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There is some confusion in the literature as to the meaning of shadow prices in linear programming. This note is an attempt at clarification.

In general (except for here calling it the objective function instead of preference !) we follow Geary and McCarthy (1964)*, usually with indication of page but with a slight but obvious change in notation.

We place ourselves at the start in the simplest conceivable situation economically and mathematically (the latter in postulating that in the Primal the problem is one of maximum, the constraint vector \underline{b} having all elements positive and all constraints potential inequalities, i. e. all with sign " \leq " and no "=", which is complicating).

Then (p. 49) in the objective function u -

$$(1) \quad u = \sum_{j=1}^n a_j x_j,$$

- if the x_j are quantities and a_j are actual prices u can be regarded as the value of production. The a_j may be taken as positive. The Primal and the Dual are set out on p. 28. On pp. 49 seq. it is shown that when u is as defined above the solution of the Dual in the variables y_i (namely $\eta_1, \eta_2, \dots, \eta_n$, some of which may be zero) are the shadow prices. But prices of what? Clearly they are hypothetical

* Elements of Linear Programming with Economic Applications by R. C. Geary and M. D. McCarthy. Charles Griffin and Co. Ltd, London.

because they obtain in an optimal situation, i. e. a situation which does not exist. They are related to (i) the objective function u in the sense that if u were other they would be different and to (ii) the m original constraints.

It is important to realize that both the Primal and the Dual can be solved in terms of their respective original variables and without reference to the slack variables. Including slacks there are $(m + n)$ variables in both Primal and Dual. In general in the optimal solution there are m variables with positive values and n with zero values in the Primal, and n positive and m zero in the Dual. Again in general there are r original variables, $r \leq m$, $r \leq n$, in both Primal and Dual. It is these variables that link the solutions, leading ultimately to the celebrated result $u_M = v_m$ (p. 32). In fact (p. 119 seq.) if we renumber the Primal variables so that the solution variables are $(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_r)$ and reorder the Primal constraints so that in the solution all slacks are zero on the first r rows, the solution vector \bar{z} , is given by -

$$(2) P \bar{z} = \beta$$

With analogously obvious change of notation for the Dual (primes indicating line vectors) -

$$(3) \eta' P = \alpha$$

The solution of which $(\eta_1, \eta_2, \dots, \eta_r)$ gives the optimal, in general positive, values of the original variables in the Dual (p. 31).

We now confine our attention to the Primal

and its Simplex solution, on the following lines. Assume that we have a feasible basis with m variables with positive values (in general) and n variables with zero values - both the m and the n can, and normally will, contain both original and slack variables. Transition from basis to basis until the optimal basis is reached is effected by changing variables one at a time, symbolically variable numbered k (previously with value zero) comes into the new basis in place of variable k' (previously with positive value) which assumes the value zero. The fundamental Simplex theorem is -

$$(4) \quad u_k = u_0 + x_k (a_k - c_k),$$

where u_k is the new value of the objective function (i. e. after the introduction of the variable number k) and u_0 the old value. In

(4) x_k is positive - actually it is the value of the variable numbered k in the new basis - so that we must arrange that $(a_k - c_k)$ is positive in order to increase the value of u . The value of c_k is given on p. 36. In practice $(a_k - c_k)$ is calculated for all n variables with value zero at the u_0 stage. The process ends when all the excluded $(a_k - c_k)$ are non-positive; this final stage is telling us (from (4)) that the introduction of a new variable will not increase the value of the objective function. At any basis stage the variable with the largest positive $(a_k - c_k)$ is selected for inclusion in the next basis. The variable numbered k' to go out is that which preserves non-zero the m solution values of x in the new basis. Given k , k' is in general unique.

At each basis, with m positive-valued variables changing all the time, the objective function u has not changed its character

in the present economic interpretation of being a gross value, the x_j retaining their character of quantities. We are making a marginal statement at (4) : in the transition from one basis to the next by variable change we increase or decrease the value of the objective function by $x_k(a_k - c_k)$. Hence $(a_k - c_k)$ has the character of a price or, perhaps more accurately, of a profit (since it can have a negative value). In an appendix we show the sense in which c_k can be regarded as a "cost" of commodity k. It may suffice to note that behaviour is rational in introducing commodity k with a determinable quantity and price which may be deemed a shadow price because it is not a real price, in general.

We recall that the optimal basis of the Primal consists of m positive variables of which r are original and (m - r) are slacks. The r linear simultaneous equations for finding optimal ³ are given by (2). By reference to system (5.1) (p. 34) in these r equations the slacks x_{r+1} are zero and therefore not in the basis. In system (4.1) (p. 28) the r potential inequalities have become equalities; the constraints have become active. The remaining (m - r) constraints have slacks with positive values and cease to have economic significance.

It is in connection with active constraints that shadow prices become important. What happens to the objective function if each of the r constraint constants in the optimal solution is increased by a quantity δ so small as never to change the identity of the variables in the optimal basis? Consider the data for Example II (pp. 72-73). Entries in final basis 3 of the Simplex show that original variables in this optimum are x_1 and x_3 with values given by -

$$x_1 + x_3 = 15$$

$$3x_1 + 10x_3 = 100$$

giving $x_1 = 50/7$ and $x_3 = 55/7$ as shown. Increases in x_1 and x_3 , say Δx_1 and Δx_3 consequent on increasing the right side by δ and 0 are given by $\Delta x_1 = 10\delta/7$, $\Delta x_3 = -3\delta/7$, so that the increase Δu in the optimal value of u , namely u_M , would be $2\Delta x_1 + 3\Delta x_3 = 11\delta/7$. The $11/7$ will be seen to be the value of c in the final basis under variable 5, i. e. the slack which was originally associated with the equation of which we increased the constant by δ . The reader may like to verify that by increasing the right side by 0 and δ the resulting increase in u_M is $1/7$, the value of c in the final basis under variable 7. This result is quite general. Note that $c = -(a - c)$ since coefficient a in the objective function is zero for slack variables; note also that these c values are independent of the b vector, in this case $\{15, 100\}$.

The active constraints in economic problems have names, thus "the skilled labour constraint", "the capital constraints," and each constraint with its constraint constant may be expressed in different units, e. g. "1,000 men" for skilled labour, "£ million" for capital. It is now clear that an increase of one unit in constraint numbered i , i. e. originally with slack variable x_{n+i} but ultimately found active, i. e. with $x_{n+i} = 0$ in the final basis, will increase the objective function by the c (or $-(a - c)$) entry in the column of the Simplex in the final basis headed $(n + i)$. So, these c values in the slack variable columns are termed the shadow prices of the relevant constraint element.

A proposition of great algebraic elegance is that, when all constraints in the Primal are \leq , that $(a - c)$ line of the final basis of the Primal with signs changed (i. e. $(c - a)$) are the solutions of the Dual. Thus in Example II (pp. 72-73) the optimal

solution of the Dual is

$y_1 = 11/7$, $y_3 = 1/7$, $y_5 = 2/7$, $y_7 = 5/7$ (as can easily be verified) - with y_2 , y_4 and y_6 zero, i. e. out of the final basis. The numbering will be noted in relation to the column numbers of the Simplex of the Primal. Original variables in the Dual are y_1 , y_2 , y_3 ; the positive solution values for y_5 and y_7 merely tell us that the corresponding Dual constraints are inactive.

Up to now we have assumed that all constraints in the Primal were of the type \leq . In Example III (p. 73 seq.), two of the three constraints (p. 75) are $=$, only one being \leq . In this case only one of the shadow prices namely $y_3 = 0.12$ appears on the $(c - a)$ line of the final basis of the Primal. It is shown in the text, however, by the δ analysis exemplified earlier that increases of unity in each of the three constraint constants result in increases of η_1 , η_2 , η_3 (the solution of the Dual) in u_M , the optimal value of the Primal. This happens also when the $=$ sign in the two Primal constraints is replaced by \leq , so long as the constraints are active. Intuitively this must be so, since when a " \leq " constraint is active, it is effectively an " $=$ " constraint. If it is inactive, the corresponding η is zero.

Conclusion

Actual prices exist whatever the economic model; in linear programming shadow prices depend on the objective function chosen. Thus in Ireland if the objective function were real GNP the unskilled labour constraint might be inactive, so that its slack would be in the final basis, i. e. with a positive value and shadow price zero. On the other hand if the objective function were

employment (to be maximized) shadow price of capital would presumably be positive in the sense that the capital constraint would be active. The shadow price of capital measures the amount of extra employment resulting from an increased unit of capital available. Shadow prices are the values of $(a - c)$ (or of $(c - a)$) in the various bases of the Simplex of the Primal. In all bases except the final the $(a - c)$ with positive values indicate the variables which it would be profitable (vis-a-vis the objective function) to bring into the next basis on one's route to the optimum. Incidentally there are obviously many routes to the summit but, in general, only a single summit. So the shadow prices have only an ephemeral existence - except at the optimal. Here the $(c - a)$ values (positive) in the columns for the slack variables, i. e. the x_{n+i} , $m \geq i > 0$ give the shadow prices associated with the active constraints. The $(c - a)$ values in the final basis of order j , $1 \leq j \leq n$ are also economically significant. For instance if the (negative) value of $(a - c)$ is small it is indicating that the corresponding variable might be introduced with small loss to the absolute maximum, a fact which may be politically important. Unquestionably, however, it is the shadow prices of the constraints which are the most significant. We suggest the following interpretation.

If the objective function is a value of some kind (to be maximized) the maximum value is constrained only by the active constraint maxima - the active b_i . If these could be increased the optimal value would be increased. The shadow prices are essentially useful here in showing the constraint limits

which should be increased. This is effected by comparison of actual and shadow prices. The planner will carefully examine the practicability of increasing the constraint maxima for which the shadow prices exceed actual prices.

This is also the viewpoint of D. Simpson in his realistic planning model which is based on linear programming.* He finds the skilled labour constraint active and so has a shadow price. This shadow price is, however, less than actual average earnings. He adds "one might conclude that it would be worthwhile expanding labour training activities to convert unskilled to skilled workers". But is the actual cost of training a skilled worker equivalent to average earnings?

Arising in the same model are shadow sector output costs. The planner here will be interested in comparing these with the actual price per unit of sector outputs (in this case £1 since the units are measured in units of £1 of some ^{multiple} denomination thereof). In the simplex tableau these shadow sector output costs would be found under the slack (in this case, surplus) activities associated with the sector output constraints. Strictly speaking they give the amount by which the objective, household consumption, is increased or decreased by the production of an extra unit of each sector's output. At the optimum, the objective will be decreased by any such extra production, consequently these shadow sector output costs are the opportunity costs of outputs.

* David Simpson ; "A Medium Term Planning Model for Ireland." ESRI Paper No. 41, 1968.

It may well be that an important element of arbitrariness in the setting up of a linear programming model may be reduced by attention to shadow prices in the manner indicated in the last two paragraphs. The maximum found u_M is partly a function of the active constraint maxima but in the derivation of all constraints there may be elements of uncertainty which can be resolved only by experimentation. Why not seek an optimum of optima?

Appendix : The Interpretation of $(a_k - c_k)$

$(a_k - c_k)$ is the marginal value of a unit of x_k to the objective function, and insofar as it is numerically the difference between a gain and a cost it has the character of a profit. We are talking here in the context of a maximising problem. a_k represents the gain to the objective function associated with a unit of x_k - perhaps it is the selling price of x_k . c_k represents the cost associated with producing a unit of x_k ; this cost being the direct inputs per unit of x_k multiplied by their respective shadow prices.

Thus, in matrix form and using the notation of the Appendix c (p. 122) from equation c.7 we have the matrix form of $(a_k - c_k)$ as $(\alpha_2' - \alpha_1' P_1^{-1} P_2)$ where the 1 subscript refers to basis variables and the 2 subscript to non-basis variables.

From the last term in c.7. it is seen that

$\alpha_1' P_1^{-1}$ is the shadow price vector giving the shadow prices of the resources, these are non zero for the resources whose associated slacks are at zero level. So the c_k terms, in the vector $\alpha_1' P_1^{-1} P_2$ consist of the shadow prices vector multiplied by P_2 , where P_2 is the matrix of column vectors of direct inputs required in the production of the excluded activities.

$-c_k$ can be interpreted in another way. In the centre of each successive tableau in the simplex procedure, are figures giving the marginal rates of substitution. These give the quantities of the included activities which would have to be given up in order to produce a unit of excluded activity. These figures are