X\,,\,w_3^{(k)}\right]\right]\Big)$$
$$-\;i\frac{\epsilon}{2}\left\{w_1^{(k)}\,,\left[X\,,\,w_2^{(k)}\right]\right\} + i\frac{\epsilon}{2}\left\{w_2^{(k)}\,,\left[X\,,\,w_1^{(k)}\right]\right\} \tag{5.75}$$

with operators $w_\mu = \sigma_\mu\otimes\mathbf{1} + \mathbf{1}\otimes\sigma_\mu$ obeying

$$[w_j\,,\,w_k] \;=\; 2i\epsilon_{jk\ell}\,w_\ell \tag{5.76}$$
$$\{w_j\,,\,w_k\} \;=\; \sigma_j\otimes\sigma_k\,+\,\sigma_k\otimes\sigma_j\,+\,i\epsilon_{jk\ell}\,(\sigma_\ell\otimes\mathbf{1} - \mathbf{1}\otimes\sigma_\ell)\ . \tag{5.77}$$

In the Schrödinger picture, the local spin states ρ_N evolve in time according to the dual generator $\mathbb{L}^\star_N = \left(\mathbb{H}^\star_N + \mathbb{D}^\star_N\right)$ where

$$\begin{aligned} \mathbb{H}^\star_N[\rho_N] &= -i\eta \sum_{k=-N}^{N} \left[w_3^{(k)}\,,\,\rho_N\right]\ , \quad \mathbb{D}^\star_N[\rho_N] = \sum_{k=-N}^{N} \left(\mathbb{D}^{(k)}\right)^\star [\rho_N]\ , \\ \left(\mathbb{D}^{(k)}_N\right)^\star [\rho_N] &= \sum_{\mu,\nu=1}^{3} M_{\mu\nu}\left(w_\nu^{(k)}\,\rho_N\,w_\mu^{(k)} - \frac{1}{2}\left\{w_\mu^{(k)}\,w_\nu^{(k)}\,,\,\rho_N\right\}\right) \\ &= \frac{1}{2}\sum_{\mu=1}^{2}\left[w_\mu^{(k)},\left[\rho_N, w_\mu^{(k)}\right]\right] + \gamma\left[w_3^{(k)},\left[w_3^{(k)},\rho_N\right]\right] \\ &+ i\frac{\epsilon}{2}\left\{w_1^{(k)},\left[\rho_N, w_2^{(k)}\right]\right\} - i\frac{\epsilon}{2}\left\{w_2^{(k)},\left[\rho_N, w_1^{(k)}\right]\right\} - 2\epsilon\left\{w_3,\rho_N\right\}\ . \end{aligned}$$

In terms of the operators w_μ, the microscopic state $\rho_N^{(\beta)}$ in (5.62) is the tensor product of N density matrices of the form

$$\frac{1}{4\cosh^2(\frac{\eta\beta}{2})}\,\exp\left(-\frac{\eta\beta}{2}w_3\right)\ .$$

Expanding the exponential and using (5.77) with $j = k = 3$ one gets:

$$\rho_N^{(\beta)} = \bigotimes_{k=-N}^{N}\frac{1}{4}\left(\mathbf{1} - \epsilon\, w_3^{(k)} + \epsilon^2\sigma_3^{(k)}\otimes\sigma_3^{(k)}\right)\ , \qquad \epsilon = \tanh\left(\frac{\beta\eta}{2}\right)\ .$$

By explicit computation one then checks that $\mathbb{L}^\star_N\left[\rho_N^{(\beta)}\right] = 0$, whence the microscopic local states are left invariant by the microscopic dissipative dynamics. This fact is one of the two conditions for applying the results of the previous Chapter. The other condition is that the action of the local generator $\mathbb{L}_N$ maps into itself the linear span K; it is possible to show that

$$\mathbb{L}_N\left[x_i^{(k)}\right] = \sum_{j=1}^{8}\mathcal{L}_{ij}x_j^{(k)}\ ,$$

with $\mathcal{L} = \mathcal{H} + \mathcal{D}$, $\mathcal{H}$ being as before, and

$$\mathcal{D} = -2\begin{pmatrix} (1+\xi)\mathbf{1}_4 & -B(\epsilon) \\ 2\epsilon\mathbf{1}_4 + B(\epsilon) & (3+\xi)\mathbf{1}_4 + C \end{pmatrix}\ , \tag{5.78}$$

$\mathbf{1}_4$ being the 4×4 identity matrix and

$$B(\epsilon) = \begin{pmatrix} 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \\ \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}\ , \quad C = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}\ .$$

Finally, as for the first model, it is sufficient to explicitly write the generator of the quasi-free mesoscopic semigroup emerging from the above microscopic dissipative dynamics in the language of creation an annihilation operators:

Proposition 5.3. *In terms of annihilation and creation operators* $a_i^\#$*,* $i = 1, 2, 3, 4$*, the mesoscopic Lindblad generator reads* $\mathbb{L} = \mathbb{H} + \mathbb{D}$*, where the action of* $\mathbb{H}$ *and* $\mathbb{D}$ *on displacement operators* $D(z)$ *is as in (5.65) and (5.66), where the Kossakowski matrix is now*

$$K^{(\beta)} = \frac{2}{\epsilon}\begin{pmatrix} A_\beta & B_\beta \\ B_\beta & A_\beta \end{pmatrix} ,$$

$$A_\beta = \begin{pmatrix} (1+\epsilon)(1+\xi) & 0 & 0 & 0 \\ 0 & (1-\epsilon)(1+\xi) & 0 & 0 \\ 0 & 0 & (1+\epsilon)(3+\xi) & 0 \\ 0 & 0 & 0 & (1-\epsilon)(3+\xi) \end{pmatrix} ,$$

$$B_\beta = \begin{pmatrix} \epsilon^2(1+\epsilon) & 0 & -c\epsilon(1+\epsilon) & 0 \\ 0 & \epsilon^2(1-\epsilon) & 0 & -c\epsilon(1-\epsilon) \\ -c\epsilon(1+\epsilon) & 0 & (\epsilon+1)(1-c^2) & 0 \\ 0 & -c\epsilon(1-\epsilon) & 0 & (1-\epsilon)(1-c^2) \end{pmatrix} ,$$

again with $\epsilon = \tanh(\eta\beta/2)$*,* $c = \sqrt{1-\epsilon^2}$*.*

The proof follows the same steps as the ones discussed for the previous model.

Though the details are different, the structure of the Kossakowski matrix is similar to the one in Model 1, so that again the Hamiltonian contribution $\mathbb{H}$ to the mesoscopic Lindblad generator commutes with the dissipative one. Moreover, also in this case, the off-diagonal elements of the Kossakowski matrix statistically couple the mesoscopic operators $a_{1,3}^\#$, $a_{2,4}^\#$ referring to different chains.

Given the results of the previous Section, one can now study whether the mesoscopic dissipative time-evolutions in Model 1 and 2 can give rise to mesoscopic entanglement between the two independent chains, and, if yes, analyze the fate of the generated entanglement in the course of time.

5.3.1 Entanglement Dynamics: Model 1

In this case the entanglement criterion (5.54) can be studied analytically: we will show that the two spin chains can indeed become mesoscopically entangled, and relate the behaviour of these bath-induced quantum correlations to the squeezing parameters, the parameter γ and the temperature associated to the initial microscopic state. For sake of simplicity, we shall further set $\delta = J_0 = \eta = 1$, since these parameters do not play any role in the discussion that follows.

The criterion for Model 1 In this model for initial symmetrically squeezed states $r_1 = r_3 = r$, or one-mode squeezed initial state, $r_1 = r$, $r_3 = 0$, an explicit analytic formula for the criterion in (5.54) can be derived. The first step is to find the evolution of the reduced covariance matrix at every time t, in the language of creation and annihilation operators. Theorem 4.2 and Lemma 5.1 gives:

$$\Phi_t\left[D(z)\right] = \mathrm{e}^{-\frac{1}{2}(Z,\tilde{\mathcal{Y}}_t Z)}\, D(z_t)\,, \tag{5.79}$$

with:

$$Z_t = \mathrm{e}^{t\widetilde{\mathcal{L}}^{tr}}\,\widetilde{Z}\ , \qquad \mathrm{e}^{t\widetilde{\mathcal{L}}^{tr}} = \Sigma_3\mathcal{M}^\dagger e^{t\mathcal{L}^{tr}}(\mathcal{M}^\dagger)^{-1}\Sigma_3\ , \qquad \widetilde{\mathcal{Y}}_t = \widetilde{\Sigma}^{(\beta)}_{0,0} - \left(\mathrm{e}^{t\widetilde{\mathcal{L}}^{tr}}\right)^\dagger \widetilde{\Sigma}^{(\beta)}_{0,0}\mathrm{e}^{t\widetilde{\mathcal{L}}^{tr}},$$

and

$$\mathrm{e}^{t\widetilde{\mathcal{L}}^{tr}} = e^{-t}\begin{pmatrix} \cosh(\gamma t) & 0 & -\epsilon\sinh(\gamma t) & c\sinh(\gamma t)\\ 0 & \cosh(\gamma t) & c\sinh(\gamma t) & \epsilon\sinh(\gamma t)\\ -\epsilon\sinh(\gamma t) & c\sinh(\gamma t) & \cosh(\gamma t) & 0\\ c\sinh(\gamma t) & \epsilon\sinh(\gamma t) & 0 & \cosh(\gamma t)\end{pmatrix}\otimes\begin{pmatrix}\mathrm{e}^{i\omega t} & 0\\ 0 & \mathrm{e}^{-i\omega t}\end{pmatrix}.$$

As a result, the evolution of the covariance matrix for the four modes reads as follows:

$$\widetilde{G}(t) = \left(\mathrm{e}^{t\tilde{\mathcal{L}}^{tr}}\right)^\dagger \widetilde{\Sigma}^{(\beta)}_{r_1,r_3}\,\mathrm{e}^{t\tilde{\mathcal{L}}^{tr}} + \widetilde{\Sigma}^{(\beta)}_{0,0} - \left(\mathrm{e}^{t\tilde{\mathcal{L}}^{tr}}\right)^\dagger \widetilde{\Sigma}^{(\beta)}_{0,0}\,\mathrm{e}^{t\tilde{\mathcal{L}}^{tr}}\ .$$

In order to construct the reduced matrix for the two relevant modes under investigation, it is sufficient to look at the block structure of formula (5.46) and to collect the corresponding entries:

$$\widetilde{G}_{red}(t) = \begin{pmatrix}\widetilde{G}_{11}(t) & \widetilde{G}_{13}(t)\\ \widetilde{G}_{13}(t) & \widetilde{G}_{33}(t)\end{pmatrix}\ ,$$

where one has $\widetilde{G}_{13} = (\widetilde{G}_{13})^\dagger$. All the four matrices are diagonal in the same basis, thus the criterion depends just on their eigenvalues. For the two mentioned cases of symmetrically squeezed or one-mode squeezed initial state, the quantity in (5.54) signalling separability takes the following explicit form:

$$\begin{aligned} S_S(t) &= \frac{(\epsilon^2-1)^2}{16\epsilon^4} + \sinh^2(r)\left[\left(\frac{1}{2\epsilon^2}-\frac{1}{2}\right)\left(\frac{y_\epsilon(t)}{\epsilon} - y_\epsilon^2(t)\right) - 2\left(1+\frac{1}{\epsilon^2}\right)y_3^2(t)\right] + \\ &+ \sinh^4(r)\left[\left(\frac{y_\epsilon(t)}{\epsilon} - y_\epsilon^2(t) + 4y_3^2(t)\right)^2 - 4\frac{y_3^2(t)}{\epsilon^2}\right]\ , \end{aligned} \tag{5.80}$$

$$\begin{aligned} S_A(t) &= \frac{(\epsilon^2-1)^2}{16\epsilon^4} + \sinh^2(r)\left[\left(\frac{1}{4\epsilon^2}-\frac{1}{4}\right)\left(\frac{y_1(t)-y_1^2(t)}{\epsilon^2} + y_2(t) - \epsilon^2 y_2^2(t)\right) + \right. \\ &- \left. y_3^2(t)\left(\frac{1}{2}+\frac{1}{2\epsilon^2}\right)\right]\ , \end{aligned} \tag{5.81}$$

where

$$y_1(t) = \frac{e^{-2t}}{2}\left(\cosh(2\gamma t)+1\right)\ , \qquad y_2(t) = \frac{e^{-2t}}{2}\left(\cosh(2\gamma t)-1\right)\ , \tag{5.82}$$

$$y_3(t) = \frac{e^{-2t}}{2}\sinh(2\gamma t)\ , \qquad y_\epsilon(t) = \frac{y_1(t)}{\epsilon} + \epsilon y_2(t)\ . \tag{5.83}$$

The behaviour in time of the logarithmic negativity E, introduced in (5.55) can similarly be computed; its behaviour is shown in Fig.5.1 for different values of the dissipative parameter γ appearing in the Kossakowski matrix and fixed initial temperature $T = \frac{1}{\beta}$. Since similar results hold for both symmetrically squeezed and one-mode

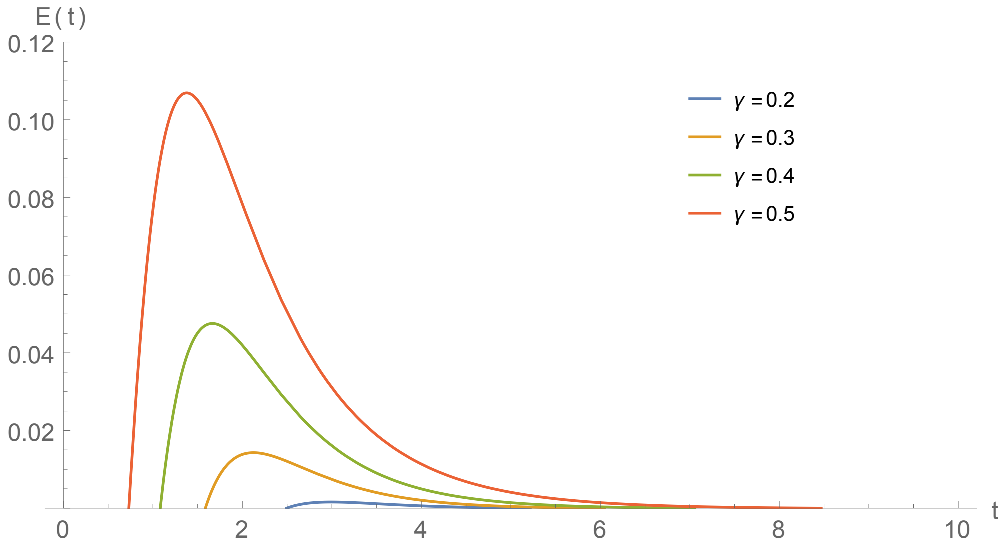

Figure 5.1: Model 1: behaviour in time of the logarithmic negativity E for different values of γ at fixed temperature $T = 0.1$, for a symmetrically squeezed initial state with $r_1 = r_3 = r = 1$.

squeezed initial states, only the graphs relative to the former case are shown. From the behaviour of E, one clearly sees that the two infinite spin chains get entangled by the dynamics. Since the Hamiltonian does not contain coupling terms, this entanglement is solely due to the mixing effects of the environment within which the two spin chains are embedded. Moreover, the amount of created entanglement increases as the dissipative parameter γ gets larger, while a non-zero entanglement appears earlier in time.

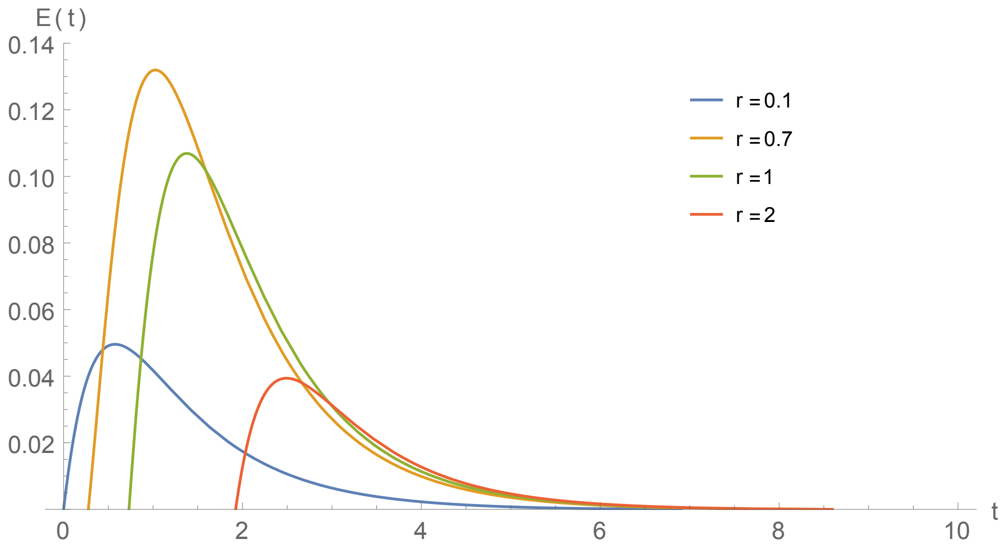

Figure 5.2: Model 1: behaviour in time of the logarithmic negativity E for different values of the squeezing parameter $r = r_1 = r_3$, at fixed temperature $T = 0.1$ and dissipative parameter $\gamma = 1/2$.

Also the amount of squeezing plays an essential role; while a non-vanishing squeezing appears necessary to create quantum correlations, too much squeezing decreases the maximum value of E. It also influences the time at which entanglement is first

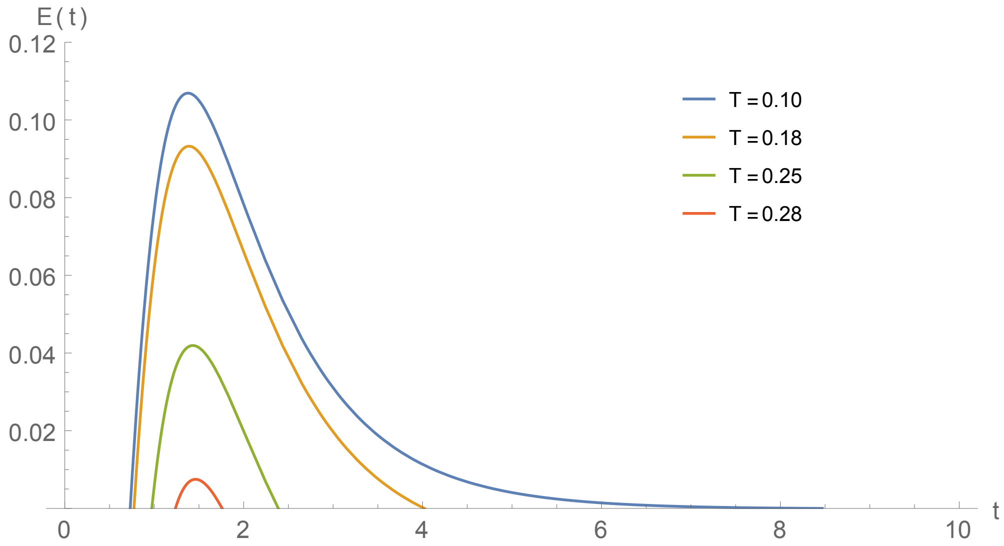

Figure 5.3: Model 1: behaviour in time of the logarithmic negativity E for different values of the temperature T, at fixed dissipative parameter $\gamma = 1/2$ and squeezing $r_1 = r_3 = r = 1$.

generated. Further, for fixed T and γ, there is a value of the squeezing parameter r allowing for a maximal value of E. All this is explicitly shown in Fig.5.2. Finally, the effect of the temperature is displayed in Fig.5.3, for fixed dissipative and squeezing parameters. One sees that increasing the temperature, the maximum of the logarithmic negativity E decreases, indicating that there exists a critical temperature T_C, above which no entanglement is possible. The explanation of this result can be traced to the behaviour of the quantity S appearing in the separability criterion in (5.54). Looking at the expressions (5.80)), and (5.81), for large temperatures, *i.e.* for ϵ small, all terms but those proportional to $1/\epsilon^4$ can be neglected, obtaining in the two cases:

$$S_S(t) \sim \frac{1}{16\epsilon^4}\left(1 + 8\sinh^2(r)\left(y_1(t) - y_1^2(t)\right)\right) \ ,$$
$$S_A(t) \sim \frac{1}{16\epsilon^4}\left(1 + 4\sinh^2(r)\left(y_1(t) - y_1^2(t)\right)\right) \ ,$$

with $y_1(t)$ still given by (5.82). Notice that since $y_1(t) < 1$ for $t > 0$, these two quantities are always positive; therefore, there must be a finite "critical temperature" T_C beyond which entanglement is no longer present.

This result is further illustrated by Fig. 5.4, where the points in the (r, T) plane with non-vanishing mesoscopic entanglement are highlighted. These figures show two regions, the dark ones associated with a non-vanishing maximal value of E, the brighter ones with vanishing maximal value of E and therefore no entanglement. The line separating the two regions determines the "critical temperature" T_C, above which entanglement among the two chains is not possible, as a function of the squeezing parameter; it is defined implicitly by the condition $\max\big(E(r,T)\big) = 0$, where the maximization is over all times.

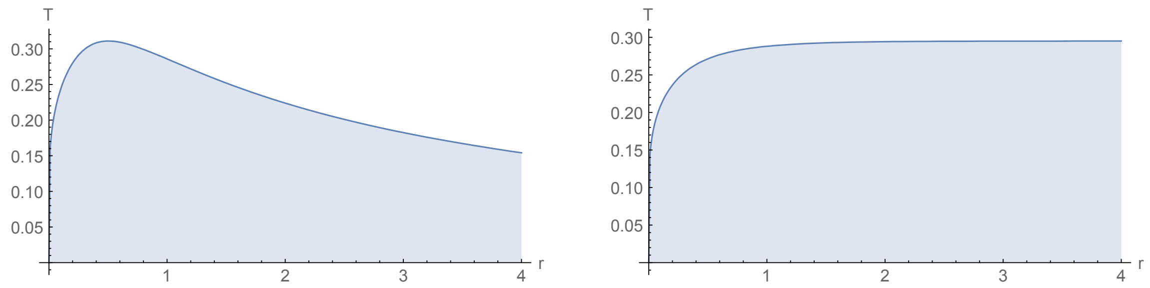

Figure 5.4: Model 1: entanglement phase diagrams for the symmetrically squeezed state $r = r_1 = r_3$ (left) and one-mode squeezed state $r = r_1$, $r_3 = 0$ (right), with $\gamma = 1/2$; the line separating the two regions gives the behaviour of the critical temperature T_C as a function of r.

Entanglement Sudden Birth and Sudden Death

The time behaviour of the logarithmic negativity E reported in Fig.'s 5.1,5.2,5.3 shows the phenomena of the so-called "sudden birth" and "sudden death" of entanglement [78], *i.e.* the sudden generation of entanglement only after a finite time since the starting of the dynamics, and the abrupt vanishing of it at a later, finite time. These two effects can be analyzed in detail as function of the temperature T of the initial state.

Let us first consider the phenomenon of sudden death and accordingly look at the large t behaviour of the evolved initial Gaussian state. The asymptotic state of the dynamics generated by (5.65) and (5.66) is thermal, with a reduced covariance matrix in the modes a_1, a_3 given by:

$$\widetilde{G}_{red}^{\infty} := \lim_{t\to\infty} \widetilde{G}_{red}(t) = \frac{1}{2\epsilon}\mathbf{1}_4 \ .$$

Positivity of the asymptotic state requires (*c.f.* (5.42)):

$$\widetilde{G}_{red}^{\infty} + \frac{i}{2}\tilde{\sigma} \geq 0 \ , \quad \tilde{\sigma} = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \ , \tag{5.84}$$

where $\tilde{\sigma}$ is the symplectic matrix in the reduced a_1, a_3 representation. This condition assures also the positivity of the partially transposed state, since $\widetilde{G}_{red}^{\infty}$ is left invariant by this transformation. In fact, the large time asymptotic limit of the lowest eigenvalue $\lambda_{min}(t)$ of the covariance $\widetilde{G}_{red}(t)$ is given by $\lambda_{min}^{\infty} = 2^{-1}(1/\epsilon - 1)$, which is always strictly positive, except at zero temperature ($\epsilon = 1$) when it vanishes. Therefore, when $T > 0$, the bath generated entanglement must always vanish in finite times, since $\lambda_{min}(t)$, from being negative, should become strictly positive for $t \to \infty$. Only at $T = 0$ the created entanglement may vanish asymptotically.

In order to study the phenomenon of sudden birth of entanglement, one has to analyze the behaviour of the logarithmic negativity E in a right neighborhood of $t = 0$. Let us consider first the case of the symmetrically squeezed initial state. Using (5.80), one checks that

$$\lim_{t\to 0^+} S_S(t) = \frac{(1-\epsilon^2)^2}{16\epsilon^4} \geq 0 \ .$$

This result already shows that only at zero temperature ($\epsilon = 1$) there is the possibility of having generation of entanglement as soon as the dynamics starts. In fact, at $T = 0$ one has:

$$S_S^{T=0}(t) = \sinh^4(r)\Big(e^{-8t} - 2e^{-6t}\cosh(2\gamma t) + e^{-4t}\Big) - e^{-4t}\sinh^2(2\gamma t)\sinh^2(r) \ . \quad (5.85)$$

Since its first derivative with respect to t vanishes at $t = 0$, one needs to study the behaviour of its second derivative:

$$\frac{d^2}{dt^2}S_S^{T=0}(t)\Big|_{t=0} = 8\big[\sinh^4(r)(1-\gamma^2) - \sinh^2(r)\gamma^2\big] \ .$$

Since $S_S^{T=0}(t) = 0$, there can be entanglement generation as soon as $t > 0$ only if this quantity is negative, *i.e.* only when $\sinh^2(r) < \gamma^2/(1-\gamma^2)$. In the opposite case, as well as for $T > 0$, entanglement generation can occur only through the sudden creation phenomenon.

Similarly, in the case of a single mode squeezed initial state, $r_1 = r$, $r_3 = 0$, from (5.81), we have:

$$\lim_{t\to 0^+} S_A(t) = \frac{(1-\epsilon^2)^2}{16\epsilon^4} \geq 0 \ .$$

Therefore, also in this case, the system may become entangled as soon as $t > 0$ only at zero temperature. Indeed, one has

$$S_A^{T=0}(t) = -\sinh^2(r)\frac{e^{-4t}\sinh^2(2\gamma t)}{16} \ , \quad (5.86)$$

which is always negative, vanishing only at $t = 0$, so that indeed entanglement is created as soon as $t > 0$. On the other hand, the phenomenon of sudden creation of entanglement always occurs for $T > 0$.

Concerning the behaviour of the critical temperature T_C for large squeezing parameter r, the first graph of Fig. 5.4 suggests a vanishing value for T_C, while that on the right a constant value, independent from r. Indeed, in the first case, recalling the result (5.85) above, one sees that for $T = 0$ and $\gamma = 1/2$, *i.e.* the largest admissible value for the dissipative parameter γ, one gets for large r:

$$S_S^{T=0}(t) \simeq e^{4(r-t)}\Big(1 - e^{-3t}\Big)\Big(1 - e^{-t}\Big) \ , \quad (5.87)$$

which is always non negative. This means that in the limit $r \to \infty$, no entanglement is created at any time when $T = 0$. The critical temperature T_C must therefore approach zero in the same limit.

Instead, in the other case one finds that for large squeezing parameter:

$$S_A(t) \simeq e^{2r}\, g(t,T) \ , \quad (5.88)$$

where $g(t,T)$ is the function multiplying $\sinh^2(r)$ in (5.81). One can show that this function takes negative values for some t, *i.e.* entanglement is generated, only for temperatures below a certain fixed value $\bar{T}$, which can be computed only numerically. As shown by the graph in the right part of Fig.4, the critical temperature is thus always non vanishing, reaching the asymptotic value $\bar{T}$ for large squeezing.

5.3.2 Entanglement Dynamics: Model 2

While in Model 1 the microscopic dynamics is generated by a Lindblad term involving contributions from both chains and also different sites, the dissipative generator (5.71) of Model 2 contains only single chain Lindblad operators, and further without any statistical coupling between different sites. This model is the many-body generalization of a two-qubit system studied in [79], where entanglement between the two qubits was shown to occur through a purely mixing mechanism induced by the presence of off-diagonal contributions of the form $(\sigma_\mu \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \sigma_\nu)$ in the dissipative generator $\mathbb{D}[\cdot]$. In fact, the entangling power of the model depends entirely on the strength of the statistical coupling of the otherwise independent qubits.

Similarly, in Model 2, mesoscopic entanglement can be dissipatively generated among the two chains in the large N limit. Unfortunately, in this case manageable analytic expressions for the logarithmic negativity are not available, so that the behaviour of E can be studied only numerically. For simplicity, in the following discussion we have set $\eta = 1$, since this parameter can be reabsorbed into a redefinition of the temperature.

As in Model 1, some initial squeezing is necessary in order for the dynamics to generate entanglement; further, the amount of created entanglement decreases as the dissipative parameter ξ entering the Kossakowski matrix (5.73) gets larger. This is explicitly shown by the behaviour of the graphs in Fig.5.5 and Fig. 5.6, where the phenomena of sudden birth and sudden death of entanglement are also visible as in Model 1.

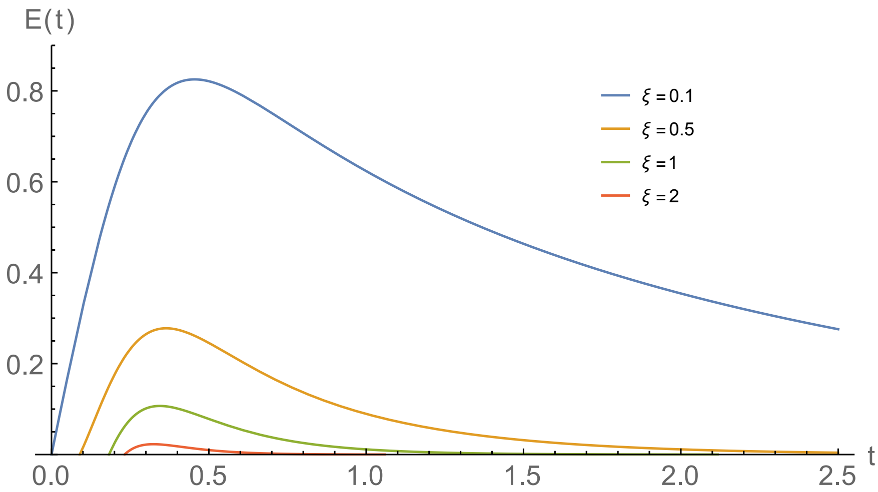

Figure 5.5: Model 2: behaviour in time of the logarithmic negativity E for different values of the dissipative parameter ξ, at fixed temperature $T = 0.1$ and squeezing $r = r_1 = r_3 = 1$.

These graphs (and the ones below) refer to the choice of a symmetrically squeezed initial state; similar results hold also in the case of one-mode squeezed initial states.

The dependence on the initial state temperature T is instead depicted in Fig. 5.7, for fixed ξ and squeezing parameter. Also in this case, one sees that increasing the temperature, the maximum of the logarithmic negativity E decreases, indicating that there exists a critical temperature T_C, above which no entanglement is possible; the

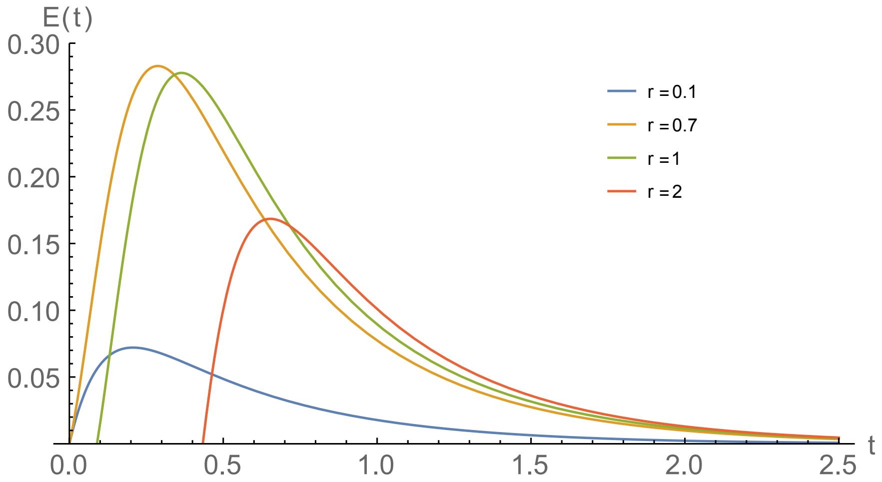

Figure 5.6: Model 2: behaviour in time of the logarithmic negativity E for different values of the temperature T, for $\xi = 1/2$ and squeezing $r = r_1 = r_3 = 1$.

behaviour of T_C as function of the squeezing parameter r is given by phase diagrams very similar to those in Fig. 5.4.

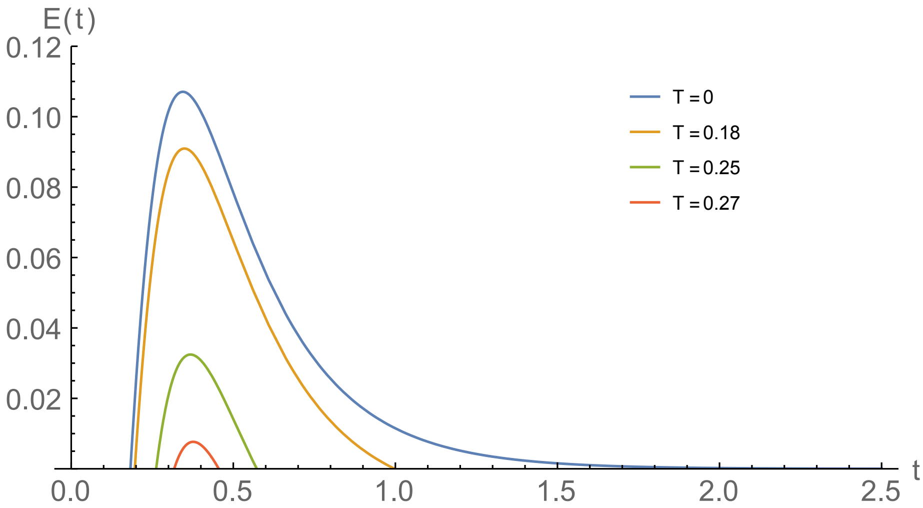

Figure 5.7: Model 2: behaviour in time of the logarithmic negativity E for different values of the temperature T, for $\xi = 1/2$ and squeezing $r = r_1 = r_3 = 1$.

However, unlike in Model 1, asymptotic entanglement is now possible. Indeed, setting the parameter $\xi = 0$ and decreasing the initial temperature T, one sees that the two chains not only get mesoscopically entangled at finite time, but remarkably, the generated mesoscopic entanglement persists for longer times. This behaviour is clearly shown by the plots in Fig. 5.8, where the time behaviour of the logarithmic negativity is reported for a symmetrically squeezed initial state: in the case of zero temperature, one sees that the generated mesoscopic entanglement persists for arbitrary long times.

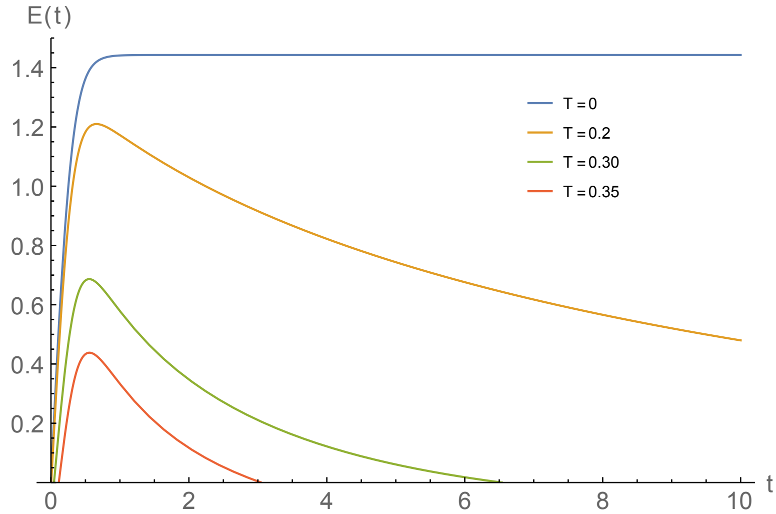

Figure 5.8: Model 2: behaviour in time of the logarithmic negativity E for different values of the temperature T, for $\xi = 0$ and squeezing $r = r_1 = r_3 = 1$.

In this Chapter, we studied the quantum dynamics of the fluctuation operators in a many-body system composed by two, non-interacting spin-1/2 chains, immersed in a common, weakly coupled external environment. This model can be thought as describing two one-dimensional lattice systems immersed in a common bath, where at each site of the lattices one has equal atoms, that, for low temperatures, can be described as two-level systems, considering just their ground state and first excited state.
The system behaves as an open quantum systems, so that noise and dissipation are expected to occur. Nevertheless, even with large number of particles, these phenomena are not able to spoil the quantum character of suitable chosen, two-chain fluctuation operators. Actually, despite the decohering and mixing-enhancing effects usually induced by the presence of the environment, the two chains can get entangled by the emergent, open mesoscopic dynamics, through a purely dissipative mechanism.

We have studied in details the fate of the generated entanglement in the course of time and of its dependence on the strength of the coupling with the environment and on the temperature of the starting microscopic many-body state: despite its inevitable dissipative action, the environment can nevertheless sustain non vanishing quantum correlations among the two chains even for very large times, provided the temperature of the initial state is sufficiently low.

Chapter 6

Outlook

Thanks to nowadays technology, experimental control of many-body quantum systems has become feasible, together with the possibility of exploiting this kind of systems in mesoscopic quantum devices or protocols. However, this asks for a theoretical description and a mathematical control of the dynamics of the relevant degrees of freedom of such systems. Because of the large amount of microscopic constituents, these degrees of freedom can only consist in coarse-grained collective observables, that provide a statistical description of the many-body system as a whole.

Among these, macroscopic averages, though giving relevant information, are not able to capture quantum collective behaviours and provide a classical description of the system. In order to pinpoint quantum features in large systems, the focus must be turned to fluctuations; indeed, it is in these deviations from the average that many-body systems retain signatures of their microscopic quantum character.

In actual experiments, based on mesoscopic devices involving many-body systems, due to the large number of particles, a complete isolation of the system from its surrounding seems hardly achievable. This thesis work has focused upon more realistic descriptions of the dynamics of these large systems, adding, to the usual Hamiltonian evolution, noise and dissipation caused by the presence of an external environment.

In particular, the dynamics of an open lattice system with Gaussian fluctuations has been derived, in the case of dissipative Markovian evolution with clustered mixing-effects, showing that the emerging dissipative dynamics of fluctuations consists in a semi-group of completely positive unital maps, that preserve the collective Gaussian character of the mesoscopic system.

Equipped with this result, it has been possible to show that the collective fluctuations of two non-interacting spin-$\frac{1}{2}$ chains can become entangled through a noisy mechanism induced by the presence of a common environment, usually responsible for decoherence and mixing-enhancing effects. Remarkably, for sufficient low temperatures, these peculiar quantum collective correlations among two mesoscopic systems, survive for arbitrarily long times. Similar results have also been shown to hold for cold atoms in two one-dimensional optical lattices immersed in a common environment [80], systems that are nowadays experimentally accessible, and can be theoretically modelled as chains of harmonic oscillators.

The mechanism of environment induced entanglement generation has been previously known only for systems involving few qubits or oscillator modes; the thesis shows that this phenomenon is at work also in the case of many-body systems provided that the correct collective observables are considered. Very importantly, thanks to nowadays technology, these results could be experimentally tested and verified.
Generalization of the presented results are possible and worth to be studied: for instance, the existence of similar results as in of Chapter 4 could be investigated for wider classes of dissipative short-range dynamics than the one studied there. Furthermore, also of interest is the formulation of general assumptions and conditions for the dissipative dynamics of fluctuations to exist and to be of Gaussian type.
From a different point of view, the presence of long-range mixing effects among particles, induced by the presence of an external environment, represents an open problem, and some interesting results were obtained in the mean-field approximation [66, 67]. Unitary mean-field dynamics has been widely studied and shown to provide qualitative and quantitative good descriptions of some many-body features, as, for instance, in the case of BCS model for superconductivity [81, 82]. More precisely, considering a lattice supporting at each site a d-level system, a (two-body interaction) mean-field unitary dynamics is the large N limit of the dynamics generated by a Hamiltonian of the following type

$$H_N^{mf} = \sum_{\mu,\nu=1}^{d^2-1} \frac{1}{N_T} \sum_{k,h=-N}^{N} C_{\mu\nu} v_\mu^{(k)} v_\nu^{(h)} ,$$

where $\{v_\mu\}_{\mu=1}^{d^2}$ is a Hermitian basis of the single-site algebra, with $v_{d^2} = \mathbf{1}$, and C a Hermitian matrix. As it can be seen in the above equation, this dynamics accounts for site-to-site interaction that does not depend on the distance between the sites; all microscopic constituents of the many-body system interact among themselves with the same strength, which is vanishing in the large N limit. Such a structure mimics the presence of long-range interaction in a large system, giving, in many instances, good approximations of its dynamical behaviour.
Especially in view of practical applications, the extension of such a mean-field approximation from the purely reversible Hamiltonian dynamics to the dissipative one generated by Lindblad type master equations should provide more physically grounded descriptions. In order to describe long-range environment induced mixing-effects one extends the structure of H_N^{mf} above, to dissipators in the following way:

$$\mathbb{D}_N^{mf}[\cdot] = \sum_{\mu,\nu=1}^{d^2-1} D_{\mu\nu} \frac{1}{N_T} \sum_{k,h=-N}^{N} \left(\left[v_\mu^{(k)}, \cdot \right] v_\nu^{(h)} + v_\mu^{(k)} \left[\cdot, v_\nu^{(h)} \right] \right) ,$$

with $D \geq 0$. In analogy with the Hamiltonian case, this generator models statistical coupling effects among couples of sites of the lattice system, due to the presence of the environment, that do not depend on the distance between sites, and whose strength is vanishing in the large N limit.
A relevant advantage of such simplified generators is that they allow for a full analytical treatment of the limiting dynamics as carried out in [66, 67]; there, the assumption of time-invariance of the reference state ω was relaxed, so that the system is free to change its collective phase (in the sense of Chapter 2), exhibiting new interesting features. For instance, the time-evolution of microscopic operators, like a single site operator, loses its dissipative character, converging, in the large N limit, to a state-dependent automorphism α_t^ω (even if no Hamiltonian contribution is present in the

generator of the microscopic dynamics), without, in general, the semi-group composition law of microscopic dynamics. Indeed, while for any finite N the dynamical maps obey $e^{(t-s)\mathbb{D}_N^{mf}} \circ e^{s\mathbb{D}_N^{mf}} = e^{t\mathbb{D}_N^{mf}}, \forall\, t \geq s \geq 0$, in the large N limit one has $\alpha^\omega_{t-s} \circ \alpha^\omega_s \neq \alpha^\omega_t$. Such a peculiar effect is strictly related to the interplay between the presence of long-range mixing-effects and the time-dependence of macroscopic observables. They give rise to memory effects that manifest themselves in the time non-locality of the generator [83] of the collective dynamics, as seen by local observables.
Instead, from the point of view of non-local, collective fluctuations, the large N limit of the microscopic dynamics is richer and retains more information about the initial generator $\mathbb{D}_N^{mf}$. From a physical point of view, the higher sensitivity of these operators to the microscopic generator shows the existence of weak but far reaching dynamical correlations between microscopic constituents. These are so weak that can not be detected by any local measure on the many-body system, but still have non-negligible effects on collective observables; their existence is uniquely revealed by fluctuation operators.
Since in absence of a time-invariant microscopic state, macroscopic observables are time-evolving degrees of freedom, the definition of fluctuation operators also becomes time-dependent. It follows that the emerging collective bosonic degrees of freedom obey commutation relations evolving with time, leading to complicated structures where the algebraic and dynamical features intertwine.
This dissipative extension of mean-field evolutions can be considered in all those instances where the many-body system is showing long-range correlations; in particular, an interesting model is the already cited BCS model. A Josephson junction between two superconductors making a ring may be described by a Hamiltonian coupling that leads to a macroscopic Josephson current [84]. Remarkably, a current can be established without any Hamiltonian, by embedding the superconductors in a suitably engineered environment that results in an appropriate mean-field dissipative dynamics. Cooper pairs can indeed pass from one superconductor to the other, just because of environment induced mixing effects.

The main purpose of this thesis work has been to show the possibility of collective quantum correlations to be generated, in infinite systems, by means of environment induced noisy mechanisms. Nevertheless, more in general, this work has also aimed at motivating the fact that, whenever mesoscopic quantum features are involved, as when clouds of cold atoms get entangled, coherent superpositions of Bose-Einstein condensate are established and so on, there, quantum fluctuations and their dynamics provide the most natural and sensible physical description.

Appendix A

Useful Tools

Proposition A.1. *Given Definition 3.2, one has,* $\forall r, s \in \mathbb{R}^n$,

$$\lim_{N\to\infty} \|[F_N(q_r), W_N(s)] - i\,[F_N(q_r), F_N(q_s)]\,W_N(s)\| = 0\,.$$

Proof. Because of Definition 3.2, one has

$$[F_N(q_r), W_N(s)] = F_N(q_r)e^{iF_N(q_s)} - e^{iF_N(q_s)}F_N(q_r)\,,$$

that, collecting an exponential on the right, can be written as

$$[F_N(q_r), W_N(s)] = \left(F_N(q_r) - e^{iF_N(q_s)}F_N(q_r)e^{-iF_N(q_s)}\right)e^{iF_N(q_s)}\,.$$

Using

$$\mathrm{e}^{x}\,y\,\mathrm{e}^{-x} = \sum_{n=0} \frac{1}{n!}\mathbb{K}_x^n[y]\ ,\quad \mathbb{K}_x^n[y] = \left[x\,,\,\mathbb{K}_x^{n-1}[y]\right]\ ,\quad \mathbb{K}_x^0[y] = y\ ,\quad \forall\,x\,,\,y \in \mathcal{A}\ , \tag{A.1}$$

the content of the round brackets is reshaped in the following way

$$\left(F_N(q_r) - e^{iF_N(q_s)}F_N(q_r)e^{-iF_N(q_s)}\right) = -\sum_{n=1}^{\infty}\frac{i^n}{n!}\mathbb{K}^n_{F_N(q_s)}[F_N(q_r)]\,;$$

all terms $n > 1$ are vanishing in the large N limit because of the fact that

$$\mathbb{K}^n_{F_N(q_s)}[F_N(q_r)] = \sum_{k=-N}^{N}\frac{1}{\sqrt{N_T^{n+1}}}\left[q_s^{(k)}\,,\,\left[q_s^{(k)}\,,\,\cdots\,\left[q_s^{(k)}\,,\,q_r^{(k)}\right]\right]\cdots\right]\ ,$$

since operators at different lattice sites commute, and the sum $n \geq 2$ is bounded in norm by

$$\left\|\sum_{n=2}^{\infty}\frac{i^n}{n!}\mathbb{K}^n_{F_N(q_s)}[F_N(q_r)]\right\| \leq \frac{\|q_r\|}{\sqrt{N_T}}\sum_{n=2}^{\infty}\frac{2^n\|q_s\|^n}{n!} \leq \frac{\|q_r\|}{\sqrt{N_T}}e^{2\|q_s\|}\,,$$

which is indeed vanishing in the $N \to \infty$ limit.
Only the first term of the series is relevant and one has

$$[F_N(q_r), W_N(s)] \sim i\,[F_N(q_r), F_N(q_s)]\,W_N(s)\,,$$

the error vanishing in norm, thus proving the thesis.

□

Proposition A.2. *Given an operator $M_N(x)$ as in Definition 3.1, and $W_N(r)$ as in equation* (3.29)*, one has*

$$\lim_{N\to\infty} \left\| \left[e^{iM_N(x)}, W_N(r)\right] \right\| = 0\,,$$

$\forall\, r \in \mathbb{R}^n$.

Proof. Using twice the algebraic relation

$$\left[e^{iA}, B\right] = \int_0^1 dy \frac{d}{dy}\left(e^{iyA} B e^{i(1-y)A}\right) = -i \int_0^1 dy\; e^{iyA}\,[B, A]\, e^{i(1-y)A}\,,$$

one has

$$\left[e^{iA}, e^{iB}\right] = -i \int_0^1 dy\; e^{iyA}\left[e^{iB}, A\right] e^{i(1-y)A} =$$
$$= -\int_0^1 dy \int_0^1 dz\; e^{iyA} e^{izB}\,[A, B]\, e^{i(1-z)B} e^{i(1-y)A}\,.$$

Therefore, considering the norm and the fact that the exponential operators are unitaries for $A = A^\dagger$ and $B = B^\dagger$, one obtains the following norm-bound

$$\left\| \left[e^{iA}, e^{iB}\right] \right\| \leq \|[A, B]\|\;.$$

Recalling Definitions 3.1, 3.2, with $A = M_N(x)$ and $B = F_N(q_r)$, one gets

$$\lim_{N\to\infty} \left\| \left[e^{iM_N(x)}, W_N(r)\right] \right\| \leq \lim_{N\to\infty} \|[M_N(x), F_N(q_r)]\|\;.$$

Because of the locality of the algebra, one has that

$$[M_N(x), F_N(q_r)] = \frac{1}{N_T^{3/2}} \sum_{k=-N}^{N} \left[x^{(k)}, q_r^{(k)}\right]\;,$$

and therefore, the following norm-bound holds

$$\|[M_N(x), F_N(q_r)]\| \leq \frac{1}{\sqrt{N_T}} 2\|x\|\|q_r\|\,,$$

showing that in the limit $N \to \infty$ the commutator goes to zero, being, $\forall\, r \in \mathbb{R}^n$, $\|x\|, \|q_r\| < \infty$.

□

Appendix B

Proof of Lemma 4.3

We shall prove that, given a time-dependent Hermitean matrix M_t and its exponential $N_t = \mathrm{e}^{iM_t}$, then

$$\dot{N}_t := \frac{\mathrm{d}N_t}{\mathrm{d}t} = O_t\, N_t \ , \quad O_t := \sum_{k=1}^{\infty} \frac{i^k}{k!} \mathbb{K}_{M_t}^{k-1}\left[\dot{M}_t\right] \ , \tag{B.1}$$

where

$$\mathbb{K}_A^n\,[B] := \left[A\,,\,\mathbb{K}_A^{n-1}\,[B]\right] \ , \quad \mathbb{K}_A^0\,[B] = B \ .$$

Indeed, given matrices A and B, one has

$$\mathrm{e}^{iA}\,B\,\mathrm{e}^{-iA} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{\left[A\left[A\,,\cdots\left[B\,,\,A\right]\cdots\right]\right]}_{n\ times} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\,\mathbb{K}_A^n\,[B] \ .$$

Then, $[N_t\,,\,M_t] = 0$ and $N_t N_t^\dagger = N_t^\dagger N_t = 1$ imply $N_t M_t N_t^\dagger = M_t$ and $\dot{N}_t\,N_t^\dagger = -N_t\,\dot{N}_t^\dagger$. Therefore,

$$N_t\,\dot{M}_t\,N_t^\dagger \,-\, \dot{M}_t \,= -\,\dot{N}_t\,M_t\,N_t^\dagger \,-\, N_t\,M_t\,\dot{N}_t^\dagger = \left[M_t\,,\,\dot{N}_t\right]N_t^\dagger \ .$$

Furthermore, since, for $n \geq 1$, $\mathbb{K}_A^n[B] = \left[A\,,\,\mathbb{K}_A^{n-1}[B]\right]$, it follows that

$$N_t\,\dot{M}_t\,N_t^\dagger - \dot{M}_t = \sum_{n=1}^{\infty} \frac{i^n}{n!}\mathbb{K}_{M_t}^n[\dot{M}_t] = \left[M_t\,,\,O_t\right] = \left[M_t\,,\,\dot{N}_t\right]N_t^\dagger \ ,$$

where $O_t = \sum_{k=1}^{\infty} \frac{i^k}{k!}\mathbb{K}_{M_t}^{k-1}[\dot{M}_t]$. Then, using again that $[N_t\,,\,M_t] = 0$, one obtains

$$\left[M_t\,,\,O_t\,N_t\right] = \left[M_t\,,\,\dot{N}_t\right] \ .$$

In order to show that $\dot{N}_t = O_t N_t$, consider the orthogonal eigenvectors $|m_a(t)\rangle$ of M_t with eigenevalues $m_a(t)$. Then, if $m_a(t) \neq m_b(t)$, the previous equality yields

$$\langle m_a(t)|O_t N_t|m_b(t)\rangle = \langle m_a(t)|\dot{N}_t|m_b(t)\rangle \ .$$

On the other hand if $|m_a(t)\rangle$ and $|m_b(t)\rangle$ correspond to a same (real) eigenvalue $m(t)$, then one uses that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle m_a(t)|m_b(t)\rangle\Big) = \langle \dot{m}_a(t)|m_b(t)\rangle \,+\, \langle m_a(t)|\dot{m}_b(t)\rangle \ ,$$

to deduce that also in such a case

$$\begin{aligned}\langle m_a(t)|O_t\, N_t|m_b(t)\rangle &= i\,\langle m_a(t)|\dot{M}_t|m_b(t)\rangle\, \mathrm{e}^{im(t)}\,\delta_{ab} = i\dot{m}(t)\,\mathrm{e}^{im(t)}\,\delta_{ab} \\ &= \langle m_a(t)|\dot{N}_t|m_b(t)\rangle\ .\end{aligned}$$

Appendix C

The Lindblad Generator on the Algebra of Fluctuations

C.1 Action of the Lindblad Generator on Finite N Fluctuations

The first step that has to be taken for the analysis of the dissipative dynamics of Weyl operators $W(r)$ is the study of the action of the local Lindbald generators satisfying (4.20), on the pre-Weyl operators in (3.29) of Definition 3.2; such characterisation is contained in the following Proposition.

Proposition C.1. *Given the real linear span K, consisting of single-site Hermitian operators and a generator $\mathbb{L}_N$ satisfying (4.20), the action of the latter on $W_N(r) = \mathrm{e}^{i(r,F_N)}$ is such that*

$$\lim_{N\to+\infty}\left\|\mathbb{L}_N\left[W_N(r)\right] - \left(\frac{i}{\sqrt{N_T}}\sum_{k=-N}^{N}\sum_{i,j=1}^{n} r_i\left(\mathcal{H}_{ij}+\mathcal{D}_{ij}\right)\,x_j\right)W_N(r)\right. \tag{C.1}$$

$$\left.+\frac{1}{2}\Big[(r,F_N)\,,\,(r,(\mathcal{H}+\mathcal{D})F_N)\Big]\;W_N(r)\;-\;S(r;N)\,W_N(r)\right\| = 0\;, \tag{C.2}$$

where

$$S(r;N) = \tfrac{1}{2}\Big(\mathbb{L}_N\left[(r,F_N)\right]\,(r,F_N)\;+\;(r,F_N)\,\mathbb{L}_N\left[(r,F_N)\right]\;-\;\mathbb{L}_N\left[(r,F_N)^2\right]\Big) \tag{C.3}$$

$$\mathbb{L}_N[(r,F_N)] = \tfrac{1}{\sqrt{N_T}}\textstyle\sum_{k=-N}^{N}\sum_{i,j=1}^{n} r_i(\mathcal{H}_{ij}+\mathcal{D}_{ij})\,x_j^{(k)} \tag{C.4}$$

$$= \Big(r,(\mathcal{H}+\mathcal{D})\,F_N\Big)\;+\;\sqrt{N_T}\,\Big(r,(\mathcal{H}+\mathcal{D})\,x_\omega\Big)\;, \tag{C.5}$$

with $x_\omega \in \mathbb{R}^n$ a real vector with components $\omega(x_i)$, $x_i \in K$.

The proof is subdivided in two lemmas concerning the large N approximation of the Hamiltonian and the dissipative terms of the Lindblad generator.

Lemma C.1. *For large N, the Hamiltonian action of the Lindblad generator can be approximated as follows:*

$$i\,[H_N\,,\,W_N(r)] \simeq \left(i\frac{1}{\sqrt{N_T}}\sum_{k=-N}^{N}\sum_{i,j=1}^{n} r_i\mathcal{H}_{ij}x_j^{(k)} - \frac{1}{2}\Big[\,(r,F_N)\;,\;(r,\mathcal{H}F_N)\,\Big]\right)\,W_N(r)\;, \tag{C.6}$$

the error vanishing in norm.

Proof. Using the notation of Definition 3.2 and the unitarity and factorisation of $W_N(r)$,

$$\begin{aligned}\left[z^{(k)},W_N(r)\right] &= \left(z^{(k)} - \mathrm{e}^{i\frac{q_r^{(k)}}{\sqrt{N_T}}}z^{(k)}\mathrm{e}^{-i\frac{q_r^{(k)}}{\sqrt{N_T}}}\right)W_N(r) = -U_{q_r^{(k)}}[z^{(k)}]W_N(r)\\ U_{q_r^{(k)}}[z^{(k)}] &= \sum_{n=1}^{\infty}\frac{i^n}{n!\left(\sqrt{N_T}\right)^n}\,\mathbb{K}^n_{q_r^{(k)}}\left[z^{(k)}\right]\;,\end{aligned}$$

with z any single-site operator and $\mathbb{K}^n_{q_r}[v_\mu]$ the multi-commutator defined by

$$\mathbb{K}^n_{q_r}\,[z] = \left[q_r,\mathbb{K}^{n-1}_{q_r}[z]\right]\;,\quad \mathbb{K}^0_{q_r}[z] = z\;. \tag{C.7}$$

Notice that $U^\dagger_{q_r^{(k)}}[z^{(k)}] = U_{q_r^{(k)}}[(z^\dagger)^{(k)}]$.

Consider the commutator with the Hamiltonian:

$$i\,[H_N\,,\,W_N(r)] = -i\sum_{k=-N}^{N}U_{q_r^{(k)}}[h^{(k)}]\,W_N(r)\;.$$

In the series expansion of $U_{q_r^{(k)}}[h^{(k)}]$, the relevant contribution is

$$\widetilde{U}_{q_r^{(k)}}[h^{(k)}] \;=\; \frac{i}{\sqrt{N_T}}\left[q_r^{(k)}\,,\,h^{(k)}\right] \;-\; \frac{1}{2N_T}\left[q_r^{(k)}\,,\,\left[q_r^{(k)}\,,\,h^{(k)}\right]\right.\;.$$

Indeed, the remaining infinite series vanishes in norm when $N\to+\infty$ as

$$\left\|\sum_{n=3}^{\infty}\frac{i^n}{n!\left(\sqrt{N_T}\right)^n}\,\mathbb{K}^n_{q_r^{(k)}}\left[h^{(k)}\right]\right\| \le \|h\|\sum_{n=3}^{\infty}\frac{2^n\|q_r\|^n}{n!\left(\sqrt{N_T}\right)^n} \le \frac{1}{N_T^{3/2}}\,\mathrm{e}^{2\|q_r\|}\|h\|\;.$$

Then, in the limit of large N, the quantity $\sum_{k=-N}^{N}U_{q_r^{(k)}}[h^{(k)}]$ behaves as

$$\sum_{k=-N}^{N}\widetilde{U}_{q_r^{(k)}}[h^{(k)}] = \sum_{k=-N}^{N}\left(\frac{i}{\sqrt{N_T}}\left[q_r^{(k)}\,,\,h^{(k)}\right] \;-\; \frac{1}{2N_T}\left[q_r^{(k)}\,,\,\left[q_r^{(k)}\,,\,h^{(k)}\right]\right]\right)\;, \tag{C.8}$$

for

$$\left\|\sum_{k=-N}^{N}\left(U_{q_r^{(k)}}[h^{(k)}] \;-\; \widetilde{U}_{q_r^{(k)}}[h^{(k)}]\right)\right\| \le \frac{1}{N_T^{1/2}}\,\mathrm{e}^{2\|q_r\|}\|h\|\,,$$

and the upper bound vanishes when $N\to+\infty$.

Using Definition 3.2 and (4.21), the first term contributing to (C.8) scales as a fluctuation. Since operators at different sites commute, it can be rewritten as

$$\frac{i}{\sqrt{N_T}}\sum_{k=-N}^{N}\left[q_r^{(k)}\,,\,h^{(k)}\right] \;=\; -\frac{i}{\sqrt{N_T}}\sum_{k,\ell=-N}^{N}\left[h^{(k)}\,,\,q_r^{(\ell)}\right]= \tag{C.9}$$

$$= -\frac{1}{\sqrt{N_T}}\sum_{k=-N}^{N}\sum_{i,j=1}^{n} r_i\mathcal{H}_{ij}x_j^{(k)} \tag{C.10}$$

$$= -(r,\mathcal{H}F_N)\,-\,\sqrt{N_T}\,(r,\mathcal{H}x_\omega)\;, \tag{C.11}$$

where $\mathcal{H}$ is the matrix with entries $\mathcal{H}_{ij}$ and $x_\omega \in \mathbb{R}^n$ has components $\omega(x_i)$.

Instead, the second term in (C.8) scales as a macroscopic observable; since operators at different sites commute, it can be rearranged as follows:

$$\sum_{k,\ell,j=-N}^{N}\frac{1}{2N_T}\left[q_r^{(j)},\left[q_r^{(\ell)},h^{(k)}\right]\right]=\frac{i}{2}\left[(r,F_N)\,,(r,\mathcal{H}F_N)\right]\;. \tag{C.12}$$

Notice that, unlike in the first term, because of the commutators, the scalar term $\sqrt{N_T}\,(r,\mathcal{H}x_\omega)$ in (C.11) does not contribute and the second term can be written in terms of the fluctuation vector $F_N=(F_N(x_1),\ldots,F_N(x_n))^{tr}$, only.

□

Lemma C.2. *For large N, the action of the dissipative part of the Lindblad generator can be approximated as follows:*

$$\mathbb{D}_N\left[e^{i(r,F_N)}\right]\sim\left(i\frac{1}{\sqrt{N_T}}\sum_{k=-N}^{N}\sum_{i,j=1}^{n}r_i\mathcal{D}_{ij}x_j^{(k)}-\frac{1}{2}\Big[(r,F_N),(r,\mathcal{D}F_N)\Big]+S(r;N)\right)e^{i(r,F_N)}, \tag{C.13}$$

where

$$S(r;N)=\frac{1}{2}\Big(\mathbb{L}_N\left[(r,F_N)\right]\,(r,F_N)\,+\,(r,F_N)\,\mathbb{L}_N\left[(r,F_N)\right]\,-\,\mathbb{L}_N\left[(r,F_N)^2\right]\Big)\;,$$

the error vanishing in norm.

Proof. The same strategy as in the proof of Lemma C.1 applied to the dissipative contribution to the Lindblad generator first yields

$$\begin{aligned}\mathbb{D}_N\left[W_N(r)\right] \;=\; & \sum_{k,\ell=-N}^{N}J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}\Big(v_\mu^{(k)}\,U_{q_r^{(k)}}[(v_\nu^\dagger)^{(k)}]\,-\,U_{q_r^{(k)}}[v_\mu^{(k)}]\,(v_\nu^\dagger)^{(\ell)}\,+\\ & -\;U_{q_r^{(k)}}[v_\mu^{(k)}]\,U_{q_r^{(k)}}[(v_\nu^\dagger)^{(k)}]\Big)\,W_N(r)\;.\end{aligned}$$

Then, by considering the expansions of the two terms in the last contribution, one shows that, apart from the first summands in each series, the rest can be estimated in norm by:

$$\left\|\sum_{k,\ell=-N}^{N}J_{k\ell}\sum_{n+m>2}\frac{i^n(-i)^m}{n!m!}\frac{\mathbb{K}^n_{q_r^{(k)}}[v_\mu^{(k)}]\mathbb{K}^m_{q_r^{(\ell)}}[(v_\nu^\dagger)^{(\ell)}]}{\sqrt{N_T^{(m+n)}}}\right\|\le\frac{1}{N_T^{3/2}}\sum_{k,\ell=-N}^{N}|J_{k\ell}|\,e^{4\|q_r\|}\|v_\mu\|\|v_\nu\|.$$

Because of the assumption (4.19) on the coefficients $J_{k\ell}$, it then follows that

$$\lim_{N\to+\infty}\left\|\sum_{k,\ell=-N}^{N}\frac{J_{k\ell}}{N_T}\left(i\left[q_r^{(k)},v_\mu^{(k)}\right]i\left[q_r^{(\ell)},(v_\nu^\dagger)^{(\ell)}\right]-U_{q_r^{(k)}}[v_\mu^{(k)}]\,U_{q_r^{(\ell)}}[(v_\nu^\dagger)^{(\ell)}]\right)\right\|=0.$$

Using similar arguments as before, one can also show that the other two contributions essentially amount to the first two terms in the series expansion; indeed,

$$\lim_{N\to+\infty}\left\|\sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}\left(\frac{1}{\sqrt{N_T}}v_\mu^{(k)}\,i\left[q_r^{(\ell)}\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right.\right.$$
$$\left.\left. -\ \frac{1}{2N_T}v_\mu^{(k)}\ \left[q_r^{(\ell)}\,,\,\left[q_r^{(\ell)}\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right]-v_\mu^{(k)}\,U_{q_r^{(k)}}[(v_\nu^\dagger)^{(k)}]\right)\right\|=0$$

$$\lim_{N\to+\infty}\left\|\sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}\left(\frac{1}{\sqrt{N_T}}\,i\left[q_r^{(k)},v_\mu^{(k)}\right]\ (v_\nu^\dagger)^{(\ell)}\right.\right.$$
$$\left.\left. -\ \frac{1}{2N_T}\left[q_r^{(k)}\,,\,\left[q_r^{(k)}\,,\,v_\mu^{(k)}\right]\right]\ (v_\nu^\dagger)^{(\ell)}-U_{q_r^{(k)}}[(v_\mu)^{(k)}]\,(v_\nu^\dagger)^{(\ell)}\right)\right\|=0\ .$$

Thus, for large N, the action of the dissipative part of $\mathbb{L}_N$ can be approximated by

$$\mathbb{D}_N\left[W_N(r)\right]\ \simeq\ \sum_{k,\ell=-N}^{N}J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2\sqrt{N_T}}\Big(i\left[v_\mu^{(k)}\,,\,q_r^{(k)}\right]\ v_\nu^{\dagger(\ell)}\ + \tag{C.14}$$
$$+\ \ i\,v_\mu^{(k)}\left[q_r^{(\ell)}\,,\,v_\nu^{\dagger(\ell)}\right]\Big)+ \tag{C.15}$$
$$+\ \sum_{k,\ell=-N}^{N}J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2N_T}\left(\ \left[q_r^{(k)},v_\mu^{(k)}\right]\ \left[q_r^{(\ell)},v_\nu^{\dagger(\ell)}\right]+\right. \tag{C.16}$$
$$\left. -\ \ \frac{1}{2}v_\mu^{(k)}\ \left[q_r^{(\ell)},\left[q_r^{(\ell)}\,,\,v_\nu^{\dagger(\ell)}\right]\right]\ +\ \frac{1}{2}\left[q_r^{(k)},\left[q_r^{(k)}\,,\,v_\mu^{(k)}\right]\right]\ v_\nu^{\dagger(\ell)}\right)\ . \tag{C.17}$$

Using Definition 3.2 and (4.21), the term in (C.14) and (C.15) that scales as $1/\sqrt{N_T}$ can be written as:

$$\frac{i}{2\sqrt{N_T}}\sum_{k,\ell=-N}^{N}J_{k\ell}\sum_{\mu,\nu=1}^{p}D_{\mu\nu}\left(\left[v_\mu^{(k)}\,,\,q_r^{(k)}\right]\ (v_\nu^\dagger)^{(\ell)}\ +\ v_\mu^{(k)}\left[q_r^{(\ell)}\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right)\ = \tag{C.18}$$
$$=i\,\mathbb{D}[(r,F_N)]=i\,(r,\mathcal{D}F_N)\ +\ i\sqrt{N_T}\,(r,\mathcal{D}x_\omega)\ , \tag{C.19}$$

where as in (C.11) $x_\omega\in\mathbb{R}^n$ is a real vector with components $\omega(x_i)$.

Concerning the term in (C.17), by using the notation in Definition 3.2 and the fact that operators at different sites commute, it can be recast in the form

$$\begin{aligned}
p_{\mu\nu}^{(k,\ell)}(r,N)\ &:=\ -\frac{1}{2N_T}v_\mu^{(k)}\left[q_r^{(\ell)}\,,\,\left[q_r^{(\ell)}\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right]\ +\ \frac{1}{2N_T}\left[q_r^{(k)}\,,\,\left[q_r^{(k)}\,,\,v_\mu^{(k)}\right]\right]\ (v_\nu^\dagger)^{(\ell)}\\
&=\ -\frac{1}{2}\,v_\mu^{(k)}\left[(r,F_N)\,,\,\left[(r,F_N)\,,\,(v_\nu^\dagger)^{(\ell)}\right]\right]\ +\\
&+\ \ \frac{1}{2}\left[(r,F_N)\,,\,\left[(r,F_N),v_\mu^{(k)}\right]\right]\ (v_\nu^\dagger)^{(\ell)}\ .
\end{aligned}$$

Since, by standard algebra,

$$-a\left[b\,,\,\left[b\,,\,c\right]\right]\,+\,\left[b\,,\,\left[b\,,\,a\right]\right]c\,=\,\left[b\,,\Big(\,a\left[b\,,\,c\right]\,+\,\left[a\,,\,b\right]c\Big)\right]\,,$$

one can finally write

$$\sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}p_{\mu\nu}^{(k,\ell)}(r;N) = -\frac{1}{2}\left[(r,F_N)\,,\,\mathbb{D}_N\left[(r,F_N)\right]\right]\ .$$

Using again (4.21) and the fact that $\mathbb{D}[1]=0$, one gets

$$\mathbb{D}_N\left[(r,F_N)\right] = \frac{1}{\sqrt{N_T}}\sum_{k=-N}^{N}\mathbb{D}_N\left[q_r^{(k)}\right] = \frac{1}{\sqrt{N_T}}\sum_{k=-N}^{N}\sum_{i,j=1}^{n} r_i\mathcal{D}_{ij}x_j^{(k)}\ .$$

Further, since $\mathbb{D}_N\left[(r,F_N)\right]$ appears inside a commutator, the scalar quantity in (C.19) does not contribute, whence

$$\sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}p_{\mu\nu}^{(k,\ell)}(r;N) = -\frac{1}{2}\left[(r,F_N)\,,\,(r,\mathcal{D}F_N)\right]\ .$$

Let us now consider the contribution in (C.16). A similar argument as before recasts it as

$$s_{\mu\nu}^{(k,\ell)}(r;N) = \frac{1}{N_T}\left[q_r^{(k)},v_\mu^{(k)}\right]\left[q_r^{(h)}\,,\,(v_\nu^\dagger)^{(h)}\right] = \left[(r,F_N)\,,\,v_\mu^{(k)}\right]\,\left[(r,F_N)\,,\,(v_\nu^\dagger)^{(\ell)}\right]\ .$$

Using the algebraic relation

$$b\Big(a\left[d\,,\,c\right]+\left[a\,,\,d\right]c\Big)+\Big(a\left[b\,,\,c\right]+\left[a\,,\,b\right]c\Big)\,d-a\left[bd\,,\,c\right]-\left[a\,,\,bd\right]c\,=\,-2\left[a\,,\,b\right]\left[d\,,\,c\right]\,,$$

we get:

$$\begin{aligned} S(r;N) \;&:=\; \sum_{k,\ell=-N}^{N} J_{k\ell}\sum_{\mu,\nu=1}^{p}\frac{D_{\mu\nu}}{2}s_{\mu\nu}^{(k,\ell)}(r;N) \\ &=\; \frac{1}{2}\Big(\mathbb{D}_N\left[(r,F_N)\right]\,(r,F_N)\,+\,(r,F_N)\mathbb{D}_N\left[(r,F_N)\right]\,-\,\mathbb{D}_N\left[(r,F_N)^2\right]\Big)\ .\end{aligned}$$

Moreover, since the Hamiltonian term of the Lindblad generator is such that

$$\mathbb{H}_N\left[(r,F_N)^2\right] = \mathbb{H}_N\left[(r,F_N)\right]\,(r,F_N)+(r,F_N)\,\mathbb{H}_N\left[(r,F_N)\right]\ ,$$

one can write $S(r;N)$ using the full Lindblad generator:

$$S(r;N) = \frac{1}{2}\Big(\mathbb{L}_N\left[(r,F_N)\right]\,(r,F_N)\,+\,(r,F_N)\,\mathbb{L}_N\left[(r,F_N)\right]\,-\,\mathbb{L}_N\left[(r,F_N)^2\right]\Big)\ .$$

□

C.2 Time-Invariant Microscopic State

In Section 4.3, we solve the time-evolution equation (4.14) under the assumption that the microscopic state ω be left invariant under the local microscopic dissipative dynamics generated by $\mathbb{L}_N$ and formally represented by the semigroup of local maps $\Phi_t^N = \exp(t\,\mathbb{L}_N)$, $t \geq 0$. Namely, we shall assume

$$\omega\Big(\Phi_t^N(X)\Big) = \omega(X) \Leftrightarrow \omega\Big(\mathbb{L}_N[X]\Big) = 0 \;, \tag{C.20}$$

for all X in local algebras $\mathcal{A}_{[-N,N]}$.
The consequences of a time-invariant ω can be appreciated by considering the expectation of the action of the Lindblad generator on a fluctuation operator. Putting together (C.9), (C.11) and (C.18), (C.19), one gets

$$\mathbb{L}_N[(r,F_N)] = \Big(r,(\mathcal{H}+\mathcal{D})F_N\Big) \;+\; \sqrt{N_T}\,\Big(r,(\mathcal{H}+\mathcal{D})\,x_\omega\Big) \;,$$

where the last quantity is a scalar multiple of the identity operator. Therefore, since fluctuation operators have vanishing mean values, $\omega(F_N(x_i)) = 0$, time-invariance of ω yields

$$\omega\Big(\mathbb{L}_N[(r,F_N)]\Big) \;=\; 0 = \sqrt{N_T}\,\Big(r,(\mathcal{H}+\mathcal{D})\,x_\omega\Big) \tag{C.21}$$

$$\mathbb{L}_N[(r,F_N)] \;=\; \Big(r,(\mathcal{H}+\mathcal{D})F_N\Big), \quad \Phi_t^N[(r,F_N)] = \Big(r,\mathrm{e}^{t(\mathcal{H}+\mathcal{D})}\,F_N\Big)\;. \tag{C.22}$$

Furthermore, consider the quantity $S(r;N)$ in (C.3); from (C.20) it follows that

$$\omega(S(r;N)) = \frac{1}{2}\omega\Big(\mathbb{L}[(r,F_N)]\,(r,F_N)\Big) \;+\; \frac{1}{2}\omega\Big((r,F_N)\,\mathbb{L}[(r,F_N)]\Big)\;.$$

Using the (C.22) and (C.5), one gets

$$\omega\Big(\mathbb{L}[(r,F_N)]\,(r,F_N)\Big) \;=\; \sum_{i,j,k=1}^{n} r_i r_j\,\Big(\mathcal{H}_{ik}+\mathcal{D}_{ik}\Big)\,\omega\Big(F_N(x_k)F_N(x_j)\Big)\;,$$

$$\omega\Big((r,F_N)\,\mathbb{L}[(r,F_N)]\Big) \;=\; \sum_{i,j,k=1}^{n} r_i r_j\,\Big(\mathcal{H}_{ik}+\mathcal{D}_{ik}\Big)\,\omega\Big(F_N(x_j)F_N(x_k)\Big)\;.$$

Then, in the limit $N \to +\infty$

$$\lim_{N\to+\infty}\omega(S(r;N)) = \Big(r,(\mathcal{H}+\mathcal{D})\,\Sigma^{(\omega)}\,r\Big)\;, \tag{C.23}$$

where $\Sigma^{(\omega)}$ is the fluctuation covariance matrix (3.39).
The proof of Proposition C.1 shows that $S(r;N)$ is the only operator involving products of operators from more than one site produced by the action of a Lindblad generator with the property (4.20) on local fluctuations. It is of the form

$$R_N = \frac{1}{N_T}\sum_{k,\ell=-N}^{N} J_{k\ell}\;a_\alpha^{(k)}\,b_\beta^{(\ell)}\;, \tag{C.24}$$

with a and b suitable single-site operators.
Despite the fact that $S(r;N)$ does not possess the shape of a macroscopic observable as those studied in (3.1), the fast decaying of $|J(k-\ell)|$ when $|k-\ell|\to\infty$, makes such an observable behave as a multiple of the identity in the large N limit. The time-invariance of ω implies the existence of the average $\omega(R_N)$ for large N, and the following lemma exhibits further properties of this kind of observables.

Lemma C.3. *Given a set of coefficients $J_{k\ell}$ such that*

$$J_{k\ell}=J(k-\ell)=J^*_{\ell k}\ ,\qquad \sum_{\ell=-\infty}^{\infty}|J_{k\ell}|=\sum_{r=-\infty}^{\infty}|J(r)|<\infty\ ,$$

and a translation invariant clustering state ω, then

$$\lim_{N\to+\infty}\|[W_N(r)\,,\,R_N]\|=0\quad \textit{and}\quad \lim_{N\to+\infty}\omega\left((R_N-R)^\dagger\,(R_N-R)\right)=0$$

for all local Weyl-like operators as in Definition 3.2 and R_N as in (C.24) *such that $R:=\lim_{N\to+\infty}\omega(R_N)$ exists.*

Proof. From the algebraic relation $\left[\mathrm{e}^{iA}\,,\,B\right]=\int_0^1 dy\frac{d}{dy}\left(\mathrm{e}^{iyA}\,B\,\mathrm{e}^{i(1-y)A}\right)$ it follows that

$$\|[W_N(r)\,,\,R_N]\|\leq\|[(r,F_N),R_N]\|\ .$$

From

$$[(r,F_N),R_N]=\frac{1}{N_T^{3/2}}\sum_{i=1}^{n}r_i\sum_{k,\ell=-N}^{N}J_{k\ell}\left(\left[x_i^{(k)}\,,\,a^{(k)}\right]\,b^{(\ell)}\,+\,a^{(k)}\,\left[x_i^{(\ell)}\,,\,b^{(\ell)}\right]\right)\ ,\quad (C.25)$$

the upper bound

$$\|[(r,F_N),R_N]\|\leq 4n\max_{1\leq i\leq n}\{|r_i|\|x_i\|\}\,\|a\|\|b\|\,\frac{1}{N_T^{3/2}}\sum_{k,\ell=-N}^{N}|J_{k\ell}|\,,$$

follows. It vanishes when $N\to+\infty$; indeed, the hypothesis on the coefficients $J_{k\ell}$ yields

$$\begin{aligned}\lim_{N\to+\infty}\frac{1}{N_T^{3/2}}\sum_{k,\ell=-N}^{N}|J_{k\ell}| &= \lim_{N\to+\infty}\frac{1}{N_T^{3/2}}\sum_{k=-N}^{N}\sum_{p=k-N}^{k+N}|J(p)|\\ &\leq \lim_{N\to+\infty}\frac{1}{N_T^{3/2}}\sum_{k=-N}^{N}\sum_{p=-\infty}^{+\infty}|J(p)|\\ &\leq \lim_{N\to+\infty}\frac{1}{\sqrt{N_T}}\sum_{p=-\infty}^{+\infty}|J(p)|=0\ .\end{aligned}$$

This proves the first result of the Lemma, the second one amounts to showing that $\lim_{N\to+\infty}\omega\left(R_N^\dagger R_N\right)=|R|^2$, where

$$\omega\left(R_N^\dagger R_N\right)=\frac{1}{N_T^2}\sum_{k_1,\ell_1=-N}^{N}\sum_{k_2,\ell_2=-N}^{N}J^*_{k_1\ell_1}\,J_{k_2\ell_2}\omega\left(b^{\dagger(\ell_1)}\,a^{\dagger(k_1)}\,a^{(k_2)}\,b^{(\ell_2)}\right)\ .$$

Using the translation invariance of ω we write

$$\omega\left(b^{\dagger(\ell_1)}\, a^{\dagger(k_1)}\, a^{(k_2)}\, b^{(\ell_2)}\right) = \omega\left(\tau^{(\ell_1-\ell_2)}\left(b^\dagger\, a^{\dagger(k_1-\ell_1)}\right)\, a^{(k_2-\ell_2)}\, b\right)\ .$$

Then, by setting $p_1 = k_1 - \ell_1$ and $p_2 = k_2 - \ell_2$, we estimate

$$\begin{aligned}\left|\omega\left(R_N^\dagger R_N\right) - \omega\left(R_N^\dagger\right)\omega\left(R_N\right)\right| &\leq \frac{1}{N_T^2}\sum_{\ell_1,\ell_2=-N}^{N}\ \sum_{p_1=-N-\ell_1}^{N-\ell_1}\ \sum_{p_2=-N-\ell_2}^{N-\ell_2} |J(p_1)|\ |J(p_2)| \times \\ &\times\left|\omega\left(\tau^{(\ell_1-\ell_2)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right| \\ &\leq \frac{1}{N_T}\sum_{h=-N_T}^{N_T}\ \sum_{p_1,p_2=-\infty}^{\infty} |J(p_1)|\ |J(p_2)| \times \\ &\times\left|\omega\left(\tau^{(h)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right|\ .\end{aligned}$$

The two infinite sums converge uniformly in the summation index h because

$$\left|\omega\left(\tau^{(h)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right| \leq 2\,\|a\|^2\,\|b\|^2\ ,$$

and because of the assumptions on the coefficients $J_{k\ell}$; therefore, the Cesàro mean (see footnote 2 of Chapter 3) yields

$$\begin{aligned}\lim_{N\to+\infty}\left|\omega\left(R_N^\dagger R_N\right) - \omega\left(R_N^\dagger\right)\omega\left(R_N\right)\right| &\leq \sum_{p_1,p_2=-\infty}^{\infty} |J(p_1)|\ |J(p_2)| \times \\ &\times \lim_{h\to\infty}\left|\omega\left(\tau^{(h)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right| + \\ + \sum_{p_1,p_2=-\infty}^{\infty} |J(p_1)|\ |J(p_2)| \lim_{h\to-\infty}&\left|\omega\left(\tau^{(h)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right|\ .\end{aligned}$$

The result follows since the clustering properties of ω give

$$\lim_{h\to\pm\infty}\left|\omega\left(\tau^{(h)}\left(b^\dagger\, a^{\dagger(p_1)}\right)\, a^{(p_2)}\, b\right) - \omega\left(b^\dagger\, a^{\dagger(p_1)}\right)\,\omega\left(a^{(p_2)}\, b\right)\right| = 0\ .$$

□

Appendix D

Properties of the Dynamical Maps on the Algebra of Fluctuations

Proposition D.1. *The maps* $\Phi_t : \mathcal{W}\left(K, \sigma^{(\omega)}\right) \to \mathcal{W}\left(K, \sigma^{(\omega)}\right)$*, are linear and unital. In particular, this means:*

$$\Phi_t\left[\alpha\, W(r) + \beta\, W(s)\right] = \alpha \Phi_t\left[W(r)\right] + \beta \Phi_t\left[W(s)\right], \ \forall \alpha, \beta \in \mathbb{C}\,, \tag{D.1}$$

$$\Phi_t\left[W(r)W(s)\right] = \Phi_t\left[W(r+s)\right] e^{-\frac{i}{2}\left(r, \sigma^{(\omega)}\, s\right)}\,, \tag{D.2}$$

$$\Phi_t\left[\mathbf{1}\right] = \mathbf{1} \tag{D.3}$$

Proof. All these properties must be proved starting from the mesoscopic limit of (4.22). The first one (D.1) is trivial and comes from the fact that α, β are scalar and from the result of theorem 4.2.
The second one amounts to show that the macroscopic observables generated by the composition of pre-Weyl operators as shown in lemma 3.1, are not time-evolving under the action of the map Φ_t^N in the large N limit.
In particular, recalling the definition of mesoscopic limit for the maps Φ_t^N (4.22) and the implications of theorem 4.2, one has to show that, $\forall\, s_1, s_2, r_1, r_2 \in \mathbb{R}^n$

$$\lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1)W_N(r_2)\right]W_N(s_2)\right) = e^{-\frac{i}{2}\left(r_1, \sigma^{(\omega)}\, r_2\right)} \lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1+r_2)\right]W_N(s_2)\right)\,.$$

Recalling the proof of Lemma 3.1, and using the fact that the maps Φ_t^N are norm-contraction, one finds that:

$$\lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1)W_N(r_2)\right]W_N(s_2)\right) = \lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1+r_2)e^{-\frac{i}{2}(r_1, T_N\, r_2)}\right]W_N(s_2)\right)\,,$$

with T_N as in (3.35). Summing and subtracting in the exponential the expectation of matrix T_N on the state ω, which is given by $\sigma^{(\omega)}$ of (3.36), one gets

$$\lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1)W_N(r_2)\right]W_N(s_2)\right) = \lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1+r_2)\left(e^{-\frac{i}{2}(r_1,T_N\,r_2)} - e^{-\frac{i}{2}\left(r_1,\sigma^{(\omega)}\,r_2\right)}\right)\right]W_N(s_2)\right) + +e^{-\frac{i}{2}\left(r_1,\sigma^{(\omega)}\,r_2\right)} \lim_{N\to\infty} \omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1+r_2)\right]W_N(s_2)\right)\ ;$$

If the second term on the right-hand side of the equality converges to zero, then the second relation (D.2) is proved since the last term in the above equation is exactly what is needed. Therefore, we focus just on the second term; using the relation

$$e^{iA} - e^{iB} = i\int_0^1 dx\, e^{ixA}\,(A-B)\,e^{i(1-x)B}\,,$$

and the fact that $\sigma^{(\omega)}$ is a multiple of the identity, one gets

$$\left(e^{-\frac{i}{2}(r_1,T_N\,r_2)} - e^{-\frac{i}{2}\left(r_1,\sigma^{(\omega)}\,r_2\right)}\right) = i\int_0^1 dx\, e^{-\frac{ix}{2}(r_1,T_N\,r_2)}e^{-\frac{i(1-x)}{2}\left(r_1,\sigma^{(\omega)}\,r_2\right)}\times \times\left(-\frac{i}{2}\left(r_1,\left[T_N-\sigma^{(\omega)}\right]\,r_2\right)\right)\,.$$

Substituting this in the term under investigation, using the Cauchy-Schwarz inequality, the Schwarz positivity, and the fact that all operators involved, but

$$Z = \left(r_1,\left[T_N-\sigma^{(\omega)}\right]\,r_2\right)\ ,$$

are unitary, one finds the bound

$$\left|\omega\left(W_N(s_1)\Phi_t^N\left[W_N(r_1+r_2)\left(e^{-\frac{i}{2}(r_1,T_N\,r_2)} - e^{-\frac{i}{2}\left(r_1,\sigma^{(\omega)}\,r_2\right)}\right)\right]W_N(s_2)\right)\right| \leq \leq \sqrt{\omega\left(W_N^\dagger(s_2)\Phi_t^N\left[Z^2\right]W(s_2)\right)}\,;$$

thus, it is left to show that

$$\lim_{N\to\infty}\omega\left(W_N^\dagger(s_2)\Phi_t^N\left[Z^2\right]W(s_2)\right) = 0\,.$$

Since Z is made of sums of average operators as those in (3.1), then it can be written as

$$Z = \frac{1}{N_T}\sum_\alpha\sum_{k=-N}^{N} o_\alpha^{(k)}\,,$$

with o_α proper single-particle operators. Thus, one can use a similar argument as the one used for the term in (4.37), in the proof theorem 4.2, to show that

$$\lim_{N\to\infty}\omega\left(W_N^\dagger(s_2)\Phi_t^N\left[Z^2\right]W(s_2)\right) = \lim_{N\to\infty}\omega\left(\Phi_t^N\left[Z^2\right]\right)\ ,$$

and, furthermore, given the time-invariance of the state

$$\lim_{N\to\infty}\omega\left(\Phi_t^N\left[Z^2\right]\right) = \lim_{N\to\infty}\omega\left(Z^2\right)\ .$$

Now, because of Theorem 3.1 on the strong convergence of macroscopic observables, and the definitions (3.35),(3.36), one has

$$\lim_{N\to\infty} \omega\left(Z^2\right) = 0\,,$$

proving the validity of relation (D.2).
In order to show (D.3), it is sufficient to substitute $r = 0$, in the action of the map (4.23), with the definitions (4.27),(4.28).

□

Theorem D.1. *The family of linear maps* $\Phi_t : \mathcal{W}\left(K, \sigma^{(\omega)}\right) \to \mathcal{W}\left(K, \sigma^{(\omega)}\right)$*, such that*

$$\mathcal{W}\left(K, \sigma^{(\omega)}\right) \ni W(r) \to \Phi_t\left[W(r)\right] = W(r_t) e^{f_r(t)}\,, \tag{D.4}$$

with $r_t = \mathcal{X}_t^{tr}\, r$*, as in (4.27), and* $f_r(t) = -(r, \mathcal{Y}_t\, r)$ *as in (4.28), forms a semi-group of completely positive unital dynamical maps.*

Proof. The semi-group property has already been shown in (4.29), and considering the previous proposition, what is left is the proof of complete positivity.
By definition, considering any m level system, whose algebra is represented by the $m \times m$ square matrices $M_m(\mathbb{C})$, we need to show that

$$\Phi_t \otimes \mathbf{1}_m \left[X^\dagger X\right] \geq 0\,, \qquad \forall m \in \mathbb{Z},$$

where X is any operator $X \in \mathcal{W}\left(K, \sigma^{(\omega)}\right) \otimes M_m(\mathbb{C})$. Considering an Hermitian basis $\{E_\mu\}_{\mu=1}^{m^2}$ of $M_m(\mathbb{C})$, any operator X can be decomposed as

$$X = \sum_{\alpha,\beta} c_{\alpha\beta}\, W(r_\alpha) \otimes E_\beta\,,$$

with $c_{\alpha\beta}$ suited coefficients. Therefore the positive element

$$X^\dagger X = \sum_{\alpha,\beta,\gamma,\delta} \bar{c}_{\alpha\beta} c_{\gamma\delta} W^\dagger(r_\alpha) W(r_\gamma) \otimes E_\beta E_\delta\,,$$

is mapped by $\Phi_t \otimes \mathbf{1}_m$, using the composition relation for Weyl in (3.15), and the action of the map (D.4), into

$$\Phi_t \otimes \mathbf{1}_m \left[X^\dagger X\right] = \sum_{\alpha\beta\gamma\delta} \bar{c}_{\alpha\beta} c_{\gamma\delta}\, F(\alpha,\gamma) W^\dagger(X_t^{tr} r_\alpha) W(X_t^{tr} r_\gamma) \otimes E_\beta E_\delta$$

with

$$\begin{aligned} F(\alpha,\gamma) = e^{\frac{i}{2}(r_\alpha, \hat{\sigma}_t\, r_\gamma)} \exp\Big(-\big((r_\alpha - r_\gamma), \mathcal{Y}_t(r_\alpha - r_\gamma)\big)\Big)\,, \\ \hat{\sigma}_t = \sigma^{(\omega)} - \mathcal{X}_t \sigma^{(\omega)} \mathcal{X}_t^{tr}\,. \end{aligned} \tag{D.5}$$

The matrix $\hat{\sigma}_t$ is an anti-symmetric matrix, thus we can construct the Weyl algebra $\mathcal{V}\left(K, \hat{\sigma}_t^{(\omega)}\right)$, such that

$$V(r_1) V(r_2) = V(r_1 + r_2) e^{-\frac{i}{2}(r_1, \hat{\sigma}_t\, r_2)}$$

with $V(r_i) \in \mathcal{V}\left(K, \hat{\sigma}_t^{(\omega)}\right)$, and consider the functional $\varphi_{\mathcal{Y}_t}$ on such algebra, giving

$$\varphi_{\mathcal{Y}_t}\left(V(r)\right) = e^{-(r, \mathcal{Y}_t r)} .$$

Putting the two things together one has

$$F(\alpha, \gamma) = \varphi_{\mathcal{Y}_t}\left(V^\dagger(r_\alpha) V(r_\gamma)\right) ,$$

and defining

$$X_{\alpha\beta} = c_{\alpha\beta} W(\mathcal{X}_t^{tr} r_\alpha) \otimes E_\beta ,$$

one can write

$$\Phi_t \otimes \mathbf{1}_m \left[X^\dagger X\right] = \sum_{\alpha,\beta,\gamma,\delta} X^\dagger_{\alpha\beta} X_{\gamma\delta}\, \varphi_{\mathcal{Y}_t}\left(V^\dagger(r_\alpha) V(r_\gamma)\right) ,$$

or extracting the action of the functional and considering the algebra $\mathcal{W}\left(K, \sigma^{(\omega)}\right) \otimes M_m(\mathbb{C}) \otimes \mathcal{V}\left(K, \hat{\sigma}_t^{(\omega)}\right)$, with $X_{\alpha\beta} \in \mathcal{W}\left(K, \sigma^{(\omega)}\right) \otimes M_m(\mathbb{C})$,

$$\Phi_t \otimes \mathbf{1}_m \left[X^\dagger X\right] = \varphi_{\mathcal{Y}_t}\left(\sum_{\alpha,\beta,\gamma,\delta} X^\dagger_{\alpha\beta} X_{\gamma\delta} \otimes V^\dagger(r_\alpha) V(r_\gamma)\right) .$$

If $\varphi_{\mathcal{Y}_t}$ is a positive functional on the algebra $\mathcal{V}\left(K, \hat{\sigma}_t^{(\omega)}\right)$, then the operator $\Phi_t \otimes \mathbf{1}_m \left[X^\dagger X\right]$ is a positive operator $\forall\, m \in \mathbb{Z}$, proving the complete positivity of maps Φ_t. Given the quasi-free character of the functional $\varphi_{\mathcal{Y}_t}$, positivity is given by

$$2\mathcal{Y}_t + \frac{i}{2}\hat{\sigma}_t \geq 0 . \tag{D.6}$$

Because of equations (4.28), one has

$$2\mathcal{Y}_t + \frac{i}{2}\hat{\sigma}_t = \Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)} - \mathcal{X}_t\left(\Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\right)\mathcal{X}_t^{tr} ; \tag{D.7}$$

Following a similar argument, used in the proof of Lemma 4.2, and recalling equations (3.39),(3.36), one has that

$$\left(\lambda, \Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\,\lambda\right) = \sum_{i,j=1}^{n} \lambda_i^* \lambda_j \lim_{N\to\infty} \omega\left(F_N(x_i) F_N(x_j)\right) ,$$

and because of the assumption $\omega \circ \Phi_t^N = \omega$, also

$$\begin{aligned}\left(\lambda, \Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\,\lambda\right) &= \sum_{i,j=1}^{n} \lambda_i^* \lambda_j \lim_{N\to\infty} \omega \circ \Phi_t^N\left(F_N(x_i) F_N(x_j)\right) = \\ &= \lim_{N\to\infty} \omega \circ \Phi_t^N\left(F_N(q_\lambda^\dagger) F_N(q_\lambda)\right) .\end{aligned}$$

being $F_N(q_\lambda) = \sum_{i=1}^n \lambda_i\, F_N(x_i)$, $\lambda_i \in \mathbb{C}$. Because of Schwartz positivity it is also true that

$$\left(\lambda, \Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\,\lambda\right) \geq \lim_{N\to\infty} \omega\left(\Phi_t^N\Big[F_N(q_\lambda^\dagger)\Big]\Phi_t^N\Big[F_N(q_\lambda)\Big]\right) ;$$

using the second relation in (C.22), one has

$$\lim_{N\to\infty} \omega\left(\Phi_t^N\left[F_N(q_\lambda^\dagger)\right]\Phi_t^N\left[F_N(q_\lambda)\right]\right) = \left(\lambda, \mathcal{X}_t\left(\Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\right)\mathcal{X}_t^{tr}\,\lambda\right),$$

thus showing that

$$\left(\lambda, \Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\,\lambda\right) \geq \left(\lambda, \mathcal{X}_t\left(\Sigma^{(\omega)} + \frac{i}{2}\sigma^{(\omega)}\right)\mathcal{X}_t^{tr}\,\lambda\right),$$

enforcing equation (D.6), and proving the complete positivity of the maps Φ_t.

□

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