A Sandwich Theorem for Natural Extensions

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Abstract

The recently introduced weak consistency notions of 2-coherence and 2-convexity are endowed with a concept of 2-coherent, respectively, 2-convex natural extension, whose properties parallel those of the natural extension for coherent lower previsions. We show that some of these extensions coincide in various common instances, thus producing the same inferences.

Keywords. 2-convex lower previsions, coherent lower previsions, natural extensions.

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1 Introduction

In a recent paper [4] we introduced two weak consistency concepts for conditional lower previsions, 2-*convexity* and 2-*coherence*, studying their basic properties in greater detail in [5]. Formally, 2-coherent and 2-convex conditional lower previsions are a broad generalisation of the 2-coherent (unconditional) lower previsions in [6, Appendix B]. Our main aim in introducing them was to explore the flexibility of the betting scheme which underlies these and other consistency concepts (starting with de Finetti's subjective probability [1]), showing the capability of these previsions of encompassing a number of different uncertainty models in a unified framework.

An important issue is also to detect which properties from stronger consistency concepts are somehow retained by either 2-convexity or 2-coherence. As shown in [4, 5], a very relevant feature of theirs is that they are endowed with, respectively, a 2-convex and a 2-coherent *natural extension*. The properties of these extensions, exemplified in Proposition 1, are formally perfectly analogous to those of the natural extension for coherent lower previsions (following Williams' coherence in the conditional framework [7]) or the convex natural extension for convex conditional previsions [2]. In particular, when finite, they allow extending a lower prevision $\underline{P}$ from its domain $\mathcal{D}$ to any larger $\mathcal{D}' \supset \mathcal{D}$. Yet, when different natural extensions can be applied to the same $\underline{P}$, the results may differ also considerably (cf. the later Example 1 in Section 3). Since 2-coherence is weaker than coherence, inferences produced by the 2-coherent natural extension will be generally vaguer than those guaranteed by the coherent natural extension, and similarly with other instances. Actually, often 2-coherent or 2-convex natural extensions will be even too vague. This points out a drawback of these weak consistency notions and is one reason why, in our view, they should not be regarded as realistic candidates for replacing coherence or convexity. Rather, we will show in this paper that they may be helpful precisely for determining the coherent natural extension, or the convex natural extension. In fact, after concisely presenting the necessary preliminary notions in Section 2, we show in Section 3 that there are significant instances where some or all of the four extensions we mentioned so far coincide. For this, the lower prevision $\underline{P}$ is initially defined on a structured set $\mathcal{X}|\mathcal{B}^{\varnothing}$ (cf. Definition 2) of conditional gambles, representing a generalisation of a vector space to a conditional environment. Hence we are considering a special, but rather common, situation. In Proposition 2 we give an alternative expression for the coherent natural extension, which is later needed and generalises a result in [6] (cf. Corollary 1). After showing how to ensure finiteness for the relevant natural extensions, Theorems 2, 3 and 4 present instances where more different extensions coincide. These results are discussed in the comment after Theorem 4 and in the concluding Section 4. Due to space constraints, some of the proofs are omitted.

2 Preliminaries

Let $\mathcal{D}$ be an arbitrary set of conditional gambles, that is, the generic element of $\mathcal{D}$ is $X|B$, with X a gamble (a bounded random variable), and B non-impossible event. A *conditional lower prevision* $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a real map which, behaviourally, determines the supremum buying price $\underline{P}(X|B)$ of any $X|B \in \mathcal{D}$. This means that an agent should be willing to buy, or to bet in favour of, $X|B$, for any price lower than $\underline{P}(X|B)$. The agent's *gain* from the transaction/bet on $X|B$ for $\underline{P}(X|B)$ is $I_B(X - \underline{P}(X|B))$. Here I_B is the indicator of event B. Its role is that of ensuring that the purchased bet is called off and the money returned to the agent iff B does not occur. In the sequel, we shall use the symbol B for both event B and its indicator I_B.

A generic consistency requirement for $\underline{P}$ asks that no finite linear combination of bets on elements of $\mathcal{D}$, with prices given by $\underline{P}$, should produce a loss (bounded away from 0) for the agent. We obtain different known concepts by imposing constraints on the number of terms in the linear combination or on their coefficients s_i:

Definition 1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a given conditional lower prevision.*

a) $\underline{P}$ is a coherent *conditional lower prevision on $\mathcal{D}$ iff, for all $m \in \mathbb{N}_0$, $\forall X_0|B_0, \ldots, X_m|B_m \in \mathcal{D}$, $\forall s_0, \ldots, s_m \geq 0$, defining $S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 0, \ldots, m\}$ and $\underline{G} = \sum_{i=1}^{m} s_i B_i(X_i - \underline{P}(X_i|B_i)) - s_0 B_0(X_0 - \underline{P}(X_0|B_0))$, it holds, whenever $S(\underline{s}) \neq \varnothing$, that $\sup\{\underline{G}|S(\underline{s})\} \geq 0$.*

b) $\underline{P}$ is 2-coherent *on $\mathcal{D}$ iff a) holds with $m = 1$ (hence there are* two *terms in $\underline{G}$).*

c) $\underline{P}$ is convex *on $\mathcal{D}$ iff a) holds with the additional convexity constraint $\sum_{i=1}^{m} s_i = s_0 = 1$.*

d) $\underline{P}$ is 2-convex *on $\mathcal{D}$ iff c) holds with $m = 1$, i.e., iff, $\forall X_0|B_0, X_1|B_1 \in \mathcal{D}$, we have that, defining $\underline{G}_{2c} = B_1(X_1 - \underline{P}(X_1|B_1)) - B_0(X_0 - \underline{P}(X_0|B_0))$, $\sup(\underline{G}_{2c}|B_0 \vee B_1) \geq 0$.*

e) $\underline{P}$ is centered*, convex or 2-convex, on $\mathcal{D}$ iff it is convex or 2-convex, respectively, and $\forall X|B \in \mathcal{D}$, we have that $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$.*

Condition a), which is Williams' coherence [7] in the structure-free version of [3], is clearly the strongest one. Convexity is a relaxation of coherence, studied in [2]. Given $\underline{P}$ on $\mathcal{D}$, the following relationships hold:

$$\begin{array}{l} \underline{P} \text{ coherent} \Rightarrow \underline{P} \text{ 2-coherent} \Rightarrow \underline{P} \text{ 2-convex} \\ \underline{P} \text{ coherent} \Rightarrow \underline{P} \text{ convex} \Rightarrow \underline{P} \text{ 2-convex.} \end{array} \tag{1}$$

The consistency concepts recalled so far can be characterised by means of axioms on the special sets $\mathcal{X}|\mathcal{B}^{\varnothing}$ defined next:

Definition 2. *Let $\mathcal{X}$ be a linear space of gambles and $\mathcal{B} \subset \mathcal{X}$ a set of (indicators of) events, such that $\Omega \in \mathcal{B}$ and $BX \in \mathcal{X}, \forall B \in \mathcal{B}, \forall X \in \mathcal{X}$. Setting $\mathcal{B}^{\varnothing} = \mathcal{B} - \{\varnothing\}$, define $\mathcal{X}|\mathcal{B}^{\varnothing} = \{X|B : X \in \mathcal{X}, B \in \mathcal{B}^{\varnothing}\}$.*

Theorem 1 (Characterisation Theorems)**.** *Let $\underline{P} : \mathcal{X}|\mathcal{B}^{\varnothing} \to \mathbb{R}$ be a conditional lower prevision.*

a) $\underline{P}$ is coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$ if and only if [3, 7]

(A1) $\underline{P}(X|B) - \underline{P}(Y|B) \leq \sup\{X - Y|B\}, \forall X|B, Y|B \in \mathcal{X}|\mathcal{B}^{\varnothing}$.

(A2) $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B), \forall X|B \in \mathcal{X}|\mathcal{B}^{\varnothing}, \forall \lambda \geq 0$.

(A3) $\underline{P}(X + Y|B) \geq \underline{P}(X|B) + \underline{P}(Y|B), \forall X|B, Y|B \in \mathcal{X}|\mathcal{B}^{\varnothing}$.

(A4) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0, \forall X \in \mathcal{X}, \forall A, B \in \mathcal{B}^{\varnothing} : A \wedge B \neq \varnothing$.

b) $\underline{P}$ is 2-coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$ if and only if (A1), (A2), (A4) and the following axiom hold [5]:

(A5) $\underline{P}(\lambda X|B) \leq \lambda \underline{P}(X|B), \forall \lambda < 0$.

c) $\underline{P}$ is convex on $\mathcal{X}|\mathcal{B}^{\varnothing}$ if and only if (A1), (A4) and the following axiom hold [2, Theorem 8]

(A6) $\underline{P}(\lambda X + (1 - \lambda)Y|B) \geq \lambda \underline{P}(X|B) + (1 - \lambda)\underline{P}(Y|B), \forall X|B, Y|B \in \mathcal{X}|\mathcal{B}^{\varnothing}, \forall \lambda \in]0, 1[$.

d) $\underline{P}$ is 2-convex on $\mathcal{X}|\mathcal{B}^{\varnothing}$ if and only if (A1) and (A4) hold [5].

Next we recall the definitions of the various natural extensions studied in this paper. The term 'natural extension', without further qualifications, will denote the coherent natural extension in Definition 3, a).

Definition 3 (Various natural extensions)**.** *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional lower prevision, and $Z|A$ a conditional gamble.*

a) Define $L(Z|A) = \{\alpha : \sup\{\sum_{i=1}^{m} s_i B_i(X_i - \underline{P}(X_i|B_i)) - A(Z - \alpha)|A \vee S(\underline{s})\} < 0$, for some $X_i|B_i \in \mathcal{D}, s_i \geq 0, i = 1, \ldots, m\}$, where $S(\underline{s}) = \vee_{i=1}^{m}\{B_i : s_i \neq 0\}$. Then, the (coherent) *natural extension of $\underline{P}$ on $Z|A$ is $\underline{E}(Z|A) = \sup L(Z|A)$.*

b) Define $L_2(Z|A)$ putting $m = 1$ in $L(Z|A)$. The 2-coherent *natural extension of $\underline{P}$ on $Z|A$ is $\underline{E}_2(Z|A) = \sup L_2(Z|A)$.*

c) Define $L_c(Z|A)$ from $L(Z|A)$, by adding the constraint $\sum_{i=1}^{m} s_i = 1$ in the 'for some' part. The convex *natural extension of $\underline{P}$ on $Z|A$ is $\underline{E}_c(Z|A) = \sup L_c(Z|A)$.*

d) Define $L_{2c}(Z|A)$ putting $m = 1$ in $L_c(Z|A)$, i.e. $L_{2c}(Z|A) = \{\alpha : \sup\{B(X - \underline{P}(X|B)) - A(Z - \alpha)|A \vee B\} < 0$, for some $X|B \in \mathcal{D}\}$. Then, the 2-convex *natural extension $\underline{E}_{2c}$ of $\underline{P}$ on $Z|A$ is $\underline{E}_{2c} = \sup L_{2c}(Z|A)$.*

The properties of these four natural extensions are analogous [2, 3, 5]. Here we state them for the 2-convex natural extension. For the properties of $\underline{E}$, $\underline{E}_2$, $\underline{E}_c$, replace $\underline{E}_{2c}$ and '2-convex' with, respectively, $\underline{E}$ and 'coherent', $\underline{E}_2$ and '2-coherent', $\underline{E}_c$ and 'convex'.

Proposition 1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ a conditional lower prevision, with $\mathcal{D} \subset \mathcal{D}^*$. If $\underline{E}_{2c}$ is finite on $\mathcal{D}^*$, then*

a) $\underline{E}_{2c}(Z|A) \geq \underline{P}(Z|A)$, $\forall Z|A \in \mathcal{D}$.

b) $\underline{E}_{2c}$ is 2-convex on $\mathcal{D}^*$.

c) If $\underline{P}^*$ is 2-convex on $\mathcal{D}^*$ and $\underline{P}^*(Z|A) \geq \underline{P}(Z|A)$, $\forall Z|A \in \mathcal{D}$, then $\underline{P}^*(Z|A) \geq \underline{E}_{2c}(Z|A)$, $\forall Z|A \in \mathcal{D}^*$.

d) $\underline{P}$ is 2-convex on $\mathcal{D}$ if and only if $\underline{E}_{2c} = \underline{P}$ on $\mathcal{D}$.

e) If $\underline{P}$ is 2-convex on $\mathcal{D}$, then $\underline{E}_{2c}$ is its smallest 2-convex extension on $\mathcal{D}^*$.

3 When do different natural extensions coincide?

Given a lower prevision $\underline{P}$ on $\mathcal{D}$, its natural extensions $\underline{E}$, $\underline{E}_2$, $\underline{E}_c$, $\underline{E}_{2c}$ will generally be different, and ordered (when finite) as follows.

Lemma 1. *Given $\underline{P} : \mathcal{D} \to \mathbb{R}$, it holds that*

$$\begin{array}{l} \underline{E} \geq \underline{E}_2 \geq \underline{E}_{2c} \\ \underline{E} \geq \underline{E}_c \geq \underline{E}_{2c}. \end{array} \tag{2}$$

Proof. It is easy to realise that (2) holds recalling (1), Definition 3 and Proposition 1. For instance, $\underline{E}_c \geq \underline{E}_{2c}$ because $\underline{E}_c$, being convex (Proposition 1, b)), is also 2-convex (cf. (1)), but then $\underline{E}_c \geq \underline{E}_{2c}$ by Proposition 1, e). □

It may also be the case that some among $\underline{E}$, $\underline{E}_2$, $\underline{E}_c$, $\underline{E}_{2c}$ are infinite. But even when being finite, they may differ considerably, as illustrated by the next simple example.

Example 1. *Let $\mathcal{D} = \{X\}$, where X may only take the values 0 and 1. Assign $\underline{P}(X) \in (0, 1)$, which is clearly coherent, hence 2-convex, on $\mathcal{D}$. Its natural extension $\underline{E}$ on $\{2X\}$ is $\underline{E}(2X) = 2\underline{P}(X)$ by (A2), because $\underline{E}$ is coherent on $\{X, 2X\}$ and coincides with $\underline{P}$ on X. However, $\underline{E}_{2c}(2X) \geq \underline{P}(X)$ by (A1) and $\underline{E}_{2c}(2X) = \underline{P}(X) < \underline{E}(2X)$ is 2-convex. This can be checked directly using Definition 1, d). (There are only two gains $\underline{G}_{2c}$ to inspect.)*

On our way to establish when more natural extensions may coincide, we preliminarily tackle two issues: derive an alternative expression for the (coherent) natural extension, and discuss how to hedge possibly non-finite extensions. We assume throughout that the lower prevision $\underline{P}$ is initially assessed on some set $\mathcal{X}|\mathcal{B}^{\varnothing}$. As for the former issue, the following proposition holds.

Proposition 2. *Let $\underline{P}$ be coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$. Then, defining*

$$\begin{array}{ll} L_1(Z|A) & = \{\alpha : \sup\{BX - A(Z-\alpha)|A \vee B\} < 0, \\ & \textit{for some } X \in \mathcal{X}, B \in \mathcal{B},\ \textit{with } \underline{P}(X|B) = 0 \text{ if } B \neq \varnothing\}, \end{array} \tag{3}$$

$L_1(Z|A) = L(Z|A)$ *and the natural extension of* $\underline{P}$ *on* $Z|A$ *is*

$$\underline{E}(Z|A) = \sup L_1(Z|A). \tag{4}$$

Proof. We prove that $L_1(Z|A) = L(Z|A)$, with $L_1(Z|A)$ defined in (3), $L(Z|A)$ in Definition 3 a); taking their suprema gives then the thesis.

i) $L_1(Z|A) \subset L(Z|A)$.

   In fact, let $\alpha \in L_1(Z|A)$. Then $\sup\{BX - A(Z-\alpha)|A \vee B\} < 0$. If $B = \varnothing$, then $BX = 0$, $A \vee B = A$ in the supremum argument, and $\alpha \in L(Z|A)$ (case $S(\underline{s}) = \varnothing$). If $B \neq \varnothing$, then $\underline{P}(X|B) = 0$ and writing the supremum as $\sup\{B(X - \underline{P}(X|B)) - A(Z - \alpha)|A \vee B\} < 0$ it appears that again $\alpha \in L(Z|A)$.

ii) $L(Z|A) \subset L_1(Z|A)$.

   Let now $\alpha \in L(Z|A)$ and, referring to the definition of $L(Z|A)$, $W = \sum_{i=1}^{m} s_i B_i(X_i - \underline{P}(X_i|B_i)) - A(Z - \alpha)$.

   If $S(\underline{s}) = \varnothing$, then $\sup\{-A(Z-\alpha)|A\} < 0$ ensures that $\alpha \in L_1(Z|A)$ (case $B = \varnothing$).

   If $S(\underline{s}) \neq \varnothing$, since $\underline{P}$ is coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$, we may apply (A2), (A3) and (A4) in Theorem 1 a) to get

$$\begin{aligned}&\underline{P}(\textstyle\sum_{i:s_i\neq 0} s_i B_i(X_i - \underline{P}(X_i|B_i))|S(\underline{s})) \geq \\ &\textstyle\sum_{i:s_i\neq 0} s_i \underline{P}(B_i(X_i - \underline{P}(X_i|B_i))|S(\underline{s})) = 0.\end{aligned} \tag{5}$$

   Define $Y = \sum_{i:s_i\neq 0} s_i B_i(X_i - \underline{P}(X_i|B_i))$. Since $\underline{P}(Y|S(\underline{s})) \geq 0$ by (5), we obtain

$$S(\underline{s})[Y - \underline{P}(Y|S(\underline{s}))] - A(Z - \alpha) \leq Y - A(Z - \alpha) = W$$

   and hence

$$\begin{aligned}&\sup\{S(\underline{s})[Y - \underline{P}(Y|S(\underline{s}))] - A(Z - \alpha)|A \vee S(\underline{s})\} \leq \\ &\sup\{W|A \vee S(\underline{s})\} < 0.\end{aligned} \tag{6}$$

   Now put $S(\underline{s}) = B$, $Y - \underline{P}(Y|B) = X$, and note that $\underline{P}(X|B) = \underline{P}(Y - \underline{P}(Y|B)|B) = \underline{P}(Y|B) - \underline{P}(Y|B) = 0$, recalling $\underline{P}(Y - c|B) = \underline{P}(Y|B) - c$, a necessary condition for coherence, at the second equality. Hence (6) may be rewritten as

$$\sup\{BX - A(Z - \alpha)|A \vee B\} < 0,$$

   which proves that $\alpha \in L_1(Z|A)$.

□

While (4) supplies a new alternative expression for $\underline{E}(Z|A)$, it is interesting to observe that it boils down to a known result in the unconditional case, formally obtained putting $\mathcal{B} = \{\Omega, \varnothing\}$, $A = \Omega$ in Proposition 2.

Corollary 1. *If $\underline{P}$ is coherent on a linear space $\mathcal{X}$, then*

$$\underline{E}(Z) = \sup\{\underline{P}(X) : X \leq Z, X \in \mathcal{X}\}. \tag{7}$$

In fact, Corollary 1 is part of the statement of Corollary 3.1.8 in [6].

Turning to the second issue, we are interested in guaranteeing that the various natural extensions considered are finite, i.e. neither $-\infty$ nor $+\infty$. Regarding $\underline{E}$ (or $\underline{E}_2$), its finiteness is ensured if the lower prevision $\underline{P}$ to be extended is coherent (or 2-coherent) [3, 5]. In the case of $\underline{E}_c$ or $\underline{E}_{2c}$, a sufficient condition [5] is that $\underline{P}(0|A) = 0$, for any additional $Z|A$ we wish to extend $\underline{P}$ to. While this condition is generally not necessary, it is nonetheless rather natural, but a 2-convex or convex $\underline{P}$ does not necessarily fulfil it. In fact, it may be the case that $0|A \in \mathcal{X}|\mathcal{B}^{\varnothing}$ and $\underline{P}(0|A) \neq 0$, which we can avoid by restricting our attention to *centered* 2-convex or convex previsions. But even doing so, *as we will*, it may happen that $0|A \notin \mathcal{X}|\mathcal{B}^{\varnothing}$ and, unlike the case of a coherent or 2-coherent $\underline{P}$, $\underline{P}(0|A) = 0$ is not the unique (2-)convex extension of $\underline{P}$. However, it holds that [2, 5]:

Proposition 3. *Let $\underline{P}$ be centered 2-convex (alternatively, centered convex) on $\mathcal{X}|\mathcal{B}^{\varnothing}$. Given $0|A \notin \mathcal{X}|\mathcal{B}^{\varnothing}$, the extension of $\underline{P}$ such that $\underline{P}(0|A) = 0$ is 2-convex (convex).*

Proposition 3 suggests that when extending a centered $\underline{P}$ from $\mathcal{X}|\mathcal{B}^{\varnothing}$ to $\mathcal{D}^* \supset \mathcal{X}|\mathcal{B}^{\varnothing}$ we could consider first extending it to the set

$$(\mathcal{X}|\mathcal{B}^{\varnothing})^+ = \mathcal{X}|\mathcal{B}^{\varnothing} \cup \{0|A : Z|A \in \mathcal{D}^*\}, \tag{8}$$

putting $\underline{P}(0|A) = 0$. Adding zeroes is harmless when considering the natural extension, in the sense of the following

Lemma 2. *Assign $\underline{P}$ on $\mathcal{X}|\mathcal{B}^{\varnothing}$ and let $\mathcal{D}^* \supset \mathcal{X}|\mathcal{B}^{\varnothing}$. Using the notation $L(Z|A)$ for the set L in Definition 3 a) when $\mathcal{D}$ there is replaced by $\mathcal{X}|\mathcal{B}^{\varnothing}$, we write $L^+(Z|A)$ instead when $\mathcal{D} = (\mathcal{X}|\mathcal{B}^{\varnothing})^+$. Then $L(Z|A) = L^+(Z|A)$, and consequently $\underline{E}(Z|A) = \sup L(Z|A) = \sup L^+(Z|A), \forall Z|A \in \mathcal{D}^*$.*

Definition 4. *Given $\mathcal{X}|\mathcal{B}^{\varnothing} \subset \mathcal{D}^*$, let $\underline{P}$ be defined on $\mathcal{X}|\mathcal{B}^{\varnothing}$, and on $(\mathcal{X}|\mathcal{B}^{\varnothing})^+$ putting $\underline{P}(0|A) = 0, \forall 0|A \in (\mathcal{X}|\mathcal{B}^{\varnothing})^+$. Then, $\underline{E}_c^+$, $\underline{E}_{2c}^+$ are the convex, respectively 2-convex natural extension of $\underline{P}$ from $(\mathcal{X}|\mathcal{B}^{\varnothing})^+$ to $\mathcal{D}^*$.*

Theorem 2. *Let $\underline{P}$ be coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}(\subset \mathcal{D}^*)$. Then, $\underline{E}(Z|A) = \underline{E}_{2c}^+(Z|A)$, $\forall Z|A \in \mathcal{D}^*$.*

Proof. By Definitions 3 d) and 4, $\underline{E}_{2c}^+(Z|A) = \sup L_{2c}^+(Z|A)$, where

$$\begin{aligned} L_{2c}^+(Z|A) = \quad & \{\alpha : \sup\{B(X - \underline{P}(X|B)) - A(Z - \alpha)|A \vee B\} < 0, \\ & \text{for some } X|B \in (\mathcal{X}|\mathcal{B}^{\varnothing})^+\}. \end{aligned}$$

We show that $L_{2c}^+(Z|A) = L(Z|A)$.

In fact, if $\alpha \in L_{2c}^+(Z|A)$, then clearly $\alpha \in L^+(Z|A)$, hence $\alpha \in L(Z|A)$, because $L^+(Z|A) = L(Z|A)$ by Lemma 2.

Conversely, let $\alpha \in L(Z|A) = L_1(Z|A)$, by Proposition 2. Then, recalling (3), two distinct situations may occur:

a) $\sup\{BX - A(Z-\alpha)|A \vee B\} < 0$, $X \in \mathcal{X}$, $B \in \mathcal{B}^{\varnothing}$, $\underline{P}(X|B) = 0$. Rewriting the supremum as $\sup\{B(X - \underline{P}(X|B)) - A(Z-\alpha)|A \vee B\} < 0$, then clearly $\alpha \in L^{+}_{2c}(Z|A)$.

b) $\sup\{-A(Z-\alpha)|A\} < 0$. Since $0|A \in (\mathcal{X}|\mathcal{B}^{\varnothing})^{+}$, the supremum may be also written as $\sup\{A(0 - \underline{P}(0|A)) - A(Z-\alpha)|A\} < 0$, from which it is patent that $\alpha \in L^{+}_{2c}(Z|A)$.

Therefore, $L^{+}_{2c}(Z|A) = L(Z|A)$. The thesis follows taking the suprema. □

Theorem 2 assures that the natural extension and the 2-convex natural extension coincide, if $\underline{P}$ is coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$. Hence the 2-coherent natural extension $\underline{E}_2$ coincides with the former ones too, being sandwiched between them by Lemma 1.

Another result of the same kind is

Theorem 3. *Let $\underline{P}$ be centered convex on $(\mathcal{X}|\mathcal{B}^{\varnothing})^{+}$. Then, it is $\underline{E}^{+}_{c}(Z|B) = \underline{E}^{+}_{2c}(Z|A)$, $\forall Z|A \in \mathcal{D}^{*}$.*

Finally, we can now establish the sandwich theorem:

Theorem 4 (*Sandwich Theorem*)**.** *Let $\underline{P}$ be coherent on $\mathcal{X}|\mathcal{B}^{\varnothing}$. Then $\underline{E}(Z|A) = \underline{E}_2(Z|A) = \underline{E}_c(Z|A) = \underline{E}_{2c}(Z|A), \forall Z|A \in \mathcal{D}^{*}$.*

Comment The Sandwich Theorem ensures that the simpler 2-convex natural extension may be enough to compute the natural extension, or the convex natural extension, in the special case that the starting set is $\mathcal{X}|\mathcal{B}^{\varnothing}$. This seems to suggest that if $\underline{P}$ is initially assessed on a structured enough set and already coherent there, only the rather weak properties of (centered) 2-convexity really matter and need to be checked when looking for a least-committal coherent extension.

4 Conclusions

The results of the previous section show that the weak consistency notion of 2-convexity may be helpful in the important inferential problem of extending coherent or convex conditional (and unconditional) lower previsions. There remains to explore how this could be exploited in operational procedures, and whether the results can be applied to more general sets of conditional gambles than those in Definition 2. In our opinion, however, the present results already supply an additional motivation for further studying the interesting notion of 2-convexity.

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