Weak Dutch Books with Imprecise Previsions

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Abstract

Uncertainty assessments for imprecise previsions based on coherence and related concepts require that the suprema of certain random numbers (interpreted as gains) are non-negative. The extreme situation that a supremum is zero represents what is called a Weak Dutch Book (WDB) in a betting interpretation language. While most of the previous dedicated literature focused on WDBs for de Finetti's coherence with precise probabilities, in this paper we analyse the properties of WDBs with imprecise previsions, notably for conditional (Williams') coherent lower previsions. We show that WDB assessments ensure a certain 'local precision' property and imply, in the agent's evaluation, some kind of 'protection' against real losses. Further, these properties vary with the consistency notion we adopt, tending to vanish with weaker ones. A generalisation of the classical strict coherence and other alternative approaches to WDBs are also discussed.

Keywords. Weak Dutch Books,(Williams') coherent lower previsions, de Finetti's coherence, strict consistency

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1 Introduction

Modelling uncertainty by means of imprecise - *lower* or *upper* - *previsions* is a very general approach, encompassing several uncertainty measures as special instances, among them (precise) previsions and probabilities. Although *coherence* is probably the most known consistency criterion for imprecise previsions [17], weaker criteria are also widespread, such as the condition of avoiding sure loss (ASL) [17], convexity, centered or not [7], and other ones. Even the definition of coherence is not unique, in a conditional setting: Walley's coherence [17] and the broader Williams' coherence (W-*coherence*) [18, 19] are the fundamental ones.

The consistency notions we mentioned share a common structure in their definitions, going back to de Finetti's approach to subjective probability [3]: they require that the supremum of a certain *gamble* (bounded random number, in this specific instance called *gain*) is non-negative, while differing as for the rules for obtaining the admissible gains. For instance, in the simplest case that a (precise) probability P is assessed on an arbitrary set of (indicators of) events $\mathcal{D}$, we say that P is dF-*coherent* on $\mathcal{D}$ if, for all positive $n \in \mathbb{N}$, for all $E_1, \ldots, E_n \in \mathcal{D}$, for all $s_1, \ldots, s_n \in \mathbb{R}$, defining

$$G = \sum_{i=1}^{n} s_i \big(E_i - P(E_i)\big), \tag{1}$$

it holds that $\max G \geq 0$. (Here $\sup G$ is attained, as the image set of G is finite.) The other instances are analogous: in the most general situations we consider, the gain is a function of a finite number of conditional gambles and of their (precise, lower or upper) previsions, and is itself a conditional gamble (cf. the later Definitions 2.1, 2.2, 2.3, 2.4). Each gain actually has the meaning of a *gain* in a *betting* or *behavioural interpretation*. Under this interpretation, the agent assessing the uncertainty evaluation considers (at least hypothetically) buying/selling operations involving a finite number of events ($E_1, \ldots, E_n$ in (1)), but more generally of conditional gambles (those the gain is a function of) at prices given by $P(E_1), \ldots, P(E_n)$ in (1) or, in general, given by their precise (P), lower ($\underline{P}$) or upper ($\overline{P}$) previsions.

With reference to the betting interpretation, the limiting situation $\max G = 0$ in (1) is termed *Weak Dutch Book* (WDB): an agent buying/selling $s_i E_i$ for $s_i P(E_i)$, with $i = 1, \ldots, n$, would at best gain nothing, but otherwise lose, from the global transaction. More generally, we shall denote Weak Dutch Book any situation where the supremum of some gain is 0, in the definition of a consistency concept for imprecise previsions.

WDBs have not been extensively investigated in the literature. Most of the research focused on WDBs for dF-coherent probabilities, concentrating in the fifties of the last century, when de Finetti's theory was getting widespread (cf. [6, 13]). Often, the perspective was that of trying to avoid WDBs, because of their counterintuitive behavioural meaning. For this, the notion of *strict coherence* was elaborated, but later essentially dropped, being subject to rather severe restrictions.

Little has instead been written as for the properties of an uncertainty assessment incurring a WDB, even less outside dF-coherence. In particular, the agent's beliefs of incurring a real loss were investigated in [2] for dF-coherent probabilities, and in [15] for (unconditional) coherent lower/upper previsions.

The purpose of this paper is to contribute to fill this gap, also extending the work in [15]. Precisely, we will investigate:

- Basic properties of assessments incurring WDBs
- The implications of a WDB assessment on the agent's outlooks of escaping a real Dutch Book (i.e., in general, of avoiding a uniformly negative loss)
- *Strict consistency*, the generalisation of strict coherence outside dF-coherence.

We shall mainly develop the theory with reference to W-coherence, since this notion is a rather general one. Moreover, as we shall demonstrate on the way, the properties ensured by W-coherence may become less significant or vacuous with weaker notions, such as convexity.

The paper is organised as follows: Section 2 contains preliminary definitions and basic results to be applied in the sequel. In Section 3, we first present simple introductory examples of WDBs, to be rediscussed later. Next we show that if a WDB is effective for a lower prevision $\underline{P}$ and some gain $\underline{G}$, then $\underline{P}$ 'tends' to be *locally precise*. Roughly speaking, we mean with this that the restriction of $\underline{P}$ on the set of the conditional gambles in $\underline{G}$ is subject to certain contraints, which may be as strong as to require that $\underline{P}$ is a precise prevision there, in the case that (the unrestricted) $\underline{P}$ is a coherent lower prevision. We analyse how these constraints vary, depending on the degree of consistency of $\underline{P}$.

In Section 4 we focus on the relationships between WDBs with W-coherent previsions and events with positive (lower) probability. The basic result is Proposition 4.1, ensuring that if $\sup(\underline{G}|B) = 0$ and $\underline{P}(E|B) = 0$, then also $\sup(\underline{G}|B \wedge E) = 0$. If $E|B$ is an atom, $\omega|B$, of a partition on which $\underline{G}|B$ is defined, it follows that $\underline{G}|B$ assumes its maximum of 0 (at least) at $\omega|B$. Related facts about maxima or suprema of WDB gains are then explored.

Section 5 is concerned with the implications of the agent's assessment on her/his beliefs about incurring a real Dutch Book. Suppose that $\sup \underline{G} = 0$. In the simpler situation that $\sup \underline{G}$ is achieved, i.e. $\max \underline{G} = 0$, this means: which is the agent's opinion about her/his losing money, i.e. about the event $(\underline{G} < 0)$? (More generally, if $\sup \underline{G}$ is not necessarily achieved, losses bounded away from zero are considered, i.e. events ($\underline{G} \leq -\varepsilon$), for any $\varepsilon > 0$.) For a dF-coherent prevision P and a gain G incurring a WDB, it is known that $P(G < 0) = 0$ if G has a maximum (more generally $P(G \leq -\varepsilon) = 0, \forall\varepsilon > 0$) [13, 15]. We show that these reassuring beliefs are replaced by increasingly weaker ones when departing more and more from dF-coherence. Our results extend those in [15] for unconditional lower/upper coherent previsions, by investigating convex previsions and the conditional case too. Moreover, a number of results are derived from a more general one, concerning zero supremum gambles, but independent of WDBs (Proposition 5.3, Corollary 5.1).

In Section 6 strict consistency is defined as a straightforward generalisation of strict coherence, and it is proven that a W-coherent assessment is strictly consistent if and only if no non-impossible conditional event is given zero lower probability (Proposition 6.1). This generalises prior results for dF-coherent probabilities in [6]. In a second part, we briefly discuss two alternative approaches to WDBs, the one presented in [16] and that based on desirability. In Section 7 we present a summarised analysis of the results obtained.

2 Preliminaries

2.1 Describing uncertainty

We recall first some familiar concepts for describing uncertainty.

A *partition* $\mathbb{P}$ of the sure event Ω is a set of pairwise disjoint events, termed (when non-impossibile) *atoms*, whose logical sum (union, in set-theoretical language) is Ω. A *random number* X is (described by) a map $X : \mathbb{P} \to \mathbb{R}$, with image set $\{X(\omega) : \omega \in \mathbb{P}, \omega \neq \emptyset\}$. The partition $\mathbb{P}$ is not unique, for instance any partition finer than $\mathbb{P}$ could equally well be selected to describe the same X: this is necessary when a single partition describes, at a time, more random numbers, initially defined on different partitions.

In the sequel we shall consider *bounded* random numbers, called *gambles*. The simplest non-trivial gamble is the *indicator* I_A of an event A: $I_A(\omega) = 1$, if $\omega \Rightarrow A$, $I_A(\omega) = 0$, if $\omega \Rightarrow \neg A$, $\forall\omega \in \mathbb{P}$, $\omega \neq \emptyset$ (the coarsest partition I_A is defined on is $\{A, \neg A\}$). Shortly, I_A is 1 (0) iff event A is true (false). Because of this one-to-one correspondence, we shall often not distinguish explicitly I_A and A, using the same symbol A for both.

Given a partition $\mathbb{P}$, call $\mathcal{L}(\mathbb{P})$ the linear space of all gambles defined on $\mathbb{P}$. The set of all (indicators of) events in $\mathcal{L}(\mathbb{P})$, i.e. the powerset of $\mathbb{P}$, is called $\mathcal{A}(\mathbb{P})$.

Given two events A, B, $B \neq \emptyset$, the *conditional event* $A|B$ can be thought of (in a logical approach) as true when A and B are true, false when A is false and B is true, undefined when B is false. Given a partition $\mathbb{P}$ and $B \in \mathcal{A}(\mathbb{P})\setminus\{\emptyset\}$, the *conditional partition* $\mathbb{P}|B$ is $\mathbb{P}|B = \{\omega|B : \omega \in \mathbb{P}\}$. Given a gamble X defined on $\mathbb{P}$, the *conditional gamble* $X|B$ is defined on $\mathbb{P}|B$ by $X|B(\omega|B) = X(\omega)$, $\forall\omega \in \mathbb{P} : \omega \Rightarrow B$ ($\omega \neq \emptyset$). Clearly, $X|\Omega = X$.

In later computations involving conditional gains we shall use, often implicitly, the following identity [2]

$$f(X_1,\ldots,X_n)|B = f(X_1|B,\ldots,X_n|B), \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $X_1,\ldots,X_n$ are gambles and B is a non-impossible event.

We remark that, for any $k \in \mathbb{R}$,

$$(X \leq k)|B = (X|B \leq k) \tag{3}$$

are two equivalent descriptions of the same conditional event. In fact, $(X \leq k)|B = \vee\{\omega \in \mathbb{P} : X(\omega) \leq k\}|B = \vee\{\omega|B \in \mathbb{P}|B : X(\omega) \leq k\} = \vee\{\omega|B \in \mathbb{P}|B : X|B(\omega|B) \leq k\} = (X|B \leq k)$.

2.2 Evaluating uncertainty

Denote with $\mathcal{D}$ an *arbitrary* non-empty set of possibly conditional gambles. In the sequel, $\mathcal{D}$ will be the domain of a (precise or imprecise) conditional or unconditional *prevision*. Any sort of prevision is then an uncertainty measure μ for the elements of $\mathcal{D}$, $\mu : \mathcal{D} \to \mathbb{R}$. When $\mathcal{D}$ is made up of events (conditional or not) only, we preferably speak of *probability* instead of prevision.

We recall now the definition of coherence for a precise prevision, termed *dF-coherence* to easily distinguish it from other coherence concepts for imprecise previsions. In the sequel $\mathbb{N}^+$ denotes $\mathbb{N} \setminus \{0\}$.

Definition 2.1. *Given $P : \mathcal{D} \to \mathbb{R}$, P is a* (conditional) dF-coherent prevision *on $\mathcal{D}$ if, $\forall n \in \mathbb{N}^+$, $\forall X_1|B_1,\ldots,X_n|B_n \in \mathcal{D}$, $\forall s_1,\ldots,s_n \in \mathbb{R}$, defining*

$$G = \sum_{i=1}^{n} s_i B_i\big(X_i - P(X_i|B_i)\big), \quad B = \bigvee_{i=1}^{n} B_i, \tag{4}$$

it holds that $\sup(G|B) \geq 0$.

If $\mathcal{D}$ is made of unconditional *gambles only, then* (4) *simplifies to*

$$G = \sum_{i=1}^{n} s_i\big(X_i - P(X_i)\big) \quad (B = \Omega), \tag{5}$$

and consequently the coherence condition reduces to $\sup G \geq 0$.

The condition of dF-coherence allows a betting (or behavioural) interpretation, where $g_i = s_i(X_i - P(X_i))$ in (5) is an *elementary gain* with *stake* s_i. It represents the agent's gain from buying (if $s_i > 0$) or selling (if $s_i < 0$) $s_i X_i$ for $s_i P(X_i)$. Thus the condition $\sup G \geq 0$ requires that no finite combination of elementary gains produces an overall uniformly negative gain to the agent. Similarly, the generic elementary gain is $s_i B_i(X_i - P(X_i|B_i))$ in (4): it incorporates the factor B_i (here B_i stands for I_{B_i}), with the meaning that the bet on $X_i|B_i$ is effective if and only if event B_i turns out to be true. Then, conditioning G on

B in $\sup(G|B) \geq 0$ means that only those values of G are evaluated that ensure that at least one of the bets on $X_i|B_i$, for $i = 1, \ldots, n$, is effective.

The other consistency concepts we recall here have a similar betting interpretation. Actually, they can be derived from dF-coherence simply by introducing constraints on the stakes s_i (and hence, in the betting interpretation, on the buying/selling operations). Their definitions and a few basic properties are laid down below, followed by comments on some common features of theirs (for more on this topic see e.g. [7, 8, 14, 17, 18, 19]). Prior to this, let us recall some properties of dF-coherent previsions to be employed later on.

Proposition 2.1 ([5, 18, 19])**.** *If P is a dF-coherent prevision on $\mathcal{D}$, then*

(*a*) *there exists a dF-coherent extension of P on any $\mathcal{D}' \supseteq \mathcal{D}$.*

Moreover, the following properties hold whenever their terms are defined:

(*b*) $P(aX + bY|B) = aP(X|B) + bP(Y|B)$, $\forall a, b \in \mathbb{R}$ *(linearity).*

(*c*) $P(AX|B) = P(A|B)P(X|A \wedge B)$, $A \wedge B \neq \emptyset$ *(product rule).*

Definition 2.2 ([8, 18, 19])**.** *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be given.*

(*a*) $\underline{P}$ *is a W-*coherent *lower prevision on $\mathcal{D}$ if, $\forall n \in \mathbb{N}$, $\forall X_0|B_0, X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_i \geq 0$, with $i = 0, 1, \ldots, n$, defining*

$$\underline{G} = \sum_{i=1}^{n} s_i B_i\big(X_i - \underline{P}(X_i|B_i)\big) - s_0 B_0\big(X_0 - \underline{P}(X_0|B_0)\big),\ B = \bigvee_{i=0}^{n} B_i,$$

it holds that $\sup(\underline{G}|B) \geq 0$.

(*b*) $\underline{P}$ *is a lower prevision that* avoids sure loss (ASL) *on $\mathcal{D}$ if, $\forall n \in \mathbb{N}^+$, $\forall X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_i \geq 0$, with $i = 1, \ldots, n$, defining*

$$\underline{G}_{ASL} = \sum_{i=1}^{n} s_i B_i\big(X_i - \underline{P}(X_i|B_i)\big), \quad B = \bigvee_{i=1}^{n} B_i,$$

it holds that $\sup(\underline{G}_{ASL}|B) \geq 0$.

W-coherence was introduced in [18]; the structure-free form in Definition 2.2 (a) was employed in [8]. In the unconditional case, it is equivalent to Walley's coherence [17, Section 2.5.4 (a)], while it includes (strictly) Walley's definition of coherence in [17, Section 7.1.4 (b)] in the conditional environment. Hence W-coherence is a very general coherence concept, while the condition of avoiding sure loss in Definition 2.2 (b) has a more ancillary role (in the theory and in this work). It was only implicitly defined in [18], and we term it here 'avoiding sure loss' because it reduces to Walley's definition of avoiding sure loss with unconditional previsions [17, Section 2.4.4 (a)]. With conditional previsions, it generalises instead what Walley names 'avoiding partial loss' [17, Section 7.1.2]. We summarise next some fundamental properties of W-coherence.

Proposition 2.2 ([8, 18, 19])**.** *Let* $\underline{P} : \mathcal{D} \to \mathbb{R}$ *be a* W*-coherent lower prevision on* $\mathcal{D}$*. Then*

(*a*) $\underline{P}$ *has a least-committal* W*-coherent extension* $\underline{E}$ *on any* $\mathcal{D}' \supseteq \mathcal{D}$*, which is termed* natural extension*:* $\underline{E} = \underline{P}$ *on* $\mathcal{D}$ *and for every* W*-coherent extension* $\underline{P}^*$ *of* $\underline{P}$ *on* $\mathcal{D}'$*,* $\underline{E} \leq \underline{P}^*$ *on* $\mathcal{D}'$*.*

(*b*) *If, for* $X|B, Y|B \in \mathcal{D}$*,* $X|B \leq Y|B$*, then* $\underline{P}(X|B) \leq \underline{P}(Y|B)$ *(monotonicity).*

(*c*) $\underline{P}(X|B) \in [\inf(X|B), \sup(X|B)]$ *(internality).*[1]

Proposition 2.3 ([19], Theorem 2 (Envelope theorem))**.** *Given* $\underline{P} : \mathcal{D} \to \mathbb{R}$*,* $\underline{P}$ *is a* W*-coherent lower prevision on* $\mathcal{D}$ *if and only if there exists a non-empty set* $\mathcal{P}$ *of* dF*-coherent previsions on* $\mathcal{D}$ *such that*

$$\underline{P}(X|B) = \min\{P(X|B) : P \in \mathcal{P}\}, \quad \forall X|B \in \mathcal{D}.$$

Apart from coherence, we shall sometimes employ the consistency notions of convexity and centered convexity for unconditional lower previsions, developed in [7].

Definition 2.3 ([7], Definitions 3.1, 3.3)**.** *Given* $\underline{P} : \mathcal{D} \to \mathbb{R}$*,*

(*a*) $\underline{P}$ *is a* convex *lower prevision on* $\mathcal{D}$ *if,* $\forall n \in \mathbb{N}^+$*,* $\forall X_0, X_1, \ldots, X_n \in \mathcal{D}$*,* $\forall s_i \geq 0$*, with* $i = 1, \ldots, n$*, and* $\sum_{i=1}^n s_i = 1$ *(convexity condition), defining*

$$\underline{G}_c = \sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - \big(X_0 - \underline{P}(X_0)\big), \tag{6}$$

it holds that $\sup \underline{G}_c \geq 0$*.*

(*b*) *Assuming* $\emptyset \in \mathcal{D}$*,* $\underline{P}$ *is a* centered convex *lower prevision on* $\mathcal{D}$ *if it is convex on* $\mathcal{D}$ *and* $\underline{P}(\emptyset) = 0$*.*

Centered convex lower previsions have more satisfactory properties than non-centered ones (which even fail to ensure the internality property) and are in this respect closer to coherent lower previsions. This appears also from the following properties [7].

Proposition 2.4 ([7], Propositions 3.4, 3.5)**.** *Let* $\underline{P} : \mathcal{D} \to \mathbb{R}$ *be a convex lower prevision on* $\mathcal{D}$*. Then*

(*a*) *if, for* $X, Y \in \mathcal{D}$*,* $X \leq Y$*, then* $\underline{P}(X) \leq \underline{P}(Y)$ *(monotonicity).*

(*b*) *Assuming* $\underline{P}$ *is also centered on* $\mathcal{D}$*, it has a least-committal centered convex extension on any* $\mathcal{D}' \supseteq \mathcal{D}$*, the* convex natural extension $\underline{E}_c$*:* $\underline{E}_c = \underline{P}$ *on* $\mathcal{D}$*, and whatever is* $\underline{P}^*$*, convex extension of* $\underline{P}$ *on* $\mathcal{D}'$*,* $\underline{E}_c \leq \underline{P}^*$ *on* $\mathcal{D}'$*.*

[1] Being also W-coherent, a dF-coherent prevision satisfies properties (b), (c) too.

Proposition 2.5 ([7], Theorems 3.3, 3.4 (Envelope theorem)). *Let* $\underline{P} : \mathcal{D} \to \mathbb{R}$ *be given. Then*

(*a*) $\underline{P}$ *is a convex lower prevision on* $\mathcal{D}$ *if and only if there exist a non-empty set* $\mathcal{P}$ *of* dF*-coherent previsions on* $\mathcal{D}$ *and a function* $\alpha : \mathcal{P} \to \mathbb{R}$ *such that*

$$\underline{P}(X) = \min\{P(X) + \alpha(P) : P \in \mathcal{P}\}, \quad \forall X \in \mathcal{D}. \tag{7}$$

(*b*) $\underline{P}$ *is a centered convex lower prevision on* $\mathcal{D}$ *if and only if* (7) *holds, with the additional condition that* $\min\{\alpha(P) : P \in \mathcal{P}\} = 0$.

Next to lower previsions, upper previsions could also be assessed. Customarily, it is possible to refer to just one type of imprecise previsions by the *conjugacy* relation: $\overline{P}$ is defined on $-\mathcal{D} = \{-X|B : X|B \in \mathcal{D}\}$, by[2]

$$\overline{P}(-X|B) = -\underline{P}(X|B). \tag{8}$$

Using (8), the consistency notions we recalled for lower previsions and their properties can be expressed for upper previsions. For instance,

Definition 2.4 ([8, 18, 19]). *Given* $\overline{P} : -\mathcal{D} \to \mathbb{R}$*,* $\overline{P}$ *is a* W-coherent upper prevision *on* $-\mathcal{D}$ *if,* $\forall n \in \mathbb{N}$, $\forall X_0|B_0, X_1|B_1, \ldots, X_n|B_n \in -\mathcal{D}$, $\forall s_i \geq 0$*, with* $i = 0, 1, \ldots, n$*, defining*

$$\overline{G} = \sum_{i=1}^{n} s_i B_i\big(\overline{P}(X_i|B_i) - X_i\big) - s_0 B_0\big(\overline{P}(X_0|B_0) - X_0\big), \quad B = \bigvee_{i=0}^{n} B_i,$$

it holds that $\sup(\overline{G}|B) \geq 0$.

Assuming $\mathcal{D}$ is negation invariant, that is, $-\mathcal{D} = \mathcal{D}$, an extreme example of W-coherent upper (lower) prevision is the upper (lower) *vacuous* prevision $\overline{P}_v(X|B) = \sup(X|B)$ ($\underline{P}_v(X|B) = \inf(X|B)$), $\forall X|B \in \mathcal{D}$. By Proposition 2.2 (c), the couple $(\underline{P}_v, \overline{P}_v)$ maximises the imprecision of the agent's assessment on any $X|B$ while being W-coherent, and may express total lack of information on $X|B$. At the other extreme, we have a dF-coherent prevision P: in fact, $\forall X|B \in \mathcal{D}$, $P(X|B) = -P(-X|B)$ by Proposition 2.1 (*b*). Thus P is self-conjugate in (8), or $P = \overline{P} = \underline{P}$.

The various gains we recalled ($G, \underline{G}, \underline{G}_{ASL}\underline{G}_c, \overline{G}$) are gambles themselves, being functions of a finite number of gambles in $\mathcal{D}$ (and, in the conditional case, of indicators of their conditioning events).

The suprema of these gains may or may not be achieved, but are necessarily maxima when $\mathcal{D}$ is made of *simple* gambles, i.e., gambles taking only finitely many distinct values. In particular, this is the case when $\mathcal{D}$ is a set of (possibly conditional) events.

We mention next some other concepts regarding gains for later use.

[2] With imprecise probabilities, (8) reduces to $\overline{P}(\neg A|B) = 1 - \underline{P}(A|B)$.

Definition 2.5. *Let* $\underline{G}$ *be the gain in Definition 2.2 (a).*

Then $\mathcal{D}_{\underline{G}} = \{X_0|B_0, X_1|B_1, \ldots, X_n|B_n\}$*, i.e.,* $\mathcal{D}_{\underline{G}} \subseteq \mathcal{D}$ *is the set of conditional gambles forming* $\underline{G}$.

The coarsest partition $\underline{G}|B$ *is defined on is termed* $\mathbb{P}_{\underline{G}}|B$*. In other words, the atoms* $\omega|B$ *of* $\mathbb{P}_{\underline{G}}|B$ *correspond to the distinct jointly possible values of* $X_0, X_1, \ldots, X_n$ *that imply* $B = \bigvee_{i=0}^{n} B_i$.

We say that $\underline{G}$ *is a* WDB gain *if* $\sup(\underline{G}|B) = 0$.

Analogous definitions apply to the other gains we considered. For instance, if $\underline{P}$ *avoids sure loss (alternatively, is convex) on* $\mathcal{D}$*,* $\underline{G}_{ASL}$ *(*$\underline{G}_c$*) is a WDB gain if* $\sup(\underline{G}_{ASL}|\bigvee_{i=1}^{n} B_i) = 0$ *(*$\sup \underline{G}_c = 0$*).*

3 Consistency constraints induced by Weak Dutch Books

We begin our investigation of WDBs with imprecise previsions presenting a simple special case in Section 3.1. The examples therein will be useful for suggesting or illustrating some of the later developments, starting with those in Section 3.2.

3.1 A special case

Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a lower probability and $\mathcal{D}_{\underline{G}} = \{E_1, E_2\} \subseteq \mathcal{D}$. Therefore the gain

$$\underline{G} = s_1\big(E_1 - \underline{P}(E_1)\big) + s_2\big(E_2 - \underline{P}(E_2)\big) \tag{9}$$

is a gamble defined on the set of all the (non-impossible) events of the partition $\mathbb{P}_{\underline{G}} = \{E_1 \wedge E_2, E_1 \wedge \neg E_2, \neg E_1 \wedge E_2, \neg E_1 \wedge \neg E_2\}$. Thus $\underline{G}$ can take (at most) the following values:

$$\underline{G}(E_1 \wedge E_2) = s_1(1 - \underline{P}(E_1)) + s_2(1 - \underline{P}(E_2)) \tag{10}$$

$$\underline{G}(E_1 \wedge \neg E_2) = s_1(1 - \underline{P}(E_1)) - s_2\underline{P}(E_2) \tag{11}$$

$$\underline{G}(\neg E_1 \wedge E_2) = -s_1\underline{P}(E_1) + s_2(1 - \underline{P}(E_2)) \tag{12}$$

$$\underline{G}(\neg E_1 \wedge \neg E_2) = -(s_1\underline{P}(E_1) + s_2\underline{P}(E_2)). \tag{13}$$

To avoid trivial instances, *we suppose throughout this section* that $s_1 \cdot s_2 \neq 0$ and that E_1, E_2 are possible events (i.e., neither $\emptyset$ nor Ω).

We present some instances of WDBs with these assumptions, and varying consistency requirements for $\underline{P}$.

Example 3.1. *Let* $\underline{P}$ *avoid sure loss on* $\mathcal{D}$*, and* $E_1 \wedge E_2 \neq \emptyset$*. Then* $s_i > 0$*,* $i = 1, 2$*, in* (9) *and* $\max \underline{G} = 0$ *if and only if* $\underline{P}(E_1) = \underline{P}(E_2) = 1$.

In fact, it is $\max \underline{G} = \underline{G}(E_1 \wedge E_2) = 0$ *if and only if* $\underline{P}(E_1) = \underline{P}(E_2) = 1$.

We might be tempted to infer from this example that WDBs are strictly related with extreme probability events. However, in the next Examples 3.2 and 3.3, E_1 and E_2 are not given extreme probability evaluations.

Example 3.2. *Let* $\underline{P}$ *avoid sure loss on* $\mathcal{D}$, $\underline{P}(E_i) \in]0,1[$, $i = 1, 2$. *Then* $\underline{G}$ *is a WDB gain if and only if* $E_1 \wedge E_2 = \emptyset$, $\underline{P}(E_1) + \underline{P}(E_2) = 1$ *and* $s_1 = s_2$.

This can be proved as follows. We first assume that $\max \underline{G} = 0$. *It is clear, from Example 3.1, that* $E_1 \wedge E_2 = \emptyset$ *is a necessary WDB condition. Given that* $\underline{G}(\neg E_1 \wedge \neg E_2) < 0$ *if* $\neg E_1 \wedge \neg E_2 \neq \emptyset$, *it remains to examine* $\underline{G}(E_1 \wedge \neg E_2), \underline{G}(\neg E_1 \wedge E_2)$. *Note for this that* $E_1 \wedge \neg E_2 \neq \emptyset$, $\neg E_1 \wedge E_2 \neq \emptyset$, *or else we would obtain* $E_1 = \emptyset$, $E_2 = \emptyset$, *respectively. Since* $\max \underline{G} = 0$, *we have to ask that* $\underline{G}(E_1 \wedge \neg E_2) \leq 0$, $\underline{G}(\neg E_1 \wedge E_2) \leq 0$, *with at least one inequality being an equality. Requiring that* $\underline{G}(E_1 \wedge \neg E_2) = 0$, *we obtain from* (11) *that*

$$s_1 = \frac{\underline{P}(E_2)}{1 - \underline{P}(E_1)} s_2. \tag{14}$$

Substituting s_1 *in* $\underline{G}(\neg E_1 \wedge E_2) \leq 0$, *we get easily* $\underline{P}(E_1) + \underline{P}(E_2) \geq 1$. *Since* $\underline{P}(E_1) + \underline{P}(E_2) \leq 1$ *is a necessary condition for ASL when* $E_1 \wedge E_2 = \emptyset$ *(cf. [17, Section 4.6.1]), we conclude that* $\underline{P}(E_1) + \underline{P}(E_2) = 1$. *This implies, from* (14), *that* $s_1 = s_2$. *Note that it also implies that* $\underline{G}(\neg E_1 \wedge E_2) = 0$, *i.e.,* $\underline{G}$ *attains its maximum value 0 at* two *atoms of* $\mathbb{P}_{\underline{G}}$. *The symmetric assumption that* $\underline{G}(\neg E_1 \wedge E_2) = 0$ *also leads to the same conclusion.*

Conversely, the assumptions $E_1 \wedge E_2 = \emptyset$, $\underline{P}(E_1) + \underline{P}(E_2) = 1$ *and* $s_1 = s_2$ *allow to conclude that* $\max \underline{G} = \max s_1(E_1 + E_2 - 1) = 0$.

It is not difficult to check that, besides the cases developed in Examples 3.1 and 3.2, a lower probability $\underline{P}$ that avoids sure loss originates a WDB on $\{E_1, E_2\}$ in a third instance only, under the assumptions of the present section, which is item (c) in the next Proposition 3.1.

When $\underline{P}$ is coherent, next to the previous cases, further WDB opportunities arise, since (precisely) one between s_1 and s_2 may be negative. We detail one such case in the next example.

Example 3.3. *Let* $\underline{P}$ *be coherent on* $\mathcal{D}$. *Let* $s_2 < 0 < s_1$, *and* $\underline{P}(E_i) \in]0,1[$, $i = 1, 2$. *Then* $\max \underline{G} = 0$ *if and only if* $E_1 \wedge \neg E_2 = \emptyset$, $\underline{P}(E_1) = \underline{P}(E_2)$ *and* $s_1 = -s_2$.

In fact, recalling (10)–(13), *we note that*

$$\underline{G}(\neg E_1 \wedge E_2) \leq \max\{\underline{G}(E_1 \wedge E_2), \underline{G}(\neg E_1 \wedge \neg E_2)\} \leq \underline{G}(E_1 \wedge \neg E_2).$$

Now, let $\max \underline{G} = 0$, *which implies* $E_1 \wedge \neg E_2 = \emptyset$ *(or else we would have* $\underline{G}(E_1 \wedge \neg E_2) > 0$, *a contradiction). This entails, on the one hand, that* $E_1 \wedge E_2 \neq \emptyset$ *and* $\neg E_1 \wedge \neg E_2 \neq \emptyset$, *on the other hand, by monotonicity of* $\underline{P}$, *that*

$$\underline{P}(E_1) \leq \underline{P}(E_2). \tag{15}$$

Moreover, $\max \underline{G} = \max\{\underline{G}(E_1 \wedge E_2), \underline{G}(\neg E_1 \wedge \neg E_2)\} = 0$ *iff both* $\underline{G}(E_1 \wedge E_2) \leq 0$ *and* $\underline{G}(\neg E_1 \wedge \neg E_2) \leq 0$, *with at least one inequality being an equality. Together with* (15), *in both cases we deduce that* $\underline{P}(E_1) = \underline{P}(E_2)$, $s_1 = -s_2$. *Conversely, let* $E_1 \wedge \neg E_2 = \emptyset$, $\underline{P}(E_1) = \underline{P}(E_2)$, $s_1 = -s_2$. *Then,* $\max \underline{G} = \max s_1(E_1 - E_2) = 0$.

Comment. In this example too, $\max \underline{G} = 0$ is achieved at two distinct atoms of $\mathbb{P}_{\underline{G}}$, given by $E_1 = E_1 \wedge E_2$ and $\neg E_2 = \neg E_1 \wedge \neg E_2$, since here $E_1 \Rightarrow E_2$. It follows that the event $(\underline{G} = 0)$ is equal to $E_1 \vee \neg E_2$, while $(\underline{G} < 0)$ is equal to $\neg E_1 \wedge E_2$. Further, note that $\underline{P}(E_1) = \underline{P}(E_2)$, while E_1 may be not equal to E_2. From this, one easily obtains that the coherent extension of $\underline{P}$ on $\neg E_1 \wedge E_2$ is unique, since $\underline{P}(E_2) \geq \underline{P}(E_1) + \underline{P}(\neg E_1 \wedge E_2) \geq \underline{P}(E_1)$ implies $\underline{P}(\neg E_1 \wedge E_2) = 0$.

It can be checked that there are further alternatives to achieve a WDB with a coherent $\underline{P}$. We omit the simple but tedious derivations and summarise the results in the next proposition.

Proposition 3.1. *Let $\mathcal{D}_{\underline{G}} = \{E_1, E_2\} \subseteq \mathcal{D}$, where E_1, E_2 are possible events, and $\underline{P}$ be coherent on $\mathcal{D}$. Let $s_1, s_2 \in \mathbb{R} \setminus \{0\}$ be either both positive or opposite, and $\underline{G}$ be given by* (9).

Then $\max \underline{G} = 0$ if and only if one of the following holds:

- *when $s_1 s_2 > 0$,*
  - (*a*) $\underline{P}(E_i)=1$, $i = 1, 2$, $E_1 \wedge E_2 \neq \emptyset$, $s_1 > 0, s_2 > 0$;
  - (*b*) $\underline{P}(E_i) \in\ ]0, 1[$, $i = 1, 2$, $E_1 \wedge E_2 = \emptyset$, $\underline{P}(E_1) + \underline{P}(E_2) = 1$, $s_1 = s_2 > 0$;
  - (*c*) $E_1 \wedge E_2 = \emptyset$, *and either* $\underline{P}(E_1) = 0$, $\underline{P}(E_2) = 1$, $0 < s_1 \leq s_2$ *or* $\underline{P}(E_1) = 1$, $\underline{P}(E_2) = 0$, $0 < s_2 \leq s_1$;
- *when $s_1 s_2 < 0$,*
  - (*d*) $\underline{P}(E_1) = \underline{P}(E_2) \in\ ]0, 1[$ *and either* $E_1 \wedge \neg E_2 = \emptyset$, $s_1 = -s_2 > 0$ *or* $\neg E_1 \wedge E_2 = \emptyset$, $s_2 = -s_1 > 0$;
  - (*e*) $\underline{P}(E_i) = 1$, $i = 1, 2$, $s_1 \geq -s_2$ *and either* $E_1 \wedge \neg E_2 = \emptyset$, $s_2 < 0 < s_1$ *or* $\neg E_1 \wedge E_2 = \emptyset$, $s_1 < 0 < s_2$;
  - (*f*) $\underline{P}(E_i) = 0$, $i = 1, 2$, $s_1 \leq -s_2$, *and either* $E_1 \wedge \neg E_2 = \emptyset$, $s_2 < 0 < s_1$ *or* $\neg E_1 \wedge E_2 = \emptyset$, $s_1 < 0 < s_2$;
  - (*g*) *either* $\underline{P}(E_1) = 0$, $\underline{P}(E_2) = 1$, $\neg E_1 \wedge E_2 \neq \emptyset$, $s_1 < 0 < s_2$ *or* $\underline{P}(E_1) = 1$, $\underline{P}(E_2) = 0$, $E_1 \wedge \neg E_2 \neq \emptyset$, $s_2 < 0 < s_1$.

3.2 Local precision properties with Weak Dutch Books

There is an interesting feature of Proposition 3.1 which attracts one's attention: in each of its WDBs, $\underline{P}$ is actually a dF-coherent probability on $\mathcal{D}_{\underline{G}}$, i.e., it is precise on $\mathcal{D}_{\underline{G}}$. This fact is rather straightforward to check by means of well-known properties of dF-coherent probabilities. Indeed, in (*a*), (*e*) and (*g*) $\underline{P}$ is the restriction on $\mathbb{P}_{\underline{G}}$ of a probability P which is dF-coherent on $\mathcal{A}(\mathbb{P}_{\underline{G}})$: put $P(E_1 \wedge E_2) = 1$ in (*a*) and (*e*), either $P(\neg E_1 \wedge E_2) = 1$ or $P(E_1 \wedge \neg E_2) = 1$ in (*g*). Instead, in (*b*) and (*c*) $\underline{P}$ is dF-coherent on $\mathbb{P}_{\underline{G}} = \{E_1, E_2, \neg E_1 \wedge \neg E_2\}$, while in (*d*) and (*f*) $\{E_1, E_2\}$ forms a monotone family (chain) of events, and dF-coherent or coherent assignments are indistinguishable there (cf. [4, Proposition 2.10]).

As we shall see now, the result is more general: it concerns all coherent lower previsions. However, there is a distinction, as for its extent, between unconditional and conditional coherence. Further, WDBs have effects also outside coherence, but they are not necessarily of the same kind, as shown for convexity by Proposition 3.5.

Let us start with a W-coherent lower prevision $\underline{P}(\cdot|\cdot)$. Its properties on those $\mathcal{D}_{\underline{G}}$ where a WDB holds are investigated in Proposition 3.2 by means of the next lemma, which does not involve WDBs explicitly.

Lemma 3.1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision. Letting $\underline{G}, B$ be as in Definition 2.2 (a), there exists a dF-coherent prevision $\tilde{P}$ on $\mathcal{D} \cup \{\underline{G}|B\}$ such that $\tilde{P} \geq \underline{P}$ on $\mathcal{D}$,*

$$\tilde{P}(X_0|B_0) = \underline{P}(X_0|B_0) \tag{16}$$

and

$$\tilde{P}(\underline{G}|B) = \sum_{i=1}^{n} s_i \big(\tilde{P}(X_i|B_i) - \underline{P}(X_i|B_i)\big)\, \tilde{P}(B_i|B) \geq 0. \tag{17}$$

Proof. By Proposition 2.3, there exists a dF-coherent prevision $\tilde{P} : \mathcal{D} \to \mathbb{R}$ such that $\tilde{P} \geq P$ on $\mathcal{D}$ and $\tilde{P}$ satisfies (16). By (2) and Proposition 2.1 (b), any dF-coherent extension of $\tilde{P}$ (still termed $\tilde{P}$) on a large enough set is such that

$$\tilde{P}(\underline{G}|B) = \sum_{i=1}^{n} s_i \tilde{P}\big(B_i(X_i - \underline{P}(X_i|B_i))|B\big) - s_0 \tilde{P}\big(B_0(X_0 - \underline{P}(X_0|B_0))|B\big). \tag{18}$$

In general, we have that, for $i = 0, 1, \ldots, n$,

$$\begin{aligned}
\tilde{P}\big(B_i(X_i - \underline{P}(X_i|B_i))|B\big) &= \tilde{P}\big(B_i\, X_i|B - B_i|B\, \underline{P}(X_i|B_i)\big) \\
&= \tilde{P}(B_i\, X_i|B) - \underline{P}(X_i|B_i)\, \tilde{P}(B_i|B) \\
&= \tilde{P}(X_i|B_i \wedge B)\, \tilde{P}(B_i|B) - \underline{P}(X_i|B_i)\, \tilde{P}(B_i|B) \\
&= \big(\tilde{P}(X_i|B_i) - \underline{P}(X_i|B_i)\big)\, \tilde{P}(B_i|B) \geq 0,
\end{aligned}$$

using (2) at the first equality, Proposition 2.1 (b) at the second, Proposition 2.1 (c) at the third, and $B_i \wedge B = B_i$ at the fourth. The final inequality ensues from $\tilde{P} \geq \underline{P}$ on $\mathcal{D}$; by (16), it is an *equality* when $i = 0$.

From (18) and this derivation, we obtain (17). □

Proposition 3.2. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision, $\underline{G}, B$ be as in Definition 2.2 (a) and such that $\underline{G}$ is a WDB gain. Suppose $B_1|B, \ldots, B_n|B \in \mathcal{D}$. Define*

$$\mathcal{D}^+_{\underline{G}} = \{X_0|B_0\} \cup \{X_i|B_i \in \mathcal{D}_{\underline{G}} : s_i \underline{P}(B_i|B) > 0, \text{ for } i = 1, \ldots, n\}.$$

Then $\underline{P}$ is dF-coherent on $\mathcal{D}^+_{\underline{G}}$.

Proof. Let $\tilde{P}$ be the dF-coherent prevision on $\mathcal{D} \cup \{\underline{G}|B\}$ in Lemma 3.1, hence satisfying $\tilde{P} \geq \underline{P}$ on $\mathcal{D}$ and (16). Since now $\sup(\underline{G}|B) = 0$, using Proposition 2.2 (c) and Footnote 1 at the first inequality and (17) then, we have

$$0 = \sup(\underline{G}|B) \geq \tilde{P}(\underline{G}|B) = \sum_{i=1}^{n} s_i\big(\tilde{P}(X_i|B_i) - \underline{P}(X_i|B_i)\big)\, \tilde{P}(B_i\,|B) \geq 0.$$

Therefore, $\sum_{i=1}^{n} s_i\big(\tilde{P}(X_i|B_i) - \underline{P}(X_i|B_i)\big)\, \tilde{P}(B_i\,|B) = 0$. Further, $\forall X_i|B_i \in \mathcal{D}^+_{\underline{G}} \setminus \{X_0|B_0\}$, we have $\tilde{P}(B_i|B) \geq \underline{P}(B_i|B) > 0$. These two facts imply that

$$\underline{P}(X_i|B_i) = \tilde{P}(X_i|B_i), \quad \forall X_i|B_i \in \mathcal{D}^+_{\underline{G}} \setminus \{X_0|B_0\}. \tag{19}$$

Since condition (16) holds too, we conclude that (19) is valid for all $X_i|B_i \in \mathcal{D}^+_{\underline{G}}$, thus $\underline{P}$ is dF-coherent on $\mathcal{D}^+_{\underline{G}}$. □

The condition $B_1|B, \dots, B_n|B \in \mathcal{D}$ in Proposition 3.2 is not overly restrictive. If it is not met, we may consider a W-coherent extension $\underline{P}'$ of $\underline{P}$ on $\mathcal{D}' = \mathcal{D} \cup \{B_i|B : i = 1, \dots, n\}$ and apply Proposition 3.2 to $\underline{P}'$ on $\mathcal{D}'$. However, the set on which $\underline{P}'$ is dF-coherent depends then on the specific extension, being minimal when selecting the natural extension.

Rather, the result is subject to a second, more significant restriction. In fact, note that the inequality $\tilde{P}(B_i|B) > 0$, needed in the proof of Proposition 3.2, is guaranteed for $X_i|B_i \in \mathcal{D}^+_{\underline{G}} \setminus \{X_0|B_0\}$ by the hypothesis $\underline{P}(B_i|B) > 0$. Thus, assuming $\underline{P}(B_i|B) > 0$ is a sufficient but not necessary condition for dF-coherence of $\underline{P}$. In other words, $\underline{P}$ may be dF-coherent on a set larger than $\mathcal{D}^+_{\underline{G}}$ when $0 = \underline{P}(B_i|B) < \tilde{P}(B_i|B)$ for some $X_i|B_i \notin \mathcal{D}^+_{\underline{G}}$.

However, the two constraints above are no longer effective in the important special case that $\underline{P}$ is *unconditional*, where $B_i = \Omega$, for $i = 0, 1, \dots, n$. Then $B = \Omega$ as well as $B_i|B = \Omega$, for $i = 0, 1, \dots, n$, and we may always add Ω to $\mathcal{D}$, since necessarily $\underline{P}(\Omega) = 1$. Hence $\mathcal{D}_{\underline{G}} = \mathcal{D}^+_{\underline{G}}$ if all the stakes s_i, for $i = 1, \dots, n$, are non-zero. The result specialises to the statement of the following proposition.

Proposition 3.3. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional coherent lower prevision. Let $\underline{G}$ be given by*

$$\underline{G} = \sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - s_0\big(X_0 - \underline{P}(X_0)\big), \tag{20}$$

with $s_0 \geq 0$, $s_i > 0$, for $i = 1, \dots, n$, $\{X_0, X_1, \dots, X_n\} = \mathcal{D}_{\underline{G}} \subseteq \mathcal{D}$, and assume that $\underline{G}$ is a WDB gain. Then $\underline{P}$ is dF-coherent on $\mathcal{D}_{\underline{G}}$.

Thus a WDB implies a very strong constraint on the restriction on $\mathcal{D}_{\underline{G}}$ of a coherent lower prevision. We shall see some implications of this fact in the sequel.

When $\underline{P}$ satisfies alternative consistency conditions, a WDB implies generally (but not necessarily) different constraints on $\underline{P}$. The situation does not really change when $\underline{P}$ avoids sure loss, as now sketched in the conditional case:

Proposition 3.4. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional lower prevision that avoids sure loss. Considering $\underline{G}_{ASL}$, B as in Definition 2.2 (b), assume that $\underline{G}_{ASL}$ is a WDB gain. Naming $\mathcal{D}_{\underline{G}_{ASL}} = \{X_1|B_1, \ldots, X_n|B_n\}$ and*

$$\mathcal{D}^{+}_{\underline{G}_{ASL}} = \{X_i|B_i : s_i\underline{P}(X_i|B_i) > 0, \textit{ for } i = 1, \ldots, n\},$$

we have that $\underline{P}$ is dF-coherent on $\mathcal{D}^{+}_{\underline{G}_{ASL}}$.

Proof. In the first part we can adapt the proof of Lemma 3.1. Results in [18] ensure that there is a dF-coherent prevision $\tilde{P} \geq \underline{P}$ on $\mathcal{D}$. Then $\tilde{P}(\underline{G}|B)$ is written as in (17) (omitting the term containing s_0 in (18) and otherwise following the same derivation giving (17)), hence $\tilde{P}(\underline{G}|B) \geq 0$.

Now take the same steps as in the proof of Proposition 3.2, recalling that $\sup(\underline{G}|B) \geq \tilde{P}(\underline{G}|B)$ is a necessary condition for $\underline{P}$ to avoid sure loss, too. □

Interestingly, assuming that $\underline{P}$ is convex on $\mathcal{D}$ gives rise to a different sort of constraint for $\underline{P}$. This exploits a result similar to Lemma 3.1.

Lemma 3.2. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional convex lower prevision. Letting $\underline{G}_c$ be as in Definition 2.3 (a), there exist a dF-coherent prevision $\tilde{P}$ on $\mathcal{D} \cup \{\underline{G}_c\}$ and $\alpha_{\tilde{P}} \in \mathbb{R}$ such that $\tilde{P} + \alpha_{\tilde{P}} \geq \underline{P}$ on $\mathcal{D}$,*

$$\tilde{P}(X_0) + \alpha_{\tilde{P}} = \underline{P}(X_0) \tag{21}$$

and

$$\tilde{P}(\underline{G}_c) = \sum_{i=1}^{n} s_i\big(\tilde{P}(X_i) - \underline{P}(X_i)\big) - \big(\tilde{P}(X_0) - \underline{P}(X_0)\big) \geq 0. \tag{22}$$

Proof. The existence of a dF-coherent prevision $\tilde{P}$ on $\mathcal{D} \cup \{\underline{G}_c\}$ and $\alpha_{\tilde{P}} \in \mathbb{R}$ such that conditions $\tilde{P} + \alpha_{\tilde{P}} \geq \underline{P}$ on $\mathcal{D}$ and (21) are satisfied is assured by Proposition 2.5 (a). Extending $\tilde{P}$ whenever needed by linearity, we have

$$\tilde{P}(\underline{G}_c) = \sum_{i=1}^{n} s_i\big(\tilde{P}(X_i) - \underline{P}(X_i)\big) - \big(\tilde{P}(X_0) - \underline{P}(X_0)\big) \geq \sum_{i=1}^{n} s_i\alpha_{\tilde{P}} - \alpha_{\tilde{P}} = 0.$$

□

The WDB case is treated in the next proposition.

Proposition 3.5. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional convex lower prevision, and $\underline{G}_c$ as in Definition 2.3 (a) be a WDB gain. Then there exist a dF-coherent prevision $\tilde{P}$ on $\mathcal{D} \cup \{\underline{G}_c\}$ and $\alpha_{\tilde{P}} \in \mathbb{R}$ such that $\underline{P} = \tilde{P} + \alpha_{\tilde{P}}$ on $\mathcal{D}^{+}_{\underline{G}_c} = \{X_0\} \cup \{X_i : s_i > 0, \textit{ for } i = 1, \ldots, n\}$.*

Proof. Consider the dF-coherent prevision $\tilde{P}$ on $\mathcal{D} \cup \{\underline{G}_c\}$ and $\alpha_{\tilde{P}} \in \mathbb{R}$ in Lemma 3.2. From $\tilde{P} + \alpha_{\tilde{P}} \geq \underline{P}$ on $\mathcal{D}$, there exist $\varepsilon_i \geq 0$ ($i = 1, \ldots, n$) such that

$$\tilde{P}(X_i) - \underline{P}(X_i) = -\alpha_{\tilde{P}} + \varepsilon_i. \tag{23}$$

Since $0 = \sup \underline{G}_c \geq \tilde{P}(\underline{G}_c) \geq 0$ by internality of $\tilde{P}$ and Lemma 3.2, we get, by (22) and (21), $0 = \tilde{P}(\underline{G}_c) = \sum_{i=1}^n s_i(\tilde{P}(X_i) - \underline{P}(X_i)) + \alpha_{\tilde{P}}$. Hence, using (23) at the second equality,

$$-\alpha_{\tilde{P}} = \sum_{i=1}^{n} s_i\big(\tilde{P}(X_i) - \underline{P}(X_i)\big) = \sum_{i=1}^{n} s_i(-\alpha_{\tilde{P}} + \varepsilon_i) = -\alpha_{\tilde{P}} + \sum_{i=1}^{n} s_i\varepsilon_i,$$

which gives $\sum_{i=1}^n s_i\varepsilon_i = 0$. Therefore $\varepsilon_i = 0$ if $s_i > 0$, implying the thesis. □

Thus, convexity of an unconditional lower prevision $\underline{P}$ on $\mathcal{D}$ implies that $\underline{P}$ has a special structure on $\mathcal{D}^+_{\underline{G}_c}$, with WDBs: for each $X_i \in \mathcal{D}^+_{\underline{G}_c}$, $\underline{P}$ differs from a dF-coherent prevision $\tilde{P}$ by the same constant $\alpha_{\tilde{P}}$. Perhaps surprisingly, if $\underline{P}$ is *centered* convex, the preceding result does not imply that $\alpha_{\tilde{P}} = 0$ in all cases, but only if $\emptyset \in \mathcal{D}^+_{\underline{G}_c}$. Instead, if $\underline{P}$ is (convex and) coherent, then necessarily, comparing Propositions 2.3 and 2.5 (b), $\alpha_{\tilde{P}} = 0$ and we re-obtain Proposition 3.3 as a special case of Proposition 3.5.[3]

Lastly, it is interesting to notice that the WDB for the convex (not necessarily centered) lower prevision $\underline{P}$ in Proposition 3.5 propagates to precisely the dF-coherent prevision $\tilde{P}$ in the same Proposition 3.5.

Corollary 3.1. *In the assumptions of Proposition 3.5, define the gain $\underline{G}_{\tilde{P}}$ for $\tilde{P}$ as in Definition 2.3 (a) replacing $\underline{P}$ with $\tilde{P}$. Then $\sup \underline{G}_{\tilde{P}} = 0$.*

Proof. It holds that $\underline{G}_{\tilde{P}} = \underline{G}_c$: this follows easily by substituting $\underline{P} = \tilde{P} + \alpha_{\tilde{P}}$ into (6) and using $\sum_{i=1}^n s_i = 1$. Then $\sup \underline{G}_{\tilde{P}} = 0$ by the hypotheses of Proposition 3.5, and clearly $\underline{G}_{\tilde{P}}$ is an admissible gain for the dF-coherence of prevision $\tilde{P}$. □

4 Further properties of Weak Dutch Books

In this section further effects of WDBs on the uncertainty assessments in $\mathcal{D}$ are investigated. In particular, their relationships with positive lower probability events are explored. We start with conditional lower previsions that are W-coherent in Section 4.1 and discuss then unconditional coherence in Section 4.2.

4.1 W-coherent lower previsions

The following proposition is a basic result for the developments in this and the next sections.

Proposition 4.1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional W-coherent lower prevision, $\underline{G}$, B as in Definition 2.2 (a) such that $\underline{G}$ is a WDB gain. Then for any conditional event $E|B \in \mathcal{D}$ such that $\underline{P}(E|B) > 0$, it holds that*

$$\sup(\underline{G}|B \wedge E) = 0.$$

[3] With *conditional* convex lower previsions, a WDB gain determines more complex constraints on $\underline{P}$, see [1, Proposition 10].

Proof. If $E|B = B|B$, i.e., if $B \Rightarrow E$, the assertion is trivial. Let then be $B \not\Rightarrow E$. Note that $\underline{P}(E|B) > 0$ implies that $E|B \neq \emptyset|B$, that is, $E \wedge B \neq \emptyset$.

Given an arbitrary real $s > 0$, consider the auxiliary gain

$$\underline{G}' = \underline{G} + sB\big(E - \underline{P}(E|B)\big).$$

By W-coherence of $\underline{P}$, $\sup(\underline{G}'|B) \geq 0$. As a first step, notice that

$$\sup(\underline{G}'|B \wedge E) \geq 0. \tag{24}$$

Indeed,

$$0 \leq \sup(\underline{G}'|B) = \max\{\sup(\underline{G}'|B \wedge E), \sup(\underline{G}'|B \wedge \neg E)\}. \tag{25}$$

For any $\omega \in \mathbb{P}_{\underline{G}'}$, with $\omega \Rightarrow B \wedge \neg E$, we have

$$\underline{G}'(\omega) = \underline{G}(\omega) - s\underline{P}(E|B) \leq \sup(\underline{G}|B) - s\underline{P}(E|B) = -s\underline{P}(E|B),$$

therefore

$$\sup(\underline{G}'|B \wedge \neg E) \leq -s\underline{P}(E|B) < 0. \tag{26}$$

From (25) and (26), we get $0 \leq \sup(\underline{G}'|B) = \sup(\underline{G}'|B \wedge E)$, hence (24) is effective.

Given this, and since $\underline{G}'|B \wedge E = \underline{G}|B \wedge E + s(1 - \underline{P}(E|B))$, we obtain

$$0 \leq \sup(\underline{G}'|B \wedge E) = \sup(\underline{G}|B \wedge E) + s\big(1 - \underline{P}(E|B)\big),$$

that is, $\sup(\underline{G}|B \wedge E) \geq -s(1 - \underline{P}(E|B))$. The inequality applying to all $s > 0$, we deduce

$$\sup(\underline{G}|B \wedge E) \geq 0.$$

On the other hand, $\sup(\underline{G}|B \wedge E) \leq \sup(\underline{G}|B) = 0$. We conclude that $\sup(\underline{G}|B \wedge E) = 0$. □

Proposition 4.1 informs us that if there is *any* event $E|B$ in $\mathcal{D}$ which is given positive lower probability, then the supremum of $\underline{G}|B \wedge E$ is zero. In particular, recall (Definition 2.5) that $\underline{G}|B$ is defined on the conditional partition $\mathbb{P}_{\underline{G}}|B$, or on any $\mathbb{P}|B$, with partition $\mathbb{P}$ finer than $\mathbb{P}_{\underline{G}}$. Hence, *if* there is $\omega|B \in \mathbb{P}|B$ such that $\underline{P}(\omega|B) > 0$, then Proposition 4.1 implies that $\sup(\underline{G}|B) = 0$ *is achieved (at least) at* $\omega|B$, i.e.,

$$\sup(\underline{G}|B) = \max(\underline{G}|B) = 0 = \underline{G}|B(\omega|B).$$

The result has the obvious constraint that $\omega|B$, and more generally $E|B$, must belong to $\mathcal{D}$. However, it may be the case that $E|B \notin \mathcal{D}$ but the assessment $\underline{P}$ implies that $E|B$ is given positive lower probability by any W-coherent extension of $\underline{P}$ on $\mathcal{D} \cup \{E|B\} = \mathcal{D}'$. Here Proposition 4.1 still applies replacing $\mathcal{D}$ with the larger set $\mathcal{D}'$, which has no effect on the given $\underline{G}|B$. One such instance is described in the next corollary.

Corollary 4.1. *Define $\underline{P}$, $\underline{G}$, B as in Proposition 4.1 and suppose again that* $\sup(\underline{G}|B) = 0$. *If $E^*|B^* \in \mathcal{D}$, $E^* \Rightarrow E \Rightarrow B \Rightarrow B^*$ and $\underline{P}(E^*|B^*) > 0$, then* $\sup(\underline{G}|B \wedge E) = 0$.

Proof. Recalling Proposition 2.2 (b) at the first weak inequality, and the necessary condition for W-coherence $\underline{P}(E|B^*) \leq \underline{P}(E|B)$ if $B \Rightarrow B^*$ (cf. [9, Proposition 13]) at the second, we get, possibly considering a W-coherent extension of $\underline{P}$ on $\mathcal{D} \cup \{E|B, E|B^*\}$,

$$0 < \underline{P}(E^*|B^*) \leq \underline{P}(E|B^*) \leq \underline{P}(E|B).$$

Thus, Proposition 4.1 may be applied, with $\mathcal{D}$ replaced by $\mathcal{D} \cup \{E|B\}$ if $E|B \notin \mathcal{D}$. □

Corollary 4.1 is useful in the common case that $\mathcal{D}$ includes the atoms of some partition $\mathbb{P}|B^*$. Very often $B^* = \Omega$, because it is customary to assess our uncertainty judgements on all alternatives of an initial unconditional universe, possibly next to some conditional assumptions. Often, $\mathbb{P}$ also describes all $X|B \in \mathcal{D}$, meaning that all X, B belong to $\mathcal{L}(\mathbb{P})$. Such a partition is *finer* than $\mathbb{P}_{\underline{G}}$ (or possibly equal to $\mathbb{P}_{\underline{G}}$ in some cases), since $\mathbb{P}_{\underline{G}}$ supports only the gambles appearing in $\underline{G}$. This implies that for any $e \in \mathbb{P}$ there is one and only one $\omega \in \mathbb{P}_{\underline{G}}$ such that $e \Rightarrow \omega$. We may thus apply Corollary 4.1 to any $e \in \mathbb{P}$ such that $e \Rightarrow \omega \Rightarrow B$ and $\underline{P}(e) > 0$ (replace $E^*|B^*$ and $E|B$ with $e = e|\Omega$ and $\omega|B$, respectively). This procedure may allow us to draw some conclusions about whether $\sup(\underline{G}|B)$ is achieved, without ever having to explicitly determine $\mathbb{P}_{\underline{G}}|B$.

Before analysing further this and some other follow-up of Proposition 4.1, we notice that this proposition and Corollary 4.1 have implications regarding events all conditional on the same event B. Since an environment where conditioning is made on the same event does not differ substantially from an unconditional setting, we continue our discussion referring to the unconditional case. This simplifies the notation and makes the understanding of some concepts more immediate.

4.2 Unconditional coherent lower previsions

In the unconditional case, Proposition 4.1 reduces to

Proposition 4.2. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional coherent lower prevision and $\underline{G}$ as in Definition 2.2 (a), with $B_i = \Omega \quad \forall B_i$. Let also $\underline{G}$ be a WDB gain. Then, for any event $E \in \mathcal{D}$ with $\underline{P}(E) > 0$, it holds that* $\sup(\underline{G}|E) = 0$.

Clearly, $\underline{G}$ in Proposition 4.2 is a gain concerning the unconditional gambles $X_0, X_1, \ldots, X_n$. However, under some specific assumptions, the WDB extends to conditionals.

Remark 4.1 (A case of WDB propagation)**.** *If E is* irrelevant *to $X_i \in \mathcal{D}$, that is if $\underline{P}(X_i|E) = \underline{P}(X_i)$, $i = 0, 1, \ldots, n$, we obtain a WDB propagation to* *conditionals in the assumptions of Proposition 4.2. In fact,*

$$0 = \sup(\underline{G}|E) = \sup\Big(\sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - s_0\big(X_0 - \underline{P}(X_0)\big)\Big|E\Big)$$
$$= \sup\Big(\sum_{i=1}^{n} s_i E\big(X_i - \underline{P}(X_i|E)\big) - s_0 E\big(X_0 - \underline{P}(X_0|E)\big)\Big|E\Big)$$

and the last expression is the supremum of a gain regarding $X_i|E$, $i = 0, 1, \dots, n$.

Similarly, it is easy to check that if $\underline{P}(\cdot)$ is coherent on $\{X_0, X_1, \dots, X_n\}$ then $\underline{P}(\cdot|E) = \underline{P}(\cdot)$ is W-coherent on $\{X_0|E, X_1|E, \dots, X_n|E\}$.

From Corollary 4.1 and the subsequent discussion, it is patent that the following corollary holds.

Corollary 4.2. *Let $\underline{P} : \mathcal{D} \rightarrow \mathbb{R}$, $\underline{G}$ be as in Proposition 4.2. Let $\mathbb{P} \subseteq \mathcal{D}$ be either $\mathbb{P}_{\underline{G}}$ or a partition finer than $\mathbb{P}_{\underline{G}}$, $e \in \mathbb{P}, \omega \in \mathbb{P}_{\underline{G}}$ be such that $e \Rightarrow \omega$ and $\underline{P}(e) > 0$. Then $\underline{G}(e) = \underline{G}(\omega) = 0$.*

It ensues from Corollary 4.2 that

Corollary 4.3. *Let $\underline{P}$, $\underline{G}$, $\mathbb{P} \subseteq \mathcal{D}$ be as in Corollary 4.2, with $\sup \underline{G} = 0$. Let $I \neq \varnothing$ be a set of indexes such that, for all $i \in I$, we have that $e_i \in \mathbb{P}$ and $\underline{P}(e_i) > 0$. Then $\underline{G}|\bigvee_{i \in I} e_i = 0$.*

Proof. The set of possible values of $\underline{G}|\bigvee_{i \in I} e_i$ is $\{\underline{G}(e_i) : i \in I\}$, which is $\{0\}$ by Corollary 4.2. □

Let us discuss now in detail the relationship between positive lower probability events and suprema of WDB gains. As before, $\underline{P} : \mathcal{D} \rightarrow \mathbb{R}$ is an unconditional coherent lower probability, $\underline{G}$ is the gain in Definition 2.2 (a), with $B_i = \Omega \quad \forall B_i$, and let $\sup \underline{G} = 0$. $\mathbb{P}$ denotes a partition finer that $\mathbb{P}_{\underline{G}}$. Further, let

$$\mathcal{P} = \{\omega \in \mathbb{P}_{\underline{G}} : \underline{P}(\omega) > 0\}, \quad \mathcal{N} = \{\omega \in \mathbb{P}_{\underline{G}} : \underline{G}(\omega) = 0\}. \tag{27}$$

The following facts are implied by the preceding results:

(a) If there exists $e \in \mathbb{P}$ such that $\underline{P}(e) > 0$, then $\underline{G}$ achieves its supremum (at least) at e: $\sup \underline{G} = \max \underline{G} = \underline{G}(e) = 0$ (Corollary 4.2).

(b) If $\sup \underline{G}$ is not achieved, then necessarily $\underline{P}(\omega) = 0$, $\forall \omega \in \mathbb{P}_{\underline{G}}$ (Corollary 4.2). Note that there may be some $E \in D$ with $\underline{P}(E) > 0$, implying (Proposition 4.2) $\sup \underline{G}|E = 0$, but $E \notin \mathbb{P}_{\underline{G}}$.

Further, checking that $\underline{P}(e) = 0$, $\forall e \in \mathbb{P} \neq \mathbb{P}_{\underline{G}}$ does not ensure that $\underline{P}(\omega) = 0$, $\forall \omega \in \mathbb{P}_{\underline{G}}$, not even when $\underline{P}$ is a dF-coherent probability P. In this case, for instance, ω may be a logical sum of infinitely many atoms $e \in \mathbb{P}$, but P may be such that $P(\omega) > 0 = P(e)$, $\forall e \in \mathbb{P}$, $e \Rightarrow \omega$.

(c) Conversely, if $\underline{P}(\omega) = 0$, $\forall \omega \in \mathbb{P}_{\underline{G}}$, then $\sup \underline{G}$ may or may not be achieved. We exemplify both instances.

Example 4.1. *Let X be defined on $\mathbb{P} = \{\omega_n : n \in \mathbb{N}^+\}$ by $X(\omega_n) = \frac{1}{n}$, $\forall n \in \mathbb{N}^+$. $\underline{P}$ is given on $\mathbb{P} \cup \{X\}$ by $\underline{P}(\omega_n) = 0$, $\forall n \in \mathbb{N}^+$, and $\underline{P}(X) = 0 = \inf X$. Hence $\underline{P}$ is coherent, being the vacuous lower prevision. The gain $\underline{G} = -(X - 0) = -X$ is such that $\mathbb{P}_{\underline{G}} = \mathbb{P}$. Moreover $\underline{G}(\omega_n) = -\frac{1}{n}$, $\forall n \in \mathbb{N}^+$, hence $\sup \underline{G} = 0$ is not attained.*

Example 4.2. *Modify Example 4.1 by adding ω_0: $\mathbb{P} = \{\omega_n : n \in \mathbb{N}\}$. Let $X(\omega_0) = 0$, $\underline{P}(\omega_0) = 0$, while X and $\underline{P}$ are defined as before elsewhere, and again $\underline{G} = -X$, so that $\mathbb{P}_{\underline{G}} = \mathbb{P}$. Then $\underline{P}$ is still vacuous, hence coherent, on $\mathbb{P} \cup \{X\}$, but now $\sup \underline{G} = \max \underline{G} = \underline{G}(\omega_0) = 0$.*

(*d*) Considering (27), $\mathcal{P} \subseteq \mathcal{N}$, by Corollary 4.2. It is $\mathcal{P} = \mathcal{N}$ in the next Example 4.3, but the inclusion may be strict, as in the previous Example 4.2, where $\mathcal{P} = \emptyset$, $\mathcal{N} = \{\omega_0\}$.

On the one hand, this means that the cardinality of $\mathcal{P}$ is a lower bound to the number of atoms of $\mathbb{P}_{\underline{G}}$ where $\sup \underline{G}$ is achieved. On the other hand, it holds that the cardinality of $\mathcal{P}$ contributes to determine that of $\mathcal{N}$, but may generally be not the only cause. We show in Example 4.4 that the choice of the stakes may be influential too.

Example 4.3. *Let $\underline{P}$ satisfy the assumptions of Proposition 3.1 (b), see also the corresponding Example 3.2. We see that $\mathbb{P}_{\underline{G}} = \{E_1 \wedge \neg E_2 = E_1, \neg E_1 \wedge E_2 = E_2, \neg E_1 \wedge \neg E_2\}$. We can easily extend $\underline{P}$ to $\mathbb{P}_{\underline{G}}$ noting that, since $\underline{P}(E_1) + \underline{P}(E_2) = 1$, $\underline{P}(\neg E_1 \wedge \neg E_2) = 0$ is its unique coherent extension on $\mathbb{P}_{\underline{G}}$. Doing so, $\mathcal{N} = \{E_1, E_2\} = \mathcal{P}$, given that $\underline{G}(\neg E_1 \wedge \neg E_2) < 0$.*

Example 4.4. *Let $\mathcal{D} = \{E_1, E_2, \neg E_2, \neg E_1 \wedge E_2\}$, with $E_1 \wedge \neg E_2 = \emptyset$, and $\underline{P} : \mathcal{D} \to \mathbb{R}$ be the vacuous lower probability. With $\mathcal{D}_{\underline{G}} = \{E_1, E_2\}$, we know from Proposition 3.1 (f) that $\max \underline{G} = \max(s_1(E_1 - 0) + s_2(E_2 - 0)) = 0$ if $s_2 < 0 < s_1$, $s_1 \leq -s_2$. Here $\mathbb{P}_{\underline{G}} = \{E_1, \neg E_2, \neg E_1 \wedge E_2\}$, $\mathcal{P} = \emptyset$, while there are one or two atoms of $\mathbb{P}_{\underline{G}}$ where $\underline{G}$ attains its maximum 0, according to whether, respectively, $s_1 < -s_2$ or $s_1 = -s_2$. In fact $\underline{G}(E_1) = s_1 + s_2 \leq 0$ if and only if $s_1 \leq -s_2$, in particular $\underline{G}(E_1) = 0$ if and only if $s_1 = -s_2$, $\underline{G}(\neg E_2) = 0$, $\underline{G}(\neg E_1 \wedge E_2) = s_2 < 0$.*

(*e*) (*dF-coherent probabilities.*) In the special case that a dF-coherent *probability* P is assessed on $\mathcal{D}$, then $\mathbb{P}_G$ is necessarily finite, hence G has a maximum.

Further, if there is a *unique* $\tilde{\omega} \in \mathbb{P}_G$ such that $G(\tilde{\omega}) = 0$, then $P(\tilde{\omega}) = 1$ (by Corollary 4.2: $P(\omega) = 0$ for all $\omega \neq \tilde{\omega}$, or else $G(\omega)$ would be zero too). Then, every event in $\mathcal{D}_G = \{E_1, \ldots, E_n\}$ has extreme probability. In fact, $\tilde{\omega} = E_1' \wedge \ldots \wedge E_n'$, where E_i' may be either E_i or $\neg E_i$. If $E_i' = E_i$, $\tilde{\omega} \Rightarrow E_i$ and $P(E_i) = 1$, otherwise $P(\neg E_i) = 1$ and $P(E_i) = 0$.

If instead P is not concentrated on just one atom of $\mathbb{P}_G$, G is not unimodal.

Testing Weak Dutch Books. If it is not known whether, given a coherent $\underline{P}$, $\sup \underline{G} = 0$ or not, we can try to rule out the possibility of a WDB by checking the sign of $\underline{G}$ at some $\omega \in \mathbb{P}_{\underline{G}}$ such that $\underline{P}(\omega) > 0$ (if any). In fact:

- if $\underline{G}(\omega) > 0$, then obviously $\sup \underline{G} > 0$;
- if $\underline{G}(\omega) < 0$, then $\sup \underline{G} > 0$ by Corollary 4.2.

This method is very simple, but allows no conclusion when $\underline{G}(\omega) = 0$. In fact, it is clearly possible that $\underline{G}(\omega) = 0$ and $\sup \underline{G} = 0$, but even when $\underline{G}(\omega) = 0$ for all $\omega \in \mathbb{P}_{\underline{G}}$ such that $\underline{P}(\omega) > 0$, $\sup \underline{G}$ may be strictly positive. The next example illustrates this case.

Example 4.5. *Let $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$, X, Y two gambles in $\mathcal{L}(\mathbb{P})$ defined by $X(\omega_1) = Y(\omega_1) = -1$, $X(\omega_2) = Y(\omega_2) = 1$, $X(\omega_3) = \alpha > \beta = Y(\omega_3)$. The dF-coherent P defined on $\mathbb{P}$ by $P(\omega_1) = P(\omega_2) = \frac{1}{2}$, $P(\omega_3) = 0$ has a unique dF-coherent extension on $\mathbb{P} \cup \{X, Y\}$ given by the expectations of X, Y, respectively, i.e., $P(X) = \sum_{i=1}^{3} P(\omega_i)X(\omega_i) = 0$, and similarly $P(Y) = 0$. Defining $G = X - Y$, we have $\mathbb{P}_G = \mathbb{P}$, $\mathcal{P} = \{\omega_1, \omega_2\}$, $G(\omega_1) = G(\omega_2) = 0$, but $\max G = G(\omega_3) = \alpha - \beta > 0$.*

5 Vulnerability to Dutch Books in the agent's evaluations

To introduce the main topic of this section, let us first consider the simplest WDB instance: an agent assesses a dF-coherent *probability* P such that $\max G = 0$ for some G in (1). This means that when betting in favour or against all events E_i in $\mathcal{D}_G$ with the given $P(E_i)$ and stakes s_i, the agent is certain to lose or at best gain nothing from the overall bet, whatever happens (*lose-or-draw case*). But which are the agent's beliefs about bearing a loss, i.e., about suffering from a real *Dutch Book*? It was proven in [2, Section 9.5.4] that then $P(G < 0) = 0$. This result is very reassuring: even though the agent cannot win, she/he is nearly sure not to lose, in the sense that the probability of bearing a loss is zero.

The same problem, of detecting the agent's loss prospects with a WDB gain, has been investigated in [15] for unconditional dF-coherent and coherent lower or upper previsions, proving the next two propositions:

Proposition 5.1. *Given a dF-coherent prevision P on $\mathcal{D}$, let the WDB gain G be as in (5). Then, (any dF-coherent extension of) P is such that*

(a) $P(G \le -\varepsilon) = 0$, $\forall \varepsilon > 0$;

(b) if in addition $X_1, \ldots, X_n$ are all simple, we also have that $P(G < 0) = 0$.[4]

[4]The dF-coherent extension of P is mentioned explicitly because (the indicators of) the events $(G \le -\varepsilon)$ and $(G < 0)$ need not belong to $\mathcal{D}$. Similar specifications will be omitted in the following propositions.

Proposition 5.2 (Coherent lower/upper previsions). *Given an unconditional coherent lower prevision* $\underline{P}$ *(alternatively, an unconditional coherent upper prevision* $\overline{P}$*) on* $\mathcal{D}$*, let the WDB gain* $\underline{G}$ *be given as in Definition 2.2* (a)*, with* $B_i = \Omega \quad \forall B_i$ *(alternatively, let* $\overline{G} = \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i) - s_0(\overline{P}(X_0) - X_0)$ *be a WDB gain). This implies that*

(a) $\underline{P}(\underline{G} \leq -\varepsilon) = 0$ *(*$\underline{P}(\overline{G} \leq -\varepsilon) = 0$*),* $\forall \varepsilon > 0$*;*

(b) *if* $X_0, X_1, \ldots, X_n$ *are all simple,* $\underline{P}(\underline{G} < 0) = 0$ *(*$\underline{P}(\overline{G} < 0) = 0$*).*

Propositions 5.1 and 5.2 entail the following important facts:

- With dF-coherent *previsions* (rather than probabilities), the WDB gain may fail to achieve its supremum of 0. If this is the case, the agent's assessment implies that, for any $\varepsilon > 0$, the event $(G \leq -\varepsilon)$ is given zero probability. This means that *any loss bounded away from zero* is given zero probability by the agent. When $X_0, X_1, \ldots, X_n$ are simple (in particular, when they are (indicators of) events), there is $\varepsilon > 0$ such that $(G \leq -\varepsilon) = (G < 0)$, and the stronger implication that *any loss* must be given zero probability applies.

- The situation is the same with coherent lower/upper *previsions* (versus probabilities), as for the losses that are given an uncertainty evaluation of zero. However, the fundamental dissimilarity with precise assessments arises that this evaluation is now a *lower* probability.

With reference to the facts just described, in the sequel we shall term *Dutch Book* the occurrence - whatever happens - of a loss bounded away from 0, or simply of a loss, when $X_0, X_1, \ldots, X_n$ in the WDB gain are simple.

It is significant that the lower probability, but no longer the probability, of a Dutch Book is zero with coherent *imprecise* previsions. It means that with imprecise evaluations we have a generally weaker feeling that a Dutch Book will be avoided.

In this section, we analyse further the implications of a WDB assessment for the agent's Dutch Book evaluations. Precisely:

(a) We show that several such implications are entailed by a more general property of zero supremum random numbers, not explicitly involving WDBs. In particular, we deduce from this property a result analogous to Proposition 5.2 for *conditional* W-coherent lower previsions.

(b) We discuss further the agent's evaluations of Dutch Books. The strength of opinions about avoiding losses is dependent on the degree of consistency of the initial uncertainty assessment, and becomes even weaker with convex or centered convex previsions.

The starting point for (a) is the following general property:

Proposition 5.3. *Let* $X|B, Z|B : \mathbb{P}|B \to \mathbb{R}$ *be two conditional random numbers, with* $\sup(X|B) = 0$, $\sup(Z|B) < +\infty$. *Suppose that there exist* $\varepsilon > 0, \delta > 0$ *such that*

$$X|B(\omega|B) \leq -\varepsilon \quad \textit{if and only if} \quad Z|B(\omega|B) \geq -\delta, \quad \forall \omega|B \in \mathbb{P}|B.$$

Then there exists $\bar{s} > 0$ *such that,* $\forall s \in]0, \bar{s}]$,

$$\sup(X + sZ|B) < 0. \tag{28}$$

Proof. Define

$$A^+ = \{\omega|B \in \mathbb{P}|B : X|B(\omega|B) > -\varepsilon\} = \{\omega|B \in \mathbb{P}|B : Z|B(\omega|B) < -\delta\},$$
$$A^- = \{\omega|B \in \mathbb{P}|B : X|B(\omega|B) \leq -\varepsilon\} = \{\omega|B \in \mathbb{P}|B : Z|B(\omega|B) \geq -\delta\}.$$

The set A^+ is never empty, since $\sup(X|B) = 0$, whilst A^- might be empty or not. In any case, $A^+ \cap A^- = \varnothing$, $A^+ \cup A^- = \mathbb{P}|B$.

Now take any $\omega|B \in \mathbb{P}|B$. Two alternatives (at most, in the case $A^- \neq \varnothing$) may occur:

(a) $\omega|B \in A^+$.

Since then $Z|B(\omega|B) < -\delta$, we get $\forall s > 0$

$$X + sZ|B(\omega|B) = X|B(\omega|B) + sZ|B(\omega|B) < \sup(X|B) - s\delta = -s\delta;$$

it ensues that

$$\sup_{A^+}(X + sZ|B) \leq -s\delta < 0, \quad \forall s > 0. \tag{29}$$

(b) ($A^- \neq \varnothing$ and) $\omega|B \in A^-$.

Now $X|B(\omega|B) \leq -\varepsilon$, while $Z|B(\omega|B) \geq -\delta$.

$(b1)$ *If* $Z|B(\omega|B) = -\delta$,

$$X + sZ|B(\omega|B) \leq -\varepsilon - s\delta < -s\delta, \quad \forall s > 0. \tag{30}$$

$(b2)$ *If* $Z|B(\omega|B) > -\delta$, then $\sup(Z|B) + \delta > 0$. Therefore, defining

$$\bar{s} = \frac{\varepsilon}{\sup(Z|B) + \delta}, \tag{31}$$

we have that $\bar{s} > 0$. Further, for any $s \in]0, \bar{s}]$,

$$\begin{aligned} X + sZ|B(\omega|B) &= X|B(\omega|B) + sZ|B(\omega|B) \\ &= X|B(\omega|B) + s\big(Z|B(\omega|B) + \delta\big) - s\delta \\ &\leq -\varepsilon + \bar{s}(\sup(Z|B) + \delta) - s\delta \\ &= -\varepsilon + \varepsilon - s\delta = -s\delta. \end{aligned}$$

From the derivations above we may conclude that:

- If, for any $\omega|B \in \mathbb{P}|B$, case (a) or possibly $(b1)$ apply, while $(b2)$ does not, i.e. if A^- is either empty or such that $Z|B(\omega|B) = -\delta, \forall \omega|B \in A^-$, any arbitrary positive real number may be chosen as $\bar{s}$. This satisfies (28) (cf. (29),(30)).
- If, for at least one $\omega|B \in \mathbb{P}|B$, case $(b2)$ applies, $\bar{s}$ is given by (31). This ensures that (28) holds when $s \in]0, \bar{s}]$.

□

Proposition 5.3 specialises into the next corollary when $Z|B$ is *simple*, i.e., assumes finitely many distinct values. Note that the statement of the corollary is derived from that of Proposition 5.3 by putting $\delta = 0$ there.

Corollary 5.1. *Let $X|B, Z|B : \mathbb{P}|B \to \mathbb{R}$ be two conditional random numbers, with $\sup(X|B) = 0$ and $Z|B$ simple. If there exists $\varepsilon > 0$ such that*

$$X|B(\omega|B) \leq -\varepsilon \quad \textit{if and only if} \quad Z|B(\omega|B) \geq 0, \quad \forall \omega|B \in \mathbb{P}|B,$$

then there is $\bar{s} > 0$ such that, $\forall s \in]0, \bar{s}]$, $\sup(X + sZ|B) < 0$.

Proof. If $\min(Z|B) \geq 0$, then, $\forall \delta > 0$, $\forall \omega|B \in \mathbb{P}|B$,

$$Z|B(\omega|B) \geq -\delta \quad \text{iff} \quad Z|B(\omega|B) \geq 0 \quad \text{iff} \quad X|B(\omega|B) \leq -\varepsilon, \tag{32}$$

and Proposition 5.3 may be applied to give the thesis.

If $\min(Z|B) < 0$, then the event $B \wedge (Z < 0)$ is non-impossible. Hence we may consider the gamble $Z|B \wedge (Z < 0)$, which is simple and thus has a negative maximum that we term $-\gamma$:

$$\max(Z|B \wedge (Z < 0)) = -\gamma < 0.$$

Hence, $\forall \delta \in]0, \gamma[$, $\forall \omega|B \in \mathbb{P}|B$, it holds that $Z|B(\omega|B) \geq -\delta$ iff $Z|B(\omega|B) \geq 0$. Using the hypothesis, we can extend this equivalence to get (32) and apply again Proposition 5.3. □

Corollary 5.1 plays a key role in the proof of the following proposition, generalising Proposition 5.2.

Proposition 5.4. *Given a W-coherent lower prevision $\underline{P}$ (alternatively, a W-coherent upper prevision $\overline{P}$) on $\mathcal{D}$, let $\underline{G}, B$ be as in Definition 2.2 (a) such that $\underline{G}$ is a WDB gain (let $\overline{G}, B$ be as in Definition 2.4 such that $\overline{G}$ is a WDB gain). Then, for any W-coherent extension of $\underline{P}$ (still termed $\underline{P}$)*

(a) $\underline{P}(\underline{G}|B \leq -\varepsilon) = 0$ *(*$\underline{P}(\overline{G}|B \leq -\varepsilon) = 0$*)*, $\forall \varepsilon > 0$*;*

(b) *if $\mathcal{D}_{\underline{G}}$ ($\mathcal{D}_{\overline{G}}$) is made of simple conditional gambles, $\underline{P}(\underline{G}|B < 0) = 0$ ($\underline{P}(\overline{G}|B < 0) = 0$).*

Proof. Proof of (a). Let $\underline{P}$ be a W-coherent lower prevision. By contradiction, suppose that $\underline{P}(\underline{G}|B \leq -\varepsilon) > 0$ is a W-coherent extension of $\underline{P}$, for some $\varepsilon > 0$. Because of this, and noting that $(\underline{G}|B \leq -\varepsilon) = (\underline{G} \leq -\varepsilon)|B$ (see (3)), we obtain applying Definition 2.2 (a)

$$\sup\Big(\underline{G} + sB\big(I_{(\underline{G}\leq-\varepsilon)} - \underline{P}(\underline{G} \leq -\varepsilon|B)\big)|B\Big) \geq 0, \quad \forall s \geq 0. \tag{33}$$

However, defining

$$Z|B = B\big(I_{(\underline{G}\leq-\varepsilon)} - \underline{P}(\underline{G} \leq -\varepsilon|B)\big)|B,$$

we deduce that, for all $\omega|B \in \mathbb{P}|B$,

- if $\underline{G}|B(\omega|B) > -\varepsilon$, then $Z|B(\omega|B) = -\underline{P}(\underline{G} \leq -\varepsilon|B) < 0$;
- if $\underline{G}|B(\omega|B) \leq -\varepsilon$, then $Z|B(\omega|B) = 1 - \underline{P}(\underline{G} \leq -\varepsilon|B) \geq 0$.

Therefore, $\underline{G}|B(\omega|B) \leq -\varepsilon$ if and only if $Z|B(\omega|B) \geq 0$, $\forall\omega|B \in \mathbb{P}|B$. Recalling also that $\sup(\underline{G}|B) = 0$ by assumption, Corollary 5.1 may be applied to $\underline{G}|B$ and $Z|B$. It implies that there is $\bar{s} > 0$ such that

$$\sup\Big(\underline{G} + \bar{s}B\big(I_{(\underline{G}\leq-\varepsilon)} - \underline{P}(\underline{G} \leq -\varepsilon|B)\big)|B\Big) < 0,$$

contradicting (33). Thus, necessarily, the first equality in (a) holds.

To prove the second, by (8) write $\overline{G}$ as a gain concerning the conjugate $\underline{P}$ of $\overline{P}$, which is a lower prevision W-coherent on $-\mathcal{D} = \{-X|B : X|B \in \mathcal{D}\}$:

$$\overline{G} = \sum_{i=1}^{n} s_i B_i\big(-X_i - \underline{P}(-X_i|B_i)\big) - s_0 B_0\big(-X_0 - \underline{P}(-X_0|B_0)\big).$$

Then apply the first equality of (a).

Proof of (b). It is sufficient to observe that $X_0|B_0, X_1|B_1, \ldots, X_n|B_n$ being simple, $\underline{G}|B$ $(\overline{G}|B)$ is simple too. Therefore there exists $\varepsilon > 0$ such that $(\underline{G}|B \leq -\varepsilon) = (\underline{G}|B < 0)$ (and similarly for $\overline{G}|B$). □

The extent of Proposition 5.4 is perfectly analogous to that of Proposition 5.2: when $\sup(\underline{G}|B) = 0\,(\sup(\overline{G}|B) = 0)$, the agent's beliefs on $\mathcal{D}$, if elicited by a W-coherent lower or upper prevision, imply that her/his lower probability of a real Dutch Book is 0.

However, the upper probability of a Dutch Book is not necessarily 0, and may reasonably be even 1. Consider for this the next simple example (extending Example 1 in [15]).

Example 5.1. *Let* $\mathcal{D} = \{X|B\}$, *where* $X|B$ *is a simple non-constant conditional gamble. Put* $\underline{P}(X|B) = \min(X|B)$. *If* $n = 0$, $s_0 > 0$ *in Definition 2.2* (a), *the gain* $\underline{G}|B = -s_0 B(X - \min(X|B))|B = -s_0(X|B - \min(X|B))$ *is such that* $\max(\underline{G}|B) = 0$. *It is coherent to assess* $\overline{P}(\underline{G}|B \leq -\varepsilon) = 1$, *for any* $\varepsilon > 0$ *such that* $\min(\underline{G}|B) \leq -\varepsilon < 0$. *In fact (cf. Footnote 2), this is equivalent* *to extending* $\underline{P}$ *on* $\mathcal{D}' = \{X|B\} \cup \{(\underline{G}|B > -\varepsilon) : \varepsilon \in]0, -\min(\underline{G}|B)]\}$*, letting* $\underline{P}(\underline{G}|B > -\varepsilon) = 0$*. The extension is W-coherent, being the vacuous lower prevision on* $\mathcal{D}'$*.*

Since there is $\varepsilon' \in]0, -\min(\underline{G}|B)]$ *such that* $(\underline{G}|B \leq -\varepsilon') = (\underline{G}|B < 0)$*, it is coherent to assume* $\overline{P}(\underline{G}|B < 0) = 1$*. It is also intuitively sound: the only starting assignment on* $\mathcal{D}$*,* $\underline{P}(X|B) = \min(X|B)$*, is lower vacuous and compatible with the absence of any significant information on* $X|B$*. Quite reasonably then, the agent's opinion about bearing or not a Dutch Book may be vacuous too.*

We point out that Example 5.1 shows that also the upper probability about bearing any loss ($\overline{P}(\underline{G}|B \leq -\varepsilon)$) may be non-zero, and even 1.

From these remarks and Proposition 5.4, we may conclude that the (conditional) *p-box* of a WDB gain $\underline{G}|B$ for a W-coherent $\underline{P}$ has a special structure, as for its lower distribution function $\underline{F}(x) = \underline{P}(\underline{G}|B \leq x)$, $x \in \mathbb{R}$. $\underline{F}$ is a single-step function, identically equal to 0 for any $x < 0$, to 1 for any $x \geq 0$. On the contrary, the upper distribution function $\overline{F}(x) = \overline{P}(\underline{G}|B \leq x)$, $x \in \mathbb{R}$, is essentially unconstrained and need not coincide with $\underline{F}(x)$ if $(\underline{G}|B \leq x)$ is a non-trivial event.

Given that $\overline{P}(\underline{G}|B < 0)$ need not be 0 when $\underline{G}|B$ is a simple gamble, one may wonder whether it is at least *always* possible to put $\overline{P}(\underline{G}|B < 0) = 0$ or more generally (for an arbitrary $\underline{G}|B$) $\overline{P}(\underline{G}|B < -\varepsilon) = 0$, if wished. The answer is negative even in an unconditional environment, as shown next.

Proposition 5.5. *Let* $\underline{P} : \mathcal{D} \to \mathbb{R}$ *be an unconditional coherent lower prevision, and* $\underline{G}$ *defined by* (20)*, with* $s_i > 0$*,* $i = 1, \ldots, n$*, be a WDB gain. Then*

(*a*) *if* $\mathcal{D}_{\underline{G}} = \mathcal{D}$*, it is coherent to put* $\overline{P}(\underline{G} \leq -\varepsilon) = 0$*,* $\forall \varepsilon > 0$*;*

(*b*) *otherwise this choice may be incoherent.*

Proof. By conjugacy, $\overline{P}(\underline{G} \leq -\varepsilon) = 1 - \underline{P}(\underline{G} > -\varepsilon)$, hence we may equivalently evaluate the coherence of the extension $\underline{P}(\underline{G} > -\varepsilon) = 1$.

Proof of (*a*). If $\mathcal{D} = \mathcal{D}_{\underline{G}}$, then $\underline{P}$ is a dF-coherent prevision on $\mathcal{D}$ by Proposition 3.3. As such, it allows a dF-coherent extension (still termed $\underline{P}$) on $\mathcal{D} \cup \{\underline{G} \leq -\varepsilon, \underline{G} > -\varepsilon\}$. Since necessarily $\underline{P}(\underline{G} \leq -\varepsilon) = 0$ by Proposition 5.1, $\underline{P}(\underline{G} > -\varepsilon) = 1$ by dF-coherence of $\underline{P}$.

Proof of (*b*). A counterexample suffices to prove (*b*). For this, a simple idea is to fix ε such that $(\underline{G} > -\varepsilon) = (\underline{G} = 0)$, and that this event already belongs to $\mathcal{D}$, but is not given lower probability 1. Let then $\mathcal{D} = \{E_1, E_2, E_1 \vee \neg E_2\}$, with $E_1 \wedge \neg E_2 = \emptyset$ (hence $E_1 \Rightarrow E_2$), $\underline{P}(E_1) = \underline{P}(E_2) = p \in]0, 1[$, $\underline{P}(E_1 \vee \neg E_2) = 1 - \varepsilon$, $\varepsilon \in]0, 1 - p[$.

- $\underline{P}$ is a coherent lower probability on $\mathcal{D}$. In fact, $\underline{P}$ is the lower envelope (on $\mathcal{D}$) of the dF-coherent probabilities P_1, P_2:

	E_1	$\neg E_1 \wedge E_2$	E_2	$\neg E_2$	$E_1 \vee \neg E_2$
P_1	p	0	p	$1 - p$	1
P_2	p	ε	$p + \varepsilon$	$1 - (p + \varepsilon)$	$1 - \varepsilon$
$\underline{P}$	p		p		$1 - \varepsilon$

- On $\mathcal{D}_{\underline{G}} = \{E_1, E_2\}$, $\underline{P}$ may induce a WDB for a convenient $\underline{G}$: this is precisely the case of Example 3.3 in Section 3.1.
- Yet, $(\underline{G} = 0)$ cannot be given lower probability 1, because here $(\underline{G} = 0) = E_1 \vee \neg E_2 \in \mathcal{D}$ (cf. the comment following Example 3.3) and $\underline{P}(E_1 \vee \neg E_2) = 1 - \varepsilon$.

□

It is easy to check that a result analogous to Proposition 5.1 applies to dF-coherent conditional previsions:

Proposition 5.6. *Let P be a dF-coherent prevision on $\mathcal{D}$, and let G, B be as in Definition 2.1. Then*

(*a*) $P(G|B \leq -\varepsilon) = 0$, $\forall \varepsilon > 0$*;*

(*b*) *if $X_1|B_1, \ldots, X_n|B_n$ are simple gambles,* $P(G|B < 0) = 0$.

The proof replicates that of Proposition 5.4 with obvious substitutions (replace $\underline{G}$, B, $\underline{P}$ with G, $\bigvee_{i=1}^{n} B_i$, P, respectively) and is therefore omitted. Thus, an agent assessing a precise rather than an imprecise coherent evaluation is more confident to avoid a Dutch Book, and assumes that this is an event of zero (precise, rather than lower) probability.

The strength of the agent's beliefs about avoiding a proper Dutch Book depends therefore on the degree of consistency of her/his uncertainty assessments.

In fact, weaker concepts than (W-)coherence may allow for even weaker implications about the occurrence of Dutch Books. In particular, centered convexity is a notion not too far from coherence for many aspects. Yet, in the case of a WDB with a centered convex lower prevision $\underline{P}$, the agent may have no real convictions that a Dutch Book associated with the gain $\underline{G}$ will be avoided, as not even $\underline{P}(\underline{G} \leq -\varepsilon)$ may be forced to be zero. We show this in the next example.

Example 5.2. *Let $\mathcal{D} = \{\emptyset, E_1, \neg E_1 \wedge E_2, E_2\}$, with $E_1 \Rightarrow E_2$ ($E_1 \neq E_2$). Given $\underline{P}$ such that $\underline{P}(\emptyset) = 0$, $\underline{P}(E_1) = \underline{P}(E_2) = p \in \,]0, 1[$, $\underline{P}$ is clearly centered convex (even coherent) on $\{\emptyset, E_1, E_2\}$. Consider first the following question: which are the possible assignments for $\underline{P}(\neg E_1 \wedge E_2)$ such that $\underline{P}$ is a centered convex lower probability on $\mathcal{D}$? The answer is:*

$$\underline{P}(\neg E_1 \wedge E_2) \in [0, \min\{p, 1 - p\}]. \tag{34}$$

In fact, $0 = \underline{P}(\emptyset) \leq \underline{P}(\neg E_1 \wedge E_2) \leq \underline{P}(E_2) = p$, by Definition 2.3 (b) and Proposition 2.4 (a). Moreover $\underline{P}(\neg E_1 \wedge E_2) \leq \underline{P}(\neg E_1) \leq 1 - \underline{P}(E_1)$: the first inequality is again due to Proposition 2.4 (a), the second holds because $\underline{P}$, being centered convex, avoids sure loss [7, Proposition 3.5 (e)] and therefore the condition $\underline{P}(\neg E_1) \leq 1 - \underline{P}(E_1)$ applies [17, Section 4.6.1]. Further, the bounds in (34) can be achieved: this can be seen using Proposition 2.5. For instance, if $1 - p \leq p$, $\underline{P}$ is centered convex on $\mathcal{D}$ putting $\underline{P}(\neg E_1 \wedge E_2) = 1 - p$, as $\underline{P}$ is then the lower envelope on $\mathcal{D}$ of $P_1 + \alpha_1$ and $P_2 + \alpha_2$, with P_1, P_2 dF-coherent probabilities, $\alpha_1 = 0$, $\alpha_2 = p$ (as in the following table).

	E_1	$\neg E_1 \wedge E_2$	E_2	$\emptyset$	$\neg E_2$
$P_1 = P_1 + \alpha_1$	p	$1-p$	1	0	0
P_2	0	0	0	0	1
$P_2 + \alpha_2$	p	p	p	p	$1+p$
$\underline{P}$	p	$1-p$	p	0	

It can be checked similarly that the extensions $\underline{P}(\neg E_1 \wedge E_2) = 0$ *and, alternatively and if* $p < 1-p$, $\underline{P}(\neg E_1 \wedge E_2) = p$ *ensure centered convexity of* $\underline{P}$ *on* $\mathcal{D}$ *too. Moreover,* $\underline{P}(\neg E_1 \wedge E_2)$ *may take any value in* $[0, \min\{p, 1-p\}]$ *by [10, Lemma 1].*

Now consider the admissible gains for $\underline{P}$ *on* $\mathcal{D}_{\underline{G}} = \{E_1, E_2\}$*. Among such gains, we find the WDB gain* $\underline{G}$ *in Example 3.3 (which, putting* $s_1 = -s_2 = 1$*, reduces to a gain for testing convexity of* $\underline{P}$*). Further, we know (see the comment following Example 3.3) that the event* $(\underline{G} < 0)$ *is equal to the event* $\neg E_1 \wedge E_2$*. From all this, we deduce that:*

- *If* $\underline{P}$ *is originally assessed only on* $\{\emptyset, E_1, E_2\}$*, then* $\underline{P}(\underline{G} < 0) = \underline{P}(\neg E_1 \wedge E_2)$ *may be given by* (34) any *value in* $[0, \min\{p, 1-p\}]$ *while preserving centered convexity.*

- *If* $\underline{P}$ *is already assessed on* $\mathcal{D}$*, with* $\underline{P}(\neg E_1 \wedge E_2) \neq 0$ *while satisfying* (34)*, then its convex natural extension* $E_{\underline{c}}$ *on* $(\underline{G} < 0)$ *is* not *0 (being* $\underline{P}(\neg E_1 \wedge E_2)$ *itself). This follows from Proposition 2.4* (*b*)*.*

Thus, if an uncertainty assessment incurs a WDB, the agent's opinion about avoiding a real Dutch Book depends on the degree of precision of the consistency notion the assessment satisfies. The self-protection offered by the dF-coherence is maximal, whilst it becomes vacuous with convexity. Even if centered, convexity does not guarantee that $E_{\underline{c}}(\underline{G} \leq -\varepsilon)$ may be 0.

6 Hedging Weak Dutch Books

Next to their axiomatic meaning, the definitions of dF-coherence, W-coherence and other ones recalled in this paper may be given a betting interpretation which, in our view, should represent more a way of eliciting beliefs than a real-life betting scheme. It is anyway hard to explain why WDBs should be acceptable, on the basis of pure betting argumentations. In fact, the problem of hedging WDBs was perceived soon after de Finetti's theory of coherence became widespread enough. Because of this, the oldest hedging idea was (and still largely is) confined to the realm of (precise) probabilities.

6.1 Strict consistency

The idea we just mentioned is the most obvious, and radical, solution: redefine coherence so that WDBs are ruled out. The modified coherence concept is termed (nowadays) *strict coherence*, and replaces the condition $\sup G \geq 0$ with $\sup G > 0$, for any admissible $G \neq 0$.

The approach may be straightforwardly extended to a generic uncertainty measure μ to which some betting scheme applies:

Definition 6.1. *Let $\mu : \mathcal{D} \to \mathbb{R}$ be an uncertainty measure, whose consistency requires that* $\sup(G|B) \geq 0$ *for any conditional gain $G|B$ admissible according to certain rules. Then μ is* strictly consistent *if, for each such $G|B$, either $G|B = 0$ or* $\sup(G|B) > 0$.

Strict coherence for (precise) probabilities was discussed as early as the mid-fifties of the last century in [6, 13]. Both papers identify a condition necessary for strict coherence, and [6] proves its sufficiency, jointly with some conditions for non-strict coherence. In today's language, we would express the strict coherence condition for a probability P by asking that $P(E) = 1$ if and only if $E = \Omega$. Note that this is equivalent to requiring that $P(E) = 0$ if and only if $E = \emptyset$ (this is the form we shall employ later). Kemeny in [6] referred to conditional events too, but the approach to conditional coherence in [6] was still not quite focused. The case of conditional coherence was hinted in [18] and is dealt with in the next proposition, characterising strict consistency for W-coherent lower previsions, which we term strict W-coherence.

Proposition 6.1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision. Then,*

(a) If $\underline{P}$ is strictly W-coherent on $\mathcal{D}$,

$$\underline{P}(A|B) > 0, \text{ for all events } A|B \in \mathcal{D}, A|B \neq \emptyset|B. \tag{35}$$

(b) If $\underline{P}$ is not strictly W-coherent on $\mathcal{D}$ and, for any WDB gain $\underline{G}|B \neq 0$ as in Definition 2.5, there exists $\varepsilon > 0$ such that $(\underline{G}|B \leq -\varepsilon) \in \mathcal{D}$ is non-impossible,

$$\exists\, A|B \in \mathcal{D}, A|B \neq \emptyset|B : \underline{P}(A|B) = 0. \tag{36}$$

Proof. Proof of (a). By contradiction, let $A|B \in \mathcal{D}$ be such that $A|B \neq \emptyset|B$ and $\underline{P}(A|B) = 0$ (hence $A \wedge B \neq B$, otherwise $\underline{P}(A|B) = 1$). Then a WDB arises for $\underline{G}|B = -B(A - 0)|B$ (it is a special case of Example 5.1). As a consequence $\underline{P}$ is not strictly W-coherent on $\mathcal{D}$, contradicting the assumption.

Proof of (b). Let $\underline{P}$ be not strictly W-coherent (but W-coherent) on $\mathcal{D}$. Hence, there exists a conditional gain $\underline{G}|B \neq 0$ such that $\sup(\underline{G}|B) = 0$ (Definition 6.1). Take $\varepsilon > 0$ such that $(\underline{G}|B \leq -\varepsilon) \in \mathcal{D}$ is non-impossible. Then, by Proposition 5.4, $\underline{P}(\underline{G}|B \leq -\varepsilon) = 0$. This proves the thesis, with $(\underline{G}|B \leq -\varepsilon)$ playing the role of $A|B$ in (36). □

When $\mathcal{D}$ has a special structure, consistency of $\underline{P}$ on $\mathcal{D}$ may be characterised by some axioms. In such a case, strict consistency may be also characterised by the additional axiom (35), by Proposition 6.1. For instance, consider an unconditional coherent $\underline{P}$ defined on $\mathcal{L}(\mathbb{P})$. The set $\mathcal{L}(\mathbb{P})$ is large enough to satisfy the assumptions of Proposition 6.1 (b). Coherence of $\underline{P}$ is equivalent there to $\underline{P}(X) \geq \inf X, \forall X \in \mathcal{L}(\mathbb{P})$, and to its positive homogeneity and superlinearity

[17, Theorem 2.5.5], hence strict coherence is characterised by the additional axiom (35) (where $B = \Omega$). For lower previsions defined on $\mathcal{L}(\mathbb{P})$ or other special sets, the characterisation of strict W-coherence in Proposition 6.1 gets explicitly closer to results known for dF-coherence [6, 13]. This time *any* non-impossible event must be given a lower rather than precise *positive* probability.

However, the crucial point is that even W-coherence does not significantly relax the tight constraints of strict coherence already known for dF-coherence. Suppose that $\mathcal{D} \supseteq \mathbb{P}$, with $\mathbb{P}$ a partition of (unconditional) events. Then $\mathbb{P}$ can be at most denumerable for strict W-coherence. In fact, given a W-coherent $\underline{P}$, since by Proposition 2.3 there exists a dF-coherent $P \geq \underline{P}$ on $\mathbb{P}$, positivity of $\underline{P}(\omega)$ for more than countably many $\omega \in \mathbb{P}$ implies $P(\omega) > 0$ for the same atoms. This is known to contradict dF-coherence of P.

Thus, strict (W-)coherence is confined to a denumerable environment, but already Shimony observed in [13] that a finite setting would often be more appropriate. This would avoid unbalanced evaluations, i.e., those concentrating most of the probability in the denumerable case on a finite number, hence on 'few', atoms.

Because of its severe constraints, strict coherence has generally not been considered a satisfactory solution against WDBs for precise assessments, nor can it play a more significant role with W-coherence, as we have just seen.

6.2 Other approaches

Alternative ways of tackling WDBs are based rather on the interpretation of coherence for lower previsions. Take the most investigated case of dF-coherent previsions: instead of saying that $P(X)$ is the price an agent would accept for, indifferently, either buying or selling X, we may interpret $P(X)$ as both an infimum selling price and a supremum buying price for X. That is, the agent would be committed to buy (sell) X at any price lower (higher) than $P(X)$, but not necessarily at $P(X)$. In this way the agent would avoid *in practice* any WDB. For, separating the bets where the agent buys some gambles from those on gambles to be sold, a WDB gain G can be written as

$$G = \sum_{i=1}^{m} s_i\big(X_i - P(X_i)\big) + \sum_{j=1}^{r} t_j\big(P(X_j) - X_j\big),$$

with $m + r \in \mathbb{N}^+$, $s_1, \ldots, s_m, t_1, \ldots, t_r \geq 0$ (and not all zero). Then, for all $\varepsilon > 0$, the agent would be obliged to accept bets guaranteeing the net gain

$$G_\varepsilon = \sum_{i=1}^{m} s_i\big(X_i - (P(X_i) - \varepsilon)\big) + \sum_{j=1}^{r} t_j\big((P(X_j) + \varepsilon) - X_j\big).$$

Clearly, $\sup G_\varepsilon = \sup G + \varepsilon(\sum_{i=1}^{m} s_i + \sum_{j=1}^{r} t_j) = \varepsilon(\sum_{i=1}^{m} s_i + \sum_{j=1}^{r} t_j) > 0$, $\forall \varepsilon > 0$. Thus the agent avoids the WDB.

The argument just described is essentially due to [16], which discusses dF-coherent probabilities. However, the underlying interpretation of suprema buying/selling prices goes back to [17] and applies to lower/upper previsions too. Hence the argument could be generalised to imprecise evaluations.

While this solution does not really cancel WDB gains, as they remain possibilities to consider at least at a theoretical evaluation level, a second alternative based on *desirability* is more sophisticated with respect to this. In fact, here a coherent set of desirable gambles $\mathfrak{D}$ is defined by means of axioms. One of these axioms, termed *avoiding partial loss*, requires that if a gamble X is such that $\sup X \leq 0$, $X \neq 0$, then $X \notin \mathfrak{D}$. Accepting this axiom clearly rules out WDBs (i.e., a WDB gain does not belong to $\mathfrak{D}$). Yet, a weaker notion of desirability has also been considered in the literature which relaxes precisely this axiom, thus not necessarily ruling out WDBs. Hence, the agent may choose whether or not WDB gains may be considered (at least marginally) desirable. In both instances, a correspondence between coherent sets of desirable gambles and unconditional coherent lower previsions may be set up. See [11, 12] for more information on desirability. See also [19, Section 5.2] for a discussion about the avoiding partial loss axiom and its relaxation in a conditional setting.

Lastly, a further dimension has to be hinted, that of *real-world* betting. It is easy to understand that WDBs have very little room here, if any. In fact, theoretical uncertainty models only partly fit with the needs of professional bettors (and institutions acting as bettors in some sense, like insurance companies). The prices for buying/selling gambles proposed by such agents cannot be dF-coherent previsions, since the expected gain from any dF-coherent combination of bets is known to be zero. Coherent lower or upper previsions are more adequate, with some restrictions (see the discussion in [15, Section 4]). However, a professional agent will typically aim at positive enough expected gains. Thus even bets whose net gain is expected to be positive, but too close to zero, could be ruled out, even when the gain might be desirable in a theoretical view. It is easily realised then that a WDB gain will not be considered, its expectation being non-positive. We may conclude that the occurrence of WDBs in real-world is essentially occasional.

7 Conclusions

In this paper we have explored properties of imprecise uncertainty assessments incurring WDBs, according to their consistency requirements. By contrast, much of the previous work focused on dF-coherent probabilities. This study is justified also because, as appears from the late Section 6.1, even weakening dF-coherence to Williams' coherence does not broaden significantly the range of strict consistency, the radical alternative for avoiding WDBs. Thus a WDB is something to coexist with, also in an imprecise setting.

On the other hand, the occurrence of WDBs is not an issue if the various consistency definitions (W-coherence, convexity, ...) are viewed axiomatically. Note that even classical probabilities defined by Kolmogorov's axioms may incur WDBs, without this fact ever being noticed within that theory. The same argument applies to other non-behavioural theories of uncertainty.

Moving from precise to imprecise assessments reveals an interesting feature of WDBs, i.e., their tendency to 'local precision' investigated in Section 3. This surprising property cannot obviously be detected with assessments which are already precise, while the demonstrated dependence of local precision on the required degree of consistency is something one can more easily account for.

Other explored facets of WDBs regard their relationships with positive (lower) probability events (Section 4), and their implications on the agent's beliefs about suffering from real losses (Section 5). It seems that several such results do not generalise to weaker requirements than W-coherence. For instance, the proofs of Propositions 4.1 and 5.4 have no straightforward adaptation to convex previsions, because they do not ensure the convexity constraint in Definition 2.3 for some relevant gains (such as $\underline{G}'$ in the proof of Proposition 4.1).

Actually, a more detailed study of WDBs in such instances could be part of future work, as well as an investigation of the connection between WDBs and other notions, such as arbitrage or desirability.

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