

# Supplementary Information for “Correlations in quantum thermodynamics: Heat, work, and entropy production”

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## ABSTRACT

Here we report the details of the examples.

## Details of example I

### State of the system

Here we obtain the exact state of the total system up to the second order in the interaction coupling  $\lambda$ . After calculating the interaction-picture Hamiltonian  $\tilde{H}_{\text{int}}^{(\lambda)}(\tau) = U_0^\dagger(\tau) H_{\text{int}}^{(\lambda)} U_0(\tau)$  and the corresponding evolution operator  $\tilde{U}_\lambda(\tau) = \mathbb{T} e^{-i \int_0^\tau ds \tilde{H}_{\text{int}}^{(\lambda)}(s)}$ , one can read the state of the combined system from

$$\rho_{SB}^{(\lambda)}(\tau) = U_0(\tau) \tilde{U}_\lambda(\tau) \rho_{SB}(0) \tilde{U}_\lambda^\dagger(\tau) U_0^\dagger(\tau) \quad (1)$$

as

$$\begin{aligned} \rho_S^{(\lambda)}(\tau) = & \rho_S^{(0)}(\tau) + \lambda^2 \left\{ \sigma_+ \rho_S^{(0)}(\tau) \sigma_- \sum_k |f_k|^2 |\eta(\omega_0, \omega_k, \tau)|^2 \bar{n}(\omega_k, \beta) + \right. \\ & \left. + \sigma_- \rho_S^{(0)}(\tau) \sigma_+ \sum_k |f_k|^2 |\eta(\omega_0, \omega_k, \tau)|^2 (\bar{n}(\omega_k, \beta) + 1) \right. \\ & \left. - \sum_k |f_k|^2 \left( \xi^*(\omega_0, \omega_k, \omega_k, \tau) \rho_S^{(0)}(\tau) \sigma_+ \sigma_- + \xi(\omega_0, \omega_k, \omega_k, \tau) \sigma_+ \sigma_- \rho_S^{(0)}(\tau) \right) (\bar{n}(\omega_k, \beta) + 1) \right. \\ & \left. - \sum_k |f_k|^2 \left( \xi(\omega_0, \omega_k, \omega_k, \tau) \rho_S^{(0)}(\tau) \sigma_- \sigma_+ + \xi^*(\omega_0, \omega_k, \omega_k, \tau) \sigma_- \sigma_+ \rho_S^{(0)}(\tau) \right) \bar{n}(\omega_k, \beta) \right\} + O(\lambda^3), \quad (2) \end{aligned}$$

and similarly for bath  $B$ ,

$$\begin{aligned} \rho_B^{(\lambda)}(\tau) = & \rho_B^\beta + i\lambda \left( \text{Tr}[\rho_S(0) \sigma_+] e^{i\omega_0 \tau} \sum_k f_k^* \eta^*(\omega_0, \omega_k, \tau) [\rho_B^\beta, a_k] + \text{Tr}[\rho_S(0) \sigma_-] e^{-i\omega_0 \tau} \sum_k f_k \eta(\omega_0, \omega_k, \tau) [\rho_B^\beta, a_k^\dagger] \right) \\ & + \lambda^2 \left( \text{Tr}[\rho_S(0) \sigma_- \sigma_+] \sum_{kk'} \left\{ f_k^* f_{k'} \eta^*(\omega_0, \omega_k, \tau) \eta(\omega_0, \omega_{k'}, \tau) a_k \rho_B^\beta a_{k'}^\dagger \right. \right. \\ & \left. \left. - f_{k'} f_k^* \xi^*(\omega_0, \omega_{k'}, \omega_k, \tau) e^{i\tau(\omega_k - \omega_{k'})} a_{k'}^\dagger a_k \rho_B^\beta - f_{k'}^* f_k \xi(\omega_0, \omega_{k'}, \omega_k, \tau) e^{-i(\omega_k - \omega_{k'})\tau} \rho_B^\beta a_k^\dagger a_{k'} \right\} \right. \\ & \left. + \text{Tr}[\rho_S(0) \sigma_+ \sigma_-] \sum_{kk'} \left\{ f_k f_{k'}^* \eta(\omega_0, \omega_k, \tau) \eta^*(\omega_0, \omega_{k'}, \tau) a_k^\dagger \rho_B^\beta a_{k'} \right. \right. \\ & \left. \left. - f_{k'}^* f_k \xi(\omega_0, \omega_{k'}, \omega_k, \tau) e^{-i\tau(\omega_k - \omega_{k'})} a_{k'}^\dagger a_k \rho_B^\beta - f_{k'} f_k^* \xi^*(\omega_0, \omega_{k'}, \omega_k, \tau) e^{i(\omega_k - \omega_{k'})\tau} \rho_B^\beta a_k^\dagger a_{k'} \right\} \right) + O(\lambda^3), \quad (3) \end{aligned}$$

where

$$\eta(\omega_0, \omega_k, \tau) = \int_0^\tau ds e^{i(\omega_0 - \omega_k)s}, \quad (4)$$

$$\xi(\omega_0, \omega_{k'}, \omega_k, \tau) = \int_0^\tau ds_1 e^{i(\omega_0 - \omega_{k'})s_1} \eta^*(\omega_0, \omega_k, s_1), \quad (5)$$

$\bar{n}(\omega, \beta)$  shows the Planck distribution or the mean quanta number in a mode with frequency  $\omega$  [equation (54) of the main text], and  $\rho_S^{(0)}(\tau) = U_S(\tau)\rho_S(0)U_S^\dagger(\tau)$  is the unperturbed state of  $S$ , in which  $U_S(\tau) = e^{-i\tau H_S}$  (with  $H_S = \omega_0\sigma_z/2$ ) is the free-system evolution.

In the continuum- $\omega$  limit,  $\sum_k \rightarrow \int_0^\infty d\omega$ , we can find the dynamical equation of  $\rho_S^{(\lambda)}(\tau)$ . We differentiate the continuum version of equation (2) in which we take  $\tau \rightarrow \infty$  in the integrals of the RHS (long-time limit). In the long-time, weak-coupling limit we have  $\tau \rightarrow \infty$  and  $\lambda \rightarrow 0$  such that  $\lambda^2\tau = \text{const}$ . This differentiation yields the Lindblad-type equation (51) of the main text.

### Calculating thermodynamic properties

Using the following notation for the states of the system and the bath:

$$\rho_S^{(\lambda)}(\tau) = \rho_S^{(0)}(\tau) + \lambda^2 \rho_S^{(2)}(\tau) + O(\lambda^3), \quad (6)$$

$$\rho_B^{(\lambda)}(\tau) = \rho_B^\beta + \lambda \rho_B^{(1)}(\tau) + \lambda^2 \rho_B^{(2)}(\tau) + O(\lambda^3), \quad (7)$$

the effective Hamiltonians of  $S$  and  $B$  can be computed up to  $O(\lambda^3)$  as

$$H_S^{(\text{eff})}(\tau) = H_S + \lambda \text{Tr}_B \left[ \rho_B^{(1)}(\tau) H_{\text{int}}^{(\lambda)} \right] - \lambda \alpha_S \text{Tr} \left[ \rho_S^{(0)}(\tau) \otimes \rho_B^{(1)}(\tau) H_{\text{int}}^{(\lambda)} \right], \quad (8)$$

$$H_B^{(\text{eff})}(\tau) = H_B + \text{Tr}_S \left[ \rho_S^{(0)}(\tau) H_{\text{int}}^{(\lambda)} \right] - \lambda \alpha_B \text{Tr} \left[ \rho_S^{(0)}(\tau) \otimes \rho_B^{(1)}(\tau) H_{\text{int}}^{(\lambda)} \right]. \quad (9)$$

We obtain

$$\text{Tr}_B \left[ \rho_B^{(1)}(\tau) H_{\text{int}}^{(\lambda)} \right] = 2\lambda \sum_k |f_k|^2 \left( i\rho_{10} \sigma_- \int_0^\tau ds e^{i\omega_k s} e^{i(\omega_0 - \omega_k)s} + \text{h.c.} \right),$$

$$\text{Tr}_S \left[ \rho_S^{(0)}(\tau) H_{\text{int}}^{(\lambda)} \right] = 2\lambda \sum_k (f_k^* \rho_{10} e^{i\omega_0 \tau} a_k + \text{h.c.}) =: \lambda H_B^{(1)}(\tau),$$

$$\text{Tr} \left[ \rho_S^{(0)}(\tau) \otimes \rho_B^{(1)}(\tau) H_{\text{int}}^{(\lambda)} \right] = 8\lambda |\rho_{10}|^2 \sum_k |f_k|^2 \frac{1 - \cos[(\omega_0 - \omega_k)\tau]}{(\omega_0 - \omega_k)} =: \lambda H_B^{(2)}(\tau),$$

where ‘‘h.c.’’ denotes Hermitian conjugate. The energy of the bath then becomes

$$\begin{aligned} \mathbb{U}_B^{(\lambda)}(\tau) &= \text{Tr} \left[ \rho_B^{(\lambda)}(\tau) H_B^{(\text{eff})}(\tau) \right] \\ &= \mathbb{U}_B^{(0)} + \lambda^2 \left( -\alpha_B \text{Tr} \left[ \rho_B^\beta H_B^{(2)}(\tau) \right] + \text{Tr} \left[ \rho_B^{(1)}(\tau) H_B^{(1)}(\tau) \right] + \text{Tr} \left[ \rho_B^{(2)}(\tau) H_B \right] \right) + O(\lambda^3), \end{aligned} \quad (10)$$

which gives

$$\begin{aligned} d\mathbb{U}_B^{(\lambda)}(\tau) &= d\mathbb{Q}_B(\tau) + d\mathbb{W}_B(\tau) \\ &= \lambda^2 \left( \text{Tr} \left[ d\rho_B^{(2)}(\tau) H_B \right] + \text{Tr} \left[ d\rho_B^{(1)}(\tau) H_B^{(1)}(\tau) \right] + \text{Tr} \left[ \rho_B^{(1)}(\tau) dH_B^{(1)}(\tau) \right] - \alpha_B \text{Tr} \left[ \rho_B^\beta dH_B^{(2)}(\tau) \right] \right) + O(\lambda^3). \end{aligned} \quad (11)$$

After some straightforward algebra we can see that

$$\text{Tr} \left[ d\rho_B^{(2)}(\tau) H_B \right] = 8 \left[ (\bar{n}(\omega_k, \beta) + 1) \rho_{00} - \bar{n}(\omega_k, \beta) \rho_{11} \right] \sum_k |f_k|^2 \frac{\omega_k}{(\omega_0 - \omega_k)} \sin[(\omega_0 - \omega_k)\tau] d\tau, \quad (12)$$

$$\text{Tr} \left[ \rho_B^{(1)}(\tau) dH_B^{(1)}(\tau) \right] = 8|\rho_{10}|^2 \sum_k |f_k|^2 \frac{\omega_0}{(\omega_0 - \omega_k)} \sin[(\omega_0 - \omega_k)\tau] d\tau, \quad (13)$$

$$\text{Tr} \left[ d\rho_B^{(1)}(\tau) H_B^{(1)}(\tau) \right] = -8|\rho_{10}|^2 \sum_k |f_k|^2 \frac{\omega_k}{(\omega_0 - \omega_k)} \sin[(\omega_0 - \omega_k)\tau] d\tau, \quad (14)$$

$$\text{Tr} \left[ \rho_B^\beta dH_B^{(2)}(\tau) \right] = 8|\rho_{10}|^2 \sum_k |f_k|^2 \sin[(\omega_0 - \omega_k)\tau] d\tau. \quad (15)$$

Hence

$$dU_B^{(\lambda)}(\tau) = 8\lambda^2 \sum_k |f_k|^2 \sin[(\omega_0 - \omega_k)\tau] \left[ |\rho_{10}|^2(1 - \alpha_B) + \frac{\omega_k}{(\omega_0 - \omega_k)} [(\bar{n}(\omega_k, \beta) + 1)\rho_{00} - \bar{n}(\omega_k, \beta)\rho_{11}] \right] d\tau + O(\lambda^3). \quad (16)$$

For the entropy we have

$$dS_B^{(\lambda)}(\tau) = -\text{Tr} \left[ d\rho_B^{(\lambda)}(\tau) \log \rho_B^\beta \right] - \text{Tr} \left[ d\rho_B^{(\lambda)}(\tau) \left( \log \rho_B^{(\lambda)}(\tau) - \log \rho_B^\beta \right) \right], \quad (17)$$

where the first term has already been computed as

$$\begin{aligned} -\text{Tr} \left[ d\rho_B^{(\lambda)}(\tau) \log \rho_B^\beta \right] &= \lambda^2 \beta \text{Tr} \left[ d\rho_B^{(2)}(\tau) H_B \right] + O(\lambda^3) \\ &\stackrel{(12)}{=} 8\lambda^2 \beta \left[ (\bar{n}(\omega_k, \beta) + 1)\rho_{00} - \bar{n}(\omega_k, \beta)\rho_{11} \right] \sum_k |f_k|^2 \frac{\omega_k}{(\omega_0 - \omega_k)} \sin[(\omega_0 - \omega_k)\tau] d\tau + O(\lambda^3). \end{aligned} \quad (18)$$

In order to evaluate the second term of equation (17) we only need to take care of the contribution of order  $\lambda$ . We use the following integral form for the logarithm of an operator<sup>1</sup>:

$$\log A = \int_0^\infty dx \left[ \frac{I}{1+x} - (xI + A)^{-1} \right], \quad (19)$$

to obtain

$$\begin{aligned} \log \rho_B^{(\lambda)}(\tau) - \log \rho_B^\beta &= \int_0^\infty dx \left[ (xI + \rho_B^\beta)^{-1} - (xI + \rho_B^{(\lambda)}(\tau))^{-1} \right] \\ &= \lambda \int_0^\infty dx (xI + \rho_B^\beta)^{-1} \rho_B^{(1)}(\tau) (xI + \rho_B^\beta)^{-1} + O(\lambda^2), \end{aligned} \quad (20)$$

where we have used the identity<sup>1</sup>

$$(A + B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - O(B^3) \quad (21)$$

to write

$$(xI + \rho_B^{(\lambda)}(\tau))^{-1} = (xI + \rho_B^\beta)^{-1} + (xI + \rho_B^\beta)^{-1} \left( \rho_B^\beta - \rho_B^{(\lambda)}(\tau) \right) (xI + \rho_B^\beta)^{-1} + O(\lambda^2)$$

and equation (7).

To ease notation, we introduce  $O_\tau = a^\dagger(h_\tau) - a(h_\tau)$ , with

$$a(h_\tau) = i\rho_{10} \sum_k f_k^* e^{i\omega_k \tau} \eta(\omega_0, \omega_k, \tau) a_k,$$

where we have followed the shorthand introduced in equation (77) of the main text to define the vector  $h_\tau = \{h_k(\tau)\}$ , with  $h_k(\tau) = -i\rho_{10}^* f_k e^{-i\omega_k \tau} \eta^*(\omega_0, \omega_k, \tau)$ . Thus we can rewrite  $\rho_B^{(1)}(\tau)$  as

$$\rho_B^{(1)}(\tau) = [O_\tau, \rho_B^\beta], \quad (22)$$

whence

$$-\text{Tr} \left[ d\rho_B^{(\lambda)}(\tau) \left( \log \rho_B^{(\lambda)}(\tau) - \log \rho_B^\beta \right) \right] = -\lambda^2 \int_0^\infty dx \text{Tr} \left[ [dO_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} [O_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} \right] + O(\lambda^3). \quad (23)$$

Considering the spectral decomposition  $\rho_B^\beta = \sum_n r_n |n\rangle\langle n|$ , one can see

$$\text{Tr} \left[ [dO_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} [O_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} \right] = -\sum_{n,m} \langle n | dO_\tau | m \rangle \langle m | O_\tau | n \rangle \frac{(r_n - r_m)^2}{(x + r_n)(x + r_m)}, \quad (24)$$

which yields

$$\begin{aligned}
\int_0^\infty dx \text{Tr} \left[ [dO_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} [O_\tau, \rho_B^\beta] (xI + \rho_B^\beta)^{-1} \right] &= \sum_{n,m} (r_m - r_n) \log \frac{r_n}{r_m} \langle n | dO_\tau | m \rangle \langle m | O_\tau | n \rangle \\
&= \text{Tr} \left[ \rho_B^\beta \left( [O_\tau, \log \rho_B^\beta] dO_\tau + [dO_\tau, \log \rho_B^\beta] O_\tau \right) \right] \\
&= 2\beta \sum_k \omega_k \text{Re} [h_k(\tau) dh_k^*(\tau)] \\
&= 8\beta |\rho_{10}|^2 \sum_k |f_k|^2 \frac{\omega_k}{(\omega_0 - \omega_k)} \sin[(\omega_0 - \omega_k)\tau] d\tau.
\end{aligned}$$

Thus, noting equation (17), we obtain

$$dS_B^{(\lambda)}(\tau) = 8\lambda^2 \beta \sum_k |f_k|^2 \frac{\omega_k \sin[(\omega_0 - \omega_k)\tau]}{(\omega_0 - \omega_k)} \left[ (\bar{n}(\omega_k, \beta) + 1) \rho_{00} - \bar{n}(\omega_k, \beta) \rho_{11} - |\rho_{10}|^2 \right] d\tau + O(\lambda^3). \quad (25)$$

Now combining equations (16) and (25), the pseudo-temperature  $T_B^{(\lambda)}(\tau)$  reads as

$$\begin{aligned}
T_B^{(\lambda)}(\tau) &= \frac{dU_B^{(\lambda)}(\tau)}{dS_B^{(\lambda)}(\tau)} \\
&= \frac{1}{\beta} \frac{\sum_k |f_k|^2 \frac{\omega_k \sin[(\omega_0 - \omega_k)\tau]}{(\omega_0 - \omega_k)} \left[ (\bar{n}(\omega_k, \beta) + 1) \rho_{00} - \bar{n}(\omega_k, \beta) \rho_{11} - |\rho_{10}|^2 + |\rho_{10}|^2 [\alpha_B(\omega_k - \omega_0) + \omega_0] / \omega_k \right]}{\sum_k |f_k|^2 \frac{\omega_k \sin[(\omega_0 - \omega_k)\tau]}{(\omega_0 - \omega_k)} \left[ (\bar{n}(\omega_k, \beta) + 1) \rho_{00} - \bar{n}(\omega_k, \beta) \rho_{11} - |\rho_{10}|^2 \right]}. \quad (26)
\end{aligned}$$

If we go to the continuum- $\omega$  limit, take the  $\tau \rightarrow \infty$  limit, and use the identity

$$\lim_{\tau \rightarrow \infty} \frac{\sin(x\tau)}{\pi x} = \delta(x), \quad (27)$$

we obtain

$$\begin{aligned}
\lim_{\tau \rightarrow \infty} T_B^{(\lambda)}(\tau) &= \frac{1}{\beta} \frac{\left[ (\bar{n}(\omega_0, \beta) + 1) \rho_{00} - \bar{n}(\omega_0, \beta) \rho_{11} \right]}{\left[ (\bar{n}(\omega_0, \beta) + 1) \rho_{00} - \bar{n}(\omega_0, \beta) \rho_{11} - |\rho_{10}|^2 \right]} \\
&= \frac{1}{\beta} \left[ 1 + \frac{|\rho_{10}|^2}{\bar{n}(\omega_0, \beta) (\rho_{00} - \rho_{11}) + \rho_{00} - |\rho_{10}|^2} \right]. \quad (28)
\end{aligned}$$

Let us now study system  $S$ . Since we are interested in thermalization we consider the solution to the Lindblad equation equation (51) of the main text, which is given by

$$\rho_S^{(\lambda)}(\tau) = \frac{1}{2} \begin{bmatrix} 1 + z(0)e^{-\tilde{\gamma}\tau} + \tanh(\beta\omega_0/2) (e^{-\tilde{\gamma}\tau} - 1) & (x(0) - iy(0)) e^{-\tilde{\gamma}\tau/2 - i\omega_0\tau} \\ (x(0) + iy(0)) e^{-\tilde{\gamma}\tau/2 + i\omega_0\tau} & 1 - z(0)e^{-\tilde{\gamma}\tau} - \tanh(\beta\omega_0/2) (e^{-\tilde{\gamma}\tau} - 1) \end{bmatrix}, \quad (29)$$

where  $\tilde{\gamma} = \gamma \coth(\beta\omega_0/2)$  and  $(x(0), y(0), z(0))$  are the initial components of the Bloch vector. We can explicitly compute  $dS_S^{(\lambda)}(\tau)$  using the eigenvalues of  $\rho_S^{(\lambda)}(\tau)$ ,  $(1/2)(1 \pm \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)})$ , as

$$\begin{aligned}
dS_S^{(\lambda)}(\tau) &= -\frac{1}{2} \log \left( \frac{1 + \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}}{1 - \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}} \right) d \left( \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)} \right) \\
&= -\frac{1}{2} \log \left( \frac{1 + \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}}{1 - \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}} \right) \frac{\tilde{\gamma} (x^2(\tau) + y^2(\tau)) - \gamma z(\tau) - \tilde{\gamma} z^2(\tau)}{\sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}} d\tau. \quad (30)
\end{aligned}$$

The energy of this system is

$$\begin{aligned}
U_S^{(\lambda)}(\tau) &= \text{Tr} \left[ \rho_S^{(\lambda)}(\tau) H_S^{(\text{eff})}(\tau) \right] \\
&= \frac{\omega_0}{2} \text{Tr} \left[ \rho_S^{(\lambda)}(\tau) \sigma_z \right] + \lambda (1 - \alpha_S) \left( \text{Tr} \left[ \rho_S^{(\lambda)}(\tau) \sigma_+ \right] \text{Tr} \left[ \rho_B^{(\lambda)}(\tau) a(f) \right] + \text{Tr} \left[ \rho_S^{(\lambda)}(\tau) \sigma_- \right] \text{Tr} \left[ \rho_B^{(\lambda)}(\tau) a^\dagger(f) \right] \right) \\
&= \frac{\omega_0}{2} z(\tau) + 2\lambda^2 (1 - \alpha_S) (x^2(0) + y^2(0)) e^{-\tilde{\gamma}\tau} \sum_k |f_k|^2 \frac{1 - \cos[(\omega_0 - \omega_k)\tau]}{(\omega_0 - \omega_k)} + O(\lambda^3), \quad (31)
\end{aligned}$$

where we used equation (56) of the main text for  $\rho_S^{(\lambda)}(\tau)$  and  $\rho_B^{(\lambda)}(\tau) = \rho_B^\beta + \lambda \rho_B^{(1)}(\tau) + O(\lambda^2)$ . Recalling equation (55) of the main text, the expression above can be differentiated as follows:

$$\begin{aligned} dU_S^{(\lambda)}(\tau) &= -\frac{\omega_0}{2} \gamma e^{-\tilde{\gamma}\tau} \left( \coth(\beta \omega_0/2) z(0) + 1 \right) d\tau + \frac{\gamma(1-\alpha_S)}{\pi |f(\omega_0)|^2} (x^2(0) + y^2(0)) e^{-\tilde{\gamma}\tau} \sum_k |f_k|^2 \sin[(\omega_0 - \omega_k)\tau] d\tau \\ &\stackrel{\omega\text{-continuum}}{=} -\frac{\omega_0}{2} \gamma e^{-\tilde{\gamma}\tau} \left( \coth(\beta \omega_0/2) z(0) + 1 \right) d\tau. \end{aligned} \quad (32)$$

As a result, the inverse pseudo-temperature becomes

$$\frac{1}{T_S^{(\lambda)}(\tau)} = -\frac{1}{2} \log \left( \frac{1 + \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}}{1 - \sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)}} \right) \frac{\frac{1}{2} \coth(\beta \omega_0/2) (x^2(0) + y^2(0)) + z(\tau) \left( \coth(\beta \omega_0/2) z(0) + 1 \right)}{\sqrt{x^2(\tau) + y^2(\tau) + z^2(\tau)} (\omega_0/2) \left( \coth(\beta \omega_0/2) z(0) + 1 \right)}, \quad (33)$$

which yields

$$\lim_{\tau \rightarrow \infty} \frac{1}{T_S^{(\lambda)}(\tau)} = \beta \left[ 1 - \frac{(x^2(0) + y^2(0)) \coth(\beta \omega_0/2)}{2(z(0) + \tanh(\beta \omega_0/2))} \right]. \quad (34)$$

Thus, similarly to the case of  $\lim_{\tau \rightarrow 0} T_B^{(\lambda)}(\tau)$ , in this case too the pseudo-temperature  $T_S^{(\lambda)}(\tau)$  behaves as expected if there is no initial coherence ( $\rho_{10} = 0$ , or equivalently,  $x(0) = y(0) = 0$ ).

## Details of example II

If we expand  $\rho_B^{(\lambda)}(\tau) =: \rho_B^\beta + \lambda \rho_B^{(1)}(\tau) + \lambda^2 \rho_B^{(2)}(\tau) + O(\lambda^3)$ , we obtain

$$\rho_B^{(1)}(\tau) = \langle \sigma_z \rangle_S \left[ \sum_k (g_k(\tau) a_k^\dagger - g_k^*(\tau) a_k), \rho_B^\beta \right], \quad (35)$$

$$\rho_B^{(2)}(\tau) = (1/2) \left\{ \sum_{kk'} (g_k(\tau) a_k^\dagger - g_k^*(\tau) a_k) (g_{k'}(\tau) a_{k'}^\dagger - g_{k'}^*(\tau) a_{k'}), \rho_B^\beta \right\} - \sum_k (g_k(\tau) a_k^\dagger - g_k^*(\tau) a_k) \rho_B^\beta \sum_{k'} (g_{k'}(\tau) a_{k'}^\dagger - g_{k'}^*(\tau) a_{k'}). \quad (36)$$

Since we need to compute the entropy  $\mathbb{S}_B^{(\lambda)}(\tau) = -\text{Tr}[\rho_B^{(\lambda)}(\tau) \log \rho_B^{(\lambda)}(\tau)]$ , we shall need to calculate  $\log \rho_B^{(\lambda)}(\tau)$  up to  $O(\lambda^3)$ . In order to do so, we use the following identity<sup>1</sup>:

$$\begin{aligned} \log(A_0 + \lambda A_1 + \lambda^2 A_2) &= \log A_0 + \lambda \int_0^\infty dx (A_0 + xI)^{-1} A_1 (A_0 + xI)^{-1} \\ &\quad - \lambda^2 \int_0^\infty dx \left[ (A_0 + xI)^{-1} A_1 (A_0 + xI)^{-1} A_1 (A_0 + xI)^{-1} - (A_0 + xI)^{-1} A_2 (A_0 + xI)^{-1} \right] + O(\lambda^3) \\ &=: L_0 + \lambda L_1 + \lambda^2 L_2 + O(\lambda^3). \end{aligned} \quad (37)$$

Replacing the terms of  $\rho_B^{(\lambda)}(\tau)$  in equation (37) yields

$$L_0 = \log \rho_B^\beta, \quad (38)$$

$$L_1(\tau) = \beta \langle \sigma_z \rangle_0 \sum_k \omega_k (g_k(\tau) a_k^\dagger + g_k^*(\tau) a_k). \quad (39)$$

Hence

$$\begin{aligned} \mathbb{S}_B^{(\lambda)}(\tau) &= -\text{Tr} \left[ \left( \rho_B^\beta + \lambda \rho_B^{(1)}(\tau) + \lambda^2 \rho_B^{(2)}(\tau) \right) \left( L_0 + \lambda L_1(\tau) + \lambda^2 L_2(\tau) \right) \right] + O(\lambda^3) \\ &= -\text{Tr}[\rho_B^\beta L_0] - \lambda \left( \text{Tr}[\rho_B^\beta L_1(\tau)] + \text{Tr}[\rho_B^{(1)}(\tau) L_0] \right) - \lambda^2 \left( \text{Tr}[\rho_B^\beta L_2(\tau)] + \text{Tr}[\rho_B^{(1)}(\tau) L_1(\tau)] + \text{Tr}[\rho_B^{(2)}(\tau) L_0] \right) \\ &\quad + O(\lambda^3). \end{aligned} \quad (40)$$

From this relation we obtain

$$\begin{aligned} dS_B^{(\lambda)}(\tau) = & -\lambda \left( \text{Tr}[\rho_B^\beta dL_1(\tau)] + \text{Tr}[d\rho_B^{(1)}(\tau)L_0] \right) - \lambda^2 \left( \text{Tr}[\rho_B^\beta dL_2(\tau)] + \text{Tr}[d\rho_B^{(1)}(\tau)L_1(\tau)] + \text{Tr}[\rho_B^{(1)}(\tau)dL_1(\tau)] \right. \\ & \left. + \text{Tr}[d\rho_B^{(2)}(\tau)L_0] \right) + O(\lambda^3). \end{aligned} \quad (41)$$

This expression has some irrelevant (i.e., vanishing) terms. This can be seen through the identity  $dS(\tau) = -\text{Tr}[d\rho \log \rho]$ , from whence

$$dS_B^{(\lambda)}(\tau) = -\lambda \text{Tr}[d\rho_B^{(1)}(\tau)L_0] - \lambda^2 \left( \text{Tr}[d\rho_B^{(1)}(\tau)L_1(\tau)] + \text{Tr}[d\rho_B^{(2)}(\tau)L_0] \right) + O(\lambda^3). \quad (42)$$

One can see from the identity  $\text{Tr}[[A, B]f(B)] = 0$  (for any  $A, B$ , and function  $f$ ) that here

$$\text{Tr}[d\rho_B^{(1)}(\tau)L_0] \stackrel{(35), (38)}{=} 0. \quad (43)$$

Thus equation (42) reduces to

$$\begin{aligned} dS_B^{(\lambda)}(\tau) = & -\lambda^2 \left( \text{Tr}[d\rho_B^{(1)}(\tau)L_1(\tau)] + \text{Tr}[d\rho_B^{(2)}(\tau)L_0] \right) + O(\lambda^3), \\ = & 4\beta\lambda^2(1 - \langle \sigma_z \rangle_S^2) d\Delta(\tau). \end{aligned} \quad (44)$$

## References

1. F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications* (Springer, Cham, 2014).