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ORBITS OF PRIMITIVE k-HOMOGENOUS GROUPS ON $(n-k)$-PARTITIONS WITH APPLICATIONS TO SEMIGROUPS

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Abstract. The purpose of this paper is to advance our knowledge of two of the most classic and popular topics in transformation semigroups: automorphisms and the size of minimal generating sets. In order to do this, we examine the k-homogeneous permutation groups (those which act transitively on the subsets of size k of their domain X) where $|X| = n$ and $k < n/2$. In the process we obtain, for k-homogeneous groups, results on the minimum numbers of generators, the numbers of orbits on k-partitions, and their normalizers in the symmetric group. As a sample result, we show that every finite 2-homogeneous group is 2-generated.

Underlying our investigations on automorphisms of transformation semigroups is the following conjecture:

> *if a transformation semigroup S contains singular maps, and its group of units is a primitive group G of permutations, then its automorphisms are all induced (under conjugation) by the elements in the normalizer of G in the symmetric group.*

For the special case that S contains all constant maps, this conjecture was proved correct, more than 40 years ago. In this paper, we prove that the conjecture also holds for the case of semigroups containing a map of rank 3 or less. The effort in establishing this result, suggests that further improvements might be a great challenge. This problem and several additional ones on permutation groups, transformation semigroups and computational algebra, are proposed in the end of the paper.

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1. Introduction

Let $X = \{1, \dots, n\}$. Our context is *permutation groups* (subgroups of the *symmetric group* S_n) and *transformation semigroups* (subsemigroups of the *full transformation monoid* T_n) over the set X.

The *rank* of a transformation $t \in T_n$ is the size of its image, and is denoted by $\mathrm{rank}(t)$. The *kernel* of t is the partition of X in which two points lie in the same part if and only if they have the same image under t.

We say that a partition $P = (A_1, \dots, A_k)$ of a set X, where the parts are listed in decreasing order of cardinality, is of *type* $(|A_1|, \dots, |A_k|)$ and thus the type of P is itself a partition of the integer n. A partition having k parts will be called a *k-partition*, by analogy with the terminology k-subset.

Given sets $A, B \subseteq T_n$ and a transformation $t \in T_n$, we denote by $\langle A \rangle$ the semigroup generated by A. We will abuse notation by writing $\langle A, B \rangle$ rather than $\langle A \cup B \rangle$, and $\langle A, t \rangle$, rather than $\langle A \cup \{t\} \rangle$.

Our group-theoretic investigations concern the minimal generating sets of permutation groups $G \leq S_n$ and their orbits on $(n-k)$-partitions, for $k \leq n/2$. For example, in the case $k = 1$, we examine the orbits of G on $(n-1)$-partitions, which is equivalent to the study of the orbits of G on 2-sets (a slight variation of the key concept of *orbitals*). Our main results in this section fit the following template (with m and m' appearing in Tables 2-6):

Theorem template. *Let $k \leq \frac{n}{2}$ and let $G \leq S_n$ be a primitive k-homogenous group.*

- *G has at most m orbits on the set of $(n-k)$-partitions;*
- *the smallest number of elements needed to generate G is m'.*

In particular, we show that a 2-homogeneous group is 2-generated. This result seems to have been unnoticed before. For example, the computer algebra system GAP [44] provides 2-element generating sets for these groups in only about one-third of all cases.

In Theorem 3.2, we determine the 3-homogeneous groups whose orbits on $(n-3)$-partitions coincide with those of their normalizers. Our list is complete except for one unresolved family.

These preliminary results on groups are then used to extract information about transformation semigroups. McAlister and Levi proved the following theorems.

Theorem 1.1. [80] *Let $G \leq S_n$ and $t \in T_n$ be any map of rank $n-1$. Then $\langle G, t \rangle$ generates all rank $n-1$ transformations in T_n if and only if the group G has only one orbit on the $(n-1)$-partitions of $\{1, \dots, n\}$ (equivalently, G is 2-homogeneous).*

Theorem 1.2. [60] *Let $A_n \le G \le S_n$ and let $t \in T_n \setminus S_n$. Then the automorphism group of $\langle G, t\rangle$ is isomorphic to S_n.*

We generalize these results as follows.

Theorem 1.3. *Let t be a singular map in T_n, and suppose that t has kernel type $(l_1, \dots, l_k)$, with $k \ge n/2$; let G be a group having only one orbit on partitions of this type.*

(a) The automorphisms of $\langle t, G\rangle$ are those induced under conjugation by the elements of the normalizer of G in S_n:

$$Aut(\langle t, G\rangle) \cong N_{S_n}(G);$$

(b) If $k \le n-2$, then $\langle G, t\rangle$ is generated by 3 elements.

(c) Let A be a set of rank k maps such that $\langle A, G\rangle$ contains all maps of rank at most k and A has minimum size among the sets with that property. Then $|A|$ is given in Table 1.

rank	*partition type*	$\lvert A\rvert$
$n-1$	$(2, 1, \dots, 1)$	1
$n-2$	$(2, 2, 1, \dots, 1)$	2
	$(3, 1, \dots, 1)$	$O(n)$
$n-3$	$(4, 1, \dots, 1)$	144
	$(3, 2, 1, \dots, 1)$	5
	$(2, 2, 2, 1, \dots, 1)$	3
$n-4$	$(5, 1, \dots, 1)$	15
	other	5
k $(n/2 \le k \le n-5)$	*any*	$p(k)$

TABLE 1. Generating all maps of rank k

Table 1 should be read as follows: if a group G has, for example, one orbit on partitions of type $(4, 1, \dots, 1)$, then there is a set A of size 144 such that $\langle G, A\rangle$ contains all maps of rank at most $n-3$, but no smaller set will suffice.

In Theorem 1.3, we require the group of units to be transitive on partitions with the same type as the kernel of the singular map t. In the next theorem this condition is replaced by the weaker requirement that G is $|Xt|$-homogeneous, i.e., transitive on subsets with cardinality equal to the rank of t.

Here, and elsewhere in the paper, the notation $\mathrm{PXL}(2,q)$ means the 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ (for q an odd prime power square) containing $\mathrm{PSL}(2,q)$ as a subgroup of index 2, different from $\mathrm{PGL}(2,q)$; that is, it is generated by $\mathrm{PSL}(2,q)$ and an element which is the product of a diagonal and a field automorphism of order 2. (When $q = 9$, this group is better known as M_{10}, the point stabiliser in the Mathieu group M_{11}.)

Theorem 1.4. *Let G be a primitive group with just one orbit on $(n-k)$-sets, where $1 \le k \le n/2$. Let $t \in T_n$ be a rank $(n-k)$ map.*

(a) $\mathrm{Aut}\,(\langle G,t\rangle) \cong N_{S_n}(\langle G,t\rangle)$.

(b) For $k \ge 3$, the list of 3-homogeneous groups that satisfy

$$\mathrm{Aut}\,(\langle G,t\rangle) \cong N_{S_n}(G)$$

is the following:

(i) $G = N_{S_n}(G)$, *that is,*
- S_n.
- $\mathrm{P\Gamma L}(2,q)$ *for* $k = 3$.
- $\mathrm{AGL}(d,2)$ *for* $k = 3$.
- $\mathrm{A\Gamma L}(1,8)$, M_{11} *($k = 4$)*, M_{11} *(degree 12, $k = 3$)*, M_{12} *($k = 5$)*, $2^4 : A_7$, $M_{22} : 2$ *($k = 3$)*, M_{23} *($k = 4$)*, M_{24} *($k = 5$), and* $\mathrm{A\Gamma L}(1,32)$ *($k = 4$)*.

(ii) $G = A_n$;

(iii) $G = \mathrm{AGL}(1,8)$, $\mathrm{PGL}(2,8)$, $\mathrm{PGL}(2,9)$, M_{10}, $\mathrm{PSL}(2,11)$, M_{22}, $\mathrm{PXL}(2,25)$, *or* $\mathrm{PXL}(2,49)$, *with* $k = 3$ *and* $\lambda = (4,1,\dots,1)$.

The list is complete with the possible exception of the groups $\mathrm{PXL}(2,q)$ *for* $q \ge 169$.

(c) Let $A \subseteq T_n$ be a set of rank $n-k$ maps such that $\langle G,A\rangle$ generates all maps of rank at most $n-k$, and suppose A has minimum size among the sets with that property. Then the size of A is bounded by the values in Table 2.

As said, the two results above are just sample theorems. For more detailed results we refer the reader to the sections below.

The results above on automorphisms rely on the fact that the semigroups under consideration contain all constant maps. This is in line with the overwhelming majority of past papers on automorphisms of transformation semigroups. Our next theorem, however, goes a step beyond this requirement by providing a result on automorphisms of semigroups containing a primitive group of permutations, but possibly without constant maps.

Rank $n-k$	$\|A\|$	Sample k-homogeneous groups attaining the bound for $\|A\|$	Minimum number of generators for a primitive k-homogeneous group
$n-1$	$\frac{(n-1)}{2}$	C_p, D_p (n odd prime)	$\frac{C\log n}{\sqrt{\log\log n}}$
$n-2$	$O(n^2)$	Example 2.1	2
$n-3$	$O(n^3)$	$\mathrm{PSL}(2,q), \mathrm{P\Gamma L}(2,q)$	2
$n-4$	12160	$\mathrm{P\Gamma L}(2,32)$ $(n=33)$	2
$n-5$	77	M_{24} $(n=24)$	2
$n-k$ $(k\geq 5)$	$p(k)$	S_n, A_n	2

TABLE 2. Number of rank $n-k$ maps needed to together with a k-homogeneous group G generate all the maps of rank not larger than $n-k$.

Theorem 1.5. *Let $S \leq T_n$ be a semigroup with primitive group of units $G \leq S_n$. If there exists in S a map of rank 3 or less, then all the automorphisms of S are induced under conjugation by permutations of the normalizer of G in S_n.*

The semigroup-theoretic content of the paper belongs to the general area of investigating how recent results on groups, chiefly the classification of finite simple groups and detailed study of the structure of almost simple groups, can help the study of semigroups. (For other papers on this line of research, see for example [1, 4, 5, 6, 7, 9, 10, 11, 12, 13, 23, 25, 26, 61, 64, 65, 80, 82, 92] and the references therein.) The typical object in this field is a semigroup generated by a set of non-invertible transformations $A \subseteq T_n \setminus S_n$ and a group of permutations G contained in S_n. In this paper we are mainly concerned with the description of automorphisms and minimal generating sets, for semigroups having special given group of units.

The *rank* of a semigroup S, denoted by $\operatorname{rank} S$, is the least number of elements in S needed to generate S (not to be confused with the rank of an element of S). It is well-known that a finite full transformation semigroup, on at least 3 points, has rank 3, while a finite full partial transformation semigroup, on at least 3 points, has rank 4 (see [50, Exercises 1.9.7 and 1.9.13]). The problem of determining the minimum number of generators of a semigroup is classical, and has been studied extensively; see, for example, [3, 6, 24, 27, 42, 51, 56, 66, 87] and the references therein. Given the importance of idempotent generated semigroups illustrated by the Erdos/Howie famous twin results (see

[38, 49] and also [2, 22]) the related notion of *idempotent rank* appeared as natural and has also been widely investigated; the same can be said about the concepts of *relative rank* and *nilpotent rank*; see [21, 28, 36, 40, 46, 47, 48, 63]. One of the goals of this paper is to contribute to this line of research.

Another classic topic in semigroup theory is the description of the automorphisms of semigroups. After the pioneer work of Schreier [89] and Mal'cev [76], proving that the group of automorphisms of T_n is isomorphic to S_n, a long sequence of new results followed (for example, [7, 8, 13, 14, 15, 16, 17, 18, 19, 20, 58, 59, 60, 67, 75, 90, 91, 92, 93] and the references therein). In addition to the general interest of studying automorphisms of mathematical structures, the description of automorphisms of semigroups turned out to be a key ingredient in Plotkin's *universal algebraic geometry* [84] and [30, 31, 32, 39, 57, 77, 78, 79, 85, 86].

Here, we use advances in permutation group theory during the last couple of decades to contribute to this line of research, by finding the automorphisms of semigroups with given group of units.

An outline of the contents of the paper follows.

In Section 2 we prove the main theorems about the minimum number of generators of primitive groups, and we also give estimates on the number of orbits of primitive groups on $(n-k)$-partitions, for $k \geq n/2$. In Section 3 we tackle the problem of independent interest of classifying the permutation groups in which all orbits on $(n-k)$-partitions are invariant under the normalizer. In Section 4 we apply the results proved in the previous sections to describe automorphisms and ranks of semigroups in which its group of units has just one orbit on the kernel type of t. In Section 5 we consider similar problems, but for semigroups whose group of units has just one orbit on the image of t. Section 6 contains some comments on the normalizers of 2-homogeneous or primitive groups. Section 7 contains the description of the automorphisms of a semigroup with primitive group of units and a map of rank at most 3.

The paper ends with a section of open problems.

2. Group theory

Let $1 \leq k \leq n/2$, and G a k-homogeneous group. The aim of this section is to calculate exact or asymptotic bounds for the numbers of orbits of G on the set of $(n-k)$-partitions, and for the minimal numbers of generators of G.

The case in which least can be said is $k = 1$. The *rank* $r(G)$ of a transitive permutation group G (acting on $X = \{1, \ldots, n\}$) is the number of G-orbits on ordered pairs from $\{1, \ldots, n\}$. To handle the case of $k = 1$, we need a slightly different parameter, the number $n_2(G)$ of G-orbits on the set of 2-subsets of $\{1, \ldots, n\}$. Clearly

$$(r(G) - 1)/2 \leq n_2(G) \leq r(G) - 1;$$

the lower bound holds when G has odd order (since then no pair of points can be interchanged by an element of G), and the upper bound when all the orbitals of G are self-paired. Note that $r(G) \leq n$, with equality if and only if G is regular. In particular, a primitive group G has $r(G) = n$ if and only if n is prime and G is cyclic of order n. We thus see that $n_2(G) \leq n - 1$ for transitive groups G; equality is realised for an elementary abelian 2-group acting regularly, but for primitive groups of degree greater than 2 we have $n_2(G) \leq (n - 1)/2$, with equality only for the cyclic and dihedral groups of odd prime degree. This follows because if G is primitive but not cyclic or dihedral of prime degree, then all non-trivial orbits of a point stabiliser have size at least 3.

Theorem 2.1. *Let $G \leq S_n$ be a 1-homogeneous (that is, transitive) permutation group. Then*

(a) G has $n_2(G)$ orbits on the set of $(n - 1)$-partitions;
(b) the smallest number of elements needed to generate G is at most

$$\frac{Cn}{\sqrt{\log n}} \qquad \textit{if } G \textit{ is imprimitive,}$$

$$\frac{C \log n}{\sqrt{\log \log n}} \qquad \textit{if } G \textit{ is primitive,}$$

where C is a universal constant.

Proof. All $(n-1)$-partitions have one part with two elements while all other parts are singletons. Therefore the group G has as many orbits on the set of $(n - 1)$-partitions as on the set of 2-sets.

McIver and Neumann [81] showed that every subgroup of S_n can be generated by $\lfloor n/2 \rfloor$ elements if $n \neq 3$, and by 2 if $n = 3$. This bound is best possible for arbitrary subgroups, but for transitive or primitive subgroups has been improved in [71, 72] to the statements in the theorem. Moreover, these bounds are essentially best possible. ☐

Next, we are going to prove that the minimum number of generators of any 2-homogeneous finite group is 2. (It is worth observing that we could not find this fact in the literature; we are grateful to Colva Roney-Dougal and Andrea Lucchini for independently confirming it.) The proof uses the following result proved by Lucchini and Menegazzo [70]. Here $d(G)$ denotes the least number of elements needed to generate G.

Theorem 2.2 ([70]). *Let G be a non-cyclic finite group having a unique mimimal normal subgroup M. Then $d(G) = \max\{2, d(G/M)\}$.*

Corollary 2.3. *If G is a finite 2-homogeneous permutation group, then $d(G) = 2$.*

Proof. Observe that any 2-homogeneous group has a unique minimal normal subgroup, which is either elementary abelian or simple, by Burnside's theorem for 2-transitive groups [33, Theorem 4.3].

If G is almost simple, then it satisfies the conditions of Theorem 2.3, with M being the simple socle. Since, in the case of socle $\mathrm{PSL}(d, q)$, the group G contains no graph automorphisms, we have $d(G/M) \leq 2$ in all cases, so $d(G) = 2$.

If G is affine, its unique minimal normal subgroup M is elementary abelian, and the quotient H is a linear group; the relevant groups can be found in [33, 37]. If the linear group has normal subgroup $\mathrm{SL}(d, q)$, $\mathrm{Sp}(d, q)$ ($d > 1$) or $G_2(q)$, then another application of Theorem 2.2 shows that $d(H) = 2$, whence $d(G) = 2$. For 1-dimensional semi-affine groups, the linear group is metacyclic, and the result is clear. The finitely many cases remaining can be dealt with case by case: in each case, explicit generators for the linear group are known, and where more than two are given it suffices to show that the corresponding linear group can be generated by two elements. The groups (apart from the sharply 2-transitive group of degree 59^2 with linear group $\mathrm{SL}(2, 5) \times C_{29}$, which is clearly 2-generated), are within reach of GAP; the computation can be speeded up by taking the first potential generator to belong to a set of conjugacy class representatives. □

Regarding the number of orbits on $(n - k)$-partitions, we start with large values of k.

Theorem 2.4. *Suppose that G is a permutation group of degree n which is k-homogeneous, where either $6 \leq k \leq n/2$, or $k = 5$, $n \geq 25$, or $k = 4$, $n \geq 34$. Then:*

(a) G has $p(k)$ orbits on the set of $(n-k)$-partitions, where p is the partition function;
(b) there is one orbit on partitions of each possible type.

Proof. It follows from the Classification of Finite Simple Groups and known results about 4- and 5-homogeneous groups that any G under the assumptions of the theorem is S_n or A_n. The second assertion is a well-known fact about A_n and S_n. For the first, given a partition of $\{1, \dots, n\}$ with $n-k$ parts, for $k \leq n/2$, subtracting one from the size of each part gives a partition of k after discarding values of 0. Conversely, every partition of k arises in this way. The result now follows directly from part (b). □

The numbers of orbits of the finitely many k-homogeneous groups other than symmetric or alternating groups for $k = 5$ and $k = 4$ can be computed.

Theorem 2.5. *Let G be a k-homogeneous group of degree n (where $k = 4$ or 5 and $n \geq 2k$), other than S_n or A_n. Then the number of orbits of G on $(n-k)$-partitions are given in Tables 3 and 4 below.*

Remark We are grateful to Robin Chapman for independent confirmation of the values for $\mathrm{P\Gamma L}(2, 32)$ in Table 3.

Degree	9	9	11	12	23	24	33
Group	$\mathrm{PSL}(2,8)$	$\mathrm{P\Gamma L}(2,8)$	M_{11}	M_{12}	M_{23}	M_{24}	$\mathrm{P\Gamma L}(2,32)$
$(5,1,\dots)$	1	1	2	1	2	1	3
$(4,2,1,\dots)$	4	2	3	2	4	2	112
$(3,3,1,\dots)$	4	2	2	2	3	2	82
$(3,2,2,1,\dots)$	12	4	8	3	11	3	2772
$(2,2,2,2,1,\dots)$	5	3	6	5	18	7	9191
Total	26	12	21	13	38	15	12160

TABLE 3. Orbits of 4-homogeneous groups on $(n-4)$-partitions

The situation is very different for the 2- and 3-homogeneous groups, to which we now turn. The main difference is that there are infinitely many such groups (apart from the symmetric and alternating groups), so there is no reason why the number of orbits on $(n-k)$-partitions should be bounded (and indeed it is not; it can grow as a polynomial in n, whose degree depends on k and on the partition considered).

Theorem 2.6. *Let G be a 2-homogeneous permutation group on the set $\{1, \dots, n\}$. Then the number of G-orbits on the set of partitions of type $(3, 1, \dots, 1)$ is $O(n)$, and the number of orbits on the set of partitions of type $(2, 2, 1, \dots, 1)$ is $O(n^2)$.*

Proof. Since each 2-set lies in $n-2$ sets of size 3, G has at most $n-2$ orbits on 3-sets. Also, for any 2-set, there are at most $\binom{n-2}{2}$ 2-sets disjoint from it, so there are at most this many orbits on $(2, 2, 1, \dots, 1)$ partitions. □

Degree	12	24
Group	M_{12}	M_{24}
$(6,1,\ldots)$	2	2
$(5,2,1,\ldots)$	2	3
$(4,3,1,\ldots)$	2	3
$(4,2,2,1,\ldots)$	5	8
$(3,3,2,1,\ldots)$	5	8
$(3,2,2,2,1,\ldots)$	8	22
$(2,2,2,2,2,1,\ldots)$	6	31
Total	30	77

Table 4. Orbits of 5-homogeneous groups on $(n-5)$-partitions

The bound on the number of orbits is best possible, as the next example shows.

Example 2.1. Let p be a prime congruent to $-1 \pmod{12}$. Let G be the group of order $p(p-1)/2$ consisting of all maps of the field of integers mod p of the form $x \mapsto ax + b$, where a is a non-zero square. Its normalizer is the group of order $p(p-1)$, consisting of all maps of the above form for arbitrary non-zero a.

The group G is 2-homogeneous. The $(p-2)$-partitions have type $(3,1,\ldots,1)$ or $(2,2,1,\ldots,1)$. Since $|G|$ is coprime to 6, no element of G except the identity fixes such a partition, and so the number of orbits is

$$\frac{\binom{p}{3} + 3\binom{p}{4}}{p(p-1)/2} = \frac{3p^2 - 11p + 10}{12}.$$

Of these, $(p-2)/3$ are on partitions of type $(3,1,\ldots,1)$, and $(p-2)(p-3)/4$ are on partitions of type $(2,2,1,\ldots,1)$.

Using arguments like those in the proof of Theorem 2.6, we may obtain a corresponding theorem for 3-homogeneous groups.

Theorem 2.7. *Let G be a 3-homogeneous permutation group on the set $\{1,\ldots,n\}$. Then the number of G-orbits on the set of $(n-3)$-partitions is $O(n)$ for partitions of type $(4,1,\ldots,1)$, $O(n^2)$ for partitions of type $(3,2,1,\ldots,1)$, and $O(n^3)$ for partitions of type $(2,2,2,1,\ldots,1)$.*

In fact we can say more. From CFSG, we know that, if G is 3-homogeneous, then one of the following holds:

- $\mathrm{PSL}(2,q) \leq G \leq \mathrm{P\Gamma L}(2,q)$, for some prime power q;
- $G = \mathrm{AGL}(d,2)$ for some d;
- G is one of finitely many exceptions.

In the first case, the order of G is $O(n^3)$, so the number of orbits on partitions of type $(2,2,2,1,\ldots,1)$ will be $\Omega(n^3)$. However, in the other two cases, the number of orbits is bounded by a constant, independent of d in the second case. This is clear for the third case, so consider the second. Suppose we have an $(n-3)$-partition of $\{1,\ldots,n\}$. Then the set of points lying in parts of size greater than 1 has cardinality at most 6, and so these points lie in an affine subspace of dimension at most 5. The group is transitive on affine subspaces of any given dimension, and the stabiliser of such a subspace has only a bounded number of orbits on its subsets of size at most 6. The number of orbits for this type can be calculated by looking at $\mathrm{AGL}(5,2)$. We find that the number of orbits on $\mathrm{AGL}(d,2)$ on $(n-3)$-partitions is 12 for $d \geq 5$. The numbers of orbits on partitions of the different types is given in Table 5, with the same conventions as earlier.

Degree	2^d $(d \geq 5)$	16	8
Group	$\mathrm{AGL}(d,2)$	$\mathrm{AGL}(4,2)$	$\mathrm{AGL}(3,2)$
$(4,1,\ldots)$	2	2	2
$(3,2,1,\ldots)$	3	3	2
$(2,2,2,1,\ldots)$	7	6	3
Total	12	11	7

TABLE 5. Orbits of $\mathrm{AGL}(d,2)$ on $(n-3)$-partitions

Similar data can be produced for any finite number of the other 3-homogeneous groups. Table 6 gives a selection of 3-homogeneous groups of degree $n \geq 7$, which includes all the sporadic examples, all 4-homogeneous groups, and all examples with $n \leq 10$ apart from S_n and A_n.

3. Orbits of normalizers

In this section we will be interested in the following questions:

(a) Given an orbit of the k-homogeneous group G on $(n-k)$-partitions, what is the subgroup of the normalizer of G, in S_n, which fixes that orbit?

Degree	Group	$(4,1,\ldots)$	$(3,2,1,\ldots)$	$(2,2,2,1,\ldots)$	Total
8	AGL$(1,8)$	2	10	11	23
	AΓL$(1,8)$	2	4	5	11
	PSL$(2,7)$	3	4	7	14
	PGL$(2,7)$	2	3	5	10
9	PSL$(2,8)$	1	4	7	12
	PΓL$(2,8)$	1	2	3	6
10	PGL$(2,9)$	2	5	12	19
	M_{10}	2	5	9	14
	PΓL$(2,9)$	2	4	8	14
11	M_{11}	1	2	4	7
12	M_{11}	2	4	6	12
	M_{12}	1	1	3	5
16	$2^4 : A_7$	2	4	10	16
22	M_{22}	2	5	11	18
	$M_{22} : 2$	2	4	10	16
23	M_{23}	1	2	3	6
24	M_{24}	1	1	2	4
33	PΓL$(2,32)$	1	16	127	144

TABLE 6. Orbits of 3-homogeneous groups on $(n-3)$-partitions

(b) In particular, for which groups is it the case that every orbit on $(n-k)$-partitions is invariant under the normalizer, that is, the action of $N_{S_n}(G)/G$ on the set of orbits is trivial?

Note that the questions are well-posed, since $N_{S_n}(G)/G$ acts on the set of orbits.

If G is an alternating group, then each of its orbits is stabilised by the symmetric group. For $k \geq 4$, any other k-homogeneous group is equal to its normalizer, except for PGL$(2,8)$ with $n = 9$. This group is 5-homogeneous, and so has the same orbits on partitions of type $(5,1,1,1,1)$ as its normalizer. Computation shows that this is not the case for other types of 5-partitions. So we have the following theorem.

Theorem 3.1. *Let $k \geq 4$ and $n \geq 2k$, and let G be a k-homogeneous group of degree n. Then G has the same orbits on $(n-k)$-partitions of any given type as its normalizer, except in the case of* PGL$(2,8)$*, for* *which this assertion holds for partitions of type* $(5,1,1,1,1)$ *but for no other types.*

We next consider the case $k = 3$. For a partition type λ, we say that the pair (G, λ) is *closed* if each orbit of G on λ-partitions is invariant under $N_{S_n}(G)$. Note that there are three types of partition to be considered, namely $(4,1,\ldots,1)$, $(3,2,1,\ldots,1)$, and $(2,2,2,1,\ldots,1)$. Note also that (G, λ) is trivially closed if $G = N_{S_n}(G)$.

The main theorem of this section is the following (recall the notation $\mathrm{PXL}(2,q)$ defined before Theorem 1.4).

Theorem 3.2. *Suppose that* G *is a* 3*-homogeneous subgroup of* S_n*, and* λ *a type of* $(n-3)$*-partitions. Then* (G, λ) *is closed if one of the following holds:*

(a) $G = N_{S_n}(G)$*;*
(b) $G = A_n$*;*
(c) $\lambda = (4,1,\ldots,1)$ *and* $G = \mathrm{AGL}(1,8)$*,* $\mathrm{PGL}(2,8)$*,* $\mathrm{PGL}(2,9)$*,* M_{10}*,* $\mathrm{PSL}(2,11)$*,* M_{22}*,* $\mathrm{PXL}(2,25)$*, or* $\mathrm{PXL}(2,49)$*.*

No other 3*-homogeneous groups appear in a closed pair, with the possible exception of* $\mathrm{PXL}(2,q)$ *for* $q \geq 169$*.*

This theorem answers question (b) at the beginning of this section, with the exception of the groups $\mathrm{PXL}(2,q)$ referred to in its statement. Concerning question (a) the situation may be much more complicated as the following example shows.

Example 3.1. Let $n = 17$, and let G be the 3-homogeneous group $\mathrm{PSL}(2,16)$. The normalizer of G in S_n is $\mathrm{P\Gamma L}(2,16) = G:4$, with one intermediate subgroup $G:2$. Table 7 gives the number of G-orbits on the 14-partitions of various types, and the numbers with each of the three possible stabilisers.

Partition	$(4,1,\ldots,1)$	$(3,2,1,\ldots,1)$	$(2,2,2,1,\ldots,1)$	Total
Orbits	3	19	72	94
Stabiliser G	0	12	60	72
Stabiliser $G:2$	2	6	10	18
Stabiliser $G:4$	1	1	2	4

TABLE 7. Stabilisers of orbits of $\mathrm{PSL}(2,16)$

For the group $G = \mathrm{PSL}(2,2^p)$, with p prime and $p > 3$, the situation is much simpler: no $(2^p - 2)$ partition can be fixed by an element outside G, and so every orbit has stabiliser G. (This also shows, for example, that the numbers of orbits for $\mathrm{PSL}(2,32)$ are five times those for $\mathrm{P\Gamma L}(2,32)$ given in Table 6.)

We now give the proof of Theorem 3.2.

Proof. We begin by listing the 3-homogeneous groups.

(a) S_n, A_n.
(b) (Some) subgroups of $\mathrm{P\Gamma L}(2,q)$ containing $\mathrm{PSL}(2,q)$, for q a prime power ("some" means "all" if and only if q is even or congruent to 3 mod 4).
(c) $\mathrm{AGL}(d,2)$.
(d) Finitely many "sporadic" examples: $\mathrm{AGL}(1,8)$, $\mathrm{A\Gamma L}(1,8)$, M_{11}, M_{11} (degree 12), M_{12}, $2^4 : A_7$, M_{22}, $M_{22} : 2$, M_{23}, M_{24}, and $\mathrm{A\Gamma L}(1,32)$.

We remark that, of these, the groups which are equal to their normalizers (and so fall in the first case in the Theorem) are:

(a*) S_n.
(b*) $\mathrm{P\Gamma L}(2,q)$.
(c*) $\mathrm{AGL}(d,2)$.
(d*) All except $\mathrm{AGL}(1,8)$ and M_{22}.

Now type (a) are always closed. Type (c), and also type (d) with the exception of $\mathrm{AGL}(1,8)$ and M_{22}, are equal to their normalizers, so are trivially closed. For the remaining cases in (d), computation shows that, if $G = \mathrm{AGL}(1,8)$ (degree 8) or $G = M_{22}$ (degree 22), and λ is a partition type of rank $n-3$, then (G,λ) is closed if and only if $\lambda = (4,1,\ldots,1)$.

In fact, the numbers of orbits on the three types of partitions for G and its normalizer are $(2,10,11)$ and $(2,4,5)$ for $G = \mathrm{AGL}(1,8)$, and $(2,5,11)$ and $(2,4,10)$ for $G = M_{22}$. Note that M_{22} comes very close: only two orbits of each of the other two types are fused by $M_{22} : 2$.

It remains to deal with type (b). So, let q be a prime power, and G a 3-homogeneous subgroup of $\mathrm{P\Gamma L}(2,q)$ that contains $\mathrm{PSL}(2,q)$.

We will first consider G containing $\mathrm{PGL}(2,q)$. Let $\lambda = (4,1,\ldots,1)$. The partitions of this type correspond naturally to 4-subsets.

Orbits of $\mathrm{PGL}(2,q)$ on 4-tuples are parametrised by *cross ratio*: there is some flexibility about the definition, but we will take the cross ratio of $(\infty,0,1,a)$ to be a. The 24 orderings of a 4-set give rise to a set of 6 (or, in special cases, fewer) cross ratios of the form

$$\{z, 1/z, 1-z, 1/(1-z), z/(z-1), (z-1)/z\}.$$

So $\mathrm{GF}(q) \setminus \{0,1\}$ is partitioned into sets of 6 (or fewer) cross ratios corresponding to orbits of $\mathrm{PGL}(2,q)$ on 4-sets.

Now $\mathrm{P\Gamma L}(2,q)$ is generated by $\mathrm{PGL}(2,q)$ and the *Frobenius map* $x \mapsto x^p$, where q is a power of p, say $q = p^t$. Thus there is a cyclic group of order t permuting the orbits of G as well as their corresponding sets of cross ratios. To show that (G, λ) is not closed for any proper subgroup G of $\mathrm{P\Gamma L}(2,q)$ containing $\mathrm{PGL}(2,q)$, it suffices to find a set of six cross-ratios such that the cyclic group of order t generated by the Frobenius map acts regularly on the orbit containing this set, since different subgroups of a regular group have different orbits. Hence we have to find such a set which is fixed only by the identity element of this cyclic group.

Suppose that we have a 6-set $\{z, 1/z, 1-z, 1/(1-z), z/(z-1), (z-1)/z\}$ which is fixed by a non-trivial power of the Frobenius map; we can assume that this has the form $x \mapsto x^{p^u}$ where u divides t. We find all pairs (z, u) for which this holds, and show that for $q > 9$ there exists z with the required property. We put $t = uv$ and $p^u = r$, so that $q = r^v$ and the map under consideration has fixed field $\mathrm{GF}(r)$. Now, for every z,

$$z^r \in \{z, 1/z, 1-z, 1/(1-z), z/(z-1), (z-1)/z\}.$$

There are six possibilities:

- $z^r = z$. Then $z \in \mathrm{GF}(r)$.
- $z^r \in \{1/z, 1-z, z/(z-1)\}$. In each of these cases, we find that $z^{r^2} = z$, so $z \in \mathrm{GF}(r^2)$. So we may assume that $q = r^2$.
- $z^r \in \{1/(1-z), (z-1)/z\}$. In these cases, we find that $z^{r^3} = z$; so we may assume that $q = r^3$.

Only one of these possibilities can hold. We may assume that $q \neq r$, so that z can be chosen so that the first possibility does not hold.

Suppose that $q = r^2$. Now the above argument shows that the $r^2 - r$ elements outside $\mathrm{GF}(r)$ satisfy one of the three equations $z^r = 1/z$, $z^r = 1 - z$, or $z^r = z/(z-1)$. These are polynomials of degrees $r+1$, r, $r+1$ respectively; so $3r + 2 \geq r^2 - r$, giving $r \leq 4$. Now $\mathrm{PGL}(2,4) \cong A_5$ falls under case (a); and computation shows that $(\mathrm{PGL}(2,9), \lambda)$ is closed but $(\mathrm{PGL}(2,16), \lambda)$ and $(\mathrm{PGL}(2,16):2, \lambda)$ are not. (Both the last two groups have three orbits on 4-sets, but $\mathrm{PGL}(2,16):4$ has only two.)

Now suppose that $q = r^3$. We argue similarly to say that the elements outside $\mathrm{GF}(r)$ satisfy one of the two equations $z^r = 1/(1-z)$ or $z^r = (z-1)/z$, both polynomials of degree $r+1$. Thus $r^3 - r \leq 2(r+1)$, with only the solution $r = 2$. The pair $(\mathrm{PGL}(2,8), \lambda)$ is closed, since $\mathrm{PGL}(2,8)$ is 4-homogeneous.

For the other two partition types, the argument is less elegant. We will next consider type $\lambda = (3, 2, 1, \ldots, 1)$.

In this case, each orbit has a representative in which the 3-set is $\{\infty, 0, 1\}$, by 3-transitivity of G. The elements of $\mathrm{PGL}(2, q)$ which map this set to itself are the maps $z \mapsto f(z)$, where $f(z)$ is one of the six linear fractional expressions which came up in our discussion of cross ratio. Moreover, all three points are fixed by the Frobenius map.

Suppose that p is odd. Take $x \in \mathrm{GF}(p) \setminus \{0, 1\}$ and y in no proper subfield of $\mathrm{GF}(q)$, and consider the partition as above whose 2-set is $\{x, y\}$.

The points fixed by the above transformations are $x = -1$, $x = 2$, $x = \frac{1}{2}$, and x a primitive 6th root of 1. If $p \neq 3$ or 7 we can choose x to satisfy none of these, so there is only one partition containing parts $\{\infty, 0, 1\}$ and $\{x, y'\}$ in the orbit. But if we choose y to be a primitive element, then it is not fixed by any power of the Frobenius map, so this set is in a regular orbit of this map.

If $p = 3$, then $x = 2$ is fixed by three maps $z \mapsto 1/z$, $z \mapsto 1 - z$, and $z \mapsto z/(z-1)$. So if the orbit is fixed by the Frobenius map, then y must satisfy $y^r \in \{1/y, 1-y, y/(y-1)\}$. There are at most $3r+3$ such elements. So $r^v - r \leq 3r - 3$, whence $r = 2$. But the computer establishes that not all $\mathrm{PGL}(2, 9)$-orbits are fixed by $\mathrm{P\Gamma L}(2, 9)$.

Suppose that $p = 7$. A similar but easier argument applies, since each of 2, 4 and 6 is fixed by just a single element, so we find $r^v - r \leq r + 1$, which is impossible.

Lastly we have the case $p = 2$. We may assume that $t > 2$, since $\mathrm{PGL}(2, 4) \cong A_5$. Now if $y \neq 1/x, 1-x, x/(x-1)$, then only the identity in $\mathrm{PGL}(2, q)$ fixes this partition. Choosing y to be a primitive element of $\mathrm{GF}(q)$ shows that the group generated by the Frobenius map acts regularly on the orbit of this partition.

We now consider partitions of type $(2, 2, 2, 1, \ldots, 1)$. We can assume that an orbit we are considering contains the partition $\{\infty, 0\}$, $\{1, a\}$ and $\{b, c\}$.

Suppose first that $p > 2$, and take $a = 2$. The three linear fractional transformations fixing the pair of sets making up the first two cycles are $z \mapsto 2/z$, $z \mapsto (z-2)/(z-1)$, and $z \mapsto 2(z-1)/(z-2)$, have among them at most six fixed points, namely $\pm\sqrt{2}$, $1 \pm \sqrt{-1}$, and $2 \pm \sqrt{-2}$; so there is a point a fixed by none of these. If we take b and c to be linearly independent over $\mathrm{GF}(p)$, then only the identity in $\mathrm{PGL}(2, q)$ fixes the three sets; and if we take $b \neq c^p$, then we find that they are not fixed by any power of the Frobenius map.

If $p = 2$, the argument is similar. If t is even, then we have a subfield GF(4); if t is divisible by 3, a subfield GF(8). If neither of these occurs, then no field automorphism can fix a partition of type λ, since only permutations of order dividing 48 can do so.

It remains to consider groups not containing $\mathrm{PGL}(2, q)$. If q is not a square, then any subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$ but not $\mathrm{PGL}(2, q)$ must lie inside $\mathrm{P\Sigma L}(2, q)$. Moreover, we may assume that $G = \mathrm{P\Sigma L}(2, q)$, since any other subgroup is contained in a group twice as large which does itself contain $\mathrm{PGL}(2, q)$. This case only arises if $q \equiv 3 \pmod 4$, since otherwise $\mathrm{P\Sigma L}(2, q)$ is not 3-homogeneous.

If the $\mathrm{P\Sigma L}(2, q)$-orbit of a partition P is fixed by $\mathrm{P\Gamma L}(2, q)$, then a partition in that orbit must be fixed by an element of $\mathrm{P\Gamma L}(2, q) \setminus \mathrm{P\Sigma L}(2, q)$, since then its stabiliser will be twice as large, and the orbit the same size. We can assume that such an element has 2-power order, and all its cycles have the same size (since $\mathrm{P\Sigma L}(2, q)$ has odd order). This excludes type $(3, 2, 1, \ldots, 1)$, so we have to consider the other two types. Moreover, q is an odd power of p, so these maps do not involve field automorphisms.

First consider type $(4, 1, \ldots, 1)$, so we are looking for an element fixing a 4-set, acting on it as either a double transposition or a 4-cycle. By 3-homogeneity, we can consider 4-sets of the form $\{\infty, 0, 1, a\}$. For a double transposition, there are three possibilities:

- $(\infty, 0)(1, a)$: $z \mapsto a/z$ does this. Its determinant is $-a$, which is a nonsquare if and only if a is a square.
- $(\infty, 1)(0, a)$: $z \mapsto (z - a)/(z - 1)$ does this. Its determinant is $-1 + a$, which is a nonsquare if and only if $1 - a$ is a square.
- $(\infty, a)(0, 1)$: $z \mapsto (az - a)/(z - a)$ does this. Its determinant is $-a^2 + a$, which is a nonsquare if and only if $a(a-1)$ is a square.

Now the product of these three numbers is $-a^2(a - 1)^2$, which is a nonsquare. So 0 or 2 of them are squares for every a. Indeed, it is well-known (from the construction of the Paley design) that there are $(q + 1)/4$ elements a for which a and $1 - a$ are both non-squares. Now we consider 4-cycles. Up to inversion, there are three possibilities:

- $(\infty, 0, 1, a)$: $z \mapsto 1/(cz+1)$, where $1/(c+1) = a$ and $ac+1 = 0$; these equations have a unique solution $a = 2$.
- $(\infty, 0, a, 1)$: $z \mapsto a/(cz+1)$, where $a/(ac+1) = 1$ and $c+1 = 0$; the solution is $a = \frac{1}{2}$.
- $(\infty, 1, 0, a)$: $z \mapsto (z-1)/(z+c)$, where $-1/c = a$ and $a+c = 0$; the solution is $a = -1$.

So, if every orbit is accounted for, we have $(q+1)/4 \leq 3$, so $q = 7$ or $q = 11$. It can be checked (by hand or by computer) that (PSL, $(2,7), \lambda)$ for type $\lambda = (4,1,\ldots,1)$ is not closed, but $(\mathrm{PSL}(2,11), \lambda)$ is.

Now consider the type $(2,2,2,1,\ldots,1)$. This time we can assume that the partitioned 6-set is $\{\{\infty,0\},\{1,a\},\{b,c\}\}$, and it is fixed by an involution, which fixes one or all of the 2-sets. If there is a cycle $(\infty,0)$, then 1 maps to a, b or c, and we find that the product of any two of a,b,c is the third. (For example, if $(1,a)$ is a cycle, then the map is $z \mapsto a/z$, and $a/b = c$.)

In the remaining case, we have four triple transpositions to consider, namely $(\infty,1)(0,a)(b,c)$, $(\infty,a)(0,1)(b,c)$, $(\infty,b)(0,c)(1,a)$, or $(\infty,c)(0,b)(1,a)$. The third and fourth are equivalent under interchange of b and c. We have:

- $(\infty,1)(0,a)(b,c)$: we find $1-a = (1-b)(1-c)$.
- $(\infty,a)(0,1)(b,c)$: we find $(a-b)(a-c) = a(1+a)$.
- $(\infty,b)(0,c)(1,a)$: we find $a(b-c) = a-b$.

In each case, given a and b, there is only one choice of c, so $q \leq 5$, a contradiction.

The remaining class to be considered are the groups $\mathrm{PXL}(2,q)$. As mentioned earlier, we have checked by computer the odd prime power squares up to 121, and found that $\mathrm{PXL}(2,q)$ acting on $(4,1,\ldots,1)$ partitions is closed for $q = 9$, 25 and 49, but not for $q = 81$ or 121. □

4. Groups having only one orbit in a given kernel type

The remainder of this paper is dedicated to the application to semigroup theory of the results found above. We want to describe the structure (elements, ranks, automorphisms, congruences, regularity, idempotent generation, etc.) of semigroups generated by a k-homogenous subgroup of S_n and some singular maps of rank larger than $n/2$. We will use several times the well known fact ([50, p.11]) that if S is a finite semigroup and $a \in S$, then there exists a natural number ω such that a^ω is idempotent.

In this section we are going to study the semigroups generated by a singular transformation t, such that $\mathrm{rank}(t) \geq n/2$, and a permutation group that has only one orbit on the kernel type of t.

We start by noting the following. Suppose that the kernel of t has type $(l_1,\ldots,l_k)$ and m is the largest natural such that $l_m > 1$. Then G must be $(\sum_{i=1}^m l_i)$-homogeneous and hence, given that the rank of t is k, the group must be p-homogeneous, for some $p \in \{n-k+1,\ldots,2(n-k)\}$. The smallest value of p is attained if the kernel type is $(n-k+1,1,\ldots,1)$, and the largest value is attained for kernel type $(2, \dots, 2, 1, \dots, 1)$. Therefore, for any practical considerations we might assume that our groups are $(n-k+1)$-homogeneous.

Theorem 4.1. *Let t be a singular map in $T(X)$, with $X = \{1, \dots, n\}$, and suppose that t has kernel type $(l_1, \dots, l_k)$, with $k \geq n/2$; let G be a group having only one orbit in the partitions of that type. Let E denote the set of idempotents of $\langle t, G\rangle \setminus G$. Then*

$$\langle t, G\rangle \setminus G = \langle t, S_n\rangle \setminus S_n = \langle E, t\rangle.$$

The proof of this theorem will follow from a sequence of lemmas. Throughout this section $t \in T_n$ will be a rank k map of kernel type $(l_1, \dots, l_k)$, and $G \leq S_n$ will be a $(n-k+1)$-homogeneous group having only one orbit on the partitions of type $(l_1, \dots, l_k)$.

We now introduce some notation. Given the rank and the kernel type of t we have

$$t = \begin{pmatrix} A_1 & \cdots & A_k \\ a_1 & \cdots & a_k \end{pmatrix},$$

where $|A_i| = l_i$ (for all $i \in \{1, \dots, k\}$).

Throughout this section we will assume that the fixed map t has kernel $T = (A_1, \dots, A_k)$ of type $(l_1, \dots, l_k)$.

Observe that for every $g, h \in G$ we have

$$g^{-1}th = \begin{pmatrix} A_1 g & \cdots & A_k g \\ a_1 h & \cdots & a_k h \end{pmatrix}.$$

Since $k \geq n/2$ and the group is $(n-k+1)$-homogeneous, it follows that the group is also k-homogeneous. Thus given any k-set Y contained in X, there exists $th \in \langle G, t\rangle$ such that $Xth = Y$. Similarly, given any partition $Q = (B_1, \dots, B_k)$ of X of type $(l_1, \dots, l_k)$, since G has only one orbit on the partitions of this type, it follows that there exists $g \in G$ such that $\{A_1, \dots, A_k\}g = \{B_1, \dots, B_k\}$ and hence the kernel of $g^{-1}t$ is $(B_1, \dots, B_k)$. This proves the following lemma.

Lemma 4.2. *Given any partition Q of type $(l_1, \dots, l_k)$ and any k-set $Y \subseteq X$, there exist $g, h \in G$ such that $\ker(g^{-1}th) = Q$ and $Xg^{-1}th = Y$.*

The previous result shows that $\langle G, t\rangle$ has rank k maps of every possible image and kernel. The next result provides the anlogous result for idempotents.

Lemma 4.3. *Given any partition Q of type $(l_1, \dots, l_k)$ and any k-set $Y \subseteq X$ such that Y is a transversal for Q, there exists an idempotent $e \in \langle t, G\rangle$ such that $\ker(e) = Q$ and $Xe = Y$.*

Proof. By the previous lemma we know that there exist $g, h \in G$ such that $\ker(g^{-1}th) = Q$ and $Xg^{-1}th = Y$. Since Y is a transversal for Q, it follows that there exists $k \in G$, namely $k := hg^{-1}$, such that $\mathrm{rank}(tkt) = \mathrm{rank}(t)$. Therefore, every element in $\langle tk \rangle := \{(tk)^i \mid i \in \mathbb{N}\}$, the monogenic semigroup generated by tk, has the same rank as t. Since every finite semigroup contains an idempotent, we conclude that $\langle tk \rangle$ contains an idempotent, say $(tk)^\omega$, and hence $g^{-1}(tk)^\omega g = g^{-1}(thg^{-1})^\omega g$ is also an idempotent with kernel Q and image Y. The result follows. ☐

In order to increase the readability of the arguments we introduce some notation. Given a partition $P = (A_1, \ldots, A_k)$ of X, and a transversal $S = \{a_1, \ldots, a_k\}$ for P, where $a_i \in A_i$ (for all i), we represent A_i by $[a_i]_P$ and the pair (P, S) induces an idempotent mapping defined by $[a_i]_P e = \{a_i\}$. Conversely, every idempotent can be so constructed from a partition and a transversal. With this notation, we can write the idempotent

$$e = \begin{pmatrix} [a_1]_P & \ldots & [a_k]_P \\ a_1 & \ldots & a_k \end{pmatrix}$$

in the more compact form $e = ([a_1]_P, \ldots, [a_k]_P)$. This notation extends to $e = ([\underline{a_1}, b]_P, [a_2]_P, \ldots, [a_k]_P)$ when $b \in [a_1]_P$ and $[a_i]_P e = \{a_i\}$. By $([a_1], \ldots, [\underline{a_i}, b], \ldots, [a_k])$ we denote the set of all idempotents $e \in T_n$ with image $\{a_1, \ldots, a_k\}$ and such that the $\ker(e)$-class of a_i contains (at least) two elements: a_i and b, where the underlined element (in this case a_i) is the image of the class under e.

Lemma 4.4. *Let $q_1, q_2 \in \langle G, t \rangle$ be two maps of rank k such that $Xq_1 = \{b_1, \ldots, b_k\}$ and $Xq_2 = \{b_2, \ldots, b_{k+1}\}$. Then there exists an idempotent $e \in \langle G, t \rangle$ such that $Xq_1 e = Xq_2$.*

Proof. By the previous lemma, given any partition of the same type as the kernel of t, and any transversal for it, there exists in $\langle t, G \rangle$ an idempotent with that partition as kernel and that transversal as image. Therefore we can pick a partition of the same type as the kernel of t with the following parts: $Q_0 = \{\{b_1, b_{k+1}, \ldots\}, \{b_2, \ldots\}, \ldots, \{b_k, \ldots\}\}$; it is clear that Xq_2 is a transversal for Q_0 and hence the idempotent $e = [[b_1, \underline{b_{k+1}}]_{Q_0}, [b_2]_{Q_0}, \ldots, [b_k]_{Q_0}]$ satisfies the desired $Xq_1 e = Xq_2$. ☐

Since given any two k-sets $Y, Z \subseteq X$, there exists a sequence of k-subsets of X, say $(Y_1, \ldots, Y_m)$, such that $|Y_i \cap Y_{i+1}| = k-1$, with $Y_1 = Y$ and $Y_m = Z$, the following result is a consequence of the application of the previous lemma as many times as needed.

Corollary 4.5. *Let $q_1, q_2 \in \langle G, t\rangle$ be two rank k maps. Then there exists a sequence of idempotents $e_1, \ldots, e_j$ such that $Xq_1e_1 \ldots e_j = Xq_2$.*

So far we showed that it is possible to use idempotents to build maps with any given kernel of the same type as the kernel of t, and as image any k-set. Now we move a step forward.

Lemma 4.6. *Let p be a map of rank k and $x, y \in Xp = \{p_1, \ldots, p_k\}$ with $x \neq y$. Denote by (xy) the transposition induced by x and y. Then there exist idempotents $e_1, e_2, e_3 \in \langle G, t\rangle$ such that $p(xy) = pe_1e_2e_3$.*

Proof. Since p is non-invertible, there exists $c \in X \setminus Xp$. By Lemma 4.3, $\langle G, t\rangle$ intersects the following sets of idempotents:

$$\begin{aligned} A &= ([p_1], \ldots, [x, \underline{c}], \ldots, [y], \ldots, [p_k]) \\ B &= ([p_1], \ldots, [y, \underline{x}], \ldots, [c], \ldots, [p_k]) \\ C &= ([p_1], \ldots, [c, \underline{y}], \ldots, [x], \ldots, [p_k]). \end{aligned}$$

Taking $e_1 \in A$, $e_2 \in B$, and $e_3 \in C$, all from $\langle G, t\rangle$, we get the desired composition $p(xy) = pe_1e_2e_3$. □

Now we can prove Theorem 4.1.

Proof. To prove the theorem, observe that $\langle t, S_n\rangle \setminus S_n$ is generated by the set $\{gth \mid g, h \in S_n\}$. If $gth \in \langle t, G\rangle$, for all $g, h \in S_n$, the result would follow.

Let Q be the partition induced by the kernel of gth and let S_1 be a transversal for Q. Since Q has the same kernel type as $\ker(t)$ it follows, by Lemma 4.3, that there exists an idempotent $e \in \langle G, t\rangle$ such that $Xe = S_1$ and the kernel of e is Q. Let S be a transversal for the kernel of t. By Corollary 4.5, there exists a sequence of idempotents such that $Xee_1 \ldots e_i = S$. Thus the map $ee_1 \ldots e_it$ has the same rank as t, and the same kernel as gth. Similarly, there are idempotents $f_1, \ldots, f_l$ such that $Xee_1 \ldots e_itf_1 \ldots f_l = Xgth$. Thus, there exists a permutation σ of the set $Xgth$ such that $ee_1 \ldots e_itf_1 \ldots f_l\sigma = gth$. Therefore,

$$gth = ee_1 \ldots e_itf_1 \ldots f_l\sigma = ee_1 \ldots e_itf_1 \ldots f_l(x_1y_1) \ldots (x_my_m),$$

and each of these transpositions can be replaced by a product of three idempotents of $\langle t, G\rangle$. This also proves that $\langle t, G\rangle \setminus G \subseteq \langle E, t\rangle$; as the converse inclusion is obvious, the result follows. □

We recall here some known facts about the semigroups $\langle t, S_n\rangle$.

Theorem 4.7. *Let $a \in T_n$ be singular and let $S = \langle a, S_n\rangle \setminus S_n$. Let $\Omega := \{1, \ldots, n\}$. Then*

(a) $S = \{b \in \mathcal{T}_n \mid (\exists g \in S_n)\ \ker(a)g \subseteq \ker(b)\}$;

(b) S is regular, that is, for all $a \in S$ there exists $b \in S$ such that $a = aba$;
(c) S is generated by its idempotents;
(d) S and $\langle g^{-1}ag \mid g \in S_n\rangle$ have the same idempotents;
(e) $S = \langle g^{-1}ag \mid g \in S_n\rangle$;
(f) the automorphisms of $\langle a, S_n\rangle$ are those induced under conjugation by the elements of the normalizer of S in $\mathcal{S}_n$,

$$Aut(\langle a, S_n\rangle) \cong N_{\mathcal{S}_n}(S);$$

(g) we also have $Aut(\langle a, S_n\rangle) \cong S_n$;
(h) all the congruences of $\langle a, S_n\rangle$ are described;
(i) if $e^2 = e \in \langle a, S_n\rangle$, $r := \mathrm{rank}(e)$, then

$$\{f \in \langle a, S_n\rangle \mid \ker(f) = \ker(e) \textit{ and } \Omega f = \Omega e\} \cong S_r.$$

(j) regarding principal ideals and Green's relations, for all $a, b \in S$, we have

$$\begin{aligned} aS = bS \quad &\Leftrightarrow \quad \ker(a) = \ker(b) \\ Sa = Sb \quad &\Leftrightarrow \quad \Omega a = \Omega b \\ SaS = SbS \quad &\Leftrightarrow \quad \mathrm{rank}(a) = \mathrm{rank}(b) \end{aligned}$$

(k) the minimum size of a generating set for $\langle a, S_n\rangle$, for $a \in T_n \setminus S_n$, is 3.
(l) the minimum size of a set A of rank k maps such that $\langle A, S_n\rangle$ generates all maps of rank at most k is $p(k)$.

Proof. Equality (a) was proved by Symons in [92]. Claims (b), (c) and (e) were proved by Levi and McFadden in [65]. Claim (d) was proved by McAlister in [80], and (together with (c)) it also implies (e).

Claim (f) follows from the general result that every automorphism of a semigroup $S \leq T_n$ containing all the constants is induced under conjugation by the normalizer of S in S_n (see [90] and also [18, 19]); since, by (a), the semigroups $\langle S_n, a\rangle$ contain all the constants, the result follows. Claim (g) was proved by Symons in [92], but is also an easy consequence from (f). In [62] Levi described all the congruences of an S_n-normal semigroup and hence described the congruences in S. Thus (h).

Statement (i) belongs to the folklore (see Theorem 5.1.4 of [43]). The results about principal ideals (j) were proved by Levi and McFadden in [65].

Claim (k) follows from the fact that S_n is generated by two elements.

Regarding (l), observe that given any rank k map t we have that all rank k maps in $\langle t, S_n\rangle$ have the same kernel type as t; conversely, every rank k map of the same kernel type of p belongs to $\langle t, S_n\rangle$. Therefore, to generate all maps of rank k a necessary and sufficient condition is that there is in A one map of each kernel type, so that $|A| = p(n)$. It is well known that the maps of rank k, for $k > 1$, generate all maps of smaller ranks. The result follows. $\square$

The previous results immediately imply the following.

Theorem 4.8. *Let t be a singular map in T_n, the full transformation monoid on $\Omega := \{1, \dots, n\}$, and suppose that t has kernel type $(l_1, \dots, l_k)$, with $k \geq n/2$; let G be a group having only one orbit in the partitions of that type. Let $S = \langle t, G\rangle \setminus G$. Then*

- *(a) $S = \{b \in \mathcal{T}_n \mid (\exists g \in S_n)\ \ker(a)g \subseteq \ker(b)\}$;*
- *(b) S is regular, that is, for all $a \in S$ there exists $b \in S$ such that $a = aba$;*
- *(c) S is generated by its idempotents;*
- *(d) S and $\langle g^{-1}ag \mid g \in G\rangle$ have the same idempotents;*
- *(e) $S = \langle g^{-1}ag \mid g \in G\rangle$;*
- *(f) the automorphisms of $\langle a, G\rangle$ are those induced under conjugation by the elements of the normalizer of G in S_n,*
$$Aut(\langle a, G\rangle) \cong N_{S_n}(G);$$
- *(g) all the congruences of S are described;*
- *(h) if $e^2 = e \in \langle a, G\rangle$, $r := \operatorname{rank}(e)$, then*
$$\{f \in \langle a, G\rangle \mid \ker(f) = \ker(e) \text{ and } \Omega f = \Omega e\} \cong \mathcal{S}_r.$$
- *(i) regarding principal ideals and Green's relations, for all $a, b \in S$, we have*
$$\begin{aligned} aS = bS &\Leftrightarrow \ker(a) = \ker(b) \\ Sa = Sb &\Leftrightarrow \Omega a = \Omega b \\ SaS = SbS &\Leftrightarrow \operatorname{rank}(a) = \operatorname{rank}(b) \end{aligned}$$
- *(j) the minimum size of a generating set for $\langle a, G\rangle$ is 3.*
- *(k) let A be a set of rank k maps such that $\langle A, G\rangle$ generates all maps of rank at most k and A has minimum size among the sets with that property. A bound for the size of the sets $A \subseteq T_n$ such that $\langle G, A\rangle$ generate all maps of rank k is given in Table 8. In the middle column is the type on which G has only one orbit.*

Proof. Claims (a)–(e) and (g)–(i) all follow from the previous theorem and Theorem 4.1.

Rank	Kernel type	$\lvert A\rvert$
$n-1$	$(2,1,\ldots,1)$	1
$n-2$	$(2,2,1,\ldots,1)$	2
	$(3,1,\ldots,1)$	$O(n)$
$n-3$	$(4,1,\ldots,1)$	144
	$(3,2,1,\ldots,1)$	5
	$(2,2,2,1,\ldots,1)$	3
$n-4$	$(5,1,\ldots,1)$	15
	other	5
k $(n/2 \leq k \leq n-5)$	any	$p(k)$

TABLE 8. Generating all maps of rank k

Regarding claim (f), observe that by (a) the semigroup $\langle a, G\rangle$ contains all the constant maps and hence, by [90, Theorem 1], its automorphisms are

$$\{\tau^g \mid g \in S_n \wedge g^{-1}\langle a, G\rangle g = \langle a, G\rangle\},$$

where, for a given $g \in S_n$, we have $\tau^g : \langle a, G\rangle \to \langle a, G\rangle$ defined by $f\tau^g = g^{-1}fg$. Note that a permutation $g \in N_{S_n}(G)$ normalizes $\langle a, G\rangle$ if and only if it normalizes G and $\langle a, G\rangle \setminus G$. Thus the automorphisms of $\langle a, G\rangle$ are the maps induced under conjugation by the elements in the normalizer $N_{S_n}(G)$ that also normalize $\langle a, G\rangle \setminus G$. Since $\langle a, G\rangle \setminus G = \langle a, S_n\rangle \setminus S_n$ it follows that every permutation of S_n normalizes $\langle a, G\rangle \setminus G$. We conclude that the automorphisms of $\langle a, G\rangle$ are all the maps

$$\{\tau^g : \langle a, G\rangle \to \langle a, G\rangle \mid g \in N_{S_n}(G)\}.$$

To prove that in fact we have $\mathrm{Aut}\,(\langle a, G\rangle \setminus G) \cong N_{S_n}(G)$ we only need to observe that a primitive group (other than a cyclic group of prime order) have trivial center (that is, only the identity in G commutes with all other elements of G).

Regarding (j), observe that every 2-homogeneous group is 2-generated (Corollary 2.3).

Finally, (k), follows from the results in the previous section, with a little care. For example, a permutation group transitive on partitions of type $(2, 2, 1, \ldots, 1)$ is 4-homogeneous, and so 3-homogeneous; so it is transitive on $(3, 1, \ldots, 1)$ partitions also. A group transitive on $(3, 2, 1, \ldots, 1)$ partitions is 5-homogeneous, and so symmetric, alternating or a Mathieu group; we refer to Table 6 for the Mathieu groups (which are transitive on partitions of this type because they are 5-transitive). ☐

5. Groups with only one orbit on the image

We turn now to semigroups $\langle t, G\rangle$, where G is transitive on the image of t (that is, G is $(n-k)$-homogeneous, where $k \geq n/2$ is the rank of t).

Theorem 5.1. *Let G be a primitive group with just one orbit on $(n-k)$-sets, where $1 \leq k \leq n/2$. Let $t \in T_n$ be a map of rank $n-k$. Then*

(a) $\langle G, t\rangle \setminus G$ and $\langle g^{-1}tg \mid g \in G\rangle$ have the same idempotents;
(b) $\mathrm{Aut}\,(\langle G, t\rangle) \cong N_{S_n}(\langle G, t\rangle)$.
(c) For $k \geq 3$, the list of 3-homogeneous groups that satisfy

$$\mathrm{Aut}\,(\langle G, t\rangle) \cong N_{S_n}(G)$$

is the following:

(i) $G = N_{S_n}(G)$, that is,
- S_n.
- $\mathrm{P\Gamma L}(2,q)$ *for* $k=3$.
- $\mathrm{AGL}(d,2)$ *for* $k=3$.
- $\mathrm{A\Gamma L}(1,8)$, M_{11} *(*$k=4$*)*, M_{11} *(degree* 12, $k=3$*)*, M_{12} *(*$k=5$*)*, $2^4 : A_7$, $M_{22} : 2$ *(*$k=3$*)*, M_{23} *(*$k=4$*)*, M_{24} *(*$k=5$*), and* $\mathrm{A\Gamma L}(1,32)$ *(*$k=4$*).*

(ii) $G = A_n$;
(iii) $G = \mathrm{AGL}(1,8)$, $\mathrm{PGL}(2,8)$, $\mathrm{PGL}(2,9)$, M_{10}, $\mathrm{PSL}(2,11)$, M_{22}, $\mathrm{PXL}(2,25)$, or $\mathrm{PXL}(2,49)$, with $k=3$, and t of type $\lambda = (4,1,\ldots,1)$.

The list is complete with the possible exception of the groups $\mathrm{PXL}(2,q)$ for $q \geq 169$.

(d) Let $A \subseteq T_n$ be a set of rank $n-k$ maps such that $\langle A, G\rangle$ generates all maps of rank at most $n-k$ and A has minimum size among the subsets of T_n with that property. Then the maximum sizes that A can have are given in Table 9.

For more precise values depending on the group chosen, see the tables in Section 2 of the paper.

Proof. McAlister [80] proved that for any group $G \leq S_n$ and any transformation $a \in T_n$, the semigroups $\langle a, G\rangle \setminus G$ and $\langle g^{-1}ag \mid g \in G\rangle$ have the same idempotents. This proves (a).

A transitive group G is said to synchronize a map t if the semigroup $\langle G, t\rangle$ contains a constant map (and hence, by transitivity, all constant maps). It is proved that primitive groups synchronize every singular

Rank $n-k$	$\lvert A\rvert$	Sample k-homogeneous groups attaining the bound for $\lvert A\rvert$	Minimum number of generators for a primitive k-homogeneous group
$n-1$	$\frac{(n-1)}{2}$	C_p, D_p (n odd prime)	$\frac{C\log n}{\sqrt{\log\log n}}$
$n-2$	$O(n^2)$	Example 2.1	2
$n-3$	$O(n^3)$	$\mathrm{PSL}(2,q), \mathrm{P\Gamma L}(2,q)$	2
$n-4$	12160	$\mathrm{P\Gamma L}(2,32)$ ($n=33$)	2
$n-5$	77	M_{24} $\big(n=24\big)$	2
$n-k$ ($k\geq 5$)	$p(k)$	S_n, A_n	2

TABLE 9. Smallest number of rank $n-k$ maps needed to together with a k-homogeneous group G generate all the maps of rank at most $n-k$.

map of rank at least $n-4$ (see [5, 9, 88]). It is also known that 2-homogeneous groups, together with any singular map, generate all the constant maps ([12, 80]). Therefore, under the assumptions of the theorem, if the primitive group G has only one orbit on the k-sets, for $n > k \geq n/2$, then G together with any rank k map t generates all the constants and hence the automorphisms of $S := \langle t, G\rangle$ are induced under conjugation by the elements in $N_{S_n}(S)$. This implies (b).

The more detailed description included in (c) follows from Theorem 3.2.

Regarding (d), we start by observing that all maps in $\langle G, t\rangle$ having the same rank as t, have also the same kernel type as t. Therefore, to generate all rank $n-k$ maps with G and a set A of rank $n-k$ maps, A must contain maps whose kernels form a transversal of the orbits of G on each kernel type. This necessary condition turns out to be sufficient for $\langle G, A\rangle$ to generate all transformations of rank at most $n-k$. In fact, given any $(n-k)$-partition P and any transversal S for P, there exists $p \in A$ and $g \in G$ such that $P = \ker(gp)$. In addition, since G has only one orbit on the k-sets, it follows that there exists $h \in G$ such that the image of gph is S. We infer that $\mathrm{rank}(phgp) = \mathrm{rank}(p)$; thus, every element in $\langle phg\rangle$ has the same rank of p and, for some natural number ω, we have that $(phg)^\omega$ is idempotent and the same holds for $e := g(phg)^\omega g^{-1}$. In addition, $\ker(e) = P$ and the image of e coincides with the image of ph which is S. Since P and S were arbitrary, it follows that $\langle A, G\rangle$ contains all rank $n-k$ idempotents of T_n. It is well known ([2]) that the rank $n-k$ idempotents generate all maps of rank at most $n-k$ and hence the result follows. □

6. On normalizers of 2-homogeneous groups

By the main theorems of the two previous sections, to compute the automorphisms of $\langle G, t\rangle$ (with G and t under the assumptions of the theorems) it is necessary to know the normalizer of $\langle G, t\rangle$ in S_n, which is contained in the normalizer of G. Therefore we provide here the normalizers of 2-homogeneous groups.

According to a theorem of Burnside, a 2-transitive group G has a unique minimal normal subgroup T, which is either elementary abelian or simple non-abelian. (If G is 2-homogeneous but not 2-transitive, it also has a unique minimal normal subgroup, which is elementary abelian.) Thus $N_{S_n}(G) \leq N_{S_n}(T)$. So, to describe the normalizers of the 2-homogeneous groups G, we only need to look within the group $N_{S_n}(T)/T$. Table 10 gives the structure of this quotient in the case when T is simple. In the table, $G(r,s,p)$ denotes the group $\langle a, b \mid a^r = b^s = 1, b^{-1}ab = a^p\rangle$. In all rows of the table except the second and fourth, $N_{S_n}(G) = N_{S_n}(T)$. In the second and fourth rows, we have $N_{S_n}(G)/T \cong N_{N(T)/T}(G/T)$, and this quotient is computed in the metacyclic group $G(r,s,p)$.

Note that there are a few small exceptions: $\mathrm{PSL}(2,2)$, $\mathrm{PSL}(2,3)$, $\mathrm{PSU}(3,2)$, $\mathrm{Sz}(2)$, $\mathrm{Sp}(4,2)$, and $R_1(3)$ are not simple. The first four of these are solvable; the fifth has a simple subgroup of index 2 isomorphic to A_6; and the last has a simple subgroup of index 3 isomorphic to $\mathrm{PSL}(2,8)$.

Now we consider the 2-homogeneous affine groups. In each case, the classification gives a subgroup H (not necessarily 2-homogeneous) which must be contained in G. The group H contains the translation group T of G, so $G = TG_0$ and $H = TH_0$. Thus, as in the other case, we have $N_{S_n}(G) \leq N_{S_n}(H)$, so again we have to compute the normalizer within the group $N_{S_n}(H)/H \cong N_{S_{n-1}}(H_0)/H_0$. Table 11 gives the structure of this quotient group. The groups $G(r,s,p)$ are the same as defined earlier. In all cases not shown in the table, the quotient is abelian, and so the normalizers of H and T coincide, and we have not listed them explicitly.

We have not attempted to make a similar classification of normalizers of primitive groups, since this problem is as difficult as finding normalizers of arbitrary transitive groups, as the following example shows.

Let $m \geq 3$, and let K be a transitive group of degree k. Let G be the wreath product $S_m \wr K$ in its power action of degree m^k. Then G is primitive, and its normalizer in the symmetric group of degree m^k is $S_m \wr N_{S_k}(K)$.

T	Degree	$N(T)/T$	Condition
A_n	n	C_2	
$\mathrm{PSL}(d,q)$	$(q^d-1)/(q-1)$	$G(r,s,p)$	$q=p^s$, p prime, $r=\gcd(q-1,d)$
$\mathrm{Sp}(2d,2)$	$2^{2d-1}\pm 2^{d-1}$	1	
$\mathrm{PSU}(3,q)$	q^3-1	$G(r,s,p)$	$q=p^s$, p prime, $r=\gcd(q+1,3)$
$\mathrm{Sz}(q)$	q^2+1	C_{2e+1}	$q=2^{2e+1}$
$R_1(q)$	q^3+1	C_{2e+1}	$q=3^{2e+1}$
M_{11}	11	1	
M_{11}	12	1	
M_{12}	12	1	
A_7	15	1	
M_{22}	22	C_2	
M_{23}	23	1	
M_{24}	24	1	
HS	176	1	
Co_3	276	1	

TABLE 10. Normalizers of almost simple 2-transitive groups

Degree	H_0	$N(H_0)/H_0$	Condition
q^n	$\mathrm{SL}(n,q)$	$G(r,s,p)$	$q=p^s$, $r=q-1$
q	$C_{(q-1)/2}$	C_{2s}	$q=p^s$ odd
q^{2n}	$\mathrm{Sp}(2n,q)$	$G(r,s,p)$	$q=p^s$, $r=q-1$

TABLE 11. Normalizers of affine groups

However, it is known (see [45]) that, if G is a primitive permutation group of degree n, the order of $N_{S_n}(G)/G$ is smaller than n with finitely many exceptions. (Note that, in the above example, the degree of the primitive group is exponential in the degree of the transitive group.)

7. Automorphisms of transformation semigroups with primitive group of units

Very early in this investigation we arrived at the conjecture that if a transformation semigroup has singular maps and its group of units is a primitive permutation group G, then all its automorphisms are induced under conjugation by the elements in the normalizer of G, in the symmetric group. We invested a long time trying to prove this conjecture, but could not decide it. The problem is probably very difficult. What we could prove is the following theorem which is the main result of this section (and one of the main results of this paper).

Theorem 7.1. *Let $S \leq T_n$ be a semigroup with primitive group of units $G \leq S_n$. If there exists in S a map of rank 3 or less, then all the automorphisms of S are induced under conjugation by permutations of the normalizer of G in S_n.*

It is worth pointing out that in parallel with the conjecture above we also arrived at the following conjecture: if S is a transformation semigroup containing singular maps and a primitive group of units, then its subsemigroup of maps of minimum rank is idempotent generated. This conjecture looks very interesting in itself, but, moreover, it might be that in the end it turns out to be connected to the automorphisms conjecture above; future researches will certainly clarify this.

In the remainder of this section we prove the theorem above.

Suppose that S has a rank 1 map; then the transitivity of G guarantees that S contains all the constant maps and the theorem is a consequence of the following result.

Theorem 7.2. *(*[90, Theorem 1]*) If S is a subsemigroup of T_n containing all the constant maps, then all the automorphisms of S are induced by the elements of the normalizer of S in S_n.*

Suppose that S has a rank 2 map; Neumann [82] proved that primitive groups synchronize all rank 2 maps and hence S contains all the constants, a case already settled. The same conclusion holds in the case G synchronizes any rank 3 map in S. Therefore we can assume that G does not synchronize the rank 3 maps in S, and S does not contain maps of rank less than 3. Thus let $G \leq S_n$ be a non-synchronizing primitive group, and $G \subset S \subset T_n$ such that the smallest rank of a transformation in S is 3. In addition, by the results of [5], if G is primitive of degree n and t has rank at least $n-4$, then G synchronizes t, and so $\langle G, t\rangle$ contains all the constant maps; so we can assume (although this is not required by our proof) that S contains no map whose rank r satisfies $n-4 \leq r \leq n-1$.

Let $D \subseteq S$ be the subsemigroup consisting of all rank 3 maps. For $t \in D$, we define $\mathrm{ord}(t)$, the order of t, as the smallest positive integer i for which i is idempotent. This is equivalent to the group theoretic order of the restriction t' of t to $\mathrm{im}(t)$ within the subgroup $\langle t' \rangle$.

In what follows, when we refer to kernels and images, we will always mean kernels and images of elements of D. Let $X = \{1, \ldots, n\}$ be the underlying set for all actions. Note that for each $x_1, x_2 \in X$, there exists a map in S whose kernel K satisfies $(x_1, x_2) \notin K$, an easy consequence of the primitivity of G.

Lemma 7.3. *Let $a, b \in D$. Then a and b have the same image if and only if there exists $u \in S$ such that $a = ub$.*

Proof. If there exists $u \in S$ such that $a = ub$, it follows that $Xa \subseteq Xb$ and hence equality holds since a and b have the same rank. Conversely, let b^ω be the idempotent power of b. Clearly b^ω is the identity in the image of b, thus it is the identity in the image of a. Therefore, $ab^\omega = a$ and hence $a = ub$, where $u = ab^{\omega-1}$. The result follows. $\square$

In the same way we prove the following lemma.

Lemma 7.4. *Let $a, b \in D$. Then a and b have the same kernel if and only if there exists $u \in S$ such that $a = bu$.*

Let J be the set of all images of maps in D, L the set of all kernels of maps in D, and F be an automorphism of $\langle G, t \rangle$. Suppose a and b are two maps in D having the same image; then $a = ub$ (for some u) so that $aF = (uF)(bF)$, and hence aF and bF have the same image too. Thus F induces a permutation of the sets in J which we will denote by F_J (and usually abuse notation writing simply F); in the same way we prove that F induces a permutation of the kernels in L, which will be denoted by F_L.

For $I \in J$; $K \in L$, we let $i_{K,I}$ stand for the unique idempotent with kernel K and image I, and if I and I' intersect in exactly one element of X, we refer to that point as $x_{I,I'}$.

We claim that with the notation above, for every automorphism F of S there exists $h \in S_n$ such that $tF = h^{-1}th$.

Let Θ be the graph with $V(\Theta) = X$, in which two distinct vertices x, x' are adjacent if there is an $I \in J$ with $x, x' \in I$. Θ is clearly vertex-primitive, with clique number and chromatic number 3, the latter can be seen by colouring with the kernel classes of any rank 3 map. As shown in [9], Θ does not contain any induced subgraph isomorphic to a 4-clique with one edge removed.

As a side remark, we observe that the graph Θ is contained in the graph $\Gamma'(S)$ from [5]. For clearly two vertices lying in some image of an element of D cannot be collapsed by anything in S. It is not clear whether the converse holds.

Our proof proceeds through the following steps.

(a) The permutation F_J is given by $I \mapsto Ih$ for some $h \in S_X$.
(b) If F_J is given by $h \in S_X$, then $tF = h^{-1}th$.

Lemma 7.5. *Let $I, I' \in J$ such that $I \cap I' = \emptyset$. Then $IF_J \cap I'F_J = \emptyset$.*

Proof. We start by claiming that there exist $K, K' \in L$ for which $\mathrm{ord}(i_{K,I} i_{K',I'}) = 3$, where $\mathrm{ord}(x)$ denotes the order of x.

Let K_1 be an arbitrary kernel. Number the elements of I, I' so that $I = \{i_1, i_2, i_3\}$, $I' = \{i'_1, i'_2, i'_3\}$ and that $(i_j, i'_j) \in K_1$ for all j.

By primitivity, we can find a kernel K_2 that separates i_1 and i'_1. Let $\sigma_1 \in S_3$ be such that $(i_1, i'_{1\sigma_1}), (i_2, i'_{2\sigma_1}), (i_3, i'_{3\sigma_1}) \in K_2$. If σ_1 is a 3-cycle, then $\mathrm{ord}(i_{K_1,I} i_{K'_2,I'}) = 3$, as required. So assume otherwise. $\sigma_1(1) \neq 1$, so σ_1 is a transposition not fixing 1, say w.l.o.g. that $\sigma_1 = (1\ 2)$.

Once again by primitivity there exists a kernel $K_3 \in L$ that separates i_3, and i'_3. Let $\sigma_2 \in S_3$ be such that $(i_1, i'_{1\sigma_2}), (i_2, i'_{2\sigma_2}), (i_3, i'_{3\sigma_2}) \in K_3$. If σ_2 is a 3-cycle, the conclusion follows again.

Otherwise, $\sigma_2 = (2\ 3)$ or $\sigma_2 = (1\ 3)$, as σ_2 does not fix 3. Assume the former. We have that $(i_1, i'_2), (i_2, i'_1), (i_3, i'_3) \in K_2$ and we also have $(i_1, i'_1), (i_2, i'_3), (i_3, i'_2) \in K_3$. Hence $i_{K_2,I} i_{K_3,I'}|_{I'} = (i'_1\ i'_3\ i'_2)$, and so $\mathrm{ord}(i_{K_2,I} i_{K_3,I'}) = 3$. The case that $\sigma_2 = (1\ 3)$ is analogous, proving the claim. $\square$

Now suppose instead that $\bar{I}, \bar{I}' \in J$ are non-disjoint, say $x \in \bar{I}' \cap \bar{I}'$. Then, for all $\bar{K}, \bar{K}' \in L$,

$$x\ i_{\bar{K},\bar{I}} i_{\bar{K}',\bar{I}'} = x,$$

and so $i_{\bar{K},\bar{I}} i_{\bar{K}',\bar{I}'}$ fixes one element of its image and hence has order at most 2.

Now let K, K' be kernels such that $i_{K,I} i_{K',I'}$ has order 3. Then the same holds for $(i_{K,I} i_{K',I'})\, F = i_{KF,IF} i_{K'F,I'F}$. By our results above, this is not possible if $IF \cap I'F \neq \emptyset$.

Lemma 7.6. *For all automorphisms F, there exists an $h \in S_X$ (necessarily from* $\mathrm{Aut}\,(\Theta)$*) such that F_J is given Ih for all $I \in J$.*

Proof. Let $I, I \in J'$, $I \neq I'$ such that $I \cap I' \neq \emptyset$. Then $|I \cap I'| = 1$, as $|I \cap I'| = 2$ implies that the induce Θ-subgraph on $I \cup I'$ is isomorphic to a 4-clique with an edge removed. Lemma 7.5 now implies that if $|I \cap I'| = 1$ then $|IF \cap I'F| = 1$.

For each $x_{I,I'}$, define h by $x_{I,I'}h = x_{IF,I'F}$. Clearly, every $x \in X$ is of this form for some I, I', so the definition is universal on X. We have to show that it is also well defined.

Let $I'' \in J, I \neq I'' \neq I'$, such that $x_{I,I'} = x_{I',I''} = x_{I,I''} \in I \cap I' \cap I''$. Suppose that $x_{IF,I'F} \neq x_{IF,I''F}$. Then $x_{IF,I'F} \neq x_{I'F,I''F}$ for otherwise $|IF \cap I''F| \geq 2$. Similarly $x_{IF,I''F} \neq x_{I'F,I''F}$. But now $x_{IF,I'F}, x_{IF,I''F}, x_{I'F,I''F}$ form a triangle in Θ that shares one edge with the triangle on the set IF. The induced subgraph on the union of these triangles is a 4-clique missing one edge. However, Θ does not contain such a subgraph, giving a contradiction.

It follows that $x_{IF,I'F} = x_{IF,I''F}$, and by repeating this argument, that h is well-defined. Clearly, such h satisfies $IF_j = Ih$. □

Lemma 7.7. *If F_J is given by right multiplication with $h \in S_X$, then $tF = h^{-1}th$ for all $t \in S$.*

Proof. Let $t \in S$, and $t' \in D$. Then $t't$ has rank 3 and hence is in D. It follows that F acts on the images of both t' and $t't$ as right multiplication by h, and hence

$$(\mathrm{im}(t'))th = (\mathrm{im}(t't))h = \mathrm{im}((t't)F) =$$

$$= \mathrm{im}((t'F)(tF)) = (\mathrm{im}(t'F))(tF) = (\mathrm{im}(t'))h(tF),$$

and so $(\mathrm{im}(t'))t = (\mathrm{im}(t'))h(tF)h^{-1}$. As this holds for all $t' \in D$, t and tF have the same right action on J. Comparing $t'_1, t'_2 \in D$ with images $I, I' \in J$ such that $I \cap I' = \{x\}$, it then follows that $xt = xh(tF)h^{-1}$. As we can find such $t', t \in D$ for every x, $t = h(tF)h^{-1}$, and the result follows. □

Theorem 7.1 now immediately follows from the last two lemmas, noticing that if conjugation with h induces an automorphism on S, then h necessarily has to be in the normalizer of G in S_n. □

Cases in which Theorem 7.1 applies but Sullivan's Theorem [90] does not involve a primitive group G and a map t of rank 3 not synchronized by G. Examples include:

- wreath products $S_3 \wr S_n$ in the product action, and primitive subgroups of these;
- the automorphism groups of the Heawood, Tutte–Coxeter and Biggs–Smith graphs, acting on the edge sets of the graphs (these are described in detail in [5]);
- two primitive actions of the Mathieu group M_{12} with degree 495 (also in [5]).

In all these examples except $S_3 \wr S_2$ and the group of the Heawood graph, there are semigroups containing singular maps with rank greater than 3 as well as maps of rank 3. The last example has some interesting features which we now explain.

The Mathieu group M_{12} has two different primitive actions on a set of size 495 (`PrimitiveGroup(495,3)` and `PrimitiveGroup(495,5)` in the GAP numbering). Each of these is the automorphism group of a graph of valency 6, in which the closed neighbourhood of a vertex consists of three triangles with a common vertex. In both cases, the full automorphism group of the graph is $\mathrm{Aut}\,(M_{12}) = M_{12} : 2$. If we take the triangles in one graph to be the lines of a geometry whose points are the vertices, we obtain a partial linear space; the spaces obtained from the two graphs are duals of each other. Let Γ_1 and Γ_2 be these two graphs. We can construct Γ_2 from Γ_1 by taking the triangles of Γ_1 as vertices, two vertices adjacent if the triangles intersect, and *vice versa*.

As well as having clique number 3, each graph has chromatic number 3, and hence has an endomorphism onto a triangle; since in each case the automorphism group is transitive on triangles, every triangle is the image of an endomorphism.

Let S_1 and S_2 be the semigroups of endomorphisms of Γ_1 and Γ_2 respectively. By Theorem 7.1, we have

$$\mathrm{Aut}\,(S_1) \cong \mathrm{Aut}\,(S_2) \cong M_{12} : 2.$$

Any element of S_1 must map triangles of Γ_1 to triangles, and so induces a map on the vertices of Γ_2; so we have an action of S_1 on this set. This action is not faithful; also, its image contains all the constant maps, and so it cannot consist of endomorphisms of Γ_2. However, it would be interesting to know more about it; in particular, what does the semigroup generated by S_2 and the image of S_1 look like?

8. Problems

If the following question has an affirmative answer (as we conjecture), then the list in Theorem 3.2 is complete.

Problem 1. *Is it true that for $G = \mathrm{PXL}(2,q)$, $q \geq 169$ and $\lambda = (4,1,\dots,1)$, no pair (G,λ) is closed?*

The next problem looks within reachable boundaries.

Problem 2. *Prove for 2-homogeneous groups an analogous of Theorem 3.2.*

Unlike the previous, the next problem is certainly extremely difficult.

Problem 3. *Prove for primitive groups an analogous of Theorem 3.2.*

The next problem was introduced in Section 3, but only some remarks were given. A full solution is still out there.

Problem 4. *Given an orbit of the k-homogeneous group G on $(n-k)$-partitions, what is the subgroup of the normalizer of G, in S_n, which fixes that orbit?*

The results on normalizers of 2-homogeneous groups suggest the following generalization.

Problem 5. *Let G be a family of primitive groups that has been classified (e.g., primitive groups of rank 3). Build for the groups in that family a table similar to Table 10, describing the normalizers of these groups in the symmetric groups of the same degree.*

In order to give a sharper version of Theorem 5.1 (2), it would be useful to classify the primitive groups having a 2-homogeneous normalizer in S_n.

Problem 6. *Classify the primitive groups $G \leq S_n$ such that $N_{S_n}(G)$ is 2-homogeneous.*

We state now the main conjecture introduced in the previous section.

Problem 7. *Let G be a primitive group and $t \in T_n \setminus S_n$. Then all the automorphisms of $\langle G, t\rangle$ are induced (under conjugation) by the elements in $N_{S_n}(\langle G, t\rangle)$.*

If t has rank at least $n-4$, then we know (by [5]) that $\langle G, t\rangle$ contains all the constant maps and hence (by [90]) the conjecture above holds. The results of the previous section solve the conjecture for rank at most 3. The problem is open for semigroups with maps of rank between 4 and $n-5$.

The following problem was also mentioned in the previous section.

Problem 8. *Let G be a primitive group and $t \in T_n \setminus S_n$. Then the transformation of minimum rank in S are generated by idempotents.*

In [60] it is proposed the problem of finding the groups that can be the normalizers in S_n of some semigroup $S \subseteq T_n$. The main theorems of this paper provide some answers for that question, but we would like to propose the following conjecture.

Problem 9. *Is it true that a group G is the normalizer in S_n of a semigroup $S \leq T_n$ if and only if G is the normalizer in S_n of some group $H \leq S_n$?*

Let us say that a group G is *normalizer-binding* if the automorphisms of $\langle G, t\rangle$ are induced by elements in the normalizer of G, for every singular map t. We believe that the following conjecture is likely to be true:

Problem 10. *Show that the property of being normalizer-binding is closed upwards (that is, if G is normalizer-binding of degree n, and $G \leq H \leq S_n$, then H is normalizer-binding).*

Our final problem asks for a sharper version of Theorem 5.1, (2).

Problem 11. *For every pair (G, λ), where $G \leq S_n$ is a k-homogeneous group and λ is an $(n-k)$-partition of n, classify the groups $N_{S_n}(\langle G, t\rangle)$, where t is any map whose kernel has type λ.*

8.1. **Enhancing GAP.** As shown above, every 2-homogeneous group is 2-generated. However the GAP library of 2-transitive groups contains default sets of generators that in the major part of the cases have size larger than 2.

Problem 12. *Produce a library of generating sets of size 2 for all the degree n and k-homogeneous primitive groups in the GAP library (for $k \geq 2$).*

Slightly connected to the previous problem is the following.

Problem 13.
- *(a) Include in GAP a very effective function to find the homogeneity of a given permutation group.*
- *(b) Include in GAP a very effective function to find representatives for the orbits of a given permutation group on k-sets.*
- *(c) Include in GAP a very effective function to find representatives for the orbits of a given permutation group on a given k-partition.*

The next problem deals again with GAP libraries.

Problem 14. *Let G be a k-homogeneous degree n primitive group in the GAP library of primitive groups. Produce a minimal set A of degree n transformations of rank k such that $\langle G, A\rangle$ generates all the transformation of rank at most k.*

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