# Model-Free Plant Tuning

Franco Blanchini, Gianfranco Fenu, Giulia Giordano, and Felice Andrea Pellegrino

***Abstract*—Given a static plant described by a differentiable input-output function, which is completely unknown, but whose Jacobian takes values in a known polytope in the matrix space, this paper considers the problem of tuning (i.e., driving to a desired value) the output, by suitably choosing the input. It is shown that, if the polytope is robustly nonsingular (or has full rank, in the nonsquare case), then a suitable tuning scheme drives the output to the desired point. The proof exploits a Lyapunov-like function and applies a well-known game-theoretic result, concerning the existence of a saddle point for a min-max zero-sum game. When the plant output is represented in an implicit form, it is shown that the same result can be obtained, resorting to a different Lyapunov-like function. The case in which proper input or output constraints must be enforced during the transient is considered as well. Some application examples are proposed to show the effectiveness of the approach.**

***Index Terms*—Lyapunov methods, min-max theorem, robust control, uncertain systems.**

## I. Introduction

FOR several types of systems with a large number of inputs and outputs (such as electrical networks, power generation systems, electronic circuits, systems for heat generation and transmission, flow networks in general), stability is not a critical issue, while steady-state tuning is very important and, at the same time, difficult to achieve. In fact, often the plant model is unknown, hence plant tuning requires a frustrating trial-and-error approach: when attempting to set an output to the desired value, the unknown interactions among the variables can unpredictably drive the other outputs out of tune.

In a large electrical network, e.g., the voltages provided by the generators can be controlled so as to guarantee that some target nodes have the appropriate voltage level. The number of generators corresponds to the number of degrees of freedom: if it is not smaller than the number of the target nodes, then the desired voltage levels can be straightforwardly obtained by setting the generators at the proper voltage. Or, better, they *could* be straightforwardly obtained, *if the network parameters were known*. However, most often, the network parameters (such as impedances) depend on the load, which in general is not exactly known and is subject to unpredictable variations.

This paper considers the problem of tuning a *static* plant, described by a system of nonlinear equations: the inputs of the plant need to be chosen so as to drive the outputs to the desired level, yet the system equations are unknown and only qualitative information on the system Jacobian is available.

The main result shows that the robust tuning problem can be solved by means of a proper tuning law, provided that: i) the Jacobian matrix of the input-output function is included in a compact and convex set of matrices (the case of the inclusion in a polytope of matrices is especially considered), and ii) all the elements of this set are either right invertible, in the nonsquare case, or nonsingular, in the square case.

The proposed robust, model-free approach to plant tuning is obtained based on a Lyapunov approach and on the well-known min-max theorem [27]. This game-theoretic result has already been exploited in the context of robust control via Lyapunov methods [23], [24] (see also [8], [14], [21], [30]). However, the problem faced here is different, since a static plant is considered and the devised control law is actually a tuning law.

More precisely, consider the unknown function $y = g(u)$ and assume the only available information is that its Jacobian $G_u$ belongs to a polytope (or, more in general, to any convex and compact set) of matrices. In order to drive $y$ to zero, a *tuning scheme* can be adopted based on an auxiliary control variable $v = \dot{u}$, the derivative of the original control $u$. Hence, $u$ is the time integral of the new decision variable $v$: this ensures both continuity of $u$, which is fundamental in the proposed tuning setup, and zero steady-state error.

The *state* of the tuning system is then the control variable itself. *Robust* tuning is ensured, since the solution is devised based on a Lyapunov-like function. The technique relies on the existence of a saddle point of a suitable min-max problem, which has to be solved on-line to determine the control action.

The considered problem is related to other methods previously adopted for parameter tuning [4], [20] in which the goal is optimizing the performance and/or identifying the parameters. Here performance is not a concern: the only aim is to reach the target output.

Also, it is worth mentioning the iterative learning control technique [2], [15], which is specifically aimed at determining the input function of a dynamic system so that the output function matches a desired reference. In principle, the scheme

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proposed here could be seen as an iterative (continuous-time) learning process for a static nonlinear plant.

The problem could also be approached, in principle, by means of multidimensional extremum-seeking techniques [31], [26], [29]; indeed the goal is achieved when $\|g(u)\|^2 = 0$, hence the problem could admit an extremum-seeking formulation. Indeed, there are interesting connections with robust optimization (see [9] for an extensive survey).

The substantial novelty of the proposed method with respect to the existing techniques is that, based on a Lyapunov approach [21], [30], it exploits the robust nonsingularity of the Jacobian of $g$ to ensure convergence. We therefore believe that the method has potential future development in the previously mentioned areas of robust optimization and learning.

The contributions of the paper are the following.

1) An automatic tuning strategy based on an auxiliary control variable is proposed; this auxiliary control variable is the derivative of the original control (hence, the state of the system is the control variable itself).
2) Assuming, without restriction, $y = 0$ as the target, a Lyapunov-like positive-definite function of the output variable is considered. If the Jacobian takes values in a robustly nonsingular polytope (or, more generally, in any convex and compact set) of matrices, the proposed robust control strategy drives the Lyapunov-like function to 0.
3) The control, based on a min-max principle, requires the solution of a convex optimization problem on-line.
4) When bounds on the output variables need to be considered during the transient, a suitably adapted Lyapunov-like function can be employed, whose sublevel sets are tailored to match the shape of the constraint set.
5) When bounds on the input variables need to be considered (both during the transient and at steady state), the problem can be solved by means of a suitable reparametrization.
6) In some important cases, the (unknown) input-output function has an implicit form and the polytopic bounds on the Jacobian are available for the inverse transformation only. This problem can be solved as well by exploiting an integral formula [25] and considering a different Lyapunov-like positive-definite function.
7) A maximum tuning speed can be assigned by constraining the norm of the auxiliary control signal. Under suitable choices of such a norm, the convex optimization problem amounts to Quadratic Programming (Euclidean norm) or Linear Programming ($\infty$-norm).
8) Examples are provided to illustrate the technique, both in the explicit and in the implicit case.

Some of the results proposed here have been preliminarily presented in the conference paper [10], where only the case of plants given in an explicit form has been considered.

## II. Motivating Examples

*Explicit Case:* Consider the flow network represented in Fig. 1, where there is no buffer capacity at the nodes. Vector $y = [y_1 \ \ y_2 \ \ y_3 \ \ y_4]^\top$ represents the relative output flow at the four nodes, with respect to the flow reference $\bar{r}$; the flow corresponding to each link is operated by a variable $u_k$, $u = [u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6]^\top$, and its value is given by an *unknown* function $\phi_k(u_k)$. It is only known that $\phi_k(u_k)$ are increasing for all $k = 1, \ldots, 6$. This situation is typical in channel (or pipe) networks, in which the flows are regulated by locks (or valves): the control variable is then the lock opening fraction, while the corresponding flow is not known exactly; however, it is absolutely reasonable to assume that the flow functions $\phi_k(\cdot)$ are strictly increasing. Given the flow reference $\bar{r}$, the corresponding model output is

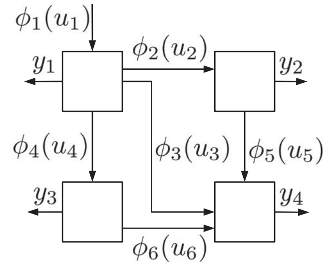


Fig. 1. The flow network problem.

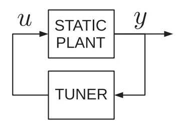


Fig. 2. The automatic tuning scheme.

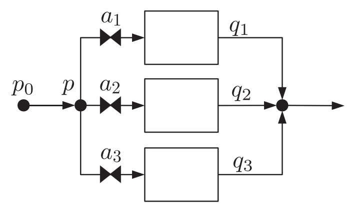


Fig. 3. The thermal regulation problem.

$$y = B\phi(u) - \bar{r},$$

where $B$ is the incidence matrix of the network graph. In the case of Fig. 1, the incidence matrix is

$$B = \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

For such a system, in order to drive $y$ to zero, a robust tuner of the form represented in Fig. 2 is sought. The only available information is given by upper and lower bounds on the derivatives of the functions $\phi_k(\cdot)$, no matter how "conservative": any bound of the form $\epsilon \le \phi_k'(\cdot) \le \mu$, with small $\epsilon$ and large $\mu$, is suitable.

*Implicit Case:* In some cases, the unknown input-output map cannot be characterized in an *explicit* form. Consider, e.g., the thermal regulation problem in Fig. 3. The flow $q_k$ along each of the three branches is a strictly increasing function of both the valve opening fraction $a_k$ and the input pressure $p$ at the branching point:

$$q_k = \phi_k(p, a_k), \quad k = 1, 2, 3.$$

The pressure $p$ itself is a function of the overall flow: typically,

$$p = p_0 - \Psi(q_1 + q_2 + q_3).$$

Therefore, ensuing an attempt to increase the flow in pipe 1, $p$ automatically decreases, hence the flows in pipes 2 and 3 decrease as well. So, if the automatic tuning system is required to regulate the flows $q_k$ to desired values $\bar{q}_k$, no explicit relation is available between the flows $q_k$ and the valve opening fractions $a_k$. However, an *implicit* representation can be derived as follows. Note that $a_k$ can be expressed as

$$a_k = \psi_k(p, q_k), \quad k = 1, 2, 3,$$

where $\psi_k(p, q_k)$ is strictly decreasing in the first argument and increasing in the second. Then, if the expression for the pressure is replaced, a function

$$a_k = \psi_k(p_0 - \Psi(q_1 + q_2 + q_3), q_k)$$

is achieved, which is increasing in $q_k$ for all $k = 1, 2, 3$. The previous reasoning can be generalized to the case of $N$ branching pipes, obtaining the functions

$$a_k = \psi_k\left(p_0 - \Psi\left(\sum_{i=1}^{N} q_i\right), q_k\right), \quad k = 1, \ldots, N.$$

Again, the only available information consists in bounds on the derivatives: $\epsilon \leq \psi_k'(\cdot) \leq \mu$.

# III. Problem Formulation and Preliminaries

## A. A Saddle-Point Theorem for Min-Max Games

Here some well-established results from game theory are recalled. Consider a *polytope of matrices*, i.e., a set

$$\mathcal{M} = \Big\{ M = \sum_{k=1}^{r} M_k \alpha_k, \ \sum_{k=1}^{r} \alpha_k = 1, \ \alpha_k \geq 0,$$
$$M_k \in \mathbb{R}^{p \times m}, \ \forall\, k = 1, \ldots, r \Big\}. \tag{1}$$

Then, given a vector $y \in \mathbb{R}^p$ and a convex and compact set $\mathcal{V} \subset \mathbb{R}^m$, the two problems

$$\nu^- \doteq \max_{M \in \mathcal{M}} \min_{v \in \mathcal{V}} y^\top M v, \tag{2}$$

$$\nu^+ \doteq \min_{v \in \mathcal{V}} \max_{M \in \mathcal{M}} y^\top M v \tag{3}$$

have a game-theoretic interpretation [5], [7]. Two players, the Maximizer and the Minimizer, respectively choose $M \in \mathcal{M}$ and $v \in \mathcal{V}$, with the conflicting goals of maximizing and minimizing (respectively) the expression $y^\top M v$. In version (2) ("the Maximizer plays first"), the decision of the Minimizer is based on the knowledge of the decision of the Maximizer, while in version (3) ("the Minimizer plays first"), the decision of the Maximizer takes into account the decision of the Minimizer (see, e.g., [17, Ch. 9, pp. 271–272]). In general, playing first is a disadvantage (due to the absence of knowledge about the opponent's choice), hence

$$\nu^- \leq \nu^+.$$

The following well-known result holds.

*Theorem 1:* If $\mathcal{M}$ and $\mathcal{V}$ are compact and convex sets, then there exist $v^* \in \mathcal{V}$ and $M^* \in \mathcal{M}$ such that

$$\nu^- = \nu^+ = \nu^* = y^\top M^* v^*. \tag{4}$$

A proof can be found in [27].

The pair $(M^*, v^*) \in \mathcal{M} \times \mathcal{V}$, called a *saddle point* of the min-max game, might not be unique in general, while the value $\nu^*$ is unique [5], [7].

Since both $M^*$ and $v^*$ depend on $y$, the following functions can be defined:

$$\Phi(y) \ = v^* \ \text{ the minimizer value in (4)}, \tag{5}$$

$$\Psi(y) = M^* \ \text{ the maximizer value in (4)}. \tag{6}$$

When $M^*$ and $v^*$ are not unique, the ambiguity can be resolved by taking the minimum-Euclidean-norm element.

## B. Problem Statement

Consider the following problem.

*Problem 1:* Given the static plant

$$y = g(u), \tag{7}$$

where $g : \mathbb{R}^m \to \mathbb{R}^p$, $p \leq m$, assume that $g(\bar{u}) = 0$ for some *unknown* $\bar{u}$ and that the following inclusion holds:

$$G_u \doteq \left[\frac{\partial g}{\partial u}\right] \in \mathcal{M}, \tag{8}$$

where $G_u$ is the Jacobian of $g$ and $\mathcal{M}$ is a known polytope (or any convex and compact set) of matrices. Find a dynamic algorithm such that, as $t \to \infty$,

$$y(t) \to 0, \tag{9}$$

$$u(t) \to \bar{u}, \tag{10}$$

where $\bar{u}$ solves the equation

$$0 = g(u). \tag{11}$$

$\diamond$

# IV. Problem Solution

To solve the tuning problem, remember that condition (8) is the only available information for control purposes. First of all, consider Problem 1 under the assumption that there are as many outputs as inputs, hence $p = m$.

## A. Case $p = m$

The next definition is fundamental [6].

*Definition 1 Robust Nonsingularity:* The polytope $\mathcal{M}$ is robustly nonsingular if any matrix in $\mathcal{M}$ is nonsingular. $\diamond$

The following standing assumption is considered.

*Assumption 1:* The family $\mathcal{M}$ is robustly nonsingular. $\diamond$

Section VI-A will illustrate how this condition can be checked, under suitable assumptions on the matrix structure. Consider a control scheme of the form

$$\dot{u}(t) = v(t), \tag{12}$$

$$v(t) = \Phi(y(t)), \tag{13}$$

$$y(t) = \text{ measured output.} \tag{14}$$

For both technical and practical reasons, the control derivative is deliberately bounded as follows:

$$v(t) \in \mathcal{V} = \{v : \|v\| \leq \xi\}, \tag{15}$$

where $\xi > 0$ and $\| \cdot \|$ is any norm.

The main result is the following.

*Theorem 2:* Under Assumption 1, Problem 1 can be solved by means of a control scheme of the form (12)–(14), with $v$ bounded as in (15). ■

The constructive proof is reported in the following subsection.

## B. Proof of Theorem 2

*Pretend*, for the moment, that the Jacobian $G_u$ is available to the controller; hence, instead of the control action (13), assume to be able to implement a control $v(t) = \Phi(y(t), G_u)$.

Consider the Lyapunov-like positive definite function

$$V(y) = \frac{1}{2} y^\top y,$$

whose Lyapunov derivative is

$$\dot{V} = y^\top \dot{y} = y^\top G_u \dot{u} = y^\top G_u v. \tag{16}$$

Then, being $G_u$ invertible, take the "pseudo" control

$$v = -\gamma(y) G_u^{-1} y, \tag{17}$$

where $\gamma(y) > 0$ is a suitable continuous scalar function, to get

$$\dot{V} = -\gamma(y) y^\top y < 0, \quad \text{for} \quad y \neq 0. \tag{18}$$

The continuous function $\gamma(\cdot)$ can be chosen so as to ensure

$$\|v\| = \|\gamma(y) G_u^{-1} y\| \leq \xi, \tag{19}$$

therefore achieving the following preliminary result.

*Proposition 1:* The "pseudo" control (17) satisfies (15) and asymptotically drives $y(t)$ to 0. ■

The existence of control (17) which satisfies (15) (or, equivalently, (19)) implies that the following result holds.

*Proposition 2:* Given $y \in \mathbb{R}^p$, for all $G_u \in \mathcal{M}$ there exists $v$, $\|v\| \leq \xi$, such that $\dot{V} \leq -\gamma(y) y^\top y$. ■

This is equivalent to writing

$$\max_{G_u \in \mathcal{M}} \min_{\|v\| \leq \xi} y^\top G_u v \leq -\gamma(y) y^\top y.$$

In view of Theorem 1, being the two sets compact and convex, it is also true that

$$\min_{\|v\| \leq \xi} \max_{G_u \in \mathcal{M}} y^\top G_u v \leq -\gamma(y) y^\top y.$$

Hence, Proposition 2 is equivalent to the following.

*Proposition 3:* Given $y \in \mathbb{R}^p$, there exists $v$, $\|v\| \leq \xi$, such that, for all $G_u \in \mathcal{M}$, $\dot{V} \leq -\gamma(y) y^\top y$. ■

The control vector can be taken as in (5), $v^* = \Phi(y)$, to achieve the "true" discontinuous control law

$$v(t) = \Phi(y(t)),$$

as in (13). Then, for all $G_u$

$$\dot{V} = y^\top G_u \Phi(y) = y^\top G_u v^*(y) \leq -\gamma(y) y^\top y. \tag{20}$$

This ensures exponential convergence, since

$$\frac{\dot{V}}{V} \leq -2\gamma(y), \tag{21}$$

with a continuous $\gamma(y) > 0$.

Therefore, if $\Phi(y)$ in (13) is taken as in (5), the control (12)–(14) guarantees that $y(t) \to 0$, and (9) is proved.

Since $u$ is the integral function of $v$, it is a continuous function. Moreover, being $y = g(u)$ invertible, $u(t) \to \bar{u}$, where $\bar{u}$ is the solution of $g(\bar{u}) = 0$, which proves (10).

Hence, Theorem 2 is proved. ■

The following corollary shows that the control $v$ can be scaled. It will be useful for considerations reported later, in Sections IV-C and VI-C.

*Corollary 1:* The control $v$ in (15) can be equivalently scaled as $\|v\| \leq \xi(y)$, where $\xi(\cdot)$ is any positive definite function of $y$, and the result in Theorem 2 follows without any modification. ■

*Proof:* The min-max problem can be restated for each $y$ as follows

$$\min_{\|v\| \leq \xi(y)} \max_{G_u \in \mathcal{M}} y^\top G_u v. \tag{22}$$

Function $\gamma(y)$ should still satisfy the condition (19), which now becomes $\|\gamma(y) G_u^{-1} y\| \leq \xi(y)$. The inequality (21) still holds for the new $\gamma(y)$. ■

*Remark 1:* The dynamics of the output $y$ can be described by $\dot{y} = G_u v$. If invertibility of $g$ is assumed, so that $u = g^{-1}(y)$, $y$ is represented by a driftless system [16], [18], for which several stabilizability results are available; these results, however, do not apply to the present case, since the model is assumed to be completely unknown. Yet, there are some analogies: also the strategy proposed here resorts to a discontinuous control, as it must be done for driftless systems [16], [18]. $\diamond$

## C. Case $p < m$

If the number of outputs is lesser than the number of inputs, Assumption 1 needs to be changed as follows.

*Definition 2 Robust right invertibility:* The polytope $\mathcal{M}$ is robustly right invertible if any matrix in $\mathcal{M}$ is right invertible. $\diamond$

*Assumption 2:* The family $\mathcal{M}$ is robustly right invertible. $\diamond$

Then, (17) can be modified by simply replacing the inverse with the pseudo inverse:

$$v = -\gamma(y) G_u^\perp y. \tag{23}$$

Along the same reasoning as in the previous subsection, the same conclusions can be reached. It is worth underlining that, in this case, there are in general multiple solutions to $g(u) = 0$ and the final value $u$ will depend on the initialization.

There is only one issue here, concerning the boundedness of $u$. Due to the lack of invertibility, the set of solutions $u$ of $y = g(u) = 0$ may be unbounded, as in the case of the "unknown" function $y = a_1 u_1 + a_2 u_2 + b$.[1] To fix the problem, take $\gamma(y) = \gamma = \text{const}$ and $\xi_0 > \gamma \|G_u^\perp\|$, for all $G_u \in \mathcal{M}$. Then $\xi(y)$, defined in Corollary 1, can be taken as $\xi(y) = \xi_0 \|y\|$.

*Proposition 4:* According to Corollary 1, let $v^*$ be the minimizer strategy in (22), such that

$$\|v^*\| \leq \xi(y) = \xi_0 \|y\|. \tag{24}$$

Then $u(t)$ is bounded and has a finite limit $\bar{u} = \lim_{t\to\infty} u(t)$.

*Proof:* Condition (21) becomes $\dot{V} \leq -2\gamma V$ and implies exponential convergence of $y(t)$ to zero. Due to (24), $v(t)$ converges to 0 exponentially as well. Then

$$u(t) = u(0) + \int_0^t v(\tau) d\tau$$

is bounded and has a finite limit. ■

*Remark 2:* The problematic case in which the number of outputs is greater than the number of inputs has been excluded from the formulation of Problem 1, because, if $p > m$, a solution to $g(u) = 0$ does not exist in general. Typically, in this case, it is possible to choose a suitable function $h(y)$ of $y$ and drive $h(y)$ to zero. ◇

### D. Existence of a Stationary Point, Local Convergence, and Constraints

***Local Convergence:*** Requiring that the condition $G_u \in \mathcal{M}$, where $\mathcal{M}$ is robustly invertible, holds globally can be too demanding in some cases. Yet, such a condition may be assured locally. For instance, given $\rho > 0$, one can consider the closed set

$$\mathcal{U}_\rho = \{u : \|g(u)\| \leq \rho\},$$

including in its interior the point $\bar{u}$ for which $g(\bar{u}) = 0$. Since the Lyapunov-like function $V(y) = y^\top y/2 = \|g(u)\|^2/2$ is nonincreasing, $\mathcal{U}_\rho$ is positively invariant.

The assumptions can be weakened by requiring $G_u \in \mathcal{M}$ in this set. Convergence $u(t) \to \bar{u}$ is then guaranteed for all initial conditions $u(0) \in \mathcal{U}_\rho$.

***Existence of a Stationary Point:*** The existence of $\bar{u}$ such that $g(\bar{u}) = 0$ *has been assumed*. However, if $\mathcal{U}_\rho$ is compact, the existence of $\bar{u} \in \mathcal{U}_\rho$ is granted. To get a proof (for $m = p$) consider the "pseudo" control (17), to get

$$\dot{u} = -\gamma(g(u)) G_u^{-1} g(u).$$

The set $\mathcal{U}_\rho$ is compact, positively invariant, and isomorphic to the closed ball $\|y\| \leq \rho$, hence it includes a stationary point $\bar{u}$ (see for instance [12]) for which $-\gamma(g(\bar{u})) G_u^{-1} g(\bar{u}) = 0$, which implies $g(\bar{u}) = 0$ (being $G_u^{-1}$ nonsingular).

Also, the global assumption $G_u \in \mathcal{M}$ (with $\mathcal{M}$ robustly invertible) implies that $\|g(u)\|$ is radially unbounded; hence, the Hadamard-Caccioppoli theorem (see [1]) guarantees the existence of a stationary point $\bar{u}$ and its uniqueness.

[1]Given a particular solution $[\bar{u}_1 \ \bar{u}_2]^\top$, the set of all the solutions has the form $[\bar{u}_1 \ \bar{u}_2]^\top + \theta[-a_2 \ a_1]^\top$, for an arbitrary $\theta$.

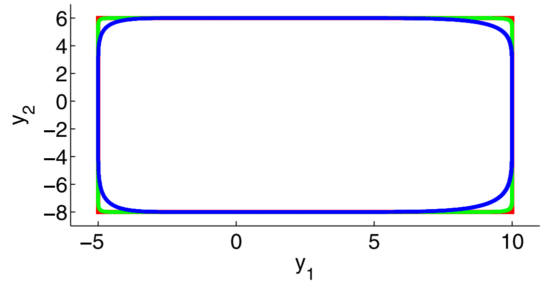


Fig. 4. The constraint set $\mathcal{Y} = \{-5 \leq y_1 \leq 10, \ -8 \leq y_2 \leq 6\}$ (red) and the 1-level sets of $V_{2q}(y)$, with $q = 4$ (blue) and $q = 20$ (green).

***Output Constraints:*** The flexibility offered by the choice of the Lyapunov-like function may be exploited to handle the presence of output constraint of the form

$$y(t) \in \mathcal{Y} = \{y \in \mathbb{R}^p : y^- \leq y \leq y^+\},$$

where vectors $y^- < 0$ and $y^+ > 0$ componentwise. The constraints are satisfied if $u(0) \in \mathcal{U}_\rho$, where $g(\mathcal{U}_\rho) \subseteq \mathcal{Y}$.

This can be quite a small portion of $\mathcal{Y}$. However, the smooth function $V_{2q} : \mathbb{R}^p \to \mathbb{R}$ can be adopted [13]:

$$V_{2q}(y) = \sqrt[q]{\sum_{i=1}^{p} \left[ \sigma_{2q}\left(\frac{y_i}{y_i^-}\right) + \sigma_{2q}\left(\frac{y_i}{y_i^+}\right) \right]}$$

where

$$\sigma_{2q}(\xi) = \begin{cases} 0 & \text{if} \quad \xi \leq 0 \\ \xi^{2q} & \text{if} \quad \xi > 0 \end{cases}$$

This function is smooth, positively homogeneous of the second order ($V_{2q}(\lambda y) = \lambda^2 V_{2q}(y)$) and its 1-level set, $\mathcal{N}[V_{2q}, 1] = \{y \in \mathbb{R}^p : V_{2q}(y) \leq 1\}$, converges to $\mathcal{Y}$ for large $q$ (as is apparent in Fig. 4; see [12] , [13] for details).

***Input Constraints:*** It is also possible to consider input constraints, such as, for instance, positivity constraints $u(t) \geq 0$ (componentwise), assuming that $g(\bar{u}) = 0$ for some positive $\bar{u}$. A first "brutal" approach is that of saturating the derivative $v_i$ when $u_i$ becomes zero: namely, if $u_i = 0$, then $v_i := \max\{0, v_i\}$. However, it is well known that this saturation introduces chattering, which would be absolutely undesirable in a tuning context.

Alternatively, the problem can be reparameterized as

$$u_i = \omega(w_i), \tag{25}$$

where $\omega : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\lim_{w_i \to -\infty} \omega(w_i) = 0$ and $\lim_{w_i \to +\infty} \omega(w_i) = +\infty$. A typical example is the exponential function. The new parameter $w = [w_1 \ldots w_m]^\top$ is then the new control action and

$$y = g(\omega(w)),$$

where $\omega(w)$ has to be intended componentwise, has Jacobian $G_w = G_u \Omega(w)$, where $\Omega(w) = \text{diag}\{\omega'(w_1), \ldots, \omega'(w_m)\}$.

Local convergence can be assured, with the only difference that the min-max problem has to be restated by considering, instead of $G_u \in \mathcal{M}$, the new constraint

$$G_w \in (\mathcal{M}\Omega(w)) = \{M\Omega(w) : M \in \mathcal{M}\}.$$

Note that the new constraint depends on $w$. Nonsingularity is locally assured, since matrix $M \in \mathcal{M}$ (nonsingular by assumption) is being multiplied by a positive diagonal matrix.

A similar approach can be pursued when dealing with both upper and lower bounds on the input, such as $0 \le u \le \hat{u}$. In this case, the reparametrization can be achieved by a function of the form

$$\omega(w_i) = \frac{\hat{u}_i}{\pi}\left[\arctan(w_i) + \frac{\pi}{2}\right],$$

such that $\lim_{w_i \to -\infty} \omega(w_i) = 0$ and $\lim_{w_i \to +\infty} \omega(w_i) = \hat{u}_i$.

## V. The Implicit Version of the Problem

In several cases, information is available exclusively on the Jacobian of the inverse function. This means that, although the decision variable is $u$ and the output is $y = h^{-1}(u)$, the only available information is on the function $h(\cdot)$. Assuming that $p = m$, the implicit invertible map is

$$h(y) = u \tag{26}$$

and its Jacobian $H_y$ is known to be of the form

$$H_y \doteq \left[\frac{\partial h}{\partial y}\right] \in \mathcal{H}, \tag{27}$$

where $\mathcal{H}$ is a polytope (or any convex and compact set) of matrices. Again, the goal is to drive $y$ to 0.

*Remark 3:* Given a polytope of matrices, the family of their inverses is not, in general, a polytope, and it may even be non-convex. For instance, take $0 \le \alpha \le 1$ and consider the polytopic family of matrices $H$ and their inverses:

$$H(\alpha) = \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix}, \quad H^{-1}(\alpha) = \frac{1}{1+\alpha^2}\begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}$$

The family $H^{-1}(\alpha)$, $0 \le \alpha \le 1$, is not convex. Hence, attempting to solve the problem by inverting the Jacobian and applying the strategy described in the previous section may be infeasible. ◇

To solve the problem directly, without inversion, a different Lyapunov-like function can be adopted and the following relation [25, Exercise 3.9, p. 156] can be exploited:

$$h(y) - h(0) = \left[\int_0^1 \frac{\partial h}{\partial y}(\sigma y) d\sigma\right] y \doteq \bar{H}_y y. \tag{28}$$

*Proposition 5:* The inclusion (27) of the Jacobian $H_y$ in $\mathcal{H}$ implies that matrix $\bar{H}_y$ in (28) belongs to $\mathcal{H}$ as well. ■

*Proof:* Define the finite sum (with increment $1/m$)

$$\Sigma_m \doteq \sum_{k=1}^{m} \frac{1}{m}\frac{\partial h}{\partial y}\left(\frac{ky}{m}\right).$$

This is a convex combination of elements $\frac{\partial h}{\partial y}(\frac{ky}{m}) \in \mathcal{H}$, with coefficients $1/m$. Hence $\Sigma_m \in \mathcal{H}$, because $\mathcal{H}$ is convex. By definition, the integral is the limit:

$$\bar{H}_y = \int_0^1 \frac{\partial h}{\partial y}(\sigma y) d\sigma = \lim_{m\to\infty} \Sigma_m \in \mathcal{H},$$

because $\mathcal{H}$ is closed. ■

Consider the following Lyapunov-like function

$$V(y) = \|h(y) - h(0)\|^2/2,$$

which is not available for control implementation, because $h(0)$ is not known. Its derivative is

$$\dot{V} = [h(y) - h(0)]^\top \frac{\partial h}{\partial y}\dot{y}. \tag{29}$$

Unfortunately, it is possible to decide $u$ only, and not $y$. However, if (26) is differentiated,

$$\frac{\partial h}{\partial y}\dot{y} = \dot{u},$$

then, denoting as before $\dot{u} = v$,

$$\dot{V} = [h(y) - h(0)]^\top v.$$

As mentioned earlier, $h(0)$ is *unknown*. However, from (28) it follows that

$$\dot{V} = y^\top \bar{H}_y^\top v. \tag{30}$$

Since Proposition 5 assures that $\bar{H}_y \in \mathcal{H}$, a polytopic set, this is exactly the same situation as in the previous case. Hence, the previous saddle-point considerations hold likewise: it is enough to consider any $H_y \in \mathcal{H}$ instead of $G_u \in \mathcal{M}$.

Now, if

$$\Phi(y) \doteq v^* \tag{31}$$

is the saddle-point decision of the minimizer,

$$\min_{v\in\mathcal{V}} \max_{H_y\in\mathcal{H}} y^\top H_y^\top v = \max_{H_y\in\mathcal{H}} \min_{v\in\mathcal{V}} y^\top H_y^\top v = y^\top (H_y^*)^\top v^*, \tag{32}$$

the control scheme is

$$\dot{u}(t) = v(t), \tag{33}$$
$$v(t) = \Phi(y(t)), \tag{34}$$
$$y(t) = \text{measured output}, \tag{35}$$

where $y$ and $u$ are related by $h(y) = u$.

*Theorem 3:* If $p = m$, $h(y) = u$ and $H_y \in \mathcal{H}$, robustly nonsingular, then Problem 1, with (8) replaced by (27), can be solved with a control scheme of the form (33)–(35), with $v$ bounded as in (15) and $\Phi$ defined as in (31). ■

*Proof:* Almost identical to that of Theorem 2. ■

*Remark 4:* In the implicit version of the problem, the case $p < m$ is not quite significant. Consider, e.g., $p = 1$ and $m = 2$. It would be $h_1(y) = u_1$ and $h_2(y) = u_2$. The only reasonable possibility is that $h_1 = h_2$ and $u_1 = u_2$, otherwise there would be an inconsistency. Hence, the problem can be reduced to the case $p = m$ by just throwing $h_2$ away. ◇

## VI. Implementation of the Scheme

For implementing the scheme, two steps are required.

1) **Off-line** Checking the robust nonsingularity (or rank completeness) of the polytope of matrices.
2) **On-line** Computing the tuning law.

### A. Checking Robust Nonsingularity (or Rank Completeness)

In the case $p = m$, before implementing the tuning scheme it is necessary to make sure that the given polytope of matrices $\mathcal{M}$ is robustly nonsingular.

Checking nonsingularity is a hard problem in general [6], especially for high-dimensional systems. For reasonable instances, however, this task can be computationally tractable and noteworthy solutions are available, as shown next.

*Proposition 6 [11], [22] Rank One Generating Matrices:* If

$$M = \sum_{i=1}^{r} d_i M_i, \quad d_i^- \le d_i \le d_i^+,$$

where $M_i$ are rank-one matrices, then robust nonsingularity is equivalent to all the vertex determinants having the same sign:

$$\det \left[ \sum_{i(\pm)} d_i^{\pm} M_i \right] > 0 \ (\text{or} < 0),$$

where the sum means, with an abuse of notation, that the coefficients $d_i$ are taken on the extrema of their intervals, obtaining $2^r$ possible combinations.[2] ■

*Proposition 7 Interval Matrices:* Consider an interval matrix $M$, having entries

$$M_{ij}^- \le M_{ij} \le M_{ij}^+.$$

Then robust nonsingularity is equivalent to all the vertex determinants having the same sign. ■

Conversely, for $p \ne m$, in general it is necessary to check that all the matrices of the family have full rank. One obvious possibility is checking if there is at least one full size square nonsingular submatrix.

For particular systems, nonsingularity can be inferred from the structure. Consider, for instance, flow systems, such as that in Fig. 1, where $g(u) = B\phi(u)$. The Jacobian matrix is $G_u = B\text{diag}\{\phi_1'(u_1), \phi_2'(u_2), \ldots, \phi_m'(u_m)\}$, with $\phi_k'$ strictly positive for all $k = 1, \ldots, m$. Since the incidence matrix of a connected graph with at least one external connection has full row rank, $G_u$ has full rank, or is nonsingular in the case $p = m$.

### B. Computing the Tuning Law $\Phi$

The control law is implemented by computing $v$ as in (13) and then computing the input $u$ by means of an integrator: $u(t) = \int_0^t v(\tau)d\tau$. The continuity of $u$, fundamental for the tuning problem, is ensured even if $v = \Phi(y)$ is not continuous. The scheme needs the measure of the output $y$ only.

[2]For instance, if $r = 2 : (d_1^-, d_2^-), (d_1^+, d_2^-), (d_1^-, d_2^+), (d_1^+, d_2^+)$.

To compute the control law (13), it is necessary to solve the min-max problem (2)–(3) and then derive the control by means of (5)–(6). To derive the control strategy, the problem is analyzed from the point of view of the maximizer, i.e., it is investigated $\Psi(y)$ as defined in (6), under the assumption that $v \in \mathcal{V} = \mathcal{S}_\xi$, the $\xi$-ball of the Euclidean norm $\|\cdot\|_2$.

*Proposition 8:* Assume that $v \in \mathcal{V} = \mathcal{S}_\xi$. Then the saddle point decision $M^* \in \mathcal{M}$ in (6), maximizing (4), can be obtained by solving the optimization problem

$$M^* \in \arg\min_{M \in \mathcal{M}} \|y^\top M\|_2. \tag{36}$$

■

*Proof:* It follows from the game-theoretic interpretation of the min-max problem. The existence of a saddle point $(v^*, M^*)$ implies that the maximizer $M \in \mathcal{M}$ can choose its action assuming that the decision of the minimizer $v$ will be based on the knowledge of its choice. For any given $M$ chosen by the maximizer, the minimizer will select $v(M) \in \mathcal{S}_\xi$ in order to minimize the scalar product $y^\top M v$:

$$v(M) = \arg\min_{\|v\|_2 \le \xi} y^\top M v = -\arg\max_{\|v\|_2 \le \xi} y^\top M v,$$

which is the vector of length $\xi$ in the opposite direction of $y^\top M$, namely

$$v(M) = -\xi \frac{M^\top y}{\|M^\top y\|_2}.$$

Then

$$y^\top M v(M) = -\xi \frac{y^\top M M^\top y}{\|M^\top y\|_2} = -\xi \|M^\top y\|_2.$$

Hence, in view of the "$-$" sign, the best strategy for the maximizer is minimizing the norm $\|M^\top y\|_2$.

*Remark 5:* The minimizer in (36) can be nonunique for some $y$. For instance, consider the family

$$M = \begin{bmatrix} 1 & 1 \\ -\alpha & 1 \end{bmatrix}, \quad 1 \le \alpha \le 2,$$

and $y^\top = [1\,0]$. Quite interestingly, however, the quantity $y^\top M^*$ is unique, given $y$, as will be seen later. ◇

Proposition 8 can be extended to the case of any norm. Given a norm $\|\cdot\|$, define $\|\cdot\|_*$, the dual norm, as [27]

$$\|x\|_* = \max_{\|z\| \le 1} z^\top x.$$

For instance, $\|\cdot\|_\infty$ has dual $\|\cdot\|_1$, while the dual of $\|\cdot\|_2$ is $\|\cdot\|_2$ itself.

*Proposition 9:* Assume that $v \in \mathcal{V}$, where $\mathcal{V}$ is the $\xi$-ball of any norm. Then a saddle point decision $M^* \in \mathcal{M}$ in (6), maximizing (4), can be obtained by solving the optimization problem

$$M^* \in \arg\min_{M \in \mathcal{M}} \|y^\top M\|_*. \tag{37}$$

■

*Proof:* Again, for any given $M$ chosen by the maximizer, the minimizer will select

$$v(M) \in \arg\min_{\|v\|\leq\xi} \; y^\top M v = -\arg\max_{\|v\|\leq\xi} y^\top M v.$$

Hence, again, the best strategy for the maximizer is minimizing the norm $\|M^\top y\|_*$. ■

Once the saddle-point strategy $M^*$ for the maximizer has been established, the control strategy is decided as

$$v^* = \Phi(y) \in -\arg\max_{\|v\|\leq\xi} y^\top M^* v.$$

### C. Computational Issues

A polytope of matrices has elements of the form

$$M = \sum_{k=1}^{r} M_k \alpha_k$$

with $M_k \in \mathbb{R}^{p\times m}$ for all $k = 1, \ldots, r$ and

$$\alpha \in \mathcal{A} = \left\{ \hat{\alpha} \; : \; \sum_{k=1}^{r} \hat{\alpha}_k = 1, \;\; \hat{\alpha}_k \geq 0 \right\}.$$

In general, it may be assumed that vector $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a polytope. Some relevant cases might be of interest.

1) If the control is bounded by the Euclidean norm, then problem (36) is a standard minimum-Euclidean-norm problem.
2) If the control is bounded by the $\infty$ norm, then problem (37) is a linear programming problem (in fact, $\|\cdot\|_*$ is $\|\cdot\|_1$).

In the Euclidean norm case,

$$\alpha^*(y) \in \arg\min_{\alpha\in\mathcal{A}} \left\| \sum_{k=1}^{r} \alpha_k y^\top M_k \right\|_2 \tag{38}$$

must be computed and then, denoting by $z_k(y) \doteq M_k^\top y$,

$$\Phi(y) = v(\alpha^*) = -\xi \frac{\sum_{k=1}^{r} \alpha_k^* z_k(y)}{\|\sum_{k=1}^{r} \alpha_k^* z_k(y)\|_2}. \tag{39}$$

This control is not continuous at $y = 0$, and introduces chattering on $v$ (though not on $u$). Yet, a sampled data implementation may introduce ripples on $u$. To face this issue, however, the bound $\xi > 0$ introduced before can be rediscussed. Indeed, it is possible to consider a set $\mathcal{V}_y$, function of $y$, given by $\|v\|_2 \leq \xi(y)$, where $\xi(y) > 0$ converges to zero as $y \to 0$ (as in Corollary 1). In particular, taking

$$\xi(y) = \zeta \left\| \sum_{k=1}^{r} \alpha_k^* z_k(y) \right\|_2, \tag{40}$$

for some positive $\zeta$, provides

$$\Phi(y) = v(\alpha^*) = -\zeta \sum_{k=1}^{r} \alpha_k^* M_k^\top y. \tag{41}$$

This choice introduces regularity in the system. Indeed, $\Phi$ is continuous at $y = 0$ ($\Phi(0) = 0$). As can be seen in simulations, this control introduces a nice "smooth" behavior. Finally, note that $\alpha_k^* z_k(y)$ is the smallest Euclidean norm in a polytope, hence $\Phi(y) = v(\alpha^*)$ is uniquely defined [19].

In the infinity norm case, (37) is considered, which becomes

$$\alpha^*(y) \in \arg\min_{\alpha\in\mathcal{A}} \left\| \sum_{k=1}^{r} \alpha_k y^\top M_k \right\|_1.$$

This problem can be solved via linear programming

$$\begin{aligned} \min_{\alpha} \quad & \bar{1}^\top z^+ + \bar{1}^\top z^- \\ & \text{s.t.} \\ & \sum_{k=1}^{r} \alpha_k y^\top M_k = z^+ - z^-, \\ & z^+ \geq 0, \quad z^- \geq 0 \quad \text{(componentwise)}, \end{aligned}$$

where $\bar{1}^\top = [1\; 1\; \ldots 1]$.

In the case of linear programming, $\alpha^*$ might be nonunique. To solve ambiguities, the minimum-Euclidean-norm $\alpha$ inside the set of optimal values can be taken. Assuming $\|v\|_\infty \leq \xi$, the minimizer decision is

$$v(\alpha^*) = -\xi \text{sign} \left[ \sum_{k=1}^{r} \alpha_k^* z_k^\top(y) \right], \tag{42}$$

where vector sign$[\cdot]$ is the componentwise sign function.

*Remark 6:* Function $v(t)$ is not continuous in general (although continuity of its integral value $u(t)$ is assured anyway). The resulting differential equations have discontinuous right hand side and their solutions can be defined in the frame of differential inclusions [3]. In the case of the $\infty$-norm, the proposed control (42) produces a bang-bang type of strategy. This may be of interest in some practical situations, for instance in the case of fluid valves or locks. Valves are often actuated at a constant opening speed $v \in \{-\xi, 0, \xi\}$, and this fits nicely with the suggested control. ◇

## VII. Static Versus Dynamic Plants

The proposed technique has been applied to *static* plants. If the plant is dynamic, e.g., of the form

$$\dot{x}(t) = A[-x(t) + g(u(t))], \quad y(t) = x(t),$$

with matrix $A$ representing the dynamics, then the problem of driving $x$ to 0 could be faced in principle by means of a static state-feedback control $u = \Phi(x)$, without the considered integral action $\dot{u} = v$ and $v = \Phi(x)$.

However, this control would not solve the considered tuning problem. Indeed, since $g(u)$ is unknown, a robust control function $\Phi$ would not be continuous in general [23], [24], [8], [21], [30], [14]. The condition $x(t) \to 0$ can be achieved because the dynamics of the plant regularizes the signal, producing a continuous $x$ (roughly, because the transfer function from the input $g$ to $x$ is strictly proper). Unfortunately, *the main tuning goal* (i.e., *driving* $u(t)$ *to* $\bar{u}$ *such that* $g(\bar{u}) = 0$) *would not be accomplished* if $\Phi$ is discontinuous.

In the case of static plants, the situation is even worse. The min-max strategy requires a pure integrator. For instance, a proportional-integral action would not be suitable, because it would introduce discontinuities in both $u$ and $y$.

In addition, an integrator is necessary to have a zero steady-state error (even for an affine plant $y = Gu + r$ with unknown matrix $G \in \mathcal{M}$ nonsingular and unknown $r$).

On the other hand, introducing the integrator might be troublesome if the plant approximately modeled as a static one is actually dynamic (for instance, a singularly perturbed system [21], [30]): the coupling between the integrator and the plant dynamics might produce oscillations and even instability. In this case, to preserve closed-loop stability, the scheme needs to be "slow enough" compared with the plant dynamics, so that a suitable time-scale separation between the tuner and the plant dynamics is ensured. In practice, this is achieved by taking the scaling function $\xi(y)$ in Corollary 1 sufficiently small, which means that the tuning speed is slow enough and the plant dynamics is not excited. A possible choice is

$$\|v\| \leq \xi(y) = \xi_0 \|y\|, \tag{43}$$

with $\xi_0 > 0$ small enough. We conjecture that taking $\xi_0$ small compared to the system time constants can ensure stability, although we do not have a general proof so far (and we believe it would not be a trivial achievement).

The conjecture is supported by the analysis of the scalar case. In view of Proposition 3, under nonsingularity assumptions, the proposed scheme would assure convergence to 0 of the system $\dot{y} = G(t)v$ for arbitrary time-varying $G(t) \in \mathcal{M}$, since the derivative (20) is negative. Assume that the scalar plant has dynamics

$$\tau \dot{y} = -y + g(u), \tag{44}$$

with time constant $\tau > 0$, and $g$ is an unknown sector-bounded scalar function $g : \mathbb{R} \to \mathbb{R}$ with derivative bounded as

$$g'(u) = D(1 + \Delta), \quad |\Delta| \leq \eta < 1,$$

which assures nonsingularity. In this simple case, it is not difficult to see that the min-max control subject to (43) is

$$v^* = \xi_0 |y| (-\text{sign}(y)) = -\xi_0 y.$$

Writing the derivative of (44) and assuming $\dot{u} = v = v^*$ yields

$$\tau \ddot{y} + \dot{y} - g'(u) v = \tau \ddot{y} + \dot{y} + \xi_0 D(1 + \Delta) y = 0.$$

According to the theory of absolute stability for sector-bounded nonlinearities [25, Ch. 10], this expression is equivalent to the loop between the linear time invariant system having transfer function

$$y = \frac{\xi_0 D}{\tau s^2 + s + \xi_0 D} \nu \doteq F(s) \nu$$

and the operator $\nu = -\Delta(t) y$ with $|\Delta| \leq \eta$. Stability holds if the $\mathcal{H}_\infty$ norm of the transfer function is bounded as

$$\sup_{\omega \geq 0} |F(j\omega)| < \frac{1}{\eta},$$

where $j = \sqrt{-1}$. Note that $1/\eta > 1$, so $|F(j0)| = 1 < 1/\eta$. The condition is fulfilled at all frequencies $\omega > 0$ (i.e., there are no resonance peaks) provided that

$$\xi_0 < \frac{1}{2D\tau}, \tag{45}$$

because in this case the modulus $|F(j\omega)|$ is decreasing. Hence, condition (45) ensures absolute stability.

The application of this technique requires the knowledge of an upper bound of the time constant $\tau$. It is so far unclear how to generalize this result, extending (45) to the multidimensional case.

## VIII. Examples

### A. The Flow Problem

Reconsider the flow problem of Fig. 1, whose equations are

$$y = B\phi(u) - \bar{r},$$

where $\phi(u)$ is a vector of strictly increasing smooth functions $\phi(u) = [\phi_1(u_1)\ \phi_2(u_2) \ldots, \phi_6(u_6)]^\top$. The Jacobian is

$$B\text{diag}\{\phi_1'(u_1), \phi_2'(u_2), \ldots, \phi_6'(u_6)\} = BD$$

where $D$ is a diagonal matrix with positive diagonal entries, hence $BD$ has full rank. The bounds on the derivatives are

$$0.5 \leq D_i \leq 5, \quad i = 1, \ldots, 6.$$

The min-max strategy is very simple. For each $y$, consider

$$\min_{0.5 \leq D_i \leq 5} \|y^\top BD\| = \min_{0.5 \leq D_i \leq 5} \sqrt{\sum_{i=1}^{6} [y^\top B]_i^2 D_i^2}.$$

The minimum is clearly achieved at $D_i = 0.5$ for all $i$:

$$y^\top BD^* = 0.5 y^\top B.$$

For simulation purposes, functions of the form

$$\phi_i(u_i) = \alpha_i u_i + \beta_i \arctan(u_i)$$

have been considered, but any strictly increasing function would work. The coefficients are

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\alpha_i$ | 12 | 8 | 6 | 6 | 5 | 5 |
| $\beta_i$ | 0.8 | 0.5 | 0.4 | 0.3 | 0.2 | 0.4 |

The target flow is

$$\bar{r} = [\ 5\ \ 5\ \ 5\ \ 5\ ].$$

The initial control value is

$$u(0) = [\ 2\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ ]^\top,$$

which corresponds to the initial relative flow

$$y(0) = [\ -1.0568\ \ -1.7644\ \ -4.0785\ \ 11.7854\ ]^\top.$$

The transient achieved by adopting the strategy (42), with $\xi = 5$, is represented in Fig. 5. The transient obtained by adopting strategy (41), with $\zeta = 5$, is reported in Fig. 6 and is smoother.

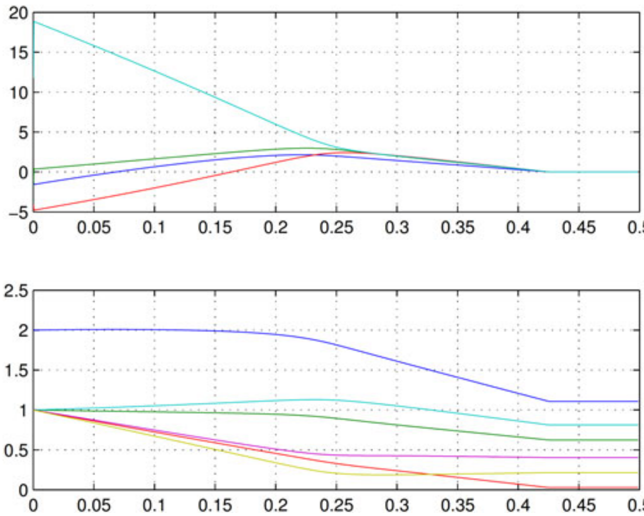


Fig. 5. The transients of the relative flow $y$ (top) and of the control $u$ (bottom) obtained by applying strategy (42).

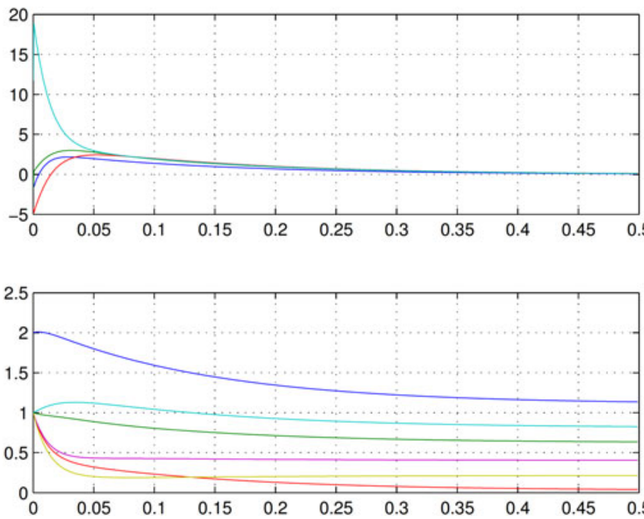


Fig. 6. The transients of the relative flow $y$ (top) and of the control $u$ (bottom) obtained by applying strategy (41).

## B. The Heating System

Reconsider now the system in Fig. 3. As discussed, a qualitative implicit model of the form

$$u_k = \psi_k(p, y_k)$$

must be handled, where $u_k = a_k$ and $y_k = q_k$, $k = 1, 2, 3$. Functions $\psi_k(\cdot,\cdot)$ are decreasing in $p$ and increasing in $y_k$. Moreover,

$$p = p_0 - \Psi\left(\sum_{i=1}^{3} y_i\right),$$

with $\Psi$ increasing. The Jacobian of this inverse relation is of the form

$$H(\alpha,\beta,\gamma,\delta,\mu,\nu) = \begin{bmatrix} \alpha+\beta & \alpha & \alpha \\ \mu & \mu+\nu & \mu \\ \gamma & \gamma & \gamma+\delta \end{bmatrix},$$

where Greek letters denote positive quantities: $\alpha = -\frac{\partial\psi_1}{\partial p}\Psi'$, $\beta = \frac{\partial\psi_1}{\partial y_1}$, $\mu = -\frac{\partial\psi_2}{\partial p}\Psi'$, $\nu = \frac{\partial\psi_2}{\partial y_2}$, $\gamma = -\frac{\partial\psi_3}{\partial p}\Psi'$, $\delta = \frac{\partial\psi_3}{\partial y_3}$.

Assuming a lower and an upper bound on all these variables, $0 < b_l < b_u$, the Jacobian is robustly non singular. The minimum norm problem for the maximizer is $\min \|y^\top H^\top\|$:

$$\min_{b_l \le \alpha,\beta,\gamma,\delta,\mu,\nu \le b_u} [\alpha(y_1+y_2+y_3)+\beta y_1]^2 + [\mu(y_1+y_2+y_3)+\nu y_2]^2 + [\gamma(y_1+y_2+y_3)+\delta y_3)]^2 . \tag{46}$$

This is a standard linear-quadratic constrained problem. Once the optimal is found, $H^* = H(\alpha^*,\beta^*,\gamma^*,\delta^*,\mu^*,\nu^*)$, the two strategies can be simulated: (39), such that $v = -\xi\frac{H^*y}{\|H^*y\|}$, and the "more gentle" (41), such that $v = -\zeta H^* y$.

The reported simulations are based on the following assumptions and data. The input pressure $p_0$ is constant and equal to $2.5 \cdot 10^5$ Pa. The output pressure $p_1$ is constant as well and equal to $1.1 \cdot 10^5$ Pa. The pressure drop in the first pipe is modeled as

$$p_0 - p = 2\rho\frac{L}{D}fV^2 = 2\rho\frac{L}{D}f\left(\frac{y_1+y_2+y_3}{S}\right)^2 \tag{47}$$

where $\rho = 10^3$ Kg/m$^3$ is the fluid density, $L = 5$ m is the length of the pipe, $D = 0.03$ m its diameter, $f = 2.5 \cdot 10^{-3}$ is a friction coefficient, $V$ is the fluid velocity, and $S$ is the cross sectional area of the pipe. The cross sectional areas of the three branches are taken as $S_1 = S_2 = S_3 = \dfrac{S}{3}$ (thus $y_k = V_k S/3$) and the pressure drop in each of the branches is assumed to be due to the valve only (thus, the drop due to the friction in the pipes is neglected). More precisely:

$$\frac{p-p_1}{\rho} = \frac{1}{2}\frac{\sigma}{u_k}V_k^2 = \frac{9}{2S^2}\frac{\sigma}{u_k}y_k^2 \tag{48}$$

where $0 < u_k \le 1$ is the valve opening fraction and $\sigma = 0.15$ is a valve flow coefficient that models the pressure drop due to the fully opened valve. From the (48), it follows that

$$u_k = \frac{9}{2S^2}\sigma y_k^2 \frac{\rho}{p-p_1} \doteq \psi_k(p, y_k) \tag{49}$$

and from (47)

$$p = p_0 - 2\rho\frac{L}{D}f\left(\frac{y_1+y_2+y_3}{S}\right)^2 \doteq p_0 - \Psi\left(\sum_{i=1}^{3} y_i\right).$$

By taking $b_l = 5 \cdot 10^{-1}$ and $b_u = 5 \cdot 10^3$, plant tuning amounts to finding the minimizer $H^*$ of (46) for each measured output vector $y(t) = [y_1(t)\ y_2(t)\ y_3(t)]^\top$ and applying the control (12), where $v$ is chosen according to (39) or (41). The valve opening fraction at time $t = 0$ is set as $u_1(0) = 0.8$, $u_2(0) = 0.1$ and $u_3(0) = 0.3$. The target flows for the three pipes were set as $\bar{q}_1 = 2.5 \cdot 10^{-3}$m$^3$/s, $\bar{q}_2 = 4.1 \cdot 10^{-3}$m$^3$/s and $\bar{q}_3 = 1.5 \cdot 10^{-3}$m$^3$/s. Note that, if the target point is shifted to zero, the actual output variables become $y_k(t) = q_k(t) - \bar{q}_k$.

The flow transients and the input (i.e., the valve opening fractions) transients achieved by adopting the strategy (39), with $\xi = 2 \cdot 10^{-3}$, are represented in Fig. 7. The flow transients and the input (i.e., the valve opening fractions) transients obtained

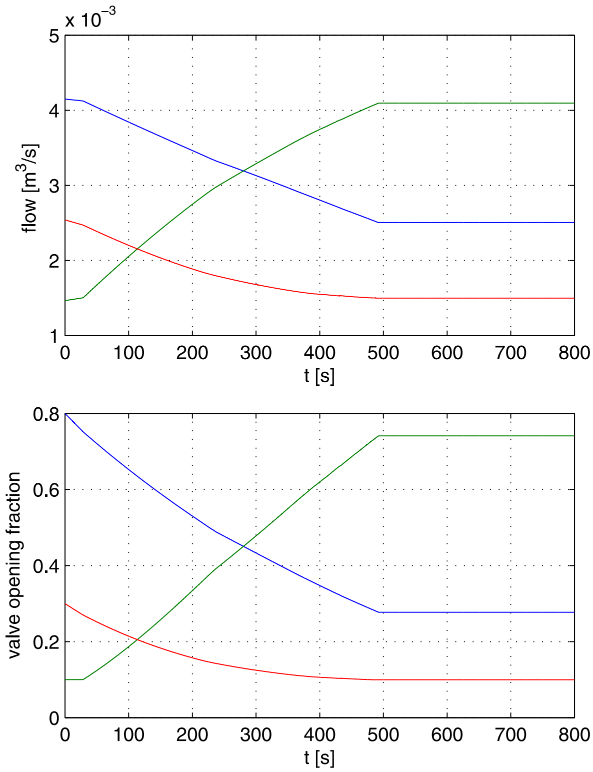


Fig. 7. Heating system: the transients of the flows (top) and of the valve opening fractions (bottom) obtained by applying strategy (39).

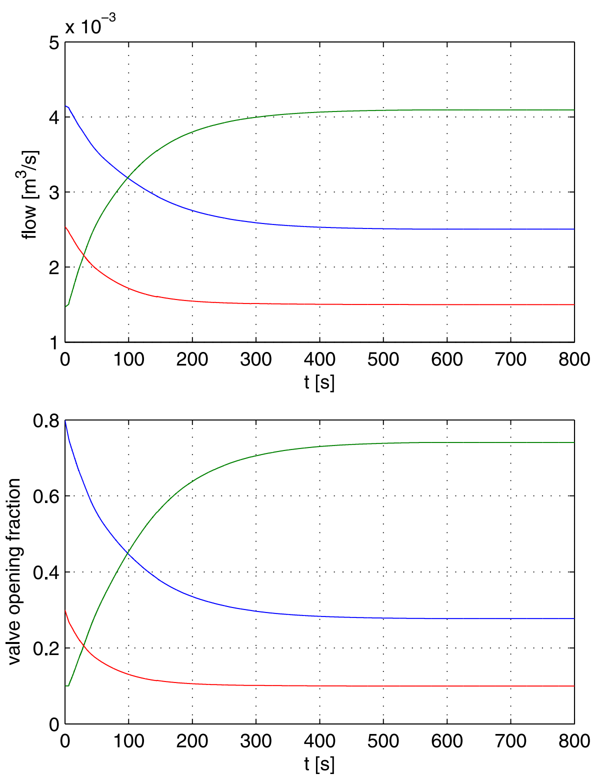


Fig. 8. Heating system: the transients of the flows (top) and of the valve opening fractions (bottom) obtained by applying strategy (41).

by applying (41) with $\zeta = 10$ are reported in Fig. 8. In both cases, a transient of 800 s of the ZOH sampled system, with sampling time of 0.1 s, is reported.

## IX. Concluding Discussion

Plant tuning is often a frustrating operation because, due to the lack of reliable models, it requires trial-and-error procedures. It has been shown that, under suitable assumption on the Jacobian of the unknown plant model, tuning can be performed by means of an automatic procedure. Both an explicit and an implicit model representation have been considered; in both cases, the robust solution is based on a Lyapunov approach and exploits a well known saddle point result for min-max games. It has been also shown that constraints on both the input and the output variables can be easily dealt with, by means of a reparametrization of the problem (in the case of input constraints) or by adopting a different Lyapunov-like function, tailored so that its 1-level set is arbitrarily close to the constraint set (in the case of output constraints).

The case in which the Jacobian of the static function is included in a known polytope has been mainly considered. However, cases of unstructured (norm-bounded) uncertainties can be dealt with as well, by considering classes of convex bounding sets more general than polytopes.

It is worth underlining that the proposed method can be adopted, in general, for the solution of systems of nonlinear equations, $g(u) = 0$, with guaranteed convergence. The simulations proposed in the example section certify this fact, since the value of $u$ assuring the desired output was unknown before running the simulation and has been found by means of the proposed procedure. Clearly, global convergence is assured because a bound is known for the Jacobian, which is not in general true when solving nonlinear equations.

The digital implementation of the scheme deserves further investigation. Other interesting problems, not considered so far, include the case in which the set $\mathcal{M}$ is either dependent on $u$, $\mathcal{M}(u)$, or $y$, $\mathcal{M}(y)$, or can change due to different working conditions or failures.

Since the problem can be formulated in terms of linear and quadratic programming problems, having an efficient numerical solution, the problem dimensionality does not seem a crucial issue. Hence, the proposed approach can tackle systems of a very large scale, where automatic tuning can be a big deal. We believe that the results of this paper can be applied to several tuning problems, concerning for instance nonlinear flow networks, power networks, and industrial plants. In fact, numerical simulations have evidenced that the method can be applied to larger plants (in terms of number of inputs and outputs) than those presented here to illustrate the technique.[3] Numerical tests have also revealed that, sometimes, numerical issues may arise due to ill-conditioning: although nonsingularity is ensured, the Jacobian bounding polytope includes matrices that are close to singularity. Such a phenomenon seems more likely to occur

[3]See http://users.dimi.uniud.it/~franco.blanchini/plantuning.zip for testing codes available on-line.

when the number of inputs is equal to the number of outputs. This issue is left as a future research direction.

## References


[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*. Cambridge, U.K.: Cambridge University Press, 1993.

[2] H.-S. Ahn, Y. Q. Chen, and K. L. Moore, "Iterative learning control: Brief survey and categorization," *IEEE Trans. Syst., Man, Cybern. C: Appl. Rev.*, vol. 37, pp. 1099–1121, 2007.

[3] J. P. Aubin and A. Cellina. *Differential Inclusions*. Berlin, Germany: Springer-Verlag, 1984.

[4] K. J. Åström, "Theory and applications of adaptive control—A survey," *Automatica*, vol. 19, no. 5, pp. 471–486, 1983.

[5] T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory, Volume 23 of Classics in Applied Mathematics*. Philadelphia, PA, USA: Soc. Ind. Appl. Math. (SIAM), 1999 (reprint of 2nd (1995) ed.).

[6] B. R. Barmish, *New Tools for Robustness of Linear Systems*. New York, NY, USA: Macmillan, 1994.

[7] D. Bauso, "Game theory: Models, numerical methods and applications," *Found. Trends Syst. Control*, vol. 1, no. 4, pp. 379–522, 2014.

[8] F. Blanchini, "The gain scheduling and the robust state feedback stabilization problems," *IEEE Trans. Autom. Control*, vol. 45, no. 11, pp. 2061–2070, 2000.

[9] H. G. Beyer and B. Sendhoff, "Robust optimization—A comprehensive survey," *Comput. Methods Appl. Mech. Eng.*, vol. 196, no. 33–34, pp. 3190–3218, 2007.

[10] F. Blanchini, G. Fenu, G. Giordano, F. A. Pellegrino, "Plant tuning: A robust Lyapunov approach," *Proc. 54th IEEE Conf. Decision Control*, Osaka, Japan, Dec. 15–18, 2015.

[11] F. Blanchini and G. Giordano, "Piecewise-linear Lyapunov functions for structural stability of biochemical networks," *Automatica*, vol. 50, no. 10, pp. 2482–2493, 2014.

[12] F. Blanchini and S. Miani, "Set-theoretic methods in control," in *Systems & Control: Foundations & Applications*. 2nd ed. Basel, Switzerland: Birkhäuser, 2015.

[13] F. Blanchini, S. Miani, and W. Ukovich, "Control of production-distribution systems with unknown inputs and system failures," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1072–1081, 2000.

[14] F. Blanchini and R. Pesenti, "Min-max control of uncertain multi-inventory systems with multiplicative uncertainties," *IEEE Trans. Autom. Control*, vol. 46, no. 6, pp. 955–959, 2001.

[15] D. A. Bristow, M. Tharayil, and A. G. Alleyne, "A survey of iterative learning control," *IEEE Control Syst.*, vol. 26, no. 3, pp. 96–114, June 2006.

[16] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, R. W. Brockett, R. S. Millman, and H. J. Sussman, Eds., Boston, MA, USA: Birkhauser, 1983, pp. 181–191.

[17] A. E. Bryson Jr., Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. New York, NY, USA: Taylor & Francis, 1975.

[18] J.-M. Coron, "Global asymptotic stabilization for controllable systems without drift," *Math. Control, Signals, Syst.*, vol. 5, pp. 295–312, 1991.

[19] A. Dax, "The smallest point of a polytope," *J. Optim. Theory Appl.*, vol. 64, no. 2, pp. 429–432, 1990.

[20] A. Fradkov, "A scheme of speed gradient and its application in problems of adaptive control," *Autom. Remote Control*, vol. 40, no. 9, pp. 1333–1342, 1980.

[21] R. A. Freeman and P. V. Kokotović, Robust nonlinear control design. State-space and Lyapunov techniques, *Systems and Control: Foundations and Applications*. Boston, MA, USA: Birkhäuser, 1996.

[22] G. Giordano, C. C. Samaniego, E. Franco, F. Blanchini, "Computing the structural influence matrix for biological systems," *J. Math. Biol.*, vol. 72, no. 7, pp. 1927–1958, 2016.

[23] S. Gutman and G. Leitmann, "Stabilizing control for linear systems with bounded parameter and input uncertainty," *Optimization Techniques Modeling and Optimization in the Service of Man Part 2*, Lecture Notes in Computer Science.New York, NY, USA: Springer, pp. 729–755, 1976.

[24] S. Gutman, "Uncertain dynamical systems—A Lyapunov min-max approach," *IEEE Trans. Autom. Control*, vol. 24, no. 3, pp. 437–443, 1979.

[25] H. Khalil. *Nonlinear Systems*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[26] S. Z. Khong, D. Nešić, Y. Tan, C. Manzie, "Unified frameworks for sampled-data extremum seeking control: Global optimisation and multi-unit systems," *Automatica*, vol. 49, no. 9, pp. 2720–2733, 2013.

[27] D. G. Luenberger, *Optimization by Vector Space Methods*. New York, NY, USA: Wiley, 1969.

[28] A. M. Meılakhs, "Design of stable control systems subject to parametric perturbation," *Autom. Remote Control*, vol. 39, no. 10, pp. 1409–1418, 1979.

[29] D. Nešić, A. Mohammadi, C. Manzie, "A framework for extremum seeking control of systems with parameter uncertainties," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 435–448, 2013.

[30] Z. Qu, *Control of Nonlinear Uncertain Systems*. New York, NY, USA: Wiley, 1998.

[31] Y. Tan, D. Nešić and I. Mareels, "On non-local stability properties of extremum seeking control," *Automatica*, vol. 42, no. 6, pp. 889–903, 2006.


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