FRACTAL, GROUP THEORETIC, AND RELATIONAL STRUCTURES ON CANTOR SPACE

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Abstract

Cantor space, the set of infinite words over a finite alphabet, is a type of metric space with a 'self-similar' structure. This thesis explores three areas concerning Cantor space with regard to fractal geometry, group theory, and topology.

We first find results on the dimension of intersections of fractal sets within the Cantor space. More specifically, we examine the intersection of a subset E of the n-ary Cantor space, $\mathcal{C}_n$ with the image of another subset F under a random isometry. We obtain almost sure upper bounds for the Hausdorff and upper box-counting dimensions of the intersection, and a lower bound for the essential supremum of the Hausdorff dimension.

We then consider a class of groups, denoted by $V_n(G)$, of homeomorphisms of the Cantor space built from transducers. These groups can be seen as homeomorphisms that respect the self-similar and symmetric stucture of $\mathcal{C}_n$, and are supergroups of the Higman-Thompson groups V_n. We explore their isomorphism classes with our primary result being that $V_n(G)$ is isomorphic to (and conjugate to) V_n if and only if G is a semiregular subgroup of the symmetric group on n points.

Lastly, we explore invariant relations on Cantor space, which have quotients homeomorphic to fractals in many different classes. We generalize a method of describing these quotients by invariant relations as an inverse limit, before characterizing a specific class of fractals known as Sierpiński relatives as invariant factors. We then compare relations arising through edge replacement systems to invariant relations, detailing the conditions under which they are the same.

Declarations

I, Casey Donoven, hereby certify that this thesis, which is approximately 32,000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2012 and as a candidate for the degree of Ph.D in September 2013; the higher study for which this is a record was carried out in the University of St Andrews between 2012 and 2016.

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Contents

Chapter 1

Cantor Space: An Introduction

Cantor space is a topological and metric space, that exhibits many self-similar properties. We explore the consequences of this regularity in fractal geometry, group theory, and topology.

We begin by defining Cantor space and its topology and metric before giving several constructions of equivalent representations. These representations allow the structure of Cantor space to be displayed visually and provide alternative approaches to its study.

1.1 The Cantor Space $\mathcal{C}_n$

Our first description of Cantor space, and main focus for the introduction, is perhaps the least visual but most useful of the three constructions we present, giving a foundation on which to build a topological space and metric space.

1.1.1 Definitions

Let $n \in \mathbb{N}$ with $n \geq 2$ and let $X_n = \{0, 1, \ldots, n-1\}$. The n-ary *Cantor set* is then the set of infinite words over the alphabet X_n, denoted by $\mathcal{C}_n = X_n^\omega$. This is also known the full one-sided shift space when equipped with the shift map, see [30]. To establish consistent notation, we use lower case Roman letters as individual letters in the n-ary alphabet X_n and lower case Greek letters as words (finite and infinite) over the alphabet. We will usually state whether or not a word is finite but it should generally be clear from context.

The set of all finite words over the alphabet X_n is denoted by X_n^*, which includes the empty word ϵ. The length of a word $\alpha \in X_n^*$, denoted by $|\alpha|$, is the number of letters that make up α (which is 0 for the empty word). We will use X_n^k to denote the set of words of length k.

Let α be a finite word of length $m > 0$ and let $k \in \mathbb{N}$ be such that $0 < k \leq m$. We us the notation α_k to represent the kth letter of α and $\alpha|_k = \alpha_1\alpha_2 \ldots \alpha_k$ to represent the word formed by the first k letters of α. A *prefix* of α is any word consisting of the first l letters of α (such as $\alpha|_l$), for some $0 \leq l \leq m$, and a *suffix* of α is a word consisting of the last l letters of α. Note that l may be 0 or m, which implies that the each finite word is its own prefix and suffix while the empty word is a prefix and suffix of every word. The definition of prefix and suffix (and the notation $\alpha|_k$) also extends to infinite words.

We write $\alpha \preccurlyeq \beta$ if α is a prefix of β and $\alpha \perp \beta$ if neither α nor β are prefixes of one another. The *meet* of two finite or infinite words, α and β, denoted by $\alpha \wedge \beta$ is the longest word γ such that $\gamma \preccurlyeq \alpha$ and $\gamma \preccurlyeq \beta$. Note that $\preccurlyeq$ forms a partial order on X_n^* and $\alpha \wedge \beta$ is the greatest lower bound of α and β within this partial order. Also note that if $\alpha \preccurlyeq \beta$, then $\alpha \wedge \beta = \alpha$.

Let $\beta \in X_n^*$ be a finite non-empty word and $m > 0$ be finite. The finite word β^m is β concatenated with itself m times and the infinite word $\overline{\beta} \in \mathcal{C}_n$ is β concatenated with itself countably many times. When not clear from context, we use the symbol $\|$ to mean the concatenation of letters or words, e.g. $\beta\|\beta = \beta^2$. We especially use this notation when the concatenated objects are the images of functions.

An infinite word $\alpha \in \mathcal{C}_n$ is *periodic* if there exists a word $\beta \in X_n^*$ such that $\alpha = \overline{\beta}$. A word $\alpha \in \mathcal{C}_n$ is *eventually periodic* if there exist words $\gamma, \beta \in X_n^*$ such that $\alpha = \gamma\overline{\beta}$. The eventually periodic words (which include periodic words) are also known as *rational* words and the set of *irrational* words is simply their complement within $\mathcal{C}_n$. The sets of rational and irritional words form a partition of $\mathcal{C}_n$ that is preserved under certain actions, which includes actions by transducers from Chapter 2.

The *lexicographical order* $\leq_{lex}$ on $X_n^* \cup \mathcal{C}_n$ is a total order such that for $\alpha, \beta \in X_n^* \cup \mathcal{C}_n$, $\alpha \leq_{leq} \beta$ if and only if

- $\alpha \preccurlyeq \beta$, or
- $\alpha_k < \beta_k$ where k is the least integer such that $\alpha_k \neq \beta_k$.

Usually, we refer to the lexicographical order when comparing finite words or infinite words, but not from one type to the other.

1.1.2 Cantor Topology

Let $\alpha \in X_n^*$. Then we define the *cone*

$$[\alpha] = \{\beta \in \mathcal{C}_n \mid \alpha \preccurlyeq \beta\}.$$

These sets are also known by other of terms, including *cylinder sets* and *intervals*. I will often use the term cone when describing $\mathcal{C}_n$ as words and interval when refering to the fractal construction of Cantor sets described in Subsection 1.2.

The containment of cones can easily be described with respect to the prefix partial order on X_n^*:

- $[\alpha] \subseteq [\beta]$ if and only if $\alpha \succcurlyeq \beta$;
- $[\alpha] \cap [\beta] = \emptyset$ if $\alpha \perp \beta$.

The set of all cones $\{[\alpha] \mid \alpha \in X_n^*\}$ forms a basis of open sets for the Cantor topology on $\mathcal{C}_n$. Brouwer's characterization of Cantor spaces (see [28] for details) shows that $\mathcal{C}_n$ and $\mathcal{C}_m$ are homeomorphic for every $n, m \geq 2$. Its statement is given as Theorem 1.1.

Theorem 1.1. *The Cantor space $\mathcal{C}_2$ is the unique (up to homeomorphism) non-empty totally disconnected compact metric space without isolated points.*

With this characterization, we will generally mean $\mathcal{C}_n$ for some n when we refer to Cantor space.

We also provide an alternative method for showing that $\mathcal{C}_n$ and $\mathcal{C}_m$ are homeomorphic, which we describe in greater detail. Specifically, we construct a homeomorphism from $\mathcal{C}_n$ to $\mathcal{C}_2$, and by composing one of these homeomorphisms with the inverse of another, one can produce a homeomorphism between $\mathcal{C}_n$ and $\mathcal{C}_m$. The homeomorphism described in the following proof can be well-described by transducers, which will be introduced in Chapter 3.

Proposition 1.2. *$\mathcal{C}_n$ is homeomorphic to $\mathcal{C}_2$.*

Proof. Let $n > 2$. We build the homeomorphism from $\mathcal{C}_n$ to $\mathcal{C}_2$ using a system of replacement rules for the letters in X_n. Let $p_n : X_n \to X_2^*$ be defined by $p_n(0) = 0$, $p_n(i) = 1^i 0$ for $0 < i < n-1$, and $p_n(n-1) = 1^{n-1}$. The homeomorphism is simply applying p_n to each letter in words from $\mathcal{C}_n$. Specifically, let $P : \mathcal{C}_n \to \mathcal{C}_2$ be defined by $P(\alpha) = p(\alpha_1) \| p(\alpha_2) \| \ldots$.

We first show that P is a bijection. For injectivity, let $\alpha \neq \beta$ be infinite words in $\mathcal{C}_n$ and let $k \in \mathbb{N}$ be the smallest integer such that $\alpha_k \neq \beta_k$. This implies that $p(\alpha_1) \| \ldots \| p(\alpha_{k-1}) = p(\beta_1) \| \ldots \| p(\beta_{k-1})$ but $p(\alpha_k) \neq p(\beta_k)$ and, more importantly, $p(\alpha_k) \perp p(\beta_k)$. Therefore, $P(\alpha) \neq P(\beta)$.

For surjectivity, we show that for every $\gamma \in \mathcal{C}_2$, $\gamma \in \{p(i) | i \in X_n\}^\omega$, i.e. the image of C_n under P. Let $\gamma \in \mathcal{C}_2$ and let $\{i_j\}_{j \geq 1}$ be the increasing sequence of natural numbers such that i_j corresponds to the position of the jth 0 in γ. Specifically, $\gamma_{i_j} = 0$ for all j and $\gamma_k = 1$ for all $k \notin \{i_j\}$. Then $\rho = \gamma_{i_j+1}\gamma_{i_j+2}\ldots\gamma_{i_{j+1}} \in X_2^*$ is equal to $1^m 0$ where $m = i_{j+1} - i_j - 1$. Note that m may be zero. This shows that $\rho = p(n-1)^d \| p(r)$ where $m = d(n-1) + r$ and $r < n-1$. (If $d = 0$, then $\rho = p(r)$.) The same arguments hold for $\gamma|_{i_1}$, and since γ is the concatenation of the words in between consecutive zeros, $\gamma \in \{p(i) | i \in X_n\}^\omega$.

To show that P is continuous, let $\alpha \in X_2^*$ and consider the basic open set $[\alpha]$. If α ends in a zero, i.e. $\alpha_{|\alpha|} = 0$, then by the previous arguments $\alpha \in \{p(i) | i \in X_n\}^*$. Let

$\beta \in X_n^*$ be the unique word such that $\alpha = p(\beta_1)\| \ldots \|p(\beta_{|\beta|})$. Then the preimage of $[\alpha]$ is $[\beta]$ which is open. If α does not end in a zero, then let $m \in \mathbb{N}$ be such that α ends in a string of m consecutive ones. Again let $m = d(n-1) + r$ with $r < n-1$. If $r = 0$, then $\alpha \in \{p(i)|i \in X_n\}^*$ and the preimage of $[\alpha]$ is a basic open set. If $r \neq 0$, then $[\alpha] = [\alpha 0] \cup [\alpha 10] \cup \ldots \cup [\alpha 1^{n-2-r} 0] \cup [\alpha 1^{n-1-r}]$. The preimage of each cone in the finite union is a basic open set, and therefore the preimage of $[\alpha]$ is open. $\square$

It is important to note that there exists a homeomorphism $\phi_\alpha : \mathcal{C}_n \to [\alpha]$, for each $\alpha \in X_n^*$, where the topology on $[\alpha]$ is the subspace topology. Let $\beta \in \mathcal{C}_n$. Define $\phi_\alpha(\beta) = \alpha\beta$. It is straightforward to see that this is a bijection, where removing the prefix α describes the inverse function. Also, the preimage of a cone contained in $[\alpha]$ is a cone in $\mathcal{C}_n$, implying that ϕ_α is continuous. This hints at the self-similar nature of Cantor space, with subsets of $\mathcal{C}_n$ being homeomorphic images of the whole.

The Cantor topology can be described in an equivalent way as the product topology over countably infinite copies of X_n equipped with the discrete topology. Mapping a tuple $(x_1, x_2, x_3, \ldots)$ to the word $x_1 x_2 x_3 \ldots$ forms a homeomorphism between the two spaces, as it takes basic open sets in the product space to finite unions of cones in $\mathcal{C}_n$, and dually, the preimage of a cone in $\mathcal{C}_n$ is a finite intersection of basic open sets in the product space. In this way, it is easy to show that the Cantor topology is compact and Hausdorff, as it is homeomorphic to the product of compact and Hausdorff spaces, namely the discrete topology on a finite set.

Cones can also form useful partitions of $\mathcal{C}_n$. Consider $A = \{[x] \mid x \in X_n\}$. The set of cones, A, is a partition of $\mathcal{C}_n$ as every element in $\mathcal{C}_n$ has a single letter prefix contained in X_n. This demonstrates how the Cantor set is a disjoint union of cones, reinforcing the idea that $\mathcal{C}_n$ has a self-similar structure. Since each cone is simply another copy of the Cantor set, each can be broken down further. These partitions of $\mathcal{C}_n$ into cones will be used in each of the following chapters, especially when building groups of homeomorphisms of Cantor space that map cones to cones.

1.1.3 Cantor Metrics

Let $n \geq 2$ and $r \in \mathbb{R}$ be such that $0 < r < \frac{1}{n}$. For two distinct infinite words $\alpha \neq \beta \in \mathcal{C}_n$, let $k = |\alpha \wedge \beta|$, i.e. the length of the longest common prefix shared by α and β. We then define the Cantor metric $d(\cdot, \cdot)$ such that for $\alpha, \beta \in \mathcal{C}_n$, $d(\alpha, \beta) = r^k$. Also, $d(\alpha, \beta) = 0$ when $\alpha = \beta$. The longer a prefix two words share, the closer they are. The following lemma shows d is not only a metric, but an ultrametric.

Lemma 1.3. *The Cantor set $\mathcal{C}_n$ with metric d is an ultrametric space.*

Proof. The distance between two points is always positive and is zero if and only if two words have every letter in common, i.e. they are the same point. Clearly $d(\alpha, \beta) = d(\beta, \alpha)$, so what is left to show is that $d(\alpha, \beta) \leq \max\{d(\alpha, \gamma), d(\gamma, \beta)\}$ for all $\alpha, \beta, \gamma \in \mathcal{C}_n$.

Let $d(\alpha, \beta) = r^k$ meaning that $\alpha|_k = \beta|_k$ but $\alpha_{k+1} \neq \beta_{k+1}$. If $\gamma|_k \neq \alpha|_k$, then $d(\alpha, \gamma) > r^k$ and $d(\gamma, \beta) > r^k$, satisfying the condition. If $\gamma|_k = \alpha|_k$, then $\gamma_{k+1} \neq \alpha_{k+1}$ or $\gamma_{k+1} \neq \beta_{k+1}$, meaning that $\max\{d(\alpha, \gamma), d(\gamma, \beta)\} = r^k$. □

The Cantor topology is induced by this metric as well. In particular, the set of all open balls in the metric is exactly the usual basis of open sets (the set of all cones).

Lemma 1.4. *Let $\delta \in (0, 1)$ and $\alpha \in \mathcal{C}_n$. Then $B_\delta(\alpha) = \{\beta \in \mathcal{C}_n \mid d(\alpha, \beta) < \delta\}$ is a cone.*

Proof. Let $k \in \mathbb{N}$ be the number such that $r^{k-1} > \delta \geq r^k$. Let $\beta \in [\alpha|_k]$. Since $d(\beta, \alpha) \leq r^k$, $\beta \in B_\delta(x)$. However, consider $\gamma \notin [\alpha|_k]$. Then γ does not share a k-letter prefix with α and $d(\alpha, \gamma) > r^k$, so $\gamma \notin B_\delta(x)$. Therefore, $B_\delta(\alpha) = [\alpha|_k]$. □

1.2 Equivalent Representations of Cantor Space

We now give two equivalent representations of Cantor space: as the boundary of an infinite graph and as a fractal subset of the unit interval. We use these to provide a visual description of the structre of $\mathcal{C}_n$.

1.2.1 The Infinite Tree $\mathcal{T}_n$

A *graph* G is a set of vertices $V(G)$ with a set of edges $E(G)$, where each edge in $E(G)$ is a subset of V with cardinality two. If $u, v \in V$ and $e = \{u, v\} \in E(G)$, we say that vertex u is *adjacent* to vertex v and are *connected* to one another by edge e. A *path* is a sequence of distinct edges $e_1, e_2 \ldots e_k$, where $k \geq 1$, and a sequence of vertices $v_1, v_2, \ldots, v_{k+1}$ that are also distinct (except for possibly v_1 and v_{k+1}) such that $v_i \neq v_{i+1}$ and $e_i = \{v_i, v_{i+1}\}$ for each i. In other words, the vertices v_i and v_{i+1} are adjacent and are connected by edge e_i. Often, we just refer to the vertices (or edges) of a path, when the other sequence is clear. A *cycle* is a path such that $v_1 = v_{k+1}$, i.e. a path that 'starts' and 'ends' at the same vertex. A *tree* is a graph without any cycles. To help distinguish edges, labels may be assigned to edges through a function $f : E(G) \to L$, where L is a set of labels.

The *rooted infinite n-ary tree*, $\mathcal{T}_n$, is a graph with vertices X_n^*. Two vertices $\alpha, \beta \in X_n^*$ are adjacent if and only if there exists an $x \in X_n$ such that $\alpha = \beta x$ or $\beta = \alpha x$. For $\alpha \in X_n^*$, the set $\{\alpha x \mid x \in X_n\} \subseteq V(\mathcal{T}_n)$ is known as the set of *children* of α with $\alpha|_{|\alpha|-1}$ being the *parent* of α. To further the analogy, any prefix of α, including the empty word, is called an *ancestor* of α, and any word with prefix α a *descendant* of α.

The empty word, ϵ, is called the *root*. It has no parent vertex and is adjacent to n vertices, namely its children, X_n. Every non-empty word $\alpha \in X_n^*$ is adjacent to $n + 1$ vertices, its parent and its n children. The mth level of $\mathcal{T}_n$ refers to the m-letter vertices, of which there are n^m.

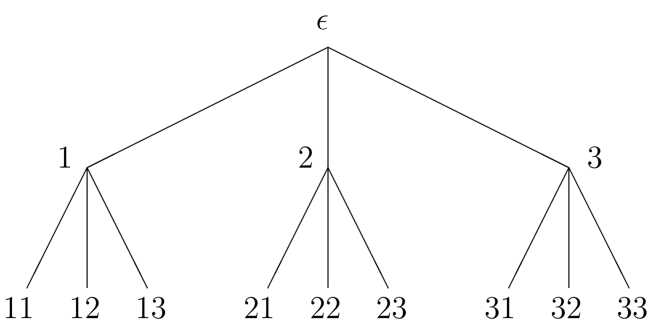

Figure 1.1: Levels 0, 1, and 2 of the infinite rooted ternary tree, $\mathcal{T}_3$.

The tree $\mathcal{T}_n$ also has a self-similar nature. Let $\alpha\mathcal{T}_n$ be the subgraph of $\mathcal{T}_n$ composed of α, its decendants, and all edges connecting them. This subtree is isomorphic to $\mathcal{T}_n$ by the isomorphism $\psi_\alpha : \mathcal{T}_n \to \alpha\mathcal{T}_n$ which takes vertex β to vertex $\alpha\beta$.

We think of $\mathcal{C}_n$ as being the boundary of $\mathcal{T}_n$. Each infinite word $\alpha \in \mathcal{C}_n$ corresponds to the unique infinite path ϵ, $\alpha|_1$, $\alpha|_2 \ldots$ from the root descending through the tree. The usual basis for Cantor space then consists of sets of paths going through a particular vertex. Specifically, let $\beta \in X_n^* = V(\mathcal{T}_n)$. Then $[\beta]$ corresponds to infinite paths that pass through vertex β. We use this characterization in Chapter 2 to give an equivalent description of the group of isometries of $\mathcal{C}_n$ as the automorphism group of $\mathcal{T}_n$.

1.2.2 Middle Thirds Cantor Set

We now consider Cantor space as the attractor of a self-similar iterated function system. Let D be a closed subset of a metric space X and let $f : D \to D$. The function f is a *contraction* if there exists $c \in \mathbb{R}$ with $0 < c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in D$. If equality holds and $d(f(x), f(y)) = cd(x, y)$, then f is known as a *contracting similarity.* An *iterated function system*, or IFS, is a finite set I (with $|I| \geq 2$) of contracting functions on a metric space X and an *attractor* of an IFS is a set $A \subseteq X$ such that $A = \bigcup_{f \in I} f(A)$. In 1981, Hutchinson showed in [26] that an IFS has a unique non-empty compact attractor.

Theorem 1.5. *Let I be an iterated function system of contractions on metric space X. Then there exists a unique non-empty compact set $A \subseteq X$ such that $A = \bigcup_{f \in I} f(A)$*

The unique attractor of an IFS is called *self-similar* if the IFS consists of contracting similiarities. We now define an IFS whose self-similar attractor is homeomorphic to Cantor Space.

Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined by $f_0(x) = \frac{1}{3}x$ and $f_1 : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) = \frac{1}{3}x + \frac{2}{3}$. We denote the fractal attractor of the IFS $I = \{f_0, f_1\}$ by $C_{2,\frac{1}{3}}$, which is also known as the *middle thirds Cantor set.*

To construct $C_{2,\frac{1}{3}}$, let $E_0 = [0, 1]$ and define $E_{m+1} = f_0(E_m) \cup f_1(E_m)$. Then $E_1 =$

$[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on. This forms a nested sequence of compact sets, i.e. $E_{m+1} \subset E_m$ for all m. The middle thirds Cantor set is then $C_{2,\frac{1}{3}} = \bigcap_m E_m$.

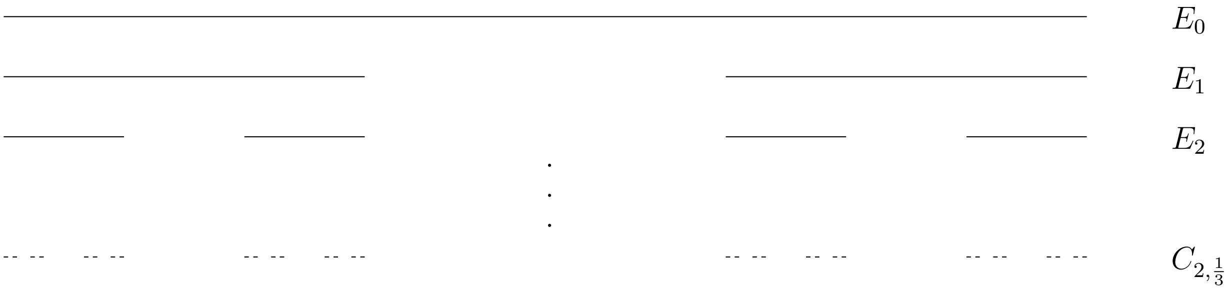

Figure 1.2: The middle thirds Cantor Set: $C_{2,\frac{1}{3}} = \cap E_m$

Note that as $C_{2,\frac{1}{3}} = f_0(C_{2,\frac{1}{3}}) \cup f_1(C_{2,\frac{1}{3}})$, the middle thirds Cantor set is the unique fractal attractor of I.

Another way of describing the set $C_{2,\frac{1}{3}}$ is as those points in the unit interval which have a base 3 expansion not using the number 1. Points may have multiple ternary expansions, such as $1/3 = 0.1\overline{0} = 0.0\overline{2}$, but those in the middle thirds Cantor set have at least one expansion without a 1. We can therefore write each point $x \in C_{2,\frac{1}{3}}$ as the sum $x = \sum_i x_i \frac{2}{3^i}$ where x_i is either 0 or 1. This way of expressing points in the Cantor set is unique as expansions differing in the kth digit are at least $1/3^k$ apart.

We show that the middle thirds Cantor set and Cantor space are homeomorphic by constructing a bi-Lipschitz map between them. Note that this also shows that the metrics on each space are strongly equivalent. Two metrics d_1 and d_2 on the space X are *strongly equivalent* if there exists real numbers $c_1, c_2 > 0$ such that $c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$, for all $x, y \in X$. In the case of Cantor space, the Euclidean metric on $\mathbb{R}$ restricted to $C_{2,\frac{1}{3}}$ is strongly equivalent to the Cantor metric on $\mathcal{C}_2$ with $r = 1/3$ (see Section 1.1) projected onto $C_{2,\frac{1}{3}}$ via the inverse of the bi-Lipschitz function we now construct.

Lemma 1.6. *The middle thirds Cantor set $C_{2,\frac{1}{3}}$ and Cantor space $\mathcal{C}_2$ are homeomorphic and the Euclidean metric on $\mathbb{R}$ restricted to $C_{2,\frac{1}{3}}$ is strongly equivalent to the Cantor metric on $\mathcal{C}_2$ with $r = 1/3$.*

Proof. Define $g : \mathcal{C}_2 \to C_{2,\frac{1}{3}}$ by $g(\gamma) = \sum_{i=1}^{\infty} \gamma_i \frac{2}{3^i}$ for $\gamma \in \mathcal{C}_2$. Let $\alpha, \beta \in \mathcal{C}_2$ be distinct points and let $k = |\alpha \wedge \beta|$. Setting $r = 1/3$ in the Cantor metric, we get $d(\alpha, \beta) = 1/3^k$. Note that $\sum_{i=m}^{\infty} \frac{2}{3^i} = \frac{1}{3^{m-1}}$ and after canceling the first k terms in $|g(\alpha) - g(\beta)|$, we have

$$\frac{1}{3}\frac{1}{3^k} \leq |\sum_{i=k+1}^{\infty} \alpha_i \frac{2}{3^i} - \sum_{i=k+1}^{\infty} \beta_i \frac{2}{3^i}| \leq \frac{1}{3^k}$$

and therefore $\frac{1}{3} d(\alpha, \beta) \leq |g(\alpha) - g(\beta)| \leq d(\alpha, \beta)$.

This shows that g is bi-Lipschitz, and therefore a homeomorphism from $\mathcal{C}_2$ to $C_{2,\frac{1}{3}}$ and that the metrics are strongly equivalent. □

The IFS used to create $C_{2,\frac{1}{3}}$ can be varied to create a range of attractors homeomorphic to $\mathcal{C}_n$. Let $n \geq 2$ and $r < 1/n$. For $0 \leq k \leq n-1$, define $f_k : \mathbb{R} \to \mathbb{R}$ to be

$$f_k(x) = rx + k\frac{1-r}{n-1}.$$

The set of contracting similiarities $\{f_0, f_1 \ldots, f_{n-1}\}$ forms an IFS much like the previous one. Again, there exists a unique self-similar set $C_{n,r}$ such that $C_{n,r} = \cup f_k(C_{n,r})$ and $C_{n,r}$ is homeomorphic to $\mathcal{C}_n$. This time, $C_{n,r}$ is made up of n copies of itself scaled by r, with an equal amount of space between each copy. An example is shown in Figure 1.3

Figure 1.3: The attractor $C_{3,\frac{1}{5}}$.

In Chapter 2, we define and discuss the fractal dimension of subsets of Cantor space with the usual metric. The dimensions we will use are Hausdorff dimension and box-counting dimension but we will mention, however, that for 'nice' self-similar sets such as $C_{n,r}$, the Hausdorff and box-counting dimensions are equal to the similarity dimension.

The *similarity dimension* for the attractor of the self-similar IFS $\{g_0, g_1, \ldots, g_{m-1}\}$ is the unique number s that satisies $\sum_i r_i^s = 1$, where r_i is the contraction ratio of similarity g_i. In the case of $C_{n,r}$, the similarity dimension is $\log n / \log r$. For a more thorough discussion, see [17].

Chapter 2

Codimension Formulae

In this chapter, we consider Cantor space as a metric space in order to examine fractal subsets of $\mathcal{C}_n$. In particular, we examine the fractal dimension of the intersection of subsets of $\mathcal{C}_n$. These results exhibit a form similar to the classical codimension formula of manifolds and its extensions by Kahane [27] and Matilla [33] to fractal subsets of $\mathbb{R}^n$. Material in this chapter has been published in [15].

2.1 Introduction

The classical codimension formula describes the dimension of the intersection of two manifolds embedded in $\mathbb{R}^n$ in general relative position, which we will define later. The formula states that for manifolds E and F, the dimension of $E \cap \sigma(F)$, where σ is a rigid motion in $\mathbb{R}^n$, is 'often' given by

$$\dim(E \cap \sigma(F)) = \max\{\dim E + \dim F - n, 0\} \tag{2.1}$$

and 'typically' no more than this value. 'Often' and 'typical' can be made precise in terms of a natural measure on the group of rigid motions on $\mathbb{R}^n$.

In this section, we discuss results by Kahane [27] and Matilla [33] that extend the classical codimension formula to the intersection of fractals in $\mathbb{R}^n$ and the necessary tools to find codimension formulae in $\mathcal{C}_n$.

2.1.1 Fractal Dimension

Both Kahane and Matilla use Hausdorff dimension in their extensions of the codimension formula, which we now describe. See [17] for details on the definitions and theorems in this subsection.

Let X be a metric space with metric $d(\cdot,\cdot)$. The *diameter* of a subset $A \subseteq X$ is denoted by $\operatorname{diam}(A)$. Then, for $\delta > 0$, a *δ-cover* of a subset $F \subseteq X$ is a countable collection of sets $\{O_i\}_i$ such that $F \subseteq \bigcup O_i$ and $\operatorname{diam}(O_i) \leq \delta$ for all i.

Letting $s \geq 0$, the *δ-approximation to the s-dimensional Hausdorff measure* of a set $F \subseteq X$ is defined as

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum \operatorname{diam}(O_i)^s \mid \{O_i\}_i \text{ is a } \delta\text{-cover of } F\Big\}.$$

The *Hausdorff measure* of F is the limit of $\mathcal{H}^s_\delta(F)$ as δ goes to 0:

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

For every non-empty set F, there is a particular dimension s for which $\mathcal{H}^t(F) = \infty$ for $0 \leq t < s$ and $\mathcal{H}^t(F) = 0$ for $t > s$. The *Hausdorff dimension* of F is this value of s and is defined specifically as:

$$\dim_H F = \sup\{s \mid \mathcal{H}^s(F) = \infty\} = \inf\{s \mid \mathcal{H}^s(F) = 0\}.$$

Note that $\mathcal{H}^{\dim_H(F)}(F)$ may be infinite or zero or take a positive finite value.

Using the construction of the Cantor set as a self-similar subset of $\mathbb{R}$ in Chapter 1 with n similar parts of ratio r, the Hausdorff dimension of $C_{n,r}$ is $-\log n/\log r$. The

Euclidean metric and the metric on Cantor space are equivalent, implying the Hausdorff dimension of Cantor space is also $-\log n/\log r$.

Calculating upper bounds for the Hausdorff dimension of a set F generally involves finding δ-covers that form good approximations of F. Lower bounds are often much more difficult to obtain, and several methods have been developed to tackle this problem. One such method, known as the potential theoretic method, was used by Kahane and Mattila to find codimension formulae in $\mathbb{R}^n$.

The proof of the potential theoretic method utilizes Frostman's Lemma, which shows that the Hausdorff measure satisfies a useful relationship with the radius of balls in the metric space.

Lemma 2.1 (Frostman's Lemma). *Let F be a Borel subset of X with $0 < \mathcal{H}^s(F) \leq \infty$. Then there exists a compact subset $E \subseteq F$ and a constant $c > 0$ such that $0 < \mathcal{H}^s(E) < \infty$ and $\mathcal{H}^s(E \cap B(x,r)) \leq cr^s$ for all $x \in X$ and $r > 0$.*

The potential theoretic method uses the s-energy of mass distributions to bound the Hausdorff dimension. A *mass distribution* is a positive finite measure with bounded support and the s-energy of a mass distribution μ on X is the following integral:

$$I_s(\mu) = \int\int \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{d(x,y)^s}.$$

Theorem 2.2 (Potential Theoretic Method). *Let F be a subset of X.*

1. *If there is a mass distribution μ supported on F such that $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_H(F) \geq s$.*

2. *If F is a Borel set with $\mathcal{H}^s(F) > 0$, then there exists a mass distribution μ supported on F such that $I_t(\mu) < \infty$ for all $0 < t < s$.*

Before exploring codimension formulae involving Hausdorff dimension, we introduce the box-counting dimension, another fractal dimension we use to describe the intersection of sets in $\mathcal{C}_n$.

Let $\delta > 0$ and define $N_\delta(F)$ to be the least number of balls of radius δ needed to cover $F \subset X$. The *box-counting dimension* is then defined as

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

This limit need not exist, so the upper box-counting dimension and lower box-counting dimension are then defined as the upper and lower limits respectively, and denoted by $\overline{\dim}_B$ and $\underline{\dim}_B$.

There are many equivalent definitions of box-counting dimension. However, the ultrametric structure of Cantor space make this one particularly useful. Since, $\mathcal{C}_n$ is an ultrametric space, balls of radius δ are basic open sets.

2.1.2 Fractal Codimension Formulae in $\mathbb{R}^n$

Although both Kahane and Mattila find codimension formulae using the potential theoretic method, they use different constructions to build mass distributions and achieve slightly different results. They also use different groups of transformations of $\mathbb{R}^n$ to define general relative position.

Let $E, F \subseteq \mathbb{R}^n$. We say that something holds for sets E and F in *general relative position* if:

- the truth of the statement is invariant under simultaneous transformation of E and F by any isometry of $\mathbb{R}^n$, and

- the statement holds for fixed E and $\sigma(F)$ for almost all isometries σ with respect to the natural measure on the group of isometries.

The first condition concerns the 'relative position' of E and F and the second is saying that 'general' refers to the usual measure on isometries. We extend the usual notion of general relative position to subsets of an arbitrary metric space X and group of transformations of X (e.g. similarity transformations) with a given measure.

Mattila's codimension formula is formulated with respect to the group of similarity transformations of $\mathbb{R}^n$, decomposed into the orthogonal group $O(n)$, the group of translations in $\mathbb{R}^n$, and a scaling factor r. To highlight the similarity and differences with our results, we present a specific case of the formulae for the group of isometries of $\mathbb{R}^n$, effectively setting the scaling factor r equal to 1.

The measure on the group of isometries that Mattila uses is the product of a measure on the group of translations and a measure on $O(n)$. The n-dimensional Lebesgue measure, $\mathcal{L}^n$, forms a measure on the group of translations, while there exists a left Haar measure on $O(n)$, denoted θ_n. A left Haar measure is a measure on a topological group satisfying certain conditions. The most important of which is that for all Borel subsets S of the topological group, $\theta_n(S) = \theta_n(gS)$ for all g, where $gS = \{gs | s \in S\}$. See [37] for more details on Haar measures.

Notationally, let the function $f_z(x) = x + z$ be the translation in $\mathbb{R}^n$ by z.

Theorem 2.3 (Mattila). *Let $s, t > 0$, $s + t > n$, and $t > (n+1)/2$. If E and F are Borel sets in $\mathbb{R}^n$ with $\mathcal{H}^s(E) > 0$ and $\mathcal{H}^t(F) > 0$, then for θ_n almost all $g \in O(n)$*

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim(E \cap (f_z \circ g)F) \geq s + t - n\}) > 0$$

The proof builds mass distributions on the intersection of E and F by taking slices of a product measure supported on $E \times F$. For $n \geq 2$, it is not known whether the condition $t > (n+1)/2$ is necessary. In fact, the more general formula with respect to the group of similarity transformations does not require the dimension of F to be greater than $(n+1)/2$.

Kahane's codimension formula is given with respect to an arbitrary closed subgroup of the general linear group on $\mathbb{R}^n$ with a transitivity condition.

Theorem 2.4 (Kahane)**.** *Let G be a closed subgroup of the general linear group on $\mathbb{R}^n$ which is transitive in $\mathbb{R}^n \setminus \{0\}$ and let μ be a Haar measure on G. Also let $s + t > n$ and E and F be Borel sets in $\mathbb{R}^n$ such that $\mathcal{H}^s(E) > 0$ and $\mathcal{H}^t(F) > 0$. Then for μ almost all $g \in G$,*

$$\mathcal{L}^n(\{z \in \mathbb{R}^n \mid \dim_H(E \cap (f_z \circ g)F) \geq s + t - n\}) > 0$$

Kahane's proof builds mass distributions on $E \cap (f_z \circ g)F$ by convolving measures that approximate Hausdorff measure supported on compact subsets of E and F. Our codimension formulae in Cantor space will also use Frostman's Lemma and the potential theoretic method to bound Hausdorff dimension from below, building random measures using martingales.

2.1.3 Fractal Codimension Formulae in $\mathcal{C}_n$

Like Mattila, we find codimension formulae in $\mathcal{C}_n$ with respect to the group of isometries of $\mathcal{C}_n$, denoted by $\operatorname{Iso}\mathcal{C}_n$, with respect to the Cantor metric d defined in Section 1.1. This group has a convenient description using the infinite n-ary tree discussed in Chapter 1 and supports a natural probability measure, making it easy to describe a random element. We formalize this statement in this subsection.

Lemma 2.5. *The group $\operatorname{Iso}\mathcal{C}_n$ is the group of automorphisms of the infinite n-ary tree, $\mathcal{T}_n$, acting on the boundary of $\mathcal{T}_n$, under the natural identification.*

Proof. Let $\alpha, \beta \in \mathcal{C}_n$ with $d(\alpha, \beta) = r^k$. This implies that $\alpha|_k = \beta|_k$ but $\alpha|_{k+1} \neq \beta|_{k+1}$, both as prefixes and as vertices in $\mathcal{T}_n$. Let ϕ be an automorphism of $\mathcal{T}_n$. Then $\phi(\alpha)|_k = \phi(\beta)|_k$ and $\phi(\alpha)_{k+1} \neq \phi(\beta)_{k+1}$. Therefore, $d(\phi(\alpha), \phi(\beta)) = r^k$ which implies that ϕ is also an isometry. This shows that $\operatorname{Aut}(\mathcal{T}_n) \subseteq \operatorname{Iso}\mathcal{C}_n$.

Let ψ be an isometry of $\mathcal{C}_n$ and consider $\psi([\gamma])$ where $\gamma \in X_n^*$ and $|\gamma| = k$. Note that $[\gamma] = B(\alpha, r^k)$ for some $\alpha \in \mathcal{C}_n$ such that $\alpha|_k = \gamma$. Since ψ preserves distance,

$$\begin{aligned}\psi([\gamma]) &= \psi(B(\alpha, r^k)) \\ &= B(\psi(\alpha), r^k) \\ &= [\psi(\alpha)|_k].\end{aligned}$$

This shows that ψ maps cones at level k, represented by vertices in $\mathcal{T}_n$, to other cones at level k, showing that ψ induces a level preserving action on $\mathcal{T}_n$. Moreover, from a similar argument, it is clear that for each $x \in X_n$, $\psi([\gamma x]) = [\psi(\beta)]$ for some $\beta \in \mathcal{C}_n$ such that $\beta|_{k+1} = \gamma x$. Since $\beta \in [\gamma]$, this implies that ψ preserves adjacencies and must therefore be a graph endomorphism. However, ψ is invertible and ψ^{-1} is also an isometry, implying that ψ^{-1} also maps vertices to vertices and edges to edges preserving adjacencies. Therefore, ψ induces an automorphism of $\mathcal{T}_n$ □

We define the probability measure on Iso $\mathcal{C}_n$ in terms of the group of automorphisms of $\mathcal{T}_n$. Each automorphism acts by 'twisting' the tree at a set of vertices (corresponding to cones in $\mathcal{C}_n$), perhaps infinitely many, rearranging the children of each node into a new permutation. An example automorphism of $\mathcal{T}_3$ is shown in 2.1.3.

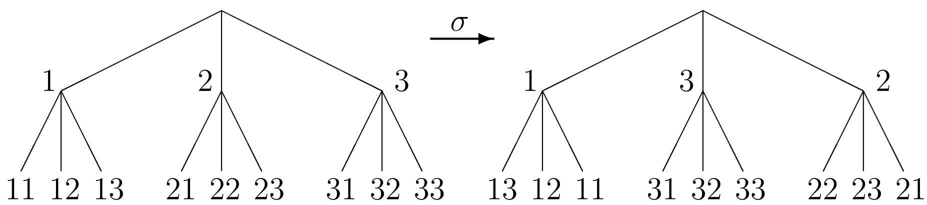

Figure 2.1: An automorphism σ acting on two levels of the ternary Cantor space

For the remainder of this chapter, we will use the word 'interval' in lieu of 'cone' to refer to those words with a common prefix. This is to correspond directly with the construction of Cantor set as a self-similar subset of $\mathbb{R}$, for visual effect and notational simplicity. In general, we use capital I and J to represent intervals. We also use I' and J' to denote the parent intervals of I and J respectively, i.e. if interval I corresponds to cone $[\alpha|_k]$, then I' corresponds to the cone $[\alpha|_{k-1}]$.

An interval *at level* k corresponds to a cone with a common prefix of length k and the set of all intervals at level k is denoted by U_k. We use similar notation to refer to the set of intervals at a specific level needed to cover a subset $F \in \mathcal{C}_n$,

$$U_k(F) = \{I \in U_k | I \cap F \neq \emptyset\}.$$

Note that $U_k = U_k(\mathcal{C}_n)$.

The natural invariant probability space $(\mathrm{Iso}\,\mathcal{C}_n, \mathcal{F}, \mathbb{P})$ on the isometries of $\mathcal{C}_n$ is defined as follows. For each k let π be an admissible permutation of the intervals of U_k (i.e. one that is achievable by some $\sigma \in \mathrm{Iso}\,\mathcal{C}_n$) and let $\mathcal{I}_\pi$ be the set of all isometries $\sigma \in \mathrm{Iso}\,\mathcal{C}_n$ such that $\sigma(I) = \pi(I)$ for all $I \in U_k$. Let $\mathcal{F}_k$ be the finite sigma-algebra consisting of finite unions of all such $\mathcal{I}_\pi$. We define a probability $\mathbb{P}$ on $\mathcal{F}_k$ by ascribing equal probability to each $\mathcal{I}_\pi$, so that $\mathbb{P}(\mathcal{I}_\pi) = n^{-k(k+1)/2}$, and extending to $\mathcal{F}_k$. These sigma-algebras form an increasing sequence and we define $\mathcal{F} = \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_k)$ for the sigma-algebra generated by their union and extend $\mathbb{P}$ to $\mathcal{F}$ in the usual way. Note that for $I, J \in U_k$ and $\sigma \in \mathrm{Iso}\,\mathcal{C}_n$, $\mathbb{P}\big(\sigma(I) = J\big) = n^{-k}$.

We can now formulate our main results as the following theorem. Recall that the *essential supremum* of a function $f : X \to \mathbb{R}$ with respect to measure μ, denoted by $\mathrm{esssup}_{x \in X}\{f\}$, is the least upper bound b such that $\mu\big(\{x | f(x) > b\}\big) = 0$.

Theorem 2.6. *Let $E, F \subset \mathcal{C}_n$ be Borel sets. Then for a random isometry $\sigma \in \mathrm{Iso}\,\mathcal{C}_n$:*

(*i*) *almost surely* $\overline{\dim}_B(E \cap \sigma(F)) \leq \max\big\{\overline{\dim}_B E + \overline{\dim}_B F + \log n / \log r,\ 0\big\}$,

(*ii*) *almost surely* $\dim_H(E \cap \sigma(F)) \leq \max\big\{\dim_H E + \overline{\dim}_B F + \log n / \log r,\ 0\big\}$,

(*iii*) $\mathrm{esssup}_{\sigma \in \mathrm{Iso}\mathcal{C}_n}\big\{\dim_H(E \cap \sigma(F))\big\} \geq \max\big\{\dim_H E + \dim_H F + \log n / \log r,\ 0\big\}$.

Note the similarity between these codimension formulae and the previous ones, when the dimension n of the ambient space $\mathbb{R}^n$ is replaced by $-\log n / \log r$, the dimension of $\mathcal{C}_n$

2.2 Upper Box-Counting Dimension: Upper Bound

In this section, we bound the upper box-counting dimension of the intersection of a subset of $\mathcal{C}_n$ with a random image of another subset.

Theorem 2.7. *Let $E, F \subset \mathcal{C}_n$. Then, almost surely,*

$$\overline{\dim}_B(E \cap \sigma(F)) \leq \max\left\{\overline{\dim}_B E + \overline{\dim}_B F + \frac{\log n}{\log r}, 0\right\}. \tag{2.2}$$

Proof. First note that

$$U_k\big(E \cap \sigma(F)\big) \subset U_k\big(E\big) \cap U_k\big(\sigma(F)\big) = U_k\big(E\big) \cap \sigma\big(U_k(F)\big).$$

For $k \geq 0$ and $J \in U_k(F)$, consider the indicator function $\chi_J : \operatorname{Iso}\mathcal{C}_n \to \{0,1\}$ such that $\chi_J(\sigma) = 1$ when $\sigma(J) \in U_k(E)$ and $\chi_J(\sigma) = 0$ otherwise. Then

$$\big|U_k\big(E \cap \sigma(F)\big)\big| \leq |U_k\big(E\big) \cap \sigma\big(U_k(F)\big)| = \sum_{J \in U_k(F)} \chi_J(\sigma).$$

A random automorphism σ maps interval $J \in U_k$ onto a particular interval $I \in U_k$ with probability n^{-k}. Therefore, for all $J \in U_k$ we have

$$\mathbb{E}\big(\chi_J(\sigma)\big) = n^{-k}|U_k(E)|.$$

This implies

$$\mathbb{E}\big(\big|U_k\big(E \cap \sigma(F)\big)\big|\big) \leq \sum_{J \in U_k(F)} \mathbb{E}\big(\chi_J(\sigma)\big) = n^{-k}|U_k(E)||U_k(F)|.$$

Assume that $\overline{\dim}_B E + \overline{\dim}_B F + \log n/\log r > 0$ (otherwise there is nothing to prove since $\overline{\dim}_B(E \cap \sigma(F)) \geq 0$). Take a and b in $\mathbb{R}$ such that $a > \overline{\dim}_B E$ and $b > \overline{\dim}_B F$. From the definition of upper box-counting dimension, there exist $c_1, c_2 > 0$ such that, for all $k \geq 0$,

$$|U_k(E)| \leq c_1 r^{-ka} \quad \text{and} \quad |U_k(F)| \leq c_2 r^{-kb}. \tag{2.3}$$

Setting $c = c_1 c_2$, for all $k > 0$,

$$\mathbb{E}\Big(\big|U_k\big(E \cap \sigma(F)\big)\big|\Big) \leq c r^{-k(a+b+\log n/\log r)} = c r^{-kd},$$

where $d = a + b + \log n/\log r > 0$. Let $\epsilon > 0$. Then

$$\mathbb{E}\Big(\sum_{k=0}^{\infty} r^{k(d+\epsilon)}\big|U_k\big(E \cap \sigma(F)\big)\big|\Big) \leq \sum_{k=0}^{\infty} c r^{k\epsilon} < \infty.$$

Thus, almost surely, there exists a random $C < \infty$ such that

$$\sum_{k=0}^{\infty} r^{k(d+\epsilon)}\big|U_k\big(E \cap \sigma(F)\big)\big| \leq C,$$

so

$$\big|U_k\big(E \cap \sigma(F)\big)\big| \leq C r^{-k(d+\epsilon)}$$

for all $k \geq 0$. When calculating upper box-counting dimension it is enough to consider coverings by intervals of lengths r^{-k} for $0 \leq k < \infty$ (see [17]), so

$$\overline{\dim}_B(E \cap \sigma(F)) \leq d + \epsilon = a + b + \log n/\log r + \epsilon.$$

Taking ϵ arbitrarily small and a and b arbitrarily close to $\overline{\dim}_B E$ and $\overline{\dim}_B F$ gives (2.2). □

Cleary, the upper box-counting dimension of a set cannot be negative, and with only a minor variation in the proof of Theorem 2.7, we show what happens in the case that $\overline{\dim}_B E + \overline{\dim}_B F + \frac{\log n}{\log r} < 0$.

Theorem 2.8. *Let $E, F \subset \mathcal{C}_n$ be such that $\overline{\dim}_B E + \overline{\dim}_B F + \frac{\log n}{\log r} < 0$. Then, almost surely, $E \cap \sigma(F) = \emptyset$.*

Proof. Take a and b in $\mathbb{R}$ such that $a > \overline{\dim}_B E$, $b > \overline{\dim}_B F$, and $a + b + \log n / \log r < 0$. Let $d = a + b + \log n / \log r < 0$ and let ϵ satisfy $0 < \epsilon < -d$. By the arguments in the proof of Theorem 2.7, there exists a $c > 0$ such that

$$\mathbb{E}\Big(\sum_{k=0}^{\infty} r^{k(d+\epsilon)} \big|U_k\big(E \cap \sigma(F)\big)\big|\Big) \leq \sum_{k=0}^{\infty} c r^{k\epsilon} < \infty.$$

Thus, almost surely, there exists a random $C < \infty$ such that

$$\sum_{k=0}^{\infty} r^{k(d+\epsilon)} \big|U_k\big(E \cap \sigma(F)\big)\big| \leq C,$$

which implies that

$$r^{k(d+\epsilon)} \big|U_k\big(E \cap \sigma(F)\big)\big| \leq C$$

for all k, that is

$$\big|U_k\big(E \cap \sigma(F)\big)\big| \leq C r^{-k(d+\epsilon)} \ \to \ 0.$$

The number of intervals intersection $E \cap \sigma(F)$ is an integer so $\big|U_k\big(E \cap \sigma(F)\big)\big| = 0$ for all $k > K$ for some K almost surely. This shows that $E \cap \sigma(F) = \emptyset$ almost surely. □

2.3 Hausdorff Dimension: Upper Bound

We will now obtain an upper bound for the Hausdorff dimension of the intersections. Rather than working with Hausdorff measures, it is convenient to use an equivalent definition based on coverings of subsets of $\mathcal{C}_n$ by intervals (cones). Let $\mathcal{U} = \bigcup_{k=0}^{\infty} U_k$ denote the collection of intervals (which is the usual basis of open sets on Cantor space) and recall that $\operatorname{diam}(\cdot)$ denotes the diameter of a set with respect to the Cantor metric

$d(\cdot,\cdot)$. For $s \geq 0$, $\delta > 0$ and $A \subset \mathcal{C}_n$, define the δ-premeasures by

$$\mathcal{M}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(I_i)^s : A \subset \bigcup_{i=1}^{\infty} I_i, \mathrm{diam}(I_i) \leq \delta, I_i \in \mathcal{U}\right\}$$

and let

$$\mathcal{M}^s(A) = \lim_{\delta\to 0} \mathcal{M}^s_\delta(A).$$

Then $\mathcal{M}^s$ is a Borel measure on $\mathcal{C}_n$.

Lemma 2.9. *For all $A \subset \mathcal{C}_n$, $\mathcal{M}^s(A) = \mathcal{H}^s(A)$. In particular,*

$$\dim_H(A) = \sup\{s : \mathcal{M}^s(A) > 0\} = \inf\{s : \mathcal{M}^s(A) = 0\}.$$

Proof. Cleary $\mathcal{H}^s(A) \leq \mathcal{M}^s(A)$ for all A, since any admissible cover for $\mathcal{M}^s$ is an admissible cover for $\mathcal{H}^s$. For the opposite inequality, note that the diameter of any set $O \subset \mathcal{C}_n$ equals that of the smallest interval I of $\mathcal{U}$ that contains O. Thus replacing any covering set O by the corresponding interval I does not change the diameters involved in the definitions of the measures, so $\mathcal{M}^s(A) \leq \mathcal{H}^s(A)$. □

Theorem 2.10. *Let $E, F \subset \mathcal{C}_n$. Almost surely*

$$\dim_H(E \cap \sigma(F)) \leq \max\left\{\dim_H E + \overline{\dim}_B F + \frac{\log n}{\log r}, 0\right\}. \tag{2.4}$$

Proof. Let a and b be real numbers that satisfy $a > \dim_H E$ and $b > \overline{\dim}_B F$. Then there exists $c > 0$ such that for all $k \geq 0$

$$|U_k(F)| \leq c r^{-kb}.$$

By Lemma 2.9, for all $\delta > 0$ we can find a countable set of intervals $\{I_i\}_i \subseteq \mathcal{U}$ such that $E \subset \bigcup_i I_i$, $\mathrm{diam}(I_i) \leq \delta$, and $\sum_i \mathrm{diam}(I_i)^a \leq 1$. Taking only those intervals I_i that intersect $\sigma(F)$ non-trivially, gives a δ-cover of $E \cap \sigma(F)$ and therefore, for $s > 0$,

$$\mathcal{M}^s_\delta(E \cap \sigma(F)) \leq \sum_i \{\text{diam}(I_i)^s : \sigma^{-1}(I_i) \cap F \neq \emptyset\}.$$

Taking the expectation,

$$\mathbb{E}\big(\mathcal{M}^s_\delta(E \cap \sigma(F))\big) \leq \sum_i \text{diam}(I_i)^s \mathbb{P}(\sigma^{-1}(I_i) \cap F \neq \emptyset).$$

If $I_i \in U_k$, then $\text{diam}(I_i) = r^k$, so

$$\begin{aligned}
\mathbb{P}(\sigma^{-1}(I_i) \cap F \neq \emptyset) &= n^{-k}|U_k(F)| \\
&\leq c n^{-k} r^{-kb} \\
&= c \ \text{diam}(I_i)^{-(b+\log n/\log r)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\big(\mathcal{M}^s_\delta(E \cap \sigma(F))\big) &\leq c \sum_i \text{diam}(I_i)^{s-(b+\log n/\log r)} \\
&= c \sum_i \text{diam}(I_i)^a \text{diam}(I_i)^{s-(a+b+\log n/\log r)} \\
&\leq c \sum_i \text{diam}(I_i)^a \delta^{s-(a+b+\log n/\log r)} \\
&\leq c \delta^{s-(a+b+\log n/\log r)}
\end{aligned}$$

provided that $s - (a + b + \log n/\log r) > 0$. Taking $\delta = 2^{-k}$ and summing,

$$\mathbb{E}\Big(\sum_{k=1}^{\infty} \mathcal{M}^s_{2^{-k}}(E \cap \sigma(F))\Big) \leq c \sum_{k=1}^{\infty} 2^{-k(s-(a+b+\log n/\log r))} < \infty.$$

This implies that, almost surely,

$$\sum_{k=1}^{\infty} \mathcal{M}^s_{2^{-k}}(E \cap \sigma(F)) < \infty$$

so

$$\mathcal{M}^s(E \cap \sigma(F)) = \lim_{\delta \to 0} \mathcal{M}^s_\delta(E \cap \sigma(F)) = 0.$$

In particular, by Lemma 2.9, $\dim_H(E \cap \sigma(F)) \leq s$ almost surely, provided that $s > a + b + \log n / \log r$. This holds for a and b arbitrarily close to $\dim_H E$ and $\overline{\dim}_B F$, giving (2.4). □

Note that if, as often happens, either E or F is sufficiently regular to have equal Hausdorff and upper box-counting dimensions, then we get $\dim_H$ throughout inequality (2.4).

Again, minor changes to Theorem 2.10 show that $E \cap \sigma(F) = \emptyset$ almost surely if $\dim_H E + \overline{\dim}_B F + \log n / \log r < 0$, resulting in the following theorem.

Theorem 2.11. *Let $E, F \subset \mathcal{C}_n$ be such that $\dim_H E + \overline{\dim}_B F + \frac{\log n}{\log r} < 0$. Almost surely $E \cap \sigma(F) = \emptyset$.*

Proof. Let a, b, and ϵ be positive real numbers such that $a > \dim_H E$, $b > \overline{\dim}_B F$, and $\epsilon = -(a + b + \frac{\log n}{\log r}) > 0$. Then, by the definition of upper box-counting dimension, there exists $c > 0$ such that for all $k \geq 0$

$$|U_k(F)| \leq c r^{-kb}.$$

By Lemma 2.9, for each $k \in \mathbb{N}$, we can find a 2^{-k}-cover of E by intervals $I_i \in \mathcal{U}$ such that $\operatorname{diam}(I_i) \leq 2^{-k}$ and $\sum_i \operatorname{diam}(I_i)^a \leq 1$. Then, for an lth level interval I_i, we see

$$\begin{aligned}
\mathbb{P}(\sigma^{-1}(I_i) \cap F \neq \emptyset) &\leq n^{-l} |U_l(F)| \\
&\leq c n^{-l} r^{-lb} \\
&= \operatorname{diam}(I_i)^{-(b + \log n / \log r)}.
\end{aligned}$$

We can now take the expectation,

$$\begin{aligned}
\mathbb{E}(|\{I_i|\sigma^{-1}(I_i)\cap F\neq\emptyset\}|) &= \sum_i \mathbb{P}(\sigma^{-1}(I_i)\cap F\neq\emptyset) \\
&\leq \sum_i c\ \mathrm{diam}(I_i)^{-(b+\log n/\log r)} \\
&= \sum_i c\ \mathrm{diam}(I_i)^a \mathrm{diam}(I_i)^{-(a+b+\log n/\log r)} \\
&\leq c2^{-k\epsilon}\sum_i \mathrm{diam}(I_i)^a \\
&\leq c2^{-k\epsilon}.
\end{aligned}$$

Since $|\{I_i|\sigma^{-1}(I_i)\cap F\neq\emptyset\}|$ is an integer,

$$\begin{aligned}
\mathbb{P}(E\cap\sigma(F)\neq\emptyset) &\leq \mathbb{P}(|\{I_i|\sigma^{-1}(I_i)\cap F\neq\emptyset\}|\geq 1) \\
&\leq \mathbb{E}\big(|\{I_i|\sigma^{-1}(I_i)\cap F\neq\emptyset\}|\big) \\
&\leq c2^{-k\epsilon}.
\end{aligned}$$

This is true for all $k\in\mathbb{N}$, so $E\cap\sigma(F)=\emptyset$ almost surely. □

2.4 Hausdorff Dimension: Lower Bound

In this section we obtain a lower bound for the essential supremum of $\dim_H(E\cap\sigma(F))$ where σ is a random isometry. To achieve this we put Frostman-type measures on E and F and define a measure martingale that converges to a measure on $E\cap\sigma(F)$. By examining the s-energy of this measure we obtain a lower bound for the dimension that occurs with positive probability. The bulk of the calculation is devoted to showing that the martingales are $\mathcal{L}^2$-bounded.

2.4.1 Martingales

We begin this section with a brief introduction to martingales, as we use them to produce a lower bound for the Hausdorff dimension of the intersection of sets in Cantor space. See [38] for more details.

The following definition of conditional expectation is technical and we provide a simple explanation afterward.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space with Ω a set, $\mathcal{G}$ a sigma-algebra over Ω, and $\mathbb{P}$ a probability measure on the sets of $\mathcal{G}$. Also let X be a random variable on this space and $\mathcal{H}$ be a sub-sigma-algebra of $\mathcal{G}$. Then there exists a random variable Y such that:

- Y is $\mathcal{H}$ measurable;
- $\mathbb{E}(|Y|) < \infty$;
- for every set $H \in \mathcal{H}$, $\int_H Y \mathrm{d}\mathbb{P} = \int_H X \mathrm{d}\mathbb{P}$.

Furthermore, if Y' is a second random variable that satisfies these three conditions, then $Y = Y'$ almost surely. We say Y is the *conditional expectation* of X given $\mathcal{H}$, which we write $\mathbb{E}(X|\mathcal{H})$.

The sub-sigma-algebra $\mathcal{H}$ represents a collection of events inside Ω that provides some information on the outcome of X. For example, consider $(\operatorname{Iso} \mathcal{C}_n, \mathcal{F}, \mathbb{P})$, the probability space defined in Section 2.1. Then $\mathcal{F}_m$, the sigma-algebra that is the power set of allowable permutations of the mth level of $\mathcal{T}_n$ by its automorphisms, is a (finite) sub-sigma-algebra of $\mathcal{F}$. If we take the conditional expectation given $\mathcal{F}_m$ of a random variable $X(\sigma)$ with $\sigma \in \operatorname{Iso} \mathcal{C}_n$, it is as if we have knowledge of the action of σ on the first m levels of the infinite tree.

The *tower property* allows one to consolidate nested conditional expectations and is given in the following lemma.

Lemma 2.12. *Let X be a random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and $\mathcal{G}_1$ and $\mathcal{G}_2$ be sub-sigma-algebras of $\mathcal{G}$ such that $\mathcal{G}_2$ is a sub-sigma-algebra of $\mathcal{G}_1$. Then*

$$\mathbb{E}\big(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2\big) = \mathbb{E}(X|\mathcal{G}_2).$$

We can now define martingales. A sequence $\{Y_m, \mathcal{G}_m\}_{m=1}^{\infty}$ of random variables Y_m and sigma-algebras $\mathcal{G}_m$ such that Y_m is measurable with respect to $\mathcal{G}_m$ and $\mathcal{G}_m$ a sub-sigma algebra of $\mathcal{G}_{m+1}$ for all m is a *martingale* if it satisfies the following two conditions for all m:

(i) $\mathbb{E}(|Y_m|) < \infty$;

(ii) $\mathbb{E}(Y_{m+1}|\mathcal{F}_m) = Y_m$.

When the sigma-algebras are clear, we simply refer to the martingale as Y_m.

We say a martingale Y_m is $\mathcal{L}^2$-bounded if there exists a $C > 0$ such that $\mathbb{E}(Y_m^2) < C$ for all m. Under certain conditions, martingales will converge to random variables with desirable properties.

Theorem 2.13. *Let Y_m be an non-negative martingale. Then there is a non-negative random variable Y such that $Y_m \to Y$ almost surely. Moreover, if Y_m is $\mathcal{L}^2$ bounded, then $\mathbb{E}(Y) = \mathbb{E}(Y_m)$ for all m, and therefore $E(Y_1) > 0$ implies that $\mathbb{P}(Y \neq 0) > 0$.*

See [16] for proof.

In this section, we use martigales to build random set functions on $\mathcal{C}_n$ as the limits of martingales, which we then extend to measures.

2.4.2 Lower Bound of Hausdorff Dimension

Throughout this subsection, E, F will be Borel subsets of $\mathcal{C}_n$ and $0 < a < \dim_H E$ and $0 < b < \dim_H F$. Eventually we will take a and b arbitrarily close to the respective dimensions.

Lemma 2.14. *There exist probability measures μ and ν, with compact support contained in E and F respectively, and positive constants c_E and c_F such that for all $k \geq 0$ and $I \in U_k$,*

$$\mu(I) \leq c_E r^{ka} \quad \textit{and} \quad \nu(I) \leq c_F r^{kb}. \tag{2.5}$$

Proof. By Frostman's Lemma (Lemma 2.1), there are probability measures μ and ν, such that $\mu(A) \leq c_E \mathrm{diam}(A)^a$ and $\nu(A) \leq c_F \mathrm{diam}(A)^b$ for all $A \subset \mathcal{C}_n$. If $I \in U_k$, then $\mathrm{diam}(I) = r^k$ so the conclusion follows. □

Let $k \in \mathbb{N}$ and let μ and ν be probability measures given by Lemma 2.14. For all $A \in U_k$ and $l \geq k$ define a random variable

$$\tau_l(A) = n^l \sum_{I \in U_l(A)} \mu(I)\nu(\sigma^{-1}(I)). \tag{2.6}$$

Note that $\tau_l(A)$ is $\mathcal{F}_l$-measurable, where $\mathcal{F}_l$ is the sigma-algebra generated by the isometries defined at the lth level, see Section 2.1.

We will show that $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is an $\mathcal{L}^2$-bounded martingale and that the limits of these martingales give rise to an additive set function on $\mathcal{U} = \bigcup_{k=0}^{\infty} U_k$ and thus a measure on $\mathcal{C}_n$.

Lemma 2.15. *Let $A \in U_k$. Then $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is a non-negative martingale.*

Proof. Let $l \geq k+1$. For each $I \in U_l$, we write $I' \in U_{l-1}$ for the parent interval of I, i.e. if I corresponds to cone $[\rho|_l]$ for some $\rho \in \mathcal{C}_n$, then I' corresponds to cone $[\rho|_{l-1}]$ and therefore $I \subsetneq I'$. Then

$$\mathbb{E}\big(\tau_l(A)|\mathcal{F}_{l-1}\big) \;=\; n^l \sum_{I \in U_l(A)} \mu(I)\mathbb{E}\big(\nu(\sigma^{-1}(I))|\mathcal{F}_{l-1}\big). \tag{2.7}$$

Conditional on $\mathcal{F}_{l-1}$, $\sigma^{-1}(I)$ is equally likely to be any of the n children of $\sigma^{-1}(I')$ so

$$\mathbb{E}\big(\nu(\sigma^{-1}(I))|\mathcal{F}_{l-1}\big) = n^{-1}\nu(\sigma^{-1}(I')).$$

Partitioning the sum (2.7) over the intervals I' at the $(l-1)$th level gives

$$\begin{aligned} n^l \sum_{I' \in U_{l-1}(A)} \sum_{\substack{I \subset I' \\ I \in U_l}} \mu(I)\mathbb{E}\big(\nu(\sigma^{-1}(I))|\mathcal{F}_{l-1}\big) &= n^l \sum_{I' \in U_{l-1}(A)} \sum_{\substack{I \subset I' \\ I \in U_l}} n^{-1}\mu(I)\nu(\sigma^{-1}(I')) \\ &= n^{l-1} \sum_{I' \in U_{l-1}(A)} \mu(I')\nu(\sigma^{-1}(I')) \\ &= \tau_{l-1}(A). \end{aligned}$$

Clearly $\tau_l(A) \geq 0$ for all l, so $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is a non-negative martingale. □

In proving $\mathcal{L}^2$-boundedness, we use the following inequality.

Lemma 2.16. *Let $x_1, x_2, \ldots, x_n \geq 0$ be real numbers. Then*

$$n \sum_{i \neq j} x_i x_j \quad \leq \quad (n-1) \sum_{i,j} x_i x_j \tag{2.8}$$

Proof. Young's Inequality implies that $x_i x_j \leq \frac{1}{2}x_i^2 + \frac{1}{2}x_j^2$ for each pair i and j. By summing over all pairs such that $i \neq j$, we see that

$$\sum_{i \neq j} x_i x_j \leq (n-1) \sum_i x_i^2$$

and therefore

$$\begin{aligned} n \sum_{i \neq j} x_i x_j &= \sum_{i \neq j} x_i x_j + (n-1) \sum_{i \neq j} x_i x_j \\ &\leq (n-1) \sum_{i=1}^{n} x_i^2 + (n-1) \sum_{i \neq j} x_i x_j \\ &= (n-1) \sum_{i,j} x_i x_j. \end{aligned}$$

□

Lemma 2.17. *Assume that $a + b > -\log n/\log r$. There is a constant c_0 such that for all $A \in U_k$ and $l \geq k$,*

$$\mathbb{E}\big(\tau_l(A)^2\big) \leq c_0 \mu(A) r^{k(a+b+\log m/\log r)}. \tag{2.9}$$

In particular, the martingale $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is $\mathcal{L}^2$-bounded.

Proof. Let $A \in U_k$. We will first bound $\mathbb{E}\big(\tau_l(A)^2 | \mathcal{F}_{l-1}\big)$ in terms of $\tau_{l-1}(A)$ where $l \geq k+1$, to obtain (2.13) below. As before, we make the convention that $I' \in U_{l-1}$ is the parent interval of $I \in U_l$.

The expectation of $\tau_l(A)^2$ conditional on $\mathcal{F}_{l-1}$ breaks down into three sums: over pairs of distinct intervals with distinct parents, over pairs of distinct intervals with the same parent, and pairs of intervals which are the same. This gives

$$\begin{aligned}
\mathbb{E}\big(\tau_l(A)^2 | \mathcal{F}_{l-1}\big) &= n^{2l} \sum_{I,J \in U_l(A)} \mu(I)\mu(J)\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) \\
&= n^{2l} \sum_{\substack{I',J' \in U_{l-1}(A) \\ I' \neq J'}} \sum_{\substack{I \subset I' \\ J \subset J'}} \mu(I)\mu(J)\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) \qquad (2.10) \\
&\quad + n^{2l} \sum_{I' \in U_{l-1}(A)} \sum_{\substack{I,J \subset I' \\ I \neq J}} \mu(I)\mu(J)\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) \qquad (2.11) \\
&\quad + n^{2l} \sum_{I' \in U_{l-1}(A)} \sum_{I \subset I'} \mu(I)^2\mathbb{E}\Big(\nu(\sigma^{-1}(I))^2\big|\mathcal{F}_{l-1}\Big). \qquad (2.12)
\end{aligned}$$

We estimate the expectation terms in (2.10), (2.11), and (2.12) separately.

Case 1: The sum in (2.10) is over intervals $I, J \in U_l$ with different parent intervals, $I', J' \in U_{l-1}$ respectively. This implies that, given the sub-sigma-algebra $\mathcal{F}_{l-1}$, that $\nu(\sigma^{-1}(I))$ and $\nu(\sigma^{-1}(J))$ are independent random variables, and therefore

$$\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) = \mathbb{E}\Big(\nu(\sigma^{-1}(I))\big|\mathcal{F}_{l-1}\Big)\mathbb{E}\Big(\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big).$$

Given $\mathcal{F}_{l-1}$, $\sigma^{-1}(I)$ is equally likely to be any one of the n intervals $I_0 \in U_l$ that are children of $\sigma^{-1}(I')$, so

$$\mathbb{E}\Big(\nu(\sigma^{-1}(I))\big|\mathcal{F}_{l-1}\Big) = \sum_{\substack{I_0 \subset \sigma^{-1}(I') \\ I_0 \in U_l}} \frac{\nu(I_0)}{n} = \frac{\nu\big(\sigma^{-1}(I')\big)}{n},$$

with a similar expression for the term involving $\sigma^{-1}(J)$. The expected value in (2.10) then becomes

$$\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) = \frac{\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J'))}{n^2}.$$

Case 2: The sum in (2.11) is over two disjoint intervals with the same parent interval, $I' \in U_{l-1}$. The pair of intervals, $\sigma^{-1}(I)$ and $\sigma^{-1}(J)$, is equally likely to be any of the $n(n-1)$ pairs of distinct children I_0 and J_0 of $\sigma^{-1}(I') \in U_{l-1}$, and using (2.8),

$$\begin{aligned}
\mathbb{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) &= \sum_{\substack{I_0, J_0 \subset \sigma^{-1}(I') \\ I_0 \neq J_0}} \nu(I_0)\nu(J_0)\frac{1}{n(n-1)} \\
&\leq \sum_{I_0, J_0 \subset \sigma^{-1}(I')} \nu(I_0)\nu(J_0)\frac{1}{n^2} \\
&= \frac{\nu(\sigma^{-1}(I'))^2}{n^2}.
\end{aligned}$$

Case 3: The sum in (2.12) is over intervals I with parent interval I', and $\sigma^{-1}(I)$ is equally likely to be any of the n children of $\sigma^{-1}(I')$, say I_0. Combining this with the inequality $\nu(I_0) \leq c_F r^{lb}$ from (2.5),

$$\begin{aligned}
\mathbb{E}\Big(\nu(\sigma^{-1}(I))^2\big|\mathcal{F}_{l-1}\Big) &= \sum_{I_0 \subset \sigma^{-1}(I')} \nu(I_0)^2 n^{-1} \\
&\leq \sum_{I_0 \subset \sigma^{-1}(I')} c_F r^{lb} \nu(I_0) n^{-1} \\
&= \frac{c_F r^{lb}\nu(\sigma^{-1}(I'))}{n}.
\end{aligned}$$

Incorporating these three cases in (2.10)–(2.12) and using that $\mu(I) \leq c_E r^{la}$ for every $I \in U_l$,

$$\begin{aligned}
\mathbb{E}\big(\tau_l(A)^2|\mathcal{F}_{l-1}\big) \leq n^{2l} & \sum_{I',J'\in U_{l-1}(A)} \sum_{\substack{I\subset I'\\ J\subset J'}} \mu(I)\mu(J)\frac{\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J'))}{n^2} \\
&+ \ n^{2l} \sum_{I'\in U_{l-1}(A)} \sum_{I\subset I'} \mu(I) c_E r^{la} \frac{c_F r^{lb}\nu(\sigma^{-1}(I'))}{n} \\
&= n^{2(l-1)} \sum_{I',J'\in U_{l-1}(A)} \mu(I')\mu(J')\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J')) \\
&+ \ c_E c_F r^{la} r^{lb} n^l n^{l-1} \sum_{I'\in U_{l-1}(A)} \mu(I')\nu(\sigma^{-1}(I')) \\
&= \tau_{l-1}(A)^2 + c\tau_{l-1}(A) r^{l(a+b+\log n/\log r)},
\end{aligned} \tag{2.13}$$

where $c = c_E c_F$.

We apply this inequality inductively (working backwards) to bound $\mathbb{E}\big(\tau_l(A)^2|\mathcal{F}_k\big)$ where $A \in U_k$. Assume that for some j with $k \leq j \leq l-1$,

$$\mathbb{E}\big(\tau_l(A)^2|\mathcal{F}_j\big) \leq \tau_j(A)^2 + c\tau_j(A) \sum_{i=j+1}^{l} r^{i(a+b+\log n/\log r)}; \tag{2.14}$$

when $j = l-1$ this is just (2.13). Using the tower property for conditional expectation, inequalities (2.14), (2.13) (with j playing the role of l), and that τ_j is a martingale, we have

$$\begin{aligned}
\mathbb{E}\big(\tau_l(A)^2|\mathcal{F}_{j-1}\big) =& \mathbb{E}\big(\mathbb{E}(\tau_l(A)^2|\mathcal{F}_j)|\mathcal{F}_{j-1}\big) \\
&\leq \mathbb{E}\big(\tau_j(A)^2|\mathcal{F}_{j-1}\big) + c\mathbb{E}\big(\tau_j(A)|\mathcal{F}_{j-1}\big) \sum_{i=j+1}^{l} r^{i(a+b+\log n/\log r)} \\
&\leq \tau_{j-1}(A)^2 + c\tau_{j-1}(A) \sum_{i=j}^{l} r^{i(a+b+\log n/\log r)},
\end{aligned}$$

for the inductive step. Taking $j = k$ in (2.14) and recalling that $a + b + \log n/\log r > 0$, we conclude that

$$\mathbb{E}\big(\tau_l(A)^2|\mathcal{F}_k\big) \leq \tau_k(A)^2 + c_1\tau_k(A)r^{k(a+b+\log n/\log r)}, \tag{2.15}$$

where c_1 does not depend on l, k or A.

With $A \in U_k$ as before, we take unconditional expectations of this inequality, and use (2.6) and (2.5):

$$\begin{aligned}
\mathbb{E}\big(\tau_l(A)^2\big) \leq & \mathbb{E}\big(\tau_k(A)^2\big) + c_1\mathbb{E}\big(\tau_k(A)\big)r^{k(a+b+\log n/\log r)} \\
&= n^{2k}\mu(A)^2\mathbb{E}\big(\nu(\sigma^{-1}(A))^2\big) + c_1 n^k\mu(A)\mathbb{E}\big(\nu(\sigma^{-1}(A))\big)r^{k(a+b+\log n/\log r)} \\
&= n^{2k}\mu(A)^2\sum_{I\in U_k}\nu(I)^2 n^{-k} + c_1 n^k\mu(A)\sum_{I\in U_k}\nu(I)n^{-k}r^{k(a+b+\log n/\log r)} \\
&\leq c_E c_F n^k\mu(A)\sum_{I\in U_k}\nu(I)r^{k(a+b)} + c_1\mu(A)r^{k(a+b+\log n/\log r)} \\
&\leq c_0\mu(A)r^{k(a+b+\log n/\log r)},
\end{aligned} \tag{2.16}$$

where $c_0 = c_E c_F + c_1$. □

We now use the τ_l to obtain a limiting measure. First let $A \in \sigma(U_k)$, the sigma-algebra of subsets of $\mathcal{C}_n$ generated by the kth level intervals, so A is a (finite) union of intervals in U_k. For $l \geq k$ define

$$\tau_l(A) = n^l \sum_{I\in U_l(A)} \mu(I)\nu(\sigma^{-1}(I)).$$

Note that when $A \in U_k$ this coincides with the definition of $\tau_l(A)$ given by (2.6). For all k and all $A \in \sigma(U_k)$, $\{\tau_l(A), \mathcal{F}_l\}_{l\geq k}$ is a martingale since it is a finite sum of martingales. Thus $\tau_l(A)$ converges almost surely to a random variable on the sigma-algebra $\mathcal{F} = \sigma(\bigcup_{k=0}^{\infty}\mathcal{F}_k)$, so we may define, for all $A \in \bigcup_{k=0}^{\infty}\sigma(U_k)$,

$$\tau(A) = \lim_{l\to\infty}\tau_l(A), \tag{2.17}$$

the limit existing almost surely for all $A \in \bigcup_{k=0}^{\infty}\sigma(U_k)$ simultaneously.

Let $A, B \in \bigcup_{k=0}^{\infty} \sigma(U_k)$ be disjoint, so that $A, B \in \sigma(U_k)$ for some k. Then, for $l \geq k$, $\tau_l(A \cup B) = \tau_l(A) + \tau_l(B)$. Taking limits gives $\tau(A \cup B) = \tau(A) + \tau(B)$, so almost surely, τ is a finitely additive set function on $\bigcup_{k=0}^{\infty} \sigma(U_k)$. Since $\{\tau_l(\mathcal{C}_n), \mathcal{F}_l\}_{l \geq 0}$ is a non-negative martingale, $\tau_l(\mathcal{C}_n) < \infty$ almost surely. By the Hahn–Kolmogorov extension theorem, see [14], almost surely τ has a unique extension to $\sigma\big(\bigcup_{k=0}^{\infty} \sigma(U_k)\big)$, i.e. τ is a random Borel measure on $\mathcal{C}_n$.

Proposition 2.18. *The support of τ is contained in $E \cap \sigma(F)$, with $\tau(\mathcal{C}_n) < \infty$ almost surely and $\tau(\mathcal{C}_n) > 0$ with positive probability. Moreover, for all $k \geq 0$ and $A \in U_k$,*

$$\mathbb{E}\big(\tau(A)^2\big) \leq c_0 \mu(A) r^{k(a+b+\log n/\log r)}. \tag{2.18}$$

Proof. Let $x \notin E \cap \sigma(F)$ but $x \in \mathcal{C}_n$. Since μ and ν have support on compact subsets of E and F respectively, either $x \notin \operatorname{supp}(\mu)$ or $\sigma^{-1}(x) \notin \operatorname{supp}(\nu)$. Without loss of generality, assume $x \notin \operatorname{supp}(\mu)$. Then there exists an open neighborhood of x that does not intersect $\operatorname{supp}(\mu)$, which we may take to be an interval $A \in U_k$ for some k. Then by (2.6), for all $l \geq k$, $\tau_l(A) = 0$, so $\tau(A) = 0$ and x is not in the support of τ.

Since $\{\tau_l(\mathcal{C}_n), \mathcal{F}_l\}_{l \geq 0}$ is a non-negative martingale $0 \leq \tau(\mathcal{C}_n) < \infty$ almost surely, and, since it is $\mathcal{L}^2$-bounded, $\tau(\mathcal{C}_n) > 0$ with positive probability. Since $\mathcal{L}^2$-bounded martingales converge in $\mathcal{L}^2$, (2.18) follows from (2.9). □

Recall that the s-energy of a measure υ is defined as

$$I_s(\upsilon) = \int\int \frac{\mathrm{d}\upsilon(x)\mathrm{d}\upsilon(y)}{d(x,y)^s}.$$

To use the potential theoretic method in Section 2.1, we find the expected value of $I_s(\tau)$, where τ is the random measure on $E \cap \sigma(F)$ constructed above.

Lemma 2.19. *Let $0 < s < a + b + \log n/\log r$. Then*

$$\mathbb{E}\Big(\int\int \frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big) < \infty.$$

Proof. For x and y in Cantor space, note that $[x \wedge y]$ is the largest interval containing both x and y.

$$\begin{aligned}
\mathbb{E}\Big(\int\int \frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big) &\leq \mathbb{E}\Big(\sum_{k=0}^{\infty}\sum_{I\in U_k}\int\int_{[x\wedge y]=I}\frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big) \\
&\leq \sum_{k=0}^{\infty}\sum_{I\in U_k}\mathbb{E}\Big(r^{-ks}\int\int_{[x\wedge y]=I}\mathrm{d}\tau(x)\mathrm{d}\tau(y)\Big) \\
&\leq \sum_{k=0}^{\infty} r^{-sk}\sum_{I\in U_k}\mathbb{E}\big(\tau(I)^2\big) \\
&\leq c_0\sum_{k=0}^{\infty} r^{-sk}\sum_{I\in U_k}\mu(I)r^{k(a+b+\log n/\log r)} \\
&\leq c_0\sum_{k=0}^{\infty} r^{k(a+b+\log n/\log r-s)} \\
&< \infty,
\end{aligned}$$

since $a+b+\log n/\log r - s > 0$. □

Our final theorem now follows from the potential theoretic characterization of Hausdorff dimension.

Theorem 2.20. *Let E and F be Borel subsets of $\mathcal{C}_n$. For all $\epsilon > 0$,*

$$\dim_H(E\cap\sigma(F)) > \dim_H E + \dim_H F + \frac{\log n}{\log r} - \epsilon \tag{2.19}$$

with positive probability.

Proof. Let $0 < a < \dim_H E$, $0 < b < \dim_H F$ and $0 < s < a + b + \log n/\log r$. From Lemma 2.19, the s-energy of τ, $I_s(\tau)$, is finite almost surely. Provided that $\tau(\mathcal{C}_n) > 0$, which happens with positive probability by Proposition 2.18, then by Theorem 2.2

$$\dim_H(E\cap\sigma(F)) \geq s.$$

By choosing a and b sufficiently close to $\dim_H E$ and $\dim_H F$ and s close to $a + b +$

$\log n/\log r$, we obtain (2.19) for any given $\epsilon > 0$. □

We may rephrase Theorem 2.20 as follows, with the case of equality coming from Theorem 2.10.

Corollary 2.21. *Let E and F be Borel subsets of $\mathcal{C}_n$. Then*

$$\operatorname{esssup}_{\sigma\in \mathrm{Iso}\mathcal{C}_n}\{\dim_H(E\cap\sigma(F))\} \geq \dim_H E + \dim_H F + \frac{\log n}{\log r}.$$

Equality holds if either $\dim_H E = \overline{\dim}_B E$ *or* $\dim_H F = \overline{\dim}_B F$.

It is natural to ask whether the lower bound in Corollary 2.21 occurs with positive probability rather than just as an essential supremum. The following example shows that this is not true in general.

Example 2.22. The following construction shows that for all $0 < a, b < -\log n/\log r$ with $a+b+\log n/\log r > 0$ there exist Borel sets E and F such that $\dim_H E = \dim_B E = a$ and $\dim_H F = \dim_B F = b$ and

$$\mathbb{P}\left\{\dim_H(E\cap\sigma(F)) \geq \dim_H E + \dim_H F + \frac{\log n}{\log r}\right\} = 0.$$

For each integer $i > 1/a$, choose some interval $I_i \in U_i$ and construct a Borel set $E_i \subset I_i$ such that $\dim_H E_i = \dim_B E_i = a - 1/i$. This can be done in a number of ways including using a Cantor-type construction starting with I_i but varying slightly the number of children intervals at each stage to get the required dimension. (Imagine deleting several of the children at each to decrease the overall dimension of the subset.) We may further ensure that $|U_k(E_i)| \leq r^{-ka} = m^{-ka\log r/\log n}$ for all $k \geq i$. Let $E = \bigcup_{i>1/a} E_i$, so $\dim_H E = a$.

In the same way, let $F = \bigcup_{j>1/b} F_j$, where $F_j \subset I_j$ for some $I_j \in U_j$ for $j > 1/b$, and $\dim_H F_j = \dim_B F_j = b-1/j$, with $|U_k(F_j)| \leq m^{-kb\log r/\log m}$ for all $k \geq j$, so $\dim_H F = b$.

By Theorem 2.7 or Theorem 2.10, for each $i > 1/a$ and $j > 1/b$,

$$\dim_H(E_i \cap \sigma(F_j)) \le \max\big\{a + b + \log n/\log r - 1/i - 1/j, 0\big\}$$

with probability 1. Let $\epsilon > 0$. Since $E \cap \sigma(F) = \bigcup_{i>1/a} \bigcup_{j>1/b} E_i \cap \sigma(F_j)$,

$$\begin{aligned}
\mathbb{P}\big(\dim_H(E \cap \sigma(F)) > a + b + \log n/\log r - \epsilon\big) \\
&\le \sum_{1/i+1/j<\epsilon} \mathbb{P}\big(\dim_H(E_i \cap \sigma(F_j)) > a + b + \log n/\log r - \epsilon\big) \\
&\le \sum_{1/i+1/j<\epsilon} \mathbb{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) \\
&\le \sum_{j\ge i>1/\epsilon} \mathbb{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) + \sum_{i\ge j>1/\epsilon} \mathbb{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big). \qquad (2.20)
\end{aligned}$$

For $j \ge i$, by construction E_i is contained in at most $n^{-ja\log r/\log n}$ intervals of U_j, so

$$\begin{aligned}
\mathbb{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) &\le \mathbb{P}\big(E_i \cap \sigma(I_j) \neq \emptyset\big) \\
&\le n^{-ja\log r/\log n}\big/n^j \\
&= n^{-j(1+a\log r/\log n)}.
\end{aligned}$$

Thus, since $1 + a\log r/\log n > 0$, the left hand sum of (2.20) is at most

$$\begin{aligned}
\sum_{i>1/\epsilon} \sum_{j\ge i} n^{-j(1+a\log r/\log n)} &\le c_1 \sum_{i>1/\epsilon} n^{-i(1+a\log r/\log n)} \\
&\le c_2 n^{-(1+a\log r/\log n)/\epsilon},
\end{aligned}$$

where, provided that ϵ is sufficiently small, c_1 does not depend on i and ϵ and c_2 does not depend on ϵ. With a similar estimate of $c_3 n^{-(1+b\log r/\log n)/\epsilon}$ for the right hand sum of (2.20) we conclude that

$$\lim_{\epsilon\to 0} \mathbb{P}\big(\dim_H(E \cap \sigma(F)) > a + b + \log n/\log r - \epsilon\big) = 0.$$

Nevertheless, if the E and F are of positive Hausdorff measure in their dimensions we can achieve the lower bound with positive probability.

Proposition 2.23. *Let E and F be Borel subsets of $\mathcal{C}_n$ and suppose that $\mathcal{H}^a(E) > 0$ and $\mathcal{H}^b(F) > 0$ where $a = \dim_H E$ and $b = \dim_H F$. Then*

$$\mathbb{P}\left\{ \dim_H(E \cap \sigma(F)) \geq \dim_H E + \dim_H F + \frac{\log m}{\log r} \right\} > 0. \tag{2.21}$$

Proof. In this case, the inequalities (2.5) of Lemma 2.14 hold for suitable constants c_E and c_F where a and b actually equal the dimensions of E and F, respectively. The arguments of Section 4 then goes through without the need to approximate these dimensions. The probability for which (2.19) holds is then just the probability that $\tau(\mathcal{C}_n) > 0$ which will not depend on $\epsilon > 0$, so taking ϵ arbitraily small gives (2.21). □

Chapter 3

Thompson's Group V and Family

In 1965, Richard Thompson described several new groups in some unpublished notes, now collectively known as Thompson's groups. They are commonly represented as groups of homeomorphisms of Cantor space and quotients thereof. See Cannon, Floyd, and Perry [13] for an extensive background on these groups, their presentations, and important characteristics. The Thompson groups fit inside many different families of groups, such as those described in [20], [24], and [36], and we will explore one such family, the family of groups $V_n(G)$. We build these groups using transducers and describe some isomorphism results, specifically characterizing for which finite permutation groups G is the group $V_n(G)$ isomorphic to V_n. The results in this chapter can be found in [9].

3.1 Homeomorphisms as Transducers

A transducer, or finite state machine, is a tool from formal language theory that can be used to create functions from $\mathcal{C}_m$ to $\mathcal{C}_n$. In this section, we define several groups of homeomorphisms of Cantor space consisting of elements generated by transducers. We begin by defining transducers and stating some of their properties, before describing the Thompson Groups. Our description of transducers follows that of Grigorchuk, Nekrashevych, and Suschanskiĭ in [23].

3.1.1 Definitions and Examples

A *transducer* T is a tuple $\langle X, Y, Q, \lambda, \pi \rangle$ with

- $X = \{0, 1, \ldots, m-1\}$, a finite input alphabet,
- $Y = \{0, 1, \ldots, n-1\}$, a finite output alphabet,
- $Q = \{q_0, q_1, \ldots, q_{p-1}\}$, a finite set of internal states,
- $\lambda : X \times Q \to Y^*$, a rewrite rule, and
- $\pi : X \times Q \to Q$, a transition function.

To avoid trivial cases, we require $m, n \geq 2$.

Effectively, a transducer T will read words (finite or infinite) over the alphabet X and rewrite them as words over the alphabet Y, according to the rewrite rule and transition function, λ and π respectively. The transducer rewrites each letter according to its current state, and after it processes one letter, it may transition to another internal state.

In order for a transducer to receive an input word, it must have a current state. An *initialized transducer* is a transducer in a specified state and we denote a transducer T in state q by T_q. Often, we will drop the word 'initialized' when it is clear from context. Given a transducer T, we interpret the initialized transducer T_q as a function, $T_q : X^* \cup X^\omega \to Y^* \cup Y^\omega$.

The transformation induced by T is defined as follows. Let $\alpha = \alpha_1 \alpha_2 \ldots \in X^* \cup X^\omega$ with $|\alpha| \geq 2$. Then

$$T_q(\alpha) = \lambda(\alpha_1, q) \| T_{\pi(\alpha_1, q)}(\alpha_2 \alpha_3 \ldots).$$

After T_q rewrites the first letter of α, the transducer transitions to the next state (which depends on α_1 and the initial state q), and then rewrites the second letter, and so on.

Example 3.1. As an example, consider the transducer A with $X = Y = \{0, 1\}$ and $Q = S_2$, the symmetric group on two points with elements $\{id, (0\ 1)\}$. Transducers with group elements as states will be used extensively throughout this chapter. The rewrite rule λ is simply defined to be the action of the state q on letter i, i.e. $\lambda(i,q) = q(i)$. The transition function π is defined by $\pi(0,q) = id$ and $\pi(1,q) = (0\ 1)$, so that it does not depend on the state q. Consider an input of 0110 given to A_{id}:

$$\begin{aligned} A_{id}(0110) =& \lambda(0,id) \| A_{\pi(0,id)}(110) \\ =& 0 \| A_{id}(110). \end{aligned}$$

The transducer reads a 0, outputs a 0, and processes the rest of the word after transitioning to the next internal state (which happens to be the same state). The rest of the calculation is as follows:

$$\begin{aligned} A_{id}(0110) =& 0 \| A_{id}(110) \\ &= 0 \| \lambda(1,id) \| A_{\pi(1,id)}(10) \\ &= 01 \| A_{(0\ 1)}(10) \\ &= 01 \| \lambda\big(1,(0\ 1)\big) \| A_{\pi(1,(0,1))}(0) \\ &= 010 \| A_{(0\ 1)}(0) \\ &= 010 \| \lambda\big(0,(0\ 1)\big) \\ &= 0101. \end{aligned}$$

A transducer is called *synchronous* when λ maps $X \times Q$ into Y, i.e. λ has a single letter output. The transducer A from Example 3.1 is synchronous and therefore preserves the length of words given as an input. A transducer is *asynchronous* if it is not synchronous.

Note that often we will consider transducers with the same input and output alphabets, X and Y. When this is explicit, we will drop the output alphabet from the tuple that describes the transducer. For example, the previous transducer A would be denoted by $\langle X, S_2, \lambda, \pi \rangle$.

Transducers are often represented as directed graphs with the states Q as vertices and labeled edges representing the functions λ and π. Let $T = \langle X, Y, Q, \lambda, \pi \rangle$ with $\lambda(i, q_0) = j$ and $\pi(i, q_0) = q_1$. In the graph representation of T, there is a directed edge from vertex q_0 to vertex q_1 with the label i/j, representing the transition from q_0 to q_1 with input i and output j.

The graph representing the transducer A from Example 3.1 is shown in Figure 3.1. The vertices are $Q = S_2 = \{id, (0\ 1)\}$ with 2 edges starting from each vertex, one for each letter in the input alphabet. Note that an input (the left label) of 0 returns to state id and 1 leads to state (0 1). The output (the right label) is simply the input acted upon by the current state, the initial vertex of the edge in question.

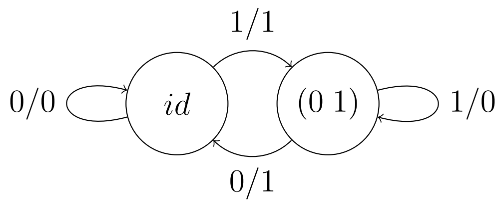

Figure 3.1: The transducer A

Often, it is useful to give an entire word in X^*, rather than a single letter, as an input to the rewrite rule and transition function, λ and π. As is standard in the literature, we will extend the definitions of λ and π to accommodate this notion. For $\alpha \in X^*$,

$$\lambda(\alpha, q) = T_q(\alpha),$$

as if the transducer had acted on each letter of α individually. Note that this implies that $\lambda(\epsilon, q) = \epsilon$, where ϵ is the empty word. Likewise,

$$\pi(\alpha, q) = \pi(\alpha_n \ldots \pi(\alpha_2, \pi(\alpha_1, q)) \ldots),$$

which is the state reached by T after having processed all of the word α. This allows us to extend the iterative definition of T_q to include processing any length prefix of an input word, rather than restricting to processing a single letter at a time, i.e. for $\alpha, \beta \in X_n^*$,

$$T_q(\alpha\beta) = \lambda(\alpha, q) \| T_{\pi(\alpha,q)}(\beta).$$

3.1.2 The Rational Group

In this subsection, we will define the rational group on n letters, $\mathcal{R}_n$, which is essentially the group of all invertible transformations of Cantor space induced by transducers. We begin by exploring the composition of transducers and examining transducers with equivalent actions on $\mathcal{C}_n$.

Transducers, like any functions, can be composed when the range and domain match appropriately. Let R and S be transducers such that the output alphabet of R is the same as the input alphabet of S. Then R and S can be composed to form a new transducer.

Lemma 3.2. *Let $R = \langle X, Y, Q_R, \lambda_R, \pi_R \rangle$ and $S = \langle Y, Z, Q_S, \lambda_S, \pi_S \rangle$. Define $T = \langle X, Z, Q_R \times Q_S, \lambda_T, \pi_T \rangle$ such that for $i \in X$ and $(q_r, q_s) \in Q_R \times Q_S$, the functions λ_T and π_T are defined by*

$$\lambda_T\big(i, (q_r, q_s)\big) = \lambda_S\big(\lambda_R(i, q_r), q_s\big)$$

and

$$\pi_T\big(i, (q_r, q_s)\big) = \big(\pi_R(i, q_r), \pi_S(\lambda_R(i, q_r), q_s)\big).$$

Then $S_{q_s} \circ R_{q_r}(\alpha) = T_{(q_r,q_s)}(\alpha)$ for all $\alpha \in X^* \cup X^{\mathbb{N}}$.

Proof. Let α be a word over X, $q_r \in Q_R$, and $q_s \in Q_S$. Then

$$\begin{aligned}(S_{q_s} \circ R_{q_r})(\alpha) =& S_{q_s}\big(R_{q_r}(\alpha)\big) \\ =& S_{q_s}\big(\lambda_R(\alpha_1, q_r) \| R_{\pi_R(\alpha_1,q_r)}(\alpha_2\alpha_3\ldots)\big) \\ =& \lambda_S(\lambda_R(\alpha_1, q_r), q_s) \| (S_{\pi_S(\lambda_R(\alpha_1,q_r),q_s)} \circ R_{\pi_R(\alpha_1,q_r)})(\alpha_2\alpha_3\ldots).\end{aligned}$$

However, doing the same calculation with T reveals

$$\begin{aligned}T_{(q_r,q_s)}(\alpha) =&\lambda_T\big(\alpha_1,(q_r,q_s)\big)\|T_{\pi_T\left(\alpha_1,(q_r,q_s)\right)}(\alpha_2\alpha_3\ldots)\\=&\lambda_S(\lambda_R(\alpha_1,q_r),q_s)\|T_{\left(\pi_R(\alpha_1,q_r),\pi_S(\lambda_R(\alpha_1,q_r),q_s)\right)}(\alpha_2\alpha_3\ldots).\end{aligned}$$

This shows that $S_{q_s}\circ R_{q_r}$ and $T_{(q_r,q_s)}$ behave in the same manner, i.e. rewriting letters in the same way and transitioning to corresponding states, and are therefore equivalent. □

When the input and output alphabets of a transducer are the same, i.e. $X = Y = \{0, 1, \ldots, n-1\}$, the transducer can be seen as a function from $\mathcal{C}_n$ into itself. As implied by the previous lemma, these transducers form a semigroup under composition.

Proposition 3.3. *Let $\mathcal{A}_n$ be the set of all transducers with input and output alphabets both equal to $X_n = \{1, 2\ldots, n\}$. Then $\mathcal{A}_n$ is a semigroup under composition.*

Distinct transducers may induce equivalent transformations of $\mathcal{C}_n$, even if they transform finite words in different ways. We explore this in the following example.

Example 3.4. Consider the one state 'identity' transducer acting on a two letter alphabet, $I = \langle X_2, \{id\}, \lambda_I, \pi_I\rangle$ where $\lambda_I(x, id) = x$ and $\pi_I(x, id) = id$. Clearly, $I_{id}(\alpha) = \alpha$ for any finite or infinite word over X_2. We compare this to the transducer $R = \langle X_2, \{q, a, b\}, \lambda_R, \pi_R\rangle$. Figure 3.2 defines the functions λ_R and π_R visually.

The transducer R remembers the previous input by moving to state a if the input was 0 and state b if the input was 1. The transducer then writes the 'stored' digit as it leaves states a and b. For finite words, R_q deletes the last letter, since the rewrite rule lags one behind the current letter. However, infinite words are mapped to themselves, meaning that R_q and I_{id} act equivalently on infinite words.

When transducers have the same action on infinite words, we call them ω-equivalent. We can then describe $\mathcal{S}_n$, the semigroup of transformations of $\mathcal{C}_n$ induced by transducers, as the quotient of the semigroup of all transducers acting on $\mathcal{C}_n$ by ω-equivalence.

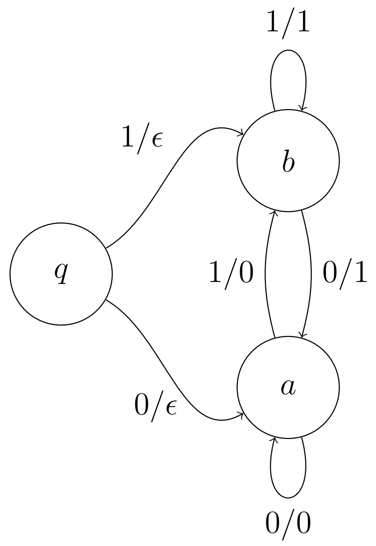

Figure 3.2: The transducer R

The *rational group on* n *letters* is the subgroup of $\mathcal{S}_n$ consisting of invertible transformations. Denoted by $\mathcal{R}_n$, this group was first introduced by Grigorchuk, Nekrashevych, and Suschanskiĭ in [23]. We present several key ideas from [23] as Theorem 3.5.

Theorem 3.5. *For* $n, m \geq 2$*, each transformation of* $\mathcal{C}_n$ *in* $\mathcal{R}_n$ *is a homeomorphism and* $\mathcal{R}_n$ *and* $\mathcal{R}_m$ *are conjugate.*

3.1.3 Thompson Groups and Generalizations

In 1965, Richard R. Thompson described in some unpublished notes several new groups, now collectively known as Thompson's groups F, T, and V. Our description of these groups will not be the same as their original description, but as homeomorphisms of binary Cantor space and as subgroups of the Rational Group on two letters. This embedding is explored in [23], but we will give a more explicit characterization of these homeomorphisms as transducers. We begin with Thompson's group V.

Let L and R be finite partitions of $\mathcal{C}_2$ into cones such that L and R have the same cardinality, and let $f : L \to R$ be a bijection. We can then define a function $v : \mathcal{C}_2 \to \mathcal{C}_2$ such that $v(\alpha\gamma) = \beta\gamma$ when $\gamma \in \mathcal{C}_2$, $[\alpha] \in L$, and $f([\alpha]) = [\beta]$. Note that v only changes a finite prefix and does not affect the rest of the infinite word.

Since L and R are partitions of Cantor space, v is a bijection, and its inverse is defined in the same manner using the inverse function $f^{-1} : R \to L$. The image of a cone under v and v^{-1} is a union of cones, and since cones are basic open sets, v and v^{-1} are continuous. This shows that v is a homeomorphism. Note that distinct pairs of partitions of Cantor space can produce the same homeomorphism of $\mathcal{C}_2$. *Thompson's group* V is the group of all such homeomorphisms of $\mathcal{C}_2$. This can be immediately extended to V_n, for homeomorphisms of $\mathcal{C}_n$, noting that $V = V_2$.

The other Thompson groups F and T can be described similarly. *Thompson's group* F is the subgroup of V consisting of those homeomorphisms that preserve the lexicographical order of $\mathcal{C}_2$. Preserving this order implies that F can also be interpreted as a group of homeomorphisms of the unit interval, using the binary expansion of the real line. *Thompson's group* T is also a subgroup of V, consisting of those homeomorphisms that preserve the lexicographical order of $\mathcal{C}_2$ up to cyclic permutation. Note that $F \lneq T \lneq V$. The group T can be described as a group of homeomorphisms of the unit circle, in the same fashion as F acts on the unit interval. The groups F_n and T_n are similarly defined as homeomorphisms of $\mathcal{C}_n$, with $F = F_2$ and $T = T_2$.

The Thompson groups F, T, and V are finitely presentable infinite groups. The groups T and V were the first known examples of such groups that are also simple, and while F is not simple, its derived subgroup is. Again, see Cannon, Floyd, and Parry [13] for an extensive background on the Thompson groups.

Each homeomorphism in V_n, and therefore in T_n and F_n, can be written as a transducer. Let $v \in V_n$ be formed from partitions L and R and bijection $f : L \to R$. Let

$$Q = \{\alpha \in X_n^* \mid \text{there exists } [\beta] \in L \text{ such that } \alpha \preccurlyeq \beta \text{ and } \alpha \neq \beta\},$$

i.e. the set of prefixes of cones in L, and consider the transducer $T = \langle X_n, Q \cup \{id\}, \lambda, \pi \rangle$ where λ and π are defined as follows for $x \in X_n$ and $\alpha \in Q$:

- $\lambda(x, \alpha) = \epsilon$ (the empty word) when $[\alpha x] \notin L$;
- $\lambda(x, \alpha) = \beta$ when $[\alpha x] \in L$ and $f([\alpha x]) = [\beta]$;
- $\lambda(x, id) = x$;
- $\pi(x, \alpha) = \alpha x$ when $[\alpha x] \notin L$;
- $\pi(x, \alpha) = id$ when $[\alpha x] \in L$;
- $\pi(x, id) = id$.

Lemma 3.6. *Let L and R be partitions of Cantor space into cones and $f : L \to R$ be a bijection. Define $v \in V_n$ and transducer T as above. Then $v = T_\epsilon$.*

Proof. Let $[\alpha] \in L$ with $|\alpha| = k$ and $f([\alpha]) = [\beta]$. Consider the action of T_ϵ on $\alpha\rho$ for $\rho \in \mathcal{C}_n$:

$$\begin{aligned} T_\epsilon(\alpha\rho) &= T_{\alpha_1}(\alpha_2 \dots \alpha_k \rho) \\ &\vdots \\ &= T_{\alpha_1 \dots \alpha_{k-1}}(\alpha_k \rho) \\ &= \beta \| T_{id}(\rho) \\ &= \beta\rho. \end{aligned}$$

This is the same action as v on $\alpha\rho$, replacing the prefix α with β. □

Although V_n is a finitely generated group, we use an infinite set of generators which we now describe to streamline our calculations. Let α and β be finite words in X_n^* such that $\alpha \perp \beta$, i.e. neither is a prefix of the other. Let v be the element of V_n such that $v(\alpha\gamma) = \beta\gamma$ and $v(\beta\gamma) = \alpha\gamma$ for all $\gamma \in \mathcal{C}_n$ and acts trivially otherwise. Provided that $[\alpha] \cup [\beta] \neq \mathcal{C}_n$, v is called a *small swap* and is denoted by $(\alpha\ \beta)$. We use this notation to draw parallels between permutations and the action of V on cones. The set of all small swaps generates V_n, i.e.

$$V = \langle(\alpha\ \beta) \mid \alpha \perp \beta \in X_n^*\rangle.$$

See [12] for details.

Thus far, we have described V_n as a group of homeomorphisms of Cantor space that can be represented as a group of tranducers. Their action on $\mathcal{C}_n$, however, only affects a finite prefix of a word and acts trivially thereafter. We now consider supergroups of V_n called $V_n(G)$, where G is a permutation group on n points, with elements whose actions extend infinitely beyond a finite prefix.

Let $\alpha \in X_n^*$ and S_n be the symmetric group acting on n points. For $g \in S_n$, consider the function $[\alpha]_g : \mathcal{C}_n \to \mathcal{C}_n$ that acts by applying the permutation g to each letter after the prefix α. Specifically, its action is defined by

$$[\alpha]_g(\alpha\gamma) = \alpha\|g(\gamma_1)\|g(\gamma_2)\|\ldots$$

for $\gamma \in \mathcal{C}_n$, and it acts trivially otherwise. Note that for nontrivial $g \in S_n$, the closure of support of $[\alpha]_g$ is $[\alpha]$. We call $[\alpha]_g$ an *iterated permutation*. For a subgroup $G \leq S_n$, the group $V_n(G)$ is defined as:

$$V_n(G) = \big\langle V_n, \{[\alpha]_g \mid \alpha \in X_n^*, g \in G\}\big\rangle.$$

Note that $V_n = V_n(\{id\})$.

The class of groups $\{V_n(G)\}$ is first explicitly studied in [20] in which Farley and Hughes study several finiteness properties, including showing that the commutator subgroup $V_n(G)'$ is a finite index simple subgroup. Of particular relevance to the work in this chapter, Farley and Hughes also demonstrate several non-isomorphism results distinguishing groups in this class. We improve upon these results by constructing isomorphisms between groups that cannot be distinguished using their methods, specifically addressing the issue of semiregular permutation groups raised in Example 7.24 in [20].

3.1.4 Statement of Results

Farley and Hughes use Rubin's Theorem in [20] to conclude that isomorphisms between $V_n(G)$ and $V_n(H)$ must be produced via conjugation by a homeomorphism of Cantor space. This allows them to make restrictions on isomorphisms by examining the dynamical structure of elements in the groups.

Proven in [35], Rubin's Theorem describes groups of homeomorphisms acting in a locally dense fashion on topological spaces. If Ω is a topological space and G is a group of self-homeomorphisms of Ω, then G is *locally dense* if and only if for every $x \in \Omega$ and open neighbourhood U of x, the set $\{g(x) \mid g \in G, g|_{\Omega \setminus U} = 1|_{\Omega \setminus U}\}$ has closure containing an open set. In other words, the orbit of x under elements of G fixing $\Omega \setminus U$ is dense in an open subset of U.

Theorem 3.7 (Rubin's Theorem)**.** *Let Ω and Σ be locally compact Hausdorff topological spaces without isolated points, let $A(\Omega)$ and $A(\Sigma)$ be the groups of self-homeomorphisms of Ω and Σ, respectively, and let $G \leq A(\Omega)$ and $H \leq A(\Sigma)$. If G and H are isomorphic and are both locally dense, then for each isomorphism $\phi : G \to H$, there is a unique homeomorphism, $\psi : \Omega \to \Sigma$, such that $\phi(g) = \psi g \psi^{-1}$ for every $g \in G$.*

We will refer to the conjugating homeomorphism ψ (which realises the isomorphism ϕ) as the *Rubin conjugator*. From Rubin's Theorem, we can conclude that isomorphisms between $V_n(G)$ and $V_n(H)$ are simply conjugation by homeomorphisms.

Lemma 3.8. *Let $G \leq S_n$ and $H \leq S_n$. If $\phi : V_n(G) \to V_n(H)$ is an isomorphism, then there exists a unique homeomorphism, ψ, from $\mathcal{C}_n$ to itself such that for every $v \in V_n(G)$ we have $\phi(v) = \psi v \psi^{-1}$.*

Proof. Cantor space is a compact Hausdorff space without isolated points and $V_n(G)$ and $V_n(H)$ are groups of self-homeomorphisms of $\mathcal{C}_n$. Let $\alpha \in \mathcal{C}_n$ and U be an open subset of $\mathcal{C}_n$ that contains α. Since $\mathcal{C}_n$ is compact, there exists a basic open set in U that contains α. This basic open set must be $[\alpha|_m]$ for some m. To show that V_n is locally dense, we show that for every $\beta \in [\alpha|_m]$ and cone $[\beta|_l]$ with $l > m$, there exists an element g in V_n such that $g|_{X \setminus U} = 1|_{X \setminus U}$ and $g(\alpha) \in [\beta|_l]$, i.e. α can be moved into every neighborhood of β by elements fixing $\mathcal{C}_n \setminus U$.

Let $\beta \in [\alpha|_m]$ and $l > m$. If $\alpha|_l = \beta|_l$, then $\alpha \in [\beta|_l]$. If not, consider the small swap $g = (\alpha|_l \ \beta|_l)$. Since $l > m$, the support of g is contained in $[\alpha|_m]$ and therefore in U. Also, $g(\alpha) \in [\beta|_l]$. This shows V_n is locally dense since each $\beta \in [\alpha|_m]$ is an accumulation point of the orbit of α under elements fixing U. The groups $V_n(G)$ and $V_n(H)$ are locally dense as well, since V_n is contained in both, and the lemma follows directly from Theorem 3.7. □

Note that this result also holds for $V_n(G)$ and $V_m(H)$ when $n \neq m$, but we focus on comparing the generalized Thompson groups acting on the same space.

In the case when G and H are conjugate subgroups of the symmetric group, a Rubin conjugator can be derived simply and explicitly.

Lemma 3.9. *Let G and H be conjugate subgroups of S_n. Then $V_n(G) \cong V_n(H)$.*

Proof. Since G and H are conjugate subgroups, there exists a permutation $g \in S_n$ such that $G^g = H$. However, conjugating permutations is equivalent to relabelling the points on which they act. It is clear that $V_n(G)$ and $V_n(H)$ will also be conjugate, and therefore isomorphic, by the same relabelling carried out on each letter of each infinite word in $\mathcal{C}_n$. The relabelling is equivalent to conjugating $V_n(G)$ by $[\epsilon]_g$, which permutes each letter of an infinite word. □

Lemma 3.9 greatly reduces the number of cases one needs to consider when solving the isomorphism problem in this family of groups. Our main results further this goal by exactly characterizing the groups G for which $V_n(G)$ is isomorphic to V_n. To state this chapter's main theorems, we need to define semiregular permutation groups. A permutation group $G \leq S_n$ is *semiregular* if and only if g has no fixed points for each non-identity element $g \in G$, i.e. $g(i) \neq i$ for all $i \in X_n$ and $g \neq id$. Examples of semiregular permutation groups acting on an appropriate number of points includes cyclic groups generated by an n-cycle acting on n points, which includes S_2, and the smallest non-cyclic example $\langle(0\ 1)(2\ 3),(0\ 2)(1\ 3)\rangle \leq S_4$.

Theorem 3.10. *Let $n \geq 2$ and $G \leq S_n$. Then $V_n(G) \cong V_n$ if and only if G is semiregular.*

Our proof given in Section 3.2 mirrors the methods of Farley and Hughes using Rubin's Theorem to show the forward implication, however, we avoid using techniques involving the more general group of germs construction (see [] and []) to give a simpler, tailored proof. We then construct explicit Rubin conjugators using transducers for the reverse implication. The transducers we develop are fairly general, allowing us to show that Theorem 3.10 can be seen as a sub-case of the following result. (Note that below we write $\mathrm{Stab}_{S_n}(R)$ for the set-wise stabilizer of R in S_n, not the pointwise stabilizer.)

Theorem 3.11. *Let $n \geq 2$ and let $H \leq S_n$ be semiregular. Let R be a set of orbit representatives of natural action of H on $X_n = \{0, 1, \ldots n-1\}$, and let $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. Then GH is a group and $V_n(GH) \cong V_n(G)$.*

3.2 Proof of Theorems 3.10 and 3.11

We begin with a discussion of semiregular permutations groups and and show the forward implication of Theorem 3.10. We then construct the Rubin conjugators using transducers, making several calculations that are useful in the main body of the proof. For small n, the isomorphism classes $V_n(G)$ can be completely described, and we do so at the end of the chapter.

3.2.1 Semiregular Permutation Groups

Let H be a semiregular subgroup of S_n and $R = \{x_1, ..., x_k\}$ be an orbit transversal of H, a set of orbit representatives of the natural action of H on X_n. Lemma 3.12 provides a way of describing elements in semiregular groups.

Lemma 3.12. *Let $i \in X_n$. Then there is a unique element $h_i \in H$ such that $h_i(i) \in R$.*

Proof. Since R contains an orbit representative for each orbit, there exists at least one element h_i such that $h_i(i) \in R$. Suppose that $h_i' \in H$ such that $h_i'(i) \in R$. Then $h_i(i) = h_i'(i)$ since the orbit representative of i is unique and therefore $h_i^{-1}h_i'(i) = i$. The only element in H with a fixed point is the identity, implying $h_i = h_i'$. □

We will use Lemma 3.12 as a definition for the element h_i of the semiregular group H given an orbit transversal R.

The following lemma demonstrates a useful relationship between $N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$ and H, alluded to in Theorem 3.11. Indeed, when H is regular, i.e. both semiregular and transitive, $N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R) \cong \mathrm{Aut}(H)$. The peculiar form of multiplication in the following lemma is not unlike twisted conjugacy [21] and arises during calculations concerning conjugating by transducers later in this chapter. Recall that $\mathrm{Stab}_{S_n}(R)$ denotes the set-wise stabilizer of R.

Lemma 3.13. *Let $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. Then $\langle G, H \rangle = G \rtimes H = GH$ and for all $h \in H$, $g \in G$, and $x \in X_n$,*

$$h_{gh(x)}ghh_x^{-1} = g.$$

Proof. Let $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. Note that since R is an orbit transversal of H, elements of $G \cap H \leq \mathrm{Stab}_{S_n}(R)$ map each $r \in R$ to itself, as r is the only point in its orbit under H in R. The only element fixing a point in H is the identity, implying that $G \cap H = \{id\}$. Among other things, this shows that $\langle G, H \rangle = G \rtimes H = GH$ and that there exists a unique decomposition $s = gh$ for each $s \in GH$ where $g \in G$ and $h \in H$.

Next, let $h \in H$, $x \in X_n$, and $r \in R$ be the orbit representative of x. Consider the action of $h_{h(x)}hh_x^{-1}$ on r,

$$h_{h(x)}hh_x^{-1}(r) = h_{h(x)}(h(x)) \in R.$$

Since r is the representative of its own orbit under H, $h_{h(x)}hh_x^{-1}$ maps r to itself. Therefore the element $h_{h(x)}hh_x^{-1} \in H$ must be the identity.

Let $g \in G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$ and $x \in X_n$. Because $g \in N_{S_n}(H)$, conjugating h_x by g gives $gh_xg^{-1} = h_y$ for some $y \in X_n$. Let $r \in R$ be the orbit representative of x and consider the action of h_y on $g(x)$,

$$\begin{aligned} h_y(g(x)) &= gh_xg^{-1}(g(x)) \\ &= gh_x(x) \\ &= g(r) \in R \end{aligned}$$

since $g \in \mathrm{Stab}_{S_n}(R)$. This shows that h_y maps $g(x)$ into R, i.e. $h_y = h_{g(x)}$. Therefore

$$h_{g(x)}gh_x^{-1} = g.$$

Putting these together gives

$$\begin{aligned} h_{gh(x)}\ gh\ h_x^{-1} &= (h_{gh(x)}\ g\ h_{h(x)}^{-1})(h_{h(x)}\ h\ h_x^{-1}) \\ &= g. \end{aligned}$$

□

When a permutation group G is not semiregular, it contains nontrivial elements with fixed points. Using these elements, we find iterated permutations $[\epsilon]_g$ that are not conjugate to elements in V_n

Lemma 3.14. *Let $g \in G \leq S_n$ and $x \in X_n$ be such that $g(x) = x$ but $g \neq id$. The element $[\epsilon]_g \in V_n(G)$ is not conjugate to any element in V_n.*

Proof. The point $\overline{x} \in \mathcal{C}_n$ is fixed under $[\epsilon]_g$. Since $g \neq id$, there exists $y \in X_n$ such that $g(y) \neq y$. Note that the words $x^{k-1}y\overline{x} \in \mathcal{C}_n$ for $k \in \mathbb{N}$ have finite nontrivial orbits under $[\epsilon]_g$ and can be found in every neighborhood of $\overline{x}$. We now show that no elements of V_n with fixed points have this property.

In order for an element $v \in V_n$ to have a fixed point, v must take some cone $[\alpha]$ to another cone $[\beta]$ that has nontrivial intersection with $[\alpha]$.

Case 1 $[\alpha] = [\beta]$: This implies that every point within the cone $[\alpha]$ is a fixed point. None of these fixed points will have points of nontrivial orbit within small open neighbourhoods.

Case 2 $[\beta] \subseteq [\alpha]$: This implies there exists a word γ such that $\beta = \alpha\gamma$. Let χ be an infinite word and note that $v^m(\alpha\chi) = \alpha\gamma^m\chi$ for all m. This shows that the only fixed point of v in $[\alpha]$ is $\alpha\overline{\gamma}$ and all other points in $[\alpha]$ lie in infinite orbits.

Case 3 $[\alpha] \subseteq [\beta]$: The element v^{-1} takes cone $[\beta]$ to $[\alpha]$. This was discussed in Case 2 and therefore $[\beta]$ contains one fixed point and the rest have infinite orbit under v^{-1}. An element and its inverse have the same orbit structure so $[\alpha]$ also contains a fixed point and points of infinite orbit.

Conjugation preserves orbit structure and since no elements of V_n have nontrivial finite orbits arbitrarily close to a fixed point, $[\epsilon]_g$ cannot be conjugate to any element of V_n. □

We can now state the following immediate corollary, which is the forward implication of Theorem 3.10.

Corollary 3.15. *Let $G \leq S_n$. If $V_n(G) \cong V_n$, then G is semiregular.*

For the remainder of this chapter, H will represent a semiregular subgroup of S_n with orbit transversal R, and for each $i \in X_n$, h_i is the element that maps i into R.

3.2.2 Rubin Conjugators

In this subsection, we construct transducers which generate the Rubin conjugators realising isomorphisms between $V_n(GH)$ and $V_n(G)$ for semiregular H and $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. They are built from the semiregular group H and a choice R of orbit transversal for the action of H. We also give some specific examples and make several general calculations to describe the Rubin conjugator's interaction with elements of $V_n(GH)$.

The proof that these transducers produce the advertised isomorphism in Theorem 3.11 is given in Section 3.2.3.

We begin by defining the synchronous transducer $A^{H,R}$ for appropriate H and R as $A^{H,R} = \langle X_n, H, \pi, \lambda \rangle$ where the rewrite function λ and transition function π are defined for all $i \in X_n$ and $g \in H$ as

$$\lambda(i, g) = g(i) \quad \text{and} \quad \pi(i, g) = h_i.$$

Note that h_i is the unique element such that $h_i(i) \in R$. When H and R are understood, we simplify the notation by writing $A^{H,R} = A$.

The inverse of the transducer, A^{-1}, can be described as $A^{-1} = \langle X_n, H, \pi', \lambda' \rangle$, where the 'inverse' rewrite function λ' and 'inverse' transition function π' are defined for all $i \in X_n$ and $g \in H$ as

$$\lambda'(i, g) = g^{-1}(i) \quad \text{and} \quad \pi'(i, g) = h_{g^{-1}(i)}.$$

It is useful (and interesting) to note that π does not depend on the current state of the transducer, i.e. $\pi(i, g) = \pi(i, h)$ for all $i \in X$ and $g, h \in H$, whereas π' does depend on the state. One way to approach this idea is to see that $\pi(i, h)$ depends on the input letter i but $\pi'(i, h)$ depends on the output of λ', $h^{-1}(i)$. This characterization of transducers is linked to synchronizing automata, which we will not explore here, but is fundamental to the classification of transducers which produce automorphisms of V_n. See [8] for more details.

Example 3.16. The transducer A described in Example 3.1 is in fact equal to $A^{S_2,\{0\}}$. The inverse is shown in Figure 3.3. Again note that the target state depends on the output of the rewrite rule and not the input.

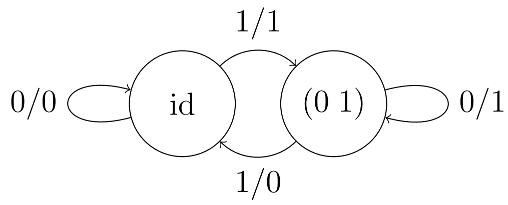

Figure 3.3: The inverse of the transducer $A^{S_2,\{0\}}$

Example 3.17. Figure 3.4 depicts another transducer which is constructed to produce a homeomorphism which will conjugate the group $V_3(G\,C_3)$ to $V_3(G)$, where the group C_3 is specifically the semiregular cyclic group of order three in S_3; $C_3 = \langle(1\ 2\ 3)\rangle$. To build our transducer, we choose $R = \{0\}$ for our orbit transversal. Note that appropriate G must stabilise $R = \{0\}$ and normalise the (normal) subgroup C_3.

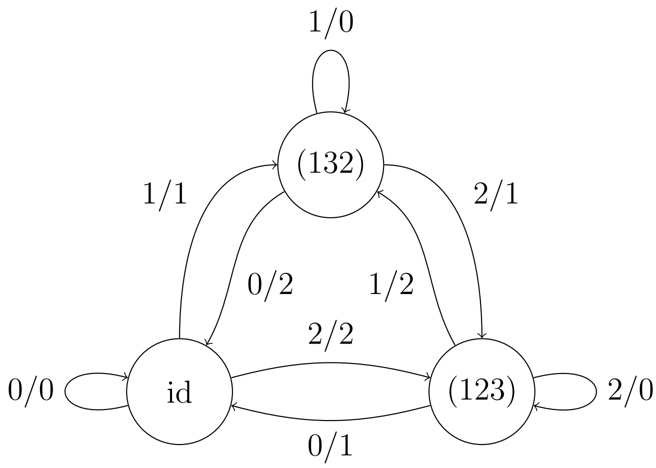

Figure 3.4: The transducer $A^{C_3,\{0\}}$.

To see that the transducers $A^{H,R}$ produce the advertised isomorphisms between $V_n(GH)$ and $V_n(G)$ for semiregular H and appropriate G, we look at where the generators of $V_n(G)$ and $V_n(GH)$ are taken under conjugation. The following lemmas are calculations to assist in computing conjugation by the homeomorphisms produced by such transducers.

We make a note of some useful notation for the following calculations. As before, for $\alpha \in X_n^*$ and $g \in G \leq S_n$, the element $[\alpha]_g \in V_n(G)$ acts on words in $\mathcal{C}_n$ iteratively as the permutation g after the prefix α and trivially otherwise. We also define a similar element of V_n denoted by $\lfloor\alpha\rfloor_g$ that acts as g precisely once, immediately after the prefix α. Let

$\beta \in \mathcal{C}_n$ and $i \in X_n$. The action is then defined as $\lfloor \alpha \rfloor_g(\alpha \| i \| \beta) = \alpha \| g(i) \| \beta$ on words in $[\alpha]$ and trivially elsewhere. A simple argument shows that $\lfloor \alpha \rfloor_g \in V_n$, as this element permutes the cones $[\alpha \| i]$ as g and fixes points without the prefix α.

Lemma 3.18. *Let $h, g \in H$. Then $A_h = \lfloor \epsilon \rfloor_{hg^{-1}} A_g$.*

Proof. Let $\alpha \in \mathcal{C}_n$, let $g, h \in H$ and consider $\lfloor \epsilon \rfloor_{hg^{-1}} A_g(\alpha)$,

$$\begin{aligned} \lfloor \epsilon \rfloor_{hg^{-1}} A_g(\alpha) &= \lfloor \epsilon \rfloor_{hg^{-1}}(g(\alpha_1) \| A_{\pi(\alpha_1, g)}(\alpha_2 \alpha_3 ...)) \\ &= h(\alpha_1) \| A_{\pi(\alpha_1, g)}(\alpha_2 \alpha_3 ...) \\ &= h(\alpha_1) \| A_{\pi(\alpha_1, h)}(\alpha_2 \alpha_3 ...) \\ &= A_h(\alpha), \end{aligned}$$

as $\pi(\alpha_1, g) = \pi(\alpha_1, h)$. □

Lemma 3.19. *Let $h, g \in H$. Then $A_h = A_g[\epsilon]_{g^{-1}h}$.*

Proof. Let $\alpha \in \mathcal{C}_n$ and let $g, h \in H$. The permutation $h_{g^{-1}h(x)} g^{-1} h \in H$ takes the letter x into the orbit transversal R. This implies that

$$h_{g^{-1}h(x)} g^{-1} h = h_x.$$

Now consider $A_g[\epsilon]_{g^{-1}h}(\alpha)$, in this case

$$\begin{aligned} A_g[\epsilon]_{g^{-1}h}(\alpha) &= A_g(g^{-1}h(\alpha_1) \| g^{-1}h(\alpha_2) \| g^{-1}h(\alpha_3) \dots) \\ &= gg^{-1}h(\alpha_1) \| h_{g^{-1}h(\alpha_1)} g^{-1}h(\alpha_2) \| h_{g^{-1}h(\alpha_2)} g^{-1}h(\alpha_3) \dots \\ &= h(\alpha_1) \| h_{\alpha_1}(\alpha_2) \| h_{\alpha_2}(\alpha_3) \dots \\ &= A_h(\alpha). \end{aligned}$$

□

The following lemma now applies to calculations in $V_n(GH)$.

Lemma 3.20. *Let $x \in X$ (so that $h_x \in H$) and $g \in N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. Then*

$$A_{h_x}[\epsilon]_g = [\epsilon]_g A_{h_{g^{-1}(x)}}.$$

Proof. Let $\alpha \in \mathcal{C}_n$, $x \in X$, and let $g \in N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$. Recall that $h_{g(x)} g h_x^{-1} = g$ from Lemma 3.13. Then,

$$\begin{aligned}
A_{h_x}[\epsilon]_g(\alpha) &= A_{h_x}(g(\alpha_1)\|g(\alpha_2)\|g(\alpha_3)\ldots) \\
&= h_x g(\alpha_1)\|h_{g(\alpha_1)} g(\alpha_2)\|h_{g(\alpha_2)} g(\alpha_3)\ldots \\
&= h_x g h_{g^{-1}(x)}^{-1} h_{g^{-1}(x)}(\alpha_1)\|h_{g(\alpha_1)} g h_{\alpha_1}^{-1} h_{\alpha_1}(\alpha_2)\|h_{g(\alpha_2)} g h_{\alpha_2}^{-1} h_{\alpha_2}(\alpha_3)\ldots \\
&= g h_{g^{-1}(x)}(\alpha_1)\|g h_{\alpha_1}(\alpha_2)\|g h_{\alpha_2}(\alpha_3)\ldots \\
&= [\epsilon]_g(h_{g^{-1}(x)}(\alpha_1)\|h_{\alpha_1}(\alpha_2)\|h_{\alpha_2}(\alpha_3)\ldots) \\
&= [\epsilon]_g A_{h_{g^{-1}(x)}}(\alpha).
\end{aligned}$$

□

3.2.3 Proof of Reverse Implication in Theorem 3.10

The following theorem shows the isomorphisms between different Thompson-like groups $V_n(G)$ for a fixed n via conjugation by $A_{id} = A_{id}^{H,R}$ constructed for a specific semiregular group H and choice of transversal R using Lemmas 3.18, 3.19, and 3.20. Note that this theorem implies Theorem 3.10.

To help the reader, we clarify our use of the transition functions π of the transducer A_{id} and π' of the inverse transducer A_{id}^{-1} in the following proof. The images $\pi(i,g)$ and $\pi'(i,g)$ are elements of the associated semiregular group H and we use $\pi(i,g)^{-1}$ and $\pi'(i,g)^{-1}$ to mean the inverses of those elements. Note that π is not invertible (it is not a bijection) and in particular, $\pi' \neq \pi^{-1}$.

Theorem 3.21. *Let $H \leq S_n$ be semiregular, R be an orbit transversal of H, and A be the usual transducer. Then for all $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$, the set GH is a group and* *the mapping* $\phi : V_n(GH) \to V_n(G)$ *defined by* $\phi(v) = A_{id} v A_{id}^{-1}$ *is an isomorphism.*

Proof. This proof is split into distinct parts describing where the generators of $V_n(GH)$ and $V_n(G)$ are taken under conjugation by A_{id} and A_{id}^{-1} respectively. The set of generators we use are the set of all small swaps and iterated permutations. Recall that a small swap $(\alpha\ \beta)$ for $\alpha \perp \beta \in X_n^*$ exchanges prefixes α and β and iterated permutation $[\rho]_g$ acts as permutation g after prefix ρ.

Case 1: Small Swaps under Conjugation

Let α and β be incomparable words in X_n^* such that $v = (\alpha\ \beta)$ is a small swap. Note that the support of $\phi(v)$, denoted by $\mathrm{supp}(\phi(v))$, is the support of v acted upon by A_{id}:

$$\mathrm{supp}\big(\phi(v)\big) = A_{id}\big(\mathrm{supp}(v)\big) = [A_{id}(\alpha)] \cup [A_{id}(\beta)].$$

Let $A_{id}(\alpha) \| \chi \in \mathcal{C}_n$ be a word in $\mathrm{supp}\big(\phi(v)\big)$. Then

$$\begin{aligned}
\phi(v)\big(A_{id}(\alpha)\|\chi\big) &= A_{id} v A_{id}^{-1}(A_{id}(\alpha)\|\chi) \\
&= A_{id} v(\alpha \| A_{\pi(\alpha,id)}^{-1}(\chi)) \\
&= A_{id}(\beta \| A_{\pi(\alpha,id)}^{-1}(\chi)) \\
&= A_{id}(\beta) \| A_{\pi(\beta,id)} A_{\pi(\alpha,id)}^{-1}(\chi) \\
&= A_{id}(\beta) \| \lfloor \epsilon \rfloor_{\pi(\beta,id)\pi(\alpha,id)^{-1}} A_{\pi(\alpha,id)} A_{\pi(\alpha,id)}^{-1}(\chi) \qquad \text{by Lemma 3.18} \\
&= A_{id}(\beta) \| \lfloor \epsilon \rfloor_{\pi(\beta,id)\pi(\alpha,id)^{-1}}(\chi).
\end{aligned}$$

This shows that $\phi(v)$ acts as $\lfloor A_{id}(\beta) \rfloor_{\pi(\beta,id)\pi(\alpha,id)^{-1}}(A_{id}(\alpha)\ \ A_{id}(\beta))$, first swapping the prefix $A_{id}(\alpha)$ with prefix $A_{id}(\beta)$ and then permuting a single level beneath prefix $A_{id}(\beta)$.

The case for β is similar. Let $A_{id}(\beta) \| \chi \in \mathcal{C}_n$ be a word in $\mathrm{supp}\big(\phi(v)\big)$. Then

$$\begin{aligned}
\phi(v)\big(A_{id}(\beta)\|\chi\big) &= A_{id} v A_{id}^{-1}(A_{id}(\beta)\|\chi) \\
&= A_{id} v(\beta \| A_{\pi(\beta,id)}^{-1}(\chi))
\end{aligned}$$

$$
\begin{aligned}
&= A_{id}(\alpha \| A^{-1}_{\pi(\beta,id)}(\chi)) \\
&= A_{id}(\alpha) \| A_{\pi(\alpha,id)} A^{-1}_{\pi(\beta,id)}(\chi) \\
&= A_{id}(\alpha) \| \lfloor \epsilon \rfloor_{\pi(\alpha,id)\pi(\beta,id)^{-1}} A_{\pi(\beta,id)} A^{-1}_{\pi(\beta,id)}(\chi) \qquad \text{by Lemma 3.18} \\
&= A_{id}(\alpha) \| \lfloor \epsilon \rfloor_{\pi(\alpha,id)\pi(\beta,id)^{-1}}(\chi).
\end{aligned}
$$

This shows that $\phi(v)$ acts as $\lfloor A_{id}(\alpha) \rfloor_{\pi(\alpha,id)\pi(\beta,id)^{-1}}(A_{id}(\beta)\ A_{id}(\alpha))$, first swapping the prefix $A_{id}(\beta)$ with prefix $A_{id}(\alpha)$ and then permuting a single level beneath prefix $A_{id}(\alpha)$.

We have now characterized the action of $\phi(v)$ on every word in its support and can conclude

$$
\phi(v) = \lfloor A_{id}(\alpha) \rfloor_{\pi(\alpha,id)\pi(\beta,id)^{-1}} \lfloor A_{id}(\beta) \rfloor_{\pi(\beta,id)\pi(\alpha,id)^{-1}} (A_{id}(\alpha)\ A_{id}(\beta)) \in V_n.
$$

Case 2: Iterated Permutations under Conjugation

Let $s \in GH$ and $\rho \in X_n^*$. This implies that there are unique $h \in H$ and $g \in G$ such that $s = gh$ and therefore $[\epsilon]_s = [\epsilon]_g[\epsilon]_h$. Now consider $\phi\big([\rho]_s\big)$. Again,

$$
\operatorname{supp}\big(\phi([\rho]_s)\big) = A_{id}\big(\operatorname{supp}([\rho]_s)\big) = [A_{id}(\rho)].
$$

Then for a word $A_{id}(\rho) \| \chi \in \operatorname{supp}(\phi([\rho]_s))$,

$$
\begin{aligned}
\phi\big([\rho]_s\big)(A_{id}(\rho)\|\chi) &= A_{id}[\rho]_s A^{-1}_{id}\big(A_{id}(\rho)\|\chi\big) \\
&= A_{id}[\rho]_s\big(\rho \| A^{-1}_{\pi(\rho,id)}(\chi)\big) \\
&= A_{id}\big(\rho \| [\rho]_s A^{-1}_{\pi(\rho,id)}(\chi)\big) \\
&= A_{id}(\rho) \| A_{\pi(\rho,id)}[\epsilon]_s A^{-1}_{\pi(\rho,id)}(\chi) \\
&= A_{id}(\rho) \| A_{\pi(\rho,id)}[\epsilon]_g[\epsilon]_h A^{-1}_{\pi(\rho,id)}(\chi) \\
&= A_{id}(\rho) \| [\epsilon]_g A_{\pi(g^{-1}(\rho),id)}[\epsilon]_h A^{-1}_{\pi(\rho,id)}(\chi) \qquad \text{by Lemma 3.20} \\
&= A_{id}(\rho) \| [\epsilon]_g A_{\pi(g^{-1}(\rho),id)h} A^{-1}_{\pi(\rho,id)}(\chi) \qquad \text{by Lemma 3.19} \\
&= A_{id}(\rho) \| [\epsilon]_g \lfloor \epsilon \rfloor_{\pi(g^{-1}(\rho),id)h\pi(\rho,id)^{-1}}(\chi) \qquad \text{by Lemma 3.18.}
\end{aligned}
$$

Summing up, we get that

$$\phi([\rho]_s) = \; [A_{id}(\rho)]_g \lfloor A_{id}(\rho) \rfloor_{\pi(g^{-1}(\rho),id)h\pi(\rho,id)^{-1}} \in V_n(G). \tag{3.1}$$

We have now shown that ϕ maps $V_n(GH)$ into $V_n(G)$ by mapping the generators of $V_n(GH)$ appropriately. To show that ϕ is onto, we consider the preimage of a generating set of $V_n(G)$.

Case 3: Small Swaps under Inverse Conjugation

Let $\alpha \in X_n^*$ and $\beta \in X_n^*$ such that $v = (\alpha\ \beta)$ is a small swap. Consider the image $\phi^{-1}(v) = A_{id}^{-1} v A_{id}$. Then

$$\mathrm{supp}\big(\phi^{-1}(v)\big) = A_{id}^{-1}(\mathrm{supp}(v)) = [A_{id}^{-1}(\alpha)] \cup [A_{id}^{-1}(\beta)].$$

For a word $A_{id}^{-1}(\alpha)\|\chi \in \mathcal{C}_n$, we see

$$\begin{aligned}
\phi^{-1}(v)(A_{id}^{-1}(\alpha)\|\chi) &= A_{id}^{-1}(\alpha\ \beta) A_{id}(A_{id}^{-1}(\alpha)\|\chi) \\
&= A_{id}^{-1}(\alpha\ \beta)(\alpha\|A_{\pi'(\alpha,id)}(\chi)) \\
&= A_{id}^{-1}(\beta\|A_{\pi'(\alpha,id)}(\chi)) \\
&= A_{id}^{-1}(\beta)\|A_{\pi'(\beta,id)}^{-1} A_{\pi'(\alpha,id)}(\chi)) \\
&= A_{id}^{-1}(\beta)\|A_{\pi'(\beta,id)}^{-1} A_{\pi'(\beta,id)}[\epsilon]_{\pi'(\beta,id)^{-1}\pi'(\alpha,id)}(\chi) \quad \text{by Lemma 3.19} \\
&= A_{id}^{-1}(\beta)\|[\epsilon]_{\pi'(\beta,id)^{-1}\pi'(\alpha,id)}(\chi).
\end{aligned}$$

This shows that $\phi^{-1}(v)$ first swaps prefix $A_{id}^{-1}(\alpha)$ with prefix $A_{id}^{-1}(\beta)$ and then acts as the permutation $\pi'(\beta, id)^{-1}\pi'(\alpha, id)$ at each level in the cone $[A_{id}^{-1}(\beta)]$.

The case for the cone $[A_{id}^{-1}(\beta)]$ is similar. For a word $(A_{id}^{-1}(\beta))\|\chi \in \mathcal{C}_n$, we see that

$$
\begin{aligned}
\phi^{-1}(v)(A_{id}^{-1}(\beta)\|\chi) =& A_{id}^{-1}(\beta\ \alpha)A_{id}(A_{id}^{-1}(\beta)\|\chi) \\
&= A_{id}^{-1}(\beta\ \alpha)(\beta\|A_{\pi'(\beta,id)}(\chi)) \\
&= A_{id}^{-1}(\alpha\|A_{\pi'(\beta,id)}(\chi)) \\
&= A_{id}^{-1}(\alpha)\|A_{\pi'(\alpha,id)}^{-1}A_{\pi'(\beta,id)}(\chi)) \\
&= A_{id}^{-1}(\alpha)\|A_{\pi'(\alpha,id)}^{-1}A_{\pi'(\alpha,id)}[\epsilon]_{\pi'(\alpha,id)^{-1}\pi'(\beta,id)}(\chi) \quad \text{by Lemma 3.19} \\
&= A_{id}^{-1}(\alpha)\|[\epsilon]_{\pi'(\alpha,id)^{-1}\pi'(\beta,id)}(\chi).
\end{aligned}
$$

This shows that $\phi^{-1}(v)$ first swaps prefix $A_{id}^{-1}(\beta)$ with prefix $A_{id}^{-1}(\alpha)$ and then acts as the permutation $\pi'(\alpha, id)^{-1}\pi'(\beta, id)$ at each level in the cone $[A_{id}^{-1}(\alpha)]$.

We have now characterized the action of $\phi^{-1}(v)$ on every word in its support and can conclude

$$\phi^{-1}(v) = [A_{id}^{-1}(\alpha)]_{\pi'(\alpha,id)^{-1}\pi'(\beta,id)}[A_{id}^{-1}(\beta)]_{\pi'(\beta,id)^{-1}\pi'(\alpha,id)}\big(A_{id}^{-1}(\alpha)\ A_{id}^{-1}(\beta)\big).$$

The permutations $\pi'(\alpha, id)$ and $\pi'(\beta, id)$ are in H, implying $\phi^{-1}(v) \in V_n(H) \leq V_n(GH)$.

Case 4: Iterated Permutations under Inverse Conjugation

Lastly, we consider the iterated permutations $[\rho]_g$ where $\rho \in X_n^*$ and $g \in G$. By using Equation (3.1) and replacing ρ with $A_{id}^{-1}(\rho)$ and setting s equal to g (i.e. $h = id$), we see that

$$\phi([A_{id}^{-1}(\rho)]_g) = [\rho]_g\lfloor\rho\rfloor_{\pi(g(A_{id}^{-1}(\rho)),id)\pi(A_{id}^{-1}(\rho),id)^{-1}}.$$

Since $\lfloor\rho\rfloor_{\pi(g(A_{id}^{-1}(\rho)),id)\pi(A_{id}^{-1}(\rho),id)^{-1}} \in V_n$ and can therfore be expressed as a product of small swaps, applying ϕ^{-1} gives

$$\phi^{-1}([\rho]_g) = [A_{id}^{-1}(\rho)]_g\ \phi^{-1}\big(\lfloor\rho\rfloor^{-1}_{\pi(g(A_{id}^{-1}(\rho)),id)\pi(A_{id}^{-1}(\rho),id)^{-1}}\big) \in V_n(HG).$$

All together, ϕ has been shown to be an onto mapping from $V_n(HG)$ to $V_n(G)$ and since the mapping is via conjugation, it is also one-to-one and a homomorphism. This demonstrates that ϕ is an isomorphism as desired. $\square$

Of particular interest is when the group $G \leq N_{S_n}(H) \cap \mathrm{Stab}_{S_n}(R)$ is the trivial group. This says

$$V_n(H) \cong V_n$$

and we have proven the reverse implication of Theorem 3.10.

3.2.4 Example Isomorphism Classes

Using Theorem 3.21, as well as some extensions of the non-isomorphism results in Section 2, we are able to distinguish several isomorphism classes of $V_n(G)$ for small n and a few are described here.

Isomorphism Classes when n=2:

The symmetric group on two points S_2 has two subgroups, the trivial group and itself. Adding the action of the trivial group to V_2 merely gives the familiar Thompson's group V, whereas $V_2(S_2)$ has the extra action of elements of the form $[\rho]_{(1\ 2)}$. One way to picture the action of these elements on the infinite tree $\mathcal{T}_2$ is as 'reflecting' or 'flipping' the entire tree beneath the node ρ. However, the symmetric group S_2 is semiregular, meaning $V_2 \cong V_2(S_2)$. The transducer A_{id} used for this isomorphism is shown in Figure 3.1. We provide two descriptions of this isomorphism, an informal heuristic argument and a formal description.

Following the action of the transducer down the infinite tree $\mathcal{T}_2$, we see that A_{id} enters the state id after travelling down the left branch (corresponding to 0) and enters the state (1 2) down the right branch (corresponding to 1). Using this picture, A_{id} will 'flip' the right half of each node in the tree. Conjugating by this will undo the infinite action of $[\rho]_{(1\ 2)}$ at node ρ, cutting it off after one level. The uniform nature of A_{id} will cancel with itself when conjugating elements of V, adding only a few additional small swaps at a finite number of locations.

From the calculations in Subsection 3.2.3, we can show the action of the isomorphism explicitly. Each element $v \in V_2$ can be described using two partitions of $\mathcal{C}_n$ into cones, L and R, and a bijection $f : L \to R$, where $v(\alpha\chi) = \beta\chi$ whenever $f([\alpha]) = [\beta]$. This description extends nicely to $w \in V_2(S_2)$, where each element requires a third list of permutations in S_2, describing the action of w on the infinite suffix of points in each cone of R. The action of w can be summarized in the notation earlier in this paper as $w = v\prod_{\beta\in R}[\beta]_{g_\beta}$. Note that g_β may be the trivial permutation, in which case $[\beta]_{g_\beta}$ is the identity. On the other hand, referring back to Section 3.2.3, it is clear that

$$\phi\big([\beta]_{(0\ 1)}\big) = \lfloor A_{id}(\beta)\rfloor_{(0\ 1)}.$$

The conjugate $\phi(v)$ is harder to describe, but it is derived in nearly the same fashion as small swaps. If α and β end in the same letter, i.e. both end in 0 or or both end in 1, then $\pi(\alpha, id) = \pi(\beta, id)$ and in particular, $\pi(\beta, id)\pi(\alpha, id)^{-1} = id$. This implies if $v(\alpha\chi) = \beta\chi$ and the last letter of α and β is the same,

$$\phi(v)\big(A_{id}(\alpha)\|\chi\big) = A_{id}(\beta)\|\chi.$$

However, when α and β end in different letters, $\pi(\beta, id)\pi(\alpha, id)^{-1} = (0\ 1)$ since there are only two elements in S_2. In this case,

$$\phi(v)\big(A_{id}(\alpha)\|\chi\big) = A_{id}(\beta_i)\|\lfloor\epsilon\rfloor_{(0\ 1)}(\chi).$$

This shows that the element $\phi(w) \in V$ is a product of three elements in V. The first corresponds to the partitions $L' = \{[A_{id}(\alpha)] | [\alpha] \in L]\}$ and $R' = \{[A_{id}(\beta)] | [\beta] \in R\}$ and the bijection $f' : L' \to R'$ where $f'([A_{id}(\alpha)]) = [A_{id}(\beta)]$ whenever $f([\alpha]) = [\beta]$. The second is the product $\prod_{\beta\in I}\lfloor A_{id}(\beta)\rfloor_{(0\ 1)}$ where $I \subseteq R$ is the subset of cones in R

such that the last letters of α and $f(\alpha)$ differ for each $[f(\alpha)] \in I$. The last element is $\prod_{\beta \in R} \lfloor A_{id}(\beta) \rfloor_{g_\beta}$. This can be generalized to higher n and other semiregular groups, but quickly becomes cumbersome and perhaps unhelpful for studying these groups in generality.

Isomorphism Classes when n=3:

For permutations on three points, there are four conjugacy classes of subgroups of S_3 with representatives: the trivial group, $S_2 := \langle (0\ 1) \rangle$, $C_3 := \langle (0\ 1\ 2) \rangle$, and S_3. It is clear that C_3 is semiregular, so $V_3 \cong V_3(C_3)$, but both S_2 and S_3 are not. However, the normalizer of C_3 is all of S_3, and S_2 stabilizes a potential orbit transversal of C_3, namely the set $\{2\}$. This fits the criteria for Theorem 3.21 and therefore $V_3(S_3) = V_3(S_2C_3) \cong V_3(S_2)$. This splits $V_3(G)$ into two distinct isomorphism classes. Both isomorphisms are built using the same transducer in Figure 3.4, perhaps with a different orbit transversal R, which can be done simply with a relabelling of the alphabet $\{0, 1, 2\}$. Note that comparing this to the case when $n = 2$ highlights the subtle yet rather intuitive fact that the number of points a permutation group acts on will significantly change orbit dynamics. In particular, $V_n(G) \cong V_n(H)$ does not imply that $V_m(G) \cong V_m(H)$, for $m \neq n$.

Isomorphism Classes when n=4:

A similar process can used to distinguish the four isomorphism classes for $V_4(G)$.

$$V_4 \cong V_4\Big(\big\langle (0\ 1)(2\ 3) \big\rangle\Big) \cong V_4\Big(\big\langle (0\ 1\ 2\ 3) \big\rangle\Big) \cong V_4\Big(\big\langle (0\ 1)(2\ 3), (0\ 2)(1\ 3) \big\rangle\Big)$$

$$V_4\Big(\big\langle (0\ 1) \big\rangle\Big) \cong V_4\Big(\big\langle (0\ 1), (0\ 1)(2\ 3) \big\rangle\Big) \cong V_4\Big(\big\langle (0\ 1), (0\ 2\ 1\ 3) \big\rangle\Big)$$

$$V_4\Big(\big\langle (0\ 1\ 2) \big\rangle\Big) \cong V_4(A_4)$$

$$V_4(S_3) \cong V_4(S_4)$$

The method of examining orbit structure in Lemma 3.14 can be used to differentiate these classes, particularly the orbits near fixed points of elements like $[\epsilon]_g$. Farley and Hughes's non-isomorphism result in [20] could also be used to distinguish these classes.

This example demonstrates the importance of the orbit structure of the permutation group used in the extension rather than the isomorphism type of the group used. Notably, there are integers $n > 2$ and $G \cong H$ subgroups of S_n with $V_n(G) \ncong V_n(H)$.

Isomorphism Classes when n=5:

The smallest examples of groups $V_n(G)$ and $V_n(H)$ for which we do not know whether they are isomorphic, occur when $n = 5$. There are 19 conjugacy classes of subgroups of S_5, nine of which form known singleton isomorphism classes of $V_5(G)$. There are also three pairs of conjugacy classes that form isomorphism classes of size two.

$$V_5 \cong V\Big(\big\langle(0\ 1\ 2\ 3\ 4)\big\rangle\Big)$$

$$V_5\Big(\big\langle(0\ 1)(2\ 3)\big\rangle\Big) \cong V\Big(\big\langle(0\ 1\ 2\ 3\ 4), (2\ 5)(3\ 4)\big\rangle\Big)$$

$$V_5\Big(\big\langle(0\ 1\ 2\ 3)\big\rangle\Big) \cong V\Big(\big\langle(0\ 1\ 2\ 3\ 4), (2\ 3\ 5\ 4)\big\rangle\Big)$$

The remaining four conjugacy classes do not form $V_5(G)$ that belong to any of the above isomorphism classes. However, they can be split into pairs, $\{A_4, A_5\}$ and $\{S_4, S_5\}$, such that it is not known whether $V_5(G) \cong V_5(H)$ within each pair, yet they are not isomorphic across the pairs. This means that there are between 14 and 16 isomorphism classes, all of size one or two.

This problem extends beyond $n = 5$, and in fact, for all $n \geq 5$, it is not known whether $V_n(A_n) \cong V_n(A_{n-1})$ or $V_n(S_n) \cong V_n(S_{n-1})$. Solving the specific case for $n = 5$ merits some attention, as a solution could possibly be extended to a general n. We discuss this problem and more in Chapter 5.

Chapter 4

Invariant Factors

In this chapter, we will examine tools useful for studying the topology of self-similar sets. In particular, we expolre quotients of Cantor space known as invariant factors, studied in [2], [3], and [4], that generalise key notions of self-similarity. It is worth noting that every self-similar set (and attractors of various other types of iterated function systems) is homeomorphic to an invariant factor.

We begin by generalizing a characterization of invariant factors as inverse limits, found in [3]. Specifically, we generalize the use of partitions of Cantor space into cones at a particular level to arbitrary partitions into cones, allowing us to state Theorem 4.6 on the relationship between this new characterization and homeomorphisms of $\mathcal{C}_n$. We then develop techniques using infinite walks on graphs to study the topology of a class of self-similar sets known as Sierpiński relatives in terms of invariant factors. Finally, we will compare another construction of quotients of Cantor space, known as edge replacement systems, [7], to invariant factors. Again, we use infinite walks on graphs to characterize edge replacement and give the conditions under which they form invariant relations.

This chapter will feature many graphs to characterize invariant factors. We will define several types of graphs that appear. A *directed multigraph* G is a set of vertices $V(G)$ and a multiset of edges $E(G)$ such that each edge e is an ordered pair in $V(G) \times V(G)$. For an edge (u, v), we say the edge starts at u, the *initial* vertex, and ends at v, the *terminal* (or *final*) vertex. We denote these by $start((u,v)) = u$ and $end((u,v)) = v$ and will often casually refer to the edge as an edge from u to v. Note that there may be duplicate edges, i.e. two edges with the same inital and terminal vertices, and *loops*, edges with the same terminal and final vertex. Two edges are *incident* if their intersection is nontrivial. Labels may be assigned to edges through a function $f : E(G) \to L$, where L is a set of labels. Usually, we refer to an edge using its label rather than the ordered pair. We define a *walk* to be a (finite or infinite) sequence of edges $e_1, e_2, \dots$ in a directed graph such that $end(e_i) = start(e_{i+1})$. Note that this differs from a path, as each vertex, and therefore each edge in a path must be distinct.

An *undirected hypergraph* Γ is a set of vertices $V(\Gamma)$ and a multiset of edges $E(G)$ such that each edge $e \in E(G)$ is a element of $\bigcup_{k\geq 2} V(G)^k$ quotiented by the relation that identifies elements that are permutations of one another. The *endpoints* of an edge $e = \{v_1, v_2, \dots v_k\}$ are the vertices contained in e. An edge in an undirected hypergraph does not have an orientation (initial or terminal vertices) but may have more than two endpoints. Note that an edge in a hypergraph may have multiple endpoints at the same vertex.

4.1 Invariant Relations

In this section, we will introduce invariant relations on Cantor space with particular emphasis on several cases. Quotients by 'simple' invariant relations can be realized as a modified inverse limit of a sequence or net of graphs, which we describe in detail first in Propositions 4.4 and 4.5.

4.1.1 Construction

An equivalence relation $\sim$ on $\mathcal{C}_n$ is called an *invariant relation* if it satisfies the following two properties:

Property 1 for all $\alpha, \beta \in \mathcal{C}_n$ and $i \in X_n$, $\alpha \sim \beta$ if and only if $i\alpha \sim i\beta$;

Property 2 the relation $\sim$ is topologically closed in $\mathcal{C}_n \times \mathcal{C}_n$.

We give the following example to illustrate an invariant relation.

Example 4.1. Let $\sim$ be an invariant relation on $\mathcal{C}_2$ such that $0\overline{1} \sim 1\overline{0}$. By Property 1, we know that $00\overline{1} \sim 01\overline{0}$ and $10\overline{1} \sim 11\overline{0}$. Furthermore, by repeated use of Property 1, we see that for all $\alpha \in X_2^*$, $\alpha 0\overline{1} \sim \alpha 1\overline{0}$.

As it turns out, the relation $\sim$ defined on $\mathcal{C}_2$ such that for $\chi, \rho \in \mathcal{C}_2$, $\chi \sim \rho$ if and only if $\chi = \rho$ or $\chi = \alpha 0\overline{1}$ and $\rho = \alpha 1\overline{0}$ for some $\alpha \in X_2^*$ (or vice versa) defines an invariant relation. The relation $\sim$ clearly satisfies Property 1 and as shown in the proof of Lemma 4.9, $\sim$ is topologically closed.

This relation is equivalent to the relation on $\mathcal{C}_2$ that arises through the binary expansion of the unit interval. Let $x, y \in [0, 1]$ have binary expansions $x = 0.\chi_1\chi_2\chi_3 \ldots = 0.\chi$ and $y = \rho_1\rho_2\rho_3 \ldots = 0.\rho$. Then $x = y$ if and only if $\chi = \rho$ or $\chi = \alpha 0\overline{1}$ and $\rho = \alpha 1\overline{0}$ for some $\alpha \in X_2^*$ or vice versa. We explore the connection between self-similar sets (such as the unit inverval) and invariant relations in Section 4.2.

The relation in Example 4.1 was described using a small number of related elements. We define the *generators* of an invariant relation $\sim$ to be

$$M_\sim = \{\alpha \in \mathcal{C}_n | \text{there exists a } \beta \in \mathcal{C}_n \text{ such that } \alpha \sim \beta \text{ and } \alpha_1 \neq \beta_1\}.$$

and we call

$$Q_\sim = M_\sim / \sim$$

the *critical points* of $\sim$. In Example 4.1, the generators of $\sim$ is the set $\{0\overline{1}, 1\overline{0}\}$ and $Q_\sim$ is a single equivalence class.

The term generator is used since each equivalence class under $\sim$ not in $Q_\sim$ begins with a common prefix and can be produced by adding a prefix to an entire equivalence class in $M_\sim$.

Lemma 4.2. *Let $q \in \mathcal{C}_n/\sim$ be an equivalence class with cardinality greater than one. Then either $q \in Q_\sim$ or there exists a $q' \in Q_\sim$ and a nonempty $\gamma \in X_n^+$ such that $\alpha \in q'$ if and only if $\gamma\alpha \in q$.*

Proof. Let $q \notin Q_\sim$ be a non-trivial equivalence class, i.e. there exists $\chi, \rho \in q$ such that $\chi \sim \rho$ and $\chi \neq \rho$. Since $q \notin Q_\sim$ each word in q begins with the same first letter. We name the common one letter prefix γ_1 and consider the equivalence class $q_1 = \{\alpha \in \mathcal{C}_n | \gamma_1\alpha \in q\}$. Either, $q_1 \in Q_\sim$ and therefore $q_1 = q'$ from the statement of the lemma, or $q_1 \notin Q_\sim$. We can then create equivalence class q_2 in the manner as q_1, deleting the common prefix γ_2 from each word in q_1. Repeating this process, we will find that $q_m \in Q_\sim$ for some finite $m \geq 1$. If we do not, we reach a contradiction in that there is one word word in q. □

For a prefix $\gamma \in X_n^*$ and equivalence class $q \in Q_\sim$, we use the notation γq to mean the equivalence class of points $\{\gamma\alpha | \alpha \in q\}$. (Note that we also use this notation for the labels of edges in graphs later this chapter. The correlation between equivalence classes and edges is deliberate and will be emphasized.)

A quotient of $\mathcal{C}_n$ by an invariant relation is called an *invariant factor*. Bandt and Retta [4] use a characterization of certain invariant factors as inverse limits to study several of their properties. We first describe the original characterization in [3] as inverse limits of a sequence of graphs equipped with a topology, and then a further generalization as inverse limits of a net of graphs. These graphs encode the connectivity of the quotient space by describing when cones share points identified under the relation.

We first define an inverse limit (or projective limit). Let $\mathcal{A}$ be a *directed poset* with partial order $\leq$, i.e. a partially ordered set such that every finite subset of $\mathcal{A}$ has an upper bound. The natural numbers form a directed poset with the usual inequality. Partitions of a set form another natural directed poset, where $P_1 \leq P_2$ if P_2 is a refinement of P_1, because every finite set of partitions has a common refinement and therefore an upper bound. A *net* is a function with a directed poset as a domain. It is usual to say that the range of a net is *indexed* by poset. A sequence is then a net with $\mathbb{N}$ as the domain.

Let $\{Y_\alpha\}$ be a set of topological spaces indexed by directed poset $\mathcal{A}$. For each pair of $A, B \in \mathcal{A}$ with $A \leq B$, let $\mu_{B,A} : Y_B \to Y_A$ be a continuous map such that:

- the map $\mu_{A,A}$ is the identity on Y_A for each $A \in \mathcal{A}$, and
- the map $\mu_{B,A} \circ \mu_{C,B} = \mu_{C,A}$ for each triple $A \leq B \leq C$ in $\mathcal{A}$.

We form the space $\Pi_{A\in\mathcal{A}} Y_A$ and let $\pi_B : \Pi_{A\in\mathcal{A}} Y_A \to Y_B$ be the projection onto the Bth coordinate. The *inverse limit* of the system $(\mathcal{A}, \mu_{B,A})$ is then the subspace

$$\varprojlim Y_A = \{y \in \Pi_A Y_A \mid \text{for each } A \leq B, \pi_A(y) = \mu_{B,A} \circ \pi_B(y)\},$$

i.e. the subspace of $\Pi_A Y_A$ where the Bth coordinate maps onto the Ath coordinate via $\mu_{B,A}$. The topology on the inverse limit is the subspace topology of the infinite product topology, which we clarify. If U is a basic open set in Y_A, then the subset of the inverse limit

$$\{y \in \varprojlim Y_A \mid \pi_A(y) \in U\}$$

is a basic open set of the subspace topology.

Let $\sim$ be an invariant relation on $\mathcal{C}_n$ such that $M_\sim$ is a finite set. We first describe the quotient $\mathcal{C}_n/\sim$ as an inverse limit of a sequence of topological spaces represented by hypergraphs, as in [3]. Let $m \geq 1$ and define G_m to be a hypergraph with vertices $V(G_m) = X_n^m$ and edges having unique labels $E(G_m) = X^{<m} \times Q_\sim$, where $X_n^{<m} = \bigcup_{0\leq k<m} X_n^k$. Let $\alpha \in X_n^{<m}$ and $q \in Q_\sim$, so that αq is a unique edge in G_m. Then the edge αq has endpoints $\alpha\beta|_m$, for each $\beta \in q$. Since q is an equivalence class of points in $\mathcal{C}_n$, the edge αq could have more than two endpoints.

We equip G_m with a topology, specifically on the set of vertices and edges of G_m. The basis for the topology on G_m is described as follows:

- each vertex as a singleton is a basic open set;
- each edge unioned with its endpoints is a basic open set.

From this, each vertex can be considered an open point and each edge a closed point, as the complement of an edge e is the union of all basic open sets not containing e.

We now define a family of homomorphisms on this sequence of graphs in order to define an inverse limit. Let the map $p_{m,k} : G_m \to G_k$ for $m \geq k$ be defined as follows:

- $p_{m,k}(\alpha) = \alpha|_k$ for $\alpha \in V(G_m)$;
- $p_{m,k}(\beta q) = \beta|_k$ for $\beta q \in E(G_m)$ with $|\beta| \geq k$;
- $p_{m,k}(\beta q) = \beta q$ for $\beta q \in E(G_m)$ with $|\beta| < k$.

Note that $p_{m,k}$ is a continuous map that projects G_m onto G_k and $p_{m,m}$ is the identity map on G_m. We can then consider the inverse limit of the topological spaces G_m. We define π_m to be the canonical projection of $\Pi_k G_k$ onto its mth coordinate.

The elements that comprise $\varprojlim G_m$ come in two flavors. Let $x \in \varprojlim G_m$. If $\pi_m(x)$ is an edge in G_m, say γq for some $q \in Q_\sim$ and $\gamma \in X_n^{<m}$, then $x = (\gamma|_1, \gamma|_2, \ldots, \gamma|_{|\gamma|}, \gamma q, \gamma q \ldots)$. If $\pi_m(x)$ is not an edge for any m, there there exists an infinite word $\alpha \in \mathcal{C}_n$ such that $x = (\alpha|_1, \alpha|_2, \alpha|_3 \ldots)$. This demonstrates how the inverse limit of G_m can be represented as the union of $\mathcal{C}_n$ with the nontrivial equivalence classes of $\mathcal{C}_n$ under $\sim$. This is, in a sense, bigger than the quotient $\mathcal{C}_n/\sim$. We instead consider the completely regular modification of $\varprojlim G_m$ in order to 'shrink' the inverse limit to the proper size.

Let X be a topological space. The *completely regular modification* of X, denoted by $\mathrm{CR}(X)$, is the quotient of X by the equivalence relation that identifies points whose images are equal to one another under all continuous functions from X to the unit interval $[0, 1] \subset \mathbb{R}$. In the case of $\varprojlim G_m$, the completely regular modification identifies points that correspond to words in $\mathcal{C}_n$ related by $\sim$ and the point that corresponds to their equivalence class. Lemma 4.3 details specifically when this occurs.

Lemma 4.3. *Two elements $\chi, \rho \in \varprojlim G_m$ are identified in $\mathrm{CR}(\varprojlim G_m)$ if and only if $\chi = ((\gamma\alpha)|_1, (\gamma\alpha)|_2, (\gamma\alpha)|_3 \ldots)$ and $\rho = (\gamma|_1, \ldots, \gamma|_{|\gamma|}, \gamma q, \gamma q \ldots)$ for some $\gamma \in X_n^*$ and $q \in Q_\sim$ such that $\alpha \in q$ or $\chi = (\alpha|_1, \alpha|_2, \alpha|_3 \ldots)$ and $\rho = (\beta|_1, \beta|_2, \beta|_3 \ldots)$ for some $\alpha, \beta \in \mathcal{C}_n$ such that $\alpha \sim \beta$.*

See [3] for more details as we omit the proof for brevity.

Let $\gamma \in X_n^*$ and $q \in Q_\sim$. Note that the equivalence class in $\mathrm{CR}(\varprojlim G_m)$ containing $\rho = (\gamma|_1, \ldots, \gamma|_{|\gamma|}, \gamma q, \gamma q \ldots)$ contains exactly itself and those points $(\alpha|_1, \alpha|_2, \ldots)$ for $\alpha \in \gamma q$. For simplicity, we denote the equivalence class containing ρ by $[\rho]$ and use ρ as the unique representative such that $\pi_k(\rho)$ is an edge in G_k for some k.

We can now state the following proposition, as proven in [3].

Proposition 4.4. *Let $\sim$ be an invariant factor with a finite set of generators. Then $\mathcal{C}_n/\!\sim\, \cong \mathrm{CR}(\varprojlim G_m)$.*

The graphs G_m characterize the connected nature of $\mathcal{C}_n/\sim$ by representing which cones corresponding to prefixes of length m share related points. This can be generalized to any partition of Cantor space into cones, rather than just those at a particular level.

Let P be a finite partition of $\mathcal{C}_n$ into cones. We now define the graph G_P. The vertices of G_P are the prefixes corresponding to the cones in P. The edges are uniquely labeled

$$\{\alpha \in X_n^* \mid \text{there exists a } [\gamma] \in P \text{ such that } \alpha \preccurlyeq \gamma \text{ and } \alpha \neq \gamma\} \times Q_\sim,$$

i.e. all of the strict prefixes of words representing cones in P. An edge αq has endpoints $\alpha\beta|_k$ for each $\beta \in q$ and the unique $k \in \mathbb{N}$ such that $[\alpha\beta|_k] \in P$. (This k exists as P is a partition of $\mathcal{C}_n$.) As with G_m, this is to represent the fact that if a cone contains a point that is related to one in another cone, that point will belong to an equivalence class, αq say, and we connect the cones in graph G_P by edge αq. An equivalence class $q \in Q_\sim$ could contain more than two points, meaning that G_P could be a hypergraph.

Note that when P is the unique partition of $\mathcal{C}_n$ into cones at level m, $G_P = G_m$. These graphs are also equipped with the same topology as before, with vertices open and edges closed.

Equipped with a partial order, the set of all finite partitions of $\mathcal{C}_n$ into cones, denoted by $\mathcal{U}$, becomes a directed poset and we can define the inverse limit of G_P. Let P and R be partitions of $\mathcal{C}_n$ into cones. Then we define $P \leq R$ if and only if for each $[\alpha] \in P$ there exists a $[\beta] \in R$ such that $\alpha \preccurlyeq \beta$, i.e. α is a prefix of β. This is equivalent to R being a refinement of P.

To construct the inverse limit, we build homomorphisms $p_{R,P} : G_R \to G_P$ when $P \leq R$. They are defined as follows for vertex β and edge γq in G_R:

- $p_{R,P}(\beta) = \alpha$ where $[\alpha] \in P$ and $\alpha \preccurlyeq \beta$;
- $p_{R,P}(\gamma q) = \alpha$ when there exists a cone $[\alpha] \in P$ such that $\alpha \preccurlyeq \gamma$;
- $p_{R,P}(\gamma q) = \gamma q$ otherwise.

The α that vertex β maps to must exist corresponding to a cone in the partition $[P]$, as R is a refinement of P. Also, the edge γq exists as an edge in G_P when there is no vertex α in G_P such that $\alpha \preccurlyeq \gamma$, since $\gamma\beta|_k$ must necessarily be a vertex in G_P for each $\beta \in q$ and corresponding $k > |\gamma|$, as P is partition.

Elements in $\varprojlim G_P$ come in two flavours, as before. First, for $R \in \mathcal{U}$, we define $\pi_R : \varprojlim G_P \to G_R$ to be the projection onto the Rth coordinate. Then for $y \in \varprojlim G_P$, either there exists an $R \in \mathcal{U}$ such that $\pi_R(y)$ is an edge in G_R, or $\pi_R(y)$ is a vertex in G_R for all $R \in \mathcal{U}$. In the first case, $\pi_R(y) = \gamma q$ for all $R \in \mathcal{U}$ such that G_R contains γq, since the edge γq is the only point in G_P that projects onto γq for all $P \geq R$. In the second case, there exists a unique $\alpha \in \mathcal{C}_n$ such that $\pi_R(y) = \alpha|_m$ for all $R \in \mathcal{U}$ where $\alpha|_m$ is a vertex in G_R. This shows that $\varprojlim G_P$ can be represented by points in $\mathcal{C}_n$ and nontrivial equivalence classes in $\mathcal{C}_n/\!\sim$, just as $\varprojlim G_k$.

We use this to show that an invariant factor is the completely regular modification of the inverse limit of the net of graphs G_P.

Proposition 4.5. *Let $\sim$ be an invariant factor with a finite set of generators. Then $\mathcal{C}_n/\!\sim \,\cong \mathrm{CR}(\varprojlim G_P)$.*

Proof. We prove this indirectly, showing that $\varprojlim G_P$ is homeomorphic to the original inverse limit of a sequence of graphs, $\varprojlim G_k$.

We define $\phi : \varprojlim G_P \to \varprojlim G_k$ on both types of elements in $\varprojlim G_P$ If $y \in \varprojlim G_P$ is such that there exists an $R \in \mathcal{U}$ where $\pi_R(y)$ is an edge γq in G_R. Then $\phi(y) = (\gamma|_1, \ldots, \gamma_{|\gamma|}, \gamma q, \gamma q, \ldots)$. However, $y \in \varprojlim G_P$ is such that $\pi_R(y)$ is a vertex for all $R \in \mathcal{U}$, then $\phi(y) = (\alpha|_1, \alpha|_2, \alpha|_3, \ldots)$, where $\alpha \in \mathcal{C}_n$ is the unique point such that for all $R \in \mathcal{U}$ there exists an $m \in \mathbb{N}$ such that $\pi_R(y) = \alpha|_m$.

From the uniqueness of the description of points in $\varprojlim G_P$ and $\varprojlim G_k$, it is clear ϕ is a bijection.

Let U be a basic open set in $\varprojlim G_P$, i.e. there exists an $R \in \mathcal{U}$ and a basic open set $A \in G_R$ such that $U = \{y \in \varprojlim G_P \mid \pi_R(y) \in A\}$. Recall that A is either a single vertex in G_R, or an edge unioned with its endpoints. If $A = \{\alpha\}$ is a singleton containing vertex $\alpha \in G_R$, then $\phi(U) = \{x \in \varprojlim G_k \mid \pi_|\alpha|(x) = \alpha\}$ which is also a basic open set.

If instead $A = \{\gamma q\} \cup \{\alpha \in G_R \mid \alpha \text{ is an endpoint of } \gamma q\}$, then

$$\begin{aligned}\phi(U) = \{(\gamma|_1, \ldots, \gamma|_|\gamma|, \gamma q, \gamma q)\} \\ \cup \{x \in \varprojlim G_k \mid \text{for each } \alpha \text{ an endpoint of } \gamma q \in G_R, \pi_{|\alpha|}(x) = \alpha\}.\end{aligned}$$

Let β be the longest word such that β is an endpoint of γq in G_R. Then $\phi(U)$ is the union of basic open sets in $\varprojlim G_k$ corresponding to the basic open set containing γq in $G_{|\beta|}$ and the basic open sets $\{\alpha\} \subset G_{|\alpha|}$ for each endpoint α of γq in G_R. This shows the images of open sets under ϕ are open.

Now let V be a basic open set in $\varprojlim G_k$. Then there exists an $m \in \mathbb{N}$ and a basic open set $B \in G_m$ such that $V = \{x \in \varprojlim G_k \mid \pi_m(x) \in B\}$. Note that G_m is equal to G_L, where L is the partition of $\mathcal{C}_n$ in cones at level m. Therefore $\phi^{-1}(V)$ is a basic open set in $\varprojlim G_P$ which corresponds to the basic open set $B \subseteq G_L$.

The bijection ϕ is thus a homeomorphism, showing that $\mathrm{CR}(\varprojlim G_P)$ is homeomorphic to $\mathrm{CR}(\varprojlim G_k)$ and therefore to $\mathcal{C}_n/\sim$ as well. □

Many homeomorphisms of Cantor space map partitions of $\mathcal{C}_n$ into cones to other partitions into cones, such as those homeomorphisms generated by transducers. Using the characterization of the invariant factors as inverse limits of graphs, we can see when a homeomorphism of $\mathcal{C}_n$ induces a homeomorphism of its quotient.

Theorem 4.6. *Let $\sim$ be an invariant relation on $\mathcal{C}_n$ such that $M_\sim$ is finite and let ϕ be a homeomorphism of $\mathcal{C}_n$ such that there exist finite partitions L and R of $\mathcal{C}_n$ into cones where $\phi([\alpha]) \in R$ for each $[\alpha] \in L$. If ϕ is also a homeomorphism of $\mathcal{C}_n/\sim$, then G_L is isomorphic to G_R.*

Proof. We show that that ϕ induces a graph isomorphism from G_L to G_R.

Let $q \in Q_\sim$ and $[\alpha], [\beta] \in L$ be such that vertices α and β in G_L are connected by edge γq for some $\gamma \in X_n^*$. This implies that there are related words $\gamma\rho, \gamma\chi \in \gamma q$ such that $\gamma\rho \in [\alpha]$ and $\gamma\chi \in [\beta]$. If ϕ is a homeomorphism of $\mathcal{C}_n/\sim$, then $\phi(\rho) \sim \phi(\chi)$, which implies that there exists a nontrivial equivalence class $\delta q'$ with $\phi(\rho), \phi(\chi) \in \delta q'$. This shows that $\phi([\alpha])$ and $\phi([\beta])$ are connected by edge $\phi(\gamma q) = \delta q'$ in G_R.

We have shown that ϕ induces a graph monomorphism, and the same argument applied to ϕ^{-1} shows that ϕ is surjective, and therefore a graph isomorphism. □

In order to apply Theorem 4.6, we state several lemmas describing the structure of the graphs G_P.

Let P and R be partitions of $\mathcal{C}_n$ into cones such that there exists a non-trivial word $\alpha \in X_n^+$ where $[\alpha\beta] \in P$ for all $[\beta] \in R$. We call the set of cones $\alpha R = \{[\alpha\beta] \mid [\beta] \in R\}$ a *subpartition* of P, since αR is a partition of the cone $[\alpha]$ and $\alpha R \subseteq P$.

As an example, consider partitions $P = \{[00], [010], [011], [1]\}$ and $R = \{[0], [10], [11]\}$. Choosing $0 \in X_n^+$, the set $0R = \{[00], [010], [011]\}$ is a subset of P and is formed by adding the prefix 0 to each cone in R. In this way, $0R$ is a subpartition of P, as $0R$ is itself a partition of $[0]$.

Note that for every finite word γ such that $[\gamma] \not\subseteq [\beta]$ for all $[\beta] \in P$, there exists a unique partition S of $\mathcal{C}_n$ such that γS is a subpartition of P. Lemma 4.7 details a decomposition of the graph G_P into pieces represented by the subpartitions of P.

Lemma 4.7. *Let $\sim$ be an invariant relation and and let P and R be finite partitions of Cantor space into cones such that there exists a word $\alpha \in X_n^*$ where αR is a subpartition of P. Then the subgraph αG_R of G_P is isomorphic to G_R, where αG_R is defined to be the subgraph of G_P consisting of vertices $\alpha\beta$ and edges $\alpha\gamma q$ for all $\beta, \gamma \in X_n^*$ and $q \in Q_\sim$ such that $\alpha\beta$ and $\alpha\gamma q$ are vertices and edges, respectively, in G_P.*

Proof. Consider the function $f : G_R \to \alpha G_R$ where $f(\beta) = \alpha\beta$ for $\beta \in V(G_R)$ and $f(\gamma q) = \alpha\gamma q$ for $\gamma q \in E(G_R)$. If $[\beta] \in R$, then

$$\begin{aligned} \beta \text{ is an endpoint of } \gamma q \text{ in } G_R &\Leftrightarrow \text{there exists a } \rho \in q \text{ such that } \beta \text{ is a prefix of } \gamma\rho \\ &\Leftrightarrow \text{there exists a } \rho \in q \text{ such that } \alpha\beta \text{ is a prefix of } \alpha\gamma\rho \\ &\Leftrightarrow \alpha\beta \text{ is an endpoint of } \alpha\gamma q \text{ in } G_P. \end{aligned}$$

This shows that f preserves adjacency within αG_R. The function f is also a bijection since it is clearly injective and is surjective from the definition of αR. □

We will also use this idea with the graph G_m. For a finite word $\alpha \in X_n^k$ of length k, define αG_m to be the subgraph of G_{m+k} consisting of edges and vertices with prefix α.

We provide the following lemma from [4] without proof for brevity.

Lemma 4.8. *The graph G_1 is connected if and only if $\mathcal{C}_n/\sim$ is connected, and furthermore, if G_1 is connected, then G_m is connected for all $m \geq 1$. Likewise G_1 is a tree if and only if $\mathcal{C}_n/\sim$ is a dendrite (a compact connected metric space containing no simple closed curves), and if G_1 is a tree, then G_m is a tree for all $m \geq 1$.*

4.1.2 Small Relations on the Binary Cantor Space

We will now discuss invariant relations on $\mathcal{C}_2$ that are small. We define a *small* invariant relation to be an invariant relation $\sim$ such that the set of generators $M_\sim$ has exactly two elements. Given a pair of elements α and β such that $\alpha_1 \neq \beta_1$, there may not exist an invariant relation such that $M_\sim = \{\alpha, \beta\}$.

Lemma 4.9. *Let $\alpha, \beta \in \mathcal{C}_2$ such that $\alpha_1 = 0$ and $\beta_1 = 1$ and hence $\alpha \neq \beta$. Then there exists a small invariant relation with generators equal to $\{\alpha, \beta\}$ if and only if for all $\gamma \in X_n^+$, $\alpha \neq \gamma\alpha$, $\alpha \neq \gamma\beta$, $\beta \neq \gamma\alpha$, and $\beta \neq \gamma\beta$.*

Proof. We begin by proving the forward implication via contradiction. Assume without loss of generality, that α is equal to $\gamma\alpha$ or $\gamma\beta$ for some non-trivial word γ.

Case 1 $\alpha = \gamma\alpha$: If $\sim$ is an invariant relation and $\alpha \sim \beta$, then $\gamma\alpha \sim \gamma\beta$. The assumption that $\alpha = \gamma\alpha$ implies that $\gamma\beta \sim \gamma\alpha \sim \alpha \sim \beta$, so the generators of $\sim$ must contain α, β, and $\gamma\beta$. Since $\gamma\alpha = \alpha$, $\gamma_1 = 0$ and therefore $\gamma\beta \neq \beta$. Also, since $\alpha \neq \beta$, we see that $\alpha = \gamma\alpha \neq \gamma\beta$. Together, these imply that α, β and $\gamma\beta$ are distinct, and $\sim$ is not small.

Case 2 $\alpha = \gamma\beta$: If $\sim$ is an invariant relation and $\alpha \sim \beta$, then $\gamma\alpha \sim \gamma\beta$. The assumption that $\alpha = \gamma\beta$ implies that $\gamma\alpha \sim \gamma\beta \sim \alpha \sim \beta$, so the generators of $\sim$ must contain α, β, and $\gamma\alpha$. Since $\gamma\beta = \alpha$, $\gamma_1 = 0$ and therefore $\gamma\alpha \neq \beta$. If $\alpha = \gamma\alpha$, then $\alpha = \overline{\gamma}$, but $\alpha = \gamma\beta$ would then imply $\beta = \overline{\gamma}$, contradicting the assumption that $\beta_1 = 1$. Therefore, $\alpha \neq \gamma\alpha$ and $\sim$ is not small since α, β, and $\gamma\alpha$ are distinct.

For the reverse implication, consider the equivalence relation $\sim$ in which the non-trivial relations are of the form $\gamma\alpha \sim \gamma\beta$ for all $\gamma \in X_n^*$. This clearly satisfies Property 1 for invariance as adding and removing common prefixes between related elements will preserve these relations, as well as trivial relations. In order to show that $\sim$ is an invariant relation, we now need to show that $\sim$ is closed as a subset of $\mathcal{C}_n \times \mathcal{C}_n$.

Consider the set

$$S_< = \bigcup_{\gamma \in X_n^*} \{[\gamma 0] \times [\gamma 1] \setminus (\gamma\alpha, \gamma\beta)\}.$$

Note that $S_<$ consists of pairs of distinct points in $\mathcal{C}_2$ in lexicographic order that are not related to one another by $\sim$. Define

$$S_> = \bigcup_{\gamma \in X_n^*} \{[\gamma 1] \times [\gamma 0] \setminus (\gamma\beta, \gamma\alpha)\}$$

similarly for the reverse order. Both $S_<$ and $S_>$ are the union of open sets, so the complement of their union,

$$(S_< \cup S_>)^c = \{(\gamma\alpha, \gamma\beta), (\gamma\beta, \gamma\alpha) | \gamma \in X_n^*\} \cup \{(\delta, \delta) | \delta \in \mathcal{C}_2\},$$

is a closed set. Therefore, the set of related pairs

$$\{(\rho, \chi) \in \mathcal{C}_2 \times \mathcal{C}_2 | \rho \sim \chi\} = (S_< \cup S_>)^c \cup \{(\delta, \delta) | \delta \in \mathcal{C}_2\}$$

is a closed set. □

The following three lemmas describe the size of the automorphism groups of G_m for small invariant relations. Note that by applying Lemma 4.8 to small invariant relations, we see that G_m is a tree for all m.

Recall from Chapter 3 that $\lceil \epsilon \rceil_{(0\,1)}$ refers to the homeomorphism (and isometry) of $\mathcal{C}_2$ that replaces all 0s with 1s and vice versa in every word in $\mathcal{C}_2$, whereas $\lfloor \epsilon \rfloor_{(0\,1)}$ refers to the homeomorphism (and isometry) that simply swaps the finite prefixes 0 and 1 and acts trivially otherwise on each word in $\mathcal{C}_2$. We extend these definitions here, to include acting on finite words in the same manner as infinite words (swapping all 0s and 1s or just the first letter) to produce graph automorphisms.

Lemma 4.10. *Let $\sim$ be a small invariant relation with generators α and β and $Q_\sim = \{q\}$ with $\alpha_1 = 0$ and $\beta_1 = 1$. If $\alpha = \lceil \epsilon \rceil_{(0\,1)}(\beta)$, then $|\mathrm{Aut}(G_k)| = 2$ for all k.*

Proof. Let $\gamma, \delta \in X_2^k$ be distinct vertices in G_k such that $\gamma \leq_{lex} \delta$ and that they are connected by edge ρq for some $\rho \in X_n^m$. The implies that $\gamma = \rho\alpha|_{k-m}$ and $\delta = \rho\beta|_{k-m}$. Consider vertices $\lceil \epsilon \rceil_{(0\,1)}(\gamma)$ and $\lceil \epsilon \rceil_{(0\,1)}(\delta)$ and edge $\lceil \epsilon \rceil_{(0\,1)}(\rho)q$. Notice that

$$\begin{aligned} \lceil \epsilon \rceil_{(0\,1)}(\gamma) &= \lceil \epsilon \rceil_{(0\,1)}(\rho) \| \lceil \epsilon \rceil_{(0\,1)}(\alpha_{k-m}) \\ &= \lceil \epsilon \rceil_{(0\,1)}(\rho) \| \beta_{k-m} \end{aligned}$$

and that

$$
\begin{aligned}
[\epsilon]_{(0\,1)}(\delta) &= [\epsilon]_{(0\,1)}(\rho) \| [\epsilon]_{(0\,1)}(\beta_{k-m}) \\
&= [\epsilon]_{(0\,1)}(\rho) \| \alpha_{k-m}.
\end{aligned}
$$

This implies that vertices $[\epsilon]_{(0\,1)}(\gamma)$ and $[\epsilon]_{(0\,1)}(\delta)$ of G_k are connected by edge $[\epsilon]_{(0\,1)}(\rho)\ q$. The homeomorphism $[\epsilon]_{(0\,1)}$ therefore induces an automorphism of G_k.

We will show that this is the only non-trivial automorphism of G_k using induction. The graph G_1 consists of two vertices 0 and 1 connected by edge q. This has two automorphisms, the trivial automorphism and the one induced by $[\epsilon]_{(0\,1)}$. Now assume that G_{k-1} only has the two aforementioned automorphisms.

The graph G_k comprises of $0G_{k-1}$ and $1G_{k-1}$ and edge q with two endpoints, one in $0G_{k-1}$ and one in $1G_{k-1}$. Any automorphism of G_k must fix edge q (perhaps swapping its endpoints) since q is the unique edge in G_k that, when removed, produces two disjoint graphs with 2^{k-1} vertices apeice.

Firstly, suppose that ϕ is an automorphism of G_k that fixes the endpoints of edge q. Then ϕ must also map $0G_{k-1}$ to $0G_{k-1}$ and therefore ϕ restricted to $0G_{k-1}$ produces an automorphism of G_{k-1}. Specifically let $\nu_i : 0G_{k-1} \to G_{k-1}$ be the map that takes $i\gamma$ to γ. Then $\nu_0(0\gamma) \mapsto \nu_0 \circ \phi(0\gamma)$ for $\gamma \in X_n^{k-1}$ is an automorphism of G_{k-1} that fixes a point, namely $\nu_0(\alpha|_k)$. Since G_{k-1} has only two automorphisms, one of which moves every point, ϕ acts trivially on $0G_{k-1}$. The automorphism ϕ must also act trivially on $1G_{k-1}$ for the same reasons, and ϕ is therefore the trivial automorphism.

Secondly, suppose that ϕ is an automorphism of G_k that maps the endpoints of q to one another, i.e. ϕ swaps $\alpha|_k$ and $\beta|_k$. Then ϕ must also map $0G_{k-1}$ to $1G_{k-1}$, and so induces an automorphism of G_{k-1} as before. Now, however, the automorphism $\nu_1(1\gamma) \mapsto \nu_1 \circ \phi(1\gamma)$ for $\gamma \in X_n^{k-1}$ takes $\nu_0(\alpha)$ to $\nu_1(\beta)$. This shows that the automorphism induced by ϕ moves a point off of itself, since $\nu_0(\alpha) \neq \nu_1(\beta)$. Again, there is only one automorphism that does this, $[\epsilon]_{(0\,1)}$. This also holds when ϕ maps $1G_{k-1}$ to $0G_{k-1}$, so ϕ is equivalent to

$[\epsilon]_{(0\ 1)}$ because it swaps the first letter (mapping $0G_{k-1}$ to $1G_{k-1}$ and vice versa) and acts as $[\epsilon]_{(0\ 1)}$ on the suffixes. □

Lemma 4.11. *Let $\sim$ be a small invariant relation with generators α and β and critical point q with $\alpha_1 = 0$ and $\beta_1 = 1$. If $\alpha = \lfloor\epsilon\rfloor_{(0\ 1)}(\beta)$, then $|\mathrm{Aut}(G_k)| = 2$ for all k.*

Proof. The proof of this lemma is nearly identical to the proof of Lemma 4.10. Let $\gamma, \delta \in X_n^k$ be distinct vertices in G_k such that $\gamma \prec \delta$ and that they are connected by edge ρq for some $\rho \in X_n^m$. The implies that $\gamma = \rho\alpha|_{k-m}$ and $\delta = \rho\beta|_{k-m}$. Consider vertices $\lfloor\epsilon\rfloor_{(0\ 1)}(\gamma)$ and $\lfloor\epsilon\rfloor_{(0\ 1)}(\delta)$ and edge $\lfloor\epsilon\rfloor_{(0\ 1)}(\rho)q$. Notice that

$$\lfloor\epsilon\rfloor_{(0\ 1)}(\gamma) = \lfloor\epsilon\rfloor_{(0\ 1)}(\rho)\|\alpha_{k-m}$$

and that

$$\lfloor\epsilon\rfloor_{(0\ 1)}(\delta) = \lfloor\epsilon\rfloor_{(0\ 1)}(\rho)\|\beta_{k-m}.$$

This implies that the vertices $[\epsilon]_{(0\ 1)}(\gamma)$ and $[\epsilon]_{(0\ 1)}(\delta)$ of G_k are connected by edge $[\epsilon]_{(0\ 1)}(\rho)q$. The homeomorphism $\lfloor\epsilon\rfloor_{(0\ 1)}$ therefore induces an automorphism of G_k.

We will show that this is the only non-trivial automorphism of G_k using induction. The graph G_1 consists of two vertices 0 and 1 connected by edge q. This has two automorphisms, the trivial automorphism and the one induced by $\lfloor\epsilon\rfloor_{(0\ 1)}$. Now assume that G_{k-1} only has the two aforementioned automorphisms.

Any automorphism of G_k must fix edge q, perhaps swapping its endpoints. Firstly, suppose that ϕ is an automorphism of G_k that fixes the endpoints of edge q. Then ϕ must map $0G_{k-1}$ to $0G_{k-1}$ and ϕ restricted to $0G_{k-1}$ produces an automorphism of G_{k-1}. As before, $\nu_0(0\gamma) \mapsto \nu_0 \circ \phi(0\gamma)$ for $\gamma \in X_n^{k-1}$ is an automorphism of G_{k-1} that fixes a point, namely $\nu_0(\alpha)$. Since G_{k-1} has only two automorphisms, one of which moves every point, ϕ acts trivially on $0G_{k-1}$. The automorphism ϕ must also act trivially on $1G_{k-1}$ for the same reasons, and is ϕ therefore the trivial automorphism.

Secondly, suppose that ϕ is an automorphism of G_k that maps the endpoints of q to one another, i.e. ϕ maps $\alpha|_k$ to $\beta|_k$. Then ϕ must also map $0G_{k-1}$ to $1G_{k-1}$, and induces an automorphism of G_{k-1} as before. Now, however, the automorphism $\nu_0(0\gamma) \mapsto \nu_1 \circ \phi(1\gamma)$ for $\gamma \in X_n^{k-1}$ takes $\nu_0(\alpha)$ to $\nu_1(\beta)$. This shows that the automorphism induced by ϕ fixes a point, since $\nu_0(\alpha) = \nu_1(\beta)$. Again, only the trivial automorphism does this. This also holds when ϕ maps $1G_{k-1}$ to $0G_{k-1}$, so ϕ is equivalent to $\lfloor\epsilon\rfloor_{(0\,1)}$ because it swaps the first letter (mapping $0G_{k-1}$ to $1G_{k-1}$ and vice versa) and acts trivially on the suffixes. □

Lemma 4.12. *Let $\sim$ be a small invariant relation with generators α and β and critical point q with $\alpha_1 = 0$ and $\beta_1 = 1$. If α is not equal to $\lceil\epsilon\rceil_{(0\,1)}(\beta)$ or $\lfloor\epsilon\rfloor_{(0\,1)}(\beta)$, then there exists a $m \in \mathbb{N}$ for which $|\mathrm{Aut}(G_k)| = 1$ for all $k \geq m$ and $|\mathrm{Aut}(G_k)| = 2$ for all $k < m$.*

Proof. Let m be the minimal integer such that $\alpha|_m$ is not equal to $\lceil\epsilon\rceil_{(0\,1)}(\beta|_m)$ nor $\lfloor\epsilon\rfloor_{(0\,1)}(\beta|_m)$. Note that $m \geq 3$ since this is satisfied for any allowed pair of two letter prefixes of α and β. From Lemmas 4.10 and 4.11, we can see that $|\mathrm{Aut}(G_k)| = 2$ for all $k < m$.

Similar to the arguments in the previous lemmas, automorphisms of G_m will either swap $0G_{m-1}$ and $1G_{m-1}$ or map them to themselves. Let ϕ be an automorphism of G_m that maps $0G_{m-1}$ to $0G_{m-1}$. This implies that ϕ fixes $\alpha|_m$, the endpoint of q in $0G_{m-1}$. The automorphism induced by ϕ on G_{m-1} when restricted to $0G_{m-1}$, $\nu_0(0\gamma) \mapsto \nu_0 \circ \phi(0\gamma)$, fixes a point. The automorphism fixing a point in G_{m-1} is the trivial automorphsim, implying that ϕ fixes every point in $0G_{m-1}$. The same arguments hold for $1G_{m-1}$, and therefore the only automorphism of G_m that does not swap $0G_{m-1}$ and $1G_{m-1}$ is the trivial automorphism.

Now suppose that ϕ maps $0G_{m-1}$ to $1G_{m-1}$. Using the projections ν_0 and ν_1, ϕ induces an automorphism of G_{m-1}, specifically $\nu_0(0\gamma) \mapsto \nu_1 \circ \phi(1\gamma)$ for $\gamma \in X_n^{m-1}$. However, this induced automorphism of G_{m-1} maps $\nu_0(\alpha|_k)$ to $\nu_1(\beta|_k)$. Our condition on k implies that $\alpha|_{m-1}$ is equal to $\lceil\epsilon\rceil_{(0\,1)}(\beta|_{m-1})$ or $\lfloor\epsilon\rfloor_{(0\,1)}(\beta|_{m-1})$.

If $\alpha|_{m-1}$ is equal to $[\epsilon]_{(0\,1)}(\beta|_{m-1})$, then $\nu_0(\alpha|_m) \neq [\epsilon]_{(0\,1)}(\nu_1(\beta|_m))$, and $\nu_0(\alpha|_m)$ and $\nu_1(\beta|_m)$ are non-isomorphic vertices in G_{m-1}. However, if $\alpha|_{m-1}$ is equal to $\lfloor\epsilon\rfloor_{(0\,1)}(\beta|_{m-1})$, then $\nu_0(\alpha|_m) \neq \lfloor\epsilon\rfloor_{(0\,1)}(\nu_1(\beta|_m))$, and $\nu_0(\alpha|_m)$ and $\nu_1(\beta|_m)$ are non-isomorphic vertices in G_{m-1}. This is contradiction, and therefore ϕ, a non-trivial isomorphism, cannot exist and so $|\mathrm{Aut}(G_m)| = 1$.

Now inductively assume that $|\mathrm{Aut}(G_{l-1})| = 1$ for some $l > m$. For the same reasons as G_m, the only automorphism of G_l that maps $0G_{l-1}$ and $1G_{l-1}$ to themselves is the trivial automorphism. If ϕ is an automorphism that maps $0G_{l-1}$ to $1G_{l-1}$, then there must be an automorphism of G_{l-1} that takes $\nu_0(\alpha|_{l-1})$ and $\nu_1(\beta|_{l-1})$. The only automorphism of G_{l-1} is trivial, however. Since $\nu_0(\alpha|_{l-1}) \neq \nu_1(\beta|_{l-1})$, ϕ cannot exist and therefore $|\mathrm{Aut}(G_l)| = 1$. We conclude that $|\mathrm{Aut}(G_l) = 1|$ for all $l \geq m$. □

We collect the results of Lemmas 4.10, 4.11, and 4.12 in the following corollary.

Corollary 4.13. *Let $\sim$ be a small invariant relation with generators α and β and critical point q with $\alpha_1 = 0$ and $\beta_1 = 1$. Then $|\mathrm{Aut}(G_k)|$ is either equal 1 or 2 and $|\mathrm{Aut}(G_k)| = 2$ for all k if and only if α is equal to $[\epsilon]_{(0\,1)}(\beta)$ or $\lfloor\epsilon\rfloor_{(0\,1)}(\beta)$.*

As a consequence of Corollary 4.13, we can describe when homeomorphisms of $\mathcal{C}_2$ that preserve partitions into equal sized cones, e.g. isometries of Cantor space, are homeomorphisms of the quotient by small invariant relations.

Theorem 4.14. *Let $\sim$ be a small relation with generators α and β. If α is equal to $[\epsilon]_{(0\,1)}(\beta)$ or $\lfloor\epsilon\rfloor_{(0\,1)}(\beta)$, then there is precisely one non-trivial isometry of $\mathcal{C}_2$ that is a homeomorphism of $\mathcal{C}_2/\sim$. If α is not equal to $[\epsilon]_{(0\,1)}(\beta)$ nor $\lfloor\epsilon\rfloor_{(0\,1)}(\beta)$, then there are no non-trivial isometries of $\mathcal{C}_2$ that are homeomorphisms of $\mathcal{C}_2/\sim$.*

Proof. Let $\alpha = [\epsilon]_{(0\,1)}(\beta)$. Then $[\epsilon]_{(0\,1)}$, which is an isometry, is a homeomorphism of $\mathcal{C}_2/\sim$. Suppose that ϕ is a non-trivial isometry of $\mathcal{C}_2$ that is not equal to $[\epsilon]_{(0\,1)}$. Let k be the minimal integer such that there exists a cone $[\gamma]$ with $|\gamma| = k$ and $\phi([\gamma]) \neq [\epsilon]_{(0\,1)}([\gamma])$. Then action of ϕ on the partition of $\mathcal{C}_2$ into cones at level k is not a graph automorphism, since it is not equal to either of its two automorphisms, and therefore ϕ is not a homeomorphism of $\mathcal{C}_2/\sim$.

The proof is the same when $\alpha = \lfloor\epsilon\rfloor_{(0\ 1)}(\beta)$.

Now let α not be equal to $\lceil\epsilon\rceil_{(0\ 1)}(\beta)$ nor $\lfloor\epsilon\rfloor_{(0\ 1)}(\beta)$. By similar arguments in Lemma 2.5 in Chapter 2, every isometry of $\mathcal{C}_2$ induces an action on words of length k, as they map cones at level k to one another. However, no non-trivial isometry ϕ of Cantor space is a homeomorphism of $\mathcal{C}_2/\sim$, since every non-trivial isometry will induce a non-trivial action on the graphs G_k for sufficiently large k but G_k has trivial automorphism group. □

Theorem 4.14 essentially shows that the group of isometries of $\mathcal{C}_2$ intersected with the group of homeomorphisms of $\mathcal{C}_2/\sim$ is nearly trivial.

4.2 Sierpiński Relatives

Attractors of iterated functions systems (see Section 1.2) can be viewed topologically as quotients of Cantor space and, under certain conditions, are invariant factors. In this section, we examine a particular collection of self-similar attractors known as Sierpiński relatives and fractal squares. We use infinite walks on finite graphs to describe the invariant relations whose quotients form these fractal sets.

4.2.1 Background and Definitions

We begin by providing a sufficient condition for iterated function systems to produce an attractor homeomorphic to an invariant factor, before defining the attractors known as Sierpiński relatives and fractal squares.

To differentiate terms in a sequence of infinite words, we use the notation ${}_i\alpha$ to represent the ith term in a sequence of words in $\mathcal{C}_n$, as α_i means the ith letter of α. Although this can be difficult, we hope we are explicit and detailed enough to prevent confusion.

Lemma 4.15. *Let $S = \{f_0, f_1, \ldots, f_{n-1}\}$ be an IFS consisting of contracting bi-Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. Then the attractor F of the IFS is homeomorphic to an invariant factor.*

Proof. For a finite word $\alpha \in X_n^*$, define $f_\alpha = f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_{|\alpha|}}$. Let $\pi_S : \mathcal{C}_n \to F$ be defined by $\pi_S(\beta) = \lim_{m\to\infty} f_{\beta|_m}(F)$. Since f_i is contracting for all i, $\pi_S(\beta)$ is a single point in F. Define the relation $\sim$ on $\mathcal{C}_n$ by $\alpha \sim \beta$ if and only if $\pi_S(\alpha) = \pi_S(\beta)$. Then $F \cong \mathcal{C}_n/\sim$.

What remains to be proven is that $\sim$ is an invariant relation. Since equality in $\mathbb{R}^d$ is an equivalence relation, so is $\sim$. Suppose that $\alpha \sim \beta$. Then $\pi_S(\alpha) = \pi_S(\beta)$ and therefore $f_i(\pi_S(\alpha)) = f_i(\pi_S(\beta))$ for each i. However,

$$\begin{aligned} f_i(\pi_S(\alpha)) =& f_i(\lim_{m\to\infty} f_{\alpha|_m}(F)) \\ =& \lim_{m\to\infty} f_i \circ f_{\alpha|_m}(F) \\ =& \lim_{m\to\infty} f_{i\alpha|_m}(F) \\ =& \pi_S(i\alpha). \end{aligned}$$

This implies $\pi_S(i\alpha) = \pi_S(i\beta)$ and therefore $i\alpha \sim i\beta$.

A similar argument works to remove a common prefix. Let $i\alpha$ and $i\beta$ be infinite words over X_n such that $i\alpha \sim i\beta$. Note that reverse the previous equalities implies that $\pi_S(i\alpha) = f_i(\pi_S(\alpha))$ and therefore $f_i^{-1}(\pi_S(i\alpha)) = \pi_S(\alpha)$. Since $i\alpha \sim i\beta$, we can see that $\pi_S(i\alpha) = \pi_S(i\beta)$ and $f_i^{-1}(\pi_S(i\alpha)) = f_i^{-1}(\pi_S(i\beta))$. This implies that $\pi_S(\alpha) = \pi_S(\beta)$ and therefore $\alpha \sim \beta$. This shows that $\sim$ satisfies Property 1.

Let ${}_i\alpha$ and ${}_i\beta$ $(i = 1, 2, \ldots)$ be convergent sequences of infinite words in $\mathcal{C}_n$ such that ${}_i\alpha \to \alpha$ and ${}_i\beta \to \beta$ and ${}_i\alpha \sim {}_i\beta$ for all i. This implies that for all $k \in \mathbb{N}$, there exists an M such that for all $m \geq M$, $({}_m\alpha)|_k = \alpha|_k$ and $({}_m\beta)|_k = \beta|_k$. Therefore, $f_{\alpha|_k}(S) \cap f_{\beta|_k}(S) \neq \emptyset$ for all k, since there exists related points in the cones $[\alpha|_k]$ and $[\beta|_k]$, namely ${}_m\alpha$ and ${}_m\beta$. The limit $\lim_{k\to\infty} f_{\alpha|_k}(S) \cap f_{\beta|_k}(S)$ is thus a nonempty singleton and contains both $\pi_S(\alpha) = \lim_{k\to\infty} f_{\alpha|_k}(S)$ and $\pi_S(\beta) = \lim_{k\to\infty} f_{\beta|_k}(S)$. Therefore, $\alpha \sim \beta$ and $\sim$ is closed. □

Determining the invariant relation of a given IFS is as simple (or complex) as finding Q, the equivalence classes of infinite words that have different single letter prefixes. In the context of the attractor F of a given IFS, this amounts to describing the overlap between $f_i(F)$ and $f_j(F)$ for each pair i and j, which corresponds to infinite words beginning with i and j respectively.

We examine a collection of IFSs of some interest, arising from self-similar contractions of a unit square. Let $D = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ be a closed unit square subdivided into an $m \times m$ grid of subsquares. We label these subsquares $D_0, D_1, \ldots, D_{m^2-1}$ from left to right, then top to bottom, i.e. the top row consists of D_0 through D_{m-1} and the bottom row is $D_{m(m-1)}$ through D_{m^2-1}. Let $f_i : D \to D_i$ be the contracting similarity that maps D onto D_i without rotation or reflection, i.e. $f_i(x) = \frac{1}{m}x + v_i$ where $v_i \in \mathbb{R}^2$ is the bottom left corner of D_i. We extend the naming convention of subsquares of D so that for a finite word $\alpha \in X^*_{m^2}$, $D_\alpha = f_\alpha(D)$.

Let $I = \{i_0, i_1, \ldots i_{n-1}\}$ be a non-empty subset of $X_{m^2} = \{0, 1, \ldots m^2 - 1\}$ with $1 < n \leq m^2$. The set of functions $\{f_{i_0}, f_{i_1}, \ldots f_{i_{n-1}}\}$ forms an iterated function system whose unique attractor is self-similar.

The topology of such fractals, deemed *fractal squares*, is examined by Lau, Luo, and Rao in [32], where they give a method for classifying a given fractal square into one of three types: ones that are totally disconnected, ones that are made up of disjoint line segments, and those with connected components that are not line segments. Our work seeks to expand these results, or provide the tools to do so, by considering contracting similarities that include rotation and reflection.

For each element of I, let g_{i_j} be an element of the symmetry group of D, i.e. the dihedral group with eight elements. We can now form a new IFS $\{h_{i_1}, h_{i_2}, \ldots h_{i_n}\}$ where $h_{i_j} = f_{i_j} \circ g_{i_j}$. These fractal squares are known as *Sierpiński relatives* and are a prominent subset of those self-similar sets studied by Falconer and O'Connor in [18]. For a given subset I, Falconer and O'Connor's work calculates the number of distinct fractal sets that can be produced using both the contractions and the symmetry group. If the subsquares of D corresponding to the indices in I form a symmetric subset, then the number of unique fractal attractors is less than the number of possible iterated function systems $\{h_{i_1}, h_{i_2}, \ldots h_{i_n}\}$ as some distinct IFSs will have identical attractors.

While Falconer and O'Connor ennumerate the unique attractors possible, they do not explore the topological properties of said sets. We provide a method to describe the invariant relation that produces these Sierpiński relatives, using infinite walks on graphs. For more information on Sierpiński relatives, including many pictures, see [34].

4.2.2 D as an Invariant Factor

We will describe Sierpiński relatives as quotients of the Cantor space I^ω by invariant relations. We begin by examining D as an invariant factor, since it is a Sierpiński relative itself. Specifically, we consider the IFS $\{f_0, f_1 \ldots f_{m^2-1}\}$, when $I = X_{m^2}$, and the contractions map the square without rotation or reflection. Let α be a word in $\mathcal{C}_{m^2}$. As before, we define $f_{\alpha|_k} = f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_k}$ and $\pi_D(\alpha) = \lim_{k\to\infty} f_{\alpha|_k}(D)$. The invariant relation defining D is then $\alpha \sim \beta$ if and only if $\pi_D(\alpha) = \pi_D(\beta)$.

To characterize the invariant relation $\sim$ that defines D, we describe a directed graph T_m with edges labeled by ordered pairs of letters in X_{m^2}. By concatenating the letters in consecutive labels, infinite walks on this graph produce points in $\mathcal{C}_{m^2}$ that are equivalent under $\sim$. See Figure 4.2 for a rough depiction of T_m and Figure 4.3 for the specific case of T_2. Relating the overlap of the self-similar components tiles to a directed graph was also expressed in [29]

The graph, T_m, has six vertices with labels $\{-, |, \diagup, \diagdown^*, {}_*\diagdown, init\}$. These labels will visually represent how subsquares of D arranged with respect to one another.

Let $\alpha, \beta \in \mathcal{C}_{m^2}$ be words such that $\alpha \leq_{lex} \beta$, $\alpha \sim \beta$, and $\alpha_1 \neq \beta_1$. Since $\pi_D(\alpha) \in D_{\alpha|_k}$ and $\pi_D(\beta) \in D_{\beta|_k}$ for all k, $D_{\alpha|_k} \cap D_{\beta|_k} \neq \emptyset$. Two distinct subsquares of D can non-trivially overlap on either an edge or a corner. This will happen in one of five cases because we require $\alpha \leq_{lex} \beta$:

Case 1: subsquare $D_{\alpha|_k}$ is directly above $D_{\beta|_k}$, represented by vertex $-$;

Case 2: subsquare $D_{\alpha|_k}$ is directly left of $D_{\beta|_k}$, represented by vertex $|$;

Case 3: subsquare $D_{\alpha|_k}$ is above and left of $D_{\beta|_k}$, represented by vertex $\diagup$;

Case 4: subsquare $D_{\alpha|_k}$ is above and right of $D_{\beta|_k}$, represented by vertex $\diagdown^{*}$;

Case 5: subsquare $D_{\alpha|_k}$ is below and left of $D_{\beta|_k}$, represented by vertex ${}_{*}\diagdown$.

To further illustrate the arrangement of subsquares, we refer to Figure 4.1, displaying subsquares when $m = 2$, to give examples of each case. Cases 1 and 2 correspond to subsquares that overlap on an horizontal edge (D_{02} and D_{20}) and vertical edge (D_{01} and D_{10}), respectively. Case 3 happens when subsquares overlap on a corner (D_{03} and D_{30}) as do Case 4 (D_{03} and D_{20}) and Case 5 (D_{03} and D_{10}) occur when subsquares overlap on a corner. Note that the Cases 4 and 5 only differ due to the requirement that $\alpha \leq_{lex} \beta$. The vertices of the graph T represent these cases, with the labels visually representing the orientation of the subsquares with respect to one another.

00	01	10	11
02	03	12	13
20	21	30	31
22	23	32	33

Figure 4.1: Subsquares D_{ij} of D

We now begin the construction of the graph T_m, displayed in Figure 4.2. Note that we have consolidated edges and left out some of the conditions on the ordered pairs for clarity in the figure.

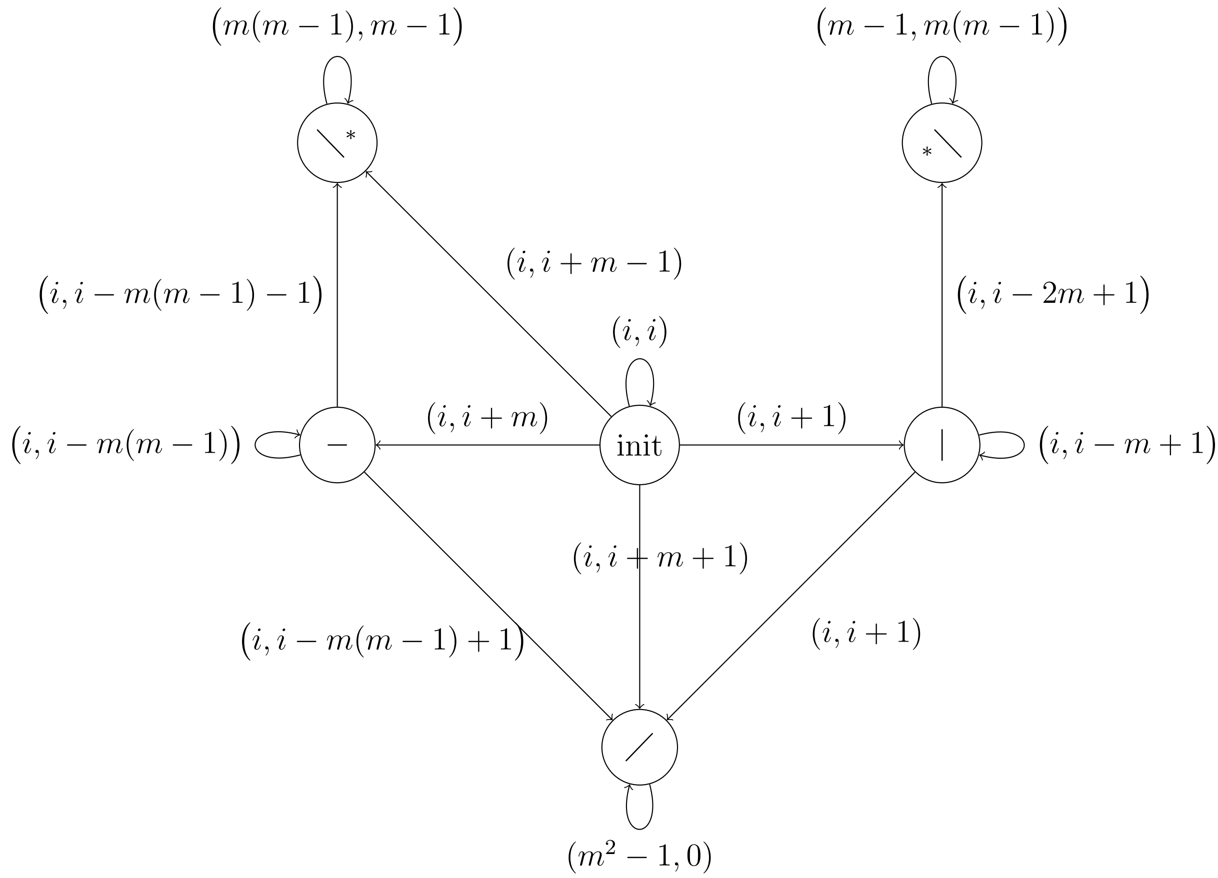

Figure 4.2: The directed graph T_m

Let $\alpha, \beta \in \mathcal{C}_n$ such that $\alpha \leq_{lex} \beta$. We treat each case of the potential arrangement of subsquares $D_{\alpha|_k}$ and $D_{\beta|_k}$ individually.

Suppose that $D_{\alpha|_k} = D_{\beta|_k}$, i.e. $\alpha|_k = \beta|_k$. Then one directed edge with label (i, j) leaves the vertex *init* (for initial) for each ordered pair of letters $(i, j) \in X_{m^2} \times X_{m^2}$ such that $i \leq j$ and $D_{\alpha|_k i} \cap D_{\beta|_k j} \neq \emptyset$. Note that this happens precisely when $D_i \cap D_j \neq \emptyset$. The edge terminates at the vertex representing the orientation of $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ with respect to one another.

- If $i = j$, then $D_{\alpha|_k i} = D_{\beta|_k j}$ and the edge terminates at vertex *init*. (This edge is a loop.)

- If $j = i + m$ and $i \leq m^2 - m$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 1 and the edge terminates at vertex $-$.

- If $j = i + 1$ and $i \not\equiv m - 1 \mod m$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 2 and the edge terminates at vertex $|$.

- If $j = i + m + 1$, $i \not\equiv m - 1 \mod m$, and $i \leq m^2 - m$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 3 and the edge terminates at vertex $\diagup$.

- If $j = i + m - 1$, $i \not\equiv 0 \mod m$, and $i \leq m^2 - m$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 4 and the edge terminates at vertex $\diagdown^*$.

We use $\diagdown^*$ here to represent the fact that the first number in the pair, i, represents the higher and more rightward subsquare. The last case, Case 5, does not appear yet, since this can only occur when considering subsquares D_α with $|\alpha| \geq 2$. Thus far we have $m^2 + 2m(m-1) + 2(m-1)^2$ edges starting at *init*. When visually representing T_m as a directed graph, we will consolidate each duplicate edge with distinct labels into one edge with multiple labels for simplicity.

Suppose $D_{\alpha|_k}$ and $D_{\beta|k}$ are directly above and below one another, i.e. belong to Case 1. The edges originating at state $-$ labeled (i, j) represent the orientation and overlap of subsquares $D_{\alpha|_k i}$ and $D_{\beta|_k j}$. Recall the assumption that $\alpha \prec \beta$ and note the association of i to α and j to β.

- If $i = j + m(m-1)$ and $j \leq m - 1$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 1 and the edge terminates at vertex $-$. (This edge is a loop.)

- If $i = j + m(m-1) - 1$ and $1 \leq j \leq m - 2$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 3 and the edge terminates at vertex $\diagup$.

- If $i = j + m(m-1) + 1$ and $j \leq m - 2$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 4 and the edge terminates at vertex $\diagdown^*$.

Suppose $D_{\alpha|_k}$ and $D_{\beta|k}$ are directly to the left and right of one another, i.e. belong to Case 2. The edges originating at state $|$ labeled (i, j) represent the orientation and overlap of subsquares $D_{\alpha|_k i}$ and $D_{\beta|_k j}$.

- If $i = j + m - 1$ and $j \equiv 0 \mod m$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 2 and the edge terminates at vertex $|$. (This edge is a loop.)

- If $i = j - 1$, $j \equiv 0 \mod m$, and $j \neq 0$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 3 and the edge terminates at vertex $\diagup$.

- If $i = j + 2m - 1$, $j \equiv 0 \mod m$, and $j \neq m(m-1)$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 5 and the edge terminates at vertex ${}_*\diagdown$.

Here, we use ${}_*\diagdown$ since the first number in the pair, i, (when concatenated with $\alpha|_k$) represents the lower and more leftward subsquare.

Suppose $D_{\alpha|_k}$ and $D_{\beta|k}$ overlap on a corner and $D_{\alpha|_k}$ is above and to the left of $D_{\beta|_k}$, i.e. they belong to Case 3. The edges originating at state $\diagup$ labeled (i, j) represent the orientation and overlap of subsquares $D_{\alpha|_k i}$ and $D_{\beta|_k j}$. Note that in this situation, there is only one edge that will represent overlapping subsquares of $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ since there is exactly one such pair of subsquares.

- If $i = m^2 - 1$ and $j = 1$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 3 and the edge terminates at vertex $\diagup$. (This edge is a loop.)

Suppose $D_{\alpha|_k}$ and $D_{\beta|k}$ overlap on a corner and $D_{\alpha|_k}$ is above and to the right of $D_{\beta|_k}$, i.e. they belong to Case 4. The edges originating at state $\diagdown^*$ labeled (i, j) represent the orientation and overlap of subsquares $D_{\alpha|_k i}$ and $D_{\beta|_k j}$. Note that in this situation, there is only one edge that will represent overlapping subsquares of $D_{\alpha|_k i}$ and $D_{\beta|_k j}$, since there is exactly one such pair of subsquares.

- If $i = m(m-1)$ and $j = m - 1$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 4 and the edge terminates at vertex $\diagdown^*$. (This edge is a loop.)

Suppose $D_{\alpha|_k}$ and $D_{\beta|k}$ overlap on a corner and $D_{\alpha|_k}$ is below and to the left of $D_{\beta|_k}$, i.e. they belong to Case 5. The edges originating at state ${}_*\diagdown$ labeled (i, j) represent the orientation and overlap of subsquares $D_{\alpha|_k i}$ and $D_{\beta|_k j}$. Note that in this situation, there is only one edge that will represent overlapping subsquares of $D_{\alpha|_k i}$ and $D_{\beta|_k j}$, since there is exactly one such pair of subsquares.

- If $i = m-1$ and $j = m(m-1)$, then $D_{\alpha|_k i}$ and $D_{\beta|_k j}$ belong to Case 5 and the edge terminates at vertex ${}_*\diagdown$. (This edge is a loop.)

To show that infinite walks in T_m correspond to related words in the quotient that defines the square, we use the following lemma on finite walks in T_m.

Lemma 4.16. *Let $k \geq 1$ and $\alpha, \beta \in X_n^k$. Then $D_\alpha \cap D_\beta \neq \emptyset$ if and only if $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_k, \beta_k)$ is a walk in T_m starting at the vertex init.*

Proof. This proof is almost entirely based on the construction of T_m.

Firstly, $D_{\alpha_1} \cap D_{\beta_1} \neq \emptyset$ if and only if (α_1, β_1) is the label of an edge starting at the vertex *init*. This edge ends at the vertex that represents the orientation of subsquares D_{α_1} and D_{β_1}.

Inductively assume that $D_{\alpha|_m} \cap D_{\beta|_m} \neq \emptyset$ for some $m < k$ and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \ldots (\alpha_m, \beta_m)$ is a walk starting at the vertex *init* and ending at the vertex v that represents the orientation of subsquares $D_{\alpha|_m}$ and $D_{\beta|_m}$. Again, we have $D_{\alpha|_{m+1}} \cap D_{\beta|_{m+1}} \neq \emptyset$ if and only if there is an an edge labeled $(\alpha_{m+1}, \beta_{m+1})$ from vertex v. This edge ends at the vertex that represents the orientation of subsquares $D_{\alpha|_{m+1}}$ and $D_{\beta|_{m+1}}$. □

Theorem 4.17. *Let $D = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the attractor of the IFS $\{f_0, f_1, \ldots, f_{m^2-1}\}$ consisting of m^2 contracting similarities without rotation and reflection. Then $D \cong \mathcal{C}_{m^2}/\sim$, where $\sim$ is the invariant relation for which $\alpha \sim \beta$ if and only if the sequence $(\alpha_1, \beta_1), (\alpha_2, \alpha_2), (\alpha_3, \beta_3) \ldots$ is an infinite walk in T_m starting at init.*

Proof. Let $\sim$ be the invariant relation such that $D \cong \mathcal{C}_{m^2}$. Then

$$\begin{aligned} \alpha \sim \beta &\Leftrightarrow D_{\alpha|_k} \cap D_{\beta|_k} \neq \emptyset \text{ for all } k \\ &\Leftrightarrow (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots \text{ is a walk in } T_m \text{ starting at } \textit{init}. \end{aligned}$$

□

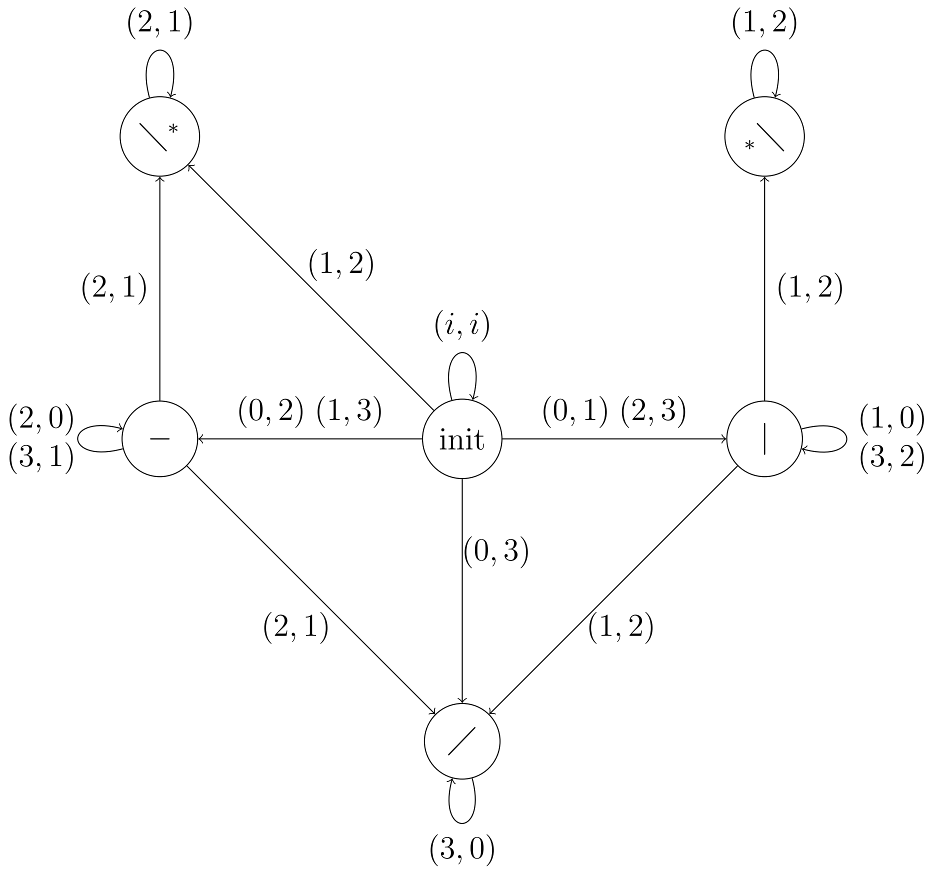

Figure 4.3: The directed graph T_2

To demonstrate the use of T_m, we consider the case when $m^2 = 4$ (i.e. two by two subsquares) and calculate a few equivalence classes and find them as infinite walks in the graph T_2, displayed in Figure 4.3. The center of the unit square, $(1/2, 1/2)$ in Cartesian coordinates, lies in each of the largest four subsquares, D_0, D_1, D_2, and D_3. There are exactly four infinite words in $\mathcal{C}_4$ that correspond to this point, one for each of the subsquares. These are $0\overline{3}$, $1\overline{2}$, $2\overline{1}$, and $3\overline{0}$.

Consider the words $0\overline{3}$ and $1\overline{2}$. Since subsquares D_0 and D_1 share a vertical edge, the edge $(0, 1)$ that starts at *init* terminates at $|$. Then, it is clear that the subsquares $D_{0(3^k)}$ and $D_{1(2^k)}$ also share a vertical edge for all $k \geq 0$, namely the lowest part of the same edge shared by D_0 and D_1 of length $2^{-(k+1)}$. In the graph T_2, this is represented by loop labeled $(3, 2)$ starting and terminating at vertex $|$. Any other pair of words representing the center also has an infinite walk in T_2.

4.2.3 Sierpiński Relatives as Invariant Factors

Thus far, we have described a graph whose infinite walks correspond to an invariant relation that defines the entire unit square. We will now give graphs for Sierpiński relatives by generalizing T_m.

We begin by considering the IFSs of the form $H = \{h_i | i \in X_{m^2}\}$ where $h_i = f_i \circ g_i$ and $g_i \in \mathrm{Sym}(D)$ is an element of the symmetry group of the square. The IFS H forms an iterated function system whose attractor is still D, but the invariant relation defined from this IFS is distinct from the previous one defined by T_m.

We define the graph $T_{m,H}$ such that infinite walks correspond to an invariant relation in the same fashion as T_m. The vertices of $T_{m,H}$ represent the relative position and orientation of two subsquares of D, including how the subsquares have been rotated and reflected. The vertices are $\mathrm{Sym}(D) \times \mathrm{Sym}(D) \times \big(\{-, |, \diagup, \diagdown^*, {}_*\diagdown\} \cup \{init\}\big)$. (Note that not all of these vertices are necessary. Some may be inaccessible, i.e $T_{m,H}$ may be disconnected. This will in fact happen quite often.) The vertex (g, g', v) for $v \in \{-$ $-, |, \diagup, \diagdown^*, {}_*\diagdown\}$ corresponds to two subsquares in a relative position represented visually as v (as in T_m) with the first having the same orientation as $g(D)$ and the second as $g'(D)$. The vertex *init* corresponds to two identical subsquares. We will not take into account the orientation of the subsquare in this case, as it is not necessary. Two words $\gamma\alpha$ and $\gamma\beta$ are related if and only if $\alpha \sim \beta$, so we ignore the orientation of subsquare D_γ and focus on the generators of $\sim$.

Let $\alpha, \beta \in X^*_{m^2}$ such that D_α and D_β are in a relative position defined above for some vertex (g, g', v) or *init* of $T_{m,H}$. Then there is an edge labeled (i, j) originating from v if and only if $D_{\alpha i} \cap D_{\beta j} \neq \emptyset$. We give the following example to illustrate this process.

Example 4.18. Let $\alpha, \beta \in X^*_{m^2}$ be words such that subsquares D_α and D_β share a horizontal edge displayed visually in Figure 4.4. Note that D_α is above D_β and D_α has been rotated $-\pi/2$, and D_β has been reflected about a vertical axis. This is represented by vertex $(g, g', -)$ in $T_{2,H}$, where $g = (0\ 1\ 3\ 2)$ and $g' = (0\ 1)(2\ 3)$ and H is an appriopriate list of elements of Sym(D).

$\alpha 2$	$\alpha 0$
$\alpha 3$	$\alpha 1$
$\beta 1$	$\beta 0$
$\beta 3$	$\beta 2$

Figure 4.4: Subsquare D_α above D_β

Since subsquares $D_{\alpha 3}$ and $D_{\beta 1}$ intersect along a horizontal line, there is an edge beginning at vertex $(g, g', -)$ labeled $(3, 1)$ and ending at $(gg_3, g'g_1, -)$.

Note that $D_{\alpha 3}$ is in position 2 (the position of D_2 within D) and $D_{\beta 1}$ is in position 0. In the graph T_m, this corresponds to the edge $(2, 0)$ starting at vertex $-$, and so in $T_{m,H}$, edge $(g(2), g'(0))$ originates at $(g, g', -)$. We use this idea to construct edges in $T_{m,H}$ by transforming edges in T_m.

For each edge (a, b) in T_m originating at vertex $v \in \{-, |, \diagup, \diagdown^*, {}_*\diagdown\}$ and ending at vertex w, there is an edge (i, j) in $T_{m,H}$ with initial vertex (g, g', v) and terminal vertex $(gg_i, g'g_j, w)$ where $(i, j) = (g(a), g(b))$.

We construct edges from *init* in $T_{m,H}$ likewise. For each edge (a, b) with $a \neq b$ in T_m originating at vertex *init* and ending at vertex v, there is an edge (a, b) in $T_{m,H}$ with initial vertex *init* and terminal vertex (g_a, g_b, v). Also, for each loop (a, a) in T_m starting and ending at vertex *init*, there is a loop (a, a) in $T_{m,H}$ from *init* to itself.

We can now state the following corollary of Theorem 4.17, adapted to include reflections and rotations.

Corollary 4.19. *Let $D = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be the attractor of the IFS $H = \{h_0, h_1, \ldots, h_{m^2-1}\}$ consisting of m^2 contracting similarities, where $g_i \in \mathrm{Sym}(D)$ and $h_i = f_i \circ g_i$. Then $D \cong \mathcal{C}_{m^2}/\sim$, where $\sim$ is the invariant relation for which $\alpha \sim \beta$ if and only if the sequence $(\alpha_1, \beta_1), (\alpha_2, \alpha_2), (\alpha_3, \beta_3) \ldots$ is an infinite walk in $T_{m,H}$ starting at init.*

The proof of Corollary 4.19 follows similarly to the proof of Theorem 4.17. The only difference is that the vertices now track the relative position and orientation of pairs of subsquares of D.

The graph $T_{m,H}$ can be quite large, even for simple examples, so that displaying it visually is not helpful. However, we will now describe a restriction of the graphs T_m and $T_{m,H}$ to the fractal subsets of square known as Sierpiński relatives.

Let $I = \{i_1, i_2, \ldots i_n\}$ be a subset of X_{m^2}. As defined before, a Sierpiński relative is the fractal attractor of an IFS of the form $H_I = \{h_i | i \in I\} \subseteq H$. To calculate the invariant relation of this IFS, we simply restrict the graph $T_{m,H}$ to a subgraph corresponding to indices in I. Specifically, we define the graph $T_{m,H,I}$ to be the subgraph of $T_{m,H}$ containing every vertex of $T_{m,H}$ but only those edges with labels contained in $I \times I$.

Recall the projection $\pi_H : \mathcal{C}_n \to F$, where H is an IFS and F is its fractal attractor, defined by $\pi_H(\alpha) = \lim_{k\to\infty}(h_{\alpha|_k}(F))$. As stated in Lemma 4.15, this defines an invariant relation $\sim$ such the quotient $\mathcal{C}_n/\sim$ is the fractal attractor F. Since H_I is contained in H, $\pi_{H_I}(\alpha) = \pi_H(\alpha)$ for all $\alpha \in I^\omega$. Let $\sim$ be the invariant relation defined by IFS H and $\sim_I$ be the invariant relation defined by IFS H_I. The projections show that for all $\alpha, \beta \in I^\omega$, $\alpha \sim \beta$ if and only if $\alpha \sim_I \beta$. Because of this, the graph $T_{m,H,I}$ will detect when two elements of I^ω are related, in the same way $T_{m,H}$ detects the relation $\sim$. From this, we give a further corollary of Theorem 4.17, now restricted to a subset of the original IFS.

Corollary 4.20. *Let $D = [0,1]\times[0,1] \subset \mathbb{R}^2$ be the attractor of an IFS $H = \{h_0, h_1, \ldots, h_{m^2-1}\}$ consisting of m^2 contracting similarities, where $g_i \in \mathrm{Sym}(D)$ and $h_i = f_i \circ g_i$. Also let $I \subseteq X_{m^2}$ with $1 < |I| \leq m^2$. Then the fractal attractor $F \subseteq D$ of the IFS H_I satisfies $F \cong I^\omega/\sim$, where $\sim$ is the invariant relation for which $\alpha \sim \beta$ if and only if the sequence $(\alpha_1, \beta_1), (\alpha_2, \alpha_2), (\alpha_3, \beta_3) \ldots$ is an infinite walk in $T_{m,H,I}$ starting at init.*

Note that the IFS H_I is potentially the subset of several distinct IFSs. However, since we remove those edges of $T_{m,H}$ corresponding to those indices not in I, they are inconsequential and any superset H of H_I satisfies the theorem.

Example 4.21. In this example, we consider the Sierpiński relative formed from the subset $I = \{0, 2, 3\}$ and the symmetry group elements $g_0 = g_2 = id$ and g_3 being the rotation of D by $-\pi/2$ radians, i.e. acts as the permutation (0 1 3 2). The attractor is shown in Figure 4.5.

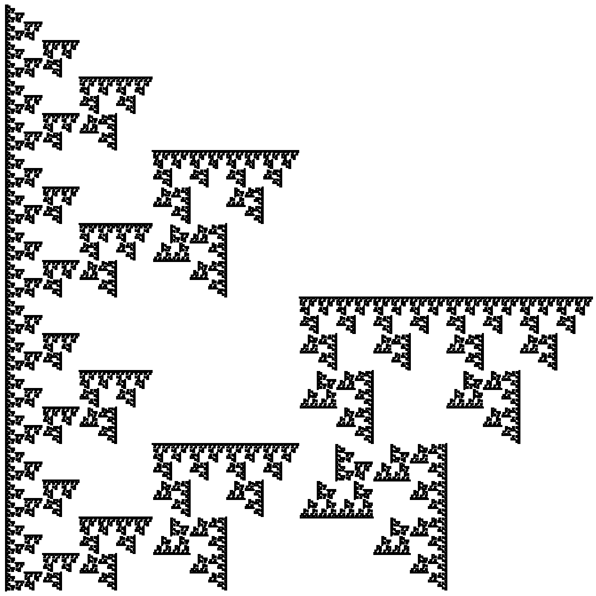

Figure 4.5: A Sierpiński relative.

Figure 4.6 shows the graph $T_{2,H,I}$ where H is an IFS with functions that map D to D_i without rotation or reflection except for $i = 3$. In this case, g_3 rotates D_3 by $-\pi/2$ radians, i.e. acts as the permutation (0 1 3 2). The subset I is equal to $\{0, 2, 3\}$, meaning we will only map into three of the four subsquares of D.

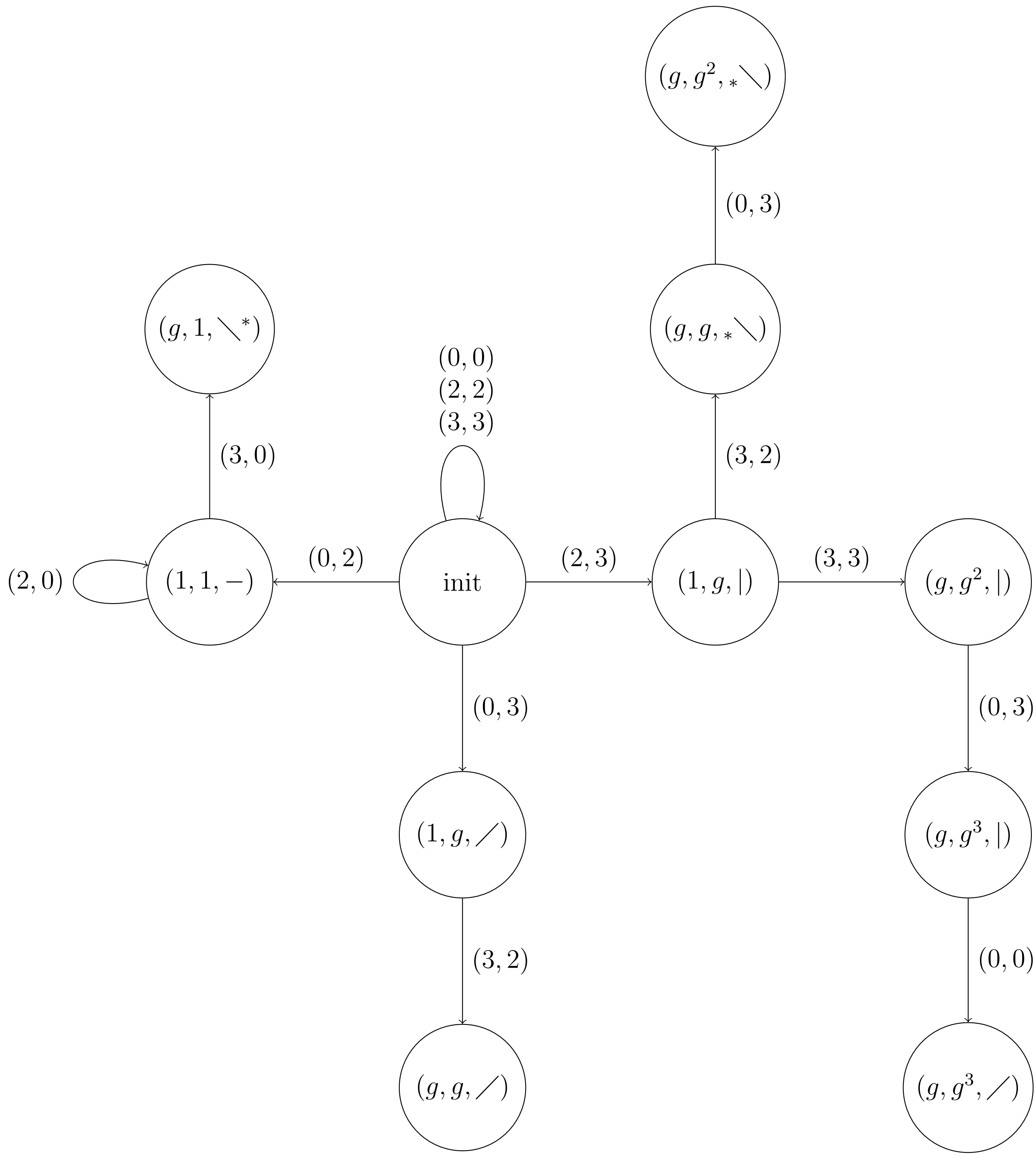

Figure 4.6: The directed graph $T_{2,H,I}$, where H and I are defined above.

Note that many walks in $T_{2,H,I}$ lead to 'dead ends' and are not an part of an infinite walk. This shows that while some subsquares that contain pieces of the fractal attractor abut one another, the parts of the subsquares that do overlap may be missing in the attractor.

4.3 Edge Replacement Systems

In this section, we compare invariant factors to edge replacement systems. Edge replacement systems were introduced by Belk and Forrest in [7] to construct groups that generalize the Thompson Groups F, T, and V. They do this by creating quotients of Cantor space that exhibit self-similar characteristics and forming groups that act on these quotients. We will focus on the relations, called gluing relations, that form the quotients in the edge replacement systems, and show the conditions under which gluing relations are invariant.

4.3.1 Construction

An edge replacement rule is a pair $e \to R$, where e is a directed (non-loop) edge and R is a finite directed multigraph. The edge replacement system (or rule) then refers to the process of replacing each of the edges of a given finite directed multigraph, G_0, with a copy of R. Repeating this process results in a sequence of graphs that can be used to define a relation on $\mathcal{C}_n$.

To be more precise, let e be a directed edge from vertex u to v with $u \neq v$. Also let R be a finite directed multigraph that contains vertices u and v. Note that e is not an edge in R. Then, given an edge g in a second directed multigraph G_0, we replace g with a copy of R in which $start(g)$ is identified with u and $end(g)$ is identified with v. Replacing each edge in G_0 with R results in another graph G_1. The graph G_m is recursively defined to be the result after having replaced each edge in G_{m-1} with R.

Let the edges of R have unique individual labels $\{0, 1, \dots n-1\}$ and the vertices of R have labels $\{u, v, r_1, \dots, r_k\}$. We call vertices u and v the *boundary* of R, whereas vertices $\{r_1, r_2, \dots, r_k\}$ are the *interior vertices*. Also let the edges of G_0 have labels $\{g_1, g_2, \dots, g_l\}$. We then define a labeling scheme for the edges and vertices in G_m. Let α be an edge label in G_{m-1}. Then the edges in G_m, formed when replacing α with a copy of R, are labeled $\{\alpha 0, \alpha 1, \dots, \alpha(n-1)\}$, where we simply concatenate the label of the edge being replaced and label of the edge in R. Therefore, the labels of the edges of G_m are $E(G_m) = \{g_i\alpha \mid g_i \in E(G_0) \text{ and } \alpha \in E(R)^m\}$

A similar process happens for vertices, however, not all vertices are replaced. The endpoints of edge $\alpha \in G_{m-1}$ retain their labels after replacing α with a copy of R. The newly formed vertices (corresponding to the interior vertices of R) have labels $\{\alpha r_1, \alpha r_2, \ldots, \alpha r_k\}$, the concatenation of edge label α with the label of the corresponding vertex of R. The unique labels of the vertices of G_m are therefore

$$V(G_m) = \{u, v\} \cup (E(R)^{<m} \times \{r_1, r_2, \ldots, r_k\}).$$

Example 4.22. The edge replacement system in Figure 4.7 is featured in [7].

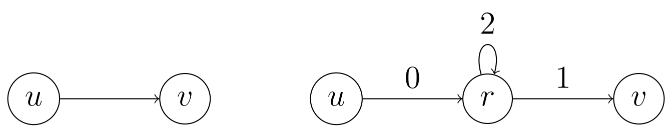

Figure 4.7: Edge e and directed multigraph R

We choose the initial graph G_0 to be a single edge. When forming G_1 and G_2, the label of the single edge is inconsequential, so we suppress it. The graph G_1 is isomorphic to R, so we show G_2 in Figure 4.8.

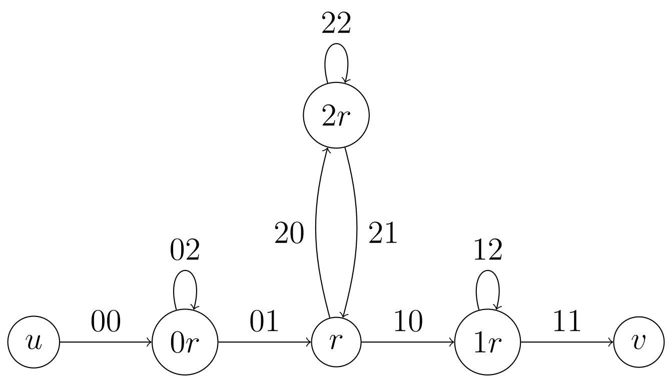

Figure 4.8: The directed multigraph G_2

This labeling scheme determines a relation $\sim$ on $\{g_1, g_2 \ldots, g_l\} \times \mathcal{C}_n$. For $\alpha, \beta \in \{g_1, g_2 \ldots, g_l\} \times \mathcal{C}_n$, $\alpha \sim \beta$ if and only if $\alpha|_m$ is incident to or equal to $\beta|_m$ as edges in G_m

for all m.

This relation defined by $e \to R$ and G_0 is called a gluing relation and is clearly reflexive and symmetric. It is not necessarily transitive, however. In order to consider gluing relations that are equivalence relations, we require that the edge replacement system $e \to R$ and G_0 be expanding, a term defined by Belk and Forrest in [7] to produce equivalence relations. An edge replacement system is *expanding* if and only if the following three properties are satisfied:

- R has at least two edges and three vertices;
- no vertex in R nor G_0 is isolated (not adjacent to another vertex);
- vertices u and v are not connected by a single edge in R

From this point onward, every edge replacement system will be assumed to be expanding. The system from Example 4.22 is expanding, and therefore the gluing relation for that edge replacement system is an equivalence relation. In fact, if one sets G_0 to be two loops from the same vertex, the appropriate quotient is homeomorphic to the Julia set associated to the polynomial $z^2 - 1$, also known as the Basilica set. This is displayed in Figure 4.9. See [6] for more details.

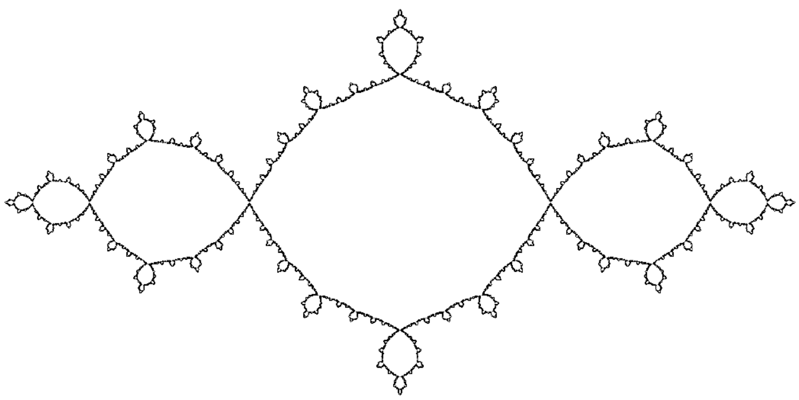

Figure 4.9: The Basilica set (created by Prokofiev, distributed under a CC-BY-SA 3.0 license).

We say that a word $\alpha \in \mathcal{C}_n$ represents a vertex γr if and only if $\alpha|_m$ is incident to γr in G_m for all m for which $\gamma r \in V(G_m)$. It is clear that if two infinite words α and β

both represent the same vertex, then $\alpha \sim \beta$. In fact, the converse of this statement is also true when $\alpha \neq \beta$.

Lemma 4.23. *Let $\alpha, \beta \in \mathcal{C}_n$ such that $\alpha \sim \beta$ and $\alpha \neq \beta$. Then there exists a vertex γr in G_m for all sufficiently large m such that α and β both represent γr.*

For brevity, we refer to [7] for the proof of this lemma. This also implies that each nontrivial equivalence class under a gluing relation corresponds to a particular vertex in some G_m.

Ultimately, we will compare invariant relations and gluing relations from edge replacement systems. However, these types of relations are on different spaces, specifically $\mathcal{C}_n$ and $E(G_0) \times \mathcal{C}_n$ respectively. To deal with this inconsistency, we set G_0 to be a single non-loop edge between two vertices so that $|E(G_0)| = 1$. We then treat the label of the single edge of G_0 as the empty word, in order to emphasize the fact that the label is inconsequential and the important information in the gluing relation is entirely contained in the infinite suffix over $\mathcal{C}_n$.

In Example 4.22, the vertex r in G_m for all $m \geq 1$ is represented by four infinite words in $\mathcal{C}_3$, $0\overline{1}$, $1\overline{0}$, $2\overline{0}$, and $2\overline{1}$. Note that the appropriate prefixes are incident as edges to r in G_1 and G_2.

The following two lemmas describe conditions under which edges are adjancent to specific vertices. The lemmas will be useful when describing words representing a vertex γr.

Lemma 4.24. *Let $\alpha \in E(G_m) = \{0, 1, \ldots, n-1\}^m$ be an edge incident to vertex $\gamma r \in V(G_m)$ and let $x \in E(R) = \{0, 1, \ldots, n-1\}$ be an edge in R. Then αx is incident to γr in G_{m+1} if and only if $start(\alpha) = \gamma r$ and x is incident to vertex u in R or $end(\alpha) = \gamma r$ and x is incident to vertex v in R.*

Proof. We begin with the forward implication. Since α is incident to γr, then either $start(\alpha) = \gamma r$ or $end(\alpha) = \gamma r$. If $start(\alpha) = \gamma r$, then replacing edge α with a copy of graph R identifies γr with vertex u. This means that only an edge y incident to u in R

can satisfy αy being incident to γr. Likewise, if $end(\alpha) = \gamma r$, the replacement identifies γr with vertex v, again meaning that only an edge y incident to v in R can satisfy αy being incident to γr.

We now consider the backward implication. If $start(\alpha) = \gamma r$ and x is incident to u in R, then replacing edge α with R indentifies γr with u. Since x is incident to u, αx is incident to γr in G_{m+1}. Likewise, if $end(\alpha) = \gamma r$ and x is incident to v in R, then replacing edge α with R indentifies γr with v. Since x is incident to v, αx is incident to γr in G_{m+1}. □

Lemma 4.25. *Let* $\alpha \in E(G_m) = \{0, 1, \ldots, n-1\}^m$ *be an edge incident to vertex* $\gamma r \in V(G_m)$ *and let* $x \in E(R) = \{0, 1, \ldots, n-1\}$ *be an edge in* R *such that* αx *is incident to* γr *in* G_{m+1}. *Then* $start(\alpha x) = \gamma r$ *if and only if* $start(x) \in \{u, v\}$ *and* $end(\alpha x) = \gamma r$ *if and only if* $end(x) \in \{u, v\}$.

Proof. Either $start(\alpha)$ or $end(\alpha)$ is γr since α is incident to γr.

Case 1 $start(\alpha) = \gamma r$: Lemma 4.24 then implies that x is incident to u. The direction of edge x with respect to u is preserved during the replacement, and since u is identified with γr, this means that $start(x) = u$ if and only if $start(\alpha x) = r$ and $end(x) = u$ if and only if $end(\alpha x) = r$.

Case 2 $end(\alpha) = r$: This case follows in the same fashion as Case 1 and implies that $start(\alpha x) = r$ if and only if $start(x) = v$ and $end(\alpha x) = r$ if and only if $end(x) = v$. □

4.3.2 Characterizing Gluing Relations

Let $e \to R$ be an expanding replacement rule with $V(R) = \{u, v, r_1, r_2, \ldots, r_k\}$ and $E(R) = \{0, 1, \ldots, n-1\}$. From this rule, we will create a new graph, Γ_R, that will describe the resulting gluing relation. The directed multigraph Γ_R will have the same vertices and twice the number of edge as R. Specifically, the vertices $V(\Gamma_R) = \{u, v, r_1, r_2, \ldots, r_k\}$ and edges $E(\Gamma_R)$ have labels $\{0_i, 0_f, 1_i, 1_f, \ldots, (n-1)_i, (n-1)_f\}$ such that for an edge x in R, the edges x_i and x_f in Γ_R satify:

- $start(x_i) = r$ and $end(x_i) = u$, where r is $start(x)$ in R;
- $start(x_f) = s$ and $end(x_f) = v$, where s is $end(x)$ in R.

In this section, we will consider walks beginning from vertices in Γ_R. A walk is a finite or infinite sequence χ of edges such that $end(\chi_i) = start(\chi_{i+1})$ for all i. Note that each vertex in Γ_R is the initial vertex of an edge, (since R is expanding) meaning that every finite walk is the beginning of an infinite walk. It is also useful to note that every edge in Γ_R ends at either u or v, the importance of which is hinted at in Lemmas 4.24 and 4.25.

Example 4.26. Recall the graph R from Example 4.22. Figure 4.10 displays Γ_R for this edge replacement system.

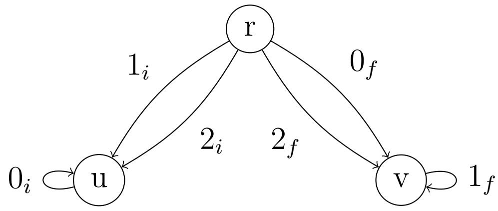

Figure 4.10: The directed muiltgraph Γ_R

Let $\Delta : E(\Gamma_R)^* \cup E(\Gamma_R)^\omega \to \cup_m E(G_m) \cup \mathcal{C}_n$ be defined by removing the subscripts i and f from each of the letters in the word. This demonstrates how finite walks in Γ_R can be mapped to edges in G_m and infinite walks to $\mathcal{C}_n$. The following lemma details the relationship between walks in Γ_R and the position and direction of edges in G_m.

Lemma 4.27. *Let ρ be an infinite walk in Γ_R starting at vertex r. If $\rho_m = x_i$ for some $x \in E(R)$, then $start(\Delta(\rho|_m)) = r$ in G_m. However, if $\rho_m = x_f$ for some $x \in E(R)$, then $end(\Delta(\rho|_m)) = r$ in G_m.*

Proof. We prove this lemma using induction. Since ρ begins at r, $start(\rho_1) = r$. From the definition of the edges in Γ_R, if $\rho_1 = x_i$, then $start(\Delta(\rho_1)) = r$ and if $\rho_1 = x_f$, then $end(\Delta(\rho_1)) = r$.

Now let $m \geq 2$ and assume that the conclusion holds for ρ_m. Either $\rho_m = x_i$ or $\rho_m = x_f$ for some $x \in E(R)$.

Case 1 $\rho_m = x_i$: By assumption, $start(\Delta(\rho|_m)) = r$. Also, since ρ is a walk and $end(\rho_m) = end(x_i) = u$, $start(\rho_{m+1}) = u$. This implies that $\Delta(\rho_{m+1})$ is incident to u. By Lemma 4.24, $\Delta(\rho|_{m+1})$ is incident to r in G_{m+1}. Furthermore, if $\rho_{m+1} = y_i$ for some y incident to u in R, then $start(\Delta(\rho_{m+1})) = u$. Together with Lemma 4.25, this shows that $start(\Delta(\rho|_{m+1})) = r$. However, if $\rho_{m+1} = y_f$ for some y incident to u in R, then $end(\Delta(\rho_{m+1})) = u$. Together with Lemma 4.25, this shows that $end(\Delta(\rho|_{m+1})) = r$.

Case 2 $\rho_m = x_f$: By assumption, $end(\Delta(\rho|_m)) = r$. Also, since ρ is a walk and $end(\rho_m) = end(x_f) = v$, $start(\rho_{m+1}) = v$. This implies that $\Delta(\rho_{m+1})$ is incident to v. By Lemma 4.24, $\Delta(\rho|_{m+1})$ is incident to r in G_{m+1}. Furthermore, if $\rho_{m+1} = y_i$ for some y incident to v in R, then $start(\Delta(\rho_{m+1})) = v$. Together with Lemma 4.25, this shows that $start(\Delta(\rho|_{m+1})) = r$. However, if $\rho_{m+1} = y_f$ for some y incident to v in R, then $end(\Delta(\rho_{m+1})) = v$. Together with Lemma 4.25, this shows that $end(\Delta(\rho|_{m+1})) = r$. □

Lemma 4.27 shows that walks of length m in Γ_R correspond to edges in G_m incident to the beginning vertex of the walk. This correspondence gives some information on the gluing relation.

Corollary 4.28. *Let ρ be an infinite walk in Γ_R starting at vertex r. Then the word $\Delta(\rho)$ represents r.*

Proof. Since $\Delta(\rho_m)$ is incident to r for every m by Lemma 4.27, $\Delta(\rho)$ represents r. □

Consider the graph Γ_R from Example 4.26 in Figure 4.10. There are four infinite walks starting at vertex r, each one following a loop at either u or v. They are, after applying the function Δ, $0\overline{1}$, $1\overline{0}$, $2\overline{0}$, and $2\overline{1}$, which are the four words that represent the vertex r.

We will now begin to show the converse of Corollary 4.28, that words representing a given vertex in $V(G_1) = V(R)$ correspond to walks in Γ_R. The next three lemmas examine words representing vertices for several cases.

Lemma 4.29. *Let ρ represent $r \in V(R)$. Then there exists an infinite walk χ in Γ_R beginning at r such that $\Delta(\chi) = \rho$.*

Proof. Since ρ represents r, ρ_1 is incident to r. If $start(\rho_1) = r$, then let $\chi_1 = (\rho_1)_i$, but if $end(\rho_1) = r$, then let $\chi_1 = (\rho_1)_f$. Doing so means that a walk in Γ_R starting with χ_1, begins at r.

By Lemma 4.24, ρ_k must be incident to either u or v for all $k \geq 2$. If $start(\rho_k) = u$ or v, then let $\chi_k = (\rho_k)_i$. Otherwise, $end(\rho_k) = u$ or v, and in this case let $\chi_k = (\rho_k)_f$. By definition, $\Delta(\chi) = \rho$. What is left to show is that $start(\rho_k) = end(\rho_{k-1})$ for all $k \geq 2$.

Case 1 $start(\rho_k) \in \{u, v\}$: Since $start(\rho_k) \in \{u, v\}$ and $\rho|_k$ is incident to r, Lemma 4.25 implies that $start(\rho|_k) = r$. Then, by Lemma 4.24, ρ_{k+1} is incident to u because $\rho|_{k+1}$ is also incident to r. If ρ_{k+1} starts at u, then $\chi_{k+1} = (\rho_{k+1})_i$ and $end(\chi_k) = start(\chi_{k+1}) = u$. If ρ_{k+1} ends at u, then $\chi_{k+1} = (\rho_{k+1})_f$ and $end(\chi_k) = start(\chi_{k+1}) = u$.

Case 2 $end(\rho_k) \in \{u, v\}$: Since $end(\rho_k) \in \{u, v\}$ and $\rho|_k$ is incident to r, Lemma 4.25 implies that $end(\rho|_k) = r$. Then, by Lemma 4.24, ρ_{k+1} is incident to v because $\rho|_{k+1}$ is also incident to r. If ρ_{k+1} starts at v, then $\chi_{k+1} = (\rho_{k+1})_i$ and $end(\chi_k) = start(\chi_{k+1}) = v$. If ρ_{k+1} ends at v, then $\chi_{k+1} = (\rho_{k+1})_f$ and $end(\chi_k) = start(\chi_{k+1}) = v$. □

Lemma 4.30. *Let r be an interior vertex of R and $\gamma \in E(G_m)$. Then $\rho \in r$ if and only if $\gamma\rho \in \gamma r$*

Proof. If γ is not a loop, then edge γ is isomorphic to G_0. Since both use the same replacement rule $e \to R$, for all $k \geq 1$, G_k is isomorphic to the subgraph of G_{m+k} induced by the edges $\gamma \| E(G_k)$. The graph isomorphism Φ is defined by $\Phi(\alpha) = \gamma\alpha$ for all $\alpha \in E(G_k)$, $\Phi(r) = \gamma r$ for the internal vertices $r \in E(R)$, $\Phi(u) = start(\gamma)$, and $\Phi(v) = end(\gamma)$. Since they are isomorphic when removing the prefix γ, $\rho|_m$ is incident to r if and only if $\gamma\rho|_{m+k}$ is incident to γr.

If γ is a loop in G_m, then the subgraph of G_{m+k} induced by the edges $\gamma \| E(R)^k$ is not isomorphic to G_k. However, Φ still maps G_k onto the aforementioned subgraph of G_{m+k}, while $\Phi(u) = \Phi(v)$ since $start(\gamma) = end(\gamma)$. Since Φ can only affect incidence to the images of u and v, it preserves incidence to the images of internal vertices and the conclusion still holds. $\square$

Lemma 4.31. *Let $\rho \in \mathcal{C}_n$. If $\rho \in u$, then for $x \in E(r)$, $x\rho \in start(x) \in V(R)$. Similarly, if $\rho \in v$, then for $x \in E(r)$, $x\rho \in end(x) \in V(R)$.*

Proof. By Lemma 4.29, if $\rho \in u$, there exists an infinite walk χ in Γ_R beginning at u such that $\Delta(\chi) = \rho$. Consider $x_i\chi$. Since $end(x_i) = u$ and χ begins at u, $x_i\chi$ is an infinite walk beginning at $start(x)$ such that $\Delta(x_i\chi) = x\rho$. Therefore, $x\rho \in start(x)$.

Likewise, if $\rho \in v$, there exists an infinite walk χ in Γ_R beginning at v such that $\Delta(\chi) = \rho$. Consider $x_f\chi$. Since $end(x_f) = v$ and χ begins at v, $x_f\chi$ is an infinite walk beginning at $end(x)$ such that $\Delta(x_f\chi) = x\rho$. Therefore, $x\rho \in end(x)$. $\square$

Now we will extend the results of previous three lemmas to show that walks in Γ_R with an appropriate prefix are equivalent to gluing relations. This theorem characterizes the connection between the graph Γ_R and the gluing relation.

Theorem 4.32. *Let $e \to R$ be an expanding replacement rule with gluing relation $\sim$ and let Γ_R be constructed as above. Then $\alpha \sim \beta$ if and only if $\alpha = \gamma\|\Delta(\rho)$ and $\beta = \gamma\|\Delta(\chi)$, where $\gamma \in E(R)^*$ and ρ and χ are infinite walks in Γ_R beginning at the same vertex.*

Proof. Recall that $\alpha \sim \beta$ implies that there exists a gluing vertex that is represented by both α and β. This gluing vertex is either of the form γr with $\gamma \in \|E(R)^*$ and r an internal vertex in $V(R)$ or it is simply u or v.

Case 1: The gluing vertex is γr. Since $\alpha, \beta \in \gamma r$, Lemma 4.30 says that $\alpha = \gamma\tau$ and $\beta = \gamma\sigma$, where $\tau, \sigma \in r$. By Lemma 4.29, there exist infinite walks ρ and χ in Γ_R such that $\tau = \Delta(\rho)$, $\sigma = \Delta(\chi)$ and τ and σ begin at the same vertex.

Case 2: The gluing vertex is u or v. This case follows directly from Lemma 4.29.

Now let ρ, χ be infinite walks in Γ_R starting at $r \in V(R)$. This implies $\Delta(\rho), \Delta(\chi) \in r$. Also let $\gamma \in E(R)^*$ with $|\gamma| = m$. If r is an internal vertex of R, using Lemma 4.30 means that $\gamma\|\Delta(\rho)$ and $\gamma\|\Delta(\chi)$ represent the gluing vertex γr and $\gamma\|\Delta(\rho) \sim \gamma\|\Delta(\chi)$.

However, if r is a boundary vertex of R, then Lemma 4.31 implies that $\gamma_m \| \Delta(\rho)$ and $\gamma_m \| \Delta(\chi)$ represent the same vertex in R, either $start(\gamma_m)$ or $end(\gamma_m)$. Repeating these arguments for $\gamma_m \| \Delta(\rho)$ and $\gamma_m \| \Delta(\chi)$, shows that $\gamma \| \Delta(\rho)$ and $\gamma \| \Delta(\chi)$ represent the same vertex. Ultimately, this implies that $\alpha = \gamma \| \Delta(\rho) \sim \gamma \| \Delta(\chi) = \beta$. □

4.3.3 Invariant Gluing Relations

The characterization of gluing relations using the graph Γ_R lets us detail the conditions under which a gluing relation is invariant. As it turns out, the only condition necessary is that the graph R contains no loops.

Consider the gluing relation in Example 4.22. The four words that represent the vertex r, $0\overline{1}$, $1\overline{0}$, $2\overline{0}$, and $2\overline{1}$, are related. In particular, this implies $2\overline{0} \sim 2\overline{1}$. However, the infinite words $\overline{0}$ and $\overline{1}$ do not represent the same vertex and are not related, and therefore $\sim$ is not an invariant relation. Whenever R has a loop, examples of related words such as these can be found, showing that the gluing relation is not invariant.

Lemmas 4.30 and 4.31 and Theorem 4.32 help show when gluing relations satisfy the first property of invariant relations, that the relations are preserved under adding and removing common prefixes. To show that gluing relations are closed, i.e. the second property of being invariant, we make use of the following three lemmas. The first demonstrates the fact that the two letter prefix of a point in Cantor space representing a vertex in R, completely detemines which vertex it represents.

Lemma 4.33. *Let α represent a vertex $r \in E(R)$. Then if $\beta = \alpha_1\alpha_2\gamma$ represents a vertex in $E(R)$, then β must also represent r.*

Proof. Since β represents a vertex in $E(R)$, there exists an infinite walk χ in Γ_R such that $\Delta(\chi) = \beta$. By Lemma 4.24, α_2 is either incident to u or v. If α_2 is incident to u, then $\chi_1 = (\alpha_1)_i$. If α_2 is instead incident to v, then $\chi_1 = (\alpha_1)_f$. In either case, α_2 determines χ_1, and therefore the vertex from which χ begins. This shows that, given $\beta = \alpha_1\alpha_2\gamma$ represents a vertex in $E(R)$, it must represent the same vertex as α as $\alpha|_2 = \alpha_1\alpha_2$. □

The next two lemmas show that the gluing relation is closed under a specific assumption that will be generalized in the final thereom.

Lemma 4.34. *Let $r \in V(\Gamma_R)$. Then the set W of infinite walks in Γ_R beginning at r is a closed set.*

Proof. Consider the set S of all finite words over $E(\Gamma_R)$ that are not finite walks beginning at r. Note that this is a countably infinite set. The set of infinite words over $E(\Gamma_R)$ that are not infinite walks beginning at r can be represented as $W^c = \cup_{\alpha \in S}[\alpha]$. Since $[\alpha]$ is open for all finite words α, W^c is open, and therefore W is closed. $\square$

Recall that we will ${}_k\alpha$ to mean the kth member of a sequence of infinite words ${}_1\alpha$, ${}_2\alpha, \ldots$.

Lemma 4.35. *Let ${}_m\alpha \to \alpha$ and ${}_m\beta \to \beta$ be convergent sequences in Cantor space such that ${}_m\alpha \sim {}_m\beta$ for all m and $\alpha_1 \neq \beta_1$. Then $\alpha \sim \beta$.*

Proof. Since ${}_m\alpha \to \alpha$ and ${}_m\beta \to \beta$, there exists a K such that for all $k, l \geq K$, ${}_k\alpha|_2 =_l \alpha|_2 = \alpha|_2$ and ${}_k\beta|_2 =_l \beta|_2 = \beta|_2$. This implies that for $k \geq K$, ${}_k\alpha|_1 \neq {}_k\beta|_1$. Since Γ_R determines the gluing relation and ${}_k\alpha$ and ${}_k\beta$ do not share a common prefix, ${}_k\alpha$ and ${}_k\beta$ must represent a vertex $r \in V(R)$. By Lemma 4.33, this shows that ${}_k\alpha$ and ${}_k\beta$ represent the same vertex in $V(R)$, for all $k \geq K$. The set of infinite walks beginning from a vertex is a closed set by Lemma 4.34, meaning that α and β represent the same vertex in $E(R)$ as ${}_k\alpha$ and ${}_k\beta$, and therefore $\alpha \sim \beta$. $\square$

Theorem 4.36. *The gluing relation $\sim$ formed by the expanding replacement rule $e \to R$ is an invariant relation if and only if R contains no loops.*

Proof. We will prove the forward implication by contradiction. Suppose that $x \in E(R)$ is a loop and $\sim$ is an invariant relation. Consider an infinite walk $x_i\rho$ in Γ_R and another, $x_f\chi$. By definition, $start(x_i) = start(x_f)$ since $start(x) = end(x)$, and therefore $\Delta(x_i\rho) \sim \Delta(x_f\chi)$. However, since $end(x_i) = u$ and $end(x_f) = v$, ρ begins at u and χ begins at v. This shows that $\Delta(\rho) \in u$ and $\Delta(\chi) \in v$, and since $e \to R$ is expanding, $\rho \not\sim \chi$. This is an example of two related elements with a common prefix that do not produce related elements when removing a common prefix. Therefore, $\sim$ is not invariant.

We will now show the backward implication, showing that the gluing relation is preserved under the removal of common prefixes and then the addition of common prefixes. Suppose that R contains no loops. For all pairs of edges x_i and x_f in Γ_R, $start(x_i) \neq start(x_f)$, since $start(x) \neq end(x)$ for all edges in R.

Let α and β be infinite words such that $\alpha_1 = \beta_1$ and $\alpha, \beta \in r \in E(R)$. Then there exists infinite walks, ρ and χ, in Γ_R beginning at the vertex r such that $\Delta(\rho) = \alpha$ and $\Delta(\chi) = \beta$. This implies $\Delta(\rho_1) = \Delta(\chi_1)$, and therefore then $\rho_1 = \chi_1$. Since removing prefixes ρ_1 and χ_1 from walks ρ and χ respectively results in two walks beginning at the same vertex in Γ_R, $\Delta(\rho_2\rho_3\ldots) \sim \Delta(\chi_2\chi_3\ldots)$, i.e. $\alpha_2\alpha_3\ldots \sim \beta_2\beta_3\ldots$

If instead α and β represent δr for some $\delta \in E(R)^+$ and $r \in V(R)$, then both α and β have δ as a prefix. This implies $\alpha = \delta\sigma$ and $\beta = \delta\tau$ for some $\sigma, \tau \in \mathcal{C}_n$. By Lemma 4.30, σ and τ both represent vertex $r \in E(R)$ and therefore $\sigma \sim \tau$.

The previous arguments show that removing the common prefixes of any two related elements of $\mathcal{C}_m$ produces elements that are still related. Futhermore, after removing $\alpha \wedge \beta$, the longest common prefix, the new pair represent a vertex of R. Lemmas 4.30 and 4.31 together imply that adding a common prefix to two elements representing the same vertex in $E(R)$ produces two new elements that again represent a common vertex (perhaps in $V(G_m)$ and not in $V(R)$), meaning that adding common prefixes preserves the gluing relation.

The last step in showing that the gluing relation is invariant is showing that $\sim$ is closed in $\mathcal{C}_n \times \mathcal{C}_n$. Let ${}_m\alpha \to \alpha$ and ${}_m\beta \to \beta$ be such that ${}_m\alpha \sim {}_m\beta$ for all m. If $\alpha_1 \neq \beta_1$, then Lemma 4.35 implies that $\alpha \sim \beta$. However, if α and β share a common non-trivial prefix γ, then $\alpha = \gamma\rho$ and $\beta = \gamma\chi$ for some $\rho, \chi \in \mathcal{C}_n$. From the previous arguments, $\alpha \sim \beta$ if and only if $\rho \sim \chi$. Since ${}_m\alpha \to \alpha$ and ${}_m\beta \to \beta$, there exists a K such that for $k \geq K$, ${}_k\alpha|_{|\gamma|} = \alpha|_{|\gamma|} = \gamma$ and ${}_k\beta|_{|\gamma}| = \beta|_{|\gamma|} = \gamma$. Removing the prefix γ from the subsequences ${}_k\alpha$ and ${}_k\beta$, where $k \geq K$, produces two new sequence that limit on ρ and

χ respectively. Using Lemma 4.35 with these newly created sequences, remembering the fact the removing common prefixes preserves relation, implies that $\rho \sim \chi$ and therefore $\alpha \sim \beta$. $\square$

The converse to this idea, expressing when an invariant relation can be formed as the gluing relation of an edge replacement system, led us to interesting yet inefficient descriptions of invariant relations. In essence, an invariant relation is a gluing relation if one can form a graph analogous to Γ_R, from which the edge replacement system can be reconstructed. They were not pursued for this thesis due to time constraints and lack of a concise theory. Characterising when invariant relations can be expressed using infinite walks on graphs merits attention, however, given its connection to not only edge replacement systems, but self-similar sets and other fractal constructions as well.

Chapter 5

Directions for Future Work

This thesis contains results stemming from the self-similar structure of Cantor space in several different areas of mathematics. In this short chapter, we will describe ideas for future research either inspired by questions arising from our results, or questions that could be approached using the techniques we have developed. This is not meant to be an exhaustive list, merely a collection of avenues of research that piqued our interests.

5.1 Codimension Formulae

In Chapter 2, we calculated codimension formulae in Cantor space using the Hausdorff and box-counting dimensions. A number of extensions of this work include exploring other fractal dimensions, defining general relative position using different groups of transformations, or simply seeking codimension formulae in other metric spaces.

We chose to work with the well-studied Hausdorff and upper box-counting dimensions mostly due to previous work on codimension formulae, drawing parallels between manifolds in $\mathbb{R}^n$, fractals in $\mathbb{R}^n$, and now fractals in $\mathcal{C}_n$. Other fractal dimensions, such as the packing dimension [17] and the Assouad dimension [22], could produce a vastly different formulae or adhere to the form of previous results. For example, the packing dimension of a set F in metric space X, $\dim_P(F)$, satisfies a useful relationship with the Hausdorff and upper box-counting dimension:

$$\dim_H(F) \leq \dim_P(F) \leq \overline{\dim}_B(F).$$

This demonstrates that some of the upper and lower bounds in Chapter 2 already apply to the packing dimension. More work would be required to replace the dimensions in the bounds with the packing dimension (if possible).

On the other hand, to describe the general relative position of sets in Cantor space, we used the natural Haar measure on the group of isometries of $\mathcal{C}_n$. Other groups of transformations of $\mathcal{C}_n$, such as the Thompson groups or the rational group, could produce independent codimension formulae, given a probability measure on the group. Conditions on the degree of transitivity of the transformation group could ensure the successful creation of $\mathcal{L}_2$ martingales to mimic the proofs of the lower bounds in Section 2.4.

The last extension of the results on Chapter 2 we discuss is to simply explore metric spaces other than $\mathcal{C}_n$ and $\mathbb{R}^n$. The p-adic numbers, $\mathbb{Q}_p$, are an excellent candidate for future research into codimension formulae, due to the similar structure of $\mathbb{Q}_p$ and $\mathcal{C}_n$. See [31] for more details. Another interesting candidate is $\mathrm{Aut}(\mathcal{T}_n)$ equipped with a metric. An investigation of fractal dimensions of subgroups of $\mathrm{Aut}(\mathcal{T}_n)$ began with Abercrombie [1] and Barnea and Shalev [5]. Defining the general relative position of two subgroups of $\mathrm{Aut}(\mathcal{T}_n)$ poses an interesting hurdle, as it could be defined in several ways, such as using random cosets or conjugation, or the group of isometries, potentially producing different codimension formula.

5.2 Thompson Group Generalisations

The clearest avenue for future research in Chapter 3 is to characterise for precisely which subgroups $G, H \leq S_n$ is $V_n(G)$ isomorphic to $V_n(H)$. As discussed at the end of Chapter 3, this is an open question for $n \geq 5$.

One thing to note about the isomorphisms we built between the Thompson-like groups

$V_n(G)$, is that we did not build explicit isomorphisms between each group in a given isomorphism class. For example, consider $V_4(G)$, and $V_4(H)$, where $G = \big\langle (0\ 1\ 2\ 3) \big\rangle$ and $H = \big\langle (0\ 1)(2\ 3), (0\ 2)(1\ 3) \big\rangle$. Both G and H are semiregular, so $V_4(G) \cong V_4(H)$, but only because they are both isomorphic to V_4. The groups G and H do not meet the criteria of Theorem 3.21 and therefore the conjugating transducers A_{id} do not apply. An investigation of this phenomenon could lead to a clearer understanding of isomorphisms between these groups.

In [10], Bleak and Lanoue proved that V_n is not isomorphic to V_m for all $n \neq m$ and Farley and Hughes [20] also describe some non-isomorphism results between $V_n(G)$ and $V_m(H)$, for select G and H. This begs the question as to whether any such isomorphism exists, or if $V_n(G) \ncong V_m(H)$ for all $n \neq m$, $G \leq S_n$, and $H \leq S_m$.

It is interesting to note that V_n is a subgroup of $V_n(G)$, yet we showed that in some cases V_n is conjugate to $V_n(G)$. One could explore this by asking, 'under what conditions does a conjugate of V_n contain V_n?' Likewise, the generalised Thompson group V_n must also contain conjugate copies of itself, using the same conjugation that takes $V_n(G)$ to V_n. The dual to the aforementioned question would be, 'under what conditions is a conjugate of V_n contained in V_n?' Research near to these questions was addressed by Bleak et. al. in [8], characterising the automorphism group of V_n. By relaxing the conditions under which a homeomorphism of $\mathcal{C}_n$ conjugates V_n to itself, one might find homeomorphisms that instead conjugate V_n into itself.

Our results pertain to whether a given group is isomorphic to V_n, but there is also an interest in understanding when a given group can be found as a subgroup of V_n. One cause for such interest comes from a conjecture of Lehnert, modified by Bleak, Matucci, and Neunhöffer in [11], relating to groups with context-free co-word problem, which were introduced by Holt, Röver, Rees, and Thomas in [25]. The Lehnert conjecture states that Thompson's group V is a universal group with context-free co-word problem, i.e. a $co\mathscr{C}\mathscr{F}$ group such that every finitely generated subgroup of V is a $co\mathscr{C}\mathscr{F}$ group, and all $co\mathscr{C}\mathscr{F}$ groups embed into V. Related to this, Farley in [19] describes a family of groups which he proves are all $co\mathscr{CF}$ groups, which includes the groups $V_n(G)$ discussed here. Farley proposes that some of these groups might be used to provide counterexamples to the Lehnert conjecture, which would occur if one can prove that one of these groups cannot embed into V. The techniques in Chapter 3 might be useful in finding transducers that could be used to embed such groups into V_n via topological conjugation.

5.3 Invariant Relations

Invariant relations, studied in Chapter 4, provide a way of characterising and generalising the topology of fractal sets. Our work mainly focused on developing tools and theory to approach future questions concerning the topology and groups of homeomorphisms of self-similar sets.

The initial motivation for studying invariant relations was to try to classify Sierpiński relatives up to homeomorphism. Their topology has been studied and categorised in [32] and [34] and we wanted to carry this further. Our results begin this process by giving a method (using finite graphs) to characterise the Sierpiński relatives as invariant factors, i.e. quotients of Cantor space. The next step would be to determine when two invariant factors are homeomorphic. While this may be difficult or nearly impossible in general, only considering invariant factors which produce Sierpiński relatives might provide enough restrictions to make an attempt feasible.

Rather than realising attractors of IFSs as invariant factors, like our results on Sierpiński relatives, one could ask the converse, whether a given invariant factor can be realised as the attractor of an IFS. This could be further narrowed to ask if the IFS can be self-similar, or even self-affine or self-conformal. Answering these questions for a class of invariant relations such as small invariant relations from Section 4.1 is an obvious starting point for such research.

In the spirit of Chapter 3 and Belk and Forest's work on edge replacement systems [7], one could consider groups of homeomorphisms of invariant factors. Theorem 4.6 establishes a connection between homeomorphisms of these quotients and graph isomorphisms. By taking an appropriate sequence (or directed set) of graph isomorphisms acting on the graphs G_P, one could potentially construct homeomorphisms of the invariant factor. While it is certainly not possible to construct the entire group of homeomorphisms of an invariant factor in this manner, one might be able to 'come close' in some sense. Restricting to transducers, an interesting topic would be investigating the group of transducers that preserve a given invariant relation (or any relation on Cantor space).

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