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Characterizations of bivariate conic, extreme value, and Archimax copulas

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Abstract: Based on a general construction method by means of bivariate ultramodular copulas we construct, for particular settings, special bivariate conic, extreme value, and Archimax copulas. We also show that the sets of copulas obtained in this way are dense in the sets of all conic, extreme value, and Archimax copulas, respectively.

Keywords: Ultramodular copula, conic copula, extreme value copula, Archimax copula

MSC: Primary 62H05, Secondary 60E05, 26B25

1 Introduction

In the literature one can find a considerable number of constructions of bivariate copulas (for example, in [14, 22, 23, 40]). Recently, some constructions based on ultramodular aggregation functions have been proposed in [28, 30].

Recall first that bivariate copulas can be characterized by supermodularity and their boundary conditions. If we have a construction leading to supermodular functions from the unit square $[0, 1]^2$ to the unit interval $[0, 1]$, and if we can find suitable constraints guaranteeing the validity of the boundary conditions of a copula then this is a construction method for bivariate copulas. Note that the composition of supermodular functions by means of an ultramodular function preserves the supermodularity [35] (compare also [13, 29]) of the inner functions. A construction method of bivariate copulas based on ultramodularity was proposed in [30], generalizing a product-based approach studied in [27, 32, 33].

The aim of this paper is to provide deeper insights into this method. After giving the definitions and notations required for the main part of the paper in Section 2, we recall a particular construction and show some well-known classes of copulas which are covered by these constructions in Section 3.

In Section 4 we propose an approach leading to a dense subset of the set of all conic copulas [24]. A construction in the context of extreme value copulas [5, 19, 40] (related to Pickands dependence functions [41]) is given in Section 5, again leading to a dense subset of the set of all extreme value copulas. In a similar way we obtain special Archimax copulas [6] (related to an additive generator of an Archimedean copula and a Pickands dependence function), the set of which is dense in the set of all Archimax copulas.

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2 Preliminaries

We shall mostly work in the n-dimensional unit cube $[0,1]^n$, equipped with the componentwise order $\leq$ induced by the linear order on $[0,1]$. For elements of $[0,1]^n$ we shall use the notations $\mathbf{x}$ and $(x_1, x_2, \ldots, x_n)$ synonymously, whichever is more convenient.

A function $f\colon [0,1]^n \to [0,1]$ is 1-*Lipschitz* (with respect to the L^1-norm) if for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \sum_{i=1}^{n} |x_i - y_i|.$$

A function $f\colon [0,1]^n \to [0,1]$ is called *supermodular* [3, 29, 35] if for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}),$$

(here $\wedge$ and $\vee$ stand for the lattice operations meet and join in $[0,1]^n$) and *ultramodular* [35] if for all $\mathbf{x}, \mathbf{y}, \mathbf{h} \in [0,1]^n$ with $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{h} \in [0,1]^n$

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \leq f(\mathbf{y} + \mathbf{h}) - f(\mathbf{y}).$$

In economics, ultramodular functions $f\colon [0,1]^n \to [0,1]$ are often said to have *non-decreasing increments* [4]. As a consequence of [35, Subsection 3.2], each ultramodular function $f\colon [0,1]^n \to [0,1]$ is necessarily supermodular.

An n-ary function $f\colon [0,1]^n \to [0,1]$ is supermodular if and only if each of its two-dimensional sections is supermodular, and it is ultramodular if and only if it is supermodular and each of its one-dimensional sections is convex [29, Propositions 2.3, 2.7].

If $n \geq 2$ and if all second partial derivatives of an n-ary function $f\colon [0,1]^n \to [0,1]$ exist, then f is ultramodular if and only if all second partial derivatives of f are non-negative [29, Corollary 2.8].

A function $A\colon [0,1]^n \to [0,1]$ is called an *(n-ary) aggregation function* [18] if it is monotone non-decreasing in each component and satisfies the two boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$.

An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is ultramodular if and only if each of its two-dimensional sections is ultramodular. Also, A is ultramodular if and only if each of its two-dimensional sections is supermodular and each of its one-dimensional sections is convex [29, Remark 2.9].

An *(n-ary) quasi-copula* [2, 10, 16] is a function $Q\colon [0,1]^n \to [0,1]$ which is 1-Lipschitz, monotone non-decreasing in each component and which satisfies

$$Q(x_1, x_2, \ldots, x_n) = x_j \quad \text{whenever } x_i = 1 \text{ for all } i \in \{1, 2, \ldots, n\} \setminus \{j\}. \tag{B1}$$

For each n-ary quasi-copula $Q\colon [0,1]^n \to [0,1]$ the inequality $W \leq Q \leq M$ holds, where the Fréchet-Hoeffding lower and upper bounds $W, M\colon [0,1]^n \to [0,1]$ are given by $W(\mathbf{x}) = \max\left(\sum_{i=1}^{n} x_i - (n-1), 0\right)$ and $M(\mathbf{x}) = \min(x_1, x_2, \ldots, x_n)$, respectively.

An *(n-ary) copula* [45] (see also [14, 40]) is a function $C\colon [0,1]^n \to [0,1]$ which satisfies the boundary conditions (B1) and

$$C(x_1, x_2, \ldots, x_n) = 0 \quad \text{whenever } x_i = 0 \text{ for some } i \in \{1, 2, \ldots, n\}, \tag{B2}$$

and which is *n-increasing*, i.e., for each box $B = \prod_{i=1}^{n} [a_i, b_i] \subseteq [0,1]^n$ its C-volume $V_C(B)$ is non-negative:

$$V_C(B) = \sum_{\mathbf{z} \in \mathrm{Ver}(B)} (-1)^{\#S(\mathbf{z})} C(\mathbf{z}) \geq 0, \tag{n-increasing}$$

where $\mathrm{Ver}(B) = \prod_{i=1}^{n} \{a_i, b_i\}$ denotes the set of vertices of B and $\#S(\mathbf{z})$ stands for the cardinality of the set $S(\mathbf{z}) = \{i \in \{1, 2, \ldots, n\} \mid z_i = a_i\}$.

It is easy to see that each copula is monotone non-decreasing in each component and 1-Lipschitz, but not necessarily symmetric. Also, each copula is a quasi-copula, but not vice versa.

An n-ary copula $C\colon [0,1]^n \to [0,1]$ is called *Archimedean* [34, 40] if there exists a continuous, strictly decreasing function $\varphi\colon [0,1] \to [0,+\infty]$ satisfying $\varphi(1) = 0$ such that for all $\mathbf{x} \in [0,1]^n$

$$C(\mathbf{x}) = \varphi^{-1}\left(\min\left(\sum_{i=1}^{n} \varphi(x_i), \varphi(0)\right)\right). \tag{1}$$

In this case, φ is called an *additive generator* of C, and it is uniquely determined by C up to a positive constant.

Observe that a continuous, strictly decreasing function $\varphi\colon [0,1] \to [0,+\infty]$ satisfying $\varphi(1) = 0$ is an additive generator of a bivariate Archimedean copula if and only if it is convex [44, Theorem 6.3.2], and of an n-ary Archimedean copula if and only if it is n-monotone [37, Theorem 2.2].

In this paper we mostly shall be concerned with bivariate copulas. Note that a function $C\colon [0,1]^2 \to [0,1]$ is a bivariate copula if and only if it is supermodular and satisfies the boundary conditions (B1) and (B2).

Bivariate Archimedean copulas $C\colon [0,1]^2 \to [0,1]$ are also *triangular norms* (*t-norms* for short) [31, 42, 44], i.e., they are symmetric, associative operations on $[0,1]$ which are monotone non-decreasing in each component and satisfy (B1). In general, each associative bivariate copula is a t-norm and each 1-Lipschitz t-norm is a bivariate copula.

Quite often we shall require a copula to be ultramodular [29, 30]. Ultramodular copulas describe the dependence structure of stochastically decreasing random vectors, and thus they are negative quadrant dependent (NQD) [40]. Moreover, they are useful in some constructions [28], as will be seen also in this paper.

Each ultramodular copula $C\colon [0,1]^n \to [0,1]$ satisfies $W \le C \le \Pi$, where the independence (or product) copula $\Pi\colon [0,1]^n \to [0,1]$ is given by $\Pi(\mathbf{x}) = x_1 x_2 \cdots x_n$, and the set of ultramodular copulas forms a compact and convex subset of $[0,1]^{[0,1]^n}$.

A copula $C\colon [0,1]^n \to [0,1]$ is ultramodular if and only if each of its horizontal and vertical sections is convex [30].

An Archimedean copula $C\colon [0,1]^n \to [0,1]$ with a two times differentiable additive generator $\varphi\colon [0,1] \to [0,+\infty]$ is ultramodular if and only if its derivative φ' is constant or $\frac{1}{\varphi'}$ is a convex function (for $n = 2$ this was shown in [30, Theorem 3.1], see also [7]).

3 Basic constructions

The following result which is a consequence of Theorem 3.1 in [29] and which generalizes [13, Theorem 5.2] will play a key role in our constructions:

Theorem 3.1. *Let $A\colon [0,1]^k \to [0,1]$ be an ultramodular k-ary aggregation function and assume that the n-ary functions $B_1, B_2, \ldots, B_k\colon [0,1]^n \to [0,1]$ are supermodular, monotone non-decreasing in each component and satisfy the properties $A\big(B_1(\mathbf{0}), B_2(\mathbf{0}), \ldots, B_k(\mathbf{0})\big) = 0$ and $A\big(B_1(\mathbf{1}), B_2(\mathbf{1}), \ldots, B_k(\mathbf{1})\big) = 1$. Then the composite function $C\colon [0,1]^n \to [0,1]$ defined by*

$$C(\mathbf{x}) = A\big(B_1(\mathbf{x}), B_2(\mathbf{x}), \ldots, B_k(\mathbf{x})\big)$$

is a supermodular n-ary aggregation function.

Proof. From [29, Theorem 3.1] it follows that C is a supermodular function. The additional hypotheses concerning $\mathbf{0}$ and $\mathbf{1}$ guarantee that C satisfies the boundary conditions of aggregation functions. Finally, because of the monotonicity of the functions A and $B_1, B_2, \ldots, B_k$, the composite C is also monotone non-decreasing in each component. □

The following construction leads to n-ary ultramodular quasi-copulas.

Proposition 3.2. *Let $C\colon [0,1]^2 \to [0,1]$ be an ultramodular bivariate copula, put $C^{[2]} = C$, and define for $n \ge 2$ an n-ary extension $C^{[n]}\colon [0,1]^n \to [0,1]$ of C inductively by*

$$C^{[n+1]}(x_1, x_2, \ldots, x_{n+1}) = C^{[2]}\left(C^{[n]}(x_1, x_2, \ldots, x_n), x_{n+1}\right).$$

Then $C^{[n]}$ is an n-ary ultramodular quasi-copula for each $n \geq 2$.

Proof. To prove this assertion by induction, note first that it holds for $n = 2$ and assume that it holds for some $n \geq 2$. Define the functions $B_1, B_2 \colon [0,1]^{n+1} \to [0,1]$ by

$$B_1(x_1, x_2, \dots, x_{n+1}) = C^{[n]}(x_1, x_2, \dots, x_n) \quad \text{and} \quad B_2(x_1, x_2, \dots, x_{n+1}) = x_{n+1}$$

which are both supermodular aggregation functions. Then, because of Theorem 3.1, $C^{[n+1]}$ is a supermodular $(n+1)$-ary aggregation function and it satisfies the boundary condition (B1) of an $(n+1)$-ary quasi-copula. Supermodularity and boundary conditions imply that $C^{[n+1]}$ is 1-Lipschitz, i.e., it is a supermodular $(n+1)$-ary quasi-copula. Finally, each section of B_1 and B_2 is convex, and hence, because of the ultramodularity of $C^{[2]}$, this fact holds also for each section of $C^{[n+1]}$, showing that $C^{[n+1]}$ is an ultramodular $(n+1)$-ary quasi-copula. □

Observe that Proposition 3.2 always leads to an n-ary quasi-copula for $n \geq 2$, but in general not to an n-ary copula (the Fréchet-Hoeffding lower bound W is a well-known counterexample).

As an immediate consequence of Theorem 3.1 and [35] we get:

Corollary 3.3. *Let $D_1, D_2, \dots D_n \colon [0,1]^2 \to [0,1]$ be bivariate copulas and assume that $D \colon [0,1]^n \to [0,1]$ is an ultramodular n-ary quasi-copula. If $f_1, \dots, f_n, g_1, \dots, g_n \colon [0,1] \to [0,1]$ are monotone non-decreasing in each component, then the function $C \colon [0,1]^2 \to [0,1]$ given by*

$$C(x,y) = D\Big(D_1\big(f_1(x), g_1(y)\big), D_2\big(f_2(x), g_2(y)\big), \dots, D_n\big(f_n(x), g_n(y)\big)\Big)$$

is supermodular and monotone non-decreasing in each component.

Now we are ready to state and prove the following result which will be fundamental for most of the constructions and characterizations in the rest of the paper:

Theorem 3.4. *Let $D_1, D_2, \dots, D_n \colon [0,1]^2 \to [0,1]$ be bivariate copulas and assume that $D \colon [0,1]^n \to [0,1]$ is an ultramodular n-ary quasi-copula. Let $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n \colon [0,1] \to [0,1]$ be monotone non-decreasing in each component such that for all $x \in [0,1]$*

$$D\big(f_1(x), f_2(x), \dots, f_n(x)\big) = D\big(g_1(x), g_2(x), \dots, g_n(x)\big) = x. \tag{2}$$

Then the function $C \colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = D\Big(D_1\big(f_1(x), g_1(y)\big), D_2\big(f_2(x), g_2(y)\big), \dots, D_n\big(f_n(x), g_n(y)\big)\Big) \tag{3}$$

is a bivariate copula.

Proof. Since (2) holds in particular for $x = 1$ and $D \leq M$ we obtain $f_i(1) = g_i(1) = 1$ for each $i \in \{1, 2, \dots, n\}$. Then for all $x \in [0,1]$ we get

$$\begin{aligned} C(x,1) &= D\Big(D_1\big(f_1(x), g_1(1)\big), D_2\big(f_2(x), g_2(1)\big), \dots, D_n\big(f_n(x), g_n(1)\big)\Big) \\ &= D\Big(D_1\big(f_1(x), 1\big), D_2\big(f_2(x), 1\big), \dots, D_n\big(f_n(x), 1\big)\Big) \\ &= D\big(f_1(x), f_2(x), \dots, f_n(x)\big) \\ &= x \end{aligned}$$

and, in complete analogy, $C(1,x) = x$. This implies $0 \leq C(0,x) \leq C(0,1) = 0$ and, similarly, $C(x,0) = 0$ for each $x \in [0,1]$, i.e., C satisfies the boundary conditions (B1) and (B2) of a bivariate copula. As a consequence of Corollary 3.3, C is a bivariate copula. □

In the constructions and representations to follow we often use *weight vectors* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ satisfying

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1.$$

In Sections 4–6 we also will require that weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$ have these two additional properties:

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n) \quad \text{are pairwise linear independent vectors;} \tag{WV1}$$

$$\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} > \dots > \frac{\alpha_n}{\beta_n} \quad \text{(where we use the convention } \frac{\alpha_i}{0} = +\infty\text{).} \tag{WV2}$$

Although these technical constraints (WV1) and (WV2) at first glance seem to be rather strong, it will turn out that they have no impact on the generality of our results.

Example 3.5. Let us keep the same notations as in Theorem 3.4, choose $D = \Pi$ and define the functions $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n \colon [0, 1] \to [0, 1]$ by $f_i(x) = x^{\alpha_i}$, $g_i(x) = x^{\beta_i}$ for some weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$. Then we obtain the bivariate copula $C \colon [0, 1]^2 \to [0, 1]$ given by

$$C(x, y) = \prod_{i=1}^{n} D_i\big(x^{\alpha_i}, y^{\beta_i}\big),$$

which was first shown to be a copula in [32, 33] (in an attempt to construct multivariate asymmetric copulas). As a special case, with $n = 2$ and $D = D_1 = \Pi$, this reduces to the bivariate copula obtained in [27]:

$$C(x, y) = x^{\alpha_1} y^{\beta_1} D_2(x^{\alpha_2}, y^{\beta_2}).$$

Example 3.6. If $D = \Pi$ and $D_1 = D_2 = \dots = D_n = W$ and, for weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$, the functions $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n \colon [0, 1] \to [0, 1]$ are defined by $f_i(x) = x^{\alpha_i}$ and $g_i(x) = x^{\beta_i}$, respectively, then the copula $C \colon [0, 1]^2 \to [0, 1]$ according to (3) in Theorem 3.4 is given by

$$C(x, y) = \Pi^{[n]}\Big(W\big(f_1(x), g_1(y)\big), W\big(f_2(x), g_2(y)\big), \dots, W\big(f_n(x), g_n(y)\big)\Big) = \prod_{i=1}^{n} \max\big(x^{\alpha_i} + y^{\beta_i} - 1, 0\big).$$

For special choices of the weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$ this leads to some well-known copulas:

(i) if $\boldsymbol{\alpha} = \boldsymbol{\beta} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ then we get

$$C(x, y) = \big(\max(\sqrt[n]{x} + \sqrt[n]{y} - 1, 0)\big)^n,$$

i.e., the Clayton copula [8, 17] with parameter $-\frac{1}{n}$, see (4.2.1) in [40, Table 4.1] (note that the family of Clayton copulas with parameters $\lambda \in [-1, +\infty]$ is a subfamily of the continuous Schweizer-Sklar t-norms $(T^{\mathbf{SS}}_{\lambda})_{\lambda \in]-\infty, +\infty]}$ given by

$$T^{\mathbf{SS}}_{\lambda}(x, y) = \big(\max\big((x^{-\lambda} + y^{-\lambda} - 1), 0\big)\big)^{-\frac{1}{\lambda}}$$

whenever $\lambda \in]-\infty, 0[\cup]0, +\infty[$, with the limit cases $T^{\mathbf{SS}}_{0} = \Pi$, and $T^{\mathbf{SS}}_{+\infty} = M$, which was originally studied in [42, 43], see also [31, 44]);

(ii) if $n = 2$, $\boldsymbol{\alpha} = (\alpha, 1 - \alpha)$ and $\boldsymbol{\beta} = (1, 0)$ then C is a DUCS copula [38, 39] given by

$$C(x, y) = \max\big(x^{\alpha} + y - 1, 0\big) \cdot x^{1-\alpha} = x \cdot \max\Big(1 - \frac{1 - y}{x^{\alpha}}, 0\Big).$$

Example 3.7. Put $D = W$, $D_1 = D_2 = \dots = D_n = \Pi$ and let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$ be weight vectors. Define the functions $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n \colon [0, 1] \to [0, 1]$ by

$$f_i(x) = 1 - \alpha_i + \alpha_i x \quad \text{and} \quad g_i(x) = 1 - \beta_i + \beta_i x.$$

Then the function $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = W^{[n]}\big(f_1(x)g_1(y), f_2(x)g_2(y), \dots, f_n(x)g_n(y)\big) = \max\left(\Big(\sum_{i=1}^{n} \alpha_i\beta_i\Big)xy + \Big(1 - \sum_{i=1}^{n} \alpha_i\beta_i\Big)(x+y-1), 0\right)$$

is a member (with parameter $\lambda = \sum_{i=1}^{n} \alpha_i\beta_i$) of the one-parameter family of bivariate copulas given in (4.2.7) in [40, Table 4.1], which is actually a subfamily (with parameters $\lambda \in [0,1]$) of the continuous Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda\in]-\infty,1]}$ [26, 46, 48] given by

$$T_\lambda^{\mathbf{SW}}(x,y) = \max(\lambda xy + (1-\lambda)(x+y-1), 0).$$

Note also that each parameter $\lambda \in [0,1[$ of this family can be attained by our construction: put $n = 2$ and choose $\alpha_1 = \lambda$, $\alpha_2 = 1-\lambda$, $\beta_1 = 1$, and $\beta_2 = 0$, in which case we have $\alpha_1\beta_1 + \alpha_2\beta_2 = \lambda$. For the parameter $\lambda = 1$ it suffices to choose $n = 1$ and $\alpha_1 = \beta_1 = 1$.

In Theorem 3.4 and in Examples 3.5–3.7 we always started with an ultramodular copula D. If we try to do the same with a copula which is not ultramodular then we sometimes obtain a copula, and sometimes not:

Example 3.8. Consider the family of continuous Hamacher t-norms $(T_\lambda^{\mathbf{H}})_{\lambda\in]-\infty,1]}$ [20, 21] (see also [31]) given by

$$T_\lambda^{\mathbf{H}}(x,y) = \begin{cases} 0 & \text{if } \lambda = 1 \text{ and } x = y = 0,\\ \frac{xy}{1-\lambda(1-x)(1-y)} & \text{otherwise.}\end{cases}$$

The subfamily of Hamacher t-norms with parameters $\lambda \in [-1,1]$ forms the family of Ali-Mikhail-Haq copulas (see [1] and (4.2.3) in [40, Table 4.1]).

Now put $D = T_1^{\mathbf{H}}$ (sometimes referred to as Hamacher product) and notice that this bivariate copula is obviously not ultramodular.

(i) If $D_1 = D_2 = W$ and if $f_1, f_2, g_1, g_2\colon [0,1] \to [0,1]$ are chosen as follows

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = \frac{2x}{x+1},$$

then (3) leads to the copula $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = \max\left(\frac{3xy + x + y - 1}{-xy + x + y + 3}, 0\right)$$

which is the member with parameter 2 of the family of copulas given by (4.2.8) in [40, Table 4.1].

(ii) If $D_1 = D_2 = \Pi$ and if $f_1, f_2, g_1, g_2\colon [0,1] \to [0,1]$ are chosen as in (i) then (3) leads to the copula $C\colon [0,1]^2 \to [0,1]$ given by $C(x,y) = \frac{2xy}{1+x+y-xy}$ which is the Ali-Mikhail-Haq copula with parameter 0.5.

(iii) If we put again $D_1 = D_2 = \Pi$ and if $f_1, f_2, g_1, g_2\colon [0,1] \to [0,1]$ are given by $f_1(x) = g_1(x) = \max(\frac{1}{2}, x)$ and $f_2(x) = g_2(x) = \min(\frac{x}{1-x}, 1)$, then the function $C\colon [0,1]^2 \to [0,1]$ obtained via (3) is given by

$$C(x,y) = \begin{cases} \frac{xy}{1-x-y+4xy} & \text{if } (x,y) \in [0,0.5]^2,\\ \frac{xy}{2x+y-2xy} & \text{if } (x,y) \in [0,0.5] \times]0.5,1],\\ \frac{xy}{x+2y-2xy} & \text{if } (x,y) \in]0.5,1] \times [0,0.5],\\ xy & \text{otherwise.}\end{cases}$$

This function is not a copula (e.g., the C-volume of the square $[0.4,0.5]^2 \subseteq [0,1]^2$ equals $-\frac{1}{252}$), but only a proper quasi-copula.

4 How to obtain conic copulas

In [24] (see also [15, 25]) a bivariate copula $C\colon [0,1]^2 \to [0,1]$ was called a *conic copula* if it is linear on any line segment in $[0,1]^2$ connecting the point $(1,1)$ with an undominated element (u,v) of the zero-set $C^{\leftarrow}(\{0\})$ of C, i.e., a point (u,v) with $C(u,v) = 0$ and $C(x,y) > 0$ whenever $x > u$ and $y > v$.

A conic copula is characterized by a continuous convex function $k_C\colon [0,1] \to [0,1]$ whose graph is contained in the set of all undominated elements of the zero-set $C^{\leftarrow}(\{0\})$ of C (and which forms a subset of the boundary of $C^{\leftarrow}(\{0\})$, see [24]).

Keep the notations of Theorem 3.4 and choose the bivariate copulas $D = W$ and $D_1 = D_2 = \cdots = D_n = M$. Consider weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ satisfying (WV1) and (WV2), and the functions $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n\colon [0,1] \to [0,1]$ given by

$$f_i(x) = 1 - \alpha_i + \alpha_i x \quad \text{and} \quad g_i(x) = 1 - \beta_i + \beta_i x.$$

Then the copula $C\colon [0,1]^2 \to [0,1]$ defined in (3) turns out to be

$$C(x,y) = \max\left(\sum_{i=1}^{n} \min(f_i(x), g_i(y)) - (n-1), 0\right).$$

Now we can prove the following result:

Proposition 4.1. *Let* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ *be weight vectors satisfying* (WV1) *and* (WV2). *Then the function* $C\colon [0,1]^2 \to [0,1]$ *given by*

$$C(x,y) = \max\left(\sum_{i=1}^{n} \min(1 - \alpha_i + \alpha_i x, 1 - \beta_i + \beta_i y) - (n-1), 0\right) \tag{4}$$

is a bivariate conic copula.

Proof. Writing $\psi_i = \frac{\alpha_i}{\beta_i}$, property (WV2) implies $1 - \psi_{i+1} + \psi_{i+1} x \geq 1 - \psi_i + \psi_i x$ for all $x \in [0,1]$. Moreover, if $\beta_i > 0$ then the equation $1 - \alpha_i + \alpha_i x = 1 - \beta_i + \beta_i y$ can be rewritten as $y = 1 - \psi_i + \psi_i x$, implying that

$$\min\bigl(1 - \alpha_i + \alpha_i x, 1 - \beta_i + \beta_i y\bigr) = \begin{cases} 1 - \alpha_i + \alpha_i x & \text{if } y \geq 1 - \psi_i + \psi_i x, \\ 1 - \beta_i + \beta_i y & \text{otherwise.} \end{cases}$$

If we write $\xi_i = \sum_{j=1}^{i} \alpha_j$, $\vartheta_i = \sum_{j=1}^{i} \beta_j$ then the copula C in (4) can be expressed as:

$$C(x,y) = \begin{cases} y & \text{if } y \in [0, 1 - \psi_1 + \psi_1 x]\,, \\ \max\bigl(\vartheta_i - \xi_i + \xi_i x + (1 - \vartheta_i) y, 0\bigr) & \text{if } y \in \,]1 - \psi_i + \psi_i x, 1 - \psi_{i+1} + \psi_{i+1} x]\,, \\ x & \text{if } y \in \,]1 - \psi_n + \psi_n x, 1]\,. \end{cases}$$

It is evident that the copula C is linear on any line segment in $[0,1]^2$ connecting the points $(1,1)$ and (u,v), where (u,v) is an undominated element of the zero-set $C^{\leftarrow}(\{0\})$ of C, i.e., C is a conic copula. □

It should be mentioned that the properties (WV1) and (WV2) of the weight factors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ are no real restrictions. The copula C in (4) does not change if we delete all the coordinates $i \in \{1, 2, \ldots, n\}$ with $\alpha_i = \beta_i = 0$ of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Further, the commutativity of the addition implies that C in (4) remains unchanged if the coordinates of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ undergo the same permutation. And finally, if $\psi_i = \psi_{i+1}$ for some index $i \in \{1, 2, \ldots, n-1\}$, we may replace the two components α_i and α_{i+1} of $\boldsymbol{\alpha}$ by $\alpha_i + \alpha_{i+1}$ and, simultaneously, the two components β_i and β_{i+1} of $\boldsymbol{\beta}$ by $\beta_i + \beta_{i+1}$.

Evidently, for the conic copulas C constructed in Proposition 4.1 the corresponding continuous convex function $k_C\colon [0,1] \to [0,1]$ is piecewise linear, and it is determined by the points

$$(1,0), \left(1 - \frac{1}{\xi_1 + \psi_1(1 - \vartheta_1)}, 1 - \frac{\psi_1}{\xi_1 + \psi_1(1 - \vartheta_1)}\right), \ldots, \left(1 - \frac{1}{\xi_n + \psi_n(1 - \vartheta_n)}, 1 - \frac{\psi_n}{\xi_n + \psi_n(1 - \vartheta_n)}\right).$$

Example 4.2. Here are two examples of conic copulas obtained by means of Proposition 4.1.

(i) For $n = 3$ choose $\boldsymbol{\alpha} = (0.3, 0.5, 0.2)$ and $\boldsymbol{\beta} = (0.1, 0.4, 0.5)$. Then we obtain $(\psi_1, \psi_2, \psi_3) = (3, 1.25, 0.4)$, $(\xi_1, \xi_2, \xi_3) = (0.3, 0.8, 1)$, and $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.1, 0.5, 1)$. The conic copula C given by (4) is related to the piecewise linear function (for a visualization see Figure 1) connecting the points $(0, \frac{3}{5})$,$(\frac{17}{57}, \frac{7}{57})$, $(\frac{2}{3}, 0)$, and $(1, 0)$.

(ii) Example 9 in [24] introduces a family of conic copulas (discussed also in [11]) which is related to piecewise linear functions connecting the points $(0, c)$, $(c, 0)$, and $(1, 0)$ with $c \in]0, 1[$. In our notation this means $n = 2$ and

$$(c, 0) = \left(1 - \frac{1}{\alpha_1 + \psi_1(1 - \beta_1)}, 1 - \frac{\psi_1}{\alpha_1 + \psi_1(1 - \beta_1)}\right),$$
$$(0, c) = \left(1 - \frac{1}{1 + \psi_2 \cdot 0}, 1 - \frac{\psi_2}{1 + \psi_2 \cdot 0}\right),$$

i.e., we obtain $(\psi_1, \psi_2) = \left(\frac{1}{1-c}, 1 - c\right)$, $\boldsymbol{\alpha} = \left(\frac{1}{2-c}, \frac{1-c}{2-c}\right)$, and $\boldsymbol{\beta} = \left(\frac{1-c}{2-c}, \frac{1}{2-c}\right)$.
Following the construction given in [24] we get

$$C(x, y) = \frac{1}{2 - c} \min\big(2x - cx, 2y - cy, \max(x + y - c, 0)\big), \tag{5}$$

while our approach leads to the equivalent formula

$$C(x, y) = \frac{1}{2 - c} \max\big(c + \min(x - c, y - cy) + \min(x - cx, y - c), 0\big). \tag{6}$$

The equivalence of the two formulas (5) and (6) can be checked case by case. If, for example, $y \geq c+(1-c)x$, then $y \geq x$ and also $x + y - c \geq x + c + (1 - c)x - c = 2x - cx$, and thus (5) yields $C(x, y) = x$. On the other hand, we have $y - cy \geq c + (1 - c)x - cy = x - c(x + y - 1) \geq x - c$ and $y - c \geq x - cx$, i.e., also in (6) we obtain $C(x, y) = x$. The other cases are checked in a similar way.

Note that, by using similar arguments as in the proof of Proposition 8 in [24], it can be shown that also the opposite claim is valid, i.e, we can formulate the following remarkable result:

Proposition 4.3. *A bivariate conic copula $C\colon [0, 1]^2 \to [0, 1]$ is singular with a support consisting of finitely many line segments if and only if there are an $n \in \mathbb{N}$ and weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^n \times [0, 1]^n$ satisfying* (WV1) *and* (WV2) *such that for all $(x, y) \in [0, 1]^2$*

$$C(x, y) = \max\left(\sum_{i=1}^{n} \min(1 - \alpha_i + \alpha_i x, 1 - \beta_i + \beta_i y) - (n - 1), 0\right).$$

Note that the set of bivariate conic copulas which can be constructed by means of Proposition 4.3 is a dense subset of the set of all bivariate conic copulas.

Proposition 4.4. *For each bivariate conic copula $C\colon [0, 1]^2 \to [0, 1]$ and for each $n \in \mathbb{N}$ there are weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^{4n} \times [0, 1]^{4n}$ satisfying* (WV1) *and* (WV2) *such that for all $(x, y) \in [0, 1]^2$*

$$\left| C(x, y) - \max\left(\sum_{i=1}^{4n} \min(1 - \alpha_i + \alpha_i x, 1 - \beta_i + \beta_i y) - (4n - 1), 0\right)\right| < \frac{1}{n}.$$

Proof. Let $C\colon [0, 1]^2 \to [0, 1]$ be a conic copula and $k_C\colon [0, 1] \to [0, 1]$ a continuous convex function whose graph is contained in the set of all undominated elements of the zero-set $C^{\leftarrow}(\{0\})$ of C.

Fix $n \in \mathbb{N}$ and choose weight vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, 1]^{4n} \times [0, 1]^{4n}$ satisfying (WV1) and (WV2) such that the conic copula $C^{(4n)}\colon [0, 1]^2 \to [0, 1]$ given by

$$C^{(4n)}(x, y) = \max\left(\sum_{i=1}^{4n} \min(1 - \alpha_i + \alpha_i x, 1 - \beta_i + \beta_i y) - (4n - 1), 0\right)$$

is related to the piecewise linear function connecting the points contained in the intersections of the graph of k_C with the $4n$ lines, each of which passes through $(1, 1)$ and one of the following points:

$$\big(0, k_C(0)\big), \big(0, \tfrac{2n-1}{2n} k_C(0)\big), \ldots, \big(0, \tfrac{1}{2n} k_C(0)\big), \big(\tfrac{1}{2n}, 0\big), \big(\tfrac{2}{2n}, 0\big), \ldots, \big(\tfrac{2n-1}{2n}, 0\big), (1, 0).$$

Since on the boundary of $[0,1]^2$ all copulas (and, in particular, C and $C^{(4n)}$) coincide, it suffices to consider the following two cases.

Case 1: $(\hat{x},\hat{y}) \in \,]0,1[^2$ and $\hat{x} \geq \hat{y}$:
then the line connecting the points $(1,1)$ and $(\hat{x},\hat{y})$ also contains the point $(\frac{\hat{x}-\hat{y}}{1-\hat{y}}, 0)$ on the x-axis. Now put

$$i^* = \inf\left\{ i \in \mathbb{N} \;\middle|\; \frac{i}{2n} \geq \frac{\hat{x}-\hat{y}}{1-\hat{y}} \right\} \quad \text{and} \quad \tilde{x} = \frac{2n-i^*}{2n}\hat{y} + \frac{i^*}{2n}.$$

Observe that $|\hat{x}-\tilde{x}| \leq \frac{1}{2n}$ and that the point $(\tilde{x},\hat{y})$ is an element of the line segment connecting the two points $\left(\frac{i^*}{2n}, 0\right)$ and $(1,1)$ on which the conic copulas C and $C^{(4n)}$ coincide by construction.

Therefore $C(\tilde{x},\hat{y}) = C^{(4n)}(\tilde{x},\hat{y})$ which, together with the 1-Lipschitz property of C and $C^{(4n)}$, implies

$$\left|C(\hat{x},\hat{y}) - C^{(4n)}(\hat{x},\hat{y})\right| \leq \left|C(\hat{x},\hat{y}) - C(\tilde{x},\hat{y})\right| + \left|C^{(4n)}(\tilde{x},\hat{y}) - C^{(4n)}(\hat{x},\hat{y})\right| \leq \frac{1}{n}.$$

Case 2: If $(\hat{x},\hat{y}) \in \,]0,1[^2$ with $\hat{x} < \hat{y}$:
if $\hat{y} \geq k_C(0) + (1-k_C(0))\hat{x}$ then we have $C(\hat{x},\hat{y}) = C^{(4n)}(\hat{x},\hat{y}) = \hat{x}$, otherwise this case can be handled in a similar way as Case 1. □

5 Extreme value copulas

Recall that a bivariate copula $C\colon [0,1]^2 \to [0,1]$ is called an *extreme value copula* (see [5, 19, 40]) if for all $(x,y) \in [0,1]^2$ and for all constants $\lambda \in \,]0,+\infty[$ we have

$$C\left(x^\lambda, y^\lambda\right) = \left(C(x,y)\right)^\lambda. \tag{7}$$

According to [22], a bivariate copula C is an extreme value copula if and only if there exists a *Pickands dependence function* [41, 47], i.e., a convex function $A\colon [0,1] \to [0.5,1]$ satisfying $\max(1-u,u) \leq A(u) \leq 1$ for each $u \in [0,1]$, such that for all $(x,y) \in [0,1]^2$

$$C(x,y) = \exp\left(\log(xy)\cdot A\left(\frac{\log x}{\log(xy)}\right)\right),$$

where the convention $\frac{0}{0} = \frac{-\infty}{-\infty} = 1$ will be used.

Proposition 5.1. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ *be weight vectors satisfying* (WV1) *and* (WV2). *Then the function* $C\colon [0,1]^2 \to [0,1]$ *given by*

$$C(x,y) = \prod_{i=1}^{n} M\left(x^{\alpha_i}, y^{\beta_i}\right) \tag{8}$$

is an extreme value copula, and the Pickands dependence function $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon [0,1] \to [0.5,1]$ *corresponding to it is given by*

$$A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) = \begin{cases} 1-x & \text{if } x \in \left[0, \frac{\beta_1}{\alpha_1+\beta_1}\right], \\ \left(\sum\limits_{j=1}^{i}\alpha_j\right)x + \left(\sum\limits_{j=i+1}^{n}\beta_j\right)(1-x) & \text{if } x \in \left]\frac{\beta_i}{\alpha_i+\beta_i}, \frac{\beta_{i+1}}{\alpha_{i+1}+\beta_{i+1}}\right], \\ x & \text{if } x \in \left]\frac{\beta_n}{\alpha_n+\beta_n}, 1\right]. \end{cases} \tag{9}$$

Proof. If $D = \Pi, D_1, D_2, \ldots, D_n = M$, if $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ are weight vectors satisfying (WV1) and (WV2), and if we define the functions $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n\colon [0,1] \to [0,1]$ by $f_i(x) = x^{\alpha_i}$ and $g_i(x) = x^{\beta_i}$, respectively, then the function $C\colon [0,1] \to [0,1]$ defined by (8) is a bivariate copula because of Theorem 3.4.

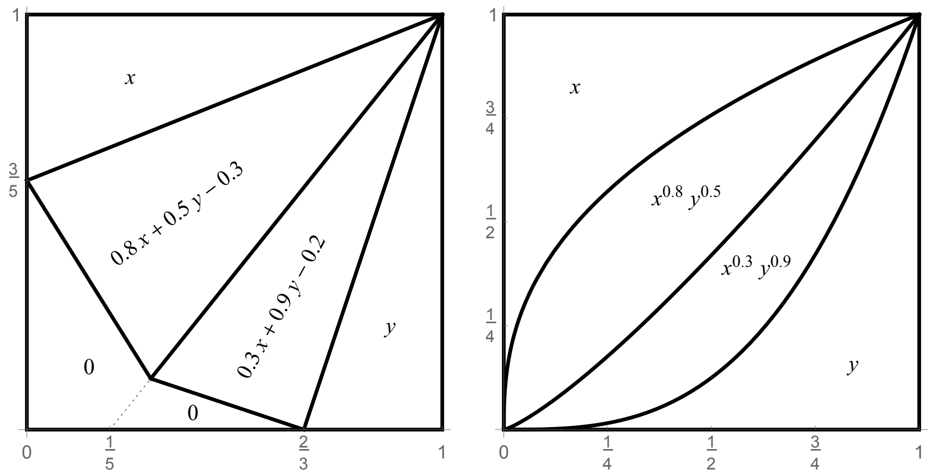

Figure 1: The conic copula in Example 4.2(i) (left), and the extreme value copula in Example 5.3(i).

Obviously, C satisfies (7) and, therefore, is an extreme value copula. Because of property (WV2) we obtain

$$C(x,y)=\begin{cases} y & \text{if } y\in\left[0,x^{\frac{\alpha_1}{\beta_1}}\right] \\ x^{\sum\limits_{j=1}^{i}\alpha_j}\,y^{\sum\limits_{j=i+1}^{n}\beta_j} & \text{if } y\in\left]x^{\frac{\alpha_i}{\beta_i}},x^{\frac{\alpha_{i+1}}{\beta_{i+1}}}\right], \\ x & \text{if } y\in\left]x^{\frac{\alpha_n}{\beta_n}},1\right]. \end{cases} \tag{10}$$

Clearly, for all $(x,y)\in[0,1]^2$ and all $\mu,\nu\in[0,1]$ we have

$$x^{\mu}y^{\nu}=\exp\Big(\log(xy)\frac{\mu\log x+\nu\log y}{\log(xy)}\Big)$$

and, putting $\frac{\log x}{\log(xy)}=u$, we obtain $\frac{\mu\log x+\nu\log y}{\log(xy)}=\mu u+\nu(1-u)$.

Now consider the function $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ given in (9) which is convex since its left derivatives in the intervals $\left]\frac{\beta_i}{\alpha_i+\beta_i},\frac{\beta_{i+1}}{\alpha_{i+1}+\beta_{i+1}}\right]$ (here we put $\frac{\beta_0}{\alpha_0+\beta_0}=+\infty$ and $\frac{\beta_{n+1}}{\alpha_{n+1}+\beta_{n+1}}=0$) are monotone non-decreasing with respect to the index i.

From (WV2) we see that for each $x\in\left]\frac{\beta_i}{\alpha_i+\beta_i},\frac{\beta_{i+1}}{\alpha_{i+1}+\beta_{i+1}}\right]$ we have $\frac{\alpha_{i+1}}{\beta_{i+1}}\le\frac{1-x}{x}<\frac{\alpha_i}{\beta_i}$. Therefore

$$\sum_{j=1}^{i}\alpha_j=\sum_{j=1}^{i}\frac{\alpha_j}{\beta_j}\beta_j\ge\frac{\alpha_i}{\beta_i}\sum_{j=1}^{i}\beta_j\ge\sum_{j=1}^{i}\beta_j\frac{1-x}{x}$$

which implies

$$A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)=\Big(\sum_{j=1}^{i}\alpha_j\Big)x+\Big(\sum_{j=i+1}^{n}\beta_j\Big)(1-x)=\Big(\sum_{j=1}^{i}\alpha_j\Big)x+\Big(1-\sum_{j=1}^{i}\beta_j\Big)(1-x)\ge 1-x.$$

Similarly,

$$1-\sum_{j=1}^{i}\alpha_j=\sum_{j=1+1}^{n}\alpha_j=\sum_{j=1+1}^{n}\psi_j\beta_j\le\psi_{i+1}.\sum_{j=1+1}^{n}\beta_j\le\sum_{j=1+1}^{n}\beta_j\frac{1-x}{x},$$

which yields

$$A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)=\Big(\sum_{j=1}^{i}\alpha_j\Big)x+\Big(\sum_{j=i+1}^{n}\beta_j\Big)(1-x)\ge x.$$

The two remaining cases are trivial: if $x\le\frac{\beta_1}{\alpha_1+\beta_1}$ then we have $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)=1-x=\max(1-x,x)$ because of $\frac{\beta_1}{\alpha_1+\beta_1}\le 0.5$. Similarly, if $x>\frac{\beta_n}{\alpha_n+\beta_n}$ then $x\ge 0.5$, i.e., $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)=x=\max(1-x,x)$.

This shows that the function $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ defines a Pickands dependence function, and it is only a matter of computation to show that it is the Pickands dependence function associated to the copula in (10). □

Again the properties (WV1) and (WV2) of the weight factors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ do not reduce the generality of the result. If $\alpha_i = \beta_i = 0$ for some $i \in \{1, 2, \dots, n\}$ then the pair (α_i, β_i) has no influence on C and we can delete it. If $\frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j}$ for some $i \neq j$ then

$$M\left(x^{\alpha_i}, y^{\beta_i}\right) \cdot M\left(x^{\alpha_j}, y^{\beta_j}\right) = M\left(x^{\alpha_i+\alpha_j}, y^{\beta_i+\beta_j}\right),$$

and the symmetry of Π allows us to rearrange the indices such that (WV2) holds.

Observe that each extreme value copula related to a piecewise linear Pickands dependence function (determined by finitely many points) can be obtained by the construction given in the proof of Proposition 5.1 using formula (10).

Moreover, the set of piecewise linear Pickands dependence functions is a dense subset of the set of all Pickands dependence functions, i.e., the construction (10) allows us to approximate each extreme value copula.

Corollary 5.2. *If $A\colon [0,1] \to [0.5,1]$ is a Pickands dependence function then, for each $n \in \mathbb{N}$, also the piecewise linear function $A^{(n)}\colon [0,1] \to [0.5,1]$ whose graph connects the $n+1$ points $(0,1)$, $(\frac{1}{n}, A(\frac{1}{n}))$, ..., $(\frac{n-1}{n}, A(\frac{n-1}{n}))$, and $(1,1)$ is a Pickands dependence function, and we have $\lim_{n\to\infty} A^{(n)}(x) = A(x)$ for each $x \in [0,1]$. Also for the corresponding extreme value copulas $C_A, C_{A^{(n)}}\colon [0,1]^2 \to [0,1]$ we have $\lim_{n\to\infty} C_{A^{(n)}}(x,y) = C_A(x,y)$ for all $(x,y) \in [0,1]^2$.*

Example 5.3. Here are two concrete examples of extreme value copulas.

(i) Take the same parameters as in Example 4.2(i), i.e., $n = 3$, $\boldsymbol{\alpha} = (0.3, 0.5, 0.2)$ and $\boldsymbol{\beta} = (0.1, 0.4, 0.5)$. The corresponding extreme value copula C given by (10) is visualized in Figure 1(right).

(ii) For $n = 3$ and $\lambda \in \,]0,1[$ choose $\boldsymbol{\alpha} = (1-\lambda, \lambda, 0)$ and $\boldsymbol{\beta} = (0, \lambda, 1-\lambda)$. The corresponding extreme value copula C given by (10) is the member with parameter λ of the family of Cuadras-Augé copulas [9] (see [40, Exercise 2.5]), i.e.,

$$C(x,y) = \begin{cases} xy^{1-\lambda} & \text{if } x \le y, \\ x^{1-\lambda}y & \text{otherwise.} \end{cases}$$

6 Archimax copulas

Archimax copulas were introduced as a joint generalization of Archimedean copulas (generated by some additive generator $\varphi\colon [0,1] \to [0,\infty]$) and extreme value copulas (related to some Pickands dependence function $A\colon [0,1] \to [0.5,1]$).

More precisely, it was shown in [6, Appendix A] that for each additive generator $\varphi\colon [0,1] \to [0,\infty]$ of an Archimedean copula C and each Pickands dependence function $A\colon [0,1] \to [0.5,1]$ the function $C_{\varphi,A}\colon [0,1]^2 \to [0,1]$ defined by

$$C_{\varphi,A}(x,y) = \varphi^{-1}\left(\min\left(\left(\varphi(x)+\varphi(y)\right) \cdot A\left(\frac{\varphi(x)}{\varphi(x)+\varphi(y)}\right), \varphi(0)\right)\right) \tag{11}$$

is a bivariate copula, and it is called the *Archimax copula related to φ and A*.

Proposition 6.1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0,1]^n \times [0,1]^n$ be weight vectors satisfying* (WV1) *and* (WV2). *Then there exists a Pickands dependence function $A\colon [0,1] \to [0.5,1]$ such that, for each Archimedean copula $C\colon [0,1]^2 \to [0,1]$*

with additive generator $\varphi\colon [0,1]\to[0,\infty]$ *the function* $D\colon [0,1]^2\to[0,1]$ *given by*

$$D(x,y)=\begin{cases} y & \text{if } \frac{\varphi(y)}{\varphi(x)}\in\left[\frac{\alpha_1}{\beta_1},+\infty\right],\\ \varphi^{-1}\left(\min\left(\left(\sum\limits_{j=1}^{i}\alpha_j\right)\varphi(x)+\left(\sum\limits_{j=i+1}^{n}\beta_j\right)\varphi(y),\varphi(0)\right)\right) & \text{if } \frac{\varphi(y)}{\varphi(x)}\in\left[\frac{\alpha_{i+1}}{\beta_{i+1}},\frac{\alpha_i}{\beta_i}\right[,\\ x & \text{if } \frac{\varphi(y)}{\varphi(x)}\in\left[0,\frac{\alpha_n}{\beta_n}\right[, \end{cases} \tag{12}$$

is an Archimax copula with respect to φ *and* A.

Proof. Let $C\colon [0,1]^2\to[0,1]$ be an Archimedean copula and $\varphi\colon[0,1]\to[0,+\infty]$ an additive generator of C, i.e., (1) holds for all $(x,y)\in[0,1]^2$.

Consider weight vectors $(\boldsymbol{\alpha},\boldsymbol{\beta})\in[0,1]^n\times[0,1]^n$ satisfying (WV1) and (WV2) and observe that the functions $f_1,f_2,\dots,f_n,g_1,g_2,\dots,g_n\colon[0,1]\to[0,1]$ defined by

$$f_i(x)=\varphi^{-1}\big(\alpha_i\varphi(x)\big)\quad\text{and}\quad g_i(x)=\varphi^{-1}\big(\beta_i\varphi(x)\big)$$

are monotone non-decreasing in each component and continuous.

Because of the associativity of C its n-ary extension $C^{[n]}$ is unique, so we may use the same notation for it, i.e., $C\colon[0,1]^n\to[0,1]$ defined by (1). Note that $C\big(f_1(x),f_2(x),\dots,f_n(x)\big)=C\big(g_1(x),g_2(x),\dots,g_n(x)\big)=x$ for all $x\in[0,1]$ and consider the function $D\colon[0,1]^2\to[0,1]$ given by

$$D(x,y)=C\Big(M\big(f_1(x),g_1(y)\big),M\big(f_2(x),g_2(y)\big),\dots,M\big(f_n(x),g_n(y)\big)\Big). \tag{13}$$

Since $f_i(x)\ge g_i(y)$ is equivalent to $\frac{\varphi(y)}{\varphi(x)}\ge\frac{\alpha_i}{\beta_i}$, we can conclude that the functions given by (12) and (13) coincide.

Putting $\frac{\varphi(x)}{\varphi(x)+\varphi(y)}=u$, it is not difficult to see that, for an arbitrary function $f\colon\mathbb{R}\to\mathbb{R}$, the equality

$$\left(\sum_{j=1}^{i}\alpha_j\right)\varphi(x)+\left(\sum_{j=i+1}^{n}\beta_j\right)\varphi(y)=(\varphi(x)+\varphi(y))\cdot f\left(\frac{\varphi(x)}{\varphi(x)+\varphi(y)}\right)$$

is equivalent to $\left(\sum\limits_{j=1}^{i}\alpha_j\right)u+\left(\sum\limits_{j=i+1}^{n}\beta_j\right)(1-u)=f(u)$.

Using similar arguments as in the proof of Proposition 5.1, we can show that the function $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon[0,1]\to[0.5,1]$ given by (9) is a Pickands dependence function. Finally, the proof in [6, Appendix A] tells us that the function D given by (13) is an Archimax copula with respect to φ and $A_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. ☐

Also in this case there is no loss of generality when assuming the validity of the properties (WV1) and (WV2) for the weight vectors $(\boldsymbol{\alpha},\boldsymbol{\beta})\in[0,1]^n\times[0,1]^n$.

Observe that Proposition 6.1 allows us to approximate any Archimax copula with arbitrary precision: given a Pickands dependence function $A\colon[0,1]\to[0.5,1]$ then, using the sequence $(A^{(n)})_{n\in\mathbb{N}}$ of Pickands dependence functions converging pointwise to A in Corollary 5.2, for each additive generator $\varphi\colon[0,1]\to[0,+\infty]$ of an Archimedean copula $C\colon[0,1]^2\to[0,1]$, the sequence $(C_{\varphi,A^{(n)}})_{n\in\mathbb{N}}$ converges pointwise to the Archimax copula $C_{\varphi,A}$.

7 Concluding remarks

Based on Theorem 3.4, we have presented constructions and representations of certain bivariate conic, extreme value, and Archimax copulas.

In Proposition 6.1 we have used the result in [6, Appendix A] to show that the function D constructed there is a copula for each Archimedean copula C. If the Archimedean copula C is also ultramodular then Theorem 3.4 tells us that D is a copula also without using [6]. This means that we have given an alternative proof for the functions represented by (11) to be copulas, provided the Archimedean copula C we start with is ultramodular.

Observe also that from [12] it follows that the set of conic copulas coincides with the set of Archimax copulas based on the additive generator of the Fréchet-Hoeffding lower bound W, and our results in Section 4 can be seen as an alternative proof of this fact.

As an interesting topic for further research in the directions of the construction and characterization of bivariate copulas as realized in this paper, one can consider the approach to construct n-ary copulas (with $n > 2$) proposed and studied in [36] based on the product copula Π and binary copulas $D_1, D_2, \dots, D_m\colon [0,1]^2 \to [0,1]$ distorted by functions $f_1, f_2, \dots, f_m, g_1, g_2, \dots, g_m\colon [0,1] \to [0,1]$, and applied to pairs $(x_i, x_j) \in [0,1]^2$ such that $\{i,j\} \subseteq \{1,2,\dots,n\}$ and $i < j$.

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