On the Synthesis of Passive Networks without Transformers

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Summary of the dissertation 'On the Synthesis of Passive Networks without Transformers' by Timothy H. Hughes

This thesis is concerned with the synthesis of passive networks, motivated by the recent invention of a new mechanical component—the inerter—which establishes a direct analogy between mechanical and electrical networks. We investigate the minimum numbers of inductors, capacitors and resistors required to synthesise a given impedance function, with a particular focus on transformerless network synthesis. The conclusions of this thesis are relevant to the design of compact and cost-effective mechanical and electrical networks for a broad range of applications. The thesis is divided into three parts.

In Part 1, we unify the Laplace-domain and phasor approach to the analysis of transformerless networks, using the framework of the behavioural approach to open and interconnected systems advocated by Willems. We show that the autonomous part of any driving-point trajectory of a transformerless network decays to zero as time passes. We then consider the trajectories of a transformerless network, which describe the permissible currents and voltages in the elements and at the driving-point terminals of the network. We show that the autonomous part of any trajectory of a transformerless network is bounded into the future, but it need not decay to zero as time passes. We then show that the value of the network's impedance at a particular point in the closed right half plane may be determined by finding a special type of network trajectory, which we call an s_0-trajectory.

In Part 2, we establish lower bounds on the numbers of inductors and capacitors required to realise a given impedance function. These lower bounds are expressed in terms of the extended Cauchy index for the impedance function, a property defined in that part. Explicit algebraic conditions are also stated in terms of a Sylvester and a Bezoutian matrix for the impedance. The lower bounds are generalised to multi-port networks. In addition, a connection is established with the properties of continued fraction expansions of real-rational functions, and the implications for network synthesis are described.

In Part 3, we first present four procedures for the realisation of a general impedance function with a transformerless network. These include two known procedures—the Bott-Duffin procedure and the Reza-Pantell-Fialkow-Gerst simplification—and two new procedures. We then show that the networks produced by the Bott-Duffin procedure, and by one of our new alternatives, contain the least possible number of reactive elements (inductors and capacitors) and resistors, for the realisation of a certain type of impedance function (called a biquadratic minimum function), among all series-parallel networks. Moreover, we show that these two procedures produce the only series-parallel networks which contain exactly six reactive elements and two resistors and which realise a biquadratic minimum function. We further show that the networks produced by the Reza-Pantell-Fialkow-Gerst simplification, and the second of our new alternatives, contain the least possible number of reactive elements and resistors for the realisation of almost all biquadratic minimum functions among the class of transformerless networks. We group the networks obtained by these two procedures into two quartets, and we show that these are the only quartets of transformerless networks which contain exactly five reactive elements and two resistors and which can realise all of the biquadratic minimum functions. Finally, we investigate the minimum number of reactive elements required for the realisation of certain impedance functions, of greater complexity than the biquadratic minimum function, with series-parallel networks.

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Declaration

I hereby declare that this dissertation is not substantially the same as any that I have submitted or will be submitting for a degree or diploma or other qualification at this or any other university. Furthermore, this dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specified explicitly in the text. I also declare that the length of this dissertation is less than 65,000 words and that the number of figures is less than 150.

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Contents

0.1 Introduction

0.1.1 Passive mechanical control

Motivation for this thesis is provided by the topic of passive mechanical control. There has been a resurgence of interest in this area since the recent invention of a new passive mechanical component, called the inerter. Since its invention, the inerter has been successfully deployed in Formula One racing [1]. Further applications of the inerter to automobile suspension [2] and railway suspension [3], motorcycle steering compensators [4, 5], vibration absorption [6], and building suspension [7] have also been identified.

An illustrative example of passive mechanical control is the design of a car suspension. Fig. 1 shows the standard quarter-car vehicle model, which comprises the sprung mass (m_s), the unsprung mass (m_u), the tyre (with stiffness k_t), together with a passive mechanism corresponding to the suspension. We may split the task of designing the suspension into two subtasks: (i) identify a suitable transfer function, relating the relative velocity of the two ends of the suspension to the force between these two ends, in order to achieve a target performance, (ii) construct a mechanism to realise the identified transfer function. Task (i) is reminiscent of a control systems design, with the caveat that the transfer function must be positive-real since the suspension is passive. Task (ii) amounts to finding a network realisation of a given positive-real function. By identifying force with current, velocity with voltage, springs with inductors, dampers with resistors, and inerters with capacitors, this task is seen to be equivalent to the classical synthesis problem of electric circuits—the realisation of a positive-real function using resistors, inductors, and capacitors only. This analogy is illustrated in Fig. 2.

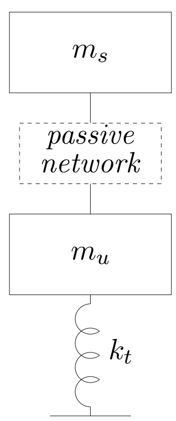


Figure 1: Quarter-car vehicle model.

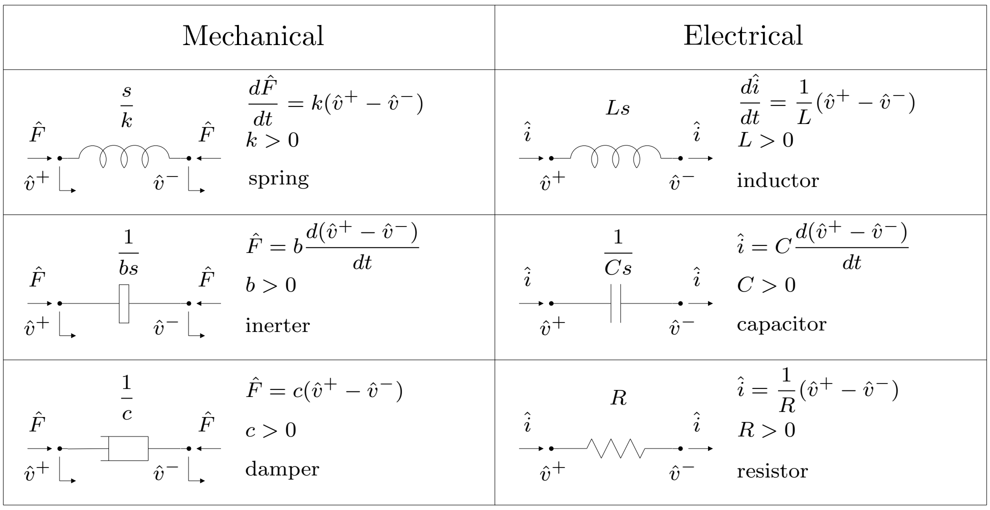


Figure 2: Passive electrical and mechanical elements.

Many open questions remain in the area of classical electric circuit synthesis. In particular, with the exception of a handful of simple cases, it is not known how to find a network of resistors, inductors, and capacitors (or, dampers, springs, and inerters) which contains the least possible number of elements for the realisation of a given transfer function. For reasons of size, cost, and reliability, it is particularly important to minimise the complexity of network realisations in mechanical applications. It follows that the search for networks which contain the least possible number of elements for the realisation of a given transfer function is of immediate relevance to the design of cost-effective and compact mechanical networks in a variety of fields.

0.1.2 History of academic developments in passive network synthesis

To place this thesis in context, we present here a brief history of the key developments in passive network synthesis.

Foster's reactance theorem [8] is arguably the first example of a systematic study which both analyses the achievable driving-point trajectories of a class of passive networks and provides a network synthesis procedure for realising such trajectories. In that paper, networks which contain only reactive elements (inductors and capacitors) are considered. It is shown that the impedance (and also the admittance) of any such network is necessarily lossless. Foster here defines a lossless function as a real-rational function whose poles and zeros are all on the imaginary axis, with the poles having (real) positive residues, and with the zeros interlacing the poles. Moreover, by considering the properties of partial fraction expansions of lossless functions, Foster showed how any lossless function may be realised by one of two series-parallel networks, each of which contains only reactive elements. These networks are now known as the *Foster forms*. Foster's reactance theorem provided the blueprint for subsequent realisation procedures: firstly, establish properties of the network trajectories which are shared by all networks in the considered class, and secondly, find a network (or a set of networks) from the considered class to realise any trajectory which satisfies these properties.

Shortly after the publication of [8], a second pair of series-parallel networks which contain only reactive elements and which can realise any given lossless function was identified by Cauer [9]. Moreover, Cauer also identified a sequence of transformations to the complex plane which allowed equivalent results to be stated for any network which comprises only two kinds of element (i.e. networks which comprise resistors and capacitors only, and networks which comprise resistors and inductors only).

The PhD thesis of Brune [10], published in 1931, extended the results of Foster to consider networks which contain resistors, transformers and reactive elements. There, the important notion of a positive-real function is first introduced, as a real-rational function which is analytic in the open right half plane and has a non-negative real part there. In that seminal thesis, it is shown that the impedance (and also the admittance) of a network which contains only resistors, transformers, and reactive elements is necessarily positive-real. Furthermore, a procedure is provided for constructing a network which contains only resistors, transformers, and reactive elements to realise any given positive-real function.

Brune's procedure is inductive, and uses results on the properties of positive-real functions to decompose the function being realised into several simpler functions for which a network realisation is evident by inspection. The procedure requires the use of ideal two-port transformers. These are devices in which the port voltages $\hat{v}_1$ and $\hat{v}_2$, and port currents $\hat{i}_1$ and $\hat{i}_2$, satisfy the relationships $\hat{v}_1/\hat{v}_2 = N_1/N_2 = -\hat{i}_2/\hat{i}_1$, where N_1/N_2 is a real number known as the turns ratio. The requirement for transformers poses issues with physical realisation since the properties of physical transformers differ considerably from their idealised counterparts. Consequently, there is a real need for procedures which avoid the use of transformers[1].

[1]A discussion of the differences between ideal and physical transformers may be found in [11, Chapter 2], where a more realistic circuit model of a physical transformer is provided. In the mechanical case, an analog to an ideal transformer is provided by an ideal lever, for which the ratio of port velocities is the inverse of the ratio of port forces and is given by the lever ratio. The properties of such an ideal lever do closely match the properties of a physical lever. However, the presence of levers poses issues whenever high lever ratios are required should there be constraints on the permissible travel between two terminals within the network—as is the case in most mechanical applications.

The first procedure which dispensed with the need for transformers was presented in a short paper in 1949 by Bott and Duffin [12]. This paper provided an alternative to that stage in Brune's procedure where transformers were used. Thus, Bott and Duffin's procedure provides a network which contains only resistors and reactive elements for realising any given positive-real function. However, the procedure uses a very large number of elements, and there has been much speculation about whether simpler procedures may exist.

A slight improvement on the Bott-Duffin procedure, which saves a single reactive element per inductive step, was provided simultaneously in the three papers [13–15]. Shortly after the publication of this improvement, it was shown by Storer [16] that the networks obtained by this improved procedure could be arrived at from the networks obtained by the Bott-Duffin procedure through a sequence of network transformations. Despite the improvement, the procedure still required a large number of elements, and speculation on the existence of yet simpler procedures has continued.

The papers of McMillan [17, 18] provide lower bounds on the number of reactive elements required to realise a given impedance function, against which these realisation procedures may be assessed. In those papers, the concept of the McMillan degree of a real-rational function is introduced, and it is shown that the number of reactive elements in a network must be greater than or equal to the McMillan degree of its impedance function. The number of reactive elements used in the procedures of Foster, Cauer, and Brune are in each case equal to the McMillan degree of the function being realised. In contrast, in the procedures of Bott and Duffin and its associated simplification, the number of reactive elements grows exponentially with the McMillan degree of the function being realised [19, Section 4]. Despite this, no further procedures for the realisation of a general positive-real function have been found which further reduce the number of reactive elements required.

In the 1960s there was a trend away from the study of electric circuits and towards the field of systems and realisation theory, a field in which there were many significant developments at the time. Techniques were developed to study electric circuits using the framework of conventional linear systems theory. One of the first complete treatments came in the paper of Youla and Tissi [20]. In contrast to earlier results (which were mostly concerned with one-port networks), the networks considered could have multiple ports. In [20], the question of the realisation of a given symmetric *bounded-real* function as the *scattering* matrix of a network containing resistors, transformers, and reactive elements was considered. The scattering matrix of a network is the mapping in the Laplace domain from the *incident excitation* $\mathbf{v}+\Lambda\mathbf{i}$ to the *reflected response* $\mathbf{v}-\Lambda\mathbf{i}$, where **v** and **i** are vector-valued functions corresponding to the Laplace transforms of the port voltages and currents respectively, and Λ is a diagonal matrix of positive (otherwise arbitrary) port-normalisation constants. As was already well known at the time, the scattering matrix of a network which contains only resistors, reactive elements, and transformers is necessarily symmetric and bounded-real. In [20], the reactance extraction technique of network analysis was introduced. By using this reactance extraction technique, together with the transformation of the Laplace-domain variable:

$$\phi(s) = \frac{s+\alpha}{s-\alpha}, \qquad \phi^{-1}(s) = \frac{\alpha(s+1)}{s-1}, \qquad \alpha > 0,$$

Youla and Tissi showed how the problem of synthesising a symmetric bounded-real matrix $S(s)$ as the scattering matrix of a network may be posed as the realisation problem of finding a symmetric matrix

$$S_a = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \tag{1}$$

with $I - S_a^T S_a$ positive semi-definite, such that

$$S(\phi^{-1}(s)) = S_{11} + S_{12}(sI - \Sigma S_{22})^{-1}\Sigma S_{12}^T. \tag{2}$$

The matrix Σ in the above equation is a signature matrix, i.e. a diagonal matrix whose diagonal entries are either $+1$ or -1. By presenting a factorisation of the matrix S_a in (1) which corresponds to the scattering matrix of a network containing only resistors and transformers, Youla and Tissi then showed how any bounded-real matrix may be realised as the scattering matrix of the network shown in Fig. 3. Here, N_r is the network containing resistors and transformers whose scattering matrix is S_a, and the values of the inductances and capacitances are related to the port-normalisation constants associated with this scattering matrix.

The recent invention of the inerter, which establishes the analogy between mechanical and electrical networks described in Subsection 0.1.1, has led to a resurgence of interest in the topic of passive network synthesis. For mechanical networks, due to cost and space limitations, there is considerable motivation for finding networks which contain the least possible number of elements for the realisation of a given impedance function. Consequently, there has been much recent research on the topic of synthesis with passive networks of restricted complexity [21–24]. In [21], networks which contain one damper, one inerter, and any number of springs are considered. Those impedances which can be realised by such networks are derived, and a set of networks is presented to realise any such impedance. The papers [22] and [23] describe those biquadratic impedance functions (i.e. functions whose McMillan degree equals two) which are realised by series-parallel networks containing five and six elements respectively. In each case, a set of networks is presented to realise any such impedance function. In [24], the theorem first stated by Reichert [25] is considered. This theorem states that the impedance function of any network which contains two reactive elements and any number of resistors (or, dampers) is also realised by a network which contains two reactive elements and at most three resistors.

This thesis represents a further contribution towards the understanding of passive network synthesis, with a focus on the number of reactive elements required for the realisation of a given positive-real function.

0.2 Structure of the dissertation

The original contributions of this thesis have been divided into three parts. Each part has been written as a stand-alone document, so the parts need not be read in order, although subsequent parts will refer to the results from the preceding parts. Definitions of many of the terms used throughout this thesis are provided in Section 1.1. In this section, we provide a short outline of each part.

Part 1 treats the topic of passive network analysis using the framework of the behavioural approach to open and interconnected systems due to Willems [26]. In that

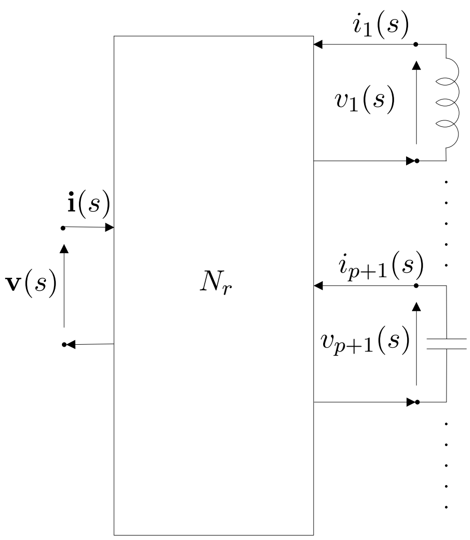


Figure 3: Network with reactive elements extracted.

part, we provide formal definitions for the class of passive networks, and the subclass of transformerless networks. We define the notions of the behaviour and the driving-point behaviour of a passive network. The driving-point behaviour is shown to be a more general notion of a network's performance than the network's impedance. We then derive several properties of the behaviour and the driving-point behaviour for a general transformerless network. Using these properties, we connect the techniques of impedance analysis and phasor analysis of networks. We also consider the connection between transformerless networks and graphs, and the implications for network analysis.

Part 2 establishes lower bounds on the individual numbers of inductors and capacitors required to realise a given positive-real function. The concept of the *extended Cauchy index* of a real-rational function is defined, and the lower bounds are expressed in terms of the extended Cauchy index of the positive-real function being realised. We also present explicit algebraic constraints on those impedance functions which may be realised by networks which contain limited number of inductors and capacitors in terms of a Sylvester matrix, a Bezoutian matrix, and a Hankel matrix associated with the impedance function. We further present a connection between the extended Cauchy index and the properties of a continued fraction expansion of a given real-rational function, and we describe the implications for network synthesis.

Part 3 investigates the number of reactive elements required by the Bott-Duffin procedure and its one known simplification. We present a new simplification to Bott-Duffin which has not appeared previously in the literature. We then show that the networks produced by the Bott-Duffin procedure contain the minimum possible number of reactive elements for the realisation of certain positive-real functions (known as *biquadratic minimum functions*) among the class of series-parallel networks. Furthermore, it is shown that the only other series-parallel networks which realise biquadratic minimum functions and which contain the same numbers of reactive elements and resistors as those of Bott and Duffin are related to the Bott-Duffin networks via a simple transformation. We further show that the networks obtained by the known simplification to Bott-Duffin contain the minimum possible number of reactive elements for the realisation of almost all biquadratic minimum functions among the broader class of transformerless networks. Moreover, we show that the network quartets produced by this known simplification, and by the new simplification we present in Part 3, are the only quartets of transformerless networks which contain exactly five reactive elements and exactly two resistors which realise all of the biquadratic minimum functions. Finally, we describe how these results extend to functions which are not biquadratic (i.e. whose McMillan degree exceeds two). In particular, for any integer r, we show the existence of positive-real functions of McMillan degree $2r$ which cannot be realised by any series-parallel network containing fewer than $4r$ reactive elements.

0.3 Notation

We will define notation specific to individual parts within the relevant part. Here, we describe notation which will be used throughout the thesis.

$\mathbb{C}$	complex plane
$\mathbb{C}_+$ ($\bar{\mathbb{C}}_+$)	open (closed) right-half complex plane
$\mathbb{C}_-$ ($\bar{\mathbb{C}}_-$)	open (closed) left-half complex plane
$\mathbb{R}$	real numbers
$j\mathbb{R}$	imaginary numbers
$\mathbb{F}$	field
$\mathbb{R}[s]$	polynomials in the indeterminate s
$\mathbb{R}[x_1, x_2, \ldots, x_n]$	polynomials in the indeterminates $x_1, x_2, \ldots, x_n$
$\mathbb{R}(s)$	real-rational functions in the indeterminate s
$\mathbb{R}_p(s)$	proper real-rational functions in the indeterminate s (i.e. bounded at $s = \infty$)
$\in$	set membership
$A \cup B$	union of sets A and B
$A \backslash B$	set A excluding set B
$\mathbb{F}^m$	m-valued column vectors with entries from the field $\mathbb{F}$ e.g. $\mathbb{R}^m[s]$ indicates the set of m-valued vectors whose entries are polynomials in the indeterminate s
$\mathbb{F}^{m \times n}$	$m \times n$ matrices with entries from the field $\mathbb{F}$ e.g. $\mathbb{R}^{m \times n}(s)$ indicates the set of $m \times n$ matrices whose entries are real-rational functions in the indeterminate s
$\bar{M}$	complex conjugate of a matrix M
M^T	transpose of a matrix M
M^*	complex conjugate transpose of a matrix M
$\langle \mathbf{x}, \mathbf{y} \rangle$	inner product $\mathbf{x}^* \mathbf{y}$ of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$
$M > 0$ (≥ 0)	Hermitian positive (semi-)definite matrix
$M < 0$ (≤ 0)	Hermitian negative (semi-)definite matrix
$A \dotplus B$	direct sum of the matrices A and B, i.e. the block diagonal matrix with diagonal blocks A and B
$\lvert M \rvert$	determinant of the matrix M
$r(M)$	rank of the matrix M
$\pi(M)$	number of strictly positive eigenvalues of the symmetric matrix M
$\nu(M)$	number of strictly negative eigenvalues of the symmetric matrix M
$\sigma(M)$	signature of the symmetric matrix M ($\sigma(M) = \pi(M) - \nu(M)$)

$M\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix}$	minor formed from rows $i_1, i_2, \dots, i_p$, and columns $j_1, j_2, \dots, j_p$, of the matrix M
I_n	$n \times n$ identity matrix
$0_{n\times m}$	$n \times m$ matrix whose entries are all zero
$\mathbf{0}_m$	m-valued column vector whose entries are all zero
$\mathrm{diag}\,(x_1, x_2, \dots, x_n)$	diagonal matrix with diagonal entries $x_1, x_2, \dots x_n$
$\Re(\alpha)$	real part of $\alpha \in \mathbb{C}$
$\Im(\alpha)$	imaginary part of $\alpha \in \mathbb{C}$
$D_j(R(s))$	greatest common divisor of all minors of order j in $R \in \mathbb{R}^{\bullet\times\bullet}[s]$
$I_{-\infty}^{+\infty}F(s)$	Cauchy index between $-\infty$ and $+\infty$ of $F \in \mathbb{R}^{\bullet\times\bullet}(s) \vert F(s) = F(s)^T$
$\gamma\,(F(s))$	extended Cauchy index of $F \in \mathbb{R}^{\bullet\times\bullet}(s) \vert F(s) = F(s)^T$ (see definition 2.3.1)
$\binom{n}{p}$	n choose p
$\mathbf{P}(x_1, x_2, \dots x_n)$	number of permanences in sign in the sequence $x_1, x_2, \dots x_n$
$\mathbf{V}(x_1, x_2, \dots x_n)$	number of variations in sign in the sequence $x_1, x_2, \dots x_n$
SP, ES, EP, PR	series-parallel, essentially series, essentially parallel, positive-real

Part 1

Passive network analysis

In this first part, we study passive networks and their behaviours, with a particular focus on the 'autonomous' components of the network behaviour, and we obtain several new results. Our approach is aligned with the behavioural approach to open and interconnected systems advocated in [26]. This enables us to rigorously connect the impedance description of a network's driving-point behaviour with a phasor analysis of its behaviour. We also present network analysis in the framework of graph theory, and we describe several key concepts from graph theory of relevance to network analysis.

It is well known that the impedance of a passive network is necessarily positive-real, and that the McMillan degree of the impedance is less than or equal to the number of reactive elements in the network. As will be show in Part 3 of this thesis, for certain impedance functions, the number of reactive elements required in fact exceeds the function's McMillan degree. In such cases, the network may possess driving-point trajectories which contain an autonomous component. The evolution of this autonomous component is completely determined by its value at a finite number of instants in time. This autonomous component cannot be inferred from an impedance description of a network's driving-point behaviour, and is overlooked in Laplace-domain descriptions of achievable network performances.

In Section 1.1, we present formal definitions of a passive network, a transformerless network, and various descriptions of a network's driving-point behaviour. These driving-point behavioural descriptions include the hybrid matrix, scattering matrix, and impedance and admittance descriptions, which feature prevalently in the literature on passive network synthesis, as well as a description as the kernel of a linear time-invariant differential operator, which we will develop in this part.

Next, in Section 1.2, we describe the application of graph theory to network analysis. We begin with some definitions of key concepts from graph theory in Subsection 1.2.1. In Subsection 1.2.2, we interpret the laws of interconnection for networks, namely Kirchhoff's current and voltage laws, in the language of graph theory, which leads us to the definition of a graph of elements, and also to a simple proof of Tellegen's theorem. Then, in Section 1.3, we present a kernel description for the behaviour of a general transformerless network in terms of the properties of the graph of elements corresponding to that network.

In Section 1.4, we will derive a kernel description of the driving-point behaviour of a general transformerless network, which we will then use to obtain conditions on the driving-point behaviour which hold for any transformerless network. Most notably, we show that if a driving-point trajectory of a transformerless network contains an autonomous component then it must decay to zero as $t \to \infty$.

The *driving-point behaviour* of a network describes the set of all permissible driving-point current and voltage trajectories for that network, while the *behaviour* of a network also describes the permissible current and voltage trajectories in each element within the network. We describe the decomposition of the behaviour of a general linear time-invariant differential system into controllable and autonomous parts, in the sense defined in [26], in Section 1.5. In Section 1.6, By considering a Smith form of the polynomial matrix $R(s)$ corresponding to the linear time-invariant differential operator $R(\frac{d}{dt})$ defining the behaviour, we obtain a parametrisation for a transformerless network's behaviour. We then use this parametrisation to describe the behaviour of a general transformerless network. In particular, we show that if a trajectory of a transformerless network contains an autonomous part then it is bounded into the future. In contrast to the autonomous parts of the driving-point trajectories, the autonomous part of a trajectory of a transformerless network need not decay to zero as $t \to \infty$.

In Section 1.7, we formalise the notion of a phasor analysis of transformerless networks through the concept of the s_0-behaviour. The s_0-behaviour corresponds to those trajectories of a network which take the form $\Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$ for some complex-valued vector $\tilde{\mathbf{b}}$ and some complex scalar s_0. One advantage of this approach is that it allows us to analyse networks using the techniques of linear algebra, rather than the polynomial algebra techniques described in Sections 1.1 to 1.6. In Section 1.7, for a given network N, and a given $s_0 \in \mathbb{C}$, we define the notion of an s_0-impedance and an s_0-admittance. We show that these respectively coincide with the impedance and admittance of N at the point s_0 whenever $s_0 \in \bar{\mathbb{C}}_+$. In addition, we show the curious result that the s_0-impedance may differ from the value of the impedance at certain points $s_0 \in \mathbb{C}_-$. In Section 1.8, we show how the notion of an s_0-behaviour may be extended in a consistent manner to cover the point at ∞.

Finally, in Section 1.9, we define the notions of subnetworks and one-port subnetworks of a transformerless network, and we show how the driving-point behaviour of a transformerless network may be obtained by considering a *graph of one-ports* corresponding to the network. This approach will be used extensively in Part 3 of this thesis.

1.1 Passive networks, trajectories, and behaviours

In this section, we provide formal definitions for the objects of interest in this thesis—namely passive networks and their driving-point behaviours.

The impedance, admittance, and hybrid matrix descriptions of the driving-point behaviour of passive networks feature most prevalently in the literature on passive networks, and our focus in subsequent parts will be on the realisation of these functions. Accordingly, we outline some of their properties in this section. We introduce the important associated concept of a positive-real function, and we describe several key properties of positive-real functions. Both the impedance and the admittance of a passive network are positive-real (when these descriptions exist), and likewise any hybrid matrix of the network is positive-real. The scattering matrix description provides the closest link with conventional linear systems and control, since it allows the interconnection laws associated with passive networks to be treated as feedback interconnections. This approach was adopted in [20], and will feature in Section 2.6. We further introduce a description of the behaviour, and the driving-point behaviour, of a passive network as the kernel of a linear time-invariant differential operator. As will be seen in subsequent sections, this provides a more general description of the driving-point behaviour of a network, since it captures not only the controllable part of the driving-point behaviour, but also the autonomous part (which is neglected in the impedance, admittance, and scattering matrix descriptions).

As discussed in Subsection 0.1.1, passive networks feature in both electrical and mechanical applications. In this thesis, we use the language associated with electrical networks, which we define first.

An electrical network is a device containing a number of terminals at which it may interact with its environment. Energy may be transferred to the network by connecting *sources* across *ports*, a port being a pair of terminals[2]. These sources provide a driving-point current through and voltage across the port at each point in time. For a network with n ports, the current and voltage provided by the sources supply a total energy to the network between time instants t_1 and t_2 of

$$\int_{t_1}^{t_2} \hat{v}^T(\tau)\hat{i}(\tau)d\tau. \tag{3}$$

[2] As noted by Willems [27], a port may instead be defined as a set of terminals that satisfies the port Kirchhoff's current law (i.e. the sum of the currents into each terminal in the set is zero). This coincides with the view of a port as a pair of terminals for networks containing only two terminals, which are the principal focus in the majority of this thesis.

Here, $\hat{v} := \begin{bmatrix} \hat{v}_{01} & \hat{v}_{02} & \ldots & \hat{v}_{0n} \end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^n$ is a vector-valued function corresponding to the time evolution of the voltages across the ports. Similarly, $\hat{i} := \begin{bmatrix} \hat{i}_{01} & \hat{i}_{02} & \ldots & \hat{i}_{0n} \end{bmatrix}^T :$ $\mathbb{R} \mapsto \mathbb{R}^n$ is a vector-valued function corresponding to the time evolution of the respective currents through the ports[3]. Each particular network will constrain the driving-point voltages and currents to follow certain paths, which we call *driving-point trajectories.* We call the network *passive* if

$$\int_{t_1}^{t_2} \hat{v}^T(\tau)\hat{i}(\tau)d\tau \geq 0,$$

for all $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$, and all driving-point trajectories which satisfy $\hat{v}(t) = \hat{i}(t) = 0$ for all $t < t_1$. This is connected with a physical property of passive networks. Specifically, the total energy supplied to the environment by a passive network which is initially at rest cannot exceed the total energy supplied to the passive network by the environment.

A network can comprise an interconnection of smaller networks of various kinds. In this thesis, we focus on networks containing only resistors, reactive elements (inductors and capacitors), and transformers. The driving-point trajectories permitted by inductors, capacitors, and resistors are listed in Fig. 2 on p. 2. A transformer is a device whose ports may be divided into two sets with corresponding currents

$$\hat{\mathbf{i}}_1 := \begin{bmatrix} \hat{i}_1 & \hat{i}_2 & \ldots & \hat{i}_{n_1} \end{bmatrix}^T,$$
$$\text{and} \quad \hat{\mathbf{i}}_2 := \begin{bmatrix} \hat{i}_{n_1+1} & \hat{i}_{n_1+2} & \ldots & \hat{i}_{n_1+n_2} \end{bmatrix}^T,$$

and with corresponding voltages

$$\hat{\mathbf{v}}_1 := \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \ldots & \hat{v}_{n_1} \end{bmatrix}^T,$$
$$\text{and} \quad \hat{\mathbf{v}}_2 := \begin{bmatrix} \hat{v}_{n_1+1} & \hat{v}_{n_1+2} & \ldots & \hat{v}_{n_1+n_2} \end{bmatrix}^T,$$

such that

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{i}}_2 \end{bmatrix} = \begin{bmatrix} 0 & T^T \\ -T & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix},$$

where $T \in \mathbb{R}^{n_2 \times n_1}$. The matrix T is commonly called the turns-ratio matrix, since it

[3]In (3), the vector-valued functions $\hat{i}$ and $\hat{v}$ are assumed to belong to a function space such that the integral is well defined. We remark that the functions $\hat{i}$ and $\hat{v}$ are denoted with hats to distinguish them from their Laplace transforms i and v. This convention is elaborated on in footnote 7.

relates to the ratio of the number of turns in the coils on a physical transformer whose behaviour is approximated by the above equation.

The current and voltage associated with any individual element within the network are constrained to follow a driving-point trajectory of that element. Moreover, the interconnection of the elements imposes further constraints, known as *Kirchhoff's current law* and *Kirchhoff's voltage law*. These laws state that there can be no accumulation of current at any point in the network, and that the sum of voltages around any closed circuit must be zero, respectively. The laws have interpretations in terms of graphs derived from the network, which we will discuss in Section 1.2.

It is straightforward to verify that any inductor, capacitor, resistor, or transformer is passive. Moreover, any network comprising an interconnection of these elements is passive [28, Section 2.3]. For the remainder of this thesis, we reserve the term *passive network* to imply a network comprising an interconnection of inductors, capacitors, resistors, and transformers. We remark that such networks are also often referred to as reciprocal networks (e.g. [28, Section 2.8]), and we will follow this convention in Part 2 of this thesis[4].

Now, let N be an n-port network with port currents $\hat{i} := \begin{bmatrix} \hat{i}_{01} & \hat{i}_{02} & \ldots & \hat{i}_{0n} \end{bmatrix}^T$ and voltages $\hat{v} := \begin{bmatrix} \hat{v}_{01} & \hat{v}_{02} & \ldots & \hat{v}_{0n} \end{bmatrix}^T$, and let N be comprised of the m passive elements $N_1, N_2, \ldots, N_m$. Further let $\hat{i}_k$ and $\hat{v}_k$ be the current and voltage for the element N_k respectively ($k = 1, 2, \ldots, m$), and let

$$\hat{\mathbf{b}} := \begin{bmatrix} \hat{i}^T & \hat{v}^T & \hat{i}_1 & \ldots & \hat{i}_m & \hat{v}_1 & \ldots & \hat{v}_m \end{bmatrix}^T. \tag{4}$$

We call $\hat{\mathbf{b}} : \mathbb{R} \mapsto \mathbb{R}^{2(m+n)}$ a *trajectory* of the network if the currents and voltages in $\hat{\mathbf{b}}$ satisfy all the element-wise and interconnection constraints imposed by the network. In other words, the vector-valued function $\hat{\mathbf{b}}$ corresponds to a physically permitted evolution of currents and voltages in the network. Such a trajectory is then associated

[4]The descriptor *reciprocal* refers to a physical property of such networks relating to the invariance of the location of applied excitation and measured response. For example, consider a two-port network comprising resistors, inductors, capacitors and transformers in which the voltages at the two ports may be varied independently of each other. If we apply a voltage $\hat{v}$ at the second port and we short circuit the first port, and we measure the current through the short circuit at the first port, then we get the same current which we would obtain at the second port if, instead, we were to apply the same voltage $\hat{v}$ at the first port and short circuit the second port.

with the driving-point trajectory

$$\hat{\mathbf{d}} := \begin{bmatrix} \hat{i} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} I_{2n} & 0_{2n\times 2m} \end{bmatrix} \hat{\mathbf{b}}.$$

Since all the element-wise constraints and interconnection constraints are linear time-invariant differential equations in the entries of the vector $\hat{\mathbf{b}}$, then the set of all trajectories of a network are given by the kernel of some linear time-invariant differential operator $R\left(\frac{d}{dt}\right)$. In other words, $\hat{\mathbf{b}}$ is a trajectory of the network N if and only if

$$R\left(\frac{d}{dt}\right)\hat{\mathbf{b}} = \mathbf{0}. \tag{5}$$

Here, $R \in \mathbb{R}^{\bullet\times\bullet}[s]$ may be obtained from the element-wise and interconnection constraints[5].

We call the set of all trajectories of a network the *behaviour*, and we call equation (5) a *kernel description* of the network behaviour[6]. Similarly, we call the set of all driving-point trajectories of a network the *driving-point behaviour*. The driving-point behaviour of a network is also equivalent to the kernel of a linear time-invariant differential operator. This follows from the elimination of the variables $\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_m$, and $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m$ from equation (5), by application of the procedures described in [26, Boxes 'Polynomial Modules and Syzygies' and 'The Fundamental Principle and the Elimination Theorem'].

As shown in [28, Section 4.4], any passive network also possesses a *hybrid matrix*. This corresponds to a partition of the driving-point currents and voltages into the two sets

$$\hat{\mathbf{w}}_1 := \begin{bmatrix} \hat{i}_{k_1} & \ldots & \hat{i}_{k_r} & \hat{v}_{k_{r+1}} & \ldots & \hat{v}_{k_n} \end{bmatrix},$$
$$\text{and} \quad \hat{\mathbf{w}}_2 := \begin{bmatrix} \hat{v}_{k_1} & \ldots & \hat{v}_{k_r} & \hat{i}_{k_{r+1}} & \ldots & \hat{i}_{k_n} \end{bmatrix},$$

such that the driving-point behaviour of the network is the solution to the linear time-

[5]To simplify the presentation in subsequent analysis, we will often consider the entries in the vector $\hat{\mathbf{b}}$ to be infinitely differentiable, which we denote by $\hat{\mathbf{b}} \in \mathscr{C}_\infty^{2(n+m)}$. This guarantees that the effect of the operator $R\left(\frac{d}{dt}\right)$ on the vector $\hat{\mathbf{b}}$ is well-defined. An alternative approach is to consider entries in $\hat{\mathbf{b}}$ to be locally integrable and to consider so-called weak solutions to the differential equations. The results presented in this thesis remain true in this alternative framework with the minor qualifications outlined in [29, Section 2.3].

[6]Note that kernel descriptions of behaviours are non-unique. In other words, in equation (5), several different linear time-invariant differential operators $R(\frac{d}{dt})$ can give rise to the same behaviour.

invariant differential equations

$$\begin{bmatrix} -P\left(\frac{d}{dt}\right) & Q\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}_1 \\ \hat{\mathbf{w}}_2 \end{bmatrix} = \mathbf{0},$$

where $P, Q \in \mathbb{R}^{n \times n}[s]$ with $|P(s)|$ not identically equal to zero. The matrix $H(s) := P(s)^{-1}Q(s)$ is referred to as a hybrid matrix, or an impedance matrix (resp. admittance matrix) if all entries in $\hat{\mathbf{w}}_1$ are voltages (resp. currents). Here, $H(s)$ may be considered as a mapping between variables in the Laplace-domain. In other words, if $\hat{\mathbf{w}}_1(t)$ and $\hat{\mathbf{w}}_2(t)$ are both zero for all $t < 0$, and if the unilateral Laplace transforms $\mathbf{w}_k(s) := \int_{0^-}^{\infty} \hat{\mathbf{w}}_k(t)e^{-st}dt$ exist $(k = 1, 2)$[7], then

$$\mathbf{w}_1 = H(s)\mathbf{w}_2.$$

A hybrid matrix of a passive network is necessarily positive-real (hereafter PR) in accordance with the following definition [28, Theorem 2.7.3]:

Definition 1.1.1 (PR)**.**
$H(s)$ is called PR if and only if the following four conditions are met:

1. $H \in \mathbb{R}^{n \times n}(s)$ for some integer n.
2. All elements of $H(s)$ are analytic in $s \in \mathbb{C}_+$.
3. $H(\xi)$ is real for all $\xi \in \mathbb{R}, \xi > 0$.
4. $H(\xi)^* + H(\xi) \geq 0$ for all $\xi \in \mathbb{C}_+$.

We also note the following theorem, which presents an alternative characterisation of PR functions:

Theorem 1.1.2 ([28], Theorem 2.7.2)**.** *Let $H \in \mathbb{R}^{n \times n}(s)$ for some integer n. Then $H(s)$ is PR if and only if*

1. *All elements of $H(s)$ are analytic in $s \in \mathbb{C}_+$.*
2. *$H(j\omega)^* + H(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $j\omega$ not a pole of any element of $H(s)$.*

[7] In this thesis, we let the unilateral Laplace transform be the mapping $\mathbf{b}(s) = \int_{0^-}^{\infty} \hat{\mathbf{b}}(t)e^{-st}dt$ from a vector-valued function $\hat{\mathbf{b}}$, to a vector-valued function $\mathbf{b}$ (in the indiscriminate s). We will refer to the function $\hat{\mathbf{b}}$ (resp. $\mathbf{b}$) as a time-domain (resp. Laplace-domain) function, and we distinguish a time-domain function from the respective Laplace-domain function by using a hat. Whenever we present the function $\mathbf{b}$, we will take it as implicit that the corresponding function $\hat{\mathbf{b}}$ satisfies $\hat{\mathbf{b}}(t) = \mathbf{0}$ for all $t < 0$.

3. *For all $\omega_0 \in \mathbb{R} \cup \infty$, if $j\omega_0$ is a pole of any element of $H(s)$ then it is at most simple, and the residue matrix at $s = j\omega_0$ is Hermitian positive semi-definite.*

Finally, a passive network N also possesses a *scattering matrix* for any choice of port normalisation constants, see e.g. [20, Section 2]. The scattering matrix of a passive network is the mapping in the Laplace domain from the *incident excitation* $\mathbf{v} + \Lambda\mathbf{i}$ to the *reflected response* $\mathbf{v} - \Lambda\mathbf{i}$, where $\mathbf{v}$ and $\mathbf{i}$ are vectors of (Laplace transformed) port voltages and currents respectively, and Λ is a diagonal matrix of positive (otherwise arbitrary) port-normalisation constants. As described in [20, Section 2], the scattering matrix of a network which comprises only resistors, reactive elements, and transformers is necessarily symmetric and *bounded-real* (as defined in [20, Section 2]).

We finish this section with some terminology. Much of the focus of this thesis is on *one-port* networks. These are networks possessing two driving-point terminals, which comprise a single port. Hereafter, we indicate the current through and voltage across the driving-point terminals by $\hat{i}$ and $\hat{v}$ respectively, as in Fig. 4. In addition, most of our focus will be on *transformerless* networks, i.e. one-port networks which do not contain transformers. We will also pay particular attending to the subclass of transformerless networks known as *series-parallel* (SP) networks[8]. Unless otherwise stated, we will assume a transformerless network N contains m elements $N_1, N_2, \ldots, N_m$ and we will denote the current through and voltage across element N_k by $\hat{i}_k$ and $\hat{v}_k$ respectively. In this manner, the behaviour of N is composed of $m+1$ currents and $m+1$ voltages, these corresponding to the currents and voltages in the m elements together with the driving-point current and voltage.

$$Z(s) = \frac{v(s)}{i(s)}$$

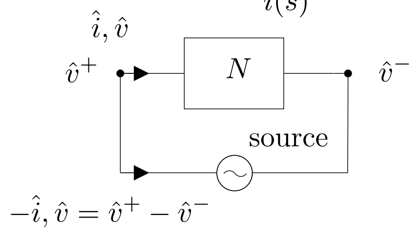


Figure 4: One-port network with source. Throughout this thesis, the driving-point terminals will be indicated with dots, and the impedance of the network will be written above the network.

[8]A formal definition of a series-parallel (SP) network is provided in Subsection 3.2.1

1.2 Transformerless networks and graphs of elements

In this section, we consider the relationship between networks and graphs, and the interpretation of Kirchhoff's laws in terms of graph-theoretic properties. The arguments presented here will be used extensively in Sections 3.5 and 3.6 of this thesis.

Treatments of passive networks in the graph theory literature tend to focus on networks containing only resistors, e.g. [30]. Conversely, treatments of passive networks from a modern systems theory viewpoint tend to omit discussions of network topology, e.g. [28]. Network topology features more prevalently in earlier literature on passive networks, e.g. [31], but these accounts tend to focus on Laplace-domain analysis which is subject to the limitations described in this part. Accordingly, we provide an outline of relevant concepts from graph theory in the next subsection. We then interpret Kirchhoff's laws in terms of a graph derived from a network in Subsection 1.2.2, which we call a graph of elements, and we use this interpretation to derive Tellegen's theorem. In Section 1.3, we will then provide a kernel description for the behaviour of a general transformerless network in terms of the graph of elements corresponding to the network.

1.2.1 Graph theory preliminaries

Questions in network analysis and synthesis may be succinctly described using the language of graph theory. Particularly relevant concepts include connectivity, trees, chord-sets, circuits and cut-sets. Accordingly, we outline many of these concepts in this section. The definitions and arguments presented here closely follow [30], with some exceptions for the convenience of our subsequent analysis.

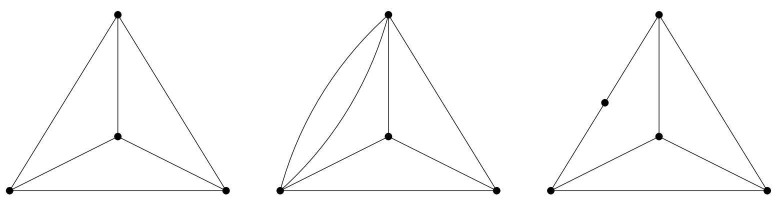


Figure 5: Graphs.

By a *graph* we mean an ordered pair (V, E) where V is a set $\{x_1, x_2, \ldots, x_n\}$ whose elements are called *vertices* and E is a set $\{y_1, y_2, \ldots, y_q\}$ of unordered pairs of vertices called *edges*, i.e. $y_k = (x_{k_1}, x_{k_2})$, $k = 1, 2, \ldots, q$. We say an edge (x_i, x_j) is incident with the vertices x_i and x_j (and, similarly, the vertices are incident with the edge, and two edges are incident if both are incident with a common vertex). Such graphs may be described by a drawing as in Fig. 5, with vertices identified by dots, and with edges identified by lines between their incident vertices.

We say a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is a *subgraph* of a graph $G = (V, E)$ if $\tilde{V}$ is a subset of V and $\tilde{E}$ is a subset of E. A *path* of a graph G from vertex x_{k_1} to x_{k_p} is a subgraph $(\tilde{V}, \tilde{E})$ of G with $\tilde{V} = \{x_{k_1}, x_{k_2}, \ldots, x_{k_p}\}$ and $\tilde{E} = \{(x_{k_1}, x_{k_2}), (x_{k_2}, x_{k_3}), \ldots, (x_{k_{p-1}}, x_{k_p})\}$, and with $x_{k_1}, x_{k_2}, \ldots, x_{k_p}$ all distinct. A *circuit* of a graph G is a subgraph $(\tilde{V}, \tilde{E})$ of G with $\tilde{V} = \{x_{k_1}, x_{k_2}, \ldots, x_{k_p}\}$, $\tilde{E} = \{(x_{k_1}, x_{k_2}), (x_{k_2}, x_{k_3}), \ldots, (x_{k_p}, x_{k_1})\}$, and with $x_{k_1}, x_{k_2}, \ldots, x_{k_p}$ again distinct[9].

A graph is called *connected* if for all pairs of vertices x_{k_1} and x_{k_2} in the graph there is a path from x_{k_1} to x_{k_2}, and *disconnected* otherwise. The graph is called *biconnected* if it is connected and remains so upon the removal of any one vertex from the graph. A connected (resp. biconnected) component of a graph G is a subgraph $\tilde{G}$ of G which is itself connected (resp. biconnected), and which is not a subgraph of any other connected (resp. biconnected) subgraph of G. It may be verified that any graph has a unique decomposition into connected (resp. biconnected) components.

Consider a partition of the vertices in a graph $G = (V, E)$ into two subsets $V^{(1)}$ and $V^{(2)}$. By a *cut* in a connected graph G we mean a subset $\tilde{E}$ of the edges of G containing each edge which is incident with one vertex in $V^{(1)}$ and one vertex in $V^{(2)}$ [30, p. 36]. It is clear that removal of the edges in a cut results in G becoming disconnected. If $\tilde{E}$ does not contain any subsets which are also cuts in G, then we call $\tilde{E}$ a *cut-set* of G.

A spanning tree (hereafter *tree*) in a connected graph G is a connected subgraph of G containing all of the vertices in G and which contains no circuit. It is straightforward to show that if G is a connected graph containing exactly n vertices, then each tree in G contains exactly $n - 1$ edges. A *chord-set* $\tilde{E}$ in a connected graph G is a subset of the edges in G for which those edges in G but not in $\tilde{E}$ are edges comprising a tree of G. It follows that if G is a connected graph comprising n vertices and q edges, then any chord-set in G must contain precisely $q + 1 - n$ edges.

The edges in a graph may each be assigned an orientation from one incident vertex to the other. We call such a graph an *oriented graph.* If the edge y_k is oriented from vertex x_{k_1} to vertex x_{k_2}, then we call x_{k_1} the *tail* vertex, and x_{k_2} the *head* vertex, of y_k.

Consider an oriented graph $G = (\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_q\})$ which contains a circuit $(\tilde{V}, \tilde{E})$ with $\tilde{V} = \{x_{k_1}, x_{k_2}, \ldots, x_{k_{p-1}}, x_{k_p}\}$, and with $\tilde{E}$ containing an edge oriented from x_{k_p} to x_{k_1}, and edges incident with vertices x_{k_i} and $x_{k_{i+1}}$ for $i = 1, \ldots, p - 1$. We

[9] We remark that these are referred to as cycles in [30].

say the edge oriented from x_{k_p} to x_{k_1}, and all edges oriented from x_{k_i} to $x_{k_{i+1}}$, are in the orientation of the circuit, while those oriented from $x_{k_{i+1}}$ to x_{k_i} are in the opposite orientation to the circuit ($i = 1, 2, \ldots, p-1$). A *circuit vector* for G is a vector $\mathbf{c}$ with dimension q (the number of edges in G), and whose entries $\mathbf{c}_i$ satisfy:

1. $\mathbf{c}_i = 1$ if $y_i \in \tilde{E}$ and is in the orientation of the circuit,
2. $\mathbf{c}_i = -1$ if $y_i \in \tilde{E}$ and is in the opposite orientation to the circuit,
3. $\mathbf{c}_i = 0$ otherwise.

Now, consider an oriented graph $G = (\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_q\})$ which contains a cut-set $(\tilde{V}, \tilde{E})$ with associated vertex sets $V^{(1)}$ and $V^{(2)}$. A *cut-set vector* for G is a vector $\mathbf{a}$ with dimension q (the number of edges in G), and whose entries $\mathbf{a}_i$ satisfy:

1. $\mathbf{a}_i = 1$ if $y_i \in \tilde{E}$ and is oriented from a vertex in $V^{(1)}$ to a vertex in $V^{(2)}$,
2. $\mathbf{a}_i = -1$ if $y_i \in \tilde{E}$ and is oriented from a vertex in $V^{(2)}$ to a vertex in $V^{(1)}$,
3. $\mathbf{a}_i = 0$ otherwise.

Any matrix in which each row corresponds to a circuit (resp. cut-set) vector we shall call a circuit (resp. cut-set) matrix.

The cut-set and circuit vectors of an oriented graph induce certain linear vector spaces over $\mathbb{F}^q$, where $\mathbb{F}$ is a field and q is the number of edges in the graph. The space spanned by all the cut-set (resp. circuit) vectors is called the *cut-set space* (resp. *circuit space*) of the graph. It is straightforward to show that any cut-set (likewise, circuit) in an oriented graph is contained wholly within a single biconnected component of the graph, hence the cut-set (resp. circuit) space is the direct sum of the cut-set (resp. circuit) spaces for each of the biconnected components.

Consider now an oriented graph G containing n vertices, q edges, and r connected components. The cut-set space has dimension $n-r$, and is the orthogonal complement in $\mathbb{F}^q$ of the circuit space of G [30, Theorem 5][10]. For the purpose of our subsequent analysis of passive networks, it is instructive to introduce pertinent basis vectors for these two spaces for the case when G is connected (i.e. $r = 1$).

Consider first the cut-set space. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a tree in G, and without loss of generality let us order the edges in G so edges in this tree are indexed from $q-n+2$ to q, i.e. the tree contains the edges $\{y_{q-n+2}, y_{q-n+3}, \ldots, y_q\}$. The removal of any single edge

[10] We note that the cut space defined in [30] coincides with the cut-set space defined here, as is evident from the proof of [30, Theorem 5].

from the tree separates it into exactly two connected components. We let the vertex sets $V^{(1),k}$ and $V^{(2),k}$ correspond to the vertices in these two connected components, with $V^{(1),k}$ containing the tail vertex for edge $y_{q-n+1+k}$ ($k = 1, 2, \dots n-1$). The cut-set in G corresponding to the vertex sets $V^{(1),k}$ and $V^{(2),k}$ thus contains only the edge $y_{q-n+1+k}$ from the tree, together with edges from the complementary chord-set. Let $\mathbf{a}^{(k)}$ be the associated cut-set vector, and let A be the $(n-1) \times q$ matrix whose kth row is equal to the transpose of the vector $\mathbf{a}^{(k)}$. Then A takes the form

$$A = \begin{bmatrix} \hat{A} & I_{n-1} \end{bmatrix},$$

which has rank $n-1$. We call such a matrix a *fundamental cut-set matrix* corresponding to the tree $\tilde{G}$.

Consider now the circuit space. Let $\bar{E} = \{y_1, y_2, \dots, y_{q-n+1}\}$ be the complementary chord-set to the tree $\tilde{G}$ in the previous paragraph. Any tree in G contains a (unique) path between any two vertices in G. Hence, there is a single circuit comprising only edge y_k from the chord-set $\bar{E}$, together with edges from the tree $\tilde{G}$, and for which y_k is in the orientation of the circuit ($k = 1, 2, \dots, q-n+1$). Let $\mathbf{c}^{(k)}$ be the associated circuit vector, and let C be the $(q-n+1) \times q$ matrix whose kth row is equal to the transpose of the vector $\mathbf{c}^{(k)}$. It follows that C takes the form

$$C = \begin{bmatrix} I_{q-n+1} & \hat{C} \end{bmatrix},$$

which has rank $q-n+1$. We call such a matrix a *fundamental circuit matrix* corresponding to the tree $\tilde{G}$. Since $AC^T = 0$ by the orthogonality of the cut-set and circuit vectors, it follows that $\begin{bmatrix} -\hat{C}^T & I_{n-1} \end{bmatrix}$ is the corresponding fundamental cut-set matrix.

1.2.2 Kirchhoff's laws, Tellegen's theorem, and graphs of elements

Consider the transformerless network N described at the end of Section 1.1. Such a network has a natural association with an oriented graph G formed by replacing each element N_k by an edge which is oriented such that the voltage across N_k is considered equal to the voltage at the tail vertex less the voltage at the head vertex. An extra edge is then added between the terminal vertices of the network, and is oriented such that the voltage across the source ($\hat{v}$ in Fig. 4) is considered equal to the voltage at the tail vertex less the voltage at the head vertex. Now, let $\hat{\mathbf{i}} := \begin{bmatrix} -\hat{i} & \hat{i}_1 & \hat{i}_2 & \dots & \hat{i}_m \end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^{m+1}$, and $\hat{\mathbf{v}} := \begin{bmatrix} \hat{v} & \hat{v}_1 & \hat{v}_2 & \dots & \hat{v}_m \end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^{m+1}$, and let $\mathbf{a}$ be a cut-set vector, and $\mathbf{c}$ a circuit vector, for G. Then Kirchhoff's current law implies $\langle \mathbf{a}, \hat{\mathbf{i}}(t_1) \rangle = 0$ for any $t_1 \in \mathbb{R}$

(since otherwise there must be either an inflow or outflow of current at one of the vertices in the vertex set $V^{(2)}$ associated with vector $\mathbf{a}$), and Kirchhoff's voltage law implies $\langle \mathbf{c}, \hat{\mathbf{v}}(t_2)\rangle = 0$ for any $t_2 \in \mathbb{R}$. Hence, $\hat{\mathbf{i}}(t_1)$ is in the circuit space of G for any $t_1 \in \mathbb{R}$, or equivalently $A\hat{\mathbf{i}} = 0$ for any fundamental cut-set matrix A of G. Furthermore, $\hat{\mathbf{v}}(t_2)$ is in the cut-set space of G for any $t_2 \in \mathbb{R}$, or equivalently $C\hat{\mathbf{v}} = 0$ for any fundamental circuit matrix C of G. Then, for any vector $\hat{\mathbf{i}}(t_1)$ satisfying Kirchhoff's current law and $\hat{\mathbf{v}}(t_2)$ satisfying Kirchhoff's voltage law,

$$\langle \hat{\mathbf{i}}(t_1), \hat{\mathbf{v}}(t_2)\rangle = 0. \tag{6}$$

This property (known as Tellegen's theorem) holds irrespective of the driving-point behaviours of the elements in the network, and for all time instances $t_1, t_2 \in \mathbb{R}$.

Motivated by the above discussion, we make the following definition of a network graph:

Definition 1.2.1 (Network graph)**.**
A *network graph* G is an oriented graph with the following additional properties:

1. The first edge (y_1) is referred to as the *source*, and is associated with two real-valued functions of a real variable, denoted $\hat{i}$ and $\hat{v}$. The vertices incident with this edge are called the *driving-point terminals* of G.

2. Each additional edge (y_{k+1}, $k = 1, 2, \ldots, m$) is associated with two real-valued functions of a real variable, denoted $\hat{i}_k$ and $\hat{v}_k$, and a linear time-invariant differential operator relating these variables of the form $p_k\left(\frac{d}{dt}\right)\hat{i}_k = q_k\left(\frac{d}{dt}\right)\hat{v}_k$, with $p_k, q_k \in \mathbb{R}[s]$.

3. The vector valued function $\hat{\mathbf{i}} := \begin{bmatrix} -\hat{i} & \hat{i}_1 & \hat{i}_2 & \ldots & \hat{i}_m \end{bmatrix} : \mathbb{R} \mapsto \mathbb{R}^{m+1}$ is such that $\hat{i}(t_1)$ is in the circuit space of G for all $t_1 \in \mathbb{R}$.

4. The vector valued function $\hat{\mathbf{v}} := \begin{bmatrix} \hat{v} & \hat{v}_1 & \hat{v}_2 & \ldots & \hat{v}_m \end{bmatrix} : \mathbb{R} \mapsto \mathbb{R}^{m+1}$ is such that $\hat{v}(t_2)$ is in the cut-set space of G for all $t_2 \in \mathbb{R}$.

We refer to the real-valued functions $\hat{i}$ and $\hat{v}$ as the current and voltage in the source respectively, and to the real-valued functions $\hat{i}_k$ and $\hat{v}_k$ as the current and the voltage in the edge y_{k+1} respectively ($k = 1, 2, \ldots, m$). Moreover, we refer to conditions 3 and 4 and Kirchhoff's current law and Kirchhoff's voltage law respectively.

As explained in the discussion at the beginning of this subsection, any transformerless network has an associated *graph of elements*, defined as follows:

Definition 1.2.2 (Graph of elements)**.**
A *graph of elements* is a network graph in which the kernel of the differential operator $\left[-p_k\left(\frac{d}{dt}\right) \quad q_k\left(\frac{d}{dt}\right)\right]$ is the driving-point behaviour of an inductor, a capacitor, or a resistor ($k = 1, 2, \ldots, m$). We will use the words element and edge interchangeably when referring to the edges in a graph of elements.

Now, let N be a transformerless network and let G be the graph of elements corresponding to N. It is then clear that any vector $\hat{\mathbf{b}}$ of the form of equation (4), whose entries satisfy the constraints imposed by G, is a trajectory of N. In the remainder of this thesis, we will refer to a trajectory (resp. driving-point trajectory, behaviour, driving-point behaviour) of a graph of elements by analogy with the trajectory (resp. driving-point trajectory, behaviour, driving-point behaviour) of a transformerless network.

We now use the relationships between the currents and voltages in a transformerless network and the circuit and cut-set spaces of the corresponding graph of elements to show the following lemma:

Lemma 1.2.3. *The driving-point behaviour of a transformerless network is unchanged by the removal of those elements which are not in any paths between the driving-point terminals of the network.*

Proof. Let G be the graph of elements corresponding to the network. Any elements which are not in any paths between the driving-point terminals of the network must correspond to edges in G which are in a different biconnected component to that which contains the source. The conclusion follows since, as described in Subsection 1.2.1, the cut-set (resp. circuit) space of a graph is the direct sum of the cut-set (resp. circuit) spaces of each of the biconnected components in the graph. □

The correspondence between a transformerless network and a graph of elements allows us to identify certain pathological networks which either contain redundant elements or which have either impedance or admittance which are identically equal to zero. We make several observations along these lines in the following remarks:

Remark 1.2.4.
It follows from Lemma 1.2.3 that no generality is lost by assuming the graph of elements corresponding to any transformerless network is biconnected. Accordingly, we will assume this to be the case throughout this thesis except in those cases where we specify otherwise. Any transformerless network whose graph of elements is not biconnected

will be familiar to the electrical engineer as a network with 'stray wires'.

Remark 1.2.5.
From Definition 1.2.2 and the subsequent discussion, the driving-point behaviour of a transformerless network depends on the network topology only through the cut-set and circuit spaces of the graph of elements corresponding to that network. Any two graphs whose cut-set and circuit spaces coincide are known as *2-isomorphic* (see, e.g. [32]). Accordingly, we consider any two transformerless networks comprised of the same elements and whose corresponding graphs of element are 2-isomorphic to be equivalent. This equivalence is typically immediately apparent to the electrical engineer. An example is provided in Fig. 6.

Remark 1.2.6.
If the two terminals of a transformerless network N are connected together, then the source in the corresponding graph of elements G forms a loop (i.e. both ends are incident with the same vertex). Hence, there is a circuit in G containing only the source, and accordingly $\hat{v} = 0$ for any driving-point trajectory of G, which implies that the impedance of N is identically zero. It follows that if the impedance of N is not identically zero then any circuit which contains the source must contain at least one other element. Similarly, if there are no paths between the driving-point terminals of N, then G contains a cut-set comprising the source alone, and accordingly $\hat{i} = 0$ for any driving-point trajectory of G, which implies that the admittance of N is identically zero. It thus follows that if the admittance of N is not identically zero then there is at least one tree in G which does not contain the source. Moreover, since G is biconnected, then either G contains only the source (in which case either $\hat{v} = 0$ or $\hat{i} = 0$ for any driving-point trajectory of G), or the number of edges in G must be no less than the number of vertices. Accordingly, if neither the impedance nor the admittance of N are identically zero, then G must contain at least two vertices and the number of edges in G must be no less than the number of vertices.

1.3 A kernel description of the behaviour of a transformerless network

In this section, we present a kernel description for the behaviour of a general transformerless network. We consider the transformerless network N described at the end of Section 1.1. From Fig. 2, the element constraint imposed by element N_k takes the

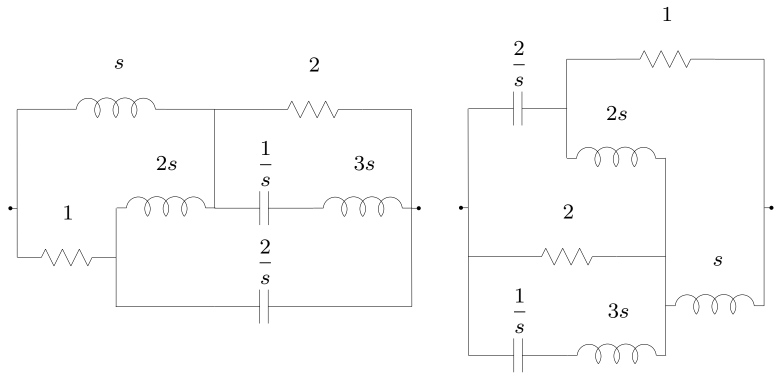


Figure 6: An example of two networks whose graphs of elements are 2-isomorphic

form

$$p_k\left(\frac{d}{dt}\right)\hat{i}_k = q_k\left(\frac{d}{dt}\right)\hat{v}_k, \tag{7}$$

where $p_k, q_k \in \mathbb{R}[s]$ and neither p_k nor q_k are identically zero ($k = 1, 2, \ldots, m$). We denote the admittance and impedance of element N_k by $Y_k(s) := q_k(s)/p_k(s)$ and $Z_k(s) := p_k(s)/q_k(s)$ respectively ($k = 1, 2, \ldots, m$).

Let us assume that neither the impedance nor the admittance of N is identically equal to zero, and let us now consider the graph of elements G corresponding to N (see Subsection 1.2.2). As discussed in Remark 1.2.4, no generality is lost in assuming that G is biconnected, so we make this assumption throughout. From Remark 1.2.6 in Subsection 1.2.2, there exists a tree in G which does not contain the source. By considering the fundamental cut-set and circuit matrices corresponding to this tree (see Subsection 1.2.1), it then follows that Kirchhoff's voltage law takes the form

$$\begin{bmatrix} 1 & \mathbf{0}_{m+1-n}^T & \hat{C}_1 \\ \mathbf{0}_{m+1-n} & I_{m+1-n} & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix} = 0, \tag{8}$$

and Kirchhoff's current law takes the form

$$\begin{bmatrix} -\hat{C}_1^T & -\hat{C}_2^T & I_{n-1} \end{bmatrix} \begin{bmatrix} -\hat{i} \\ \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \end{bmatrix} = 0. \tag{9}$$

Here, we have used the assumption that neither the impedance nor admittance of N is identically zero and the discussion in Remark 1.2.6 in Subsection 1.2.2 to conclude that $m + 1 - n \geq 0$ (since the number of edges in the graph of elements must be no less than the number of vertices), $\hat{C}_1^T \neq \mathbf{0}_{n-1}$ (since the source does not comprise a circuit on its own), and $n-1>0$ (since the graph of elements must contain at least two vertices). Furthermore, in the case $m+1-n=0$, the entries $\hat{C}_2$, $\mathbf{0}_{m+1-n}$, I_{m+1-n}, $\hat{\mathbf{v}}_1$, and $\hat{\mathbf{i}}_1$ are omitted from equations (8) and (9).

It follows that a kernel description of the behaviour of N is given by

$$\begin{bmatrix} 0 & 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{0}_{2m} & A\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{v} \\ \hat{\mathbf{r}} \end{bmatrix} = 0, \tag{10}$$

where, in the case $m+1-n>0$,

$$\mathbf{x} := \begin{bmatrix} \mathbf{0}_{m+1-n}^T & \mathbf{0}_{n-1}^T & \mathbf{0}_{m+1-n}^T & \hat{C}_1 \end{bmatrix}^T, \tag{11}$$

$$A(s) := \begin{bmatrix} P_1(s) & 0_{(m+1-n)\times(n-1)} & -Q_1(s) & 0_{(m+1-n)\times(n-1)} \\ 0_{(n-1)\times(m+1-n)} & P_2(s) & 0_{(n-1)\times(m+1-n)} & -Q_2(s) \\ 0_{(m+1-n)\times(m+1-n)} & 0_{(m+1-n)\times(n-1)} & I_{m+1-n} & \hat{C}_2 \\ -\hat{C}_2^T & I_{n-1} & 0_{(n-1)\times(m+1-n)} & 0_{(n-1)\times(n-1)} \end{bmatrix}, \tag{12}$$

$$P_1(s) := \operatorname{diag}\left(p_1(s), p_2(s), \ldots, p_{m+1-n}(s)\right), \tag{13}$$

$$Q_1(s) := \operatorname{diag}\left(q_1(s), q_2(s), \ldots, q_{m+1-n}(s)\right), \tag{14}$$

$$P_2(s) := \operatorname{diag}\left(p_{m-n+2}(s), p_{m-n+3}(s), \ldots, p_m(s)\right), \tag{15}$$

$$Q_2(s) := \operatorname{diag}\left(q_{m-n+2}(s), q_{m-n+3}(s), \ldots, q_m(s)\right), \tag{16}$$

$$\text{and} \quad \hat{\mathbf{r}} := \begin{bmatrix} \hat{\mathbf{i}}_1^T & \hat{\mathbf{i}}_2^T & \hat{\mathbf{v}}_1^T & \hat{\mathbf{v}}_2^T \end{bmatrix}^T. \tag{17}$$

For the case $m+1-n=0$, we have

$$\mathbf{x} := \begin{bmatrix} \mathbf{0}_{n-1}^T & \hat{C}_1 \end{bmatrix},$$

$$A(s) := \begin{bmatrix} P_2(s) & -Q_2(s) \\ I_{n-1} & 0_{(n-1)\times(n-1)} \end{bmatrix},$$

$$\text{and} \quad \hat{\mathbf{r}} := \begin{bmatrix} \hat{\mathbf{i}}_2^T & \hat{\mathbf{v}}_2^T \end{bmatrix}^T.$$

1.4 Properties of transformerless network driving-point trajectories

In this section, we derive properties of the driving-point trajectories of a general transformerless network. We show that it is possible for a driving-point trajectory of a transformerless network to contain an 'autonomous' component. The evolution of this autonomous component is completely determined by its value at a finite number of instants in time. This component is not made explicit by an impedance description, nor by an admittance description, of the network's driving-point behaviour, but it does feature in a kernel description. As a consequence, it is typically overlooked when network analysis is carried out in the Laplace-domain. In this section, we prove the important result that this component decays exponentially to zero as $t \to \infty$. In subsequent sections, we show how the presence of this component poses complications when analysing the impedance of a network using phasor techniques.

Our approach echoes that of [26], where the behaviour of linear time-invariant dynamical systems is studied using techniques from polynomial algebra. Here, we use results from [26] to make new statements specific to transformerless networks. We have chosen to follow the approach advocated in [26], in preference to other formalisms (e.g. [33]), due to the emphasis placed on the study of trajectories as time-domain functions rather than Laplace-domain functions. This will prove important when we consider a phasor approach to network analysis in Section 1.7. Network behaviours may alternatively be described using a state-space formalism (e.g. [28]). However, the process of constructing a state-space description of a network's behaviour is complex (see [28, Chapter 4]). Consequently, a state-space approach does not allow for a straightforward proof of the results shown here.

From the kernel description of the behaviour of a general transformerless network described in the preceding section, we will now derive properties of a kernel description of the *driving-point behaviour* of a general transformerless network. We summarise these properties in the following theorem[11]:

Theorem 1.4.1. *The driving-point behaviour of a transformerless network is the kernel of a differential operator*

$$g\left(\frac{d}{dt}\right)\begin{bmatrix}-p\left(\frac{d}{dt}\right) & q\left(\frac{d}{dt}\right)\end{bmatrix}, \tag{18}$$

[11]The proof of Theorem 1.4.1 will be presented for the case $m+1-n>0$, but is also valid in the case $m+1-n=0$. This may be verified by making the appropriate substitutions for $\mathbf{x}$, $A(s)$ and $\hat{\mathbf{r}}$.

where $g(s)$ is a polynomial in s whose roots are all in $\mathbb{C}_-$, and $p(s), q(s)$ are coprime polynomials in s with $p(s)/q(s)$ PR, this being the impedance of the network. In particular, all of the roots of $p(s)$ and $q(s)$ are in $\bar{\mathbb{C}}_-$, with those on $j\mathbb{R}$ having multiplicity at most one.

The principal contribution of Theorem 1.4.1 is the establishment of the properties of the polynomial $g(s)$ in that theorem's statement. This polynomial corresponds to an 'autonomous' component of the network driving-point behaviour, which is not captured in an impedance description of the driving-point behaviour. Such autonomous components will be discussed in greater detail in Section 1.5. As an example of the significance of this polynomial, consider the two networks shown in Fig. 7. From Fig. 2, it is clear that the driving-point behaviour of N_a is the set of solutions $\begin{bmatrix} \hat{i}_{N_a} & \hat{v}_{N_a} \end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^2$ to $\hat{i}_{N_a} = \hat{v}_{N_a}$, and so the impedance of N_a is equal to 1. As will be shown at the end of this section, the driving-point behaviour of N_b is the kernel of the differential operator

$$\left(\frac{d}{dt} + 1\right) \begin{bmatrix} -1 & 1 \end{bmatrix}. \qquad (19)$$

Hence, the impedance of N_b is also equal to 1. In contrast to N_a, the driving-point behaviour of N_b is the set of solutions $\begin{bmatrix} \hat{i}_{N_b} & \hat{v}_{N_b} \end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^2$ to $\hat{i}_{N_b}(t) = \hat{v}_{N_b}(t) + \alpha e^{-t}$, for $\alpha \in \mathbb{R}$. The value of α is fixed for a given network trajectory, and accordingly we refer to the trajectories $\hat{v}_{N_b}(t) = 0$, $\hat{i}_{N_b}(t) = \alpha e^{-t}$ $(\alpha, t \in \mathbb{R})$ as autonomous. That $g(s)$ in Theorem 1.4.1 has all roots in $\mathbb{C}_-$ implies that such autonomous trajectories will always decay to zero as $t \to \infty$ for any transformerless network[12].

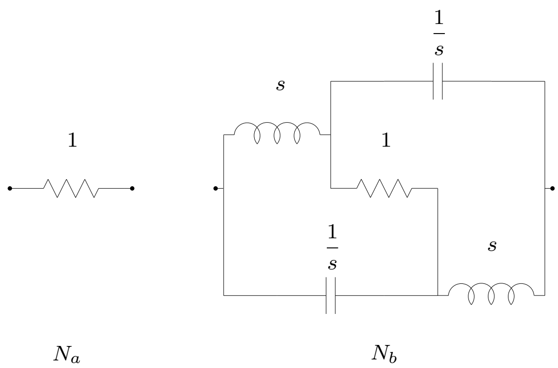


Figure 7: Two networks which have the same impedance but a different driving-point behaviour.

To show Theorem 1.4.1, it is necessary to demonstrate some properties of the matrix

[12] This will be shown formally in Section 1.5.

$A(s)$ in equation (12). These we summarise in the next lemma.

Lemma 1.4.2. *Consider the vector* $\mathbf{x}$ *in equation (11) and the matrix* $A(s)$ *in equation (12), for which*

- $P_j(s), Q_j(s)$ *are as defined in equations (13) to (16)* $(j = 1, 2)$*, where* $p_k, q_k \in \mathbb{R}[s]$ *and have no common roots in* $\bar{\mathbb{C}}_+$*, and* $p_k(s)/q_k(s)$ *is PR and not identically zero* $(k = 1, 2, \ldots, m)$.
- $\hat{C}_2 \in \mathbb{R}^{(m+1-n)\times(n-1)}$ *and* $\hat{C}_1^T \in \mathbb{R}^{n-1}$.

Then the following conditions must all hold:

1. $|A(s)|$ *is not identically zero, poles of* $A(s)^{-1}$ *are all in* $\bar{\mathbb{C}}_- \cup \infty$*, and poles on* $j\mathbb{R} \cup \infty$ *are simple.*
2. *For all* $\omega \in \mathbb{R} \cup \infty$*, if* $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ *does not have a pole at* $s = j\omega$ *then neither* $A(s)^{-1}\mathbf{x}$ *nor* $\mathbf{x}^T A(s)^{-1}$ *have poles at* $s = j\omega$.

Proof. We first show that $A(s)$ is invertible by providing an explicit form for $A(s)^{-1}$. First, note that $P_j(s)$ and $Q_j(s)$ are both diagonal and invertible in the field of real-rational matrices $(j = 1, 2)$. Next, define

$$\mathcal{Y}_j(s) := P_j(s)^{-1}Q_j(s), \tag{20}$$

$$\text{and} \quad \mathcal{Z}_j(s) := Q_j(s)^{-1}P_j(s), \tag{21}$$

for $j = 1, 2$. Then $\mathcal{Y}_j(s)$ and $\mathcal{Z}_j(s)$ are both diagonal matrices with PR entries on the diagonal (these correspond to element admittances and impedances respectively) $(j = 1, 2)$. Now, let

$$\mathcal{M}(s) := \mathcal{Y}_2(s) + \hat{C}_2^T \mathcal{Y}_1(s)\hat{C}_2, \tag{22}$$

$$\text{and} \quad \mathcal{N}(s) := \mathcal{Z}_1(s) + \hat{C}_2\mathcal{Z}_2(s)\hat{C}_2^T. \tag{23}$$

We now show that both $\mathcal{M}(s)$ and $\mathcal{N}(s)$ are invertible, and are PR in accordance with Definition 1.1.1. That $\mathcal{M}(s)$ and $\mathcal{N}(s)$ satisfy properties 1 to 3 in Definition 1.1.1 follows since entries in $\mathcal{Y}_j(s)$ and $\mathcal{Z}_j(s)$ are all PR $(j = 1, 2)$. To see that $\mathcal{M}(s)$ also satisfies condition 4 in Definition 1.1.1, let $\xi \in \mathbb{C}_+$, and note that

$$(\mathcal{M}(\xi) + \mathcal{M}(\xi)^*) = (\mathcal{Y}_2(\xi) + \mathcal{Y}_2(\xi)^*) + \hat{C}_2^T(\mathcal{Y}_1(\xi) + \mathcal{Y}_1(\xi)^*)\hat{C}_2.$$

It follows from [34, Lemma 5.1] that $(\mathcal{M}(\xi) + \mathcal{M}(\xi)^*) > 0$ for all $\xi \in \mathbb{C}_+$, and we conclude that $\mathcal{M}(s)$ is PR, and also that $|\mathcal{M}(s)|$ is not identically zero. A similar argument shows that $\mathcal{N}(s)$ is PR with $|\mathcal{N}(s)|$ not identically zero. That $\mathcal{M}(s)^{-1}$ and $\mathcal{N}(s)^{-1}$ are also PR follows since any PR matrix whose determinant is not identically zero has an inverse which is PR [34, Theorem 5.8].

Since $\mathcal{M}(s)$ and $\mathcal{N}(s)$ are both invertible then it follows by direct calculation that $A(s)^{-1}$ exists and takes the form:

$$\begin{bmatrix} \mathcal{N}(s)^{-1}Q_1(s)^{-1} & \mathcal{N}(s)^{-1}\hat{C}_2Q_2(s)^{-1} & \mathcal{N}(s)^{-1} & -\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1} \\ \hat{C}_2^T\mathcal{N}(s)^{-1}Q_1(s)^{-1} & \hat{C}_2^T\mathcal{N}(s)^{-1}\hat{C}_2Q_2(s)^{-1} & \hat{C}_2^T\mathcal{N}(s)^{-1} & \mathcal{Y}_2(s)\mathcal{M}(s)^{-1} \\ -\hat{C}_2\mathcal{M}(s)^{-1}\hat{C}_2^TP_1(s)^{-1} & \hat{C}_2\mathcal{M}(s)^{-1}P_2(s)^{-1} & \mathcal{Z}_1(s)\mathcal{N}(s)^{-1} & -\hat{C}_2\mathcal{M}(s)^{-1} \\ \mathcal{M}(s)^{-1}\hat{C}_2^TP_1(s)^{-1} & -\mathcal{M}(s)^{-1}P_2(s)^{-1} & \mathcal{M}(s)^{-1}\hat{C}_2^T\mathcal{Y}_1(s) & \mathcal{M}(s)^{-1} \end{bmatrix}.$$

Here, we have used the relationships $\mathcal{Y}_i(s) = \mathcal{Z}_i(s)^{-1}$ $(i = 1, 2)$, $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1} = \mathcal{N}(s)^{-1}\hat{C}_2\mathcal{Z}_2(s)$, and $\hat{C}_2\mathcal{M}(s)^{-1}\mathcal{Y}_2(s) = \mathcal{Z}_1(s)\mathcal{N}(s)^{-1}\hat{C}_2$, and the commutability of diagonal matrices. It follows that the poles of $A(s)^{-1}$ are the union of the poles of $\mathcal{M}(s)^{-1}$, $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}$, $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$, $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}$, $P_2(s)^{-1}\mathcal{M}(s)^{-1}$, $\mathcal{N}(s)^{-1}$, $\mathcal{Z}_1(s)\mathcal{N}(s)^{-1}$, $Q_1(s)^{-1}\mathcal{N}(s)^{-1}$, $\mathcal{Z}_2(s)\hat{C}_2^T\mathcal{N}(s)^{-1}$, and $Q_2(s)^{-1}\hat{C}_2^T\mathcal{N}(s)^{-1}$. Accordingly, to complete the proof of the present lemma, it suffices to consider the properties of the poles of these real-rational matrices, accounting for the fact that $\mathcal{Y}_j(s)$ and $\mathcal{Z}_j(s)$ are PR $(j = 1, 2)$, as are $\mathcal{M}(s)^{-1}$ and $\mathcal{N}(s)^{-1}$.

That poles of $A(s)^{-1}$ are in $\bar{\mathbb{C}}_- \cup \infty$ follows since $\mathcal{Y}_1(s)$, $\mathcal{Y}_2(s)$, $\mathcal{N}(s)^{-1}$, and $\mathcal{M}(s)^{-1}$ are PR, and $\mathcal{Y}_j(s) = P_j(s)^{-1}Q_j(s)$ where $P_j(s)$, $Q_j(s)$ are diagonal polynomial matrices for which the corresponding diagonal entries have no common roots in $\bar{\mathbb{C}}_+$ $(j = 1, 2)$. To show condition 1 of the present lemma, it remains to show that poles of $A(s)^{-1}$ on $j\mathbb{R} \cup \infty$ are simple. Here, we show that poles of $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}$, $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$, $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}$, and $P_2(s)^{-1}\mathcal{M}(s)^{-1}$ on $j\mathbb{R} \cup \infty$ are simple. That poles of $\mathcal{Z}_1(s)\mathcal{N}(s)^{-1}$, $Q_1(s)^{-1}\mathcal{N}(s)^{-1}$, $\mathcal{Z}_2(s)\hat{C}_2^T\mathcal{N}(s)^{-1}$, and $Q_2(s)^{-1}\hat{C}_2^T\mathcal{N}(s)^{-1}$ on $j\mathbb{R} \cup \infty$ are simple may be shown similarly, and completes the proof of condition 1.

Let $Y_k(s)$ be a diagonal entry of $\mathcal{Y}_1(s)$, with $p_k(s)$ and $q_k(s)$ the corresponding entries of $P_1(s)$ and $Q_1(s)$, and let $\omega \in \mathbb{R}$. Since $Y_k(s)$ is PR then poles of $Y_k(s)$ on $j\mathbb{R} \cup \infty$ are simple with real positive residue [28, Theorem 2.7.2]. Moreover, since $Y_k(s) = q_k(s)/p_k(s)$ and $p_k(s)$ and $q_k(s)$ have no common roots on $j\mathbb{R}$, then poles of $1/p_k(s)$ on $j\mathbb{R}$ are simple, and $Y_k(s)$ has a pole at $s = j\omega$ if and only if $1/p_k(s)$ has a pole there. It follows that poles of $\mathcal{Y}_1(s)$ and $P_1(s)^{-1}$ on $j\mathbb{R}$ are simple. Furthermore, the Laurent series of $\mathcal{Y}_1(s)$ and $P_1(s)^{-1}$ at $j\omega$ take the form

$$\mathcal{Y}_1(s) = \frac{U_{-1}}{s-j\omega} + U_0 + U_1(s-j\omega) + \ldots,$$
$$\text{and} \qquad P_1(s)^{-1} = \frac{\hat{U}_{-1}}{s-j\omega} + \hat{U}_0 + \hat{U}_1(s-j\omega) + \ldots,$$

for some matrices $U_{-1}, U_0, U_1, \ldots, \hat{U}_{-1}, \hat{U}_0, \hat{U}_1 \ldots \in \mathbb{C}^{(m+1-n)\times(m+1-n)}$, with $\hat{U}_{-1} = TU_{-1}$ for some non-singular diagonal matrix $T \in \mathbb{C}^{(m+1-n)\times(m+1-n)}$, and with $U_{-1} \geq 0$ by Theorem 1.1.2.

Similarly, the Laurent series of $\mathcal{Y}_2(s)$ and $P_2(s)^{-1}$ about $s = j\omega$ take the form

$$\mathcal{Y}_2(s) = \frac{V_{-1}}{s-j\omega} + V_0 + V_1(s-j\omega) + \ldots,$$
$$\text{and} \qquad P_2(s)^{-1} = \frac{\hat{V}_{-1}}{s-j\omega} + \hat{V}_0 + \hat{V}_1(s-j\omega) + \ldots,$$

for some square complex matrices $V_{-1}, V_0, V_1, \ldots, \hat{V}_{-1}, \hat{V}_0, \hat{V}_1, \ldots \in \mathbb{C}^{(n-1)\times(n-1)}$, with $\hat{V}_{-1} = RV_{-1}$ for some non-singular diagonal matrix $R \in \mathbb{C}^{(n-1)\times(n-1)}$, and with $V_{-1} \geq 0$ by Theorem 1.1.2. To determine the multiplicity of the poles of $P_2(s)^{-1}\mathcal{M}(s)^{-1}$, $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$, $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}$, and $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}$ at $j\omega$, we will examine the Laurent series at $s = j\omega$ for these functions.

From the preceding equations, it follows that the Laurent series of $\mathcal{M}(s)$ about $s = j\omega$ is

$$\mathcal{M}(s) = \frac{V_{-1} + \hat{C}_2^T U_{-1}\hat{C}_2}{s-j\omega} + \left(V_0 + \hat{C}_2^T U_0 \hat{C}_2\right) + (V_1 + \hat{C}_2^T U_1 \hat{C}_2)(s-j\omega) + \ldots. \tag{24}$$

Since $\mathcal{M}(s)^{-1}$ is PR then the Laurent series for $\mathcal{M}(s)^{-1}$ about $s = j\omega$ takes the form

$$\mathcal{M}(s)^{-1} = \frac{W_{-1}}{s-j\omega} + W_0 + W_1(s-j\omega) + \ldots, \tag{25}$$

for some matrices $W_{-1}, W_0, W_1, \ldots \in \mathbb{C}^{(n-1)\times(n-1)}$, with $W_{-1} \geq 0$, by Theorem 1.1.2. The Laurent series of $P_2(s)^{-1}\mathcal{M}(s)^{-1}$ and $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$ about $s = j\omega$ are then given by

$$P_2(s)^{-1}\mathcal{M}(s)^{-1} = \frac{\hat{V}_{-1}W_{-1}}{(s-j\omega)^2} + \frac{\hat{V}_{-1}W_0 + \hat{V}_0 W_{-1}}{s-j\omega} + (\hat{V}_{-1}W_1 + \hat{V}_0 W_0 + \hat{V}_1 W_{-1}) + \ldots, \tag{26}$$

and

$$P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1} = \frac{\hat{U}_{-1}\hat{C}_2W_{-1}}{(s-j\omega)^2} + \frac{\hat{U}_{-1}\hat{C}_2W_0 + \hat{U}_0\hat{C}_2W_{-1}}{s-j\omega} + (\hat{U}_{-1}\hat{C}_2W_1 + \hat{U}_0\hat{C}_2W_0 + \hat{U}_1\hat{C}_2W_{-1}) + \ldots, \quad (27)$$

respectively.

Moreover, from equations (24) and (25), we have

$$I_{n-1} = \frac{(V_{-1} + \hat{C}_2^T U_{-1}\hat{C}_2)W_{-1}}{(s-j\omega)^2} + \frac{(V_{-1} + \hat{C}_2^T U_{-1}\hat{C}_2)W_0 + (V_0 + \hat{C}_2^T U_0\hat{C}_2)W_{-1}}{s-j\omega} + \Big((V_{-1} + \hat{C}_2^T U_{-1}\hat{C}_2)W_1 + (V_0 + \hat{C}_2^T U_0\hat{C}_2)W_0 + (V_1 + \hat{C}_2^T U_1\hat{C}_2)W_{-1}\Big) + \ldots. \quad (28)$$

Equating coefficients of $(s-j\omega)^{-2}$ in the above equation, we find $(V_{-1}+\hat{C}_2^T U_{-1}\hat{C}_2)W_{-1} = 0_{(n-1)\times(n-1)}$. Since $V_{-1} \geq 0$ and $U_{-1} \geq 0$, this implies $V_{-1}W_{-1} = 0_{(n-1)\times(n-1)}$ and $U_{-1}\hat{C}_2W_{-1} = 0_{(m+1-n)\times(n-1)}$, and hence $\hat{V}_{-1}W_{-1} = RV_{-1}W_{-1} = 0_{(n-1)\times(n-1)}$ and $\hat{U}_{-1}\hat{C}_2W_{-1} = TU_{-1}\hat{C}_2W_{-1} = 0_{(m+1-n)\times(n-1)}$. By considering the Laurent series (26) and (27), it follows that poles of $P_2(s)^{-1}\mathcal{M}(s)^{-1}$ and $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$ on $j\mathbb{R}$ are simple. That poles of $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}$ and $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}$ on $j\mathbb{R}$ are simple then follows since the diagonal matrices $P_j(s)^{-1}$ and $Q_j(s)$ commute ($j = 1, 2$), and so $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1} = Q_2(s)P_2(s)^{-1}\mathcal{M}(s)^{-1}$ and $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1} = Q_1(s)P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$ with $Q_1 \in \mathbb{R}^{(m+1-n)\times(m+1-n)}[s]$ and $Q_2 \in \mathbb{R}^{(n-1)\times(n-1)}[s]$.

A similar argument shows that poles of $\mathcal{Z}_1(s)\mathcal{N}(s)^{-1}$, $Q_1(s)^{-1}\mathcal{N}(s)^{-1}$, $\mathcal{Z}_2(s)\hat{C}_2^T\mathcal{N}(s)^{-1}$, and $Q_2(s)^{-1}\hat{C}_2^T\mathcal{N}(s)^{-1}$ on $j\mathbb{R}$ are simple, and so we conclude that poles of $A(s)^{-1}$ on $j\mathbb{R}$ are simple. That poles of $A(s)^{-1}$ at ∞ must be simple may be shown by considering the Laurent series about the point at infinity in the preceding argument. Here, we note that $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}$ can only have a pole at $s = \infty$ if $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}$ does, and $P_2(s)^{-1}\mathcal{M}(s)^{-1}$ can only have a pole at $s = \infty$ if $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}$ does. This completes the proof of condition 1 of the present lemma.

To show condition 2, consider a vector $\mathbf{z} \in \mathbb{C}^{n-1}$ and an $\omega \in \mathbb{R}$, and suppose $\mathbf{z}^*\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = j\omega$. We will show that each of $\mathcal{M}(s)^{-1}\mathbf{z}$, $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$, $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$, $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}\mathbf{z}$, and $P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$ then do not have a pole at $s = j\omega$. A similar argument then shows that if $\mathbf{z}^*\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = \infty$ then $\mathcal{M}(s)^{-1}\mathbf{z}$, $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$, $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$, $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}\mathbf{z}$, and $P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$ do not have a pole there. Condition 2 of the present lemma then follows since from equation (11) and from the explicit form for $A(s)^{-1}$ described at the beginning of the proof of this lemma, we have

$$\mathbf{x}^T A(s)^{-1}\mathbf{x} = \hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T, \tag{29}$$

$$A(s)^{-1}\mathbf{x} = \begin{bmatrix} -\hat{C}_2^T\mathcal{Y}_1(s) & \mathcal{Y}_2(s) & -\hat{C}_2^T & I_{n-1}\end{bmatrix}^T \mathcal{M}(s)^{-1}\hat{C}_1^T, \tag{30}$$

$$\text{and} \quad \mathbf{x}^T A(s)^{-1} = \hat{C}_1 \mathcal{M}(s)^{-1}\begin{bmatrix}\hat{C}_2^T P_1(s)^{-1} & -P_2(s)^{-1} & \hat{C}_2^T\mathcal{Y}_1(s) & I_{n-1}\end{bmatrix}. \tag{31}$$

From (25), the Laurent series for $\mathcal{M}(s)^{-1}\mathbf{z}$ and $\mathbf{z}^*\mathcal{M}(s)^{-1}\mathbf{z}$ about $j\omega$ take the form

$$\mathcal{M}(s)^{-1}\mathbf{z} = \frac{W_{-1}\mathbf{z}}{s-j\omega} + W_0\mathbf{z} + W_1\mathbf{z}(s-j\omega) + \ldots, \tag{32}$$

$$\text{and} \quad \mathbf{z}^*\mathcal{M}(s)^{-1}\mathbf{z} = \frac{\mathbf{z}^*W_{-1}\mathbf{z}}{s-j\omega} + \mathbf{z}W_0\mathbf{z} + \mathbf{z}^*W_1\mathbf{z}(s-j\omega) + \ldots, \tag{33}$$

respectively, with $W_{-1} \geq 0$. Since $\mathbf{z}^*\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = j\omega$ then $\mathbf{z}^*W_{-1}\mathbf{z} = 0$, which implies $W_{-1}\mathbf{z} = 0$ since $W_{-1} \geq 0$. It follows that $\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = j\omega$. From equations (26) and (27), we then find that the Laurent series of $P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$ and $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$ about $s = j\omega$ are equal to

$$P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z} = \frac{\hat{V}_{-1}W_0\mathbf{z}}{s-j\omega} + \left(\hat{V}_{-1}W_1 + \hat{V}_0W_0\right)\mathbf{z} + \ldots, \tag{34}$$

$$\text{and} \quad P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z} = \frac{\hat{U}_{-1}\hat{C}_2W_0\mathbf{z}}{s-j\omega} + \left(\hat{U}_{-1}\hat{C}_2W_1 + \hat{U}_0\hat{C}_2W_0\right)\mathbf{z} + \ldots, \tag{35}$$

respectively. Moreover, by post-multiplying both sides of equation (28) by $\mathbf{z}$, and then equating coefficients of $(s - j\omega)^{-1}$, we obtain

$$(V_{-1} + \hat{C}_2^T U_{-1}\hat{C}_2)W_0\mathbf{z} = \mathbf{0}_{n-1}.$$

Since $V_{-1} \geq 0$ and $U_{-1} \geq 0$, this implies $V_{-1}W_0\mathbf{z} = \mathbf{0}_{n-1}$ and $U_{-1}\hat{C}_2W_0\mathbf{z} = \mathbf{0}_{m+1-n}$, and hence $\hat{V}_{-1}W_0\mathbf{z} = RV_{-1}W_0\mathbf{z} = \mathbf{0}_{n-1}$ and $\hat{U}_{-1}\hat{C}_2W_0\mathbf{z} = TU_{-1}\hat{C}_2W_0\mathbf{z} = \mathbf{0}_{m+1-n}$. From the Laurent series (34) and (35), we conclude that neither $P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$ nor $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$ have a pole at $s = j\omega$.

For $\omega \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{C}^{n-1}$, we have shown that if $\mathbf{z}\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = j\omega$, then the matrices $\mathcal{M}(s)^{-1}\mathbf{z}$, $P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$, and $P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$ cannot have a pole there either. Since $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}\mathbf{z} = Q_2(s)P_2(s)^{-1}\mathcal{M}(s)^{-1}\mathbf{z}$ and $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z} = Q_1(s)P_1(s)^{-1}\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$, with $Q_1 \in \mathbb{R}^{(m+1-n)\times(m+1-n)}[s]$ and $Q_2 \in \mathbb{R}^{(n-1)\times(n-1)}[s]$, it follows that if $\mathbf{z}\mathcal{M}(s)^{-1}\mathbf{z}$ does not have a pole at $s = j\omega$, then neither $\mathcal{Y}_2(s)\mathcal{M}(s)^{-1}\mathbf{z}$ nor $\mathcal{Y}_1(s)\hat{C}_2\mathcal{M}(s)^{-1}\mathbf{z}$ can have a pole there. Hence, from equations (29)-(31), if $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ does not have a pole at $s = j\omega$ then neither $\mathbf{x}^T A(s)^{-1}$

nor $A(s)^{-1}\mathbf{x}$ have poles at $s = j\omega$. A similar argument which considers the Laurent series about the point at ∞ shows that if $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ does not have a pole at $s = \infty$ then neither $\mathbf{x}^T A(s)^{-1}$ nor $A(s)^{-1}\mathbf{x}$ have poles at $s = \infty$. This completes the proof of condition 2. □

Lemma 1.4.2 will now be used to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. From equation (10) and [26, Boxes 'Polynomial Modules and Syzygies' and 'The Fundamental Principle and the Elimination Theorem'], the driving-point behaviour of N is the kernel of a differential operator

$$W^T\left(\frac{d}{dt}\right)\begin{bmatrix} 0 & 1 \\ \mathbf{x} & \mathbf{0}_{2(m-1)} \end{bmatrix}, \tag{36}$$

where the rows of $W(s)^T$ form a basis for the left syzygy of $\begin{bmatrix} \mathbf{x} & A(s)^T \end{bmatrix}^T$. In other words, $W(\xi)^T$ has full row rank for all $\xi \in \mathbb{C}$, and

$$W(s)^T \begin{bmatrix} \mathbf{x}^T \\ A(s) \end{bmatrix} = 0. \tag{37}$$

Since $\begin{bmatrix} \mathbf{x} & A(s)^T \end{bmatrix}^T$ is a $(2m+1)\times 2m$ polynomial matrix, then $W \in \mathbb{R}^{2m+1}[s]$, and since $W(\xi)^T$ has full row rank for all $\xi \in \mathbb{C}$, then $W(\xi) \neq \mathbf{0}_{2m+1}$ for all $\xi \in \mathbb{C}$. Moreover, this vector is unique up to scaling by a constant.

Now, let $r(s)$ be the least common multiple of all denominator polynomials in the real-rational matrix $\mathbf{x}^T A(s)^{-1}$, which is unique up to scaling by a constant. Then $\begin{bmatrix} r(s) & -r(s)\mathbf{x}^T A(s)^{-1} \end{bmatrix}^T$ is a $2m+1$ polynomial vector and $\begin{bmatrix} r(\xi) & -r(\xi)\mathbf{x}^T A(\xi)^{-1} \end{bmatrix}^T \neq \mathbf{0}_{2m+1}$ for all $\xi \in \mathbb{C}$. This follows since $r(\xi) = 0$ implies at least one entry in $r(\xi)\mathbf{x}^T A(\xi)^{-1}$ is non-zero. Moreover,

$$\begin{bmatrix} r(s) & -r(s)\mathbf{x}^T A(s)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}^T \\ A(s) \end{bmatrix} = 0.$$

Hence, $\begin{bmatrix} r(s) & -r(s)\mathbf{x}^T A(s)^{-1} \end{bmatrix}^T$ forms a basis for the left syzygy of $\begin{bmatrix} \mathbf{x} & A(s)^T \end{bmatrix}^T$, and so is an appropriate choice for the matrix $W(s)$ in equation (36). It follows that the driving-point behaviour of N is the kernel of the differential operator $D(\frac{d}{dt})$, where

$$D(s) = \begin{bmatrix} r(s) & -r(s)\mathbf{x}^T A(s)^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \mathbf{x} & \mathbf{0}_{2(m-1)} \end{bmatrix} = \begin{bmatrix} -r(s)\hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T & r(s) \end{bmatrix},$$

since $\mathbf{x}^T A(s)^{-1}\mathbf{x} = \hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T$. Now, let $\hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T =: \gamma(s)/\beta(s)$ with $\gamma, \beta \in \mathbb{R}[s]$ and coprime. Since $\mathcal{M}(s)^{-1}$ is PR and $\hat{C}_1^T \in \mathbb{R}^{n-1}\backslash\mathbf{0}_{n-1}$, then $\hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T$ is PR[13], as is its inverse. Hence, all roots of $\gamma(s)$ and all roots of $\beta(s)$ are in $\bar{\mathbb{C}}_-$, and roots on $j\mathbb{R}$ have multiplicity at most one.

Since any pole of $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ of multiplicity k must be a pole of $\mathbf{x}^T A(s)^{-1}$ of multiplicity greater than or equal to k, then $\beta(s)$ divides $r(s)$, and hence $r(s) = \alpha(s)\beta(s)$ for some polynomial $\alpha(s)$. As shown in Lemma 1.4.2, the polynomial $r(s)$ has all roots in $\bar{\mathbb{C}}_-$, and roots of $r(s)$ on $j\mathbb{R}$ have multiplicity one. Moreover, $r(s)$ has a root at $j\omega$ for $\omega \in \mathbb{R}$ if and only if $\mathbf{x}^T A(s)^{-1}$ has a pole at $s = j\omega$, which occurs if and only if $\mathbf{x}^T A(s)^{-1}\mathbf{x} = \hat{C}_1 \mathcal{M}(s)^{-1}\hat{C}_1^T$ has a pole at $s = j\omega$. This implies that roots of $r(s)$ on $j\mathbb{R}$ are all contained in the factor $\beta(s)$. Theorem 1.4.1 then follows by letting $p(s) = \gamma(s)$, $q(s) = \beta(s)$ and $g(s) = \alpha(s)$. □

As a consequence of the proof of Theorem 1.4.1, it follows that the impedance $Z(s)$ of a transformerless network may be obtained from the equation

$$Z(s) = \hat{C}_1 \left(\mathcal{Y}_2(s) + \hat{C}_2^T \mathcal{Y}_1(s)\hat{C}_2\right)^{-1} \hat{C}_1^T, \tag{38}$$

where $\hat{C}_1$, $\hat{C}_2$, $\mathcal{Y}_1(s)$, and $\mathcal{Y}_2(s)$ have the definitions given in this section.

We now apply the elimination procedure described in the preceding proof to the network N_b in Fig. 7. By identifying the element N_5 with the resistor in this network, and the elements N_1 to N_4 with the reactive elements in this network, starting with the inductor on the top-left and proceeding clockwise, we find that the behaviour of this network is the kernel of the differential operator $R(\frac{d}{dt})$, where

$$R(s) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

[13] Indeed, $X^T H(s) X$ is PR whenever $H(s)$ is PR and X is a compatible real-valued matrix. This is straightforward to verify from Definition 1.1.1.

In this case, the vector

$$W(s) = \begin{bmatrix}(1+s) & -1 & -1 & -1 & -1 & (s-1) & -1 & -s & s & 1 & (1-s)\end{bmatrix}^T$$

satisfies $W(\xi) \neq \mathbf{0}_{11}$ for all $\xi \in \mathbb{C}$, and is orthogonal to the final ten columns of $R(s)$. By pre-multiplying $R(s)$ by $W(s)^T$, we find that the driving-point behaviour of N_b is the kernel of the differential operator (19).

In Section 1.6, we will derive properties of the trajectories of a network which hold for any transformerless network by describing those solutions $\begin{bmatrix}\hat{i} & \hat{v} & \hat{\mathbf{r}}\end{bmatrix}^T$ to equation (10). Prior to that, in Section 1.5, we describe the solutions $\hat{\mathbf{b}}$ of a general linear time-invariant dynamical system with kernel description $R\left(\frac{d}{dt}\right)\hat{\mathbf{b}} = \mathbf{0}$, for $R \in \mathbb{R}^{u \times v}[s]$.

1.5 Decomposition of a behaviour into controllable and autonomous parts

In this section, we derive a parametrisation of a behaviour from a kernel description of that behaviour. This will allow us to derive properties of the trajectories of a general transformerless network in Section 1.6. The results of the present section apply to any linear time-invariant dynamical system with kernel description

$$R\left(\frac{d}{dt}\right)\hat{\mathbf{b}} = \mathbf{0}, \tag{40}$$

for $R \in \mathbb{R}^{u \times v}[s]$ (with u and v integers). We will consider solutions $\hat{\mathbf{b}} \in \mathscr{C}_\infty^v$ to the above equation[14]. In this section, we will relate the properties of the solutions $\hat{\mathbf{b}}$ to equation (40) to the Smith form of the matrix $R(s)$.

Our analysis centres on the correspondence between linear time-invariant dynamical systems and polynomial algebra, discussed in [26, Box 'Polynomial Modules and Syzygies']. We first define the concept of a *unimodular* real-rational matrix.

Definition 1.5.1 (Unimodular real-rational matrix)**.**
$F \in \mathbb{R}^{u \times u}[s]$ is called unimodular if its determinant is a non-zero constant.

From this definition, it is evident that a matrix is unimodular if and only if it is invertible with an inverse which is a polynomial matrix.

[14] Again, we remark that the results remain valid for other function spaces, e.g. the space of locally integrable solutions, with some minor qualifications as outlined in [29, Section 2.3].

We next define the *invariant polynomials* of a polynomial matrix in Lemma 1.5.2, and then the associated concept of the *Smith form* in Lemma 1.5.3. As will be seen, the solutions to (40) may be conveniently expressed in terms of a factorisation of $R(s)$ involving the Smith form together with unimodular matrices.

Lemma 1.5.2 (Invariant Polynomials)**.** *Let $R \in \mathbb{R}^{u\times v}[s]$, and let r be the maximum rank of $R(s)$ in $s \in \mathbb{C}$. Let $D_j(R(s))$ be the greatest common divisor of all minors of order j in $R(s)$, and let $\lambda_1(s) = D_1(R(s))$, and $\lambda_j(s) = D_j(R(s))/D_{j-1}(R(s))$ $(j = 2, 3, \dots, r)$. Then $\lambda_j \in \mathbb{R}[s]$ $(j = 1, 2, \dots r)$, and $\lambda_j(s)$ divides $\lambda_{j+1}(s)$ $(j = 1, 2, \dots, r-1)$. We call $\lambda_j(s)$ the* invariant polynomials *of the matrix $R(s)$ $(j = 1, 2, \dots, r)$.*

Lemma 1.5.3 (Smith form)**.** *Let $R \in \mathbb{R}^{u\times v}[s]$, and let r be the maximum rank of $R(s)$ in $s \in \mathbb{C}$, and $\lambda_j(s)$ be the invariant polynomials of $R(s)$ $(j = 1, 2, \dots, r)$. Then there exist unimodular matrices $U \in \mathbb{R}^{u\times u}[s]$ and $V \in \mathbb{R}^{v\times v}[s]$ such that*

$$U(s)R(s)V(s) = \Lambda(s), \tag{41}$$

with

$$\Lambda(s) = \begin{bmatrix} \tilde{\Lambda}(s) & 0_{r\times(v-r)} \\ 0_{(u-r)\times r} & 0_{(u-r)\times(v-r)} \end{bmatrix},$$

and

$$\tilde{\Lambda}(s) = diag\left(\lambda_1(s), \lambda_2(s), \dots, \lambda_r(s)\right).$$

Moreover, let $R_1 \in \mathbb{R}^{u\times v}[s]$. Then there exist unimodular matrices $U_1 \in \mathbb{R}^{u\times u}[s]$ and $V_1 \in \mathbb{R}^{v\times v}[s]$ such that $R_1(s) = U_1(s)R(s)V_1(s)$ if and only if R and R_1 have the same invariant polynomials.

We call $\Lambda(s)$ the Smith form *of $R(s)$.*

For proof of the above lemmas, see [35, Theorem 3 and Corollaries 1 and 2, p. 141].

In the remainder of this section, we derive a parametrisation for the kernel of the differential operator $R\left(\frac{d}{dt}\right)$ using the decomposition (41). In order to describe this parametrisation, we will first introduce a partition of these matrices.

We partition V as follows:

$$V(s) =: \begin{bmatrix} V_1(s) & V_2(s) \end{bmatrix}, \tag{42}$$

with $V_1 \in \mathbb{R}^{v\times r}[s]$ and $V_2 \in \mathbb{R}^{v\times(v-r)}[s]$, and we let

$$V_1(s) =: \begin{bmatrix} \mathbf{y}_1(s) & \mathbf{y}_2(s) & \dots & \mathbf{y}_r(s) \end{bmatrix}, \tag{43}$$

where $\mathbf{y}_j \in \mathbb{R}^v[s]$ $(j = 1, 2, \dots, r)$. We partition $V(s)^{-1}$ similarly, i.e.

$$V(s)^{-1} =: \begin{bmatrix} \hat{V}_1(s) \\ \hat{V}_2(s) \end{bmatrix}, \tag{44}$$

with $\hat{V}_1 \in \mathbb{R}^{r\times v}[s]$ and $\hat{V}_2 \in \mathbb{R}^{(v-r)\times v}[s]$. Then

$$\begin{bmatrix} \hat{V}_1(s) \\ \hat{V}_2(s) \end{bmatrix} \begin{bmatrix} V_1(s) & V_2(s) \end{bmatrix} = \begin{bmatrix} \hat{V}_1(s)V_1(s) & \hat{V}_1(s)V_2(s) \\ \hat{V}_2(s)V_1(s) & \hat{V}_2(s)V_2(s) \end{bmatrix} = \begin{bmatrix} I_r & 0_{r\times(v-r)} \\ 0_{(v-r)\times r} & I_{v-r} \end{bmatrix}. \tag{45}$$

Furthermore, we write the polynomial λ_j in terms of its irreducible factors:

$$\lambda_j(s) =: (s - s_1)^{d_{j1}} (s - s_2)^{d_{j2}} \dots (s - s_m)^{d_{jm}}, \tag{46}$$

$j = 1, 2, \dots r$. Hence, $0 \le d_{jl} \le d_{(j+1)_l}$ for $j = 1, 2, \dots, r-1$ and $l = 1, 2, \dots, m$, by Lemma 1.5.2.

The main results of this section are summarised in the next theorem. Despite being developed independently, this theorem shares similarities with results from [29]. Indeed, we have named the variables in the decomposition (47) $\hat{\mathbf{b}}_{\text{cont}}$ and $\hat{\mathbf{b}}_{\text{aut}}$ for consistency with [29]. In the following theorem, the behaviour defined by the set of solutions $\hat{\mathbf{b}}_{\text{cont}}$ to equation (50) is *controllable* in accordance with [29, Definition 5.2.2]. Moreover, the behaviour defined by the set of solutions $\hat{\mathbf{b}}_{\text{aut}}$ to equation (51) is *autonomous* in accordance with [29, Definition 3.2.1].

Theorem 1.5.4. *Let $R \in \mathbb{R}^{u\times v}[s]$, and let $U(s)$, $V(s)$, and $\Lambda(s)$ be as defined in Lemma 1.5.3. Moreover, consider the partitions of $V(s)$ and $V(s)^{-1}$ described in equations (42), (43) and (44), and let (46) be the decomposition of $\lambda_j(s)$ into irreducible factors $(j = 1, 2, \dots, r)$. Then $\hat{\mathbf{b}} \in \mathscr{C}_\infty^v$ is a solution to equation (40) if and only if $\hat{\mathbf{b}}$ takes the form*

$$\hat{\mathbf{b}} = \hat{\mathbf{b}}_{cont} + \hat{\mathbf{b}}_{aut}, \tag{47}$$

where $\hat{\mathbf{b}}_{cont}$ takes the form

$$\hat{\mathbf{b}}_{cont} = V_2\left(\frac{d}{dt}\right)\hat{\mathbf{z}}, \tag{48}$$

for some $\hat{\mathbf{z}} \in \mathscr{C}_\infty^{v-r}$*, and* $\hat{\mathbf{b}}_{aut}$ *takes the form*

$$\hat{\mathbf{b}}_{aut}(t) = \Re\left(\sum_{j=1}^{r} \sum_{l\mid\{l=1,2,\ldots,m,d_{j_l}\neq 0,\Im(s_l)\geq 0\}} \sum_{n=0}^{d_{j_l}-1} \sum_{p=0}^{n} a_{jln}\binom{n}{p}\frac{d^{n-p}\mathbf{y}_j(s)}{ds^{n-p}}(s_l)t^p e^{s_l t}\right) \quad (49)$$

for some $a_{jln} \in \mathbb{C}$*.*

In particular, $\hat{\mathbf{b}}_{cont} \in \mathscr{C}_\infty^{v}$ *satisfies*

$$\hat{V}_1\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{cont} = \mathbf{0}_r, \quad (50)$$

and $\hat{\mathbf{b}}_{aut} \in \mathscr{C}_\infty^{v}$ *satisfies*

$$\begin{bmatrix} I_{v-r} & 0_{(v-r)\times r} \\ 0_{r\times(v-r)} & \tilde{\Lambda}\left(\frac{d}{dt}\right)\end{bmatrix}\begin{bmatrix}\hat{V}_2\left(\frac{d}{dt}\right) \\ \hat{V}_1\left(\frac{d}{dt}\right)\end{bmatrix}\hat{\mathbf{b}}_{aut} = \mathbf{0}_v. \quad (51)$$

Given the complexity of the form of equation (49), we give an example at the end of this section. Prior to proving Theorem 1.5.4, we state the following lemma concerning the effect of linear time-invariant differential operators on certain vector-valued functions.

Lemma 1.5.5. *Let* $Q \in \mathbb{R}^{u\times v}[s]$*,* $\mathbf{a}_j \in \mathbb{C}^v$ $(j = 1, 2, \ldots, k)$*, and* $s_0 \in \mathbb{C}$*. Then*

$$\begin{aligned} Q\left(\frac{d}{dt}\right)\Re\Big(\Big(\mathbf{a}_0 + \mathbf{a}_1 t + \ldots + \mathbf{a}_{k-1}t^{k-1} + \mathbf{a}_k t^k\Big)e^{s_0 t}\Big) \\ = \Re\Big(\mathbf{c}_0 + \mathbf{c}_1 t + \ldots + \mathbf{c}_{k-1}t^{k-1} + \mathbf{c}_k t^k\Big)e^{s_0 t}, \quad (52)\end{aligned}$$

where

$$\begin{bmatrix}\mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{k-1} \\ \mathbf{c}_k\end{bmatrix} := \begin{bmatrix} \binom{0}{0}Q(s_0) & \binom{1}{0}\frac{dQ(s)}{ds}(s_0) & \ldots & \binom{k-1}{0}\frac{d^{k-1}Q(s)}{ds^{k-1}}(s_0) & \binom{k}{0}\frac{d^{k}Q(s)}{ds^{k}}(s_0) \\ 0 & \binom{1}{1}Q(s_0) & \ldots & \binom{k-1}{1}\frac{d^{k-2}Q(s)}{ds^{k-2}}(s_0) & \binom{k}{1}\frac{d^{k-1}Q(s)}{ds^{k-1}}(s_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \binom{k-1}{k-1}Q(s_0) & \binom{k}{k-1}\frac{dQ(s)}{ds}(s_0) \\ 0 & 0 & \ldots & 0 & \binom{k}{k}Q(s_0)\end{bmatrix}\begin{bmatrix}\mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{k-1} \\ \mathbf{a}_k\end{bmatrix}.$$

The above lemma follows from [29, Lemma 3.2.6] and the linearity of linear time-invariant differential operators. We now proceed to prove Theorem 1.5.4.

Proof of Theorem 1.5.4. Since $\hat{\mathbf{b}}$ satisfies equation (40), then

$$\Lambda\left(\frac{d}{dt}\right)V\left(\frac{d}{dt}\right)^{-1}\hat{\mathbf{b}} = U\left(\frac{d}{dt}\right)\mathbf{0}_u = \mathbf{0}_u,$$

and hence,

$$\tilde{\Lambda}\left(\frac{d}{dt}\right)\hat{V}_1\left(\frac{d}{dt}\right)\hat{\mathbf{b}} = \mathbf{0}_r.$$

We first show that $\hat{\mathbf{b}} \in \mathscr{C}_\infty^v$ has a decomposition $\hat{\mathbf{b}} = V_2\left(\frac{d}{dt}\right)\hat{\mathbf{z}} + \hat{\mathbf{b}}_{\text{aut}}$ where $\hat{\mathbf{z}} \in \mathscr{C}_\infty^{v-r}$ and $\hat{\mathbf{b}}_{\text{aut}}$ satisfies equation (51). Indeed, this follows by letting

$$\hat{\mathbf{z}} := \hat{V}_2\left(\frac{d}{dt}\right)\hat{\mathbf{b}}, \tag{53}$$

$$\text{and} \qquad \hat{\mathbf{b}}_{\text{aut}} := \left(I_v - V_2\left(\frac{d}{dt}\right)\hat{V}_2\left(\frac{d}{dt}\right)\right)\hat{\mathbf{b}}.$$

Here, we have used the relationships in (45). Defining $\hat{\mathbf{b}}_{\text{cont}}$ by equation (48), we find that $\hat{\mathbf{b}}_{\text{cont}}$ satisfies (50).

That $\hat{\mathbf{b}}_{\text{aut}}$ takes the form (49) then follows from Lemma 1.5.5, since any solution to the equation $\lambda_j\left(\frac{d}{dt}\right)\alpha = 0$ has the form

$$\alpha(t) = \Re\left(\sum_{l|\{l=1,2,\ldots,m,d_{j_l}\neq 0,\Im(s_l)\geq 0\}}\sum_{n=0}^{d_{j_l}-1} a_{jln}t^n e^{s_l t}\right),$$

for some $a_{jln} \in \mathbb{C}$. Here, the summation can be restricted to those roots s_l of $\lambda_j(s)$ which satisfy $\Im(s_l) \geq 0$ since roots of $\lambda_j \in \mathbb{R}[s]$ occur in complex conjugate pairs, and $\Re\big(at^n e^{s_l t}\big) = \Re\big(\bar{a}t^n e^{\bar{s}_l t}\big)$ We have thus shown that if $\hat{\mathbf{b}} \in \mathscr{C}_\infty^v$ is a solution to equation (40), then $\hat{\mathbf{b}}$ takes the form of equation (47), where $\hat{\mathbf{b}}_{\text{cont}}$ takes the form of equation (48), and $\hat{\mathbf{b}}_{\text{aut}}$ takes the form of equation (49).

Now, let $\hat{\mathbf{b}}$ take the form (47) where $\hat{\mathbf{b}}_{\text{cont}}$ takes the form (48) for some $\hat{\mathbf{z}} \in \mathscr{C}_\infty^{v-r}$, and $\hat{\mathbf{b}}_{\text{aut}}$ takes the form (49) for some $a_{jln} \in \mathbb{C}$. From the preceding argument, it may be verified that $R\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{\text{cont}} = \mathbf{0}_u$. Moreover, since $\tilde{\Lambda}\left(\frac{d}{dt}\right)\hat{V}_1\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{\text{aut}} = \mathbf{0}_r$, then $\Lambda\left(\frac{d}{dt}\right)\hat{V}\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{\text{aut}} = \mathbf{0}_u$, and hence $R\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{\text{aut}} = U^{-1}\left(\frac{d}{dt}\right)\Lambda\left(\frac{d}{dt}\right)V\left(\frac{d}{dt}\right)^{-1}\hat{\mathbf{b}}_{\text{aut}} = \mathbf{0}_u$. By the linearity of the differential operator $R\left(\frac{d}{dt}\right)$, it follows that equation (40) holds. This completes the proof of Theorem 1.5.4. □

We now make some remarks about the decomposition presented in Theorem 1.5.4 of particular relevance to our subsequent analysis.

Remark 1.5.6.
It follows from Theorem 1.5.4 that the location of the roots of the invariant polynomial $\lambda_r(s)$ determines the nature of the trajectories $\hat{\mathbf{b}}_{\text{aut}}$ satisfying equation (51). Indeed, these trajectories are such that $\hat{\mathbf{b}}_{\text{aut}}(t)$ is bounded as $t \to \infty$ if and only if all the roots of $\lambda_r(s) = D_r(R(s))/D_{r-1}(R(s))$ are in $\bar{\mathbb{C}}_-$, and those roots on $j\mathbb{R}$ have multiplicity one.

Remark 1.5.7.
As explained in [26, 29], the description of certain types of behaviour as controllable, and other types of behaviour as autonomous, has an intuitive appeal. Specifically, the word 'controllable' reflects the property that any 'desired' future trajectory can be achieved irrespective of the past trajectory. In other words, given any two trajectories $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ of a given controllable behaviour, and given any $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$, then there exists a trajectory $\hat{\mathbf{b}}_{\text{cont}}$ which satisfies $\hat{\mathbf{b}}_{\text{cont}}(t) = \hat{\mathbf{b}}_1(t)$ for all $(t < t_1)$ and $\hat{\mathbf{b}}_{\text{cont}}(t) = \hat{\mathbf{b}}_2(t)$ for all $(t \geq t_2)$. This follows from an appropriate choice of $\hat{\mathbf{z}}$ in (48). On the other hand, the word 'autonomous' reflects the fact that the entire trajectory $\hat{\mathbf{b}}_{\text{aut}}$ is fixed by the values of the constants $a_{jln} \in \mathbb{C}$ in (49), which are completely determined by the value of the trajectory at a finite number of instants in time.

Remark 1.5.8.
As will be illustrated in the example considered in footnote 15, the autonomous part of a behaviour is not uniquely defined. The decomposition provided in Theorem 1.5.4 provides just one example of a decomposition of a behaviour into controllable and autonomous parts. However, the decomposition described in Theorem 1.5.4 is unique. To see this, suppose otherwise. Then there exists $\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2 \in \mathscr{C}_\infty^v$ such that

$$\hat{\mathbf{b}} = \hat{\mathbf{a}}_1 + \hat{\mathbf{b}}_1 = \hat{\mathbf{a}}_2 + \hat{\mathbf{b}}_2, \tag{54}$$

with

$$\hat{V}_1\left(\frac{d}{dt}\right)\hat{\mathbf{a}}_1 = \hat{V}_1\left(\frac{d}{dt}\right)\hat{\mathbf{a}}_2 = \mathbf{0}, \tag{55}$$

and

$$\hat{V}_2\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_1 = \hat{V}_2\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_2 = \mathbf{0}. \tag{56}$$

From (54) and (56) we obtain

$$\hat{V}_2\left(\frac{d}{dt}\right)(\hat{\mathbf{a}}_1 - \hat{\mathbf{a}}_2) = \hat{V}_2\left(\frac{d}{dt}\right)(\hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_1) = \mathbf{0},$$

which together with (55) implies

$$\hat{\mathbf{a}}_1 - \hat{\mathbf{a}}_2 = V\left(\frac{d}{dt}\right)\mathbf{0} = \mathbf{0},$$

which implies $\hat{\mathbf{a}}_1 = \hat{\mathbf{a}}_2$. That $\hat{\mathbf{b}}_1 = \hat{\mathbf{b}}_2$ is then immediate from (54).

Remark 1.5.9.
Theorem 1.5.4 provides a formal proof of the statements made in Section 1.4 about the autonomous part of the driving-point trajectory of a transformerless network. From the kernel description in Theorem 1.4.1, is it readily seen that $r = 1$ and $\lambda_1(s) = g(s)$ in the notation of Theorem 1.5.4. Since the roots of $g(s)$ are all in $\mathbb{C}_-$, it follows that the autonomous part of the driving-point trajectory of a transformerless network decays exponentially to zero as $t \to \infty$.

We finish this Section with an example to illustrate equation (49) in Theorem 1.5.4. Consider the matrix

$$R\left(\frac{d}{dt}\right) = \begin{bmatrix} \frac{d}{dt}+1 & \frac{d^2}{dt^2}+\frac{d}{dt} \\ 0 & \frac{d^3}{dt^3}+5\frac{d^2}{dt^2}+8\frac{d}{dt}+4 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}+1 & 0 \\ 0 & \left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}+2\right)^2 \end{bmatrix}\begin{bmatrix} 1 & \frac{d}{dt} \\ 0 & 1 \end{bmatrix}.$$

It may then be verified that $R(s)$ can be written in Smith form as in Lemma 1.5.3 with $r = u = v = 2$,

$$V(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix},$$
$$\lambda_1(s) = (s+1)(s+2)^0,$$
$$\text{and} \quad \lambda_2(s) = (s+1)(s+2)^2,$$

and so $V_2(s)$ and $\hat{V}_2(s)$ are null matrices, hence $\hat{\mathbf{b}}_{\text{cont}} = \mathbf{0}_2$. Moreover,

$$\mathbf{y}_1(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\text{and} \quad \mathbf{y}_2(s) = \begin{bmatrix} -s \\ 1 \end{bmatrix},$$

which implies $m = 2$, $s_1 = -1, s_2 = -2$, $d_{1_1} = 1, d_{1_2} = 0, d_{2_1} = 1$, and $d_{2_2} = 2$. We thus find

$$\mathbf{y}_1(s_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{y}_2(s_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{y}_2(s_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{ and } \frac{d\mathbf{y}_2(s)}{ds}(s_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Since the second summation in equation (49) runs through the integers $l = 1, 2, \ldots, m$ for which $(s - s_l)$ is a root of the jth invariant polynomial (of degree greater than zero), and $\Im(s_l) \geq 0$, we thus obtain:

$$\hat{\mathbf{b}}_{\text{aut}}(t) = \begin{bmatrix} a_{110} + a_{210} \\ a_{210} \end{bmatrix} e^{-t} + \begin{bmatrix} 2a_{220} - a_{221} + 2a_{221}t \\ a_{220} + a_{221}t \end{bmatrix} e^{-2t}.$$

It may then be verified that $\hat{\mathbf{b}}_{\text{aut}}$ does indeed satisfy $R\left(\frac{d}{dt}\right)\hat{\mathbf{b}}_{\text{aut}} = \mathbf{0}_2$.

1.6 Properties of transformerless network trajectories

In this section, we use the parametrisation of Theorem 1.5.4 to derive properties of the behaviour of a general transformerless network. We will demonstrate that there are transformerless networks whose behaviour contains an autonomous component. Furthermore, we show that the autonomous part of the trajectory of a transformerless network is bounded for all future time.

The results of this section are summarised in the following theorem:

Theorem 1.6.1. *Let N be a transformerless network and let $\hat{\mathbf{b}}$ be a trajectory of N. Then $\hat{\mathbf{b}}$ may be written in the form*

$$\hat{\mathbf{b}} = \hat{\mathbf{b}}_{cont} + \hat{\mathbf{b}}_{st} + \hat{\mathbf{b}}_{osc} \tag{57}$$

where

$$\hat{\mathbf{b}}_{cont} = \begin{bmatrix} f\left(\frac{d}{dt}\right) q\left(\frac{d}{dt}\right) \\ f\left(\frac{d}{dt}\right) p\left(\frac{d}{dt}\right) \\ \mathbf{h}\left(\frac{d}{dt}\right) \end{bmatrix} \hat{z},$$

for some $\hat{z} \in \mathscr{C}_\infty$, in which the following four conditions all hold:

1. *$p(s)$ and $q(s)$ are as in Theorem 1.4.1. In particular, $p(s)/q(s)$ is PR and is the impedance of N.*

2. *$f \in \mathbb{R}[s]$, and all roots of $f(s)$ are in $\mathbb{C}_-$.*

3. *$\mathbf{h} \in \mathbb{R}^\bullet[s]$, and $\mathbf{h}(\xi) \neq \mathbf{0}$ for all $\xi \in \mathbb{C}$.*

4. *The maximum of the degrees of the polynomials in* $\mathbf{h}(s)$ *does not exceed the maximum of the degrees of* $f(s)p(s)$ *and* $f(s)q(s)$.

Moreover, $\hat{\mathbf{b}}_{st}$ *and* $\hat{\mathbf{b}}_{osc}$ *satisfy the following conditions:*

5. $\hat{\mathbf{b}}_{st} : \mathbb{R} \mapsto \mathbb{R}^{2(m+1)}$ *is a trajectory of* N, *and* $\hat{\mathbf{b}}_{st}(t) \to \mathbf{0}$ *as* $t \to \infty$.
6. $\hat{\mathbf{b}}_{osc} = \sum_{k=1}^{n} \hat{\mathbf{b}}_{osc}^{(k)}$ *where* $\hat{\mathbf{b}}_{osc}^{(k)}(t) = \Re\left(\begin{bmatrix}\alpha_k q(j\omega_k) & \alpha_k p(j\omega_k) & \tilde{\mathbf{r}}_k^T\end{bmatrix}^T e^{j\omega_k t}\right)$ *for some* $\omega_k \in \mathbb{R}$, $\omega_k \geq 0$, $\alpha_k \in \mathbb{C}$, *and* $\tilde{\mathbf{r}}_k \in \mathbb{C}^{2m}$, *and* $\hat{\mathbf{b}}_{osc}^{(k)}$ *is a trajectory of* N $(k = 1, 2, \ldots, n)$.

The principal contribution of Theorem 1.6.1 is the establishment of the properties of $\hat{\mathbf{b}}_{\mathrm{st}}$ and $\hat{\mathbf{b}}_{\mathrm{osc}}$, which describe the 'autonomous' part of the behaviour of a general transformerless network. Laplace-domain analyses of the behaviour of a network typically only obtain information about $\hat{\mathbf{b}}_{\mathrm{cont}}$.

Proof of Theorem 1.6.1. We consider the terminology associated with the transformerless network N introduced in Section 1.4. As described in that section, if $\hat{\mathbf{b}} := \begin{bmatrix}\hat{i} & \hat{v} & \hat{\mathbf{r}}^T\end{bmatrix}^T : \mathbb{R} \mapsto \mathbb{R}^{2(m+1)}$ is a trajectory of N then it satisfies equation (10). Our proof uses the decomposition described in Theorem 1.5.4, and we will further consider the terminology introduced in that theorem. In this case, we have $u = r = 2m+1$ and $v = 2(m+1)$ for the row dimension u, column dimension v, and rank r of the matrix $R(s)$. Hence, $\hat{\mathbf{b}}_{\mathrm{cont}} = \mathbf{c}\left(\frac{d}{dt}\right)\hat{z}$ for some $\hat{z} \in \mathscr{C}_\infty$ and $\mathbf{c} \in \mathbb{R}^{2(m+1)}[s]$, where $\mathbf{c}(s)$ satisfies $R(s)\mathbf{c}(s) = \mathbf{0}$, and $\mathbf{c}(\xi) \neq \mathbf{0}_{2(m+1)}$ for all $\xi \in \mathbb{C}$. Indeed, $\mathbf{c}(s)$ is unique up to constant scaling, and will form a basis for the right syzygy of $R(s)$. We will first show that

$$\mathbf{c}\left(\frac{d}{dt}\right) = \begin{bmatrix} f\left(\frac{d}{dt}\right) q\left(\frac{d}{dt}\right) \\ f\left(\frac{d}{dt}\right) p\left(\frac{d}{dt}\right) \\ \mathbf{h}\left(\frac{d}{dt}\right) \end{bmatrix},$$

where $f(s), p(s), q(s)$ and $\mathbf{h}(s)$ satisfy conditions 1 to 4 in Theorem 1.6.1.

Let $a(s)$ be the least common multiple of all denominator polynomials in $A(s)^{-1}\mathbf{x}$. In other words, $a(s)$ is the polynomial of least degree (unique up to constant scaling) such that $a(s)A(s)^{-1}\mathbf{x}$ is polynomial. Hence, $a(\xi)A(\xi)^{-1}\mathbf{x} \neq \mathbf{0}_{2m}$ for all $\xi \in \mathbb{C}$, and $a(s)\mathbf{x}^T A(s)^{-1}\mathbf{x}$ is polynomial. Moreover,

$$\begin{bmatrix} 0 & 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{0}_{2(m-1)} & A(s) \end{bmatrix} \begin{bmatrix} a(s) \\ a(s)\mathbf{x}^T A(s)^{-1}\mathbf{x} \\ -a(s)A(s)^{-1}\mathbf{x} \end{bmatrix} = \mathbf{0},$$

so accordingly we let

$$\mathbf{c}(s) := \begin{bmatrix} a(s) \\ a(s)\mathbf{x}^T A(s)^{-1}\mathbf{x} \\ -a(s)A(s)^{-1}\mathbf{x} \end{bmatrix}, \tag{58}$$

and hence $R(s)\mathbf{c}(s) = \mathbf{0}$ and $\mathbf{c}(\xi)$ is non-zero for all $\xi \in \mathbb{C}$. We conclude that $\hat{\mathbf{b}}_{\text{cont}} = \mathbf{c}(\frac{d}{dt})\hat{z}$ for $\hat{z} \in \mathscr{C}_\infty$. Accordingly, let $\mathbf{h}(s) = -a(s)A(s)^{-1}\mathbf{x}$, so from before we have $\mathbf{h}(\xi) \neq \mathbf{0}_{2m}$ for all $\xi \in \mathbb{C}$. We next show that the first two entries in $\mathbf{c}(s)$ take the form $f(s)q(s)$ and $f(s)p(s)$ such that conditions 1, 2 and 4 of the present theorem statement hold.

From the proof of Theorem 1.4.1, $\mathbf{x}^T A(s)^{-1}\mathbf{x} = \hat{C}_1\mathcal{M}(s)^{-1}\hat{C}_1^T = p(s)/q(s)$ where $p(s)$ and $q(s)$ are the coprime polynomials described in the statement of that theorem. Since any pole of $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ of multiplicity k must be a pole of $\mathbf{x}^T A(s)^{-1}$ of multiplicity greater than or equal to k, it follows that $q(s)$ must divide $a(s)$, so accordingly we let $a(s) = f(s)q(s)$ for $f \in \mathbb{R}[s]$, and then $a(s)\hat{C}_1\mathcal{M}(s)^{-1}\hat{C}_1^T = f(s)p(s)$. From Lemma 1.4.2, all of the roots of $a(s)$ must be in $\bar{\mathbb{C}}_-$, and roots on $j\mathbb{R}$ have multiplicity one. Moreover, for $\omega \in \mathbb{R}$, if $a(s)$ has a root at $s = j\omega$, then $A(s)^{-1}\mathbf{x}$ has a pole at $s = j\omega$, and so too does $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ by Lemma 1.4.2. Hence, $a(s)\mathbf{x}^T A(s)^{-1}\mathbf{x} = a(s)\hat{C}_1\mathcal{M}(s)^{-1}\hat{C}_1^T$ cannot have a root at $s = j\omega$ if $a(s)$ does. It follows that $\begin{bmatrix} f(j\omega)p(j\omega) & f(j\omega)q(j\omega) \end{bmatrix}^T \neq \mathbf{0}$ for all $\omega \in \mathbb{R}$, and hence all roots of $f(s)$ are in $\mathbb{C}_-$. As a further consequence of Lemma 1.4.2, since $A(s)^{-1}\mathbf{x}$ can only have simple poles at $s = \infty$, and $A(s)^{-1}\mathbf{x}$ has a simple pole at ∞ if and only if $\mathbf{x}^T A(s)^{-1}\mathbf{x}$ does, then the degree of any entry in $-a(s)A(s)^{-1}\mathbf{x}$ cannot exceed the maximum of the degrees of $a(s)$ and $a(s)\mathbf{x}^T A(s)^{-1}\mathbf{x} = a(s)\hat{C}_1\mathcal{M}(s)^{-1}\hat{C}_1^T$. This completes the proof of conditions 1, 2, 3, and 4 in Theorem 1.6.1.

Next, we show that $D_{2m+1}(R(s))/D_{2m}(R(s))$ has all roots in $\bar{\mathbb{C}}_-$, and roots on $j\mathbb{R}$ have multiplicity one. From Theorem 1.5.4, this implies that $\hat{\mathbf{b}}_{\text{aut}} : \mathbb{R} \mapsto \mathbb{R}^{2(m+1)}$ may be written in the form

$$\hat{\mathbf{b}}_{\text{aut}} = \hat{\mathbf{b}}_{\text{st}} + \sum_{k=1}^{n} \hat{\mathbf{b}}_{\text{osc}}^{(k)}, \tag{59}$$

where

$$\hat{\mathbf{b}}_{\text{osc}}^{(k)}(t) = \Re\left(\tilde{\mathbf{b}}_{\text{osc}}^{(k)} e^{j\omega_k t}\right) \tag{60}$$

for some $\omega_k \in \mathbb{R}$, $\omega_k \geq 0$, $\alpha_k \in \mathbb{C}$, and $\tilde{\mathbf{b}}_{\text{osc}}^{(k)} \in \mathbb{C}^{2(m+1)}$, $\hat{\mathbf{b}}_{\text{osc}}^{(k)}$ is a trajectory of N $(k = 1, 2, \ldots, n)$, and condition 5 of the present theorem statement holds. Here, ω_k are the imaginary axis roots of the invariant polynomials of the matrix

$$\begin{bmatrix} 0 & 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{0}_{2m} & A(s) \end{bmatrix}$$

corresponding to the differential operator in (10).

We fix $\omega \in \mathbb{R}$, and we first consider the matrix $A(s)$. From Lemma 1.4.2, $A(s)^{-1}$ has all poles in $\bar{\mathbb{C}}_- \cup \infty$, and poles on $j\mathbb{R}$ have multiplicity one. Hence, $D_{2m}(A(s))/D_{2m-1}(A(s))$ has all roots in $\bar{\mathbb{C}}_-$, and roots on $j\mathbb{R}$ have multiplicity one. Then, from Lemmas 1.5.2 and 1.5.3, there exist unimodular matrices $U \in \mathbb{R}^{2m\times 2m}[s]$, and $V \in \mathbb{R}^{2m\times 2m}[s]$ such that

$$U(s)A(s)V(s) = \Lambda(s) = \text{diag}\left(\lambda_1(s), \lambda_2(s), \ldots, \lambda_{2m}(s)\right),$$

where $\lambda_l(s)$ has all roots in $\bar{\mathbb{C}}_-$ ($l = 1, 2, \ldots, 2m$). Moreover, suppose $\lambda_l(s)$ has a root at $j\omega$ for some $l = 1, 2, \ldots, 2m$. By Lemma 1.5.2, $\lambda_l(s)$ divides $\lambda_{l+1}(s)$ for $l = 1, 2, \ldots, 2m-1$. Hence, there exists an $r \in 1, 2, \ldots, 2m$ such that $\lambda_l(s)$ has no root at $j\omega$ for $l = 1, 2, \ldots, 2m-r$, and $\lambda_l(s)$ has a root of multiplicity one at $j\omega$ for $l = 2m-r+1, 2m-r+2, \ldots, 2m$.

Since

$$\begin{bmatrix} 1 & 0 \\ 0 & U(s) \end{bmatrix} \begin{bmatrix} 0 & 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{0}_{2(m-1)} & A(s) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\mathbf{x}^T V(s) \\ 0 & 0 & V(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \mathbf{0}_{2(m-1)}^T \\ U(s)\mathbf{x} & \mathbf{0}_{2(m-1)} & \Lambda(s) \end{bmatrix}, \quad (61)$$

and both of the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & U(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\mathbf{x}^T V(s) \\ 0 & 0 & V(s) \end{bmatrix}$$

are unimodular, then $D_{2m+1}(R(s))/D_{2m}(R(s)) = D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$, where

$$\hat{R}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ g_1(s) & 0 & \lambda_1(s) & 0 & \ldots & 0 \\ g_2(s) & 0 & 0 & \lambda_2(s) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{2m}(s) & 0 & 0 & 0 & \ldots & \lambda_{2m}(s) \end{bmatrix}, \quad (62)$$

for some $g_1, g_2, \ldots g_{2m} \in \mathbb{R}[s]$. This follows from Lemmas 1.5.2 and 1.5.3. Here, $\begin{bmatrix} g_1(s) & g_2(s) & \ldots & g_{2m}(s) \end{bmatrix}^T = U(s)\mathbf{x}$ in equation (61).

Since the minor formed from the final $2m+1$ rows and columns of $\hat{R}(s)$ is equal to $\prod_{l=1}^{2m} \lambda_l(s)$, then $D_{2m+1}(\hat{R}(s))$ must divide $\prod_{l=1}^{2m} \lambda_l(s)$, and hence all of the roots of $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ are in $\bar{\mathbb{C}}_-$. Furthermore, $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ has a root at $j\omega$ only if $\lambda_{2m}(s)$ does. Moreover, if $\lambda_{2m}(s)$ has a root of multiplicity one at $j\omega$ but $\lambda_{2m-1}(s)$ does not have a root at $j\omega$, and if $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ does have a root at $j\omega$, then its multiplicity is one.

To complete the proof that roots of $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ on $j\mathbb{R}$ have multiplicity one, we finally show that $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ has a root at $j\omega$ of multiplicity one whenever $j\omega$ is a root of multiplicity one of $\lambda_l(s)$ for $l = 2m-r+1, \ldots, 2m$ but is not a root of $\lambda_l(s)$ for $l = 1, 2, \ldots, 2m-r$, with $r \geq 2$. To see this, consider a minor of order $2m+2-k$ of $\hat{R}(s)$ for some $1 \leq k \leq r$. By examining equation (62), it may be seen that this minor is the product of $2m+2-k$ polynomials. This product contains exactly one polynomial from among the pair of polynomials $g_i(s)$ and $\lambda_i(s)$ for precisely one value of i with $2m-r+1 \leq i \leq 2m$, and at least $r-k$ polynomials from amongst the polynomials $\lambda_l(s)$ for $2m-r+1 \leq l \leq 2m$, $l \neq i$. It may then be seen that $j\omega$ is a root of $D_{2m+2-k}(\hat{R}(s))$ of multiplicity $r-k+n$, where n is the multiplicity of $j\omega$ as a root of the highest common factor of the polynomials $g_{2m-r+1}(s), g_{2m-r+2}(s), \ldots, g_{2m}(s)$, and $(s^2+\omega^2)$. It thus follows that $j\omega$ is a root of $D_{2m+1}(\hat{R}(s))$ of multiplicity $r-1+n$, and is a root of $D_{2m}(\hat{R}(s))$ of multiplicity $r-2+n$. Hence, $j\omega$ is a root of $D_{2m+1}(\hat{R}(s))/D_{2m}(\hat{R}(s))$ of multiplicity one.

We have thus shown that $\hat{\mathbf{b}}_{\text{aut}}$ may be written in the form of equation (59), where $\hat{\mathbf{b}}_{\text{osc}}^{(k)}$ takes the form of equation (60) for some $\omega_k \in \mathbb{R}$, $\omega_k \geq 0$, $\alpha_k \in \mathbb{C}$, and $\tilde{\mathbf{b}}_{\text{osc}}^{(k)} \in \mathbb{C}^{2(m+1)}$, $\hat{\mathbf{b}}_{\text{osc}}^{(k)}$ is a trajectory of N $(k = 1, 2, \ldots, n)$, and condition 5 of the present theorem statement holds. It remains to complete the proof of condition 6 in the present theorem. In order to complete the proof of this condition, we consider a particular value of $k \in 1, 2, \ldots, n$, we let $\tilde{\mathbf{b}}_{\text{osc}}^{(k)} =: \begin{bmatrix} \tilde{i}_{\text{osc}}^{(k)} & \tilde{v}_{\text{osc}}^{(k)} & \tilde{\mathbf{r}}_{\text{osc}}^{(k)} \end{bmatrix}^T$, and we will show that $\tilde{i}_{\text{osc}}^{(k)} = \alpha_k q(j\omega_k)$ and $\tilde{v}_{\text{osc}}^{(k)} = \alpha_k q(j\omega_k)$ for some $\alpha_k \in \mathbb{C}$.

Since $\hat{\mathbf{b}}_{\text{osc}}^{(k)}$ is a trajectory of N, then $\hat{\mathbf{d}}_{\text{osc}}^{(k)}$ is a driving-point trajectory of N, where $\hat{\mathbf{d}}_{\text{osc}}^{(k)}(t) = \Re\left(\begin{bmatrix} \tilde{i}_{\text{osc}}^{(k)} & \tilde{v}_{\text{osc}}^{(k)} \end{bmatrix}^T e^{j\omega_k t}\right)$. Hence, $\hat{\mathbf{d}}_{\text{osc}}^{(k)}$ must be in the kernel of the differential operator (18). This implies

$$g\left(\frac{d}{dt}\right)\begin{bmatrix} -p\left(\frac{d}{dt}\right) & q\left(\frac{d}{dt}\right) \end{bmatrix}\left(\Re\left(\begin{bmatrix} \tilde{i}_{\text{osc}}^{(k)} \\ \tilde{v}_{\text{osc}}^{(k)} \end{bmatrix} e^{j\omega_k t}\right)\right) = 0,$$

and hence

$$|(g(j\omega_k)||(q(j\omega_k)\tilde{v}_{\text{osc}}^{(k)} - p(j\omega_k)\tilde{i}_{\text{osc}}^{(k)})| \cos(\omega_k t - \phi_k) = 0.$$

where ϕ_k is the argument of the complex number $g(j\omega_k)(q(j\omega_k)\tilde{v}_{\text{osc}}^{(k)} - p(j\omega_k)\tilde{i}_{\text{osc}}^{(k)})$. Then, since $g(s)$ has no roots on $j\mathbb{R}$, we require

$$q(j\omega_k)\tilde{v}_{\text{osc}}^{(k)} = p(j\omega_k)\tilde{i}_{\text{osc}}^{(k)}. \tag{63}$$

Since at most one of $q(j\omega_k)$ and $p(j\omega_k)$ can be zero by Theorem 1.4.1, then the set of solutions $\begin{bmatrix}\tilde{i}_{\text{osc}}^{(k)} & \tilde{v}_{\text{osc}}^{(k)}\end{bmatrix}^T$ to equation (63) is a linear subspace of $\mathbb{C}^2$ of dimension one. Since one solution to equation (63) is given by $\tilde{v}_{\text{osc}}^{(k)} = p(j\omega_k)$ and $\tilde{i}_{\text{osc}}^{(k)} = q(j\omega_k)$, then any solution to equation (63) must take the form $\tilde{i}_{\text{osc}}^{(k)} = \alpha_k q(j\omega_k)$ and $\tilde{v}_{\text{osc}}^{(k)} = \alpha_k p(j\omega_k)$ for some $\alpha_k \in \mathbb{C}$. This completes the proof of Theorem 1.6.1. □

Remark 1.6.2.
As a consequence of Theorem 1.6.1, if the driving-point current $\hat{i}$ and voltage $\hat{v}$ are both sinusoidally varying (i.e. $\hat{i}(t) = \Re(\tilde{i}e^{j\omega t})$ and $\hat{v}(t) = \Re(\tilde{v}e^{j\omega t})$ for some $\tilde{i}, \tilde{v} \in \mathbb{C}$ and $\omega \in \mathbb{R}$), then the absence of imaginary axis roots in the polynomial $f(s)$ implies that $\hat{z}(t) = \Re\big(\tilde{z}e^{j\omega t}\big)$ for some $\tilde{z} \in \mathbb{C}$, and then $\mathbf{h}(\frac{d}{dt})\Re\big(\tilde{z}e^{j\omega t}\big) = \Re\big(\mathbf{h}(j\omega)\tilde{z}e^{j\omega t}\big)$. In this case, the internal currents and voltages remain bounded as $t \to \infty$.

We return to the example of network N_b in Fig. 7. As discussed at the end of Section 1.4, the behaviour of this network is the kernel of the differential operator $R(\frac{d}{dt})$ with $R(s)$ as in equation (39). In this case, we find that

$$U(s) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -s & 0 & 0 & 0 & s & -1 & 0 & 0 \\
0 & 0 & 1 & -s & 0 & 1 & 0 & s & 0 & 0 & -1 \\
0 & s & -s^2 & s^3 & -1 & -(s^2+1) & s & -s^3 & -1 & 1 & (s^2+1)
\end{bmatrix},$$

$$\text{and} \quad V(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & s & (1+s) \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & s & (s^2+s+1) & (1+s) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (1+s) & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & s & 0 & 0 & s & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & s & 0 & 0 & s^2 & (s^3+s^2+s) & s \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & (1-s) \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s & (s^2+s) & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s & (s^2+s+1) & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (1-s) \end{bmatrix},$$

are unimodular matrices which transform $R(s)$ into its Smith form, as in (41). Moreover, the eleventh invariant polynomial of the matrix $R(s)$ is equal to $(s^2+1)(s+1)$, and all other invariant polynomials are equal to one. It follows from Theorem 1.5.4 that any trajectory $\hat{\mathbf{b}}$ of N_b may be written in the form

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{i} & \hat{v} & \hat{i}_1 & \hat{i}_2 & \ldots & \hat{i}_5 & \hat{v}_1 & \hat{v}_2 & \ldots & \hat{v}_5 \end{bmatrix}^T,$$

with

$$\begin{bmatrix} \hat{i} \\ \hat{v} \\ \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \\ \hat{i}_4 \\ \hat{i}_5 \\ \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{v}_4 \\ \hat{v}_5 \end{bmatrix}(t) = \left(\begin{bmatrix} 1+\frac{d}{dt} \\ 1+\frac{d}{dt} \\ 1 \\ \frac{d}{dt} \\ 1 \\ \frac{d}{dt} \\ 1-\frac{d}{dt} \\ \frac{d}{dt} \\ 1 \\ \frac{d}{dt} \\ 1 \\ 1-\frac{d}{dt} \end{bmatrix} \hat{z} \right)(t) + \beta_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{-t} + \beta_2 \begin{bmatrix} -\sin(t+\phi) \\ -\sin(t+\phi) \\ \cos(t+\phi) - \sin(t+\phi) \\ -\sin(t+\phi) \\ 0 \\ -\cos(t+\phi) \\ \cos(t+\phi) \\ -\cos(t+\phi) - \sin(t+\phi) \\ \cos(t+\phi) \\ 0 \\ -\sin(t+\phi) \\ \cos(t+\phi) \end{bmatrix}, \quad (64)$$

for some $\hat{z} \in \mathscr{C}_\infty$ and $\beta_1, \beta_2, \phi \in \mathbb{R}$. Here, β_1, β_2 and ϕ are fixed for a given trajectory, and are determined by the value of the trajectory at a finite number of instants in time[15].

[15]We note that the decomposition in Theorem 1.6.1 is not unique. To see this, consider

This concludes our study of the properties of the behaviour and the driving-point behaviour of a general transformerless network using the techniques of polynomial algebra. The principal results are those of Theorems 1.4.1 and 1.6.1. These theorems described properties of the 'autonomous' component of the driving-point behaviour and the behaviour of a general transformerless network, respectively. Such components are overlooked in a Laplace-domain analysis and are not made explicit in impedance or admittance descriptions of a network's behaviour. In the next section, we focus on trajectories $\hat{\mathbf{b}}$ of the form $\hat{\mathbf{b}}(t) = \Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$ for $s_0 \in \mathbb{C}$ and $\tilde{\mathbf{b}} \in \mathbb{C}^{2(m+1)}$. This approach is commonly termed a *phasor* analysis of passive networks, and will be used extensively in Sections 3.5 and 3.6. This approach transforms network analysis from the domain of polynomial algebra to the domain of linear algebra. However, as will be seen, care is required when a phasor approach is adopted to study the impedance of a network.

1.7 The s_0-trajectories of transformerless networks

A common analysis technique for electrical networks is the so-called *phasor* approach. In this technique, a sinusoidal excitation is applied to the network, and the steady-state response to this excitation is measured. This approach may be generalised to consider excitations in which, for example, the driving-point voltage $\hat{v} : \mathbb{R} \mapsto \mathbb{R}$ takes the form $\hat{v}(t) = \Re\left(\tilde{v}e^{s_0 t}\right)$ for some $\tilde{v}, s_0 \in \mathbb{C}$, as described in [36, Section 5.4]. In that section, it is shown that, providing $s_0 \in \bar{\mathbb{C}}_+$ and the network has no 'natural modes' on the imaginary axis, the steady-state driving-point current $\hat{i} : \mathbb{R} \mapsto \mathbb{R}$ takes the form $\Re\left(\tilde{i}e^{s_0 t}\right)$ for some $\tilde{i} \in \mathbb{C}$. Moreover, it is shown that the steady-state current $\hat{i}_k : \mathbb{R} \mapsto \mathbb{R}$ and voltage $\hat{v}_k : \mathbb{R} \mapsto \mathbb{R}$ for an element N_k in the network take the form $\hat{i}_k(t) = \Re\left(\tilde{i}_k e^{s_0 t}\right)$ and $\hat{v}_k(t) = \Re\left(\tilde{v}_k e^{s_0 t}\right)$ respectively, for some $\tilde{i}_k, \tilde{v}_k \in \mathbb{C}$. In [36, Section 5.4], this is then used to establish a connection between a Laplace-domain analysis and a time-domain analysis of the network, the connection being valid for networks which satisfy the aforementioned assumptions. One purpose of the present section is to extend such an analysis to cover those networks which violate the assumptions made in [36, Section 5.4].

In this section, we will consider trajectories of the form $\Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$ for $s_0 \in \mathbb{C}$ and $\tilde{\mathbf{b}} \in \mathbb{C}^{2(m+1)}$. One significant advantage of considering such trajectories is that it allows us to transform analysis from the domain of linear algebra over the ring of polynomials

the parametrisation in (64). By letting $\hat{z}(t) = \hat{z}_b(t) + \beta_1 t e^{-t} + (\beta_2/2)(\sin(t+\phi) - \cos(t+\phi))$, where $\hat{z}_b \in \mathscr{C}_\infty$, we obtain an equivalent parametrisation for which $\hat{i} = \left(1 + \frac{d}{dt}\right)\hat{z}_b$ and $\hat{v}$ satisfies $\hat{v}(t) = \left(\left(1 + \frac{d}{dt}\right)\hat{z}\right)(t) + 2\beta_1 e^{-t}$. It is then evident that the driving-point behaviour of N_b is given by $\hat{v}(t) = \hat{i}(t) + \alpha e^{-t}$ for $\alpha \in \mathbb{R}$, as shown in Section 1.4.

to the domain of linear algebra over the field of complex numbers. We will define an s_0-impedance and an s_0-admittance relating to such trajectories. For a transformerless network N with impedance $Z(s)$ and admittance $Y(s)$, and for a given $s_0 \in \bar{\mathbb{C}}_+$, we will show that the s_0-impedance (resp. s_0-admittance) exists if and only if $Z(s)$ (resp. $Y(s)$) does not have a pole at $s = s_0$. Moreover, if $Z(s)$ (resp. $Y(s)$) doesn't have a pole at $s = s_0$, then $Z(s_0)$ (resp. $Y(s_0)$) is equal to the s_0-impedance (resp. s_0-admittance). However, we will show the curious result that $Z(s_0)$ (resp. $Y(s_0)$) may differ from the s_0-impedance (resp. s_0-admittance) for certain $s_0 \in \mathbb{C}_-$.

We first define the notions of an s_0-trajectory and an s_0-driving-point trajectory of a network, and the associated concepts of the s_0-behaviour and s_0-driving-point behaviour.

Definition 1.7.1 (s_0-trajectory/ behaviour, s_0-driving-point trajectory/ behaviour)**.** Let N be a transformerless network, let $\hat{\mathbf{b}} : \mathbb{R} \mapsto \mathbb{R}^{2(m+1)}$ in (4) be a trajectory of N with $\hat{\mathbf{b}}(t) = \Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$ for some $s_0 \in \mathbb{C}$ and $\tilde{\mathbf{b}} \in \mathbb{C}^{2(m+1)}$, and let $\hat{\mathbf{d}} := \begin{bmatrix} I_2 & 0_{2\times 2m} \end{bmatrix} \hat{\mathbf{b}}$ be the corresponding driving-point trajectory, so $\hat{\mathbf{d}}(t) = \Re\left(\tilde{\mathbf{d}}e^{s_0 t}\right)$ for some $\tilde{\mathbf{d}} \in \mathbb{C}^2$. We call $\tilde{\mathbf{b}}$ an s_0-trajectory of N, and we call $\tilde{\mathbf{d}}$ an s_0-driving-point trajectory of N.

We call the set of all s_0-trajectories (resp. s_0-driving-point trajectories) of N the s_0-behaviour (resp. s_0-driving-point behaviour) of N.

We note that our definition of an s_0-driving-point trajectory excludes those driving-point trajectories $\hat{\mathbf{d}}$ of the form $\hat{\mathbf{d}}(t) = \Re\left(\tilde{\mathbf{d}}e^{s_0 t}\right)$ for which the network trajectory $\hat{\mathbf{b}}$ does not take the form $\Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$. As an example of such a driving-point trajectory, consider again the network N_b in Fig. 7, whose behaviour takes the form of (64). Here,

$$\text{both} \quad \hat{\mathbf{b}}_1(t) = \begin{bmatrix} 1 \\ 1 \\ t \\ 1-t \\ t \\ 1-t \\ 2t-1 \\ 1-t \\ t \\ 1-t \\ t \\ 2t-1 \end{bmatrix} e^{-t}, \text{ and } \quad \hat{\mathbf{b}}_2(t) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{-t},$$

take the form of (64) for $\hat{z} = te^{-t}$, and $\beta_1 = \beta_2 = \phi = 0$, and for $\hat{z} = \beta_2 = \phi = 0$ and $\beta_1 = 1$, respectively. We thus obtain the two driving-point trajectories $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ of N_b, where

$$\hat{\mathbf{d}}_1(t) := \begin{bmatrix} I_2 & 0_{2\times 10} \end{bmatrix} \hat{\mathbf{b}}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}, \text{ and } \hat{\mathbf{d}}_2(t) := \begin{bmatrix} I_2 & 0_{2\times 10} \end{bmatrix} \hat{\mathbf{b}}_2(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$$

From Definition 1.7.1, for $s_0 = -1$, we conclude that $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$ is an s_0-driving-point trajectory, but $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ need not be.

The following lemma permits our subsequent definition of the s_0-impedance and s_0-admittance of a transformerless network N.

Lemma 1.7.2. *For a given $s_0 \in \mathbb{C}$, the s_0-driving-point behaviour of a transformerless network is a linear subspace of $\mathbb{C}^2$ of dimension one.*

Definition 1.7.3 (s_0-impedance, s_0-admittance)**.**
Let N be a transformerless network, and let $s_0 \in \mathbb{C}$. Further let $\begin{bmatrix} \tilde{i} & \tilde{v} \end{bmatrix}^T \neq \mathbf{0}_2$ be an s_0-driving-point trajectory of N. If $\tilde{i} = 0$ (resp. $\tilde{v} = 0$) then we say the s_0-impedance (resp. s_0-admittance) of N does not exist. Otherwise, we define the s_0-impedance (resp. s_0-admittance) as the ratio $\tilde{v}/\tilde{i}$ (resp. $\tilde{i}/\tilde{v}$).

The main results of this section are summarised in the following theorem. The remainder of this section is concerned with the proof of this theorem and the proof of Lemma 1.7.2.

Theorem 1.7.4. *Let N be a transformerless network with impedance $Z(s)$ and admittance $Y(s)$, and let $s_0 \in \bar{\mathbb{C}}_+$. Then the s_0-impedance (resp. s_0-admittance) of N does not exist if and only if $Z(s)$ (resp. $Y(s)$) has a pole at $s = s_0$. Moreover, if the s_0-impedance (resp. s_0-admittance) of N exists, then it is equal to $Z(s_0)$ (resp. $Y(s_0)$).*

Proof of Lemma 1.7.2. Let us first consider the case where N is a single element. If N is an inductor with inductance L then $\tilde{v} = Ls_0\tilde{i}$, if N is a capacitor with capacitance C then $\tilde{i} = Cs_0\tilde{v}$, and if N is a resistor with resistance R then $\tilde{v} = R\tilde{i}$. Hence, both Lemma 1.7.2 and Theorem 1.7.4 are satisfied for any transformerless network which comprises a single passive element, and so we may refer to the s_0-impedance and s_0-admittance of a passive element. Now, let us consider the network N described in

Section 1.4, and let us take an $s_0 \in \mathbb{C}$. We consider s_0-trajectories of the form

$$\tilde{\mathbf{b}} =: \begin{bmatrix} \tilde{i} & \tilde{v} & \tilde{i}_1 & \ldots & \tilde{i}_m & \tilde{v}_1 & \ldots & \tilde{v}_m \end{bmatrix}^T. \tag{65}$$

Then, $\begin{bmatrix} \tilde{i}_k & \tilde{v}_k \end{bmatrix}^T$ is an s_0-driving-point trajectory for the element N_k $(k = 1, 2, \ldots, m)$. Hence, either $\tilde{v}_k = 0$ and the s_0-admittance of N_k does not exist[16], or N_k possesses an s_0-admittance $Y_k^{s_0}$ and $\tilde{i}_k = Y_k^{s_0} \tilde{v}_k$ $(k = 1, 2, \ldots, m)$.

Let us partition the elements $N_1, N_2, \ldots, N_{m+1-n}$ such that the s_0-admittances of elements $N_1, N_2, \ldots, N_{m_{1a}}$ exist, whereas no s_0-admittance exists for the remaining elements $N_{m_{1a}+1}, \ldots, N_{m+1-n}$ (i.e. $\tilde{v}_{m_{1a}+1} = \ldots = \tilde{v}_{m+1-n} = 0$). Here, either $m_{1a} = 0$, which implies that no s_0-admittance exists for the elements $N_1, \ldots, N_{m+1-n}$, or $1 \leq m_{1a} \leq m + 1 - n$. Similarly, we partition the elements $N_{m+2-n}, N_{m+3-n}, \ldots, N_m$ such that the s_0-admittances of elements $N_{m+2-n}, N_{m+3-n}, \ldots, N_{m+m_{2a}+1-n}$ exist, and no s_0-admittance exists for the elements $N_{m+m_{2a}+2-n}, \ldots, N_m$. In this case, $0 \leq m_{2a} \leq n - 1$. Then, let

$$\mathcal{Y}_{1a}^{s_0} = \operatorname{diag}\left(Y_1^{s_0}, Y_2^{s_0}, \ldots, Y_{m_{1a}}^{s_0}\right), \tag{66}$$

$$\text{and} \quad \mathcal{Y}_{2a}^{s_0} = \operatorname{diag}\left(Y_{m+2-n}^{s_0}, Y_{m+3-n}^{s_0}, \ldots, Y_{m+m_{2a}+1-n}^{s_0}\right). \tag{67}$$

Furthermore, let us partition $\tilde{\mathbf{b}}$ as:

$$\tilde{\mathbf{b}} = \begin{bmatrix} \tilde{i} & \tilde{v} & \tilde{\mathbf{i}}^T & \tilde{\mathbf{v}}^T \end{bmatrix}^T, \tag{68}$$

with

$$\tilde{\mathbf{i}} = \begin{bmatrix} \tilde{\mathbf{i}}_{1a}^T & \tilde{\mathbf{i}}_{1b}^T & \tilde{\mathbf{i}}_{2a}^T & \tilde{\mathbf{i}}_{2b}^T \end{bmatrix}^T,$$

$$\text{and} \quad \tilde{\mathbf{v}} = \begin{bmatrix} \tilde{\mathbf{v}}_{1a}^T & \tilde{\mathbf{v}}_{1b}^T & \tilde{\mathbf{v}}_{2a}^T & \tilde{\mathbf{v}}_{2b}^T \end{bmatrix}^T.$$

Here,

$$\tilde{\mathbf{i}}_{1a} := \begin{bmatrix} \tilde{i}_1 & \ldots & \tilde{i}_{m_{1a}} \end{bmatrix}^T,$$

$$\tilde{\mathbf{i}}_{1b} := \begin{bmatrix} \tilde{i}_{m_{1a}+1} & \ldots & \tilde{i}_{m+1-n} \end{bmatrix}^T,$$

$$\tilde{\mathbf{i}}_{2a} := \begin{bmatrix} \tilde{i}_{m+2-n} & \ldots & \tilde{i}_{m+m_{2a}+1-n} \end{bmatrix}^T,$$

[16]As is the case when $s_0 = 0$ and the element is an inductor. In Section 1.9, we will consider transformerless networks as interconnections of one-port subnetworks, whose s_0-admittance may not exist at other points $s_0 \in \mathbb{C}$.

$$\text{and} \quad \tilde{\mathbf{i}}_{2b} := \begin{bmatrix} \tilde{i}_{m+m_{2a}+2-n} & \dots & \tilde{i}_m \end{bmatrix}^T.$$

Then, by Kirchhoff's voltage and current laws, we have

$$\begin{bmatrix} 1 & \mathbf{0}^T_{m_{1a}} & \mathbf{0}^T_{m+1-n-m_{1a}} & \hat{C}_{11} & \hat{C}_{12} \\ \mathbf{0}_{m_{1a}} & I_{m_{1a}} & 0_{m_{1a}\times(m+1-n-m_{1a})} & \hat{C}_{21} & \hat{C}_{22} \\ \mathbf{0}_{m+1-n-m_{1a}} & 0_{(m+1-n-m_{1a})\times m_{1a}} & I_{m+1-n-m_{1a}} & \hat{C}_{31} & \hat{C}_{32} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\mathbf{v}}_{1a} \\ \tilde{\mathbf{v}}_{1b} \\ \tilde{\mathbf{v}}_{2a} \\ \tilde{\mathbf{v}}_{2b} \end{bmatrix} = \mathbf{0}, \quad (69)$$

$$\text{and} \quad \begin{bmatrix} -\hat{C}^T_{11} & -\hat{C}^T_{21} & -\hat{C}^T_{31} & I_{m_{2a}} & 0_{m_{2a}\times(n-1-m_{2a})} \\ -\hat{C}^T_{12} & -\hat{C}^T_{22} & -\hat{C}^T_{32} & 0_{(n-1-m_{2a})\times m_{2a}} & I_{n-1-m_{2a}} \end{bmatrix} \begin{bmatrix} -\tilde{i} \\ \tilde{\mathbf{i}}_{1a} \\ \tilde{\mathbf{i}}_{1b} \\ \tilde{\mathbf{i}}_{2a} \\ \tilde{\mathbf{i}}_{2b} \end{bmatrix} = \mathbf{0}, \quad (70)$$

respectively, these relationships being the same as those in equations (8) and (9), but with a finer partitioning of the matrices. Furthermore, from the preceding discussion, we require

$$\begin{aligned} \tilde{\mathbf{i}}_{1a} &= \mathcal{Y}^{s_0}_{1a}\tilde{\mathbf{v}}_{1a}, \\ \tilde{\mathbf{i}}_{2a} &= \mathcal{Y}^{s_0}_{2a}\tilde{\mathbf{v}}_{2a}, \\ \tilde{\mathbf{v}}_{1b} &= \mathbf{0}, \\ \text{and} \quad \tilde{\mathbf{v}}_{2b} &= \mathbf{0}. \end{aligned}$$

In the above, entries corresponding to $\tilde{\mathbf{i}}_{1a}$ and $\tilde{\mathbf{v}}_{1a}$ (resp. $\tilde{\mathbf{i}}_{1b}$ and $\tilde{\mathbf{v}}_{1b}$; $\tilde{\mathbf{i}}_{2a}$ and $\tilde{\mathbf{v}}_{2a}$; $\tilde{\mathbf{i}}_{2b}$ and $\tilde{\mathbf{v}}_{2b}$) are omitted when $m_{1a} = 0$ (resp. $m_{1a} = m+1-n$; $m_{2a} = 0$; $m_{2a} = n-1$)[17]. Define $\mathcal{M}^{s_0} := \mathcal{Y}^{s_0}_{2a} + \hat{C}^T_{21}\mathcal{Y}^{s_0}_{1a}\hat{C}_{21}$, which is symmetric, and let $\tilde{\mathbf{x}}^T = \begin{bmatrix} \tilde{\mathbf{v}}^T_{2a} & \tilde{\mathbf{i}}^T_{1b} \end{bmatrix}$. Following some rearrangement and elimination, we obtain

$$\begin{bmatrix} \mathcal{M}^{s_0} & -\hat{C}^T_{31} \\ -\hat{C}_{31} & 0_{(m+1-n-m_{1a})\times(m+1-n-m_{1a})} \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} -\hat{C}^T_{11} \\ \mathbf{0}_{m+1-n-m_{1a}} \end{bmatrix} \tilde{i}, \quad (71)$$

$$\text{and} \quad \begin{bmatrix} -\hat{C}_{11} & \mathbf{0}^T_{m+1-n-m_{1a}} \end{bmatrix} \tilde{\mathbf{x}} = \tilde{v}. \quad (72)$$

Now, consider an $\tilde{i} \in \mathbb{C}$, and suppose $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ are two solutions for $\tilde{\mathbf{x}}$ to equation

[17] In the remainder of this proof, we focus on the case $1 \leq m_{1a} < m+1-n$ and $1 \leq m_{2a} < n-1$. The present theorem is still valid in the four exceptional cases $m_{1a} = 0, m_{1a} = m+1-n, m_{2a} = 0$, and $m_{2a} = n-1$, as can be verified by making appropriate modifications to the arguments presented here.

(71). Further, let $\tilde{v}^{(1)}$ and $\tilde{v}^{(2)}$ be the respective solutions for $\tilde{v}$ to equation (72). Since

$$\begin{aligned}\langle \tilde{i}, \overline{\tilde{v}^{(1)} - \tilde{v}^{(2)}} \rangle &= \left\langle \tilde{i}, \overline{\begin{bmatrix} -\hat{C}_{11} & \mathbf{0}^T_{m+1-n-m_{1a}} \end{bmatrix} (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)} \right\rangle \\ &= \left\langle \begin{bmatrix} -\hat{C}^T_{11} \\ \mathbf{0}_{m+1-n-m_{1a}} \end{bmatrix} \tilde{i}, \overline{(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)} \right\rangle \\ &= \left\langle \begin{bmatrix} \mathcal{M}^{s_0} & -\hat{C}^T_{31} \\ -\hat{C}_{31} & 0_{(m+1-n-m_{1a})\times(m+1-n-m_{1a})} \end{bmatrix} \tilde{\mathbf{x}}_1, \overline{(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)} \right\rangle \\ &= \left\langle \tilde{\mathbf{x}}_1, \overline{\begin{bmatrix} \mathcal{M}^{s_0} & -\hat{C}^T_{31} \\ -\hat{C}_{31} & 0_{(m+1-n-m_{1a})\times(m+1-n-m_{1a})} \end{bmatrix} (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)} \right\rangle,\end{aligned}$$

and

$$\begin{bmatrix} \mathcal{M}^{s_0} & -\hat{C}^T_{31} \\ -\hat{C}_{31} & 0_{(m+1-n-m_{1a})\times(m+1-n-m_{1a})} \end{bmatrix} (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2) = \mathbf{0},$$

then $\tilde{i}(\tilde{v}^{(1)} - \tilde{v}^{(2)}) = 0$. Hence, if $\tilde{i} \neq 0$, then $\tilde{v}^{(1)} = \tilde{v}^{(2)}$. Suppose instead that $\tilde{i} = 0$ for all solutions to equation (71). We will show that there is a solution to (71) and (72) with $\tilde{v} \neq 0$ in this case. Given the linearity of equations (71) and (72), this will allow us to conclude that the s_0-driving-point trajectories of a network span a subspace of $\mathbb{C}^2$ of dimension one, which will complete the proof of Lemma 1.7.2.

For a matrix $M \in \mathbb{C}^{r\times r}$, we define the null-space of M as the set $\{\mathbf{x} \in \mathbb{C}^r | M\mathbf{x} = \mathbf{0}\}$, and the range-space of M as the set $\{\mathbf{y} \in \mathbb{C}^r | \mathbf{y} = M\mathbf{z} \text{ for some } \mathbf{z} \in \mathbb{C}^r\}$. As is well known, both the null-space and the range-space of M are linear subspaces of $\mathbb{C}^r$, and the null-space of M and range-space of M^* are orthogonal complements of $\mathbb{C}^r$. Moreover, if M is symmetric, then any real-valued vector in the range-space of M^* is also in the range-space of M. To see this, note that $\mathbf{y} = M^*\mathbf{z}$ with $\mathbf{y} \in \mathbb{R}^r$ and $M = M^T$ implies that $\mathbf{y} = \bar{\mathbf{y}} = M^T\bar{\mathbf{z}} = M\bar{\mathbf{z}}$.

Now, let

$$F := \begin{bmatrix} \mathcal{M}^{s_0} & -\hat{C}^T_{31} \\ -\hat{C}_{31} & 0_{(m+1-n-m_{1a})\times(m+1-n-m_{1a})} \end{bmatrix} \in \mathbb{C}^{(m+1-n-m_{1a}+m_{2a})\times(m+1-n-m_{1a}+m_{2a})},$$

which is symmetric, and let

$$G := \begin{bmatrix} -\hat{C}^T_{11} \\ \mathbf{0}_{m+1-n-m_{1a}} \end{bmatrix} \in \mathbb{R}^{m+1-n-m_{1a}+m_{2a}}.$$

If G is in the range-space of F then, given $\tilde{i} \in \mathbb{C}\backslash 0$, equation (71) may be solved for $\tilde{\mathbf{x}}$, and then equation (71) has a unique solution for $\tilde{v}$ according to the preceding argument.

If all solutions to equation (71) have $\tilde{i} = 0$, then G is not in the range-space of F, and solutions to (71) are those for which $\tilde{\mathbf{x}}$ is in the null-space of F. Since $\tilde{v} = \langle G, \tilde{\mathbf{x}} \rangle$, it follows that if $\tilde{v} = 0$ for all $\tilde{\mathbf{x}}$ in the null-space of F then G is in the range-space of F^*. Since, in addition, G is real-valued and F is symmetric, then this implies that G is in the range-space of F, a contradiction. Hence, when $\tilde{i} = 0$ there must be at least one solution to the equations (71) and (72) with $\tilde{v} \neq 0$. This completes the proof of Lemma 1.7.2. □

Proof of Theorem 1.7.4. In the proof of Lemma 1.7.2, we demonstrated the existence of network trajectories of the form $\Re\left(\tilde{\mathbf{b}} e^{s_0 t}\right)$ where $\tilde{\mathbf{b}}$ takes the form of (68) and at least one of $\tilde{i}$ and $\tilde{v}$ are non-zero. From Theorem 1.6.1, any network trajectory $\hat{\mathbf{b}} \in \mathscr{C}_\infty^{2(m+1)}$ may be written in the form (57) where $\hat{z} \in \mathscr{C}_\infty$, and conditions 1 to 6 in that theorem statement hold. Let us consider the terminology from Theorem 1.5.4. In that theorem, $\hat{z} = \hat{V}_2\left(\frac{d}{dt}\right)\hat{\mathbf{b}}$ (see equation (53)). Hence, if $\hat{\mathbf{b}}(t) = \Re\left(\tilde{\mathbf{b}} e^{s_0 t}\right)$, then from Lemma 1.5.5 we obtain $\hat{z}(t) = \left(\hat{V}_2\left(\frac{d}{dt}\right)\hat{\mathbf{b}}\right)(t) = \Re(\hat{V}_2(s_0)\tilde{\mathbf{b}} e^{s_0 t})$. Writing $\tilde{z} = \hat{V}_2(s_0)\tilde{\mathbf{b}}$, then from Theorem 1.6.1 we conclude that any s_0-driving-point trajectory may be written in the form

$$\begin{bmatrix} \tilde{i} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} q(s_0) \\ p(s_0) \end{bmatrix} f(s_0)\tilde{z},$$

when $s_0 \in \bar{\mathbb{C}}_+$, with the exception of the points $s_0 = j\omega_k$, at which

$$\begin{bmatrix} \tilde{i} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} q(j\omega_k) \\ p(j\omega_k) \end{bmatrix} (f(j\omega_k)\tilde{z} + \alpha_k),$$

and the points $s_0 = -j\omega_k$, at which

$$\begin{bmatrix} \tilde{i} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} q(-j\omega_k) \\ p(-j\omega_k) \end{bmatrix} (f(-j\omega_k)\tilde{z} + \bar{\alpha}_k).$$

In each case, the s_0-driving-point behaviour is a linear subspace of dimension one by Lemma 1.7.2. This subspace corresponds to vectors $\begin{bmatrix} \tilde{i} & \tilde{v} \end{bmatrix}^T$ satisfying $\tilde{i} = 0$ (resp. $\tilde{v} = 0$) if and only if $q(s_0) = 0$ (resp. $p(s_0) = 0$), which implies that $Z(s)$ (resp. $Y(s)$) has a pole at $s = s_0$. If, on the other hand, $q(s_0) \neq 0$ (resp. $p(s_0) \neq 0$), then any s_0-driving-point trajectory $\begin{bmatrix} \tilde{i} & \tilde{v} \end{bmatrix}^T \neq \mathbf{0}$ satisfies $\tilde{v}/\tilde{i} = p(s_0)/q(s_0) = Z(s_0)$ (resp. $\tilde{i}/\tilde{v} = q(s_0)/p(s_0) = Y(s_0)$). This completes the proof of Theorem 1.7.4. □

Note that the s_0-impedance of a network may differ from the value of the network impedance $Z(s)$ at $s = s_0$ for certain $s_0 \in \mathbb{C}_-$. For example, consider again the network N_b in Fig. 7. From the discussion at the beginning of this section, we find that, for $s_0 = -1$, the s_0-driving-point behaviour of this network is spanned by the vector $\begin{bmatrix}-1 & 1\end{bmatrix}^T$. This may alternatively be seen by considering the null-space of the matrix $R(-1)$ for $R(s)$ as in (39). Hence, the s_0-impedance of N_b for $s_0 = -1$ is equal to -1, yet the impedance $Z(s)$ of N_b is everywhere equal to one, so in particular $Z(-1) = 1$. Hence, $Z(s_0)$ differs from the s_0-impedance of N_b at the point $s_0 = -1$.

Thus far, our phasor analysis has focussed on trajectories of the form $\Re\left(\tilde{\mathbf{b}}e^{s_0 t}\right)$ for $s_0 \in \mathbb{C}$. As will be shown in the following section, the arguments may be extended to cover the point at ∞. This will allow us to determine the value of the impedance function $Z(s)$ as $s \to \infty$ (providing $Z(s)$ does not have a pole at $s = \infty$) by finding a solution to a set of linear equations in a similar manner to that of the present section.

1.8 The ∞-trajectories of transformerless networks

In Subsection 1.7.2, we considered those network trajectories in which the current $\hat{i}_k$ and voltage $\hat{v}_k$ in the element N_k took the form $\begin{bmatrix}\hat{i}_k & \hat{v}_k\end{bmatrix}^T(t) = \Re\left(\begin{bmatrix}\tilde{i}_k & \tilde{v}_k\end{bmatrix}^T e^{s_0 t}\right)$, where $\tilde{i}_k, \tilde{v}_k, s_0 \in \mathbb{C}$. In this section, we examine the nature of such trajectories in the limit as $s_0 \to \infty$.

Once again, we will consider the network N described in Section 1.4. Let us consider an $s_0 \in \mathbb{C}$ and an element N_k for some $k \in 1, 2, \ldots, m$. From equation (7), $\begin{bmatrix}\tilde{i}_k(s_0) & \tilde{v}_k(s_0)\end{bmatrix}^T$ is an s_0-driving-point trajectory of N_k if and only if

$$\begin{bmatrix}p_k(s_0) & q_k(s_0)\end{bmatrix}\begin{bmatrix}\tilde{i}_k(s_0) \\ \tilde{v}_k(s_0)\end{bmatrix} = 0. \tag{73}$$

Now, let $a_k \in \mathbb{R}[s]$ have degree equal to the maximum of the degrees of $p_k(s)$ and $q_k(s)$. Then, for $s_0 \in \mathbb{C}$ such that $a_k(s_0) \neq 0$, equation (73) is equivalent to the relationship

$$\begin{bmatrix}\frac{p_k(s_0)}{a_k(s_0)} & \frac{q_k(s_0)}{a_k(s_0)}\end{bmatrix}\begin{bmatrix}\tilde{i}_k \\ \tilde{v}_k\end{bmatrix} = 0.$$

Accordingly, we define the ∞-driving-point trajectory of the element N_k as the kernel of the vector

$$\lim_{s_0 \to \infty}\begin{bmatrix}\frac{p_k(s_0)}{a_k(s_0)} & \frac{q_k(s_0)}{a_k(s_0)}\end{bmatrix}.$$

We then define an ∞-trajectory, an ∞-driving-point trajectory, the ∞-behaviour, and the ∞-driving-point behaviour of a network N as follows.

Definition 1.8.1 (∞-trajectory/ behaviour, ∞-driving-point trajectory/ behaviour)**.** Let N be a transformerless network and let N comprise the elements $N_1, N_2, \ldots, N_m$. Let $\tilde{\mathbf{b}}$ be as in (65), and let $\tilde{\mathbf{i}} := \begin{bmatrix}-\tilde{i} & \tilde{i}_1 & \ldots & \tilde{i}_m\end{bmatrix}^T$ and $\tilde{\mathbf{v}} := \begin{bmatrix}\tilde{v} & \tilde{v}_1 & \ldots & \tilde{v}_m\end{bmatrix}^T$. We call $\tilde{\mathbf{b}}$ an ∞-trajectory of N if $\tilde{\mathbf{i}}$ satisfies Kirchhoff's current law for N, $\tilde{\mathbf{v}}$ satisfies Kirchhoff's voltage law for N, and $\begin{bmatrix}\tilde{i}_k & \tilde{v}_k\end{bmatrix}$ is an ∞-driving-point trajectory for the element N_k ($k = 1, 2, \ldots, m$). If $\tilde{\mathbf{b}}$ is an ∞-trajectory of N then we call $\tilde{\mathbf{d}} := \begin{bmatrix}I_2 & 0_{2\times 2m}\end{bmatrix}\tilde{\mathbf{b}}$ the corresponding ∞-driving-point trajectory of N. Furthermore, we call the set of all ∞-trajectories (resp. ∞-driving-point trajectories) of N the ∞-behaviour (resp. ∞-driving-point behaviour) of N.

We may then show the following lemma:

Lemma 1.8.2. *The ∞-driving-point behaviour of a transformerless network is a linear subspace of $\mathbb{C}^2$ of dimension one.*

Proof. This follows by a similar proof to that of Lemma 1.7.2. In this case, the elements $N_{m_{1a}+1}, \ldots, N_{m+1-n}$, and the elements $N_{m+2-n+m_{2a}}, \ldots, N_m$, correspond to those elements whose admittance has a pole at ∞. Furthermore, for $k = 1, 2, \ldots, m_{1a}$, and for $k = m+2-n, m+3-n, \ldots, m+m_{2a}+1-n$, we replace $Y_k^{s_0}$ with $\lim_{s\to\infty} Y_k(s)$, where $Y_k(s)$ is the admittance of N_k. □

It follows that we may define the ∞-impedance and ∞-admittance by substituting ∞ for s_0 in definition 1.7.3. Finally, we obtain the following theorem.

Theorem 1.8.3. *Theorem 1.7.4 holds in the case $s_0 = \infty$.*

Proof. Consider the network N described in Section 1.4, and let $Z(s)$ be the impedance, and $Y(s)$ the admittance, of N. Moreover, let

$$\hat{R}(s) := \begin{bmatrix} 0 & 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{0}_{2m} & \hat{A}\,(s) \end{bmatrix},$$

where

$$\hat{A}(s) := \begin{bmatrix} \hat{P}_1(s) & 0_{(m+1-n)\times(n-1)} & -\hat{Q}_1(s) & 0_{(m+1-n)\times(n-1)} \\ 0_{(n-1)\times(m+1-n)} & \hat{P}_2(s) & 0_{(n-1)\times(m+1-n)} & -\hat{Q}_2(s) \\ 0_{(m+1-n)\times(m+1-n)} & 0_{(m+1-n)\times(n-1)} & I_{m+1-n} & \hat{C}_2 \\ -\hat{C}_2^T & I_{n-1} & 0_{(n-1)\times(m+1-n)} & 0_{(n-1)\times(n-1)} \end{bmatrix},$$

$$\hat{P}_1(s) := \text{diag}\left(\hat{p}_1(s), \hat{p}_2(s), \ldots, \hat{p}_{m+1-n}(s)\right),$$
$$\hat{Q}_1(s) := \text{diag}\left(\hat{q}_1(s), \hat{q}_2(s), \ldots, \hat{q}_{m+1-n}(s)\right),$$
$$\hat{P}_2(s) := \text{diag}\left(\hat{p}_{m+2-n}(s), \hat{p}_{m+3-n}(s), \ldots, \hat{p}_m(s)\right),$$
$$Q_2(s) := \text{diag}\left(\hat{q}_{m+2-n}(s), \hat{q}_{m+3-n}(s), \ldots, \hat{q}_m(s)\right),$$

and $\hat{p}_k(s) := p_k(s)/a_k(s)$, $\hat{q}_k(s) := q_k(s)/a_k(s)$ for $k = 1, 2, \ldots, m$, where $a_k \in \mathbb{R}[s]$ has degree equal to the maximum of the degrees of $p_k(s)$ and $q_k(s)$. The remaining terms are as defined in Section 1.4. Then, from the proof of Theorem 1.6.1, $\hat{R}(s)\mathbf{b}(s) = \mathbf{0}$, where

$$\mathbf{b}(s) := \begin{bmatrix} \frac{f(s)q(s)}{a(s)} & \frac{f(s)p(s)}{a(s)} & \frac{1}{a(s)}\mathbf{h}(s)^T \end{bmatrix}^T,$$

$f(s)$, $q(s)$ and $\mathbf{h}(s)$ are as defined in Theorem 1.6.1, and $a \in \mathbb{R}[s]$ has degree equal to the maximum of the degrees of $f(s)p(s)$ and $f(s)q(s)$. In particular, $Z(s) = p(s)/q(s) = (f(s)p(s)/a(s))/(f(s)q(s)/a(s))$, and $Y(s) = 1/Z(s) = (f(s)q(s)/a(s))/(f(s)p(s)/a(s))$. Then, by condition 4 of Theorem 1.6.1, $\lim_{s\to\infty} \mathbf{b}(s)$ is finite. Hence, $\lim_{s\to\infty} \hat{R}(s)\mathbf{b}(s) = \lim_{s\to\infty} \hat{R}(s) \lim_{s\to\infty} \mathbf{b}(s) = \mathbf{0}$, and so $\lim_{s\to\infty} \mathbf{b}(s)$ is an ∞-trajectory of N. It follows that

$$\begin{bmatrix} \tilde{i} \\ \tilde{v} \end{bmatrix} = \lim_{s\to\infty} \begin{bmatrix} \frac{f(s)q(s)}{a(s)} \\ \frac{f(s)p(s)}{a(s)} \end{bmatrix}$$

must be an ∞-driving-point trajectory of N. Here, $\tilde{i} = 0$ (resp. $\tilde{v} = 0$) if and only if $\deg(q(s)) < \deg(p(s))$ (resp. $\deg(p(s)) < \deg(q(s))$), which implies that $Z(s)$ (resp. $Y(s)$) has a pole at $s = \infty$, and otherwise $\tilde{v}/\tilde{i} = \lim_{s\to\infty} Z(s)$ (resp. $\tilde{i}/\tilde{v} = \lim_{s\to\infty} Y(s)$). It is then immediate from the definition of the ∞-impedance (resp. ∞-admittance) that $\tilde{i} = 0$ (resp. $\tilde{v} = 0$) if and only if the ∞-impedance (resp. ∞-admittance) of N does not exist, otherwise the ∞-impedance (resp. ∞-admittance) of N is equal to $\tilde{v}/\tilde{i}$ (resp. $\tilde{i}/\tilde{v}$). □

This concludes our phasor analysis of transformerless networks. We have shown how, for a given transformerless network N with impedance $Z(s)$, and a given $s_0 \in \bar{\mathbb{C}}_+ \cup \infty$, the existence and value of $\lim_{s\to s_0} Z(s)$ may be determined by finding a solution to a set of linear equations in a similar manner to the conventional phasor approach to network analysis described in [36, Section 5.4]. These linear equations were obtained from the constraints due to the individual elements in the network, and from the constraints due to their interconnection (Kirchhoff's current and voltage laws).

1.9 Transformerless networks and graphs of one-ports

For the purposes of analysis, it is often useful to view a passive network as an interconnection of various subnetworks, and especially one-port subnetworks. The following definition formalises the notion of subnetworks and one-port subnetworks of a given network.

Definition 1.9.1 (Subnetworks, One-port subnetworks)**.**
Let N be a transformerless network with a source attached between the driving-point terminals as in Fig. 4. We call N_k a *subnetwork* in N if it is an interconnection of passive elements within N (which may connect to the rest of N at any number of terminals). If N_k connects to the rest of N at exactly two terminals, then we call N_k a *one-port subnetwork* in N (*one-port* for short), and we call the two terminals at which N_k connects to the rest of N the driving-point terminals of N_k.

A one-port subnetwork of a transformerless network is itself a transformerless network, and so it may be associated with a current and a voltage which lie in the driving-point behaviour of that transformerless network. This motivates the following definition of a graph of one-ports.

Definition 1.9.2 (Graph of one-ports)**.**
A *graph of one-ports* is a network graph in which the kernel of the differential operator $\begin{bmatrix} -p_k\left(\frac{d}{dt}\right) & q_k\left(\frac{d}{dt}\right) \end{bmatrix}$ is the driving-point behaviour of a transformerless network (see Theorem 1.4.1). We will use the words one-port and edge interchangeably when referring to the edges in a graph of one-ports.

We summarise certain connections between transformerless networks and graphs of one-ports in the following remarks:

Remark 1.9.3.
A transformerless network may be associated with a number of different graphs of one-ports. These correspond to any graph which is formed by partitioning the elements in the network into one-port subnetworks and identifying each such one-port with an individual edge. This is perhaps best illustrated by an example. In the network on the top left of Fig. 8, the resistor connected in parallel with the series connected inductor and capacitor in the top right of this network forms a one-port subnetwork. It is then clear that this network has the form of the network in the centre top location of Fig. 8, in which this one-port is denoted by the network N_1. The network N_1 is then shown on

the top right of Fig. 8. Underneath each of these networks we show the corresponding graphs of one-ports.

Remark 1.9.4.
We define a trajectory, a driving-point trajectory, the behaviour, and the driving-point behaviour of a graph of one-ports by analogy with their definitions for a transformerless network. Likewise, for $s_0 \in \mathbb{C}$, we may define an s_0-trajectory, an s_0-driving-point-trajectory, the s_0-behaviour, and the s_0-driving-point behaviour by analogy with their definitions for a transformerless network. Indeed, the proofs presented in Sections 1.7 and 1.8 remain valid if we replace the words 'transformerless network' with 'graph of one-ports' throughout.

Remark 1.9.5.
Following the proof of Lemma 1.2.3, it may be seen that the driving-point behaviour of a graph of one-ports is unchanged by the removal of those one-ports which are not in the biconnected component containing the source. Moreover, it is straightforward to verify that, providing the graph of elements associated with a given transformerless network is biconnected, any graph of one-ports associated with that transformerless network will also be biconnected. Similarly to Remark 1.2.4, it follows that no generality is lost in assuming any graph of one-ports associated with a transformerless network is biconnected.

As may be expected, the driving-point behaviour (resp. s_0-driving-point behaviour) for a network is the same as the driving-point behaviour (resp. s_0-driving-point behaviour) for any graph of one-ports corresponding to that network. This is made formal in the

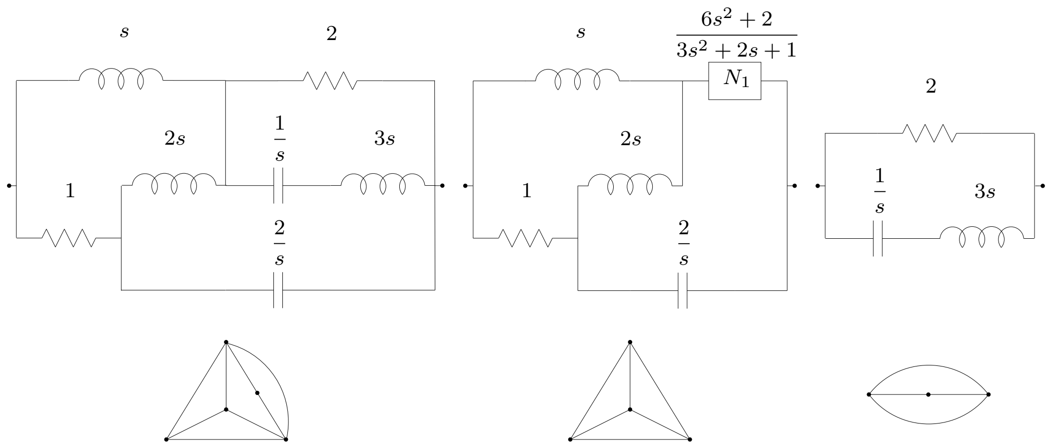


Figure 8: Example of a graph of one-ports.

following lemma:

Lemma 1.9.6. *Let N be a transformerless network, and let G be a graph of one-ports corresponding to N. Then $\hat{\mathbf{d}}$ (resp. $\tilde{\mathbf{d}}$) is a driving-point trajectory (resp. s_0-driving-point trajectory) of G if and only if $\hat{\mathbf{d}}$ (resp. $\tilde{\mathbf{d}}$) is a driving-point trajectory (resp. s_0-driving-point trajectory) of N.*

Proof. Consider the network N described at the end of Section 1.1, and let $\tilde{N}$ be a one-port in N comprising the elements $N_{m-m_1+1}, N_{m-m_1+2}, \ldots, N_m$. Further, let $\hat{G}$ be the graph of one-ports corresponding to N whose edges correspond to the one-ports $N_1, N_2, \ldots N_{m-m_1}, \tilde{N}$. We will show that $\hat{\mathbf{d}}$ is a driving-point trajectory of $\hat{G}$ if and only if it is a driving-point trajectory of N. It may similarly be shown that $\tilde{\mathbf{d}}$ is an s_0-driving-point trajectory of $\hat{G}$ if and only if it is an s_0-driving-point trajectory of N. The lemma will then follow from induction.

Firstly, let $\bar{G}$ be the graph of elements corresponding to the network N. Secondly, let $\tilde{G}$ be the graph of elements corresponding to the one-port $\tilde{N}$, let $\tilde{G}$ contain n_1 vertices, and let the edges corresponding to the elements $N_{m-m_1+1}, \ldots, N_m$ have the same orientation in $\bar{G}$ and $\tilde{G}$. Thirdly, let the edges corresponding to the elements $N_1, N_2, \ldots, N_{m-m_1}$ have the same orientation in $\bar{G}$ and $\hat{G}$. Finally, denote the two vertices corresponding to the terminals at which the one-port $\tilde{N}$ connects to the rest of N by x_1 and x_2 in the same manner in $\bar{G}$, $\tilde{G}$, and $\hat{G}$, and orient the source in $\tilde{G}$, and the edge corresponding to the one-port $\tilde{N}$ in $\hat{G}$, from x_1 to x_2. Since $\bar{G}$ (resp. $\hat{G}$; $\tilde{G}$) contains $m+1$ (resp. $m-m_1+2$; m_1+1) edges and n (resp. $n-n_1+2$; n_1) vertices, then the cut-set space of $\bar{G}$ (resp. $\hat{G}$; $\tilde{G}$) has dimension $n-1$ (resp. $n-n_1+1$; n_1-1), and the circuit space of $\bar{G}$ (resp. $\hat{G}$; $\tilde{G}$) has dimension $m-n+2$ (resp. $m-m_1-n+n_1+1$; m_1-n_1+2). We now consider a basis for each of these spaces.

For the cut-set space, we consider a so-called *node-incidence* basis. For a graph with $\hat{n}$ vertices $x_1, x_2, \ldots, x_{\hat{n}}$, this corresponds to cut-set vectors $\mathbf{a}^{(k)}$ for $k = 1, 2, \ldots, \hat{n}-1$ which correspond to partitions in which $V^{(2)}$ contains vertex x_{k+1}, and $V^{(1)}$ contains all other vertices in the network. It may be shown that these vectors are a basis for the cut-set space of this graph. We call the matrix whose kth row is the vector $\mathbf{a}^{(k)}$ a *node-incidence matrix* for the graph. Now, let $\hat{\mathbf{b}}$ in (4) be a trajectory of N, and let $\hat{\mathbf{i}}_1 := \begin{bmatrix} -\hat{i} & \hat{i}_1 & \ldots & \hat{i}_{m-m_1} \end{bmatrix}^T$, $\hat{\mathbf{i}}_2 := \begin{bmatrix} \hat{i}_{m-m_1+1} & \ldots & \hat{i}_m \end{bmatrix}^T$, $\hat{\mathbf{v}}_1 := \begin{bmatrix} \hat{v} & \hat{v}_1 & \ldots & \hat{v}_{m-m_1} \end{bmatrix}^T$, and $\hat{\mathbf{v}}_2 := \begin{bmatrix} \hat{v}_{m-m_1+1} & \ldots & \hat{v}_m \end{bmatrix}^T$. Since x_1 and x_2 are the two vertices at which $\tilde{N}$ connects to the rest of N, then Kirchhoff's current law implies

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0_{(n-n_1)\times m_1} \\ 0_{n_1-2\times m-m_1+1} & A_{32} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \end{bmatrix} = \mathbf{0}, \tag{74}$$

where

$$A_{1a} := \begin{bmatrix} A_{21} & \mathbf{0}_{n-n_1} \\ A_{11} & 1 \end{bmatrix},$$

is a node-incidence matrix for $\hat{G}$, and

$$A_{1b} := \begin{bmatrix} 1 & A_{12} \\ \mathbf{0}_{n_1-2} & A_{32} \end{bmatrix},$$

is a node-incidence matrix for $\tilde{G}$.

Let us now consider the circuit space of $\bar{G}$. First, consider the graph $\tilde{G}$ with the edge representing the source removed. From this graph, we obtain $m_1 + 1 - n_1$ independent circuit vectors for the graph $\tilde{G}$ with each vector having a zero entry for the source. Similarly, consider the graph obtained by removing the edge corresponding to the one-port $\tilde{N}$ from the graph $\hat{G}$. In this case, we can obtain $m - m_1 - n + n_1$ independent circuit vectors for the network $\hat{G}$ with each vector having a zero entry for the edge corresponding to the one-port $\tilde{N}$. Taken together, these vectors constitute $m + 1 - n$ basis vectors for the circuit space of $\bar{G}$. For the final basis vector, we take a circuit formed from the union of a path in $\hat{G}$ from vertex x_1 to x_2 which does not contain the edge corresponding to the one-port $\tilde{N}$, and a path in $\tilde{G}$ from vertex x_2 to x_1 which does not contain the source. Then, Kirchhoff's voltage law implies

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & 0_{(m-m_1-n+n_1)\times m_1} \\ 0_{(m_1+1-n_1)\times(m-m_1+1)} & C_{32} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix} = \mathbf{0}, \tag{75}$$

where

$$C_{1a} := \begin{bmatrix} C_{21} & \mathbf{0}_{m-m_1-n+n_1} \\ C_{11} & -1 \end{bmatrix},$$

is a circuit matrix for $\hat{G}$ with full row rank, and

$$C_{1b} := \begin{bmatrix} 1 & C_{12} \\ \mathbf{0}_{m_1+1-n_1} & C_{32} \end{bmatrix},$$

is a circuit matrix for $\tilde{G}$ with full row rank.

Now, let $\hat{i}_{\tilde{N}} := -A_{11}\hat{\mathbf{i}}_1 = A_{12}\hat{\mathbf{i}}_2$. Then, equation (74) is equivalent to

$$A_{1a}\begin{bmatrix}\hat{\mathbf{i}}_1\\ \hat{i}_{\tilde{N}}\end{bmatrix} = \mathbf{0}, \tag{76}$$

$$\text{and} \qquad A_{1b}\begin{bmatrix}-\hat{i}_{\tilde{N}}\\ \hat{\mathbf{i}}_2\end{bmatrix} = \mathbf{0}. \tag{77}$$

Moreover, let $\hat{v}_{\tilde{N}} := C_{11}\hat{\mathbf{v}}_1 = -C_{12}\hat{\mathbf{v}}_2$. Then equation (75) is equivalent to

$$C_{1a}\begin{bmatrix}\hat{\mathbf{v}}_1\\ \hat{v}_{\tilde{N}}\end{bmatrix} = \mathbf{0}, \tag{78}$$

$$\text{and} \qquad C_{1b}\begin{bmatrix}\hat{v}_{\tilde{N}}\\ \hat{\mathbf{v}}_2\end{bmatrix} = \mathbf{0}. \tag{79}$$

Since $\begin{bmatrix}\hat{i}_k & \hat{v}_k\end{bmatrix}^T$ is a driving-point trajectory for the element N_k ($k = 1, 2, \ldots, m$), then equations (77) and (79) imply that $\begin{bmatrix}\hat{i}_{\tilde{N}} & \hat{v}_{\tilde{N}}\end{bmatrix}^T$ is a driving-point trajectory for the one-port $\tilde{N}$. Then, equations (76) and (78) together imply that the vector

$$\begin{bmatrix}-\hat{i} & \hat{v} & \hat{i}_1 & \ldots & \hat{i}_{m-m_1} & \hat{i}_{\tilde{N}} & \hat{v}_1 & \ldots & \hat{v}_{m-m_1} & \hat{v}_{\tilde{N}}\end{bmatrix}^T$$

is a trajectory for $\hat{G}$. Hence, $\hat{\mathbf{d}}$ is a driving-point trajectory of N if and only if it is a driving-point trajectory of $\hat{G}$. That $\tilde{\mathbf{d}}$ is an s_0-driving-point trajectory of N if and only if it is an s_0-driving-point trajectory of $\hat{G}$ may similarly be seen. The present lemma then follows by induction. □

Combining Remark 1.9.4 with Lemma 1.9.6, we obtain the following theorem:

Theorem 1.9.7. *Let N be a transformerless network with impedance $Z(s)$, and let G be a graph of one-ports corresponding to N. Further, let $\tilde{\mathbf{b}}$ in (65) be an s_0-trajectory of G for some $s_0 \in \bar{\mathbb{C}}_+ \cup \infty$, with $\tilde{\mathbf{d}} = \begin{bmatrix}\tilde{i} & \tilde{v}\end{bmatrix}^T := \begin{bmatrix}I_2 & 0_{2\times 2m}\end{bmatrix}\tilde{\mathbf{b}} \neq \mathbf{0}$ the corresponding s_0-driving-point trajectory. Then $\tilde{i} = 0$ if and only if $Z(s)$ has a pole at $s = s_0$, with $Z(s_0) = \tilde{v}/\tilde{i}$ whenever $\tilde{i} \neq 0$.*

Part 2

Algebraic criteria for network realisations

A fundamental result in passive network synthesis states that the McMillan degree of a passive network's impedance is less than or equal to the number of reactive elements in the network. When these numbers are equal, we will call the network minimally-reactive. In this part, we derive constraints on the impedance of a reciprocal network (a network containing only resistors, reactive elements, and transformers) according to the individual numbers of inductors and capacitors the network contains. As will be shown, this problem is closely related to the classical algebraic problem of finding the number of common roots of two polynomials. This connection will be established by considering the reactance extraction technique, first proposed in [20]. The connection leads to explicit algebraic conditions on the parameters of a reciprocal network's impedance function according to the numbers of capacitors and inductors contained within that network. The results will be expressed in terms of a Hankel, a Sylvester, and a Bezoutian matrix corresponding to the network's impedance. An interpretation of these results will also be given in terms of a generalisation of the Cauchy index for the impedance function, which we call the extended Cauchy index. Moreover, we will establish a relationship between the extended Cauchy index and the properties of continued fraction expansions for real-rational functions, which has implications for network realisation procedures.

The content in Sections 2.1 to 2.6 of this part appeared in our paper [37]. We have maintained the structure of the argument which appeared in that paper, starting with a simple special case in Section 2.1, and building to the general result in Section 2.6. The material on continued fraction expansions in Sections 2.7 and 2.8 did not appear in that paper.

In Section 2.1, we consider one-port minimally-reactive reciprocal networks with proper impedance functions, and we relate the numbers of capacitors and inductors in the network to the numbers of positive and negative eigenvalues of the Hankel matrix for this impedance function, respectively. The results are then extended to non-proper impedance functions in Sections 2.2 and 2.3, where they are presented in terms of a Sylvester matrix associated with the impedance. Moreover, the extended Cauchy index is defined for a scalar real-rational function, and equivalent results are derived in terms of this extended Cauchy index. In Section 2.4, we introduce a Bezoutian matrix for a scalar real-rational function, and we present equivalent results to those in the preceding sections in terms of this matrix. Then, in Section 2.5, we further extend the results to include reciprocal networks which are not minimally-reactive. We show how the results generalise to the case of multi-port reciprocal networks in Section 2.6, where they are presented in terms of a Bezoutian matrix for a matrix real-rational function, and also in terms of a matrix extended Cauchy index.

We discuss the relationship between the extended Cauchy index and the properties of continued fraction expansions of real-rational functions in Section 2.7. There, this relationship is used to derive explicit algebraic formulae for the element parameters in the Cauer form networks in terms of the overall network impedance. Finally, we describe the implications of the results for the realisation of biquadratic impedance functions in Section 2.8.

2.1 Reactance extraction and the Hankel matrix

In this section, we consider one-port minimally-reactive reciprocal networks with proper impedance functions. By considering the reactance extraction technique outlined in [28, Chapter 4], we show how the number of capacitors (resp. inductors) in a one-port minimally-reactive reciprocal network whose impedance function is proper is equal to the number of positive (resp. negative) eigenvalues of the Hankel matrix for the network impedance. We recall from Section 0.3 that we denote the set of proper real-rational functions by $\mathbb{R}_p(s)$.

Let us consider a function $Z \in \mathbb{R}_p(s)$ with $\delta(Z(s)) = n$. Suppose $Z(s)$ is the impedance of a one-port network N which contains exactly p inductors and q capacitors and is minimally-reactive, so $p + q = n$. Using the procedure of reactance extraction [20], N takes the form of Fig. 9, where the subnetwork N_r possesses a hybrid matrix M such that

$$\begin{bmatrix} v \\ \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} i \\ \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix}, \tag{80}$$

where $\mathbf{i_a} := [i_1, \ldots, i_p]^T$ is the vector of (Laplace-transformed) currents through the inductors in N with corresponding voltages $\mathbf{v_a}$, $\mathbf{v_b} = [v_{p+1}, \ldots, v_{p+q}]^T$ is the vector of (Laplace-transformed) voltages across the capacitors in N with corresponding currents

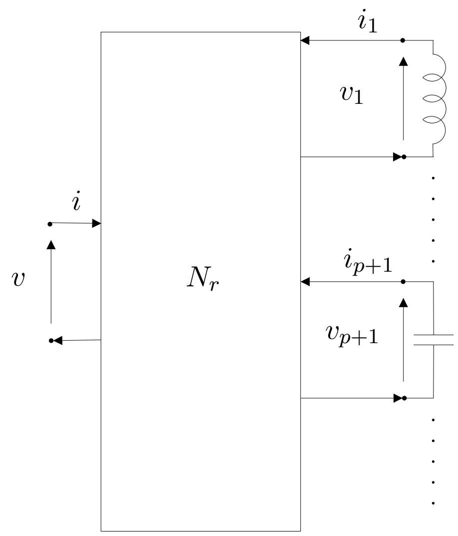


Figure 9: One-port network N with reactive elements extracted.

$\mathbf{i_b}$, and the matrix M is partitioned compatibly with the pertinent vectors. The existence of a hybrid matrix in the form (80) follows from [28, Section 4.4] and is discussed in greater detail in Section 2.5. Since N_r is a reciprocal network then, by [28, Theorem 2.8.1],

$$(1 \dotplus \Sigma)\, M = M^T\, (1 \dotplus \Sigma)\,, \tag{81}$$

where $\Sigma := (I_p \dotplus (-I_q))$. We recall that throughout this thesis, $\dotplus$ indicates the direct sum of two matrices. In other words, $I_p \dotplus (-I_q)$ is the block diagonal matrix with diagonal blocks of I_p and $-I_q$. When terminated on the reactive elements, we have

$$\begin{bmatrix} \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = -s\Lambda \begin{bmatrix} \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix},$$

where $\Lambda := \mathrm{diag}\,(L_1, \ldots, L_p, C_1, \ldots, C_q)$. Here, L_j indicates the inductance of the jth inductor, and C_k the capacitance of the kth capacitor $(j = 1, 2, \ldots, p,\ k = 1, 2, \ldots, q)$ Then it can readily be seen that $Z(s) = J + H(sI - F)^{-1}G$ where

$$F = -\Lambda^{-1} \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{82}$$

$$G = -\Lambda^{-1} \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} \in \mathbb{R}^{n \times 1}, \tag{83}$$

$$H = \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \in \mathbb{R}^{1 \times n}, \tag{84}$$

$$\text{and} \qquad J = M_{11} \in \mathbb{R}, \tag{85}$$

and, since $\Sigma^2 = I_n$, and Σ and Λ are both diagonal, from (81) we have

$$\Sigma\Lambda F = F^T \Sigma\Lambda, \tag{86}$$

$$\text{and} \qquad H^T = -\Sigma\Lambda G. \tag{87}$$

Consider now the controllability and observability matrices

$$V_c := \left[G, FG, \ldots, F^{n-1}G\right], \tag{88}$$

$$\text{and} \qquad V_o := [H^T, F^T H^T, \ldots, \left(F^T\right)^{n-1} H^T]^T. \tag{89}$$

Since $\delta\,(Z(s)) = n$, the state-space realisation (82)-(85) must be controllable and observable, and hence V_o and V_c both have rank n [20, Section 3]. Furthermore, from (86) and (87) we have

$$V_o = -V_c^T \Lambda\Sigma. \tag{90}$$

We introduce the Hankel matrix

$$\mathcal{H}_n := V_o V_c = \begin{bmatrix} h_0 & h_1 & \dots & h_{n-1} \\ h_1 & h_2 & \dots & h_n \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_n & \dots & h_{2n-2} \end{bmatrix}, \tag{91}$$

where $h_i = HF^iG$ for $i = 0, 1, 2, \dots$ are the Markov parameters, which are also directly defined from the Laurent expansion

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \dots, \tag{92}$$

in which $h_{-1} = J$. It follows from (90) that

$$\mathcal{H}_n = V_c^T \left(-\Lambda\Sigma\right) V_c. \tag{93}$$

From (93) and Sylvester's law of inertia [38, Section 1], we deduce the following:

Theorem 2.1.1. *Let $Z \in \mathbb{R}_p(s)$ with $\delta\left(Z(s)\right) = n$, and let $\mathcal{H}_n$ be as in (91) for $Z(s)$ as in (92). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p + q = n$, then $\pi(\mathcal{H}_n) = q$ and $\nu(\mathcal{H}_n) = p$.*

Finally in this section, we define the infinite Hankel matrix

$$\mathcal{H} := \begin{bmatrix} h_0 & h_1 & h_2 & \dots \\ h_1 & h_2 & h_3 & \dots \\ h_2 & h_3 & h_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{94}$$

and the corresponding finite Hankel matrices

$$\mathcal{H}_k := \begin{bmatrix} h_0 & h_1 & \dots & h_{k-1} \\ h_1 & h_2 & \dots & h_k \\ \vdots & \vdots & \ddots & \vdots \\ h_{k-1} & h_k & \dots & h_{2k-2} \end{bmatrix}, \tag{95}$$

for $k = 1, 2, \dots$. From [39, p. 206-7], $\mathcal{H}$ has finite rank equal to n and $|\mathcal{H}_n| \neq 0$. Moreover, from (93) and [35, Theorem 24, p. 343] we have the following:

Theorem 2.1.2. *Let $Z \in \mathbb{R}_p(s)$ with $\delta\left(Z(s)\right) = n$, and let $\mathcal{H}_k$ be as in (95) for $Z(s)$ as* *in (92). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q=n$, then $|\mathcal{H}_n| \neq 0$, $|\mathcal{H}_k| = 0$ for $k > n$, and*

$$q = \mathbf{P}(1, |\mathcal{H}_1|, \ldots, |\mathcal{H}_n|), \tag{96}$$

$$\textit{and} \qquad p = \mathbf{V}(1, |\mathcal{H}_1|, \ldots, |\mathcal{H}_n|). \tag{97}$$

In any sub-sequence of zero values ($|\mathcal{H}_k| \neq 0$, $|\mathcal{H}_{k+1}| = |\mathcal{H}_{k+2}| = \ldots = 0$), signs are assigned to the zero values as follows: $\text{sign}\,(|\mathcal{H}_{k+j}|) = (-1)^{\frac{j(j-1)}{2}} \text{sign}\,(|\mathcal{H}_k|)$.

Theorem 2.1.2 presents algebraic constraints on those proper impedance functions which can be realised by minimally-reactive reciprocal networks according to the individual numbers of capacitors and inductors contained in the network.

2.2 The Cauchy index and the Sylvester matrix

Here, we define the Cauchy index of a real-rational function, in addition to a Sylvester matrix corresponding to the real-rational function. We then state equivalent results to those of the preceding section in terms of this Cauchy index and Sylvester matrix.

The *Cauchy index* of $F \in \mathbb{R}(s)$ between limits $-\infty$ and $+\infty$, denoted $I_{-\infty}^{+\infty}F(s)$, is the difference between (a) the number of jumps of $F(s)$ from $-\infty$ to $+\infty$, and (b) the number of jumps of $F(s)$ from $+\infty$ to $-\infty$, as s is increased in $\mathbb{R}$ from $-\infty$ to $+\infty$. From [39, Theorem 9, p. 210], if $F \in \mathbb{R}_p(s)$, then $I_{-\infty}^{+\infty}F(s)$ is equal to the signature of the corresponding Hankel matrix. From Theorem 2.1.1 we can deduce the following:

Theorem 2.2.1. *Let $Z \in \mathbb{R}_p(s)$ be the impedance of a reciprocal network containing exactly p inductors and q capacitors and with $p+q = \delta\,(Z(s))$. Then*

$$q - p = I_{-\infty}^{+\infty}Z(s).$$

We now write

$$Z(s) = \frac{a(s)}{b(s)} =: \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}, \text{ with at least one of } a_n, b_n > 0. \tag{98}$$

Associated with $Z(s)$ are the matrices

$$\mathcal{S}_j := \left.\overbrace{\begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \cdots \\ a_n & a_{n-1} & a_{n-2} & \cdots \\ 0 & b_n & b_{n-1} & \cdots \\ 0 & a_n & a_{n-1} & \cdots \\ 0 & 0 & b_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{j \text{ columns}}\right\} j \text{ rows} \tag{99}$$

for $j = 1, 2, \ldots$. Now suppose $Z(s)$ is proper and the Laurent series for $Z(s)$ about the point at ∞ is as in (92). Then, equating with (98), multiplying both sides by $b(s)$, and equating coefficients of s, we obtain

$$\begin{aligned} h_{-1}b_n &= a_n, \\ h_{-1}b_{n-1} + h_0 b_n &= a_{n-1}, \\ &\vdots \\ h_{-1}b_0 + h_0 b_1 + \ldots + h_{n-2}b_{n-1} + h_{n-1}b_n &= a_0, \\ \text{and} \qquad h_r b_0 + h_{r+1}b_1 + \ldots + h_{r+n-1}b_{n-1} + h_{r+n}b_n &= 0, \quad (r = 0, 1, \ldots). \end{aligned}$$

Following [39, p. 214], we observe that $\mathcal{S}_{2k} = \Gamma_{2k} U_{2k}$ where

$$\Gamma_{2k} := \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ h_{-1} & h_0 & \ldots & h_{k-2} & h_{k-1} & \ldots & h_{2k-2} \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & h_{-1} & \ldots & h_{k-3} & h_{k-2} & \ldots & h_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & h_{-1} & h_0 & \ldots & h_{k-1} \end{bmatrix},$$

$$\text{and} \qquad U_{2k} := \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \ldots & b_{n-2k+1} \\ 0 & b_n & b_{n-1} & \ldots & b_{n-2k+2} \\ 0 & 0 & b_n & \ldots & b_{n-2k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b_n \end{bmatrix}.$$

Since a sequence of $k(k-1)$ pairwise row permutations carries Γ_{2k} into a block lower triangular matrix with diagonal blocks I_k and $\mathcal{H}_k$, then

$$|\mathcal{S}_{2k}| = b_n^{2k}|\mathcal{H}_k|. \tag{100}$$

It may be observed that $|\mathcal{S}_{2n}|$ is the Sylvester resultant of $a(s)$ and $b(s)$, which is well known to be non-zero if and only if $a(s)$ and $b(s)$ are coprime. Accordingly, we will refer to the matrices $\mathcal{S}_j$ in (99) as *Sylvester matrices.* If $Z \in \mathbb{R}_p(s)$, then $b_n \neq 0$, and from (100) and Theorem 2.1.2 we obtain the following:

Theorem 2.2.2. *Let $Z \in \mathbb{R}_p(s)$ with $\delta\left(Z(s)\right) = n$, and let $|\mathcal{S}_j|$ $(j = 1, 2, \ldots)$ be as in (99) for $Z(s)$ as in (98). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q = n$, then $|\mathcal{S}_{2n}| \neq 0$, $|\mathcal{S}_{2k}| = 0$ for $k > n$, and*

$$q = \mathbf{P}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2n}|),$$
$$\textit{and} \qquad p = \mathbf{V}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2n}|),$$

where, in any sub-sequence of zero values ($|\mathcal{S}_{2k}| \neq 0$, $|\mathcal{S}_{2(k+1)}| = |\mathcal{S}_{2(k+2)}| = \ldots = 0$), signs are assigned to the zero values as follows: $\text{sign}\left(|\mathcal{S}_{2(k+j)}|\right) = (-1)^{\frac{j(j-1)}{2}}\text{sign}\left(|\mathcal{S}_{2k}|\right)$.

We remark that Theorem 2.2.2 still holds when the polynomials $a(s)$ and $b(s)$ in (98) are not coprime providing we replace n with $r := \delta\left(a(s)/b(s)\right)$ in the above theorem statement. Indeed the conditions $|\mathcal{S}_{2r}| \neq 0$ and $|\mathcal{S}_{2k}| = 0$ for all $k > r$ hold if and only if the function $Z(s)$ in (98) has $\delta\left(Z(s)\right) = r$, or equivalently the polynomials $a(s)$ and $b(s)$ have exactly $n - r$ roots in common[18].

In contrast to Theorem 2.1.2, the algebraic criteria presented in Theorem 2.2.2 are directly in terms of the parameters in the impedance function, and so have a particular transparency.

[18]Considering a real-rational function as a ratio of two polynomials in which the two polynomials share common roots is not as pathological as it may initially appear. Indeed, this occurs when we consider the impedance $Z(s)$ of a transformerless network as the ratio of the polynomials in the operator whose kernel defines the driving-point behaviour of that network (i.e. the polynomials $g(s)p(s)$ and $g(s)q(s)$ in Theorem 1.4.1).

2.3 Non-proper impedances and the extended Cauchy index

We next consider the extension of the previous results to general real-rational functions (without the assumption of properness). We first introduce the following:

Definition 2.3.1.
For $F \in \mathbb{R}(s)$, we define the *extended Cauchy index* $\gamma(F(s))$ to be the difference between (a) the number of jumps of $F(s)$ from $-\infty$ to $+\infty$, and (b) the number of jumps of $F(s)$ from $+\infty$ to $-\infty$, as s is increased in $\mathbb{R}$ from any point a through $+\infty$ and then from $-\infty$ to a again, where $a \in \mathbb{R}$ is not a pole of $F(s)$.

Thus, the extended Cauchy index also considers jumps at $s = \infty$, unlike the Cauchy index. If $F(s)$ is proper, or has a pole of even multiplicity at $s = \infty$, then $\gamma(F(s)) = I_{-\infty}^{+\infty} F(s)$, and if $F(s)$ is non-proper and has a pole of odd multiplicity at $s = \infty$ then $\gamma(F(s))$ differs from $I_{-\infty}^{+\infty} F(s)$ by ± 1. Note that Definition 2.3.1 does not depend on the choice of a. It is then straightforward to verify the following lemma:

Lemma 2.3.2. *Let $F, F_1, F_2 \in \mathbb{R}(s)$. Then*

1. $\gamma(F(s)) = -\gamma(1/F(s))$.
2. *If $F(s) = F_1(s) + F_2(s)$ and $\delta(F(s)) = \delta(F_1(s)) + \delta(F_2(s))$ then $\gamma(F(s)) = \gamma(F_1(s)) + \gamma(F_2(s))$.*

In the above lemma, condition 1 follows by considering both the jumps and the crossings of zero by the function $F(s)$ as s increases in $\mathbb{R}$ from the point a through $+\infty$ and then from $-\infty$ to a again. It may readily be seen that the number of jumps in $F(s)$ from $+\infty$ to $-\infty$ plus the number of crossings of zero by $F(s)$ from positive to negative must be equal to the number of jumps in $F(s)$ from $-\infty$ to $+\infty$ plus the number of crossings of zero by $F(s)$ from negative to positive. Condition 2 in the above lemma follows by considering a partial fraction decomposition for $F(s)$.

Now, suppose that a non-proper $Z(s)$ with $\delta(Z(s)) = n$ is the impedance of a minimally-reactive reciprocal network containing p inductors and q capacitors. Then $1/Z(s)$ is (strictly) proper and is the admittance of the network. Again following [28, Section 4.4, Theorem 2.8.1], reactance extraction provides a hybrid matrix M satisfying (80) with v and i interchanged, and with (81) satisfied for $\Sigma = (-I_p \dotplus I_q)$. If we now form the Hankel matrix $\mathcal{H}_n^\dagger$ corresponding to $1/Z(s)$, we can deduce that

$$p - q = \sigma(\mathcal{H}_n^\dagger) = \gamma\left(1/Z(s)\right),$$

where we have used the same reasoning as for Theorem 2.1.1 (noting the change in sign due to the change in sign in Σ), and the fact that the extended Cauchy index for a proper real-rational function is equal to the signature of the corresponding Hankel matrix [39, p. 210]. Hence using condition 1 in Lemma 2.3.2, and combining with Theorem 2.2.1 for the case that $Z(s)$ is proper, we obtain the following result:

Theorem 2.3.3. *Let $Z \in \mathbb{R}(s)$ be the impedance of a reciprocal network containing exactly p inductors and q capacitors and with $p + q = \delta\left(Z(s)\right)$. Then*

$$q - p = \gamma\left(Z(s)\right).$$

We next consider a non-proper $Z(s)$. As in Section 2.1, we can form Hankel matrices $\mathcal{H}_k^\dagger$ corresponding to $1/Z(s)$. It can then be seen that Theorem 2.1.2 holds with $Z(s)$ replaced by $1/Z(s)$, the expressions for q and p in equations (96) and (97) interchanged, and $|\mathcal{H}_k|$ replaced everywhere by $|\mathcal{H}_k^\dagger|$. Now, if $Z(s)$ is written in the form (98), then $a_n \neq 0$, and we can define Sylvester matrices $\mathcal{S}_j^\dagger$ corresponding to $1/Z(s)$ ($j = 1, 2, \ldots$). As in Section 2.2, it follows that

$$|\mathcal{S}_{2k}^\dagger| = a_n^{2k} |\mathcal{H}_k^\dagger|. \tag{101}$$

We further note that $\mathcal{S}_{2k}^\dagger$ differs from $\mathcal{S}_{2k}$ by the interchange of row i with row $i+1$ for i odd. Therefore

$$|\mathcal{S}_{2k}^\dagger| = (-1)^k |\mathcal{S}_{2k}|. \tag{102}$$

Combining the modified form of Theorem 2.1.2 with (101) and (102), we obtain the following:

Theorem 2.3.4. *Theorem 2.2.2 (and its subsequent remark) holds for any $Z \in \mathbb{R}(s)$.*

Theorem 2.3.4 gives explicit algebraic constraints on those impedance functions which can be realised by minimally-reactive networks according to the individual numbers of capacitors and inductors contained in the networks. These constraints are directly in terms of the parameters in the impedance function, and do not rely on the impedance function being proper (as was assumed in Sections 2.1 and 2.2).

2.4 The Bezoutian matrix

We now define a Bezoutian matrix associated with a real-rational function, and we show how the results of the preceding sections may alternatively be expressed in terms of this Bezoutian matrix.

Let $Z(s) = a(s)/b(s)$ with $a, b \in \mathbb{R}[s]$ as in (98). The *Bezoutian* matrix is a symmetric matrix $\mathcal{B} = \mathcal{B}(b, a)$ whose elements $\mathcal{B}_{ij}$ satisfy

$$a(w)b(z) - b(w)a(z) =: \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{B}_{ij} z^{i-1}(z-w)w^{j-1}. \tag{103}$$

If $Z \in \mathbb{R}_p(s)$ then, for $\mathcal{H}_k$ as in (95) with $Z(s)$ written as in (92), the matrix $\mathcal{B}(b, a)$ is congruent to $\mathcal{H}_n$ [40, equation (8.58)]. It follows that $\gamma(Z(s)) = \sigma(\mathcal{H}_n) = \sigma(\mathcal{B}(b, a))$ and $\delta(Z(s)) = r(\mathcal{H}_n) = r(\mathcal{B}(b, a))$, with these relationships holding irrespective of whether $a(s)$ and $b(s)$ are coprime. If $Z(s)$ is not proper then, since $b(s)/a(s)$ is proper and $\mathcal{B}(b, a) = -\mathcal{B}(a, b)$, we have $\gamma(Z(s)) = -\gamma(1/Z(s)) = -\sigma(\mathcal{B}(a, b)) = \sigma(\mathcal{B}(b, a))$ and $\delta(Z(s)) = r(\mathcal{B}(a, b)) = r(\mathcal{B}(b, a))$. There is also a close relationship between the Bezoutian matrix and the Sylvester matrix. Let $Z(s)$ be as in (98) and let $\mathcal{B}_k$ be the matrix formed from the final k rows and columns of $\mathcal{B}(b, a)$, i.e.

$$(\mathcal{B}_k)_{ij} := \mathcal{B}_{i+n-k,j+n-k}, \tag{104}$$

for $k = 1, 2, \ldots, n$ and $i, j = 1, 2, \ldots k$. Define matrices $T, P_{11}, P_{12}, P_{21}, P_{22} \in \mathbb{R}^{k\times k}$ where

$$T := \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

and

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} := \begin{bmatrix} a_{n-k} & \ldots & a_{n-2k+1} & b_{n-k} & \ldots & b_{n-2k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & \ldots & a_{n-k} & b_{n-1} & \ldots & b_{n-k} \\ a_n & \ldots & a_{n-k+1} & b_n & \ldots & b_{n-k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & a_n & 0 & \ldots & b_n \end{bmatrix},$$

in which we put $a_j = 0$, $b_j = 0$ for $j < 0$. Then, following [40, Theorem 8.44], the matrices P_{21} and P_{22} commute and, using a Gohberg-Semencul formula [41, Theorem 5.1], we find

$$|P| = |P_{11}P_{22} - P_{12}P_{21}| = |\mathcal{B}_k||T|.$$

Since a sequence of $\sum_{j=1}^{k-1} j = k(k-1)/2$ permutations of neighbouring columns carries T into I_k, and a sequence of $2\sum_{j=1}^{k-1} j + \sum_{j=1}^{k} j = k(3k-1)/2$ permutations of neighbouring columns followed by $\sum_{j=1}^{2k-1} j = k(2k-1)$ permutations of neighbouring rows carries P into $\mathcal{S}_{2k}^T$, it follows that

$$|\mathcal{S}_{2k}| = |\mathcal{B}_k|, \tag{105}$$

for $k = 1, 2, \ldots, n$. Theorems 2.3.3 and 2.3.4 then lead to the following result.

Theorem 2.4.1. *Let $Z \in \mathbb{R}(s)$ be as in (98) with $\delta(Z(s)) = n$. Further let $\mathcal{B}_k$ be as in (104) for $\mathcal{B}_{ij}$, $\mathcal{B}(b,a)$ defined via (103). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p + q = n$, then*

$$q = \frac{1}{2}\left(\delta(Z(s)) + \gamma(Z(s))\right) = \pi(\mathcal{B}(b,a)) = \mathbf{P}(1, |\mathcal{B}_1|, \ldots, |\mathcal{B}_n|),$$
$$\textit{and} \quad p = \frac{1}{2}\left(\delta(Z(s)) - \gamma(Z(s))\right) = \nu(\mathcal{B}(b,a)) = \mathbf{V}(1, |\mathcal{B}_1|, \ldots, |\mathcal{B}_n|).$$

In any sub-sequence of zero values ($|\mathcal{B}_k| \neq 0, |\mathcal{B}_{k+1}| = |\mathcal{B}_{k+2}| = \ldots = 0$), signs are assigned to the zero values as follows: $\operatorname{sign}(|\mathcal{B}_{k+j}|) = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign}(|\mathcal{B}_k|)$.

We remark that the above theorem still holds when the polynomials $a(s)$ and $b(s)$ are not coprime providing we replace n with $r := \delta(a(s)/b(s))$ in the theorem statement.

Theorem 2.4.1 provides an alternative to Theorem 2.3.4 for the computation of algebraic criteria for minimally-reactive reciprocal network realisations. As will be seen in Section 2.6, both the Bezoutian matrix and the extended Cauchy index have a natural generalisation to matrix real-rational functions. In that section, we will show how these generalisations allow Theorem 2.4.1 to be extended to the multi-port case.

2.5 Non-minimally-reactive networks

In [20, Theorem 2], Youla and Tissi use the scattering matrix formalism to establish lower bounds on the number of capacitors and inductors which are needed in reciprocal realisations of a given scattering matrix. This result holds for networks irrespective of whether they are minimally-reactive. In this section, we derive similar lower bounds

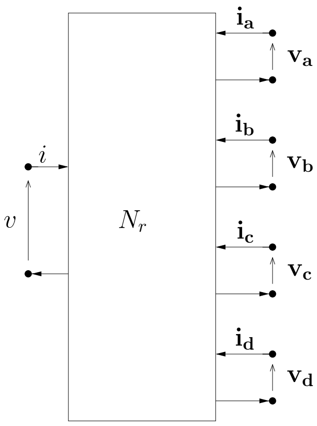


Figure 10: The network N_r obtained by removing all reactive elements from N.

for an impedance function, using the reactance extraction procedure as described in Anderson and Vongpanitlerd [28, Section 4.4]. We also provide algebraic criteria for the lower bounds in terms of the previously defined Bezoutian matrix.

Let $Z \in \mathbb{R}_p(s)$ be the impedance matrix of a one-port reciprocal network N containing exactly p inductors and q capacitors. Using the procedure in [28, Section 4.4], upon removal of the reactive elements in N we are left with the network N_r in Fig. 10 possessing a hybrid matrix M [28, equation 4.4.56] such that

$$\begin{bmatrix} v \\ \mathbf{v_a} \\ \mathbf{i_b} \\ \mathbf{i_c} \\ \mathbf{v_d} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ -M_{14}^T & -M_{24}^T & -M_{34}^T & 0 & 0 \\ -M_{15}^T & -M_{25}^T & -M_{35}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \mathbf{i_a} \\ \mathbf{v_b} \\ \mathbf{v_c} \\ \mathbf{i_d} \end{bmatrix},$$

where $(\mathbf{i_a}, \mathbf{v_a}), \ldots, (\mathbf{i_d}, \mathbf{v_d})$ are pairs of Laplace-transformed vectors of currents and voltages of dimensions p', q', $p - p'$, $q - q'$ respectively, and M is partitioned compatibly with the pertinent vectors. The network N is obtained upon terminating the ports corresponding to $(\mathbf{i_a}, \mathbf{v_a})$, $(\mathbf{i_c}, \mathbf{v_c})$ with inductors and the ports corresponding to $(\mathbf{i_b}, \mathbf{v_b})$, $(\mathbf{i_d}, \mathbf{v_d})$ with capacitors. Then we have

$$\begin{bmatrix} \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = -s \begin{bmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix} \begin{bmatrix} \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} \mathbf{v_c} \\ \mathbf{i_d} \end{bmatrix} = -s \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} \mathbf{i_c} \\ \mathbf{v_d} \end{bmatrix},$$

where $\mathcal{L}_2 := \operatorname{diag}\left(L_1, \ldots, L_{p'}\right)$, $\mathcal{C}_3 := \operatorname{diag}\left(C_1, \ldots, C_{q'}\right)$, $\mathcal{L}_4 := \operatorname{diag}\left(L_{p'+1}, \ldots, L_p\right)$

and $\mathcal{C}_5 := \mathrm{diag}\left(C_{q'+1}, \ldots, C_q\right)$. It then follows that

$$\left(s\left(\begin{bmatrix}\mathcal{L}_2 & 0\\ 0 & \mathcal{C}_3\end{bmatrix} + \begin{bmatrix}M_{24} & M_{25}\\ M_{34} & M_{35}\end{bmatrix}\begin{bmatrix}\mathcal{L}_4 & 0\\ 0 & \mathcal{C}_5\end{bmatrix}\begin{bmatrix}M_{24} & M_{25}\\ M_{34} & M_{35}\end{bmatrix}^T\right) + \begin{bmatrix}M_{22} & M_{23}\\ M_{32} & M_{33}\end{bmatrix}\right)\begin{bmatrix}\mathbf{i_a}(s)\\ \mathbf{v_b}(s)\end{bmatrix}$$
$$= \left(s\begin{bmatrix}M_{24} & M_{25}\\ M_{34} & M_{35}\end{bmatrix}\begin{bmatrix}\mathcal{L}_4 & 0\\ 0 & \mathcal{C}_5\end{bmatrix}\begin{bmatrix}-M_{14}^T\\ -M_{15}^T\end{bmatrix} + \begin{bmatrix}-M_{21}\\ -M_{31}\end{bmatrix}\right) i(s),$$

and

$$v(s) = \left(s\begin{bmatrix}M_{14} & M_{15}\end{bmatrix}\begin{bmatrix}\mathcal{L}_4 & 0\\ 0 & \mathcal{C}_5\end{bmatrix}\begin{bmatrix}M_{24} & M_{25}\\ M_{34} & M_{35}\end{bmatrix}^T + \begin{bmatrix}M_{12} & M_{13}\end{bmatrix}\right)\begin{bmatrix}\mathbf{i_a}(s)\\ \mathbf{v_b}(s)\end{bmatrix}$$
$$+ \left(s\begin{bmatrix}M_{14} & M_{15}\end{bmatrix}\begin{bmatrix}\mathcal{L}_4 & 0\\ 0 & \mathcal{C}_5\end{bmatrix}\begin{bmatrix}M_{14}^T\\ M_{15}^T\end{bmatrix} + M_{11}\right) i(s).$$

Since N_r is reciprocal then, by [28, Theorem 2.8.1],

$$\left(1\dot{+}I_{p'}\dot{+}(-I_{q'})\dot{+}(-I_{p-p'})\dot{+}I_{q-q'}\right) M = M^T \left(1\dot{+}I_{p'}\dot{+}(-I_{q'})\dot{+}(-I_{p-p'})\dot{+}I_{q-q'}\right). \quad (106)$$

which implies that all entries in M_{15}, M_{25} and M_{34} are zero. Furthermore, since $Z(s)$ is proper, we require $M_{14} = 0$. It may then be verified that $Z(s)$ has a state-space realisation with (Laplace-transformed) state vector $\begin{bmatrix}\mathbf{i_a}^T & \mathbf{v_b}^T\end{bmatrix}^T$ with dimension $n = p' + q'$, and with $Z(s) = J + H\left(sI - F\right)^{-1} G$ where

$$F = -R\begin{bmatrix}M_{22} & M_{23}\\ M_{32} & M_{33}\end{bmatrix} \in \mathbb{R}^{n\times n}, \quad (107)$$

$$G = -R\begin{bmatrix}M_{21}\\ M_{31}\end{bmatrix} \in \mathbb{R}^{n\times 1}, \quad (108)$$

$$H = \begin{bmatrix}M_{12} & M_{13}\end{bmatrix} \in \mathbb{R}^{1\times n}, \quad (109)$$

$$\text{and} \qquad J = M_{11} \in \mathbb{R}. \quad (110)$$

Here

$$R = \begin{bmatrix}R_{11} & 0\\ 0 & R_{22}\end{bmatrix},$$

with

$$R_{11} = \left(\mathcal{L}_2 + M_{24}\mathcal{L}_4 M_{24}^T\right)^{-1} \in \mathbb{R}^{p' \times p'},$$
$$\text{and} \qquad R_{22} = \left(\mathcal{C}_3 + M_{35}\mathcal{C}_5 M_{35}^T\right)^{-1} \in \mathbb{R}^{q' \times q'},$$

where R_{11} and R_{22} exist, and are positive definite, since both $\left(\mathcal{L}_2 + M_{24}\mathcal{L}_4 M_{24}^T\right)$ and $\left(\mathcal{C}_3 + M_{35}\mathcal{C}_5 M_{35}^T\right)$ are positive definite.

Let $\Sigma := \left(I_{p'} \dotplus (-I_{q'})\right)$. It is straightforward to verify that $\Sigma^2 = I_n$, $\Sigma R = R\Sigma$, and both R and Σ are symmetric. Then, from equations (106)-(110) we have $FR\Sigma = R\Sigma F^T$ and $G = -R\Sigma H^T$. Let V_c and V_o be as in (88) and (89) with $\mathcal{H}_n$ as in (91). It is straightforward to show that $V_c = -R\Sigma V_o^T$, and hence

$$\mathcal{H}_n = V_o \left(-R\Sigma\right) V_o^T.$$

From [38, Theorem 2], the number of positive and negative eigenvalues of $\mathcal{H}_n$ cannot exceed the corresponding quantities for $-R\Sigma$. Since $-R\Sigma = (-R_{11} \dotplus R_{22})$ with $-R_{11} < 0$ and $R_{22} > 0$, it follows that $-R\Sigma$ has exactly q' positive and p' negative eigenvalues. From the dimension of the state vector it follows that the McMillan degree of $Z(s)$ is no greater than $n = p' + q'$. Hence, for $\mathcal{H}_k$ as in (95), we have $\pi\left(\mathcal{H}_n\right) = \pi\left(\mathcal{H}_k\right)$ and $\nu\left(\mathcal{H}_n\right) = \nu\left(\mathcal{H}_k\right)$ for all $k \geq \delta\left(Z(s)\right)$, and so $\pi\left(\mathcal{H}_k\right) \leq q' \leq q$ and $\nu\left(\mathcal{H}_k\right) \leq p' \leq p$ for all $k \geq \delta\left(Z(s)\right)$.

Using the argument in Section 2.3 about the existence of either a proper impedance or a proper admittance, we obtain the following theorem which holds irrespective of whether the network is minimally-reactive or whether $a(s)$ and $b(s)$ are coprime:

Theorem 2.5.1. *Let $Z \in \mathbb{R}(s)$ be as in (98). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors, then*

$$q \geq \frac{1}{2}\left(\delta\left(Z(s)\right) + \gamma\left(Z(s)\right)\right) = \pi\left(\mathcal{B}\left(b, a\right)\right),$$
$$\textit{and} \qquad p \geq \frac{1}{2}\left(\delta\left(Z(s)\right) - \gamma\left(Z(s)\right)\right) = \nu\left(\mathcal{B}\left(b, a\right)\right).$$

Here, $\pi\left(\mathcal{B}\left(b, a\right)\right)$ and $\nu\left(\mathcal{B}\left(b, a\right)\right)$ can be calculated in accordance with Theorem 2.4.1 providing we replace n with $r := \delta\left(a(s)/b(s)\right)$.

From Theorem 2.5.1, we obtain lower bounds on the individual numbers of inductors and capacitors required to realise a given impedance function with a reciprocal network. These bounds are determined by algebraic criteria which are expressed in terms of the extended Cauchy index of the impedance function, and also in terms of a Bezoutian matrix related to the impedance. Equivalently, the criteria may be expressed directly in terms of the parameters in the impedance function by using the Sylvester matrix together with the relationship (105).

2.6 Multi-port networks, generalised Bezoutians, and the extended matrix Cauchy index

As will be shown in this section, the preceding results generalise in a natural way to multi-port networks. In contrast to the one-port case there is no guarantee of the existence of a proper impedance or a proper admittance function. However, from [42, Theorem 2], any reciprocal m-port network N possesses a scattering matrix description $S(s)$ where

$$\begin{bmatrix} v_1 - i_1 \\ v_2 - i_2 \\ \vdots \\ v_m - i_m \end{bmatrix} = S(s) \begin{bmatrix} v_1 + i_1 \\ v_2 + i_2 \\ \vdots \\ v_m + i_m \end{bmatrix}, \tag{111}$$

and $i_1, v_1, \ldots$ are the Laplace-transformed currents and voltages at the m ports. It is well known that $S \in \mathbb{R}_p^{m\times m}(s)$ and is symmetric [20, Section 2].

Consider the transformation

$$\phi(s) := \frac{s+\alpha}{s-\alpha}, \qquad \alpha \in \mathbb{R}, \alpha > 0, \tag{112}$$

for which

$$\phi^{-1}(s) = \frac{\alpha\,(s+1)}{s-1},$$

which maps the left half of the s-plane onto the interior of the unit circle in the ϕ-plane. Let

$$\hat{S}(s) := S(\phi^{-1}(s)).$$

It follows from [20, Section 3] that $\hat{S} \in \mathbb{R}_p^{m\times m}(s)$ is symmetric and has a realisation $\hat{S}(s) = J + H\,(sI - F)^{-1}\,G$ satisfying $J = J^T$, $\Sigma F = F^T \Sigma$, and $\Sigma G = H^T$, where $\Sigma = (I_p \dotplus (-I_q))$ with p (resp. q) the number of inductors (resp. capacitors) in N. It may then be shown that $V_o = V_c^T \Sigma$ where V_c, V_o are as in (88) and (89) for $n = p + q \geq \delta\left(\hat{S}(s)\right)$.

Consider now the infinite Hankel matrix for $\hat{S}(s)$

$$\mathcal{H} := \begin{bmatrix} W_0 & W_1 & W_2 & \dots \\ W_1 & W_2 & W_3 & \dots \\ W_2 & W_3 & W_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{113}$$

together with the finite Hankel matrices

$$\mathcal{H}_k := \begin{bmatrix} W_0 & W_1 & \dots & W_{k-1} \\ W_1 & W_2 & \dots & W_k \\ \vdots & \vdots & \ddots & \vdots \\ W_{k-1} & W_k & \dots & W_{2k-2} \end{bmatrix},$$

for $k = 1, 2, \dots$, where $W_i := HF^iG$ for $i = 0, 1, 2 \dots$ which coincide with the matrices in the Laurent series expansion of $\hat{S}(s)$ about the point at ∞

$$\hat{S}(s) = W_{-1} + \frac{W_0}{s} + \frac{W_1}{s^2} + \frac{W_2}{s^3} + \dots, \tag{114}$$

in which $W_{-1} = J$. Then, from [20, Appendix 1], $r(\mathcal{H}) = r(\mathcal{H}_k) = \delta\left(\hat{S}(s)\right)$ for all $k \geq \delta\left(\hat{S}(s)\right)$ (and indeed for all $k \geq r$ where $r \leq \delta\left(\hat{S}(s)\right)$ is the degree of the least common multiple of all denominators of $\hat{S}(s)$). Furthermore, if $\hat{S}(s)$ is symmetric, then so too is $\mathcal{H}$, and $\sigma(\mathcal{H}) = \sigma(\mathcal{H}_k)$ for all $k \geq \delta\left(\hat{S}(s)\right)$, as may be shown using results from [43, Section 3][19]. Since, in addition

$$\begin{aligned} \mathcal{H}_n &= V_o V_c \\ &= V_c^T \Sigma V_c \\ &= \begin{bmatrix} V_c^T & 0_{mn\times(m-1)n} \end{bmatrix} \begin{bmatrix} \Sigma & 0_{n\times(m-1)n} \\ 0_{(m-1)n\times n} & 0_{(m-1)n\times(m-1)n} \end{bmatrix} \begin{bmatrix} V_c \\ 0_{(m-1)n\times mn} \end{bmatrix} \end{aligned}$$

and $n \geq \delta\left(\hat{S}(s)\right)$, then from [38, Theorem 2] we see that the number of positive (resp. negative) eigenvalues of $\mathcal{H}_n$ are bounded below by the number of $+1$ (resp. -1) entries in Σ, and we may similarly show that the same is true for the numbers of positive and

[19]This follows from considering the decomposition of $\hat{S}(s)$ into partial fractions and then obtaining the Hankel matrix for $\hat{S}(s)$ by summing the Hankel matrices for each partial fraction summand. Then, [43, Lemmas 3.3 and 3.4] shows that the rank and the signature of the infinite Hankel matrix for each partial fraction summand are respectively equal to the ranks and the signatures of the finite Hankel matrices whose dimensions are compatible with $\mathcal{H}_k$ ($k \geq \delta\left(\hat{S}(s)\right)$). That $\sigma(\mathcal{H}) = \sigma(\mathcal{H}_k)$ for all $k \geq \delta\left(\hat{S}(s)\right)$ then follows from [43, Theorem A1].

negative eigenvalues of $\mathcal{H}_k$ for all $k \geq \delta\left(\hat{S}(s)\right)$. We thus obtain the following theorem:

Theorem 2.6.1. *Let $S(s)$ be the scattering matrix of a reciprocal m-port network containing exactly p inductors and q capacitors. Further let $\hat{S}(s) := S\left(\phi^{-1}(s)\right)$ for $\phi(s)$ as in (112). Then $\hat{S} \in \mathbb{R}_p^{m\times m}(s)$ is symmetric and, with $\mathcal{H}$ as in (113) for $\hat{S}(s)$ written as in (114), we have $p \geq \pi(\mathcal{H})$ and $q \geq \nu(\mathcal{H})$.*

For $\mathcal{H}$ defined as in (113) with $\hat{S} \in \mathbb{R}_p^{m\times m}(s)$ symmetric and written as in (114), $\sigma(\mathcal{H})$ is equal to the matrix Cauchy index of $\hat{S}(s)$ [43, Theorem 3.1]. To extend these results to the case of non-proper real-rational matrix functions, we introduce the following generalisation of the extended Cauchy index:

Definition 2.6.2.
For a symmetric matrix $F \in \mathbb{R}^{m\times m}(s)$, we define the extended matrix Cauchy index $\gamma(F(s))$ to be the difference between (a) the number of jumps in the eigenvalues of $F(s)$ from $-\infty$ to $+\infty$, and (b) the number of jumps in the eigenvalues of $F(s)$ from $+\infty$ to $-\infty$, as s is increased in $\mathbb{R}$ from a point a through $+\infty$ and then from $-\infty$ to a again, for any $a \in \mathbb{R}$ which is not a pole of $F(s)$.

We remark that $\gamma(F(s))$ is well-defined since the eigenvalues of $F(s)$ are defined by algebraic functions (as defined in [44, Definition 11]). Accordingly, the eigenvalues can be described locally using Laurent expansions containing fractional powers in the manner discussed in [44, Chapter II]. Then, since $F(s)$ has real eigenvalues for any real s, the local power series defining them will not possess fractional powers, hence we can define an extended Cauchy index for each eigenvalue individually and then take the sum.

Definition 2.6.2 coincides with the extended Cauchy index of Definition 2.3.1 in the scalar case. Furthermore, if $F \in \mathbb{R}_p^{m\times m}(s)$, then $\gamma(F(s))$ coincides with the matrix Cauchy index defined in [43]. It is then straightforward to show the following generalisation of Lemma 2.3.2:

Lemma 2.6.3. *Let $F, F_1, F_2 \in \mathbb{R}^{m\times m}(s)$ be symmetric. Then*

1. *$\gamma(F(s)) = -\gamma\left(F(s)^{-1}\right)$ when $F(s)^{-1}$ exists.*
2. *If $F(s) = F_1(s) + F_2(s)$ and $\delta(F(s)) = \delta(F_1(s)) + \delta(F_2(s))$, then $\gamma(F(s)) = \gamma(F_1(s)) + \gamma(F_2(s))$.*

The above lemma follows from similar considerations to those used to show Lemma 2.3.2 in the scalar case. Here, we note that if $F(s)$ is invertible then $F(s)$ and its inverse each have m real eigenvalues for all $s \in \mathbb{R} \cup \infty$ (except at poles), that these eigenvalues vary continuously with s, and that the eigenvalues of $F(s)$ are in one-to-one correspondence with those of $F(s)^{-1}$. Condition 1 then follows by considering the extended Cauchy index for each eigenvalue in turn. Similarly to the scalar case, condition 2 follows by considering a matrix partial fraction decomposition for $F(s)$.

As in the scalar case, there is a correspondence between the matrix extended Cauchy index and a matrix Bezoutian. If $F(s)$ is a symmetric matrix with a left matrix factorisation given by $F(s) = B(s)^{-1}A(s)$ ($A(s)$ and $B(s)$ need not be left coprime) then, consistently with [43, p. 665], we define the matrix Bezoutian $\mathcal{B}(B, A)$ as the symmetric matrix with block entries $\mathcal{B}_{ij}$ satisfying

$$B(z)A^T(w) - A(z)B^T(w) =: \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{B}_{ij} z^{i-1}(z-w)w^{j-1}.$$

This definition coincides with the definition in Section 2.4 in the scalar case. If $F \in \mathbb{R}_p^{m\times m}(s)$ is symmetric, and has a left matrix factorisation $F(s) = B(s)^{-1}A(s)$, then from [45, Theorem 2.1] and [43, p. 666] we have

$$\delta\left(F(s)\right) = r\left(\mathcal{B}\left(B, A\right)\right),$$
$$\text{and} \qquad \gamma\left(F(s)\right) = \sigma\left(\mathcal{B}\left(B, A\right)\right).$$

We remark that these properties hold irrespective of whether $B(s)$ and $A(s)$ are left coprime. We now show that the above equations also hold in the case when $F(s)$ is *not* proper. In this case, consider the transformation $\phi(s)$ in (112) for any $\alpha \in \mathbb{R}, \alpha > 0$ which is not a pole of $F(s)$. Then the function $\hat{F}(s) := F\left(\phi^{-1}(s)\right)$ is proper. Further, note that $\phi \in \mathbb{R}(s)$, $\phi(s)$ is bounded at $s = \infty$, and

$$\frac{d\phi}{ds} = -\frac{2\alpha}{(s-\alpha)^2},$$

which, since $\alpha > 0$, implies that $\phi(s)$ is a monotonically decreasing function of s for $s \in \mathbb{R}$ (except at $s = \alpha$), and is also decreasing at it moves through the point at $+\infty$ and returns from $-\infty$. It then follows that $\gamma\left(\hat{F}(s)\right) = -\gamma\left(F(s)\right)$. Moreover, we have $\delta\left(\hat{F}(s)\right) = \delta\left(F(s)\right)$ from the properties of the McMillan degree[20]. Suppose in addition that $F(s)$ has a left matrix factorisation $F(s) = B(s)^{-1}A(s)$, and let n be the

[20]This follows from the properties stated in [20, Section 3], since $\phi : \{\mathbb{R} \cup \infty\} \mapsto \{\mathbb{R} \cup \infty\}$ is one-to-one and onto, and so poles of $\hat{F}(s)$ are in one-to-one correspondence with poles of $F(s)$.

maximum of the degrees of the entries in the matrices $A(s)$ and $B(s)$. It follows that $\hat{F}(s)$ has a left matrix factorisation $\hat{F}(s) = \hat{B}(s)^{-1}\hat{A}(s)$ with $\hat{B}(s) = (s-1)^n B(\phi^{-1}(s))$ and $\hat{A}(s) = (s-1)^n A(\phi^{-1}(s))$. Hence,

$$\begin{aligned}
&\hat{B}(z)\hat{A}^T(w) - \hat{A}(z)\hat{B}^T(w) \\
&\qquad = (z-1)^n \left(B\left(\phi^{-1}(z)\right) A^T\left(\phi^{-1}(w)\right) - A\left(\phi^{-1}(z)\right) B^T\left(\phi^{-1}(w)\right)\right)(w-1)^n \\
&\qquad = (z-w)\mathbf{z}^T \mathcal{B}\left(\hat{B}, \hat{A}\right)\mathbf{w} \\
&\qquad = -2\alpha(z-w)\hat{\mathbf{z}}^T \mathcal{B}\left(B, A\right)\hat{\mathbf{w}},
\end{aligned}$$

for all z, w, where

$$\begin{aligned}
\mathbf{z} &:= \begin{bmatrix} 1, & z, & \ldots, & z^{n-1} \end{bmatrix}^T, \\
\mathbf{w} &:= \begin{bmatrix} 1, & w, & \ldots, & w^{n-1} \end{bmatrix}^T, \\
\hat{\mathbf{z}} &:= (z-1)^{n-1} \begin{bmatrix} 1, & \alpha\tfrac{z+1}{z-1}, & \ldots, & \left(\alpha\tfrac{z+1}{z-1}\right)^{n-1} \end{bmatrix}^T, \\
\text{and} \qquad \hat{\mathbf{w}} &:= (w-1)^{n-1} \begin{bmatrix} 1, & \alpha\tfrac{w+1}{w-1}, & \ldots, & \left(\alpha\tfrac{w+1}{w-1}\right)^{n-1} \end{bmatrix}^T.
\end{aligned}$$

Now, consider the matrices

$$T_1 := \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ \alpha & 2^1\alpha & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n-2} & \binom{n-2}{1}2^1\alpha^{n-2} & \ldots & 2^{n-2}\alpha^{n-2} & 0 \\ \alpha^{n-1} & \binom{n-1}{1}2^1\alpha^{n-1} & \ldots & \binom{n-1}{n-2}2^{n-2}\alpha^{n-1} & 2^{n-1}\alpha^{n-1} \end{bmatrix},$$

$$\text{and} \qquad T_2 := \begin{bmatrix} (-1)^{n-1} & \binom{n-1}{1}(-1)^{n-2} & \ldots & \binom{n-1}{n-2}(-1) & 1 \\ (-1)^{n-2} & \binom{n-2}{1}(-1)^{n-3} & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Since

$$T_2\mathbf{w} = \begin{bmatrix} (w-1)^{n-1} \\ (w-1)^{n-2} \\ \vdots \\ (w-1) \\ 1 \end{bmatrix} = (w-1)^{n-1} \begin{bmatrix} 1 \\ \frac{1}{w-1} \\ \vdots \\ \left(\frac{1}{w-1}\right)^{n-2} \\ \left(\frac{1}{w-1}\right)^{n-1} \end{bmatrix},$$

$$\text{and} \qquad T_1T_2\mathbf{w} = (w-1)^{n-1} \begin{bmatrix} 1 \\ \alpha\left(1+\frac{2}{w-1}\right) \\ \vdots \\ \alpha^{n-2}\left(1+\frac{2}{w-1}\right)^{n-2} \\ \alpha^{n-1}\left(1+\frac{2}{w-1}\right)^{n-1} \end{bmatrix},$$

then $\hat{\mathbf{w}} = T_1T_2\mathbf{w}$, and similarly $\hat{\mathbf{z}} = T_1T_2\mathbf{z}$. It follows that

$$\mathcal{B}\left(\hat{B},\hat{A}\right) = (T_1T_2)^T\left(-2\alpha\mathcal{B}\left(B,A\right)\right)(T_1T_2).$$

Since $\hat{F}(s)$ is proper and $\alpha > 0$, then $\gamma\left(F(s)\right) = -\gamma\left(\hat{F}(s)\right) = -\sigma\left(\mathcal{B}\left(\hat{B},\hat{A}\right)\right) = \sigma\left(\mathcal{B}\left(B,A\right)\right)$ and $\delta\left(F(s)\right) = \delta\left(\hat{F}(s)\right) = r\left(\mathcal{B}\left(\hat{B},\hat{A}\right)\right) = r\left(\mathcal{B}\left(B,A\right)\right)$. We have shown the following:

Lemma 2.6.4. *Let $F \in \mathbb{R}^{m\times m}(s)$ be symmetric with left matrix factorisation $F(s) = B(s)^{-1}A(s)$. Then*

$$\delta\left(F(s)\right) = r\left(\mathcal{B}\left(B,A\right)\right),$$
$$\textit{and} \qquad \gamma\left(F(s)\right) = \sigma\left(\mathcal{B}\left(B,A\right)\right).$$

We conclude by considering the case when a hybrid matrix description of the driving-point behaviour of N is available. By rearranging equation (111), we find

$$(I-\Sigma_e S(s))\begin{bmatrix}\mathbf{v}_\alpha \\ \mathbf{i}_\beta\end{bmatrix} = (I+\Sigma_e S(s))\begin{bmatrix}\mathbf{i}_\alpha \\ \mathbf{v}_\beta\end{bmatrix},$$

where $\mathbf{i}_\alpha$, $\mathbf{v}_\alpha$ are the Laplace-transformed vectors of current and voltage across the first m_1 ports, $\mathbf{i}_\beta$, $\mathbf{v}_\beta$ are the Laplace-transformed vectors of current and voltage across the remaining m_2 ports, and $\Sigma_e := (I_{m_1} \dotplus (-I_{m_2}))$. Hence, providing the matrix

$(I - \Sigma_e S(s))$ is invertible, we have

$$\begin{bmatrix} \mathbf{v}_\alpha \\ \mathbf{i}_\beta \end{bmatrix} = M(s) \begin{bmatrix} \mathbf{i}_\alpha \\ \mathbf{v}_\beta \end{bmatrix}, \tag{115}$$

where

$$\begin{aligned} M(s)\Sigma_e &= (I - \Sigma_e S(s))^{-1} (I + \Sigma_e S(s)) \Sigma_e \\ &= (I - \Sigma_e S(s))^{-1} (\Sigma_e S(s) - I + 2I) \Sigma_e \\ &= -\Sigma_e + 2 (I - \Sigma_e S(s))^{-1} \Sigma_e \\ &= -\Sigma_e + 2 (\Sigma_e - S(s))^{-1}, \end{aligned}$$

which is symmetric. Such a matrix Σ_e is commonly referred to as an *external signature matrix*, e.g. [46, Definition 4]. From the properties of the McMillan degree [20, Section 3], we have

$$\delta (M(s)\Sigma_e) = \delta (S(s)) = \delta \left(\hat{S}(s)\right),$$

and from Lemma 2.6.3 and the previous discussion, it is straightforward to verify that

$$\gamma (M(s)\Sigma_e) = \gamma (S(s)) = -\gamma \left(\hat{S}(s)\right).$$

Combining this with Lemma 2.6.4 and Theorem 2.6.1, we obtain the following theorem which holds irrespective of whether the network is minimally-reactive or whether $A(s)$ and $B(s)$ are left coprime:

Theorem 2.6.5. *Let $M(s)$ be the hybrid matrix of an m-port reciprocal network containing exactly p inductors and q capacitors, with current excitation at the first m_1 ports and voltage excitation at the remaining m_2 ports as in (115), and let $\Sigma_e := (I_{m_1} \dot{+} (-I_{m_2}))$. Then $M\Sigma_e \in \mathbb{R}^{m\times m}(s)$ is symmetric and, with $M(s)\Sigma_e$ written as a left matrix factorisation $M(s)\Sigma_e = B(s)^{-1}A(s)$, we have*

$$\begin{aligned} & q \geq \frac{1}{2} (\delta (M(s)\Sigma_e) + \gamma (M(s)\Sigma_e)) = \pi (\mathcal{B} (B, A)), \\ \textit{and} \quad & p \geq \frac{1}{2} (\delta (M(s)\Sigma_e) - \gamma (M(s)\Sigma_e)) = \nu (\mathcal{B} (B, A)). \end{aligned}$$

Theorem 2.6.5 provides lower bounds on the individual numbers of inductors and capacitors required to realise a given impedance function with a multi-port reciprocal network. This is a generalisation of the preceding results in this part. Indeed, Theorem 2.5.1 now follows as a special case of Theorem 2.6.5.

2.7 Continued fraction expansions and network synthesis

In this section, we derive a relationship between the extended Cauchy index and the properties of continued fraction expansions for real-rational functions, and we describe the implications for network synthesis. In particular, we use this relationship to show that the impedance function of any network which contains resistors, capacitors, and transformers only is realised by the forms of Foster and Cauer, which were first introduced in [8] and [9] respectively, and were described briefly in Subsection 0.1.2. Moreover, we provide explicit algebraic expressions for the element parameters in the Cauer forms in terms of the parameters in the corresponding network impedance function. These explicit expressions, which are provided in Theorem 2.7.4, are not apparent in the existing literature on passive network synthesis. The proof of that theorem depends on certain properties of the impedance functions which are realisable by networks containing only resistors, capacitors, and transformers. These properties are established in Lemmas 2.7.1 to 2.7.3.

We begin with the following lemma concerning the impedance function of a network which contains only resistors, capacitors, and transformers.

Lemma 2.7.1. *Let $Z(s)$ be the impedance of a one-port network containing resistors, capacitors and transformers only. Then $Z \in \mathbb{R}(s)$, $Z(s)$ is analytic in $s \in \mathbb{C}_+$, $Z(\xi) \geq 0$ for $\xi \in \mathbb{R}, \xi > 0$, and $\delta\left(Z(s)\right) = \gamma\left(Z(s)\right)$.*

Proof. That $\delta\left(Z(s)\right) = \gamma\left(Z(s)\right)$ follows from Theorem 2.5.1. The remaining conditions follow since $Z(s)$ is PR. □

Any given $H \in \mathbb{R}(s)$ possesses a partial fraction expansion

$$H(s) = H_1(s) + H_2(s) + \sum_{i=1}^{k} \frac{A_1^{(i)}}{s - \alpha_i} + \ldots + \frac{A_{n_i}^{(i)}}{(s - \alpha_i)^{n_i}},$$

where $H_1(s)$ is a polynomial in s, $H_2(s)$ is a strictly proper real-rational function with no real poles, n_i is a strictly positive integer and $A_{n_i}^{(i)} \neq 0$, and all the $\alpha_i \in \mathbb{R}$ are distinct ($i = 1, 2, \ldots, k$). From the properties of the McMillan degree [20, Section 3], it may be verified that

$$\delta\left(H(s)\right) = \deg\left(H_1(s)\right) + \delta\left(H_2(s)\right) + \sum_{i=1}^{k} n_i. \qquad (116)$$

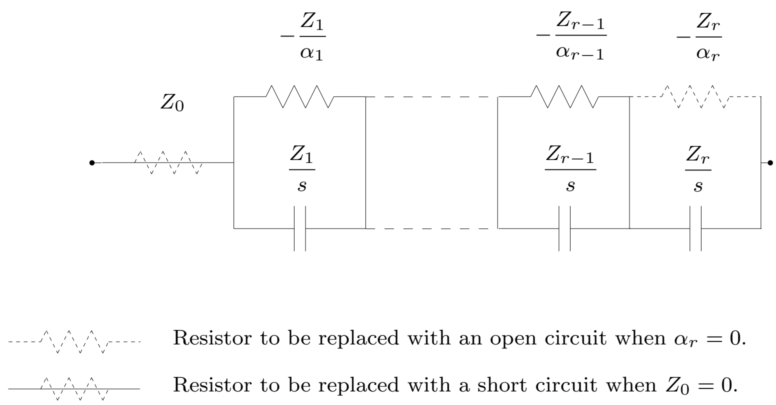


Figure 11: Foster canonical network for realising the function $Z(s)$ in (118).

Furthermore, whenever $H(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$, we have

$$\gamma\left(H(s)\right) = -\deg\left(H_1(s)\right) \bmod 2 + \sum_{i=1}^{k}(n_i \bmod 2) \text{ sign}\left(A_{n_i}^{(i)}\right). \tag{117}$$

To see this, note that $H_2(s)$ is bounded in $s \in \mathbb{R} \cup \infty$, and $H_1(s)$ is bounded in $s \in \mathbb{R}$ but has a pole at $s = \infty$ whenever $\deg\left(H_1(s)\right) > 0$. Then equation (117) follows by considering the nature of the jumps in $H(s)$ at the poles α_i $(i = 1, 2, \ldots, k)$, and at ∞. From equations (116) and (117), we see that $\gamma\left(H(s)\right) = \delta\left(H(s)\right)$ implies $\deg\left(H_1(s)\right) = \delta\left(H_2(s)\right) = 0$. Furthermore, we find that $n_i = 1$ and $A_{n_i}^{(i)} > 0$ for $i = 1, 2, \ldots, k$.

Combining the above with Lemma 2.7.1, it follows that if $Z(s)$ is the impedance of a one-port network containing resistors, capacitors, and transformers only, then $Z(s)$ has the partial fraction expansion

$$Z(s) = Z_0 + \sum_{i=1}^{r}\frac{Z_i}{s - \alpha_i}, \tag{118}$$

with $Z_0 \geq 0$, and both $Z_i > 0$ and $\alpha_i \leq 0$ for $i = 1, 2, \ldots r$ with $\alpha_i < \alpha_j$ for $i < j$. Here, $\delta\left(Z(s)\right) = \gamma\left(Z(s)\right) = r$. It is then clear that $Z(s)$ is the impedance of the *Foster form* in Fig. 11. In that figure, the dotted resistor without a continuous line through it is to be replaced with an open circuit whenever $\alpha_r = 0$, and the dotted resistor with a continuous line through it is to be replaced with a short circuit whenever $Z_0 = 0$.

Furthermore, from the partial fraction decomposition (118), we find

$$\frac{dZ}{ds} = -\sum_{i=1}^{r}\frac{Z_i}{(s - \alpha_i)^2},$$

and we see that $Z(s)$ is a monotonically decreasing function of s for $s \in \mathbb{R}$ (except at the poles α_i, $i = 1, 2, \ldots, r$). It follows that $Z(\xi) = 0$ at exactly one point $\xi \in \mathbb{R}$ between any two adjacent poles of $Z(s)$, which implies the interlacing property summarised in the next lemma.

Lemma 2.7.2. *Let $H \in \mathbb{R}(s)$, and let $H(s)$ be analytic in $s \in \mathbb{C}_+$ and satisfy $H(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$. Then $\gamma(H(s)) = \delta(H(s)) = r$ if and only if the poles of $H(s)$ (denoted α_i for $i = 1, 2, \ldots, r$), and zeros of $H(s)$ (denoted α_i' for $i = 1, 2, \ldots, r$), satisfy the interlacing property:*

$$-\infty \leq \alpha_1' < \alpha_1 < \alpha_2' < \alpha_2 < \ldots < \alpha_{r-1}' < \alpha_{r-1} < \alpha_r' < \alpha_r \leq 0.$$

We will now use the connection between the extended Cauchy index and the Sylvester matrix to show the following lemma:

Lemma 2.7.3. *Let $Z(s)$ in (98) be analytic in $s \in \mathbb{C}_+$ and let $Z(s)$ satisfy $Z(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$, and $\gamma(Z(s)) = \delta(Z(s)) = r$. Further let $\mathcal{S}_j$ be as in (99) for $j = 1, 2, \ldots$. Then*

$$\begin{aligned} &|\mathcal{S}_j| > 0 && \text{for } j = 1, 2, \ldots, 2r, \\ \text{and} \quad &|\mathcal{S}_{2r+1}| \geq 0 && \text{with } |\mathcal{S}_{2r+1}| = 0 \text{ if and only if } b_0 = 0. \end{aligned}$$

Proof. Since $Z(s)$ does not have a pole at $s = \infty$ by Lemma 2.7.2, then $Z(s)$ is proper, and hence $b_n > 0$. Furthermore, since $Z(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$, then the leading coefficient of $a(s)$ is also positive. Moreover, since $Z(s)$ is proper, then $\gamma(Z(s)) = I_{-\infty}^{+\infty} Z(s)$. From [39, Theorem 9, p. 210], it follows that the Hankel matrix $\mathcal{H}_k$ in (95) is positive definite for $k = 1, 2, \ldots, r$, where $h_0, h_1, \ldots$ are the parameters in the Laurent series for $Z(s)$ about ∞ in (92). Hence, for $k = 1, 2, \ldots, r$, we require $|\mathcal{H}_k| > 0$, which implies $|\mathcal{S}_{2k}| > 0$ by equation (100). Moreover, since $\delta(Z(s)) = r$, then $|\mathcal{H}_{r+1}| = 0$, and accordingly $|\mathcal{S}_{2(r+1)}| = 0$, by [39, Theorem 8, p. 207].

Now, note that

$$\frac{1}{sZ(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{a_n s^{n+1} + a_{n-1} s^n + \ldots + a_0 s},$$

so, in a similar fashion to the matrices $\mathcal{S}_j$ associated with the real-rational function

$Z(s)$ in (99), and providing $a_n \neq 0$, we associate the matrices

$$\hat{\mathcal{S}}_j := \left.\begin{bmatrix} \overbrace{\begin{matrix} a_n & a_{n-1} & a_{n-2} & \cdots \end{matrix}}^{j \text{ columns}} \\ \begin{matrix} 0 & b_n & b_{n-1} & \cdots \\ 0 & a_n & a_{n-1} & \cdots \\ 0 & 0 & b_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \end{bmatrix}\right\} j \text{ rows} \tag{119}$$

with the real-rational function $1/(sZ(s))$ $(j = 1, 2, \ldots)$. In the case $a_n = 0$, we instead let

$$\hat{\mathcal{S}}_j := \left.\begin{bmatrix} \overbrace{\begin{matrix} a_{n-1} & a_{n-2} & a_{n-3} & \cdots \end{matrix}}^{j \text{ columns}} \\ \begin{matrix} b_n & b_{n-1} & b_{n-2} & \cdots \\ 0 & a_{n-1} & a_{n-2} & \cdots \\ 0 & b_n & b_{n-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \end{bmatrix}\right\} j \text{ rows.} \tag{120}$$

From the interlacing property, and since $Z(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$, it is clear that $W(s) := 1/sZ(s)$ is analytic in $s \in \mathbb{C}_+$, that $W(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$, and that

$$\delta\left(\frac{1}{sZ(s)}\right) = \gamma\left(\frac{1}{sZ(s)}\right) = \tilde{r},$$

where $\tilde{r} = r + 1 - \epsilon_p - \epsilon_z$ with $\epsilon_p = 1$ if $Z(s)$ has a pole at $s = 0$ and 0 otherwise, and $\epsilon_z = 1$ if $Z(s)$ has a zero at $s = -\infty$ and 0 otherwise. Hence, by a similar argument to the preceding paragraph, we find $|\hat{\mathcal{S}}_{2k}| > 0$ for $k = 1, 2, \ldots, \tilde{r}$, and $|\hat{\mathcal{S}}_{2(\tilde{r}+1)}| = 0$.

There are now two cases to consider: (i) $\epsilon_z = 0$, and (ii) $\epsilon_z = 1$.

In case (i), $Z(s)$ does not have a zero at $s = -\infty$, and hence $a_n > 0$. Furthermore, $\tilde{r} = r + 1 - \epsilon_p$, and from equation (119) we find

$$\hat{\mathcal{S}}_{2(\tilde{r}+1)} = \begin{bmatrix} a_n & a_{n-1} \; a_{n-2} \; \cdots \\ 0 & \\ 0 & \mathcal{S}_{2\tilde{r}+1} \\ \vdots & \end{bmatrix}.$$

Since $|\hat{\mathcal{S}}_{2k}| > 0$ for $k = 1, 2, \ldots, r + 1 - \epsilon_p$, $|\mathcal{S}_{2k}| > 0$ for $k = 1, 2, \ldots, r$, and $a_n > 0$, then $|\mathcal{S}_j| > 0$ for $j \leq \max\{2r, 2(r - \epsilon_p) + 1\}$. Moreover, since $|\hat{\mathcal{S}}_{2(r+2-\epsilon_p)}| = 0$, then $|\mathcal{S}_{2r+1}| = 0$ if and only if $\epsilon_p = 1$.

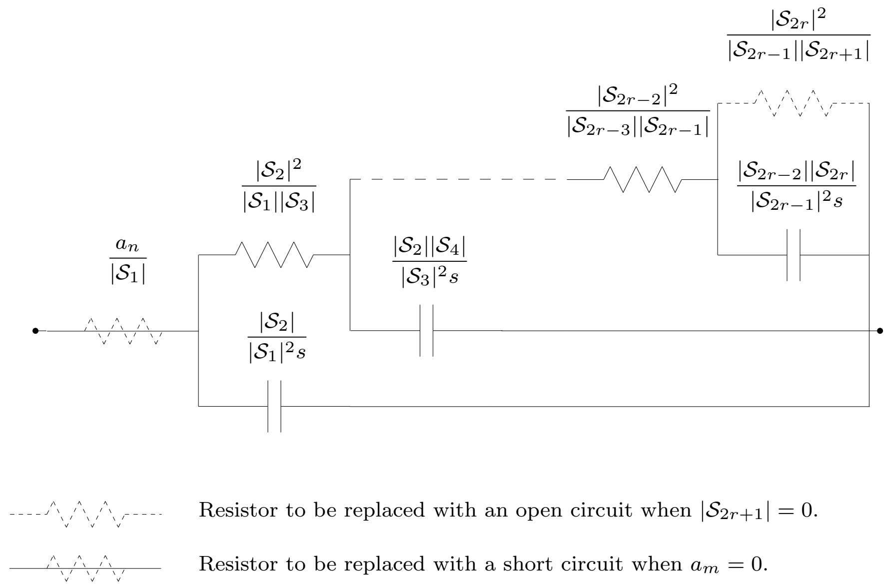


Figure 12: Network realisation for the function $Z(s)$ in (98). Here, $\mathcal{S}_j$ is as in (99) $(j = 1, 2, \ldots)$, and $r = \delta\,(Z(s))$.

In case (ii), $Z(s)$ has a zero at $s = -\infty$, so $a_n = 0$. Also, $\tilde{r} = r - \epsilon_p$, and from equation (120) we have

$$\mathcal{S}_{2(r+1)} = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \cdots \\ 0 & & & \\ 0 & & \hat{\mathcal{S}}_{2r+1} & \\ \vdots & & & \end{bmatrix},$$

which again implies that $|\mathcal{S}_j| > 0$ for $j \leq \max\{2r, 2(r - \epsilon_p) + 1\}$, and $|\mathcal{S}_{2r+1}| = 0$ if and only if $\epsilon_p = 1$.

Since $\epsilon_p = 1$ if and only if $b_0 = 0$, we conclude that $|\mathcal{S}_j| > 0$ for $j = 1, 2, \ldots, 2r$, $|\mathcal{S}_{2r+1}| \geq 0$, and $|\mathcal{S}_{2r+1}| = 0$ if and only if $b_0 = 0$. □

Finally, we will use a continued fraction expansion of $Z(s)$ to show the following theorem:

Theorem 2.7.4. *Let $Z(s)$ be the impedance of a one-port reciprocal network containing resistors, capacitors, and transformers only. Then $Z(s)$ is realised by the network in Fig. 12.*

We remark that when $b_0 = 0$ (which implies $|\mathcal{S}_{2r+1}| = 0$) the dotted resistor without a continuous line through it should be replaced by an open circuit, and when $a_n = 0$

the dotted resistor with a continuous line through it in Fig. 12 is replaced by a short circuit.

Proof. Following the caption in Fig. 12, we let $Z(s)$ be as in (98), we let $\mathcal{S}_j$ be as in (99) for $j = 1, 2, \ldots$, and we let $r = \delta(Z(s))$. From Lemma 2.7.1, $\gamma(Z(s)) = \delta(Z(s))$. To prove the present theorem, we will show that $Z(s)$ has the continued fraction expansion

$$Z(s) = u_r + \cfrac{1}{v_r s + \cfrac{1}{u_{r-1} + \cfrac{1}{v_{r-1}s + \ldots + \cfrac{1}{u_1 + \cfrac{1}{v_1 s + t}}}}}, \tag{121}$$

where

$$u_r = \frac{a_n}{|\mathcal{S}_1|}, \tag{122}$$

$$v_r = \frac{|\mathcal{S}_1|^2}{|\mathcal{S}_2|}, \tag{123}$$

and

$$u_k = \frac{|\mathcal{S}_{2(r-k)}|^2}{|\mathcal{S}_{2(r-k)-1}||\mathcal{S}_{2(r-k)+1}|}, \tag{124}$$

$$v_k = \frac{|\mathcal{S}_{2(r-k)+1}|^2}{|\mathcal{S}_{2(r-k)}||\mathcal{S}_{2(r-k+1)}|}, \tag{125}$$

for $k = 1, 2, \ldots, r-1$. Furthermore,

$$t = \frac{|\mathcal{S}_{2r-1}||\mathcal{S}_{2r+1}|}{|\mathcal{S}_{2r}|^2}, \tag{126}$$

or, equivalently,

$$t = \frac{|\mathcal{S}_{2n-1}|b_0}{|\mathcal{S}_{2n}|} \tag{127}$$

when $r = n$. By Lemma 2.7.3, it follows that $u_k, v_k > 0$ ($k = 1, 2, \ldots, r-1$), $v_r > 0$, $u_r \geq 0$ with $u_r = 0$ if and only if $a_n = 0$, and $t \geq 0$ with $t = 0$ if and only if $b_0 = 0$. It is then clear that $Z(s)$ is realised by the network in Fig. 12.

To show that $Z(s)$ has a continued fraction expansion of the form of (121), suppose $Z_k \in \mathbb{R}(s)$ with $\delta(Z_k(s)) = \gamma(Z_k(s)) = k$. Then $U_k(s) := 1/(Z_k(s) - \lim_{s\to\infty} Z_k(s))$

satisfies $\delta(U_k(s)) = -\gamma(U_k(s)) = k$ by Lemma 2.3.2, and has a pole at $s = \infty$ which must be simple as a consequence of Lemma 2.7.2. Now, consider $V_{k-1}(s) := U_k(s) - s\lim_{s\to\infty}(U_k(s)/s)$. Since $|\gamma(F(s))| \leq \delta(F(s))$ for any $F \in \mathbb{R}(s)$, and both $\delta(V_{k-1}(s)) = k-1$ and $\gamma(V_{k-1}(s)) \leq -(k-1)$ by Lemma 2.3.2, then $\delta(V_{k-1}(s)) = -\gamma(V_{k-1}(s)) = k-1$. Hence, $Z_{k-1}(s) := 1/V_{k-1}(s)$ satisfies

$$Z_{k-1}(s) = \frac{1}{\frac{1}{Z_k(s)-u_k} - v_k s}, \tag{128}$$

with

$$\begin{aligned} u_k &= \lim_{s\to\infty} Z_k(s), \\ v_k &= \lim_{s\to\infty}\left(\frac{1}{(Z_k(s)-u_k)s}\right), \\ \text{and} \qquad \delta(Z_{k-1}(s)) &= \gamma(Z_{k-1}(s)) = k-1. \end{aligned}$$

Now, let $Z(s) =: Z_r(s)$. Then $Z_r(s)$ satisfies $\delta(Z_r(s)) = \gamma(Z_r(s)) = r$ by Lemma 2.7.1, and from the preceding argument we obtain functions $Z_j(s)$ with $\delta(Z_j(s)) = \gamma(Z_j(s)) = j$ ($j = 0, 1, \ldots, r-1$). Moreover, from equation (128), we obtain

$$Z_k(s) = u_k + \frac{1}{v_k s + \frac{1}{Z_{k-1}(s)}},$$

and therefore $Z(s)$ has a continued fraction expansion of the form (121).

It remains to show that the parameters u_j, v_j ($j = 1, 2, \ldots, r$), and t, in the continued fraction expansion (121) are given by the equations (122) to (127). To see this, let $p(s)$ be the (monic) greatest common divisor of $a(s)$ and $b(s)$ in (98), let $m = n - r$ be the degree of $p(s)$, and write $Z_k(s) = a_k(s)/b_k(s)$ with

$$\begin{aligned} a_k(s) &= a_{k,m+k}s^{m+k} + a_{k,m+k-1}s^{m+k-1} + \ldots + a_{k,0}, \\ \text{and} \qquad b_k(s) &= b_{k,m+k}s^{m+k} + b_{k,m+k-1}s^{m+k-1} + \ldots + b_{k,0}, \end{aligned}$$

for $k = 0, 1, \ldots, r$, so $p(s)$ divides both $a_r(s)$ and $b_r(s)$. Then

$$Z_{k-1}(s) = \frac{a_k(s) - u_k b_k(s)}{(1 + u_k v_k s)b_k(s) - v_k s a_k(s)},$$

so let

$$a_{k-1}(s) = a_k(s) - u_k b_k(s), \tag{129}$$

$$\text{and} \qquad b_{k-1}(s) = (1 + u_k v_k s) b_k(s) - v_k s a_k(s). \tag{130}$$

Then, by induction, $p(s)$ divides both $a_j(s)$ and $b_j(s)$ $(j = 0, 1, \ldots, r)$. Moreover, we find

$$u_k = \lim_{s\to\infty} \frac{a_k(s)}{b_k(s)} = \frac{a_{k,m+k}}{b_{k,m+k}}, \tag{131}$$

and

$$v_k = \lim_{s\to\infty} \frac{b_k(s)}{s(a_k(s) - u_k b_k(s))} = \lim_{s\to\infty} \frac{b_k(s)}{s a_{k-1}(s)} = \frac{b_{k,m+k}}{a_{k-1,m+k-1}}, \tag{132}$$

for $k = 1, 2 \ldots r$.

Now, let

$$\mathcal{S}^{(k)} := \overbrace{\begin{bmatrix} b_{k,m+k} & b_{k,m+k-1} & b_{k,m+k-2} & \cdots \\ a_{k,m+k} & a_{k,m+k-1} & a_{k,m+k-2} & \cdots \\ 0 & b_{k,m+k} & b_{k,m+k-1} & \cdots \\ 0 & a_{k,m+k} & a_{k,m+k-1} & \cdots \\ 0 & 0 & b_{k,m+k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{2k+1 \text{ columns}} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}}\right\} 2k+1 \text{ rows.}$$

From equations (129) and (130), we obtain

$$a_k(s) = (1 + u_k v_k s) a_{k-1}(s) + u_k b_{k-1}(s),$$

$$\text{and} \qquad b_k(s) = v_k s a_{k-1}(s) + b_{k-1}(s),$$

which implies

$$\mathcal{S}^{(k)} = \hat{L}^{(k)} \begin{bmatrix} b_{k,m+k} & b_{k,m+k-1} & b_{k,m+k-2} & b_{k,m+k-3} & \cdots \\ 0 & a_{k-1,m+k-1} & a_{k-1,m+k-2} & a_{k-1,m+k-3} & \cdots \\ 0 & 0 & & & \\ 0 & 0 & & \mathcal{S}^{(k-1)} & \\ \vdots & \vdots & & & \end{bmatrix},$$

where

$$\hat{L}^{(k)} := \left.\overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ u_k & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & v_k & 1 & 0 & 0 & 0 & \cdots \\ 0 & u_k v_k & u_k & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & v_k & 1 & 0 & \cdots \\ 0 & 0 & 0 & u_k v_k & u_k & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{2k+1 \text{ columns}}\right\} 2k+1 \text{ rows}.$$

Proceeding by induction, we find

$$\mathcal{S}^{(r)} = LU, \tag{133}$$

where

$$U := \begin{bmatrix} b_{r,m+r} & b_{r,m+r-1} & \cdots & b_{r,m-r+2} & b_{r,m-r+1} & b_{r,m-r} \\ 0 & a_{r-1,m+r-1} & \cdots & a_{r-1,m-r+2} & a_{r-1,m-r+1} & a_{r-1,m-r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{1,m+1} & b_{1,m} & b_{1,m-1} \\ 0 & 0 & \cdots & 0 & a_{0,m} & a_{0,m-1} \\ 0 & 0 & \cdots & 0 & 0 & b_{0,m} \end{bmatrix},$$

in which $b_{k,j}, a_{k,j} = 0$ for $j < 0$. Moreover, L is the product of lower triangular matrices whose diagonal entries are all one, so L is itself lower triangular with ones on the diagonal. Then, from the Binet Cauchy theorem [35, Chapter I, Section 2] and equations (131) and (132), we obtain the relationships (122), (123), (124), and (125). This may be seen by multiplying the numerator and denominator of u_k in (131) by $a_{k,m+k} \times (b_{r,m+r} \times \Pi_{i=1}^{r-k-1} b_{k+i,m+k+i} a_{k+i,m+k+i})^2$, and also multiplying the numerator and denominator of v_k in (132) by $b_{k,m+k} \times (\Pi_{i=1}^{r-k} b_{k+i,m+k+i} a_{k+i-1,m+k+i-1})^2$, and by then factoring the resulting expressions into products of the leading principal minors of the matrix U. Moreover, since $\delta(Z_0(s)) = 0$, then $t = 1/Z_0(s) = b_0(s)/a_0(s) = b_{0,m}/a_{0,m}$, and equation (126) must hold. Finally, if $r = n$, then $m = 0$, and since $b_{k,j} = a_{k,j} = 0$ for $j < 0$, then the first $2r$ entries in the final column of U are zero. By equating the entry in the bottom right-hand corner of the matrix equation (133), we obtain the relationship (127), which must hold whenever $r = n$. □

We thus conclude that the impedance function of any given network which contains only resistors, capacitors, and transformers is also the impedance of the Cauer form network in Fig. 12. This network is minimally-reactive and transformerless. In fact, it is series-parallel (SP) in accordance with the definition in Subsection 3.2.1. Furthermore, we have derived explicit expressions for the element parameters in this network in terms of the parameters in the impedance function. These are given in Theorem 2.7.4 (taking into account its subsequent remark whenever certain of the element parameters are zero). Similar arguments allow us to conclude that any function $Z(s)$ which is the impedance of a network containing only resistors, inductors, and transformers satisfies $\gamma(Z(s)) = -\delta(Z(s))$. Furthermore, any such $Z(s)$ is also the impedance of a minimally-reactive and series-parallel network which contains only resistors and inductors. Explicit expressions for the element parameters in such a network may also be derived in a similar manner.

In other words, the network in Fig. 12 contains the least possible number of reactive elements for the realisation of $Z(s)$ in equation (98) whenever $Z(s)$ is the impedance of a network which contains only resistors, capacitors, and transformers (in which case $\delta(Z(s)) = \gamma(Z(s))$).

2.8 Biquadratic impedance functions

In this section, we consider the application of the previous results and techniques to the realisation of biquadratic PR functions, these being PR functions with McMillan degree two. Much recent literature on the topic of passive network synthesis has focussed on the realisation of such biquadratic functions [22–24]. In these papers, the biquadratic PR functions which may be realised by certain classes of networks are expressed algebraically in terms of the parameters in the impedance function. For example, in [22], the concept of a regular function is introduced, and an algebraic description of biquadratic regular functions is given.

The minors of the Sylvester matrix feature in many of these algebraic constraints. This has led several researchers to expect a connection between the Sylvester matrix and passive network synthesis. In particular, in [47], Foster hypothesised that the Sylvester determinant corresponding to a network's impedance is positive if the network contains two reactive elements of the same kind, and negative if it contains two reactive elements of different kinds. However, no proof was provided, as noted by Kalman [19]. We have provided a proof of this hypothesis in Section 2.3 of this thesis, as a special case of Theorem 2.3.4 for the case $n = 2$. In this section, we will first explicitly describe the condition obtained in that case. We will then show how the minors of the Sylvester matrix appear in the continued fractions which correspond to the impedance of certain series-parallel network realisations of regular biquadratic functions.

We begin by writing down explicitly the conditions obtained in Theorem 2.3.4 which apply to the biquadratic function. Let

$$Z(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}, \text{ with } a_j, b_j \geq 0 \text{ for } j = 0, 1, 2. \tag{134}$$

The matrix $\mathcal{S}_4$ takes the form

$$\mathcal{S}_4 = \begin{bmatrix} b_2 & b_1 & b_0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ 0 & a_2 & a_1 & a_0 \end{bmatrix}, \tag{135}$$

and we have

$$\begin{aligned} & |\mathcal{S}_2| = b_2 a_1 - b_1 a_2, \\ \text{and} \quad & |\mathcal{S}_4| = (b_2 a_1 - b_1 a_2)(b_1 a_0 - b_0 a_1) - (b_2 a_0 - b_0 a_2)^2. \end{aligned}$$

The realisability conditions implied by Theorem 2.3.4 are shown in Table 1. Note that $|\mathcal{S}_4| > 0$ implies $|\mathcal{S}_2| \neq 0$. In Table 1, it may be observed that whether the reactive elements are of the same kind, or of different kind, is determined by the sign of the determinant $|\mathcal{S}_4|$ (this determinant being equal to the *resultant* of the numerator and denominator polynomials of $Z(s)$). This fact is stated by Foster [47], but no proof is provided, as noted by Kalman [19]. Also, for the case that $|\mathcal{S}_4| > 0$, [47] differentiates the two cases in Table 1 according to $\text{sign}(b_2 a_0 - a_2 b_0)$ rather than $\text{sign}(|\mathcal{S}_2|)$, which is easily shown to be equivalent.

	$\lvert\mathcal{S}_2\rvert > 0$	$\lvert\mathcal{S}_2\rvert < 0$	$\lvert\mathcal{S}_2\rvert = 0$
$\lvert\mathcal{S}_4\rvert > 0$	$(0, 2)$	$(2, 0)$	-
$\lvert\mathcal{S}_4\rvert < 0$	$(1, 1)$	$(1, 1)$	$(1, 1)$
$\lvert\mathcal{S}_4\rvert = 0$	$(0, 1)$	$(1, 0)$	$(0, 0)$

Table 1: The number of reactive elements is given here in the form (# inductors, # capacitors) for any minimally-reactive reciprocal network whose impedance takes the form (134).

Table 1 does not contain any information about synthesis, i.e. whether a network exists which realises a given biquadratic PR function $Z(s)$, it only gives the properties that a minimally-reactive reciprocal realisation must satisfy whenever one does exist. However, we have already shown some results pertaining to the existence of such networks.

Indeed, from the results in Section 2.7, we see that minimally-reactive transformerless networks exist to realise *any* PR function in the form (134) whenever $|\mathcal{S}_4| \geq 0$.

The case $|\mathcal{S}_4| < 0$ is much more complex. In Part 3 it will be shown that there are certain biquadratic PR functions which cannot be realised by transformerless minimally-reactive networks. Those biquadratic PR functions which are realised by minimally-reactive series-parallel (SP) networks are investigated in [22]. In Section IV of that paper, the concept of a regular function is introduced in accordance with the following definition:

Definition 2.8.1 (Regular function)**.**
We say $H(s)$ is *regular* if it is PR, and if, for $\omega \in \mathbb{R} \cup \infty$, either the smallest value of $\Re(H(j\omega))$, or the smallest value of $\Re(1/H(j\omega))$, occurs at either $\omega = 0$ or at $\omega = \infty$.

In [22, Theorem 1], it is then shown that $Z(s)$ can be realised by a SP network containing at most two reactive elements if and only if $Z(s)$ is regular.

In the remainder of this section, we will show that any regular biquadratic function possesses a continued fraction expansion which corresponds to the impedance of a SP network. We also relate the parameters in this continued fraction expansion to the minors of the Sylvester matrix. As described earlier in this section, the case $|\mathcal{S}_4| \geq 0$ is covered by the results in Section 2.7. Accordingly, we consider the case $|\mathcal{S}_4| < 0$ in the following lemma. In the proof of this lemma, and in Fig. 13, we denote by

$$M \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix}$$

the minor formed from rows $i_1, i_2, \dots, i_p$, and columns $j_1, j_2, \dots, j_p$, of the matrix M.

Lemma 2.8.2. *Let $Z(s)$ be the impedance of a minimally-reactive SP network which contains exactly one inductor and exactly one capacitor. Then $Z(s)$ is realised by one of the networks in Fig. 13.*

Proof. Since $Z(s)$ is PR then $Z(s)$ takes the form of (134) with $a_j, b_j \geq 0$. Moreover, since $Z(s)$ is the impedance of a network N which contains exactly one inductor and exactly one capacitor and is minimally-reactive, then $|\mathcal{S}_4| < 0$ for $\mathcal{S}_4$ as in (135) by Theorem 2.3.4. Also, since N is SP, then $Z(s)$ is regular by [22, Theorem 1] and [48, p. 619]. There are thus four cases to consider: (i) $\min_{\omega \in \mathbb{R}\cup\infty}(Z(j\omega)) = Z(0)$, (ii) $\min_{\omega \in \mathbb{R}\cup\infty}(1/Z(j\omega)) = 1/Z(0)$, (iii) $\min_{\omega \in \mathbb{R}\cup\infty}(Z(j\omega)) = Z(\infty)$, and (iv) $\min_{\omega \in \mathbb{R}\cup\infty}(1/Z(j\omega)) = 1/Z(\infty)$. We will show that, in case (i), $Z(s)$ is realised by the network on the top left of Fig. 13.

Since $\min_{\omega\in\mathbb{R}\cup\infty} \Re(Z(j\omega)) = Z(0)$, then $Z(0)$ must be bounded, and so $b_0 \neq 0$. Let

$$u_2 := Z(0) = \frac{a_0}{b_0} \geq 0, \tag{136}$$

$$\text{and} \qquad H_1(s) := Z(s) - u_2 = \frac{a_{1,2}s^2 + a_{1,1}s}{b_2s^2 + b_1s + b_0},$$

which is PR by [10, Theorem III, Corollary 1]. Then $H_2(s) := 1/H_1(s)$ is PR by [10, Theorem I, Coroll, 1], and $H_2(s)$ has a simple pole at $s = 0$ with residue

$$v_2 := \lim_{s\to 0} sH_2(s) = \frac{b_0}{a_{1,1}} > 0. \tag{137}$$

Moreover, let

$$\frac{1}{H_3(s)} := H_2(s) - \frac{v_2}{s} = \frac{b_{1,2}s + b_{1,1}}{a_{1,2}s + a_{1,1}},$$

which is PR by [10, Theorem IV]. Then $\delta\left(H_3(s)\right) = 1$, and $\gamma\left(H_3(s)\right) = -\gamma\left(1/H_3(s)\right) = \gamma\left(v_2/s\right) - \gamma\left(H_2(s)\right) = 1 + \gamma\left(Z(s)\right)$. Since $|\mathcal{S}_4| < 0$ then $\gamma\left(Z(s)\right) = 0$ (see Table 1 and Theorem 2.2.1), and hence $\gamma\left(H_3(s)\right) = \delta\left(H_3(s)\right) = 1$. Since $H_3(s)$ is PR then $H_3(\xi) \geq 0$ for all $\xi \in \mathbb{R}, \xi > 0$. It then follows from Section 2.7 that $H_3(s)$ has a continued fraction expansion

$$H_3(s) = u_1 + \frac{1}{v_1s + t},$$

where

$$u_1 := \lim_{s\to\infty} H_3(s) = \frac{a_{1,2}}{b_{1,2}} \geq 0. \tag{138}$$

Moreover, with

$$H_4(s) := H_3(s) - u_1 = \frac{a_{0,1}}{b_{1,2}s + b_{1,1}} = \frac{1}{v_1s + t},$$

then

$$v_1 = \frac{b_{1,2}}{a_{0,1}} > 0, \tag{139}$$

$$\text{and} \qquad t = \frac{b_{1,1}}{a_{0,1}} \geq 0. \tag{140}$$

We have shown that whenever $Z(s)$ is PR with $|\mathcal{S}_4| < 0$ and $\min_{\omega\in\mathbb{R}\cup\infty}(Z(j\omega)) = Z(0)$, then $Z(s)$ has the continued fraction expansion

$$Z(s) = u_2 + \cfrac{1}{\cfrac{v_2}{s} + \cfrac{1}{u_1 + \cfrac{1}{v_1 s + t}}},$$

where u_i, v_i $(i = 1, 2)$, and t, are given by the expressions in the preceding argument. By relating the above process to manipulations on the rows of the matrix $\mathcal{S}_4$, we obtain

$$\mathcal{S}_4 = \begin{bmatrix} 1 & 0 & v_2 u_2 & v_2 \\ u_1 + u_2 & 1 & u_1 v_2 u_2 & u_1 v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & u_2 & 1 \end{bmatrix} \begin{bmatrix} b_{1,2} & b_{1,1} & 0 & 0 \\ 0 & a_{0,1} & 0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ 0 & a_{1,2} & a_{1,1} & 0 \end{bmatrix}.$$

Then, by equating suitable submatrices in the above equation and applying the Binet Cauchy theorem, we find

$$\begin{aligned} b_0 &= \mathcal{S}_4\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right), \\ b_{1,2} &= \mathcal{S}_4\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \\ b_0 a_{1,1} &= -\mathcal{S}_4\left(\begin{smallmatrix} 3 & 4 \\ 3 & 4 \end{smallmatrix}\right) = -\mathcal{S}_4\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right), \\ b_0 a_{1,2} &= -\mathcal{S}_4\left(\begin{smallmatrix} 3 & 4 \\ 2 & 4 \end{smallmatrix}\right) = -\mathcal{S}_4\left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right), \\ b_{1,1} b_0 a_{1,1} &= -\mathcal{S}_4\left(\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 3 & 4 \end{smallmatrix}\right), \\ \text{and} \qquad b_{1,2} a_{0,1} b_0 a_{1,1} &= -|\mathcal{S}_4|. \end{aligned}$$

From equations (136)-(140), we see that $Z(s)$ is realised by the network on the top left of Fig. 13. If, instead, $\min_{\omega \in \mathbb{R} \cup \infty}(1/Z(j\omega)) = 1/Z(0)$ (resp. $\min_{\omega \in \mathbb{R} \cup \infty}(Z(j\omega)) = Z(\infty)$; $\min_{\omega \in \mathbb{R} \cup \infty}(1/Z(j\omega)) = 1/Z(\infty)$) then a similar argument shows that $Z(s)$ is realised by the network on the top right (resp. bottom left; bottom right) of Fig. 13. □

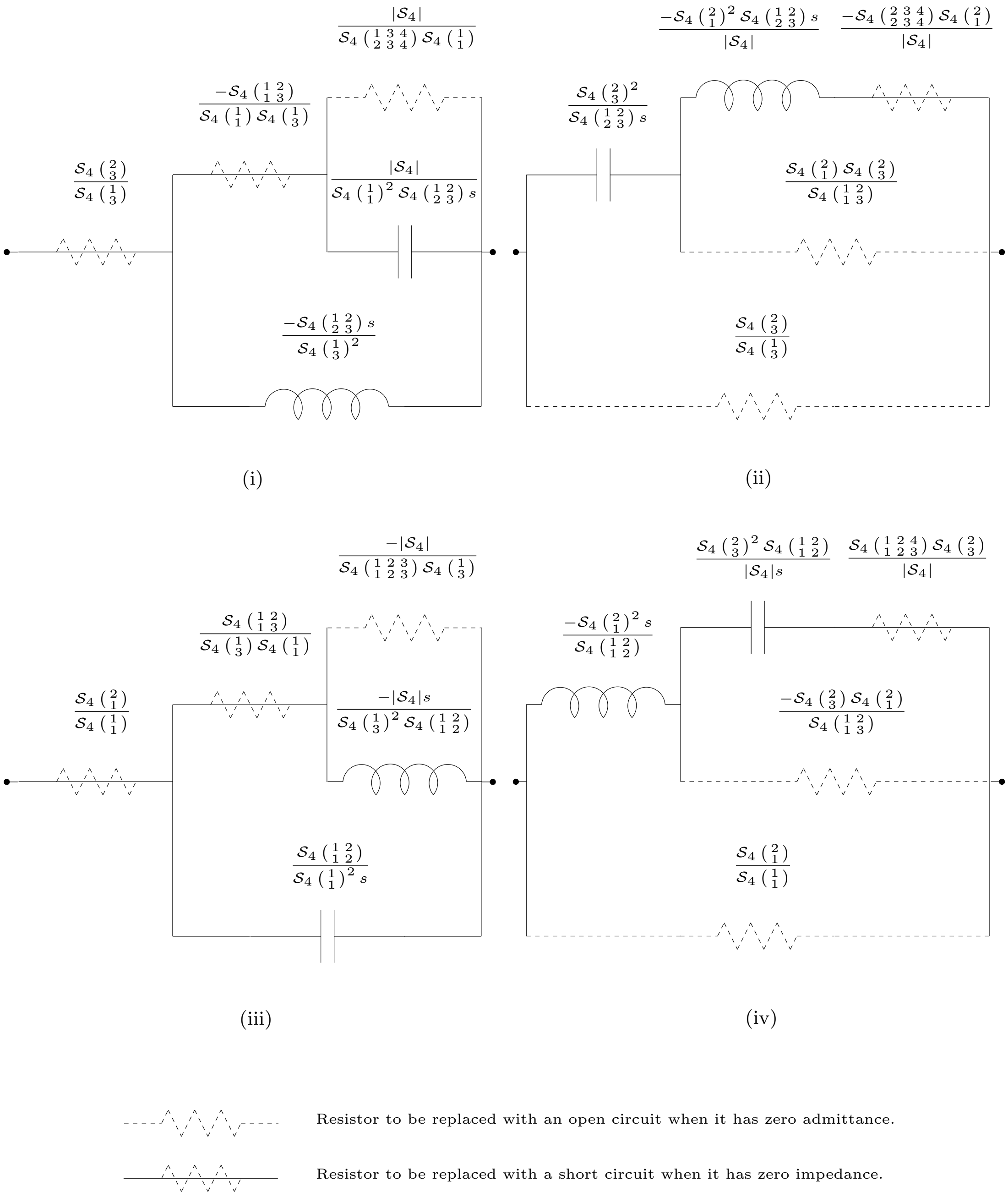


Figure 13: Realisations for the biquadratic regular function $Z(s)$. In this figure, $Z(s)$ is as in (134) with $a_j, b_j \geq 0$ $(j = 0, 1, 2)$, and $\mathcal{S}_4$ is as in (135) with $|\mathcal{S}_4| < 0$. Network (i) corresponds to the case $\min_{\omega\in\mathbb{R}\cup\infty}(Z(j\omega)) = Z(0)$, network (ii) to the case $\min_{\omega\in\mathbb{R}\cup\infty}(1/Z(j\omega)) = 1/Z(0)$, network (iii) to the case $\min_{\omega\in\mathbb{R}\cup\infty}(Z(j\omega)) = Z(\infty)$, and network (iv) to the case $\min_{\omega\in\mathbb{R}\cup\infty}(1/Z(j\omega)) = 1/Z(\infty)$.

Part 3

Minimality and uniqueness of transformerless network realisation procedures

It is well known that the impedance of a passive network is necessarily positive-real [10]. In [12], Bott and Duffin provided the first explicit construction for the realisation of a given (scalar) positive-real function as the impedance of a transformerless network. A significant remaining challenge in network synthesis is to establish the minimum numbers of reactive elements and resistors which are required to realise a given positive-real (PR) function. Indeed, it is a major puzzle that the only known procedures for the realisation of a general (scalar) PR function require apparently extravagant numbers of reactive elements and resistors. There have been no improvements on these procedures since 1954, with the simultaneous publication in [13–15] of the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure.

In this part, we describe the Bott-Duffin procedure, and the simplification of Reza-Pantell-Fialkow-Gerst. We then present a sequence of network transformations which lead to several equivalent networks to those of Bott and Duffin. In particular, we obtain a new simplification to Bott-Duffin, which produces networks which contain the same number of reactive elements, and the same number of resistors, as the Reza-Pantell-Fialkow-Gerst networks. We subsequently examine the minimality of the networks produced by these various procedures. In particular, we show how the Bott-Duffin procedure produces networks which contain the least possible number of reactive elements, and the least possible number of resistors, for the realisation of certain PR functions (called biquadratic minimum functions) among all series-parallel (SP) networks. Moreover, we show how the networks produced by the Reza-Pantell-Fialkow-Gerst simplification, and the alternatives presented in this part, contain the least possible number of reactive elements, and the least possible number of resistors, for the realisation of a biquadratic minimum function among all transformerless networks, with the exception of a few special cases. We will also provide an explicit description of these special cases. In addition, we describe all those transformerless (resp. SP) networks which realise a biquadratic minimum function and which contain the same number of reactive elements and the same number of resistors as the Reza-Pantell-Fialkow-Gerst (resp. Bott-Duffin) networks. Finally, we consider the case of impedance functions of higher McMillan degree. In particular, we prove the existence of PR functions of McMillan degree $2r$ which cannot be realised by SP networks which contain fewer than $4r$ reactive elements.

This part is structured as follows. In Section 3.1, we describe the Bott-Duffin procedure, and we derive the aforementioned equivalent networks to those of Bott and Duffin. In Section 3.2, we provide some technical preliminaries required for the subsequent sections. Specifically, we introduce a parametrisation of the biquadratic minimum function which facilitates the subsequent analysis, we provide a formal framework for classifying networks, and we provide algebraic conditions for two polynomials to have roots in common. We consider the minimality and uniqueness of the networks produced by the Bott-Duffin procedure for the realisation of biquadratic minimum functions among the class of SP networks in Sections 3.3 and 3.4. Then, in Sections 3.5 and 3.6, we consider the minimality and uniqueness of the networks produced by the Reza-Pantell-Fialkow-Gerst simplification for the realisation of biquadratic minimum functions among the broader class of transformerless networks. Finally, in Section 3.7, we consider the numbers of reactive elements required for the realisation of certain PR functions of higher McMillan degree.

Since the focus in this part is on transformerless (hence, one-port) networks, we will use the descriptor PR to imply a scalar positive-real function.

3.1 Transformerless network realisation procedures

The procedure of Bott and Duffin, and the Reza-Pantell-Fialkow-Gerst simplification to this procedure, allow one to realise a given PR function as the impedance of a transformerless network. The procedures are inductive, and involve a preliminary procedure at each stage of the inductive process known as the Foster preamble. The Foster preamble reduces a given PR function to a special type of PR function known as a minimum function. We describe the Foster preamble, and provide a formal definition for a minimum function, in Section 3.1.2. In Section 3.1.3, we then describe the Bott-Duffin procedure itself, before describing the Reza-Pantell-Fialkow-Gerst simplification in Section 3.1.4. We then present two new alternatives to Bott-Duffin in Section 3.1.5, and we show how the networks obtained from these new alternative procedures, as well as the Reza-Pantell-Fialkow-Gerst simplification, may be derived from the networks obtained by the Bott-Duffin procedure through the application of a sequence of network transformations. The first new alternative to the Bott-Duffin procedure that we present produces series-parallel (SP) networks which contain the same number of reactive elements and the same number of resistors as the networks from the Bott-Duffin procedure. This network was alluded to in our paper [49]. The second new alternative that we present produces networks which contain the same number of reactive elements and the same number of resistors as the networks obtained from the Reza-Pantell-Fialkow-Gerst simplification. This represents the simplest procedure to be identified for the realisation of a given PR function as the impedance of a transformerless network since the simultaneous publication in 1954 of the three papers [13–15], which described the Reza-Pantell-Fialkow-Gerst simplification.

First, in Subsection 3.1.1, we describe Foster's reactance theorem, the prototype for all subsequent transformerless network realisation procedures.

3.1.1 Foster's reactance theorem

The 1924 paper [8] by Foster considers the realisation problem for networks which contain only reactive elements. In that paper, it is shown that the impedance of such a network is necessarily lossless in accordance with the following definition:

Definition 3.1.1 (Lossless function)**.**
We say $H \in \mathbb{R}(s)$ is lossless if and only if all poles and zeros of $H(s)$ are on $j\mathbb{R} \cup \infty$, are simple, and alternate, with the poles having (real) positive residues.

Equivalently, it may be shown that $H(s)$ is lossless if it is both PR and maps the imaginary axis onto the imaginary axis, and moreover that $H(s)+H(-s) = 0$ whenever $H(s)$ is lossless [36, Section 9.5].

By considering the properties of partial fraction expansions for lossless functions, Foster showed how any given lossless function can be realised by two different SP networks, each of which contains only capacitors and inductors. As will be seen in the next subsection, Foster's reactance theorem may be extended to cover the realisation of a given PR function with a transformerless network (the so-called *Foster preamble*). However, for many PR functions, the Foster preamble only provides a partial realisation, leaving a special type of PR function known as a minimum function whose realisation is required.

3.1.2 The Foster preamble

The Foster preamble is a network synthesis procedure which either provides a transformerless network realisation for a given PR function, or provides a partial realisation and reduces the given PR function to a minimum function. A minimum function is defined as follows:

Definition 3.1.2 (Minimum function)**.**
$H(s)$ is a minimum function if it is PR, not identically zero, has no poles or zeros on the extended imaginary axis, and satisfies $\Re(H(j\omega_0)) = 0$ for at least one strictly positive value of ω_0 (this implies $\Im(H(j\omega_0)) \neq 0$). The value of ω_0 is called a minimum frequency.

The Foster preamble depends on the following properties of PR functions:

Theorem 3.1.3 ([10], Theorem III, Coroll. 1)**.** *If $H(s)$ is PR, and R_1 is less than or equal to $\min_{\omega\in\mathbb{R}\cup\infty} \Re(H(j\omega))$, then $H_r(s) = H(s) - R_1$ is PR.*

Theorem 3.1.4 ([10], Theorem IV)**.** *If $H(s)$ is PR and has poles at $s = j\omega_r$ for $\omega_r > 0$ with residue $k_r/2$ ($r = 1, 2, \ldots, n$), in addition to a pole at $s = 0$ with residue k_0, and a pole at $s = \infty$ with residue k_∞, then*

$$H(s) = \frac{k_0}{s} + k_\infty s + \sum_{r=1}^{n} \frac{k_r s}{s^2 + \omega_r^2} + H_r(s), \tag{141}$$

where each term is PR.

In Theorem 3.1.3, $H(s)$ is the impedance of a network with impedance $H_r(s)$ in series with a resistor with resistance R_1. Moreover, each term in the sum in (141) is the impedance of a parallel connection of an inductor and a capacitor, and the first two terms on the right hand side of equation (141) are the impedance of a capacitor and an inductor respectively. Hence, in Theorem 3.1.4, $H(s)$ is the impedance of a network with impedance $H_r(s)$ in series with a network comprising reactive elements. Similar partial network realisations can be obtained for which $H(s)$ is equal to the admittance of the network. The inductive application of these two procedures will result in a function $H_r(s)$ which is either identically zero or is a minimum function. Moreover, the McMillan degree of $H_r(s)$ cannot exceed that of $H(s)$.

3.1.3 The Bott-Duffin procedure

The procedure of Bott-Duffin provides a realisation for a given minimum function $H(s)$ as the impedance of a network which contains six reactive elements and two subnetworks whose impedances have McMillan degree at least two fewer than $H(s)$. The proof in [12] uses a generalisation of a theorem in [50], which we now describe. Let $H(s)$ be PR and $\mu > 0$, and let

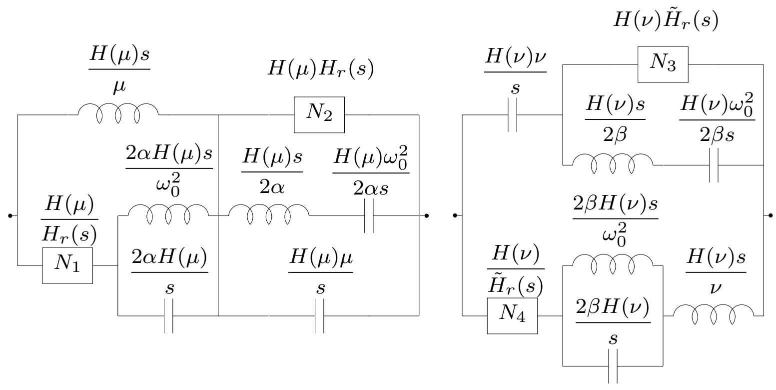
$$R(s) := \frac{\mu H(s) - sH(\mu)}{\mu H(\mu) - sH(s)}. \tag{142}$$

Then $R(s)$ is PR, and its McMillan degree does not exceed that of $H(s)$.

Bott and Duffin applied this result to the realisation problem as follows. Suppose $H(s)$ is a minimum function, and let $\omega_0 > 0$ and $X \in \mathbb{R}$ with $X \neq 0$ be such that $H(j\omega_0) = j\omega_0 X$. Consider first the case where $X > 0$. Let $\mu > 0$ be a solution to $X = H(\mu)/\mu$ (such a solution is guaranteed since the function $H(\mu)/\mu$ takes on all values between 0 and $+\infty$ as μ is varied between 0 and $+\infty$). Then the function $R(s)$ in (142) has zeros, and hence $1/R(s)$ has poles, at $s = \pm j\omega_0$. Let α be the residue of $1/R(s)$ at $s = j\omega_0$, and let $1/H_r(s) = 1/R(s) - 2\alpha s/(s^2 + \omega_0^2)$. Then $H_r(s)$ is PR and has McMillan degree at least two fewer than $H(s)$. Furthermore, by rearranging equation (142) we find

$$H(s) = \left(\frac{R(s)}{H(\mu)} + \frac{\mu}{H(\mu)s}\right)^{-1} + \left(\frac{1}{H(\mu)R(s)} + \frac{s}{H(\mu)\mu}\right)^{-1}.$$

It follows that $H(s)$ is the impedance of the network on the left of Fig. 14, where $H(\mu)H_r(s)$ and $H(\mu)/H_r(s)$ are both PR. If, on the other hand, $X < 0$, then similar considerations applied to $1/H(s)$ show that $H(s)$ is the impedance of the network on the



(a) (b)

Figure 14: The networks obtained in a single inductive step of the Bott-Duffin procedure for realisation of a minimum function $H(s)$ in the cases: (a) $X > 0$, (b) $X < 0$, where $H(j\omega_0) = j\omega_0 X$. Here, μ, α, ν and β are as defined in Section 3.1.3.

right of Fig. 14. Here $\nu > 0$ solves $\nu H(\nu) = j\omega_0 H(j\omega_0)$, and $(\nu H(s) - sH(\nu))/(\nu H(\nu) - sH(s)) = 1/\tilde{H}_r(s) + 2\beta s/(s^2 + \omega_0^2)$, where $\tilde{H}_r(s)$ is PR with McMillan degree at least two fewer than $H(s)$. In this case, $H(\nu)\tilde{H}_r(s)$ and $H(\nu)/\tilde{H}_r(s)$ are both PR. The Foster preamble and Bott-Duffin procedure may then be applied to the impedances $H(\mu)H_r(s)$ and $H(\mu)/H_r(s)$ (or $H(\nu)\tilde{H}_r(s)$ and $H(\nu)/\tilde{H}_r(s)$ when $X < 0$), and the process repeats. In this manner one can construct transformerless networks to realise *any* specified PR function[21].

We remark that for a biquadratic function the McMillan degree of $H_r(s)$, and of $\tilde{H}_r(s)$, will be zero, and hence the impedances $H(\mu)H_r(s)$ and $H(\mu)/H_r(s)$ (or $H(\nu)\tilde{H}_r(s)$ and $H(\nu)/\tilde{H}_r(s)$) are each realised by a resistor. In this case, the networks each contain six reactive elements and two resistors.

3.1.4 The Reza-Pantell-Fialkow-Gerst simplification to Bott-Duffin

Since the publication of the Bott-Duffin procedure [12], it has been a matter of speculation whether a simpler realisation procedure may exist. A slight improvement on the Bott-Duffin procedure was published simultaneously in the three papers [13–15]. We refer to this as the Reza-Pantell-Fialkow-Gerst simplification, after the authors of these three papers. In those papers, it was shown that any minimum function $H(s)$ with a minimum function at ω_0 and such that $H(j\omega_0) = j\omega_0 X$ may be realised by the network on the left of Fig. 15 in the case $X > 0$, and by the network on the right of Fig. 15 in the case $X < 0$. In that figure, α, $H(\mu)$, μ, β, $H(\nu)$, and ν are as defined in Section

[21]Indeed, these networks are also SP.

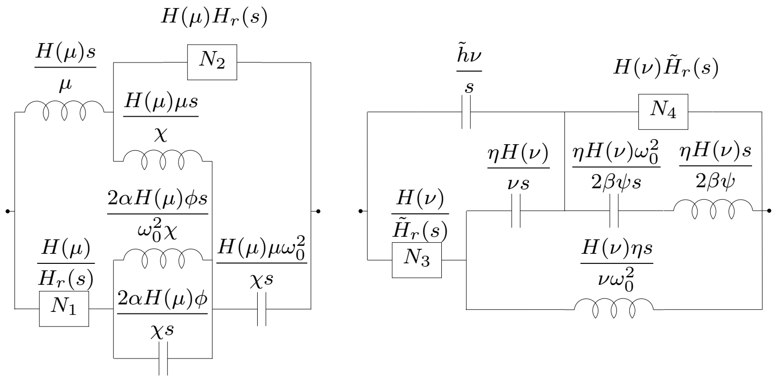


Figure 15: The networks obtained in a single inductive step of the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure for a minimum function $H(s)$ in the cases: (a) $X > 0$, (b) $X < 0$, where $H(j\omega_0) = j\omega_0 X$. Here, $\chi = \omega_0^2 + 2\alpha\mu$, $\phi = \chi + \mu^2$, $\eta = \omega_0^2 + 2\beta\nu$, $\psi = \eta + \nu^2$, and μ, α, ν and β are as defined in Section 3.1.3.

3.1.3. The simplification gives a saving of a single reactive element per inductive step relative to the Bott-Duffin procedure.

Despite ongoing speculation as to whether other simpler procedures may exist, no further general realisation procedures have appeared which produce networks which contain as few reactive elements as the Reza-Pantell-Fialkow-Gerst simplification. In the next subsection, we derive a new procedure for the realisation of a given minimum function which produces networks containing the same number of reactive elements and the same number of resistors as the Reza-Pantell-Fialkow-Gerst simplification. We further show how the networks derived by this procedure, and those networks obtained from the Reza-Pantell-Fialkow-Gerst simplification, may be obtained from the Bott-Duffin networks through a sequence of network transformations.

3.1.5 A new simplification to Bott-Duffin

In this subsection, we describe a sequence of transformations to the networks obtained by the Bott-Duffin procedure to produce several alternatives to that procedure for the realisation of a given minimum function. These alternatives include those networks obtained from the Reza-Pantell-Fialkow-Gerst simplification, as well as two original alternatives. The first of these original alternatives was alluded to in our paper [49], and obtains series-parallel networks which contain the same number of reactive elements and the same number of resistors as the Bott-Duffin networks. The second of these alternatives has not appeared previously in the literature, and obtains networks which contain the same number of reactive elements and the same number of resistors as the networks obtained by the Reza-Pantell-Fialkow-Gerst simplification.

The results in this subsection are summarised in the following theorem, whose proof occupies the remainder of this subsection.

Theorem 3.1.5. *Let $H(s)$ be a minimum function with $H(j\omega_0) = j\omega_0 X$ for $X > 0$ (resp. $X < 0$). Then $H(s)$ is realised by the networks on the top left and bottom right (resp. top right and bottom left) of Figs. 19.1, 19.2, 19.3, and 19.4.*

Proof. From Section 3.1.3, the network on the top left of Fig. 19.1 is the network obtained in a single inductive step of the Bott-Duffin procedure for the realisation of a given minimum function $H(s)$ which satisfies $H(j\omega_0) = j\omega_0 X$ with $X > 0$. We will first show the equivalence of the impedance of the network on the top left of Fig. 19.1 to the impedances of the networks on the top left of Figs. 19.2, 19.3, and 19.4.

The transformation [22, Lemma 11] shows that the network on the top left of Fig. 19.1 has the same impedance as the network on the top left of Fig. 19.2.

Consider now a trajectory for the network in Fig. 16 whose Laplace transform exists. Following a long but routine calculation using the network analysis results in Part 1, we obtain the surprising result that $i_1(s) = 0$. It follows that the connection between the points A and B in this network may be removed without affecting its impedance[22]. The two series connected capacitors resulting from the removal of this connection may then be replaced with a single capacitor. Hence, this network has the same impedance as the network on the top left of Fig. 19.4. The network on the top left of Fig. 19.4 is our original simplification to the Bott-Duffin procedure for the realisation of a minimum function $H(s)$ satisfying $H(j\omega_0) = j\omega_0 X$ with $X > 0$.

We now show how the argument presented in the preceding paragraph may be extended to show that the network on the top left of Fig. 19.2 has an equivalent impedance to the network on the top left of Fig. 19.3. This provides a new insight into the relationship between the Bott-Duffin procedure and the Reza-Pantell-Fialkow-Gerst simplification. Consider again the trajectory for the network in Fig. 16 which was described in the previous paragraph. Routine calculations based on the results in Part 1 give

$$\begin{aligned} i_3(s) &= \frac{\omega_0^2}{\omega_0^2 + 2\alpha\mu} i_4(s) \\ &= \frac{H_r(s)\omega_0^2 s}{s^3 + H_r(s)(2\alpha + \mu)s^2 + \omega_0^2 s + H_r(s)\mu\omega_0^2} \frac{v(s)}{H(\mu)}. \end{aligned}$$

[22]This follows from an argument similar to that which will be employed in the proof of Lemma 3.5.10.

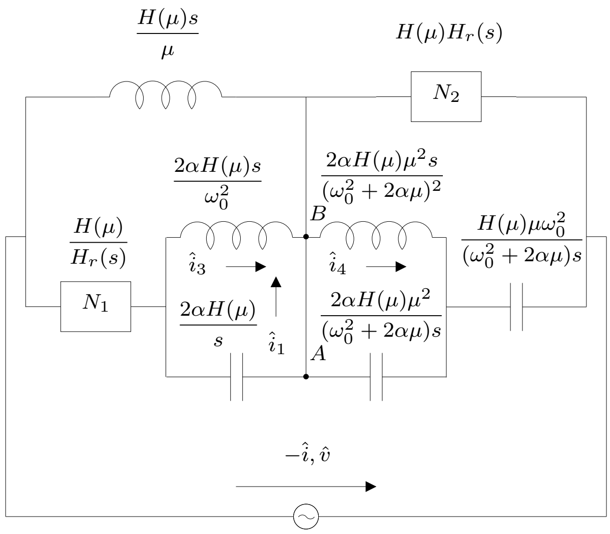


Figure 16: A series-parallel alternative to Bott-Duffin.

We may replace the inductor with current $\hat{i}_4$ with two inductors in parallel, whose admittances sum to the admittance of the original inductor, without affecting the impedance of the network. In effect, we replace this element with a current-divider. A pertinent choice is shown in Fig. 17. In this network, we find $i_1(s) = i_2(s) = 0$. It follows that the connections both between points A and B and between points B and C may be removed without affecting the impedance of this network. Again, combining all series connections of elements of the same kind, we arrive at the network on the top left of Fig. 19.3.

As an alternative to the preceding proof of the equivalence of the impedances of the networks on the top left of Figs. 19.1, 19.2, 19.3, and 19.4, it suffices to calculate the impedances directly using equation (38). In each case, we find that the impedance is equal to

$$H(\mu)\frac{s^3 + H_r(s)(2\alpha + \mu)s^2 + \omega_0^2 s + H_r(s)\mu\omega_0^2}{H_r(s)s^3 + \mu s^2 + H_r(s)(2\alpha\mu + \omega_0^2)s + \mu\omega_0^2}. \tag{143}$$

From Section 3.1.3, the network on the top right of Fig. 19.1 is the network obtained in a single inductive step of the Bott-Duffin procedure for the realisation of a given minimum function $H(s)$ which satisfies $H(j\omega_0) = j\omega_0 X$ with $X < 0$. Next, we will show the equivalence of the impedance of this network to the impedances of the networks on the top right of Figs. 19.2, 19.3, and 19.4. In this case, this follows by applying the transformation [22, Lemma 11] to a pertinent subnetwork, and then identifying points in the resulting network which are at a common voltage. Indeed, it may be verified by direct calculation that the impedances of each of the networks on the top right of Figs.

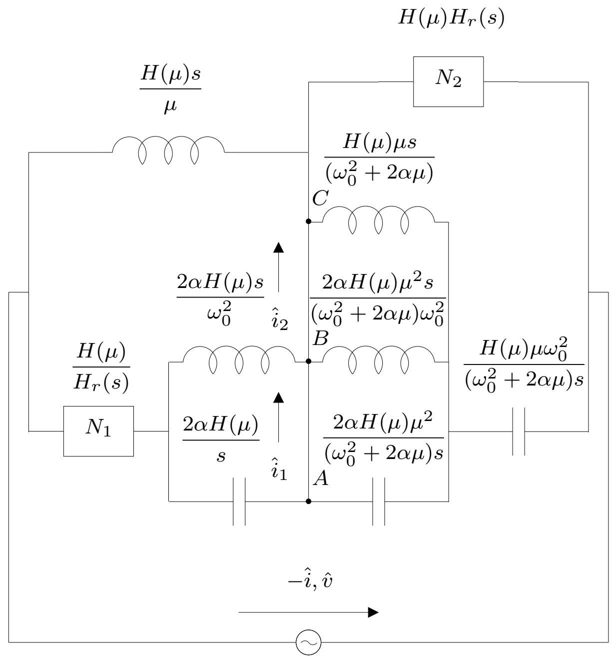


Figure 17: A network equivalent to the network in Fig. 16.

19.1, 19.2, 19.3, and 19.4 are all equal to

$$H(\nu)\frac{\tilde{H}_r(s)s^3+\nu s^2+\tilde{H}_r(s)(\omega_0^2+2\beta\nu)s+\nu\omega_0^2}{s^3+\tilde{H}_r(s)(2\beta+\nu)s^2+\omega_0^2 s+\tilde{H}_r(s)\nu\omega_0^2}. \tag{144}$$

Finally, we describe the equivalence of the impedances of the networks on the top left (resp. top right) of Figs. 19.1 to 19.4 with those on the bottom right (resp. bottom left) of those figures. As explained in [16], the network on the top left of Fig. 19.1 takes the form of a balanced bridge. This is more clearly illustrated in Fig. 18. The network on the left of that figure (which is the network on the top left of Fig. 19.1) is the interconnection of one-port subnetworks shown on the right of Fig. 18, for which

$$\frac{Z_a(s)}{Z_b(s)}=\frac{Z_c(s)}{Z_d(s)}=\frac{s\left(s^2+2\alpha sH_r(s)+\omega_0^2\right)}{\mu H_r(s)\left(s^2+\omega_0^2\right)}.$$

Then, considering a trajectory for the network in Fig. 18 (whose Laplace transform exists), it may be seen that $i_1(s)=0$, and so the connection between points A and B may be removed without altering the impedance of this network. This leads to the network on the bottom right of Fig. 19.1. A set of transformations, similar to those in the preceding paragraphs, then show that this network has equivalent impedance to the networks on the bottom right of Figs. 19.2, 19.3, and 19.4. Similar considerations

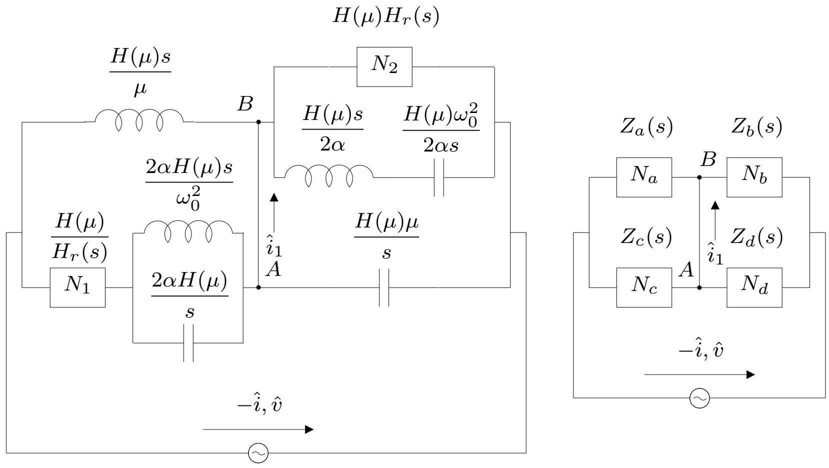


Figure 18: The network on the top left of Fig. 19.1. In the right hand figure, $Z_a(s) = H(\mu)s/\mu$, $Z_b(s) = H(\mu)H_r(s)(s^2+\omega_0^2)/(s^2+2\alpha s H_r(s)+\omega_0^2)$, $Z_c(s) = H(\mu)(s^2 + 2\alpha s H_r(s)+\omega_0^2)/(H_r(s)(s^2+\omega_0^2))$, and $Z_d(s) = H(\mu)\mu/s$.

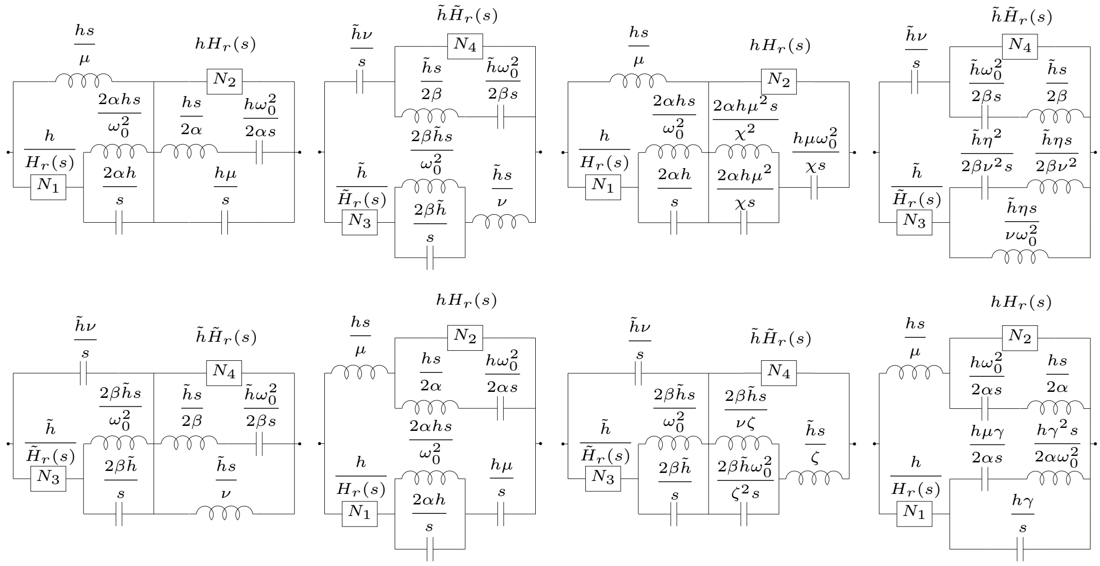
also show that the networks on the bottom left of Figs. 19.1 to 19.4 have equivalent impedance to the networks on the top right of those figures. That the impedances of the networks on the bottom right (resp. bottom left) of Figs. 19.1 to 19.4 take the form of equation (143) (resp. equation (144)) may again be verified by direct calculation. □

We remark that for a biquadratic function, the networks in Figs. 19.3 and 19.4 each contain five reactive elements and two resistors[23]. In the remainder of this part, we find those networks which realise biquadratic minimum functions and contain no more reactive elements than the networks in Fig. 19.

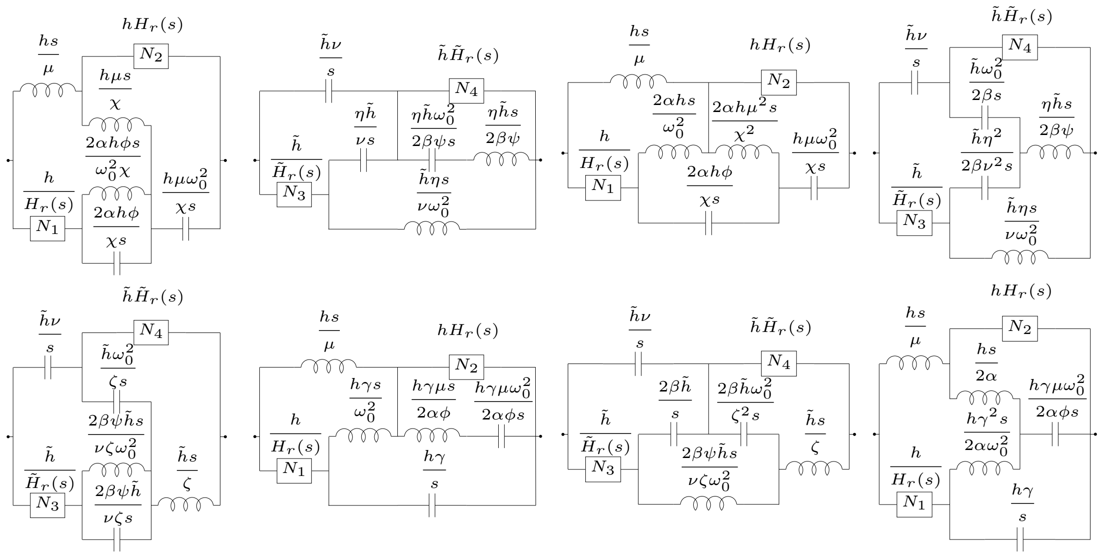
3.2 Technical preliminaries

Our focus in this part is on transformerless networks, and on SP networks in Sections 3.3 and 3.4. In particular, we consider those transformerless networks which realise a minimum function (in accordance with definition 3.1.2), with a particular focus on the biquadratic minimum function. In this section, we introduce the necessary machinery

[23]These seven element networks were identified prior to the discovery of the more general networks in Fig. 19.4 in a study of those transformerless network which contain five or fewer reactive elements and which realise a biquadratic minimum function, the results of which will be reported in Section 3.6. It was subsequently recognised that these networks could be generalised to the networks shown in Fig. 19.4, in order to provide a realisation for any given minimum function. The sequence of transformations connecting these networks to the Bott-Duffin networks was subsequently found upon studying the similarities between the Bott-Duffin networks and the networks in Fig. 19.4.



(1) Bott-Duffin (2) Series-parallel alternative



(3) Reza-Pantell-Fialkow-Gerst simplification (4) New simplification

Figure 19: The networks on the top left and bottom right (resp. top right and bottom left) of Figs. 19.1 to 19.4 realise $H(s)$ in equation (143) (resp. equation (144)). In this figure, $h = H(\mu)$, $\chi = \omega_0^2 + 2\alpha\mu$, $\gamma = \mu + 2\alpha$, $\phi = \chi + \mu^2$, $\tilde{h} = H(\nu)$, $\eta = \omega_0^2 + 2\beta\nu$, $\zeta = \nu + 2\beta$, $\psi = \eta + \nu^2$, and $H_r(s)$, μ, α, $\tilde{H}_r(s)$, ν, β are as described in Section 3.1.5.

and notation for our subsequent analysis. First, in Section 3.2.1, we provide a formal definition for a SP network, and we present a formal framework for classifying networks. Then, in Section 3.2.2, we introduce a parametrisation for a biquadratic minimum function, which couples with the network classification formalism of Section 3.2.1 and enables a neat presentation of subsequent results. Finally, in Subsection 3.2.3, we describe algebraic conditions for two polynomials to have common roots.

3.2.1 A formal framework for the classification of networks

Following [51, Definition II], we say that a network is SP if it is either a series or a parallel connection of two SP networks, or it is a single element. We say the network is essentially series (ES) whenever it is a series connection, and essentially parallel (EP) whenever it is a parallel connection, of two SP networks. In this thesis, we adopt the convention that single elements are excluded from the ES and EP classes. We write $N = N_u + N_v$ (resp. $N = N_u \cdot N_v$) whenever N is a series (resp. parallel) connection of N_u and N_v. If $N = N_u + N_v$ and $Z(s)$, $Z_u(s)$, $Z_v(s)$ are the impedances of N, N_u, N_v, then $Z(s) = Z_u(s) + Z_v(s)$. Similarly, if $N = N_u \cdot N_v$ and $Y(s)$, $Y_u(s)$, $Y_v(s)$ are the admittances of N, N_u, N_v, then $Y(s) = Y_u(s) + Y_v(s)$. We consider series (likewise, parallel) connections to be commutative as the impedances of the corresponding networks are equal. We exclude from consideration any networks containing two series or parallel connected elements of the same kind (e.g. both resistors) as the impedance of the two series or parallel connected elements may be realised by a single element of the same kind.

In a recent paper on the classification of networks containing two reactive elements [22], the usefulness of duality and frequency inversion in network enumeration problems is demonstrated, and the associated concept of a network quartet is introduced. Since our primary concern is with networks which realise minimum functions $H(s)$ for which there exists an $\omega_0 > 0$ where $\Re(H(j\omega_0)) = 0$, we will use a particular specialisation of frequency inversion with respect to ω_0. Specifically, for a network N with impedance $Z(s)$ we denote by N^i the network with impedance $Z(\omega_0^2/s)$ obtained by the procedures outlined in [22, Section III]. We will refer to N^i as the *frequency-inverted* network of N. Furthermore, providing N is *planar* (see [22, Section III]), we denote by N^d the *dual* network of N, whose impedance is $1/Z(s)$. Also, we denote by N^{di} the network $\left(N^d\right)^i = \left(N^i\right)^d$, whose impedance is equal to $1/Z(\omega_0^2/s)$. We note that the networks N^d, N^i, and N^{di} are all SP whenever N is SP, and they contain the same number of reactive elements and the same number of resistors as N. It is clear from this definition that if the impedance $Z(s)$ of N is a minimum function with ω_0 a minimum frequency, then the impedances of N^d, N^i, and N^{di} are also minimum functions with ω_0 a minimum frequency[24]. In the remainder of this part, we will take $\omega_0 > 0$ to be

[24]Here, and in the remainder of this thesis, we make use of the properties of (scalar) PR functions, lossless functions, and minimum functions available in most standard textbooks on

fixed, to define uniquely the relationship between N and N^i. Furthermore, whenever we refer to minimum functions without further clarification, we will assume that ω_0 is a minimum frequency.

We will define certain network classes, indicated by the letters $\mathcal{N}_1, \mathcal{N}_2, \ldots$, as sets of networks with a common configuration of elements and with various constraints and relationships involving the element values (e.g. Fig. 21). In particular, we define the class $\mathcal{L}$ (resp. $\mathcal{C}$) as the set of networks consisting of a single inductor (resp. capacitor) only. We then adopt the notation $\mathcal{N}_1 + \mathcal{N}_2$ (and likewise for $\mathcal{N}_1 \cdot \mathcal{N}_2$) for the network class containing all networks $N_1 + N_2$ where N_1 is from $\mathcal{N}_1$ and N_2 from $\mathcal{N}_2$. We further denote by $\mathcal{N}_1^d$ the class containing all networks N_1^d where N_1 is from $\mathcal{N}_1$ (and likewise for $\mathcal{N}_1^i$ and $\mathcal{N}_1^{di}$). We denote by $\mathcal{Q}_1$ the *quartet* corresponding to the network class $\mathcal{N}_1$ where $\mathcal{Q}_1$ is the union of $\mathcal{N}_1$, $\mathcal{N}_1^d$, $\mathcal{N}_1^i$, and $\mathcal{N}_1^{di}$. We note that certain network classes satisfy one or more of the set equalities $\mathcal{N} = \mathcal{N}^i$, $\mathcal{N} = \mathcal{N}^d$, and $\mathcal{N} = \mathcal{N}^{di}$, and so a network quartet is the union of up to (but possibly fewer than) four network classes. For example, $\mathcal{L}^d = \mathcal{L}^i = \mathcal{C}$, and so the union of $\mathcal{L}$ and $\mathcal{C}$ is a network quartet. Since a network class (likewise, a network quartet) is a set of networks, then there is a corresponding set of impedance functions realised by the networks in the set, and we say the network class (likewise, quartet) *realises* this set of functions.

In diagnosing whether the impedance of a given network has imaginary axis poles or zeros, we will use the concepts of paths and cut-sets. We say that a network has an *L-path* (resp. *C-path*) if there is a path between the external terminals consisting entirely of inductors (resp. capacitors). Similarly, a network has an *L-cut-set* (resp. *C-cut-set*) if it is possible to remove only inductors (resp. capacitors) from the network and leave the two terminals in disconnected parts. Now, consider a network N with impedance $Z(s)$ and admittance $Y(s)$. In [31, Theorem 8.3], it is shown that $Z(s)$ has a pole at $s = 0$ (resp. $s = \infty$) if and only if N has a C-cut-set (resp. L-cut-set). Furthermore, $Y(s)$ has a pole at $s = 0$ (resp. $s = \infty$) if and only if N has an L-path (resp. C-path).

We also make use of the network transformation [22, Lemma 11], which we now state. For arbitrary $Z_1(s), Z_2(s)$, and $a, b, c > 0$, the impedances of the two networks in Fig. 20 are identical providing we let $a' = ab/(a+b)$; $b' = b^2/(a+b)$; $c' = c(b/(a+b))^2$ (resp. $a = a'(a'+b')/b'$; $b = a'+b'$; $c = c'((a'+b')/b')^2$).

passive networks (e.g. [36]). In particular, we note that if $H(s)$ is PR (resp. lossless) and $\alpha > 0$, then $H(\alpha s), H(\alpha/s), 1/H(\alpha s)$, and $\alpha H(s)$ are PR (resp. lossless). In addition, if $H(s) = H_u(s) + H_v(s)$ with $H_u(s)$ and $H_v(s)$ PR (resp. lossless), then $H(s)$ is PR (resp. lossless).

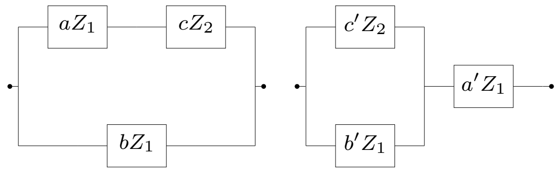


Figure 20: Two networks with identical impedance when $a' = ab/(a+b)$; $b' = b^2/(a+b)$; $c' = c(b/(a+b))^2$ (resp. $a = a'(a'+b')/b'$; $b = a'+b'$; $c = c'((a'+b')/b')^2$).

3.2.2 A parametrisation for the biquadratic minimum function

Here, we introduce a parametrisation for the biquadratic minimum function. This parametrisation couples neatly with the network classification formalism, and thus enables a neat presentation of subsequent results. Our parametrisation is:

$$H_p(s) := \alpha \frac{s^2 + \frac{\omega_0(1-W)X}{W}s + \omega_0^2 W}{s^2 + \frac{\omega_0(1-W)}{X}s + \frac{\omega_0^2}{W}}. \tag{145}$$

It may be shown that $H_p(s)$ in (145) is a biquadratic minimum function, with $H(j\omega_0) = \alpha X j$, providing the parameters W, X, α, and ω_0 satisfy the conditions $\alpha, \omega_0, W > 0$, $W \neq 1$, and $X > 0$ if $W < 1$ with $X < 0$ otherwise (see Remark 3.2.1). Moreover, any biquadratic minimum function may be written in the form (145) for some α, ω_0, W, X satisfying these constraints. It is further evident from these constraints that if $H(s)$ is a biquadratic minimum function (with minimum frequency ω_0), then $H(0) < H(\infty)$ if and only if $\Im(H(j\omega_0)) > 0$.

Remark 3.2.1.
The parametrisation in equation (145) is equivalent to the parametrisation [52, Equations (4), (8)] under the substitutions $\alpha = R$, $W = k$, and $X = X_0/R$. It then follows from [52, Theorem 8] that $H_p(s)$ in (145) is a biquadratic minimum function, with $H(j\omega_0) = \alpha X j$, providing the parameters W, X, α, and ω_0 satisfy the conditions $\alpha, \omega_0, W > 0$, $W \neq 1$, and $X > 0$ if $W < 1$ with $X < 0$ otherwise, and moreover that any biquadratic minimum function may be written in the form (145) for some α, ω_0, W, X satisfying these constraints. Our choice of parametrisation differs from [52] to couple more neatly with frequency- and magnitude-scaling considerations.

Remark 3.2.2.
The parametrisation (145) is also closely related to the canonical form for a biquadratic PR function, $Z_c(s)$, used in [22, Equation (4)]. The function $Z_c(s)$ is equivalent to $H_p(s)$ in (145) when $\alpha = \omega_0 = 1$, $\sigma_c = 0$, and $X = \sqrt{U/V}$ if $W < 1$, $X = -\sqrt{U/V}$ if $W > 1$. Here, $\sigma_c = 4UV + 2 - (1/W + W)$ as defined in [22, Section V]. Hence, $U, V > 0$

and $\sigma_c = 0$ imply $W \neq 1$. Moreover, σ_c is zero on the boundary of the shaded region in Fig. 4 of [22, Section V]. In that paper, it is shown that there are no networks which contain two or fewer reactive elements and realise an impedance $Z_c(s)$ which lies on that boundary. In this part, we show that the only impedances which lie on that boundary and are realised by transformerless networks containing three or fewer reactive elements satisfy either $W = 1/2$ or $W = 2$. Moreover, the only additional impedances which lie on that boundary and are realised by transformerless networks containing four or fewer reactive elements either satisfy $1/2 < W < 1$ together with one of the conditions $U = \sqrt{W(2W-1)}/2$ or $V = \sqrt{W(2W-1)}/2$, or satisfy $1 < W < 2$ together with one of the conditions $U = \sqrt{2-W}/2W$ or $V = \sqrt{2-W}/2W$. In other words, we conclude that the set of impedances $Z_c(s)$ on the boundary $\sigma_c = 0$ which are realised by transformerless networks containing fewer than five reactive elements is a negligibly small subset of this boundary.

3.2.3 Algebraic conditions for two polynomials to have common roots

In this subsection, we state conditions on the number of roots common to two polynomials in terms of the coefficients of these polynomials. Such conditions may be stated in terms of either a Sylvester, a Hankel, or a Bezoutian matrix associated with the two polynomials, see e.g. [40], or Sections 2.1 to 2.4. In contrast to the Hankel and Bezoutian matrices, all entries in the Sylvester matrix are coefficients of the two polynomials. This leads to a set of particularly transparent conditions, which we state in the following theorem.

Theorem 3.2.3. *Let $a, b \in \mathbb{R}[s]$ have degrees p and q respectively, and let*

$$\begin{aligned} a(s) &= a_p s^p + a_{p-1}s^{p-1} + \ldots + a_1 s + a_0, \\ \text{and} \quad b(s) &= b_q s^q + b_{q-1}s^{q-1} + \ldots + b_1 s + b_0. \end{aligned}$$

Further let $R_k(a(s), b(s))$ be the $(p+q-2k) \times (p+q-2k)$ determinant

$$R_k(a(s), b(s)) = \begin{vmatrix} a_p & a_{p-1} & \cdots & & \\ 0 & a_p & a_{p-1} & \cdots & \\ & & \vdots & & \\ b_q & b_{q-1} & \cdots & & \\ 0 & b_q & b_{q-1} & \cdots & \\ & & \vdots & & \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} p-k \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} q-k \text{ rows} \end{matrix},$$

for $k = 0, 1, 2, \ldots, \min\{p, q\} - 1$. Then $a(s)$ and $b(s)$ have at least r common roots if and only if $R_0(a(s), b(s)) = R_1(a(s), b(s)) = \ldots = R_{r-1}(a(s), b(s)) = 0$.

Proof. This follows by considering the matrices $\mathcal{S}_j$ in equation (99). In those matrices, we let $n := \max\{p, q\}$, and in the case $q > p$ (resp. $p > q$) we set the coefficients of $a(s)$ (resp. $b(s)$) whose degree exceeds p (resp. q) to zero. Then, $a(s)$ and $b(s)$ have at least r common roots if and only if $|\mathcal{S}_{2k}| = 0$ for $k = n - r + 1, n - r + 2, \ldots, n$, by Theorem 2.2.2. A suitable permutation of the rows in $\mathcal{S}_{2k}$, followed by a suitable expansion of the determinant of the matrix thus obtained, proves the present theorem. □

We will refer to the determinants $R_k(a(s), b(s))$ as *Sylvester determinants* in s.

By way of an example, consider the polynomials $a(s) = (s + \alpha)(s + \beta)^3$ and $b(s) = (s + \gamma)^2$. In this case, we find $R_0(a(s), b(s)) = (\beta - \gamma)^6(\alpha - \gamma)^2$ and $R_1(a(s), b(s)) = (\beta - \gamma)^2(4\gamma - 3\alpha - \beta)$. It follows that $a(s)$ and $b(s)$ have exactly one common root if and only if $\gamma = \alpha \neq \beta$, and exactly two common roots if and only if $\gamma = \beta$.

3.3 On the minimality of the Bott-Duffin procedure for biquadratic minimum functions

The material in this and the following section was contained in our paper [49], and is presented here with a few minor adjustments.

In this section, we will demonstrate the minimality of the networks obtained by the Bott-Duffin realisation procedure, both in number of reactive elements and in number of resistors, among SP networks realising biquadratic minimum functions. Our approach is to characterise those SP networks whose impedance (or admittance) satisfies some of the conditions of a minimum function, most notably that the real part is equal to zero at $s = j\omega_0$. We restrict our attention to networks containing at most n reactive elements, for successive cases as n is increased (Lemma 3.3.2 onwards). We begin with the following lemma, which will be used to construct lower bounds on the number of reactive elements in those SP networks which realise certain types of PR functions.

Lemma 3.3.1. *Let $H(s) = H_u(s) + H_v(s)$ where $H_u(s)$ and $H_v(s)$ are PR, and let $\omega \in \mathbb{R} \cup \infty$. Then the following three conditions must all hold:*

1. *$\Re(H(j\omega)) = 0$ if and only if $\Re(H_u(j\omega)) = 0$ and $\Re(H_v(j\omega)) = 0$.*

2. *$H(j\omega) = 0$ if and only if $\Re(H_u(j\omega)) = \Re(H_v(j\omega)) = 0$ and $\Im(H_u(j\omega)) = -\Im(H_v(j\omega))$.*

3. *$H(s)$ has a pole at $s = j\omega$ if and only if $H_u(s)$ or $H_v(s)$ has a pole at $s = j\omega$.*

Proof. This is a straightforward consequence of the definition of a PR function. □

Before stating Lemma 3.3.2, we recall that in this part we will take $\omega_0 > 0$ to be fixed, to define uniquely the relationship between the networks N and N^i.

Lemma 3.3.2. *Let N be a network containing at most one reactive element. Further let N have impedance or admittance $H(s)$ which is not identically zero and satisfies $\Re(H(j\omega_0)) = 0$. Then $H(s)$ is equal to Es/ω_0 or $E\omega_0/s$ for some $E > 0$. In particular, $H(s)$ is lossless, and N contains exactly one reactive element.*

Proof. Since $H(s)$ must be PR with McMillan degree at most one, then $H(s) = (As + B\omega_0) / (Cs + D\omega_0)$ for some $A, B, C, D \geq 0$ with at least one of A, B non-zero and at least one of C, D non-zero. It follows that $\Re(H(j\omega_0)) = (BD + AC) / \left(C^2 + D^2\right)$, and $\Re(H(j\omega_0)) = 0$ implies either $B = C = 0$ or $A = D = 0$. We conclude that $H(s)$ is equal to Es/ω_0 or $E\omega_0/s$ for some $E > 0$, which is lossless. Since $H(s)$ has McMillan degree equal to one, then N must contain exactly one reactive element. □

We remark that a lossless function is either identically zero or has at least one pole and at least one zero on $j\mathbb{R} \cup \infty$. Hence, a network containing only one reactive element cannot realise a minimum function.

We introduce the network N_1 in Fig. 21, whose impedance is equal to

$$H_1(s) := \frac{s^2 + \omega_0^2}{As^2 + B\omega_0 s + A\omega_0^2}. \tag{146}$$

We define the network class $\mathcal{N}_1$ to be the set of all such networks N_1 for $A, B > 0$ (note we exclude infinite values for parameters in network classes). We denote the corresponding quartet by $\mathcal{Q}_1$, and we note that $\mathcal{N}_1 = \mathcal{N}_1^i$.

Lemma 3.3.3. *Let N be a SP network containing at most two reactive elements. Further let N have impedance (resp. admittance) $H(s) \not\equiv 0$ which is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$. Then N contains exactly two reactive elements and at least one resistor. Furthermore, $H(j\omega_0) = 0$, and $H(s)$ takes the form (146) for some $A, B > 0$. Moreover, if N contains exactly one resistor, then N belongs to the class $\mathcal{N}_1$ (resp. $\mathcal{N}_1^d$).*

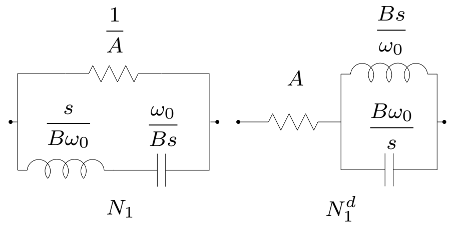


Figure 21: Quartet $\mathcal{Q}_1$. $A, B > 0$.

Proof. From Lemma 3.3.2, N must contain exactly two reactive elements. If N contains no resistors, then $H(s)$ is lossless by [8, p. 259] (see also Subsection 0.1.2). Hence, N must also contain at least one resistor. As N is not a single element, then N is either ES or EP.

Consider first the case where $H(s)$ is the impedance of N. Suppose initially N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$ by Lemma 3.3.1. Hence, both $Z_u(s)$ and $Z_v(s)$ are lossless by Lemma 3.3.2, which implies that $H(s)$ is lossless. We conclude that there are no ES networks with the required properties.

Let N be EP with $N = N_u \cdot N_v$, and $1/H(s) = Y_u(s) + Y_v(s)$ where $Y_u(s)$, $Y_v(s)$ are the admittances of N_u, N_v. If $\Im(H(j\omega_0)) \neq 0$ then $H(j\omega_0) = bj$ for some $b \in \mathbb{R}, b > 0$, hence $1/H(j\omega_0) = -j/b$ which implies $\Re(1/H(j\omega_0)) = 0$, and so a similar argument to before shows that $H(s)$ is lossless. We thus require $\Im(H(j\omega_0)) = 0$, which implies $H(j\omega_0) = 0$. It follows that $1/H(s)$ has poles at $s = \pm j\omega_0$, so the McMillan degree of $H(s)$ is at least two. Since, in addition, N contains exactly two reactive elements, we conclude that the McMillan degree of $H(s)$ is exactly two. It can then be verified directly that $1/H(s) = A + Bs/(s^2 + \omega_0^2)$ for some $A \geq 0$ and $B > 0$ (see Theorem 3.1.4). If $A = 0$ then $H(s)$ is lossless, so $H(s)$ must take the form (146) for some $A, B > 0$.

Now suppose an $H(s)$ satisfying the conditions of the present lemma is the impedance of a network N which contains exactly two reactive elements and one resistor. From before, N is EP with $N = N_u \cdot N_v$, and $1/H(s) = Y_u(s) + Y_v(s)$ where $Y_u(s)$, $Y_v(s)$ are the admittances of N_u, N_v. Since $H(j\omega_0) = 0$ then $1/H(s)$ has poles at $s = \pm j\omega_0$, and hence one of $Y_u(s)$ or $Y_v(s)$ must have poles at $s = \pm j\omega_0$ by Lemma 3.3.1. Without loss of generality, let this be $Y_u(s)$. Since $Y_u(s)$ has two poles, then N_u must contain exactly two reactive elements. Also, in order for N_v to be non-empty, then N_v must contain only the resistor. It is straightforward to show that N_u must belong to $\mathcal{L} + \mathcal{C}$ and for the product of the capacitance and the inductance to be $1/\omega_0^2$. Hence, N belongs to class $\mathcal{N}_1$.

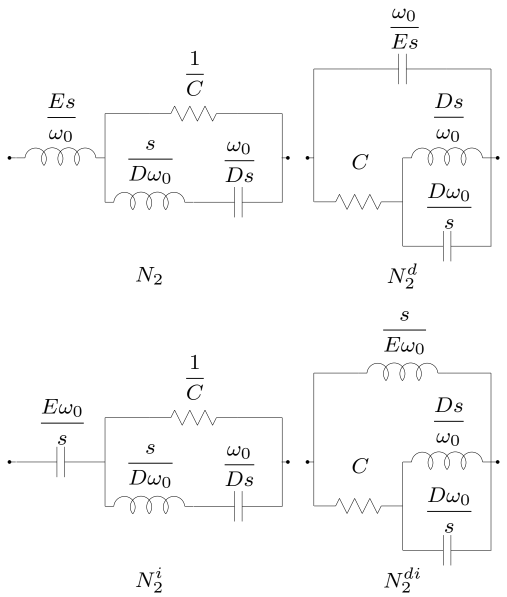


Figure 22: Quartet $\mathcal{Q}_2$. $C, D, E > 0$.

The case where $H(s)$ is the admittance of N is similar. Once again, we deduce that $\Im(H(j\omega_0)) = 0$, and $H(s)$ must take the form (146) for some $A, B > 0$. In this case we find that if N contains exactly one resistor, then N is from the class $\mathcal{N}_1^d$. $\square$

Lemma 3.3.3 shows that if a SP network contains two reactive elements and has impedance or admittance $H(s)$ where $\Re(H(j\omega_0)) = 0$, then $H(s)$ must have at least one zero on $j\mathbb{R} \cup \infty$. Again, such a network cannot realise a minimum function.

We now introduce the network N_2 in Fig. 22, which is the series connection of an inductor and a network from the class $\mathcal{N}_1$. The impedance of N_2 is equal to

$$H_2(s) := \frac{s^2 + \omega_0^2}{Cs^2 + D\omega_0 s + C\omega_0^2} + \frac{Es}{\omega_0}. \tag{147}$$

We define the network class $\mathcal{N}_2$ to be the set of all such networks N_2 for $C, D, E > 0$. We denote the corresponding network quartet by $\mathcal{Q}_2$.

Lemma 3.3.4. *Let N be a SP network containing at most three reactive elements. Further let N have impedance or admittance $H(s)$ which is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$. Then N contains exactly three reactive elements and at least one resistor. Furthermore, either $H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, or $1/H(\omega_0^2/s)$ takes the form (147) for some $C, D, E > 0$. Moreover, if N contains exactly one resistor, then N belongs to the quartet $\mathcal{Q}_2$.*

Proof. As in the proof of Lemma 3.3.3, both impedance and admittance of N satisfy the conditions on $H(s)$ in the present lemma, so without loss of generality let $H(s)$ be the impedance. That N must contain exactly three reactive elements and at least one resistor follows from Lemma 3.3.3. It follows that N is either ES or EP since N contains more than one element.

Consider first the case where N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$ from Lemma 3.3.1. Since $H(s)$ is not lossless, then at least one of $Z_u(s), Z_v(s)$ is not lossless. Without loss of generality, let this be $Z_v(s)$. Then, by Lemma 3.3.2, N_u (resp. N_v) contains at least one reactive element (resp. two reactive elements). Lemma 3.3.2 also shows that N_u must contain exactly one reactive element, and $Z_u(s)$ must take the form Es/ω_0 or $E\omega_0/s$ for some $E > 0$. Furthermore, Lemma 3.3.3 shows that N_v must contain exactly two reactive elements, and $Z_v(s)$ must take the form (146) for some $A, B > 0$. It follows that $H(s)$ or $H(\omega_0^2/s)$ must take the form (147) for some $C, D, E > 0$. Moreover, if N contains exactly one resistor, then N_v must be from $\mathcal{N}_1$, and N_u must be from $\mathcal{L}$ or $\mathcal{C}$. We conclude that N must belong to $\mathcal{N}_2$ or $\mathcal{N}_2^i$.

The case where N is EP is similar. In particular, it follows that either $1/H(s)$ or $1/H(\omega_0^2/s)$ takes the form (147) for some $C, D, E > 0$, and that N is from $\mathcal{N}_2^d$ or $\mathcal{N}_2^{di}$ if N contains exactly one resistor. □

It may be observed that $H_2(s)$ in (147) has a pole at $s = \infty$, and hence $H_2(\omega_0^2/s)$ has a pole at $s = 0$, and $1/H_2(s)$ (resp. $1/H_2(\omega_0^2/s)$) has a zero at $s = \infty$ (resp. $s = 0$). Hence, a SP network containing at most three reactive elements cannot realise a minimum function.

We now introduce the network N_3 in Fig. 23, which is the series connection of a network from the class $\mathcal{N}_1$ and a network from the class $\mathcal{N}_2^d$. The impedance of N_3 is equal to

$$H_3(s) := H_1(s) + 1/H_2(s), \tag{148}$$

where $H_1(s)$ is as in (146) for some $A, B > 0$, and $H_2(s)$ is as in (147) for some $C, D, E > 0$. We define the network class $\mathcal{N}_3$ as the set of all such networks N_3 for $A, B, C, D, E > 0$. The corresponding network quartet is denoted by $\mathcal{Q}_3$. We remark that the impedance of *any* network from $\mathcal{Q}_3$ is a minimum function (with ω_0 a minimum frequency). This may be verified by direct calculation or pertinent application of Lemma 3.3.1.

Theorem 3.3.5. *Let $H(s)$ be the impedance of a SP network N where $H(s)$ is a*

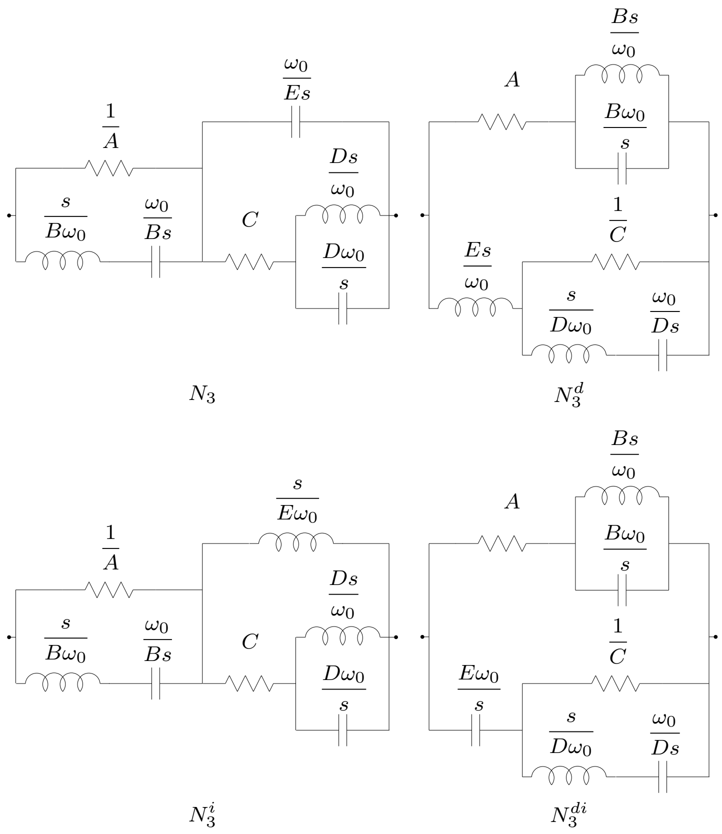


Figure 23: Quartet $\mathcal{Q}_3$. $A, B, C, D, E > 0$.

minimum function with ω_0 *a minimum frequency. Then* N *contains at least five reactive elements and at least two resistors. Furthermore, if* N *contains exactly five reactive elements, then either* $H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, *or* $1/H(\omega_0^2/s)$ *takes the form (148) where* $H_1(s), H_2(s)$ *are as in (146), (147) for some* $A, B, C, D, E > 0$. *If in addition* N *contains exactly two resistors, then* N *belongs to the quartet* $\mathcal{Q}_3$.

Proof. By definition, $\Re(H(j\omega_0)) = 0$, $\Im(H(j\omega_0)) \neq 0$, and $H(s)$ has no poles or zeros on $j\mathbb{R} \cup \infty$. Note that $1/H(s)$, $H(\omega_0^2/s)$, and $1/H(\omega_0^2/s)$ are also minimum functions with a minimum frequency at ω_0. It follows that N must contain at least four reactive elements by Lemma 3.3.4. Since N contains more than one element, it is either ES or EP.

Consider first the case where N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then, by Lemma 3.3.1, $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$, and neither $Z_u(s)$ nor $Z_v(s)$ have any poles on $j\mathbb{R} \cup \infty$. In particular, neither $Z_u(s)$ nor $Z_v(s)$ is lossless. Furthermore, either $\Im(Z_u(j\omega_0)) \neq 0$ or $\Im(Z_v(j\omega_0)) \neq 0$, so without loss of generality let $\Im(Z_v(j\omega_0)) \neq 0$. Then N_u (resp.

N_v) must contain at least two (resp. three) reactive elements and at least one resistor from Lemma 3.3.2 (resp. Lemma 3.3.3). Hence, N must contain at least five reactive elements and at least two resistors.

We recall that $Z_u(s)$ is not lossless and satisfies $\Re(Z_u(j\omega_0)) = 0$. Hence, by Lemma 3.3.3, if N_u contains exactly two reactive elements, then $Z_u(s)$ takes the form of $H_1(s)$ in (146) for some $A, B > 0$. Furthermore, $\Re(Z_v(j\omega_0)) = 0$, $\Im(Z_v(j\omega_0)) \neq 0$, and $Z_v(s)$ has no poles on $j\mathbb{R} \cup \infty$. It follows from Lemma 3.3.4 that if N_v contains exactly three reactive elements, then either $1/Z_v(s)$ or $1/Z_v(\omega_0^2/s)$ takes the form of $H_2(s)$ in (147) for some $C, D, E > 0$. Hence, either $H(s)$ or $H(\omega_0^2/s)$ must equal $H_1(s) + 1/H_2(s)$ for some $A, B, C, D, E > 0$. Moreover, if N contains exactly two resistors, then N_u must belong to $\mathcal{N}_1$ by Lemma 3.3.3, and N_v must belong to $\mathcal{N}_2^d$ or $\mathcal{N}_2^{di}$ from the proof of Lemma 3.3.4. We conclude that N must belong to $\mathcal{N}_3$ or $\mathcal{N}_3^i$.

The case where N is EP is similar. Once again, we deduce that N must contain exactly five reactive elements and at least two resistors. In this case, we find that either $1/H(s)$ or $1/H(\omega_0^2/s)$ must equal $H_1(s) + 1/H_2(s)$ for some $A, B, C, D, E > 0$, and that N is from one of the classes $\mathcal{N}_3^d$ or $\mathcal{N}_3^{di}$ if N contains exactly two resistors. □

From Theorem 3.3.5, we see that there are minimum functions realisable by networks containing five reactive elements. These are precisely the impedances of those networks in the quartet $\mathcal{Q}_3$ (for which ω_0 is a minimum frequency). The McMillan degree of the impedance of any network from this quartet is less than or equal to five.

There are minimum functions with McMillan degree equal to five which are not realised by any network in the quartet $\mathcal{Q}_3$. For example, consider the function

$$H_4(s) := \frac{8s^5 + 14s^4 + 28s^3 + 25s^2 + 23s + 8}{(s+1)^5}. \tag{149}$$

Noting that $H_4(s) + H_4(-s)$ is equal to

$$\frac{4(2s+1)(2s-1)(s^2+s+2)(s^2-s+2)(s^2+1)^2}{(s+1)^5(s-1)^5},$$

we see that $H_4(s)$ is a minimum function. In particular, $H_4(j) = -3j/4$ and $H_4(0) = H_4(\infty) = 8$. It is straightforward to verify that if $Z(s)$ is the impedance of a network in $\mathcal{Q}_3$, then $Z(0) \neq Z(\infty)$. It follows that $H_4(s)$ in (149) cannot be realised by a network from $\mathcal{Q}_3$.

Furthermore, certain networks from $\mathcal{Q}_3$ have McMillan degree less than five. For example, the function

$$\frac{11s^4 + 27s^3 + 56s^2 + 67s + 25}{(2s+1)(s+2)(11s^2 + 16s + 3)}$$

is equal to $H_1(s) + 1/H_2(s)$ for $A = 2$, $B = 5$, $C = 11/3$, $D = 20/3$, $E = 2$, and $\omega_0 = 1$. Also, the function

$$\frac{11s^3 + 16s^2 + 27s + 28}{(s+2)(11s^2 + 16s + 3)}$$

is equal to $H_1(s) + 1/H_2(s)$ for $A = 1$, $B = 5/2$, $C = 11/3$, $D = 20/3$, $E = 2$, and $\omega_0 = 1$. However, as will be shown in the next theorem, there are no networks in $\mathcal{Q}_3$ which realise a biquadratic impedance. To show this we will employ the necessary and sufficient algebraic conditions on the coefficients of two polynomials for those polynomials to have common roots which were stated in Theorem 3.2.3. In particular, for two polynomials $a(s)$ and $b(s)$ we define the *Sylvester determinants* in s: $R_0(a(s), b(s)), R_1(a(s), b(s)), \ldots, R_k(a(s), b(s))$, $k < \min\{\deg(a(s)), \deg(b(s))\}$, in accordance with Subsection 3.2.3. There it is shown that $a(s)$ and $b(s)$ have at least r common roots if and only if $R_0(a(s), b(s)) = R_1(a(s), b(s)) = \ldots = R_{r-1}(a(s), b(s)) = 0$.

Theorem 3.3.6. *Let $H(s)$ be the impedance of a SP network N where $H(s)$ is a biquadratic minimum function. Then N contains at least six reactive elements and at least two resistors.*

Proof. Let ω_0 be a minimum frequency of $H(s)$. From Theorem 3.3.5, N must contain at least five reactive elements and at least two resistors. Moreover, if N contains exactly five reactive elements, then either $H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, or $1/H(\omega_0^2/s)$ is equal to $H_3(s)$ in (148). Here, $H_3(s) = (n_u(s)/d_u(s)) + (n_v(s)/d_v(s))$ with $n_u(s) = s^2 + \omega_0^2$, $d_u(s) = As^2 + B\omega_0 s + A\omega_0^2$, $n_v(s) = Cs^2 + D\omega_0 s + C\omega_0^2$, and $d_v(s) = CEs^3 + (1 + DE)\omega_0 s^2 + CE\omega_0^2 s + \omega_0^3$, for some $A, B, C, D, E > 0$. It suffices to show that $H(s)$ so defined cannot be biquadratic.

Let $h(s)$ be the greatest common divisor of $d_u(s)$ and $d_v(s)$, and let $d_u(s) = h(s)\tilde{d}_u(s)$ and $d_v(s) = h(s)\tilde{d}_v(s)$, so

$$H(s) = \frac{n_u(s)\tilde{d}_v(s) + n_v(s)\tilde{d}_u(s)}{h(s)\tilde{d}_u(s)\tilde{d}_v(s)}.$$

Since $R_0(n_u(s), d_u(s)) = B^2\omega_0^4$ and $R_0(n_v(s), d_v(s)) = CD^2\omega_0^9$, neither of which can be zero, it follows that $n_u(s)$ and $d_u(s)$ are coprime, as are $n_v(s)$ and $d_v(s)$. By definition, $\tilde{d}_u(s)$ and $\tilde{d}_v(s)$ are coprime. This implies that $n_u(s)\tilde{d}_v(s) + n_v(s)\tilde{d}_u(s)$ and $\tilde{d}_u(s)\tilde{d}_v(s)$

are coprime. Writing $H(s) = n(s)/d(s)$ with $n(s)$ and $d(s)$ coprime, we see that

$$\begin{aligned}\deg\left(d(s)\right) &\geq \deg\left(\tilde{d}_u(s)\right) + \deg\left(\tilde{d}_v(s)\right) \\ &= \deg\left(d_u(s)\right) + \deg\left(d_v(s)\right) - 2\deg\left(h(s)\right). \end{aligned} \tag{150}$$

If $H(s)$ is biquadratic then $\deg\left(d(s)\right) = 2$, and (150) implies that $\deg\left(h(s)\right) = 2$. I.e., $d_u(s)$ and $d_v(s)$ must have at least two common roots, which implies $R_0(d_u(s), d_v(s)) = R_1(d_u(s), d_v(s)) = 0$. These determinants are both polynomials in C (of degrees two and one respectively), and, for these polynomials to be simultaneously equal to zero, we require $R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right) = A^3B^4E^2\omega_0^{10} = 0$, which is not possible[25]. We conclude that $H_3(s)$ cannot be biquadratic, and hence there are no SP networks which realise a biquadratic minimum function and which contain fewer than six reactive elements or fewer than two resistors. □

Theorem 3.3.6 proves that there are no SP networks realising biquadratic minimum functions which contain fewer reactive elements or fewer resistors than the networks obtained by the Bott-Duffin procedure (see Fig. 19.1 and the final paragraph of Section 3.1.3).

3.4 On the uniqueness of the Bott-Duffin procedure for biquadratic minimum functions

In this section, we show that any SP network which contains no more than six reactive elements and no more than two resistors and realises a minimum function belongs to one of eleven quartets. Only two of these quartets can realise biquadratic minimum functions. Those networks in these two quartets which realise biquadratic minimum functions must satisfy certain constraints among their element impedances, and are shown in Fig. 30. One of these quartets contains those networks from the Bott-Duffin procedure. The other quartet is related to it through a simple transformation.

We will use a similar approach to Section 3.3, in that we characterise those networks whose impedance (or admittance) satisfies certain pertinent constraints, these networks containing at most n reactive elements, for successive cases as n is increased.

[25] We remark that the choice of parameter C in the expression $R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right)$ is arbitrary. Indeed, $R_0(d_u(s), d_v(s))$ and $R_1(d_u(s), d_v(s)) = 0$ are both polynomials in A, B, C, D, and E. The parameter C was chosen as it is then immediately evident that $R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right)$ cannot equal zero given $A, B, D, E > 0$.

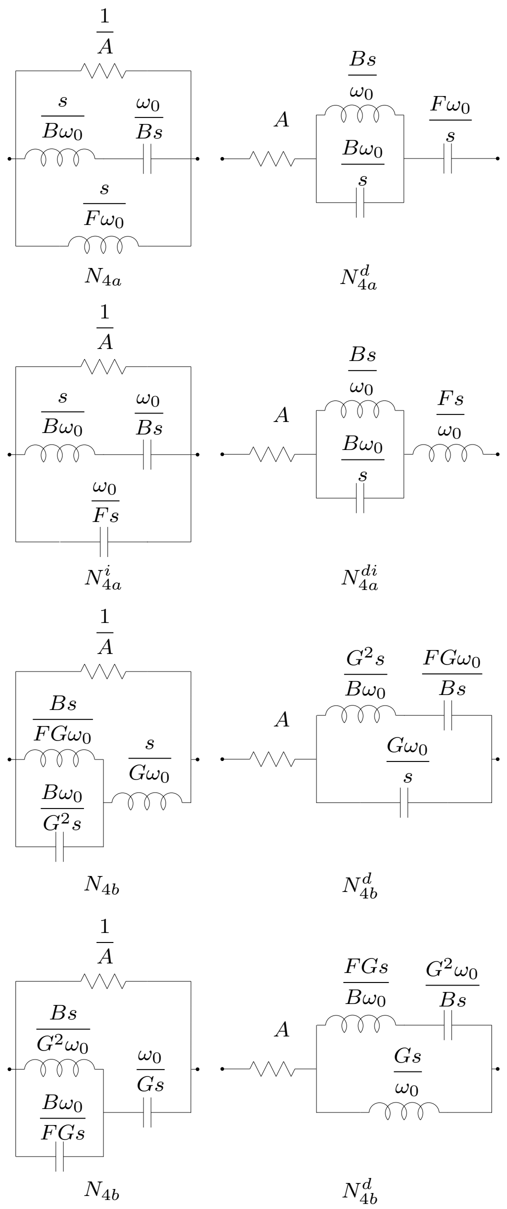


Figure 24: Quartets $\mathcal{Q}_{4a}$ and $\mathcal{Q}_{4b}$. $A, B, F > 0$, $G = B + F$.

We first introduce the networks N_{4a}, N_{4b}, and N_5 in Figs. 24 and 25. We define the network class $\mathcal{N}_{4a}$ (resp. $\mathcal{N}_{4b}$; $\mathcal{N}_5$) to be the set of all such networks N_{4a} (resp. N_{4b}; N_5) for $A, B, F > 0$, where $G = B + F$ for network N_{4b}. The corresponding network quartet is denoted by $\mathcal{Q}_{4a}$ (resp. $\mathcal{Q}_{4b}$; $\mathcal{Q}_5$). We note that the impedances of networks N_{4a} and N_{4b} are identical (with $G = B + F$) as they are related via the network transformation in [22, Lemma 11], which is described in Section 3.2.1. As in the previous section, we take $\omega_0 > 0$ to be fixed, to define uniquely the relationship between the networks N and N^i.

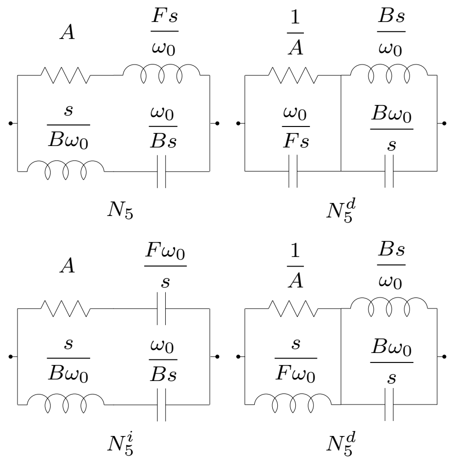


Figure 25: Quartet $\mathcal{Q}_5$. $A, B, F > 0$.

Lemma 3.4.1. *Let N be a SP network containing exactly three reactive elements and exactly one resistor. Further let N have impedance (resp. admittance) $H(s) \not\equiv 0$ which is not lossless and satisfies $H(j\omega_0) = 0$. Then N or N^i (resp. N^d or N^{di}) belongs to one of the classes $\mathcal{N}_{4a}$, $\mathcal{N}_{4b}$, or $\mathcal{N}_5$.*

Proof. Note that N is either ES or EP since it contains more than one element.

Consider first the case where $H(s)$ is the impedance of N. Suppose initially N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$ by Lemma 3.3.1. Without loss of generality, let N_u contain the resistor. In order that N_v is non-empty, N_u cannot contain all three reactive elements. Furthermore, $Z_v(s)$ is lossless, which implies $Z_u(s)$ cannot be lossless since $H(s)$ is not lossless. So, by Lemma 3.3.3, N_u contains exactly two reactive elements and $Z_u(j\omega_0) = 0$. It follows that N_v is from $\mathcal{L}$ or $\mathcal{C}$, so $Z_v(j\omega_0) \neq 0$ which implies $H(j\omega_0) \neq 0$. Hence, there are no ES networks with the required properties.

Let N be EP with $N = N_u \cdot N_v$, so $1/H(s) = Y_u(s) + Y_v(s)$ where $Y_u(s)$, $Y_v(s)$ are the admittances of N_u, N_v. Since $H(j\omega_0) = 0$ and $H(s)$ is not identically zero, then $1/H(s)$ has poles at $s = \pm j\omega_0$. So, by Lemma 3.3.1, at least one of $Y_u(s)$ or $Y_v(s)$ must contain poles at $s = \pm j\omega_0$. Without loss of generality, let this be $Y_u(s)$. Since $Y_u(s)$ contains at least two poles, N_u contains at least two reactive elements. There are three cases to consider: (i) N_u contains two reactive elements only, (ii) N_u contains two reactive elements and the resistor, (iii) N_u contains three reactive elements only.

In case (i), the only possibility is for N_u to be from $\mathcal{L} + \mathcal{C}$, and for the product of the

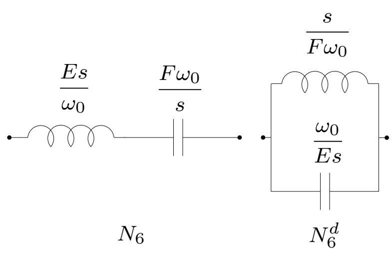


Figure 26: Quartet $\mathcal{Q}_6$. $E, F > 0$, $E \neq F$.

capacitance and inductance to be $1/\omega_0^2$. Then N or N^i must belong to $\mathcal{N}_{4a}$ or $\mathcal{N}_5$. In case (ii), Lemma 3.3.3 shows that N_u belongs to $\mathcal{N}_1$, so N or N^i once again belongs to $\mathcal{N}_{4a}$. In case (iii), the only new possibilities are those in which N_u is ES. There are just two possibilities which do not contain two reactive elements of the same kind in series or parallel. These correspond to either N or N^i belonging to $\mathcal{N}_{4b}$. The restriction $G = B + F$ is required for $Y_u(s)$ to have poles at $s = \pm j\omega_0$.

The case where $H(s)$ is the admittance of N is similar. In this case, we find that N or N^i must belong to one of $\mathcal{N}_{4a}^d$, $\mathcal{N}_{4b}^d$, or $\mathcal{N}_5^d$. □

We remark that the impedance of *any* network from the classes $\mathcal{N}_{4a}$, $\mathcal{N}_{4b}$, or $\mathcal{N}_5$ satisfies the conditions on $H(s)$ in Lemma 3.4.1. This follows by direct calculation or pertinent application of Lemma 3.3.1. A similar fact holds in the next lemma, though we will omit the proof since it is not essential to our present argument.

Lemmas 3.3.3, 3.3.4, and 3.4.1 collectively enumerate all SP networks containing no more than three reactive elements and one resistor with either impedance or admittance $H(s)$ where $H(s)$ is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$. As will be shown in Theorem 3.4.3, one such network is always present as a subnetwork of any network containing no more than six reactive elements and two resistors which realises a minimum function (with a minimum frequency at ω_0). In the case where such a subnetwork contains exactly two reactive elements and one resistor, then it may connect to a subnetwork containing exactly four reactive elements and one resistor, which we consider in the next lemma.

We introduce the networks N_6 and N_7 in Figs. 26 and 27. We define the network class $\mathcal{N}_6$ as the set of all such networks N_6 for $E, F > 0$, and $E \neq F$. Similarly, $\mathcal{N}_7$ is defined as the set of all networks N_7 for $C, D, E, F > 0$, and $E \neq F$. We denote the corresponding quartets by $\mathcal{Q}_6$ and $\mathcal{Q}_7$. We remark that if $E = F$ then the impedance of network N_6 has a zero at $j\omega_0$, and the impedance of network N_7 has a pole at $j\omega_0$.

We define further network classes as follows:

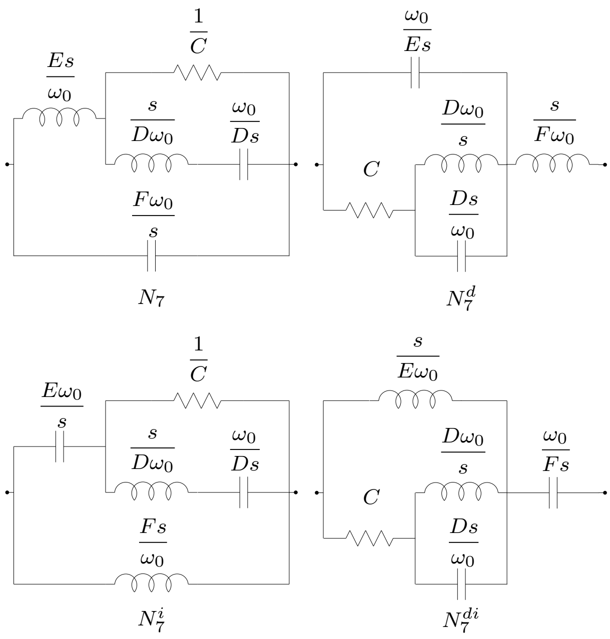


Figure 27: Quartet $\mathcal{Q}_7$. $C, D, E, F > 0$, $E \neq F$.

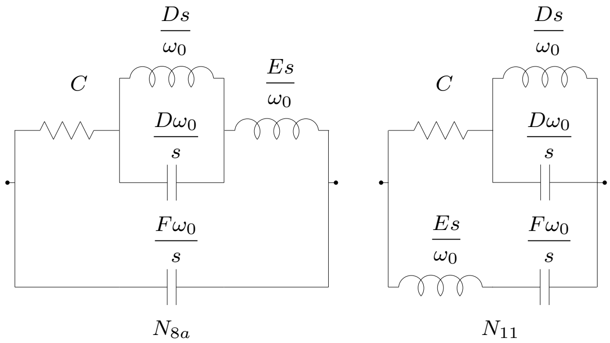


Figure 28: Network classes $\mathcal{N}_{8a}$ $(C, D, E, F > 0)$ and $\mathcal{N}_{11}$ $(C, D, E, F > 0,\ E \neq F)$.

- $\mathcal{N}_{8a}$ (resp. $\mathcal{N}_{8b}$; $\mathcal{N}_9$; $\mathcal{N}_{10}$) is equal to $\mathcal{N}_u \cdot \mathcal{C}$ for $\mathcal{N}_u$ equal to $\mathcal{N}_{4a}^{di}$ (resp. $\mathcal{N}_{4b}^{di}$; $\mathcal{N}_5^d$; $\mathcal{N}_5^{di}$);
- $\mathcal{N}_{11}$ (resp. $\mathcal{N}_{12}$) is equal to $\mathcal{N}_1^d \cdot \mathcal{N}_v$ for $\mathcal{N}_v$ equal to $\mathcal{N}_6$ (resp. $\mathcal{N}_6^d$).

We denote the corresponding quartets by $\mathcal{Q}_{8a}$, $\mathcal{Q}_{8b}$, $\mathcal{Q}_9$, $\mathcal{Q}_{10}$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{12}$, and we note that $\mathcal{N}_{11} = \mathcal{N}_{11}^i$ and $\mathcal{N}_{12} = \mathcal{N}_{12}^i$. Examples of a network N_{8a} from class $\mathcal{N}_{8a}$, and a network N_{11} from the class $\mathcal{N}_{11}$, are shown in Fig. 28.

Lemma 3.4.2. *Let N be a SP network containing exactly four reactive elements and exactly one resistor. Further let N have impedance (resp. admittance) $H(s)$ which has no poles on $j\mathbb{R} \cup \infty$ and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$. Then N or N^i*

(resp. N^d or N^{di}) is from one of the classes $\mathcal{N}_7$, $\mathcal{N}_{8a}$, $\mathcal{N}_{8b}$, $\mathcal{N}_9$, $\mathcal{N}_{10}$, $\mathcal{N}_{11}$, or $\mathcal{N}_{12}$.

Proof. Note that $H(s)$ is not lossless since it is not identically zero and it has no poles on $j\mathbb{R} \cup \infty$. Since N contains more than one element, then it is either ES or EP.

Consider first the case where $H(s)$ is the impedance of N. Suppose initially N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$ by Lemma 3.3.1. Without loss of generality, let N_u contain the resistor. Then $Z_v(s)$ is lossless, so $H(s)$ has a pole on $j\mathbb{R} \cup \infty$. Hence, there are no ES networks with the required properties.

Let N be EP and let $N = N_u \cdot N_v$, so $1/H(s) = Y_u(s) + Y_v(s)$ where $Y_u(s)$, $Y_v(s)$ are the admittances of N_u, N_v. Then $\Re(Y_u(j\omega_0)) = \Re(Y_v(j\omega_0)) = 0$ by Lemma 3.3.1. Without loss of generality, let N_u contain the resistor. Then $Y_v(s)$ is lossless, so $Y_u(s)$ cannot be lossless since $H(s)$ is not lossless. In order that N_v is non-empty, then N_u contains at most three reactive elements. Since $\Re(Y_u(j\omega_0)) = 0$ and $Y_u(s)$ is not lossless, then N_u contains at least two reactive elements by Lemma 3.3.2. If N_u contains exactly two reactive elements then $\Im(Y_u(j\omega_0)) = 0$ by Lemma 3.3.3. Hence, if $\Im(Y_u(j\omega_0)) \neq 0$, then N_u contains three reactive elements. There are three cases to consider: (i) $\Im(Y_u(j\omega_0)) = 0$ and N_u contains exactly two reactive elements, (ii) $\Im(Y_u(j\omega_0)) = 0$ and N_u contains exactly three reactive elements, (iii) $\Im(Y_u(j\omega_0)) \neq 0$ and N_u contains exactly three reactive elements.

In case (i), Lemma 3.3.3 shows that N_u belongs to $\mathcal{N}_1^d$. Since N_v contains exactly two reactive elements, then N_v is from $\mathcal{L}+\mathcal{C}$ or $\mathcal{L}\cdot\mathcal{C}$. Moreover, by Lemma 3.3.1 it follows that since $Y_u(j\omega_0) = 0$ and $1/H(s)$ has neither a pole nor zero at $s = j\omega_0$, then $Y_v(s)$ cannot have a pole or zero at $s = j\omega_0$. We conclude that N_v must be from $\mathcal{N}_6$ or $\mathcal{N}_6^d$, so N is from $\mathcal{N}_{11}$ or $\mathcal{N}_{12}$.

In case (ii), either N_u or N_u^i belongs to $\mathcal{N}_{4a}^d$, $\mathcal{N}_{4b}^d$, or $\mathcal{N}_5^d$ by Lemma 3.4.1, and N_v is from $\mathcal{L}$ or $\mathcal{C}$. The network N thus belongs to one of twelve classes ($\mathcal{N}_{4a}^d \cdot \mathcal{L}$, $\mathcal{N}_{4a}^{di} \cdot \mathcal{L}$, $\mathcal{N}_{4a}^d \cdot \mathcal{C}$, and so forth). Since $H(s)$ has no poles on $j\mathbb{R} \cup \infty$, then N must not possess an L-cut-set or C-cut-set. Then, noting that $\mathcal{N}_5^d \cdot \mathcal{L} = \left(\mathcal{N}_5^{di} \cdot \mathcal{C}\right)^i$, it follows that N or N^i must belong to $\mathcal{N}_{8a}$, $\mathcal{N}_{8b}$, $\mathcal{N}_9$, or $\mathcal{N}_{10}$.

In case (iii), N_u belongs to $\mathcal{Q}_2$ by Lemma 3.3.4, and N_v is from $\mathcal{L}$ or $\mathcal{C}$. The only possibilities which have not been covered in case (i) are those in which N_u is from $\mathcal{N}_2$ or $\mathcal{N}_2^i$. Since $H(s)$ has no poles on $j\mathbb{R} \cup \infty$, then N does not possess an L-cut-set or C-cut-set. It follows that N or N^i must be from $\mathcal{N}_7$. That $H(s)$ does not have a pole at $s = j\omega_0$ implies the restriction $E \neq F$.

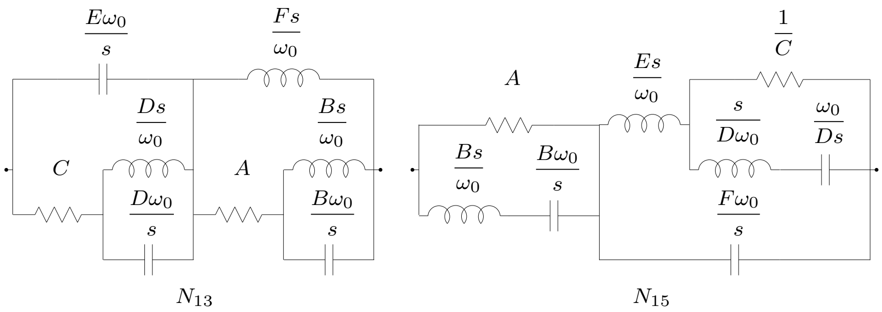


Figure 29: Network classes $\mathcal{N}_{13}$ $(A, B, C, D, E, F > 0)$ and $\mathcal{N}_{15}$ $(A, B, C, D, E, F > 0, E \neq F)$.

The case where $H(s)$ is the admittance of N is similar. In this case, we find that N or N^i belongs to one of the classes $\mathcal{N}_7^d$, $\mathcal{N}_{8a}^d$, $\mathcal{N}_{8b}^d$, $\mathcal{N}_9^d$, $\mathcal{N}_{10}^d$, $\mathcal{N}_{11}^d$, or $\mathcal{N}_{12}^d$. □

Lemmas 3.3.4 and 3.4.2 collectively enumerate all networks containing no more than four reactive elements and one resistor with impedance or admittance $H(s)$ where $H(s)$ has no poles on $j\mathbb{R} \cup \infty$, $\Re(H(j\omega_0)) = 0$, and $\Im(H(j\omega_0)) \neq 0$. It will be shown in Theorem 3.4.3 that one such network is always present as a subnetwork of any network containing no more than six reactive elements and two resistors which realises a minimum function (with ω_0 a minimum frequency).

Next, we introduce the following network classes:

- $\mathcal{N}_{13}$ (resp. $\mathcal{N}_{14a}$; $\mathcal{N}_{14b}$) is equal to $\mathcal{N}_2^d + \mathcal{N}_u$ for $\mathcal{N}_u$ equal to $\mathcal{N}_2^{di}$ (resp. $\mathcal{N}_{4a}$; $\mathcal{N}_{4b}$);
- $\mathcal{N}_{15}$ (resp. $\mathcal{N}_{16a}$; $\mathcal{N}_{16b}$; $\mathcal{N}_{17}$; $\mathcal{N}_{18}$; $\mathcal{N}_{19}$; $\mathcal{N}_{20}$) is equal to $\mathcal{N}_1 + \mathcal{N}_v$ for $\mathcal{N}_v$ equal to $\mathcal{N}_7$ (resp. $\mathcal{N}_{8a}$; $\mathcal{N}_{8b}$; $\mathcal{N}_9$; $\mathcal{N}_{10}$; $\mathcal{N}_{11}$; $\mathcal{N}_{12}$).

We denote the corresponding quartets by $\mathcal{Q}_{13}$, $\mathcal{Q}_{14a}$, $\mathcal{Q}_{14b}$, $\mathcal{Q}_{15}$, $\mathcal{Q}_{16a}$, $\mathcal{Q}_{16b}$, $\mathcal{Q}_{17}$, $\mathcal{Q}_{18}$, $\mathcal{Q}_{19}$, and $\mathcal{Q}_{20}$. Note that $\mathcal{N}_{13} = \mathcal{N}_{13}^i$, $\mathcal{N}_{19} = \mathcal{N}_{19}^i$, and $\mathcal{N}_{20} = \mathcal{N}_{20}^i$. Examples of a network N_{13} from class $\mathcal{N}_{13}$, and a network N_{15} from the class $\mathcal{N}_{15}$, are shown in Fig. 29. We remark that, with the exception of $\mathcal{Q}_{13}$, the impedance of any network from one of these quartets is a minimum function (with ω_0 a minimum frequency). The impedances of certain networks in $\mathcal{Q}_{13}$ are not minimum functions. For example, the impedance $H_{13}(s)$ of the network N_{13} in Fig. 29 satisfies $H_{13}(j\omega_0) = j/F - j/E$, and so $H_{13}(j\omega_0) = 0$ when $E = F$, in which case $H_{13}(s)$ is not a minimum function. On the other hand, it may be verified that $H_{13}(s)$ is a minimum function whenever $A, B, C, D, E, F > 0$ and $E \neq F$.

Theorem 3.4.3. *Let N be a SP network containing exactly six reactive elements and exactly two resistors. Further let N have impedance $H(s)$ which is a minimum function* *with ω_0 a minimum frequency. Then N belongs to one of the ten quartets $\mathcal{Q}_{13}$, $\mathcal{Q}_{14a}$, $\mathcal{Q}_{14b}$, $\mathcal{Q}_{15}$, $\mathcal{Q}_{16a}$, $\mathcal{Q}_{16b}$, $\mathcal{Q}_{17}$, $\mathcal{Q}_{18}$, $\mathcal{Q}_{19}$, or $\mathcal{Q}_{20}$.*

Proof. By definition, $\Re(H(j\omega_0)) = 0$, $\Im(H(j\omega_0)) \neq 0$, and $H(s)$ contains no poles or zeros on $j\mathbb{R} \cup \infty$. As in the proof of Theorem 3.3.5, $H(\omega_0^2/s)$, $1/H(s)$ and $1/H(\omega_0^2/s)$ are also minimum functions with ω_0 a minimum frequency. Since N contains more than one element, then it is either ES or EP.

Consider first the case where N is ES with $N = N_u + N_v$, so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then, as in the proof of Theorem 3.3.5, neither $Z_u(s)$ nor $Z_v(s)$ contains a pole on $j\mathbb{R} \cup \infty$, $\Re(Z_u(j\omega_0)) = \Re(Z_v(j\omega_0)) = 0$, and without loss of generality let $\Im(Z_v(j\omega_0)) \neq 0$. Then N_u (resp. N_v) contains at least two (resp. three) reactive elements and at least one resistor by Lemma 3.3.2 (resp. Lemma 3.3.3). There are three cases to consider: (i) N_u contains exactly three reactive elements and $\Im(Z_u(j\omega_0)) \neq 0$, (ii) N_u contains exactly three reactive elements and $\Im(Z_u(j\omega_0)) = 0$, (iii) N_u contains exactly two reactive elements.

In case (i), both N_u and N_v must contain exactly three reactive elements and one resistor. Since $\Re(Z_u(j\omega_0)) = 0$, $\Im(Z_u(j\omega_0)) \neq 0$, $Z_u(s)$ is not lossless, and $Z_v(s)$ satisfies these same three properties, then both N_u and N_v belong to the quartet $\mathcal{Q}_2$ by Lemma 3.3.4. Furthermore, since neither $Z_u(s)$ nor $Z_v(s)$ can have a pole on $j\mathbb{R} \cup \infty$, then neither N_u nor N_v can have an L-cut-set or C-cut-set, and so both N_u and N_v must belong to either $\mathcal{N}_2^d$ or $\mathcal{N}_2^{di}$. Since $H(s)$ has no zeros on $j\mathbb{R} \cup \infty$, then N must not have an L-path or C-path. It follows that N must be from $\mathcal{N}_{13}$.

In case (ii), both N_u and N_v must again contain exactly three reactive elements and one resistor and, similarly to case (i), N_v must be from $\mathcal{N}_2^d$ or $\mathcal{N}_2^{di}$. In this case, $Z_u(s)$ is not lossless and satisfies $Z_u(j\omega_0) = 0$, hence N_u or N_u^i is from $\mathcal{N}_{4a}$, $\mathcal{N}_{4b}$, or $\mathcal{N}_5$ by Lemma 3.4.1. Since $Z_u(s)$ cannot have a pole on $j\mathbb{R} \cup \infty$, then N_u cannot have an L-cut-set or C-cut-set. It follows that N_u or N_u^i must be from $\mathcal{N}_{4a}$ or $\mathcal{N}_{4b}$. Since $H(s)$ has no zeros on $j\mathbb{R} \cup \infty$, then N cannot have an L-path or C-path, and so N or N^i must be from $\mathcal{N}_{14a}$ or $\mathcal{N}_{14b}$.

In case (iii), since N contains exactly six reactive elements and two resistors, N_u (resp. N_v) must contain exactly two (resp. four) reactive elements and one resistor. Since $Z_u(s)$ is not lossless and $\Re(Z_u(j\omega_0)) = 0$, then N_u belongs to $\mathcal{N}_1$ by Lemma 3.3.3. Also, since $\Re(Z_v(j\omega_0)) = 0$, $\Im(Z_v(j\omega_0)) \neq 0$, and $Z_v(s)$ has no poles on $j\mathbb{R} \cup \infty$, then N_v or N_v^i must belong to one of the classes $\mathcal{N}_7$, $\mathcal{N}_{8a}$, $\mathcal{N}_{8b}$, $\mathcal{N}_9$, $\mathcal{N}_{10}$, $\mathcal{N}_{11}$, or $\mathcal{N}_{12}$ by Lemma 3.4.2. Hence, N or N^i is from one of the classes $\mathcal{N}_{15}$, $\mathcal{N}_{16a}$, $\mathcal{N}_{16b}$, $\mathcal{N}_{17}$, $\mathcal{N}_{18}$, $\mathcal{N}_{19}$, or $\mathcal{N}_{20}$.

The case where N is EP is similar. In this case, we find that N or N^i is from one of the classes $\mathcal{N}_{13}^d$, $\mathcal{N}_{14a}^d$, $\mathcal{N}_{14b}^d$, $\mathcal{N}_{15}^d$, $\mathcal{N}_{16a}^d$, $\mathcal{N}_{16b}^d$, $\mathcal{N}_{17}^d$, $\mathcal{N}_{18}^d$, $\mathcal{N}_{19}^d$, or $\mathcal{N}_{20}^d$. $\square$

From Theorems 3.3.5 and 3.4.3, any SP network which realises a minimum function and contains at most six reactive elements and at most two resistors belongs to one of eleven quartets, these being $\mathcal{Q}_3$ in Fig. 23 and the ten quartets listed in Theorem 3.4.3. As will be shown in the next theorem, only networks from $\mathcal{Q}_{14a}$ and $\mathcal{Q}_{14b}$ realise biquadratic minimum functions, and only when the impedances of their elements satisfy certain relationships. Those networks which realise biquadratic minimum functions are shown in Fig. 30.

We finally introduce the networks N_{21a} and N_{21b} in Fig. 30. We define the network class $\mathcal{N}_{21a}$ (resp. $\mathcal{N}_{21b}$) as the set of all such networks N_{21a} (resp. N_{21b}) for $B = AD/C$, $F = A/(CE)$, $D = (\sqrt{AC} - 1)(A + CE^2)/(AE)$, $A, E > 0$, $AC > 1$ (and $G = B + F$). The corresponding network quartets are denoted by $\mathcal{Q}_{21a}$ and $\mathcal{Q}_{21b}$. We remark that $\mathcal{Q}_{21a}$ contains those networks obtained by applying the Bott-Duffin procedure to a biquadratic minimum function (which correspond to the networks in Fig. 19.1 for the case of the realisation of a biquadratic minimum function). Furthermore, the networks in $\mathcal{Q}_{21b}$ are related to those in $\mathcal{Q}_{21a}$ via application of the transformation in [22, Lemma 11] (see also Section 3.2.1) to a pertinent subnetwork, and correspond to the networks in Fig. 19.2 for the case of the realisation of a biquadratic minimum function.

Theorem 3.4.4. *Let N be a SP network containing at most six reactive elements and at most two resistors. Further let N have impedance $H(s)$ which is a biquadratic minimum function with minimum frequency ω_0. Then N belongs to one of the quartets $\mathcal{Q}_{21a}$ or $\mathcal{Q}_{21b}$.*

Proof. From Theorem 3.3.6, N must contain exactly six reactive elements and exactly two resistors. It follows that N satisfies the conditions of Theorem 3.4.3, and so N belongs to one of the ten quartets listed in that theorem's statement. Let $H_k(s)$ be the impedance of a network N_k from the class $\mathcal{N}_k$ ($k = 13, 14a, 14b, 15, 16a, 16b, 17, 18, 19, 20$). As in Theorem 3.3.6, it suffices to consider the conditions for these functions to be biquadratic.

In the following, we let $H_k(s) = H_u(s) + H_v(s)$ where $H_u(s)$ and $H_v(s)$ are the impedances of the subnetworks N_u and N_v, described in the proof of Theorem 3.4.3, respectively. In each case, we will write $H_u(s) = n_u(s)/d_u(s)$ and $H_v(s) = n_v(s)/d_v(s)$ where $n_u(s)$, $d_u(s)$, $n_v(s)$, and $d_v(s)$ are polynomials in s with $n_u(s)$ and $d_u(s)$ coprime and with $n_v(s)$ and $d_v(s)$ coprime. The functions $n_u(s)$, $d_u(s)$, $n_v(s)$, and $d_v(s)$ are also

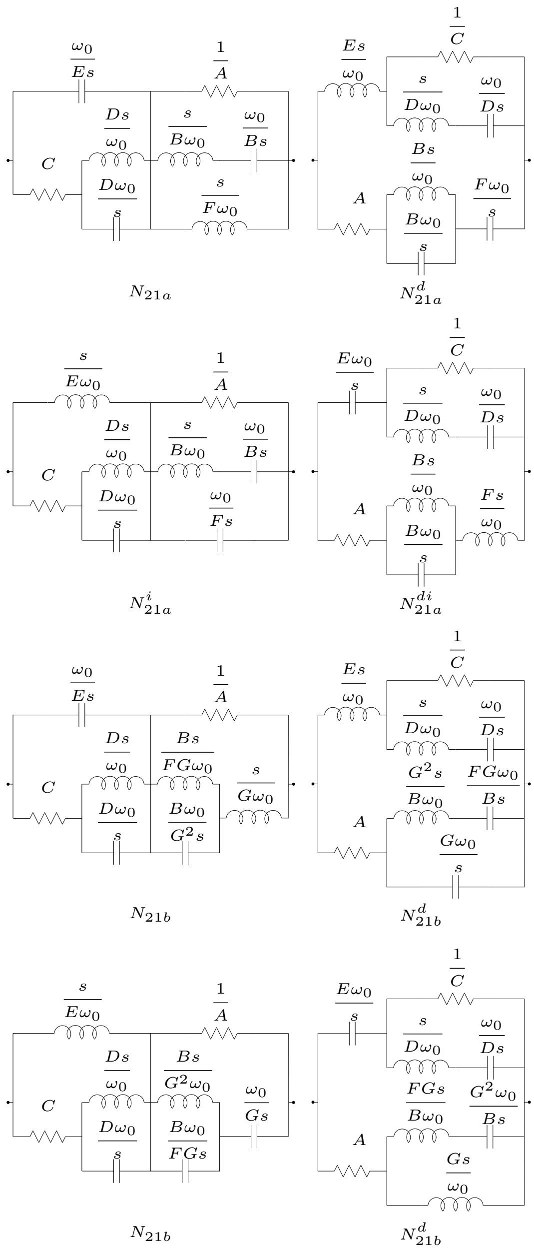


Figure 30: Quartets $\mathcal{Q}_{21a}$ and $\mathcal{Q}_{21b}$. $G = B + F$, $B = AD/C$, $F = A/(CE)$, $D = (\sqrt{AC} - 1)(A + CE^2)/(AE)$, $A, E > 0$, $AC > 1$.

polynomials in the network class parameters $A \ldots F$, hence so too are the Sylvester determinants in s involving any pair of these functions. Moreover, we find $\deg(d_u(s)) \geq 2$ and $\deg(d_v(s)) \geq 3$ in each case so, similarly to the proof of Theorem 3.3.6, we require $d_u(s)$ and $d_v(s)$ to have at least two common roots. We first show that this is not possible for $H_{13}(s)$, $H_{15}(s)$, $H_{17}(s)$, $H_{19}(s)$, and $H_{20}(s)$.

For $H_{13}(s)$, $n_u(s) = As^3 + B\omega_0 s^2 + A\omega_0^2 s$, $d_u(s) = s^3 + AF\omega_0 s^2 + (1+BF)\omega_0^2 s + AF\omega_0^3$, $n_v(s) = C\omega_0 s^2 + D\omega_0^2 s + C\omega_3$, and $d_v(s) = CEs^3 + (1+DE)\omega_0 s^2 + CE\omega_0^2 s + \omega_0^3$, for some $A, B, C, D, E, F > 0$. It was shown in the proof of Theorem 3.3.6 that $n_v(s)$ and $d_v(s)$ are coprime. That $n_u(s)$ and $d_u(s)$ are coprime follows since $R_0(n_u(s), d_u(s)) = A^2B^2F\omega_0^9 \neq 0$. As explained in the preceding paragraph, if $H_{13}(s)$ is biquadratic then $d_u(s)$ and $d_v(s)$ have two or more common roots, which implies $R_0(d_u(s), d_v(s)) = R_1(d_u(s), d_v(s)) = 0$. As explained earlier, $R_0(d_u(s), d_v(s))$ and $R_1(d_u(s), d_v(s))$ are both polynomials in A, B, C, D, E, and F, so for these to be simultaneously zero we require $R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right) = 0$. This is not possible since

$$\begin{aligned} &R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right) \\ &\qquad\qquad = B^2E^6F^6\omega_0^{30}\left((BF+DE)(B^2+(ADE)^2) + B^3DEF\right)^2, \end{aligned}$$

which cannot be zero[26]. Hence, $H_{13}(s)$ cannot be biquadratic.

For $H_{15}(s)$, $H_{17}(s)$, $H_{19}(s)$, and $H_{20}(s)$, we have $n_u(s) = s^2 + \omega_0^2$ and $d_u(s) = As^2 + B\omega_0 s + A\omega_0^2$ for some $A, B > 0$, in which case $n_u(s)$ and $d_u(s)$ are coprime. For $H_{17}(s)$, $n_v(s) = (1+DF)\omega_0 s^2 + CD\omega_0^2 s + \omega_0^3$, and

$$d_v(s) = (F(1+DE)+E)s^3 + C(1+DE)\omega_0 s^2 + (E+F)\omega_0^2 s + C\omega_0^3,$$

for some $C, D, E, F > 0$, which are coprime since $R_0(n_v(s), d_v(s)) = D^2(C^2+F^2)^2\omega_0^9$. Furthermore, we find

$$\begin{aligned} &R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right) \\ &\qquad\qquad = A^3\omega_0^{10}\left(B^2(F(1+DE)+E) + F(ADE)^2\right)^2, \end{aligned}$$

which cannot be zero. For $H_{15}(s)$, $H_{19}(s)$, and $H_{20}(s)$, we find that

$$\begin{aligned} & n_v(s) = CEF\omega_0 s^3 + F(1+DE)\omega_0^2 s^2 + CEF\omega_0^3 s + F\omega_0^4, \\ & n_v(s) = CEs^4 + DE\omega_0 s^3 + C(E+F)\omega_0^2 s^2 + DF\omega_0^3 s + CF\omega_0^4, \\ \text{and}\quad & n_v(s) = C\omega_0 s^3 + D\omega_0^2 s^2 + C\omega_0^3 s, \end{aligned}$$

respectively. For $H_{15}(s)$ and $H_{20}(s)$,

$$d_v(s) = CEs^4 + (1+DE)\omega_0 s^3 + C(E+F)\omega_0^2 s^2 + (1+DF)\omega_0^3 s + CF\omega_0^4,$$

[26] We again remark that the choice of parameter C in the expression $R_0\left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right)$ is arbitrary, and has been selected in order to provide a concise algebraic argument. Similar algebraic arguments will be made throughout the remainder of this thesis.

and for $H_{19}(s)$,

$$d_v(s) = Es^4 + C\omega_0 s^3 + (D + E + F)\omega_0^2 s^2 + C\omega_0^3 s + F\omega_0^4,$$

for some $C, D, E, F > 0$, with $E \neq F$. That $n_v(s)$ and $d_v(s)$ are coprime for $H_{15}(s)$, $H_{19}(s)$, $H_{20}(s)$ follows since

$$R_0(n_v(s), d_v(s)) = C^3 D^2 E^2 F^7 \omega_0^{16},$$
$$R_0(n_v(s), d_v(s)) = D^2 EF\omega_0^{16} \left(C^2(E - F)^2 + D^2 EF\right)^2,$$
$$\text{and} \qquad R_0(n_v(s), d_v(s)) = C^3 D^2 F \omega_0^{16},$$

respectively. In each of these three cases, we find

$$R_0 \left(R_0(d_u(s), d_v(s))(C), R_1(d_u(s), d_v(s))(C)\right) = A^6 B^4 \omega_0^{14} (E - F)^2,$$

which cannot be zero since $E \neq F$. As in the proof of Theorem 3.3.6, it follows that $d_u(s)$ and $d_v(s)$ cannot have two common roots for $H_{15}(s)$, $H_{17}(s)$, $H_{19}(s)$, and $H_{20}(s)$, and hence these functions cannot be biquadratic.

In the remaining cases, we will show that two of the network class parameters can be written as rational functions in the remaining parameters whenever $d_u(s)$ and $d_v(s)$ have at least two common roots. We then let $H_k(s) = \tilde{n}(s)/\tilde{d}(s)$ where $\tilde{n}(s)$ and $\tilde{d}(s)$ are polynomials in s and the remaining network class parameters. By then considering conditions on $\tilde{n}(s)$ and $\tilde{d}(s)$ for $H_k(s)$ to be biquadratic, we will show that $H_{16a}(s)$, $H_{16b}(s)$, and $H_{18}(s)$ cannot be biquadratic. In each of these three cases, we have $n_u(s) = s^2 + \omega_0^2$ and $d_u(s) = As^2 + B\omega_0 s + A\omega_0^2$ for some $A, B > 0$, and so $n_u(s)$ and $d_u(s)$ are coprime.

For both $H_{16a}(s)$ and $H_{16b}(s)$, we have $n_v(s) = F\omega_0 s^3 + C\omega_0^2 s^2 + (D + F)\omega_0^3 s + C\omega_0^4$, and

$$d_v(s) = EFs^4 + CE\omega_0 s^3 + (E(D + F) + 1)\omega_0^2 s^2 + CE\omega_0^3 s + \omega_0^4,$$

for some $C, D, E, F > 0$. Since $R_0(n_v(s), d_v(s)) = D^2 F^2 \omega_0^{16}$ it follows that $n_v(s)$ and $d_v(s)$ are coprime. For $d_u(s)$ and $d_v(s)$ to have two common roots, we require $R_0(d_u(s), d_v(s)) = R_1(d_u(s), d_v(s)) = 0$. These equations may be solved for C and F to give

$$C = \frac{B^2 + A^2 DE}{ABE},$$
$$\text{and} \qquad F = \frac{1}{E}.$$

We then find

$$\tilde{n}(s) = BEs^4 + A(B + DE^2)\omega_0 s^3 + (B(B + 2E) + A^2 DE)\omega_0^2 s^2 + A\left(B(1 + DE) + DE^2\right)\omega_0^3 s + (B(B + E) + A^2 DE)\omega_0^4,$$

and $\tilde{d}(s) = d_u(s)\tilde{d}_2(s)$ with $\tilde{d}_2(s) = E(Bs^2 + ADE\omega_0 s + B\omega_0^2)$. Since $R_0(\tilde{n}(s), \tilde{d}_2(s)) = A^4 B^4 D^2 E^6 \omega_0^8$ which cannot be zero, then for $H_{16a}(s)$ or $H_{16b}(s)$ to be biquadratic we require $\tilde{n}(s)$ and $d_u(s)$ to have at least two common roots. This is not possible since

$$R_0\left(R_0(\tilde{n}(s), d_u(s))(A), R_1(\tilde{n}(s), d_u(s))(A)\right) = B^{24} E^4 \omega_0^{28},$$

which cannot be zero. Hence, neither $H_{16a}(s)$ nor $H_{16b}(s)$ can be biquadratic.

A similar argument to the above shows that $H_{18}(s)$ cannot be biquadratic. In this case, we have $n_v(s) = \omega_0 s^3 + CD\omega_0^2 s^2 + (1 + DF)\omega_0^3 s$, and

$$d_v(s) = Es^4 + C(1 + DE)\omega_0 s^3 + (E(1 + DF) + F)\omega_0^2 s^2 + C\omega_0^3 s + F\omega_0^4,$$

for some $C, D, E, F > 0$. We then find

$$R_0(n_v(s), d_v(s)) = D^2 F \omega_0^{16}(C^2 + F^2)^2,$$

which cannot be zero. Furthermore, $R_0(d_u(s), d_v(s)) = R_1(d_u(s), d_v(s)) = 0$ requires

$$C = \frac{BE(B^2 + A^2 DE)}{AG}, \quad \text{and} \quad F = \frac{B^2 E}{G},$$

where $G = B^2(1 + DE) + (ADE)^2$. In this case, we find

$$\tilde{n}(s) = EGs^4 + A(G + BDE^2)\omega_0 s^3 + E(G + B(B(1 + BD) + A^2 D^2 E))\omega_0^2 s^2 + A(BDE(B + E) + G)\omega_0^3 s + B^2 E\omega_0^4,$$

and $\tilde{d}(s) = d_u(s)\tilde{d}_2(s)$ with $\tilde{d}_2(s) = E(Gs^2 + ABDE\omega_0 s + B^2\omega_0^2)$. Here, $R_0(\tilde{n}(s), \tilde{d}_2(s)) = B^2 D^2 E^6 G^2 \omega_0^8 \left(B^2(B^2 + A^2 DE) + A^2 G\right)^2$, and

$$R_0\left(R_0(\tilde{n}(s), d_u(s))(A), R_1(\tilde{n}(s), d_u(s))(A)\right) = B^{38} D^6 E^{14} \omega_0^{50}(1 + DE)^4 (B + DE^2)^2 (B(DE + 2) + 2DE^2)^4,$$

neither of which can be zero. Hence, $H_{18}(s)$ cannot be biquadratic.

We finally show that $H_{14a}(s)$ (resp. $H_{14b}(s)$) is biquadratic only when the corresponding network belongs to the class $\mathcal{N}_{21a}$ (resp. $\mathcal{N}_{21b}$). For both $H_{14a}(s)$ and $H_{14b}(s)$, we find $n_u(s) = s^3+\omega_0^2 s$, $d_u(s) = As^3+(B+F)\omega_0 s^2+A\omega_0^2 s+F\omega_0^3$, $n_v(s) = C\omega_0 s^2+D\omega_0^2 s+C\omega_0^3$, and $d_v(s) = CEs^3+(1+DE)\omega_0 s^2+CE\omega_0^2 s+\omega_0^3$, for some $A, B, C, D, E, F > 0$. From before, we see that $n_v(s)$ and $d_v(s)$ are coprime. That $n_u(s)$ and $d_u(s)$ are coprime follows since $R_0(n_u(s), d_u(s)) = B^2 F\omega_0^9$. For $d_u(s)$ and $d_v(s)$ to have two common roots, we require $R_0(d_u(s), d_v(s)) = R_1(d_u(s), d_v(s)) = 0$. These equations may be solved for B and F to give

$$B = \frac{AD}{C},$$
$$\text{and} \qquad F = \frac{A}{CE}.$$

We then have

$$\tilde{n}(s) = CEs^3 + AC\omega_0 s^2 + (AD + CE)\omega_0^2 s + AC\omega_0^3,$$

and $\tilde{d}(s) = Ad_v(s)$. Then $H_{14a}(s)$ (likewise, $H_{14b}(s)$) is biquadratic if and only if

$$R_0(\tilde{n}(s), d_v(s)) = CD^2 E\omega_0^9((ADE + A + CE^2)^2 - AC(A + CE^2)^2) = 0,$$

implying

$$D = \frac{(\sqrt{AC} - 1)(A + CE^2)}{AE}.$$

It follows that the only networks containing exactly six reactive elements and two resistors which realise a biquadratic minimum function (with minimum frequency ω_0) are those from the quartets $\mathcal{Q}_{21a}$ and $\mathcal{Q}_{21b}$. □

It may be shown that if $H(s)$ is a biquadratic minimum function with $H(0) > H(\infty)$ (resp. $H(\infty) > H(0)$), then $H(s)$ is the impedance of a network from each of the classes $\mathcal{N}_{21a}$, $\mathcal{N}_{21b}$, $\mathcal{N}_{21a}^{di}$, and $\mathcal{N}_{21b}^{di}$ (resp. $\mathcal{N}_{21a}^{d}$, $\mathcal{N}_{21b}^{d}$, $\mathcal{N}_{21a}^{i}$, and $\mathcal{N}_{21b}^{i}$), the minimum frequency being ω_0.

We conclude that any SP network which realises a biquadratic minimum function and contains no more than six reactive elements and no more than two resistors belongs to one of the two quartets in Fig. 30 (where ω_0 is the minimum frequency).

3.5 On the minimality of the Reza-Pantell-Fialkow-Gerst procedure for biquadratic minimum functions

In this section, we show that only a small subset of the biquadratic minimum functions are realised by transformerless networks containing fewer than five reactive elements. In particular, we show that the Reza-Pantell-Fialkow-Gerst simplification, and the alternative described in Section 3.1.5, contain both the minimal number of reactive elements and the minimal number of resistors for realising biquadratic minimum functions, apart from some exceptional cases. To show this, we first find those transformerless networks containing fewer than five reactive elements which can realise minimum functions, and we present networks containing the least possible numbers of reactive elements and resistors for realising each such minimum function. We go on to determine those exceptional *biquadratic* minimum functions which can be realised by transformerless networks containing fewer than five reactive elements, and again we present a network realisation for each such case. The networks we identify are precisely those networks identified in [52] and [53]. This allows us to conclude that not only do these networks contain the minimal number of passive elements, but they also contain the minimal number of reactive elements, for the realisation of certain biquadratic minimum functions among all transformerless networks.

We will adopt the phasor analysis described in Part 1 of this thesis, and we will couple this with the notion of a graph of one-ports (see Definition 1.9.2 and the discussion in Section 1.9). In Part 1, the notions of an s_0-trajectory and an s_0-driving-point trajectory of a graph of one-ports corresponding to a transformerless network are introduced. In the following lemma, we consider properties of the s_0-trajectories of a graph of one-ports corresponding to a network N whose impedance $Z(s)$ or admittance $Y(s)$ satisfies certain imaginary axis constraints.

Lemma 3.5.1. *Let N be a transformerless network with impedance $Z(s)$ and admittance $Y(s)$. Further, let G be a graph of one-ports corresponding to N, in which the edges correspond to the one-ports $N_1, \ldots, N_m$ (in addition to an edge for the source), and denote the impedance and admittance of the one-port N_k by $Z_k(s)$ and $Y_k(s)$ respectively ($k = 1, 2, \ldots m$). Consider an $\omega \in \mathbb{R} \cup \infty$, and let $\tilde{\mathbf{b}}$ in (65) be a $j\omega$-trajectory of G, with $\tilde{\mathbf{d}} := \begin{bmatrix} I_2 & 0_{2\times 2m} \end{bmatrix} \tilde{\mathbf{b}} \neq \mathbf{0}_2$ the corresponding $j\omega$-driving-point trajectory. Then the following three conditions must all hold:*

1. *$\tilde{i} = 0$ if and only if $Z(s)$ has a pole at $s = j\omega$, otherwise $Z(j\omega) = \tilde{v}/\tilde{i}$.*
2. *If $Z_k(s)$ has a pole at $s = j\omega$ then $\tilde{i}_k = 0$, otherwise $\tilde{v}_k = Z_k(j\omega)\tilde{i}_k$.*

3. If $Y_k(s)$ has a pole at $s = j\omega$ then $\tilde{v}_k = 0$, otherwise $\tilde{i}_k = Y_k(j\omega)\tilde{v}_k$.

Moreover, suppose either $\Re(Z(j\omega)) = 0$ or $\Re(Y(j\omega)) = 0$. Then the following two conditions must also hold:

4. If $\Re(Z_k(j\omega)) \neq 0$ then $\tilde{i}_k = 0$.

5. if $\Re(Y_k(j\omega)) \neq 0$ then $\tilde{v}_k = 0$.

Proof. From Subsections 1.7 to 1.9, $\begin{bmatrix}\tilde{i}_k & \tilde{v}_k\end{bmatrix}^T$ is a $j\omega$-trajectory of the one-port N_k, and hence conditions 2 and 3 hold by Theorems 1.7.4 and 1.8.3. Since, in addition, $\begin{bmatrix}\tilde{i} & \tilde{v}\end{bmatrix}^T \neq \mathbf{0}$, then condition 1 must also hold by Theorem 1.9.7.

Since the cut-set and circuit spaces of a graph are orthogonal, then $\begin{bmatrix}-\tilde{i} & \tilde{i}_1 & \dots & \tilde{i}_m\end{bmatrix}^T$ and $\begin{bmatrix}\tilde{v} & \tilde{v}_1 & \dots & \tilde{v}_m\end{bmatrix}^T$ are orthogonal, hence

$$\tilde{v}^*\tilde{i} + \tilde{i}^*\tilde{v} = \sum_{k=1}^{m} \tilde{v}_k^*\tilde{i}_k + \tilde{i}_k^*\tilde{v}_k. \tag{151}$$

If $Z(s)$ has a pole or a zero at $s = j\omega$, then $\tilde{v}^*\tilde{i} = \tilde{i}^*\tilde{v} = 0$ by condition 1 of the present lemma, otherwise $\tilde{v}^*\tilde{i} + \tilde{i}^*\tilde{v} = \Re(Z(j\omega))\,|\tilde{i}|^2 = \Re(Y(j\omega))\,|\tilde{v}|^2$. Hence, the left hand side of (151) is zero if either $\Re(Z(j\omega))$ or $\Re(Y(j\omega))$ is zero. Furthermore, if $Z_k(s)$ or $Y_k(s)$ has a pole at $s = j\omega$, then $\tilde{v}_k^*\tilde{i}_k = \tilde{i}_k^*\tilde{v}_k = 0$ from conditions 2 and 3. Otherwise, $\tilde{v}_k^*\tilde{i}_k + \tilde{i}_k^*\tilde{v}_k = \Re(Z_k(j\omega))\,|\tilde{i}_k|^2 = \Re(Y_k(j\omega))\,|\tilde{v}_k|^2$, which is non-negative since $Z(s)$ and $Y(s)$ are PR. Since all terms in the summation on the right hand side of (151) are non-negative then they must all be zero in order that their sum is zero. This proves conditions 4 and 5. □

Hence, if N is a network which realises a minimum function with minimum frequency $j\omega_0$, and $\tilde{\mathbf{b}}$ is a $j\omega_0$-trajectory of N, then any resistors in N possess zero current and zero voltage (by conditions 4 and 5 of Lemma 3.5.1). Indeed, the absence of current or voltage is a property of a much broader class of one-ports within the network. We refer to such one-ports as $i(j\omega_0)$-blocked and $v(j\omega_0)$-blocked respectively. In our argument, we will group together those one-ports which are both $\tilde{i}(j\omega_0)$-blocked and $\tilde{v}(j\omega_0)$-blocked into subnetworks, which we will refer to as ω_0-*blocked*. We summarise these concepts in the following definition:

Definition 3.5.2.
Let N be a transformerless network and let $\bar{N}$ be a one-port of N. Further, let G be

a graph of one-ports corresponding to N in which one of the edges corresponds to the one-port $\bar{N}$ and the remaining edges correspond to elements in N (in addition to an edge for the source). Moreover, consider an $\omega \in \mathbb{R} \cup \infty$, consider the $j\omega$-trajectories of G, and denote the current and voltage in the edge in G corresponding to $\bar{N}$ by $\hat{i}_{\bar{N}}(t) = \Re(\tilde{i}_{\bar{N}} e^{j\omega t})$ and $\hat{v}_{\bar{N}}(t) = \Re(\tilde{v}_{\bar{N}} e^{j\omega t})$ respectively. Then

1. We call N_k $\tilde{i}(j\omega)$-*blocked* if $\tilde{i}_{\bar{N}} = 0$ for all $j\omega$-trajectories.
2. We call N_k $\tilde{v}(j\omega)$-*blocked* if $\tilde{v}_{\bar{N}} = 0$ for all $j\omega$-trajectories.

Further, let $\hat{N}$ be a subnetwork of N. Then

3. We call $\hat{N}$ ω-*blocked* if all elements within $\hat{N}$ are both $\tilde{i}(j\omega)$-blocked and $\tilde{v}(j\omega)$-blocked, and a *maximal ω-blocked subnetwork* of N if, in addition, it is not contained within any other ω-blocked subnetworks of N.

We now make some remarks about the preceding definition of relevance to our subsequent analysis.

Remark 3.5.3.
The definition of $\tilde{i}(j\omega)$-blocked and $\tilde{v}(j\omega)$-blocked one-ports refers to a particular graph of one-ports corresponding to the transformerless network N. Now, let G be *any* graph of one-ports corresponding to N which contains an edge corresponding to the one port $\bar{N}$. It may be shown that $\bar{N}$ is an $\tilde{i}(j\omega) - blocked$ (resp. $\tilde{v}(j\omega) - blocked$) one-port subnetwork of N if and only if, for all $j\omega$-trajectories of G, the current (resp. voltage) in $\bar{N}$, denoted $\hat{i}_{\bar{N}} = \tilde{i}_{\bar{N}} e^{j\omega t}$ (resp. $\hat{v}_{\bar{N}} = \tilde{v}_{\bar{N}} e^{j\omega t}$), satisfies $\tilde{i}_{\bar{N}} = 0$ (resp. $\tilde{v}_{\bar{N}} = 0$). This may be proved similarly to Lemma 1.9.6 in Section 1.9. Whenever $\bar{N}$ is an $\tilde{i}(j\omega)$-blocked (resp. $\tilde{v}(j\omega)$-blocked) one-port in N, we will also refer to the edge in G corresponding to the one-port $\bar{N}$ as an $\tilde{i}(j\omega)$-blocked (resp. $\tilde{v}(j\omega)$-blocked) edge of G. Moreover, given the equivalence between a transformerless network and its graph of elements, we will refer to the ω-blocked subnetworks (resp. maximal ω-blocked subnetworks) of a graph of elements in an analogous manner to the ω-blocked subnetworks (resp. maximal ω-blocked subnetworks) of the corresponding network.

Remark 3.5.4.
The maximal ω-blocked subnetworks of a given network are unique. These correspond to the connected components of the graph which remains when all elements which are either not $\tilde{i}(j\omega)$-blocked or not $\tilde{v}(j\omega)$-blocked are removed from the graph of elements corresponding to the network.

We now describe certain properties of a transformerless network whose impedance satisfies certain imaginary axis constraints in terms of its ω-blocked subnetworks. We recall that throughout this part we fix $\omega_0 > 0$, to define uniquely the relationship between N and N^i.

Lemma 3.5.5. *Let N be a transformerless network, and let N have impedance $H(s)$ which is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$. Further, let G be the graph of elements corresponding to N. Then the following three conditions must hold:*

1. *All resistors in N are ω_0-blocked subnetworks.*
2. *At least two edges in G which are not $\tilde{i}(j\omega_0)$-blocked are incident at each vertex where the maximal ω_0-blocked subnetworks connect to the rest of G.*
3. *No edge in G which is not $\tilde{i}(j\omega_0)$-blocked is incident with two vertices of the same maximal ω_0-blocked subnetwork.*

If, in addition, N contains fewer than five reactive elements, then the following two conditions must also hold:

4. *There are either three or four passive elements in N which are not $\tilde{i}(j\omega_0)$-blocked, and each of these elements is reactive.*
5. *There are either one or two maximal ω_0-blocked subnetworks of N, and each of these subnetworks is a one-port.*

Proof. Let N (likewise, G) comprise the elements $N_1, \ldots, N_m$, and let the passive element N_k have impedance $Z_k(s)$ and admittance $Y_k(s)$ ($k = 1, 2, \ldots, m$). Further, let $\tilde{\mathbf{b}}$ in (65) be a $j\omega_0$-trajectory of N (likewise, G) with $\tilde{\mathbf{d}} := \begin{bmatrix} I_2 & 0_{2\times 2m} \end{bmatrix} \tilde{\mathbf{b}} \neq \mathbf{0}$ the corresponding $j\omega_0$-driving-point trajectory. Since $H(s)$ does not have a pole at $s = j\omega_0$, and $H(j\omega_0) \neq 0$, then $\tilde{i} \neq 0$, and $\tilde{v} \neq 0$ by Lemma 3.5.1. If the element N_k is a resistor, then $\Re(Z_k(j\omega_0)) \neq 0$ and $\Re(Y_k(j\omega_0)) \neq 0$, hence $\tilde{i}_k = \tilde{v}_k = 0$ by Lemma 3.5.1. It follows that condition 1 of the present lemma statement must hold. Moreover, as $H(s)$ is not lossless, then N must contain at least one resistor. Hence, there must be at least one ω_0-blocked subnetwork in N (likewise, G). Since N_k is a passive element and $\omega_0 > 0$, then $Z_k(s)$ (likewise, $Y_k(s)$) has neither a pole nor a zero at $s = j\omega_0$, and hence $\tilde{v}_k = 0$ if and only if $\tilde{i}_k = 0$, by Lemma 3.5.1 ($k = 1, 2, \ldots, m$).

Suppose we have a maximal ω_0-blocked subnetwork in G which connects to the rest of G at exactly x vertices. If at any one of these vertices the ω_0-blocked subnetwork connects to only edge in the rest of G, then the current $\hat{i}_k$ in this edge satisfies $\hat{i}_k(t) =$

$\Re(\tilde{i}_k e^{j\omega_0 t}) = 0$ by Kirchhoff's current law. Consequently, this element cannot be the source, and if instead it corresponds to a passive element then the voltage $\hat{v}_k$ in the edge satisfies $\hat{v}_k(t) = \Re(\tilde{v}_k e^{j\omega_0 t}) = 0$ (see the final sentence in the previous paragraph), which contradicts our requirement for the ω_0-blocked subnetwork to be maximal. Hence, condition 2 of the present lemma statement must hold. Now, suppose there is an edge in G outside of this maximal ω_0-blocked subnetwork which is incident with two vertices in the maximal ω_0-blocked subnetwork. Then the voltage $\hat{v}_k$ in the edge satisfies $\hat{v}_k(t) = \Re(\tilde{v}_k e^{j\omega_0 t}) = 0$ by Kirchhoff's voltage law, and similarly to before we contradict our requirement for the ω_0-blocked subnetwork to be maximal. Hence, condition 3 of the present lemma statement must also hold. It also follows that there must be at least $2x$ edges in G which are not ω_0-blocked, of which one is the source and the rest correspond to reactive elements by condition 1. As explained in Remark 1.2.4, no generality is lost in assuming G is biconnected, and so each ω_0-blocked subnetwork must connect to the rest of G at two or more vertices. Since there are fewer than five reactive elements in N, then we require $x = 2$, implying that there must be exactly three or exactly four passive elements in N which are not $\tilde{i}(j\omega_0)$-blocked. We have thus shown condition 4 of the present lemma.

It remains to show condition 5 in the present lemma statement. Suppose we have exactly y maximal ω_0-blocked subnetworks in G. By definition, these subnetworks are disconnected from each other. Moreover, in the preceding paragraph, it was shown that each of these subnetworks is incident with edges which are not ω_0-blocked at exactly two vertices, which implies that each maximal ω_0-blocked subnetwork is a one-port. Furthermore, from conditions 2 and 3, it follows that each of these subnetworks is incident with at least four edges in the rest of G. Let us sum the number of edges in G which are not ω_0-blocked and which are incident with the maximal ω_0-blocked subnetworks. In forming this sum, each edge in G which is not ω_0-blocked will be counted at most twice (since a particular edge can only be incident with two vertices). We thus find that the number of edges which are not ω_0-blocked must exceed $2y$. One of these edges is the source, and the rest correspond to reactive elements. Since there are fewer than five reactive elements in N, we conclude that $y \leq 2$, which completes the proof. $\square$

As a direct corollary to Lemma 3.5.5, we have the following:

Corollary 3.5.6. *Let N be a transformerless network with impedance $H(s)$ which is a minimum function. Then N contains at least three reactive elements.*

One way to consider the network N described in Lemma 3.5.5 is illustrated in Fig.

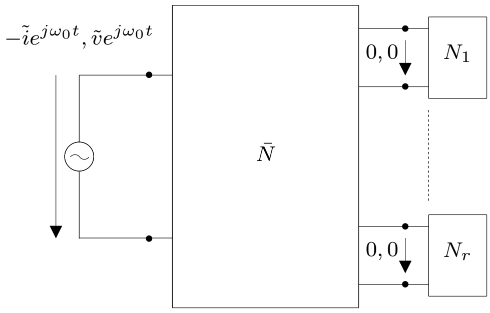


Figure 31: Network described in Lemma 3.5.5.

31. Here, whenever $\tilde{\mathbf{b}}$ in (65) is a $j\omega_0$-trajectory of N, certain one-ports $N_1, \ldots, N_r$ in N possess zero voltage and zero current. As a consequence, the one-ports $N_1, \ldots, N_r$ may be replaced by either open- or short-circuits without affecting the impedance of the network at $j\omega_0$. This will be formalised in the Lemma 3.5.10. We first make the following definition of the procedures of opening and shorting one-port subnetworks of a network, which we follow with some remarks.

Definition 3.5.7.
Let N_k be a one-port subnetwork of a network N. The operation of *opening* (resp. *shorting*) the one-port N_k corresponds to removing this one-port from the network N (resp. connecting together the driving-point terminals of this one-port in N), and then removing all the remaining elements which do not feature in any path between the driving-point terminals of N.

Remark 3.5.8.
The operation of opening a one-port subnetwork of a network N may leave no path of elements between the driving-point terminals. Similarly, shorting a one-port subnetwork of a network N may cause the driving-point terminals to be identified. In other words, these operations may result in a transformerless network which violates the assumptions made in Remark 1.2.6. In such cases, we obtain a transformerless network whose driving-point trajectories satisfy either $\hat{v} = 0$ or $\hat{i} = 0$.

Remark 3.5.9.
We define the operations of shorting and opening edges in a graph of one-ports in an analogous manner to Definition 3.5.7.

Lemma 3.5.10. *Let N be a transformerless network with impedance $H(s)$, let G be a graph of one-ports corresponding to N, and let $\omega \in \mathbb{R} \cup \infty$. If G contains a cut-set (resp. circuit) comprising the source together with $\tilde{i}(j\omega)$-blocked (resp. $\tilde{v}(j\omega)$-blocked)* *edges, then $H(s)$ has a pole (resp. zero) at $s = j\omega$.*

Now suppose $H(s)$ has neither a pole nor a zero at $s = j\omega$. Then opening certain of the $\tilde{i}(j\omega)$-blocked one-ports, and shorting certain of the $\tilde{v}(j\omega)$-blocked one-ports, will result in a transformerless network $\bar{N}$. Let the impedance of $\bar{N}$ be $\bar{H}(s)$. Then $\bar{H}(j\omega) = H(j\omega)$.

Proof. Suppose initially that G contains a cut-set which comprises the source together with $\tilde{i}(j\omega)$-blocked edges. Without loss of generality, let $N_1, N_2, \ldots, N_r$ correspond to the $\tilde{i}(j\omega)$-blocked edges in this cut-set. Then there is a cut-set vector $\mathbf{a}$ corresponding to this cut-set whose only non-zero entries correspond to the source and the edges $N_1, \ldots, N_r$. Further, let $\tilde{\mathbf{b}}$ in (65) be a $j\omega$-trajectory of G. Then $\tilde{i} = 0$ since $\tilde{i}_1 = \ldots = \tilde{i}_r = 0$, $\mathbf{a}_{r+1} = \ldots = \mathbf{a}_m = 0$, and $\begin{bmatrix} -\tilde{i} & \tilde{i}_1 & \ldots & \tilde{i}_m \end{bmatrix} \mathbf{a} = 0$ by Kirchhoff's current law. Hence, $H(s)$ has a pole at $s = j\omega$ by Theorem 1.9.7. That $H(s)$ has a zero at $s = j\omega$ if N contains a circuit comprising the source together with $\tilde{v}(j\omega)$-blocked edges may be shown similarly.

Suppose now that $H(s)$ has neither a pole nor a zero at $s = j\omega$. Further, suppose N_m is an $\tilde{i}(j\omega)$-blocked one-port in N, and let G be the graph of one-ports corresponding to N in which one of the edges corresponds to the one-port N_m, and the remaining edges correspond to the passive elements $N_1, N_2, \ldots, N_{m-1}$ (with an additional edge for the source). Further, suppose the one-port N_m and the elements $N_{m-r+1}, N_{m-r+2}, \ldots, N_{m-1}$ are removed when opening N_m. Again, let $\tilde{\mathbf{b}}$ in (65) be a $j\omega$-trajectory of G. We will show that $P_2\tilde{\mathbf{b}}$ is a $j\omega$-trajectory of the network which is obtained by opening N_m, where

$$P_2 := \begin{bmatrix} I_2 & 0_{2\times(m-r)} & 0_{2\times r} & 0_{2\times(m-r)} & 0_{2\times r} \\ 0_{(m-r)\times 2} & I_{m-r} & 0_{(m-r)\times r} & 0_{(m-r)\times(m-r)} & 0_{(m-r)\times r} \\ 0_{(m-r)\times 2} & 0_{(m-r)\times(m-r)} & 0_{(m-r)\times r} & I_{m-r} & 0_{(m-r)\times r} \end{bmatrix}.$$

First, consider the effect of removing the edge corresponding to the one-port N_m from G to form G_a. Let $\tilde{\mathbf{i}}^T := \begin{bmatrix} \tilde{i}_1 & \tilde{i}_2 & \ldots & \tilde{i}_{m-1} \end{bmatrix}$ and $\tilde{\mathbf{v}}^T := \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \ldots & \tilde{v}_{m-1} \end{bmatrix}$. Then $\begin{bmatrix} -\tilde{i} & \tilde{\mathbf{i}}^T & \tilde{i}_m \end{bmatrix}^T$ is in the circuit space of G by Kirchhoff's current law, and $\tilde{i}_m = 0$. Following Remark 1.2.4, we assume without loss of generality that G is biconnected. It follows that removal of N_m from G must not leave the network disconnected, so G contains a tree which does not contain the one-port N_m. By choosing a basis for the circuit space of G corresponding to this tree, we see that $\begin{bmatrix} -\tilde{i} & \tilde{\mathbf{i}}^T & \tilde{i}_m \end{bmatrix}^T$ is a linear sum of circuits of G which do not contain the one-port N_m. Since any circuit in G which does not contain the one-port N_m is also a circuit in G_a, then $\begin{bmatrix} -\tilde{i} & \tilde{\mathbf{i}}^T \end{bmatrix}^T$ is in the circuit space of G_a. Moreover, $\begin{bmatrix}\tilde{v} & \tilde{\mathbf{v}}^T & \tilde{v}_m\end{bmatrix}^T$ is in the cut-set space of G by Kirchhoff's voltage law, and any cut-set in G is a cut in G_a (corresponding to the same partitioning of vertices) providing the edge corresponding to the one-port N_m is removed whenever this features in a cut-set in G. Hence, $\begin{bmatrix}\tilde{v} & \tilde{\mathbf{v}}^T\end{bmatrix}^T$ is in the cut-set space of G_a (since this coincides with the space spanned by the cut vectors, as is evident from [30, Chapter II, Theorem 5]).

Now, let $\hat{N}$ be the network obtained by opening the one-port N_m in N, so $\hat{N}$ is comprised of the source together with the elements $N_1, N_2, \ldots, N_{m-r}$. Further, let us partition $\tilde{\mathbf{i}}$ as $\begin{bmatrix}\tilde{\mathbf{i}}_1^T & \tilde{\mathbf{i}}_2^T\end{bmatrix}^T$, where $\tilde{\mathbf{i}}_2 := \begin{bmatrix}\tilde{i}_{m-r+1} & \tilde{i}_{m-r+2} & \ldots & \tilde{i}_{m-1}\end{bmatrix}^T$ are the currents in those one-ports which correspond to edges in G_a which do not feature in any path between the vertices of G_a incident with the source. Partition $\tilde{\mathbf{v}}$ likewise. Since the circuit space (resp. cut-set space) of a graph is the orthogonal direct sum of the circuit spaces (resp. cut-set spaces) of the biconnected components of the graph (see Subsection 1.2.1), then $\begin{bmatrix}-\tilde{i} & \tilde{\mathbf{i}}_1^T\end{bmatrix}^T$ is in the circuit space of $\hat{N}$ and $\begin{bmatrix}\tilde{v} & \tilde{\mathbf{v}}_1^T\end{bmatrix}^T$ is in the cut-set space of $\hat{N}$. Moreover, $\begin{bmatrix}\tilde{i}_k & \tilde{v}_k\end{bmatrix}^T$ is a $j\omega$-driving-point trajectory for the element N_k $(k = 1, 2, \ldots, m-r)$. It follows that $P_2\tilde{\mathbf{b}}$ is a $j\omega$-trajectory of $\hat{N}$, and so the impedance of $\hat{N}$ at $j\omega$ is equal to $\tilde{v}/\tilde{i} = H(j\omega)$ by Theorems 1.7.4 and 1.8.3.

A similar argument shows that if N_m is a $\tilde{v}(j\omega)$-blocked one-port, if G is the graph of one-ports corresponding to N for which one edge corresponds to the one-port N_m and the remaining edges correspond to the passive elements $N_1, N_2, \ldots, N_{m-1}$ (with an additional edge for the source), if the elements $N_{m-r+1}, N_{m-r+2}, \ldots, N_{m-1}$ and the one-port N_m are removed upon shorting N_m, and if $\tilde{\mathbf{b}}$ in (65) is a $j\omega$-trajectory corresponding to G, then $P_2\tilde{\mathbf{b}}$ is a $j\omega$-trajectory for the network obtained by shorting the one-port N_m in N. The proof then follows by induction on the $\tilde{i}(j\omega)$-blocked one-ports which are opened and the $\tilde{v}(j\omega)$-blocked one-ports which are shorted in forming the network $\bar{N}$ described in the present lemma statement. □

We remark that Lemma 3.5.10 contains, as special cases relating to $\omega = 0$ and $\omega = \infty$, a similar result to that stated in Subsection 3.2.1 connecting L-paths, C-paths, L-cut-sets, and C-cut-sets to the poles of the impedance and admittance of a network at 0 and ∞. We summarise these results in the following corollary.

Corollary 3.5.11. *Let N be a transformerless network with impedance $Z(s)$ and admittance $Y(s)$. Then:*

1. *If N has a C-cut-set (resp. L-cut-set) then $Z(s)$ has a pole at $s = 0$ (resp. $s = \infty$).*

2. If N has a L-path (resp. C-path) then $Y(s)$ has a pole at $s = 0$ (resp. $s = \infty$).

Now, suppose neither $Y(s)$ nor $Z(s)$ contain a pole at $s = 0$ (resp. $s = \infty$). Then opening certain of the capacitors (resp. inductors), and shorting certain of the inductors (resp. capacitors), will result in a transformerless network. Denote the impedance of this network by $\bar{Z}(s)$. Then $Z(0) = \bar{Z}(0)$ (resp. $Z(\infty) = \bar{Z}(\infty)$).

Also, as a further consequence of Lemma 3.5.10, we may also show the following corollary, which is a slight generalisation of [52, Theorem 6]:

Corollary 3.5.12. *Let N be a transformerless network with impedance $H(s)$ which is a biquadratic minimum function. Then N contains at least two resistors, and the resistors in N must not all be located in a single one-port subnetwork of N comprised of resistors alone.*

Proof. From Section 3.2.2, any biquadratic minimum function may be written in the form of $H_p(s)$ in (145) with $W > 0$ and $W \neq 1$. In particular, $H(s)$ has neither a pole nor a zero at the points $s = 0$ and $s = \infty$, and $H(0) \neq H(\infty)$. Now, suppose all the resistors in N are in a single one-port subnetwork $\bar{N}$ of N (note that this is necessarily the case if N contains only one resistor). Let us consider the graph of one-ports G corresponding to N in which a single edge corresponds to the one-port $\bar{N}$ and the remaining edges correspond to reactive elements in N (with an additional edge for the source). Since $H(s)$ has neither a pole nor a zero at the points $s = 0$ and $s = \infty$ then from the proof of Lemma 3.5.10, by opening all the edges in G which correspond to capacitors and shorting all the edges in G which correspond to inductors, we arrive at a graph of one-ports corresponding to a transformerless network whose impedance at $s = 0$ must equal $H(0)$. Likewise, by shorting all the edges in G which correspond to capacitors and opening all the edges in G which correspond to inductors, we arrive at a graph of one-ports corresponding to a transformerless network whose impedance at ∞ must equal $H(\infty)$. Since neither of these graphs of one-ports contain any edges which correspond to reactive elements, then they must both contain only the one-port $\bar{N}$ (with the source attached between its vertices). As $\bar{N}$ contains only resistors, then its impedance is a positive constant, which implies $H(0) = H(\infty)$: a contradiction. □

We now turn our attention to the structure of those networks which contain fewer than five reactive elements and which realise a minimum function. From Theorem 3.3.5, we see that such networks cannot be series-parallel. In fact, we may extend the arguments in that part to show the following lemma:

Lemma 3.5.13. *Let N be a transformerless network which contains fewer than five reactive elements. Further let N have impedance $H(s)$ which is a minimum function. Then N cannot be a series connection, nor a parallel connection, of two one-port subnetworks.*

Proof. Suppose initially that N is a series connection of the one-ports N_1 and N_2, so $Z(s) = Z_1(s) + Z_2(s)$ where $Z(s), Z_1(s), Z_2(s)$ are the impedances of N, N_1, N_2. Then neither $Z_1(s)$ nor $Z_2(s)$ have any poles on $j\mathbb{R} \cup \infty$, and $\Re(Z_1(j\omega_0)) = \Re(Z_2(j\omega_0)) = 0$, by Lemma 3.3.1. In particular, neither $Z_1(s)$ nor $Z_2(s)$ is lossless, so both N_1 and N_2 must contain at least (and hence exactly) two reactive elements by Lemma 3.3.2. Since $\Im(Z(j\omega_0)) \neq 0$ then, without loss of generality, we require $\Im(Z_1(j\omega_0)) \neq 0$.

Now consider the function $F(s) = Z_1(s) + Z_1(-s)$, so $\Re(Z_1(j\omega)) = F(j\omega)/2$ for $\omega \in \mathbb{R} \cup \infty$, and $F(s)$ has no poles on $j\mathbb{R} \cup \infty$ since $Z_1(s)$ has no poles on $j\mathbb{R} \cup \infty$. Since N_1 has exactly two reactive elements, then $F(s)$ has McMillan degree at most four. Moreover, since $\Re(Z_1(j\omega_0)) = 0$ and $Z_1(s)$ is PR, then $F(s)$ has zeros of multiplicity two at $s = \pm j\omega_0$. It follows that $\Re(Z_1(j\omega)) \neq 0$ for $\omega \in \mathbb{R} \cup \infty \backslash \omega_0$. Since, in addition, $\Im(Z_1(j\omega_0)) \neq 0$, then $Z_1(s)$ has no zeros on $j\mathbb{R} \cup \infty$, and hence $Z_1(s)$ is a minimum function. So, by Lemma 3.5.5, N_1 must contain at least three reactive elements: a contradiction. We conclude that N cannot be a series connection of two one-ports.

The case where N is a parallel connection of two one-ports is similar, and completes the proof. □

We are now in a position to construct a complete description of those networks which contain fewer than five reactive elements and which realise a minimum function. Motivated by Lemma 3.5.5, we will describe these networks as an interconnection of reactive elements and maximal ω_0-blocked subnetworks (which are one-ports as shown in that lemma). The combination of Lemmas 3.5.1 to 3.5.13 will allow us to place further restrictions on the network. We summarise these considerations in the following theorem:

Theorem 3.5.14. *Let N be a transformerless network containing fewer than five reactive elements and with impedance $H(s)$ which is a minimum function (with a minimum frequency at ω_0). Then N takes the form of Fig. 32, in which $N_1, N_2, \ldots, N_5$ are one-ports of N and either N or N^i satisfy exactly one of the following four conditions:*

1. *N_1 contains resistors together with at most one reactive element, N_2 contains only resistors, N_3 comprises a single capacitor, N_4 and N_5 each comprise a single inductor, and $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$.*

2. *N_1 and N_2 each contain only resistors, N_3 comprises a single capacitor, N_4 comprises either a series or a parallel connection of an inductor and a capacitor, N_5 comprises a single inductor, and $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$.*

3. *N_1 contains only resistors, N_2 and N_3 each comprise a single capacitor, N_4 and N_5 each comprise a single inductor, and $Z_2(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0)) + Z_4(j\omega_0)(Z_3(j\omega_0) + Z_5(j\omega_0)) = 0$.*

4. *N_1 and N_2 each comprise a single capacitor, N_3 contains only resistors, N_4 and N_5 each comprise a single inductor, and $Z_1(j\omega_0)Z_2(j\omega_0) = Z_4(j\omega_0)Z_5(j\omega_0)$ where $Z_k(j\omega_0) + Z_l(j\omega_0) \neq 0$ for $k = 1, 2$ and $l = 4, 5$.*

Proof. Since $H(s)$ is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$, then N has a corresponding graph of one-ports G with either one or two ω_0-blocked edges (which correspond to maximal ω_0-blocked subnetworks), and either three or four additional edges corresponding to reactive elements which are not $\tilde{i}(j\omega_0)$-blocked (and an additional edge for the source). Hence, G contains at most seven edges, is biconnected (see Remark 1.2.4), and must not be series-parallel by Lemma 3.5.13. By [54, pp. 325 - 327], it must be either the complete graph on four vertices (graph G_1 in Fig. 33), or the graph obtained by replacing any single edge in this graph by either two edges in series or two in parallel (graphs G_2 and G_3 in Fig. 33). Moreover, by Lemma 3.5.13, the source must not appear in series or in parallel with any other edge. Furthermore, from Lemma 3.5.5 and its proof, the maximal ω_0-blocked one-port subnetworks must also correspond to edges which do not appear in series or in parallel with any other edge, and these edges must not be incident with each other. It follows that N takes the form of Fig. 32, in which $N_1, N_2, \ldots, N_5$ are one-ports in N, either one or two non-incident one-ports N_i $(i = 1, 2, \ldots, 5)$ are ω_0-blocked, and all other one-ports are comprised exclusively of reactive elements.

Consider first the case where N contains exactly one ω_0-blocked one-port subnetwork. No generality is lost in letting this ω_0-blocked subnetwork be either N_1 or N_3 given the symmetry of the network[27]. Furthermore, the remaining one-ports must each comprise a single reactive element.

Suppose initially that N_1 corresponds to the ω_0-blocked one-port. Then we require the two networks in Fig. 34, to have the same $j\omega_0$-impedance by Lemma 3.5.10. Furthermore, since $H(s)$ has no imaginary axis poles or zeros, then N must not contain an L-cut-set, C-cut-set, L-path or C-path. It may be verified that either N or N^i must

[27]More formally, we are using the equivalence of 2-isomorphic graphs insofar as network analysis is concerned, see Remark 1.2.5.

satisfy condition 3 in this case.

If, instead, N_3 corresponds to the ω_0-blocked one-port, then we require the two networks in Fig. 35 to have the same $j\omega_0$-impedance. Again, N must not contain an L-cut-set, C-cut-set, L-path or C-path. It may be verified that in this case either N or N^i must satisfy condition 4.

Consider next the case where the network contains exactly two ω_0-blocked one-port subnetworks. Then, without loss of generality, these must be subnetworks N_1 and N_2. Furthermore, since there must be at least three reactive elements not within these one-ports, and the network contains at most four reactive elements, then at least one of these two one-ports must contain only resistors. Without loss of generality, we may let this one-port be N_2. By Lemma 3.5.10, we require the four networks in Fig. 36 to have the same $j\omega_0$-impedance. This implies

$$Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0). \tag{152}$$

There are now three cases to consider: (i) N_3, N_4 and N_5 each comprise a single reactive element and N_1 contains resistors and at most one reactive element, (ii) N_3 comprises two reactive elements, N_4 and N_5 each comprise a single reactive element, and N_1 contains only resistors, (iii) N_4 comprises two reactive elements, N_3 and N_5 each comprise a single reactive element, and N_1 contains only resistors.

In case (i), equation (152) implies that N_4 and N_5 must comprise reactive elements of the same kind, and of different kind to N_3, and so either N or N^i must satisfy condition 1.

In case (ii), equation (152) implies that N_4 and N_5 must again comprise reactive elements of the same kind. Since we exclude from consideration those networks which contain two elements of the same kind in series or in parallel, then N_3 must contain both an inductor and a capacitor, and it may be verified that N then contains either an L-cut-set, C-cut-set, L-path or C-path. Since $H(s)$ has no imaginary axis poles or zeros, it follows by Corollary 3.5.11 that there are no networks which realise minimum functions in this case.

In case (iii), equation (152) implies that N_3 and N_5 must comprise reactive elements of different kind, In this case, excluding networks which contain two elements of the same kind in series or in parallel, it may be verified that either N or N^i must satisfy condition 2. □

If a network contains only one kind of reactive element then the impedance of that

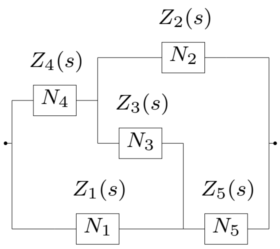


Figure 32: Network comprised of the one-ports $N_1, N_2, \ldots, N_5$.

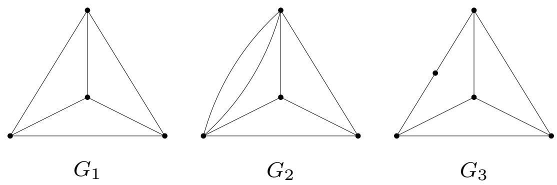


Figure 33: Non-series-parallel biconnected graphs containing seven or fewer edges.

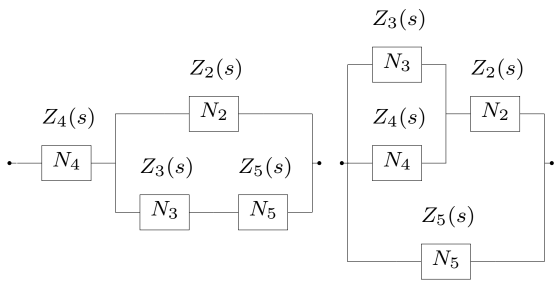


Figure 34: Networks obtained by the application of Lemma 3.5.10 to the network in Fig. 32, for the case where N_1 is an ω_0-blocked one-port. The left hand (resp. right hand) network is obtained by opening (resp. shorting) N_1.

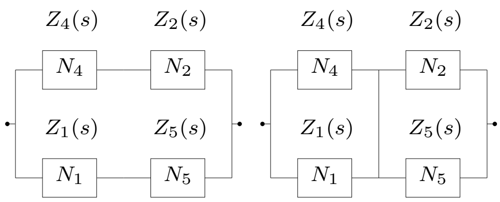


Figure 35: Networks obtained by the application of Lemma 3.5.10 to the network in Fig. 32, for the case where N_3 is an ω_0-blocked one-port. The left hand (resp. right hand) network is obtained by opening (resp. shorting) N_3.

network is realised by the Cauer form network described in Section 2.7. This realisation contains the same number of reactive elements as the McMillan degree of its impedance function, which is the least possible number of reactive elements for the realisation of that impedance. We may apply this result to each of the one-ports $N_1, N_2, \ldots, N_5$ described in Theorem 3.5.14 to show Corollaries 3.5.15 and 3.5.16. Before stating the

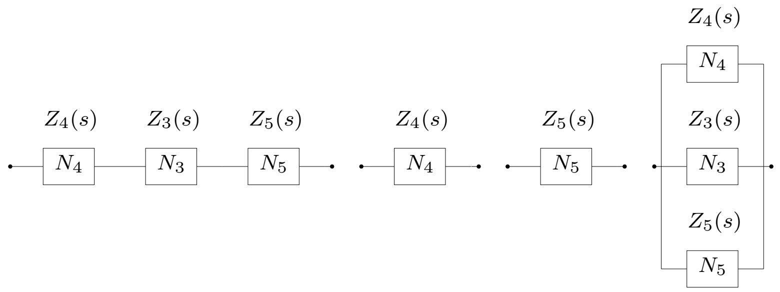


Figure 36: Networks obtained by the application of Lemma 3.5.10 to the network in Fig. 32, for the case where N_1 and N_2 are ω_0-blocked one-ports. The network on the far left (resp. second from the left; second from the right; far right) is obtained by opening N_1 and N_2 (resp. opening N_1 and shorting N_2; shorting N_1 and opening N_2; shorting N_1 and N_2).

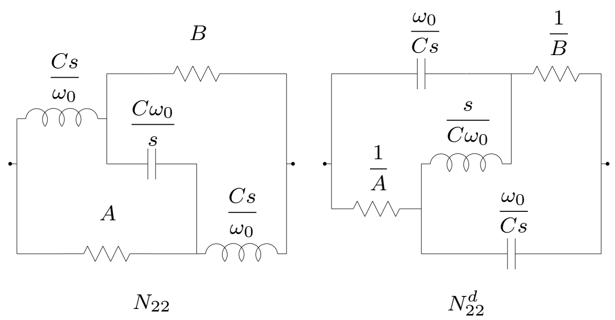


Figure 37: Quartet $\mathcal{Q}_{22}$, $A, B, C > 0$.

first of these corollaries, we introduce the network N_{22} in Fig. 37. The network class $\mathcal{N}_{22}$ is defined as the set of all such networks N_{22} for $A, B, C > 0$. We denote the corresponding network quartet by $\mathcal{Q}_{22}$.

Corollary 3.5.15. *Let N be a transformerless network containing at most three reactive elements and with impedance $H(s)$ which is a minimum function. Then $H(s)$ is realised by a network from $\mathcal{Q}_{22}$.*

Prior to stating the second corollary, we introduce further network classes as follows. Firstly, the network N_{23} (resp. N_{24}) is shown in Fig. 38 (resp. Fig. 39), and the network class $\mathcal{N}_{23}$ (resp. $\mathcal{N}_{24}$) is defined as the set of all such networks N_{23} (resp. N_{24}) for $A, B, C, D > 0$. Secondly, the network N_{25} is shown in Fig. 40, and the network class $\mathcal{N}_{25}$ is defined as the set of all such networks N_{25} for $A, B, C > 0$ and $(B-D)(C-D) > 0$. Finally, the network N_{26} (resp. N_{27}) is shown in Fig. 41 (resp. Fig. 42), and the network class $\mathcal{N}_{26}$ (resp. $\mathcal{N}_{27}$) is defined as the set of all such networks N_{26} (resp. N_{27}) for $A, C \geq 0$ and $B, D, E > 0$. The corresponding network quartets are defined in the

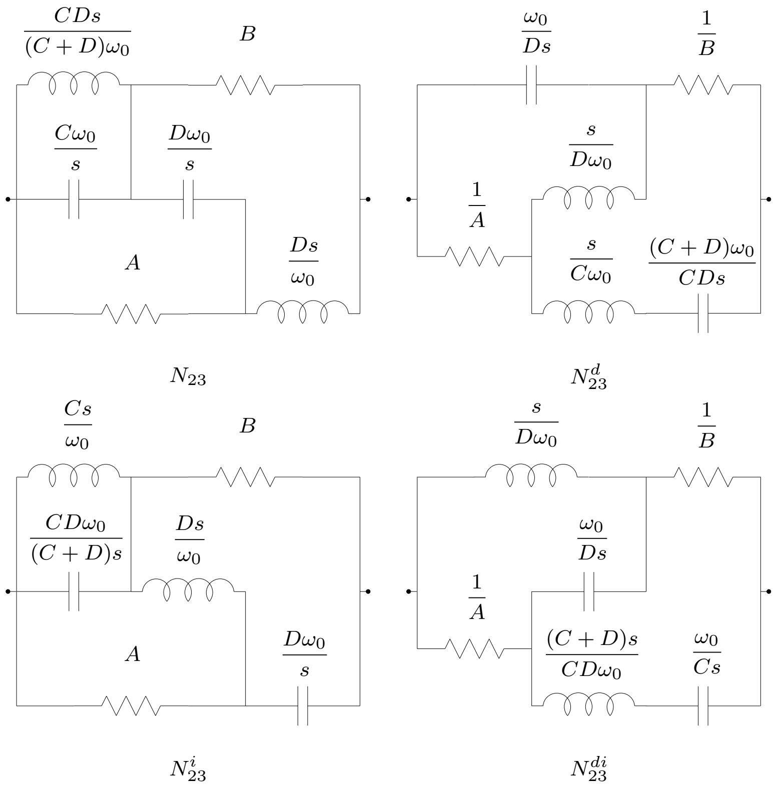


Figure 38: Quartet $\mathcal{Q}_{23}$, $A, B, C, D > 0$.

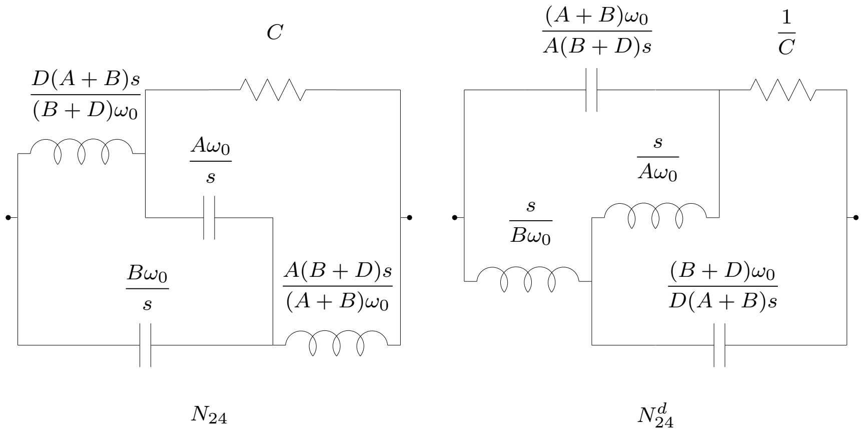


Figure 39: Quartet $\mathcal{Q}_{24}$, $A, B, C, D > 0$.

usual manner.

Corollary 3.5.16. *Let N be a transformerless network containing at most four reactive elements and with impedance $H(s)$ which is a minimum function. Then $H(s)$ is realised by a network from one of the quartets $\mathcal{Q}_{22}$ to $\mathcal{Q}_{27}$.*

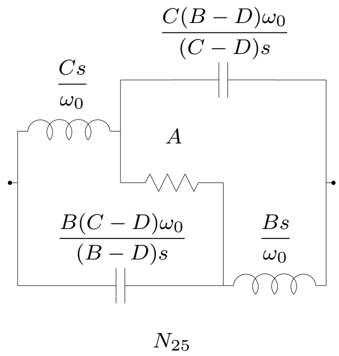


Figure 40: Quartet $\mathcal{Q}_{25}$, $A, B, C > 0$ with $(B - D)(C - D) > 0$.

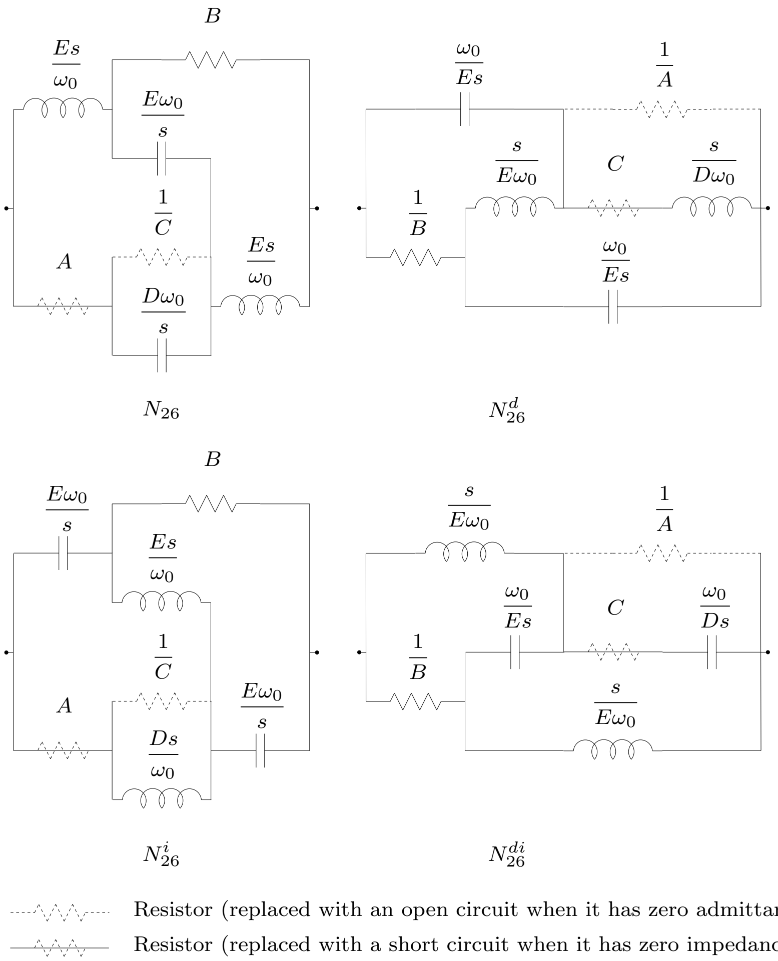


Figure 41: Quartet $\mathcal{Q}_{26}$, $A, C \geq 0$, $B, D, E > 0$.

In order to determine those biquadratic minimum functions which can be realised by transformerless networks which contain fewer than five reactive elements, it remains to check which of the networks described in Theorem 3.5.14 can realise biquadratic

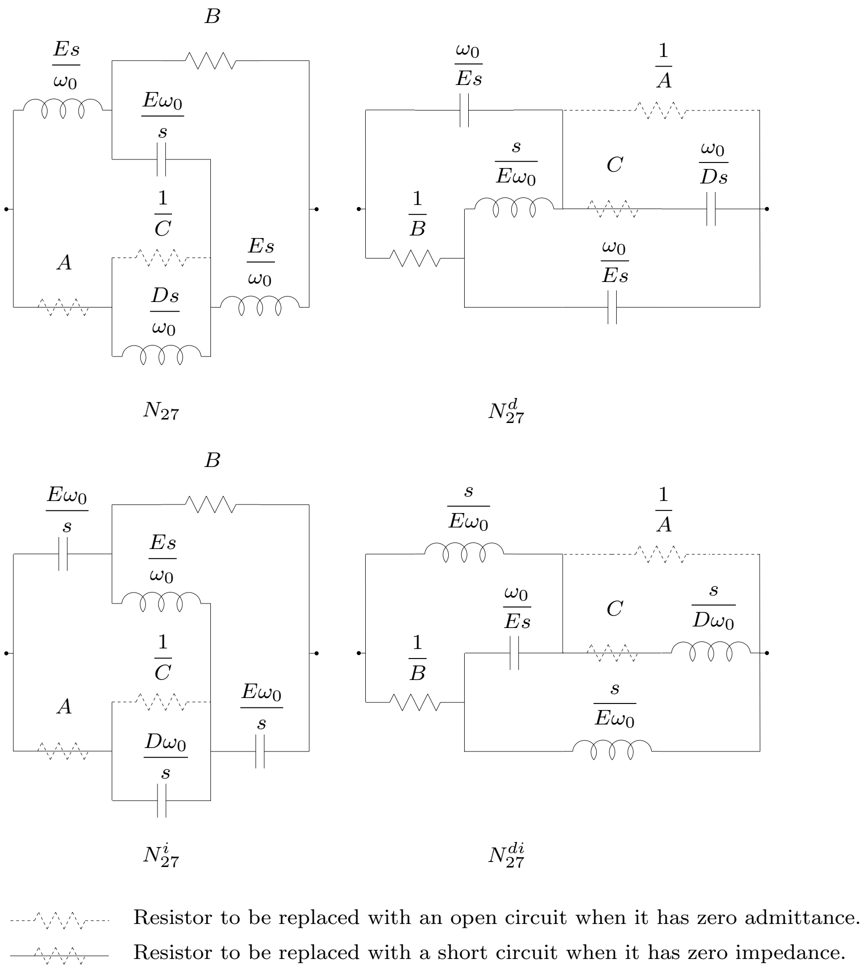


Figure 42: Quartet $\mathcal{Q}_{27}$, $A, C \geq 0$, $B, D, E > 0$.

minimum functions. Here we make use of the conditions described in Subsection 3.2.3 for two polynomials to have multiple coincident roots, and we adopt the notation $R_k(a(s), b(s))$ introduced in that subsection. We express the biquadratic minimum functions which are realised by a given network class using the parametrisation introduced in Section 3.2.2.

We first introduce network N_{28} in Fig. 43. The network class $\mathcal{N}_{28}$ is defined as the set of all networks N_{28} for $\alpha, X > 0$. It is straightforward to verify that $\mathcal{N}_{28}$ realises the set of all functions of the form $H_p(s)$ in (145) with $\alpha, X > 0$ and $W = 1/2$. We denote the corresponding quartet by $\mathcal{Q}_{28}$.

Theorem 3.5.17. *Let N be a transformerless network containing at most three reactive elements with impedance $H(s)$ which is a biquadratic minimum function (with minimum frequency ω_0). Then N contains exactly three reactive elements. More-*

over, either $H(s)$ or $1/H(s)$ takes the form of $H_p(s)$ in (145) for some $\alpha, X > 0$, with $W = 1/2$. In particular, $H(s)$ is realised by a network from $\mathcal{Q}_{28}$, and either $H(0) = 4H(\infty)$ or $H(\infty) = 4H(0)$.

Proof. Since N contains at most three reactive elements, then either N or N^i must satisfy condition 1 in the statement of Theorem 3.5.14, and the one-port subnetwork N_1 described in that condition must contain only resistors. Furthermore, $H(s)$ must be realised by a network from $\mathcal{Q}_{22}$ by Corollary 3.5.15. We first consider those circumstances in which the impedance $H_{22}(s)$ of network N_{22} in Fig. 37 takes the form of $H_p(s)$ in (145). Equating the impedance of N_{22} at $s = j\omega_0$ with $H_p(j\omega_0)$ we obtain $C = \alpha X$, and hence we require $X > 0$ which implies $0 < W < 1$ (see Remark 3.2.1). Let $A = \alpha/g_1$ and $B = \alpha/g_2$, and so we require $g_1, g_2 > 0$. Using Corollary 3.5.11, we find that $H_{22}(\infty) = H_p(\infty)$ and $H_{22}(0) = H_p(0)$ imply

$$\frac{g_1 + g_2}{g_1 g_2} = 1, \tag{153}$$

$$\text{and} \qquad \frac{1}{g_1 + g_2} = W^2. \tag{154}$$

Furthermore, from equation (38), we find that $H_{22}(s) = \alpha X n_{22}(s)/d_{22}(s)$, where $n_{22}(s) = X(g_1+g_2)s^3+(X^2 g_1 g_2+2)\omega_0 s^2+X(g_1+g_2)\omega_0^2 s+\omega_0^3$ and $d_{22}(s) = X^2 g_1 g_2 s^3 + X(g_1 + g_2)\omega_0 s^2 + (2X^2 g_1 g_2 + 1)\omega_0^2 s + X(g_1 + g_2)\omega_0^3$. For $H_{22}(s)$ to be biquadratic, we require $R_0(n_{22}(s), d_{22}(s)) = X^2\omega_0^9(g_1 - g_2)^2(1 + X^2 g_1 g_2)^4 = 0$ which, together with the conditions $g_1, g_2 > 0$ and equations (153) and (154), implies $g_1 = g_2 = 2$ and $W = 1/2$. We thus conclude that if N is from network class $\mathcal{N}_{22}$ then $H(s)$ is equal to $H_p(s)$ in (145) with $W = 1/2$, and that N is also in the class $\mathcal{N}_{28}$.

A similar argument shows that if N is from network class $\mathcal{N}_{22}^d$ then $1/H(s)$ takes the form of $H_p(s)$ in equation (145) with $W = 1/2$, and N is from the quartet $\mathcal{N}_{28}^d$. We thus conclude that either $H(s)$ or $1/H(s)$ takes the form of $H_p(s)$ in equation (145) with $W = 1/2$, and that $H(s)$ is realised by a network from $\mathcal{Q}_{28}$. □

Next, we introduce network N_{29} in Fig. 44. The network class $\mathcal{N}_{29}$ is defined as the set of all networks N_{29} for $\phi = 1 - W$, $\psi = 1 + W$, $\eta = 2W - 1$, $X = W\sqrt{2W-1}/(1-W)$, $1/2 < W < 1$, and $\alpha > 0$. It is straightforward to verify that $\mathcal{N}_{29}$ realises the set of all functions of the form $H_p(s)$ in (145) with $X = W\sqrt{2W-1}/(1-W)$, $1/2 < W < 1$, and $\alpha > 0$. We denote the corresponding quartet by $\mathcal{Q}_{29}$.

Theorem 3.5.18. *Let N be a transformerless network containing at most four reactive elements with impedance $H(s)$ which is a biquadratic minimum function. Then either*

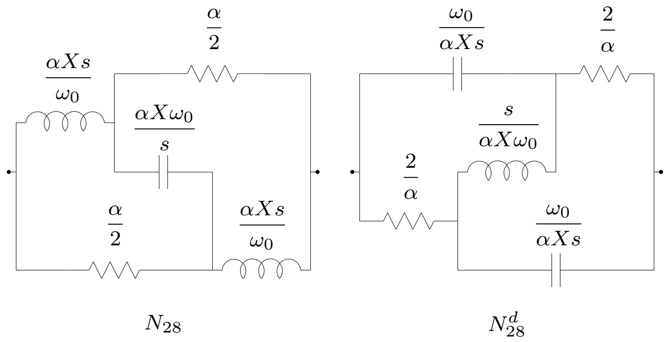


Figure 43: Quartet $\mathcal{Q}_{28}$, $X, \alpha > 0$.

$H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, or $1/H(\omega_0^2/s)$ takes the form of $H_p(s)$ in (145) for some $\alpha > 0$ with either (i) $W = 1/2$, or (ii) $1/2 < W < 1$ and $X = W\sqrt{2W-1}/(1-W)$. In particular, in case (ii), $H(s)$ is realised by a network from $\mathcal{Q}_{29}$.

Proof. Since N contains at most four reactive elements and realises a biquadratic minimum function then, by Theorem 3.5.14 and Corollary 3.5.12, either N or N^i must satisfy either condition 1 or condition 2 in that theorem's statement. Consequently, $H(s)$ is realised by a network from one of the quartets $\mathcal{Q}_{22}$, $\mathcal{Q}_{23}$, $\mathcal{Q}_{26}$, or $\mathcal{Q}_{27}$. The case where $H(s)$ is realised by a network from $\mathcal{Q}_{22}$ was covered in Theorem 3.5.17. In that case, we found that either $H(s)$ or $1/H(s)$ takes the form of $H_p(s)$ in (145) for some $\alpha, X > 0$, with $W = 1/2$. As in the proof of that theorem, we will consider those circumstances in which the impedances of the networks N_{23} in Fig. 38, N_{26} in Fig. 41, and N_{27} in Fig. 42 take the form of $H_p(s)$ in (145). Duality and frequency inversion arguments then allow us to identify the other networks from the quartets $\mathcal{Q}_{23}$, $\mathcal{Q}_{26}$, and $\mathcal{Q}_{27}$ which realise biquadratic minimum functions.

Consider first the impedance $H_{23}(s)$ of network N_{23} in Fig. 38. For $H_{23}(j\omega_0) = H_p(j\omega_0)$ we require $D = \alpha X > 0$, which implies $0 < W < 1$ (see Remark 3.2.1). Let $A = \alpha/g_1$, $B = \alpha/g_2$, and $C = \alpha X/c_2$. Then $H_{23}(\infty) = H_p(\infty)$ and $H_{23}(0) = H_p(0)$ imply

$$\frac{1}{g_2} = 1, \tag{155}$$

$$\text{and} \qquad \frac{1}{g_1 + g_2} = W^2. \tag{156}$$

Moreover, from equation (38), we find that $H_{23}(s) = \alpha X n_{23}(s)/d_{23}(s)$, where

$$n_{23}(s) = c_2 s^4 + X(g_1(1+c_2) + g_2)\omega_0 s^3 + (2(1+c_2) + X^2 g_1 g_2)\omega_0^2 s^2 + X(g_1(1+c_2) + g_2)\omega_0^3 s + (1+c_2)\omega_0^4, \tag{157}$$

and

$$d_{23}(s) = Xc_2g_2s^4 + (X^2g_1g_2(1+c_2)+c_2)\omega_0 s^3 + X(g_1(1+c_2)+g_2(1+2c_2))\omega_0^2 s^2 + (X^2g_1g_2(2+c_2)+1+c_2)\omega_0^3 s + X(1+c_2)(g_1+g_2)\omega_0^4. \quad (158)$$

For the impedance of N_{23} to be biquadratic, we require $n_{23}(s)$ and $d_{23}(s)$ to have two common roots, which implies $R_0(n_{23}(s), d_{23}(s)) = R_1(n_{23}(s), d_{23}(s)) = 0$. Here, $R_0(n_{23}(s), d_{23}(s)) = c_2\omega_0^{16}(1+c_2)(1+X^2g_1g_2)^4 f_1^2$ and $R_1(n_{23}(s), d_{23}(s)) = -c_2\omega_0^9(1+X^2g_1g_2)^2 f_2$ where f_1 and f_2 are both polynomials in c_2, g_1, g_2 and X. In particular, we require $R_0(f_1(X), f_2(X)) = c_2^6 g_2^{10}(1+c_2)^2(c_2^2g_1 + 2c_2(g_1-g_2) + g_1 - 3g_2)^2 = 0$. Taken together with equations (155) and (156) and the conditions $c_2, g_1, g_2 > 0$ and $0 < W < 1$, this implies $g_1 = (1-W^2)/W^2$, $g_2 = 1$, $c_2 = (2W-1)/(1-W)$, and $W \geq 1/2$. Then $R_0(n_{23}(s), d_{23}(s)) = 0$ and $X > 0$ imply $X = W\sqrt{2W-1}/(1-W)$. We conclude that if N is from $\mathcal{Q}_{23}$ then either $H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, or $1/H(\omega_0^2/s)$ takes the form of $H_p(s)$ in (145) for some $\alpha > 0$ with $1/2 < W < 1$ and $X = W\sqrt{2W-1}/(1-W)$. Moreover, N then belongs to the quartet $\mathcal{Q}_{29}$.

Consider next the impedance $H_{26}(s)$ of N_{26} in Fig. 41. For $H_{26}(j\omega_0) = H_p(j\omega_0)$ we require $E = \alpha X$ which again implies $X > 0$ and $0 < W < 1$. Let $A = \alpha r_1$, $B = \alpha/g_3$, $C = g_2/\alpha$, and $D = \alpha X/c_1$, so $g_3, c_1 > 0$ and $r_1, g_2 \geq 0$. For $H_{26}(\infty) = H_p(\infty)$ and $H_{26}(0) = H_p(0)$, we require

$$\frac{1+r_1g_3}{g_3} = 1, \quad (159)$$

$$\text{and} \qquad \frac{1+r_1g_2}{g_2(1+r_1g_3)+g_3} = W^2. \quad (160)$$

Moreover, from equation (38), $H_{26}(s) = \alpha X n_{26}(s)/d_{26}(s)$, where

$$n_{26}(s) = Xc_1(1+r_1g_3)s^4 + (X^2(g_2(1+r_1g_3)+g_3(1+c_1))+2r_1c_1)\omega_0 s^3 + X(X^2g_2g_3 + c_1(1+r_1g_3) + 2(1+r_1g_2))\omega_0^2 s^2 + (X^2(g_2(1+r_1g_3)+g_3)+r_1c_1)\omega_0^3 s + X(1+r_1g_2)\omega_0^4, \quad (161)$$

and

$$d_{26}(s) = X^2c_1g_3s^4 + X(X^2g_2g_3 + c_1(1+r_1g_3))\omega_0 s^3 + (X^2(g_3(1+2c_1)+g_2(1+r_1g_3)) + c_1r_1)\omega_0^2 s^2 + X(2X^2g_2g_3 + c_1(1+r_1g_3) + 1 + r_1g_2)\omega_0^3 s + X^2(g_3 + g_2(1+r_1g_3))\omega_0^4. \quad (162)$$

For $H_{26}(s)$ to be biquadratic, we require $R_0(n_{26}(s), d_{26}(s)) = R_1(n_{26}(s), d_{26}(s)) = 0$. In this case, $R_0(n_{26}(s), d_{26}(s)) = X^4\omega_0^{16}c_1(c_1^2(r_1+X^2g_3)^2+X^2(1+r_1g_2+X^2g_2g_3)^2)^2f_1^2$ and $R_1(n_{26}(s), d_{26}(s)) = -X^2c_1\omega_0^9f_2$ where f_1 and f_2 are both polynomials in c_1, r_1, g_2, g_3, and X. In particular, we require $R_0(f_1(c_1), f_2(c_1)) = X^{10}g_3^5(1-r_1g_3)((g_2(1-r_1g_3)(r_1+X^2g_3) + X^2g_3^2 + g_3r_1 - 1)^2 + g_3^2X^2)^2(g_3(3 - r_1g_3 + r_1g_2(2 - r_1g_3)) - g_2) = 0$, and $f_1 = c_1(1 - r_1g_3) + X^2g_3(g_3 - g_2(1 - r_1g_3)) = 0$. Taken together with the conditions $r_1, g_2 \geq 0$, $g_3, c_1 > 0$, and equations (159) and (160), this implies $r_1 = (g_3 - 1)/g_3$, $g_2 = g_3(4 - g_3)/(g_3 - 2)^2$, $c_1 = 2X^2g_3^2/(2 - g_3)^2$, $W = 1/2$, and $1 \leq g_3 \leq 4$, $g_3 \neq 2$. In this case, we conclude that if N is from $\mathcal{Q}_{26}$ then either $H(s)$ or $1/H(s)$ takes the form of $H_p(s)$ in (145) for some $\alpha, X > 0$, with $W = 1/2$. It may also be verified that N then belongs to the quartet $\mathcal{Q}_{65}$ which we define on p. 180.

Consider finally the impedance $H_{27}(s)$ of N_{27} in Fig. 42. In this case, $H_{27}(j\omega_0) = H_p(j\omega_0)$ implies $E = \alpha X$ and so $X > 0$ and $0 < W < 1$. Similarly to before, let $A = \alpha r_1$, $B = \alpha/g_3$, $C = g_2/\alpha$, and $D = \alpha X/x_1$, so $g_3, x_1 > 0$ and $r_1, g_2 \geq 0$. Then, from equation (38), $H_{27}(s) = \alpha X n_{27}(s)/d_{27}(s)$, where

$$\begin{aligned} n_{27}(s) = X^2(g_2(1+r_1g_3)+g_3)s^4+X(X^2g_2g_3+2(1+r_1g_2)+x_1(1+r_1g_3))\omega_0s^3 \\ +(X^2(g_2(1+r_1g_3)+g_3(1+x_1))+2r_1x_1)\omega_0^2s^2 \\ +X(x_1(1+r_1g_3)+1+r_1g_2)\omega_0^3s+r_1x_1\omega_0^4, \quad (163) \end{aligned}$$

and

$$\begin{aligned} d_{27}(s) = X^3g_2g_3s^4+X^2(g_3(1+x_1)+g_2(1+r_1g_3))\omega_0s^3 \\ +X(2X^2g_2g_3+x_1(1+r_1g_3)+1+r_1g_2)\omega_0^2s^2 \\ +(X^2(g_2(1+r_1g_3)+g_3(1+2x_1))+r_1x_1)\omega_0^3s+Xx_1(1+r_1g_3)\omega_0^4. \quad (164) \end{aligned}$$

Hence, for the impedance of N_{27} to be biquadratic, we require $R_0(n_{27}(s), d_{27}(s)) = -X^6\omega_0^{16}x_1((X^2g_3+r_1)^2x_1^2+X^2(1+r_1g_2+X^2g_2g_3)^2)^2f_1^2 = 0$, and $R_1(n_{27}(s), d_{27}(s)) = X^6\omega_0^9f_2 = 0$, where f_1 and f_2 are both polynomials in x_1, r_1, g_2, g_3, and X. We thus require $R_0(f_1(x_1), f_2(x_1)) = -X^2g_3^5((g_2(1 - r_1g_3)(r_1 + X^2g_3) - g_3r_1)^2 + g_3^2X^2)^2(g_3 - g_2(1-r_1g_3))(g_2(1-r_1g_3)^2+g_3(1+r_1g_3)) = 0$, together with $f_1 = g_3(1-r_1g_3)x_1+g_3 - g_2(1-r_1g_3) = 0$. It may be verified that these equations have no solution for $r_1, g_2 \geq 0$ and $g_3, x_1, X > 0$. We thus conclude that there are no networks in $\mathcal{Q}_{27}$ which realise a biquadratic minimum function. □

From Theorem 3.5.18, it follows that, with the exception of the biquadratic minimum functions described in that theorem statement, there are no transformerless networks

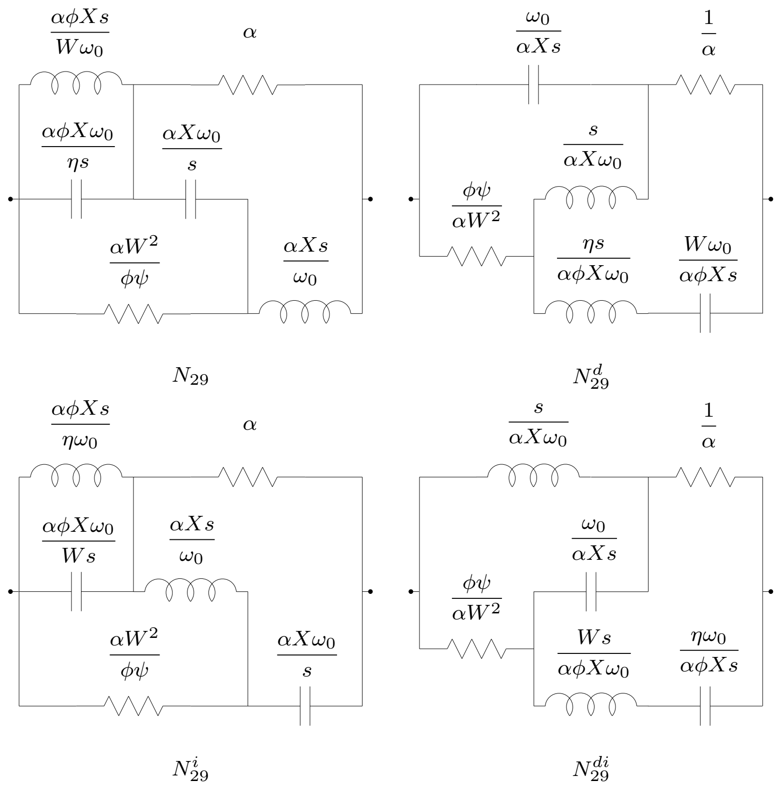


Figure 44: Quartet $\mathcal{Q}_{29}$, $\phi = 1-W$, $\psi = 1+W$, $\eta = 2W-1$, $X = W\sqrt{2W-1}/1-W$, $1/2 < W < 1$, $\alpha > 0$.

which realise a biquadratic minimum function and which contain fewer reactive elements than the networks obtained by the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure[28].

3.6 On the uniqueness of the Reza-Pantell-Fialkow-Gerst procedure for biquadratic minimum functions

In this section, we investigate those transformerless networks which realise a biquadratic minimum function and which contain the same number of reactive elements and the same total number of elements as the networks obtained by the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure. The main results in this subsection are stated in Theorems 3.6.5 and 3.6.6. Theorem 3.6.5 (resp. Theorem 3.6.6) describes those transformerless networks which contain at most five reactive elements and at

[28]It is straightforward to verify that those exceptional biquadratic minimum functions which are realised by transformerless networks containing fewer than five reactive elements correspond to the functions described in Remark 3.2.2 in Subsection 3.2.2.

most seven elements in total and which realise a minimum function (resp. a biquadratic minimum function). In particular, we show that those transformerless networks which contain at most five reactive elements and at most seven elements in total and which realise a biquadratic minimum function belong to one of eleven quartets ($\mathcal{Q}_{28}$, $\mathcal{Q}_{29}$, and $\mathcal{Q}_{65}$ to $\mathcal{Q}_{73}$). Among these, all but the four quartets $\mathcal{Q}_{70}$, $\mathcal{Q}_{71}$, $\mathcal{Q}_{72}$, and $\mathcal{Q}_{73}$, realise sets of biquadratic minimum functions of codimension one in the parameters α, X, ω_0, and W of the parametrisation $H_p(s)$ in (145). In other words, the sets of biquadratic minimum functions realised by each the quartets $\mathcal{Q}_{28}$, $\mathcal{Q}_{29}$, and $\mathcal{Q}_{65}$ to $\mathcal{Q}_{69}$, are described by a relationship between the parameters α, X, ω_0, and W, implying that the set of biquadratic minimum functions which can be realised by these quartets is negligibly small. Moreover, the only quartets to realise all of the biquadratic minimum functions are $\mathcal{Q}_{72}$, and $\mathcal{Q}_{73}$. One of these two quartets contains the networks obtained from the Reza-Pantell-Fialkow-Gerst simplification, and the second contains the networks from our alternative simplification to the Bott-Duffin procedure (which was described in Section 3.1.5).

In Section 3.5, we described those networks which contain fewer than five reactive elements and which realise a minimum function as an interconnection of maximal ω_0-blocked subnetworks and reactive elements. In Lemma 3.5.5, it was shown that all such maximal ω_0-blocked subnetworks were one-ports. As will be shown, this need not be the case for networks containing exactly five reactive elements which realise a minimum function. The following lemma extends Lemma 3.5.5 to cover such a case.

Lemma 3.6.1. *Let N be a transformerless network which contains at most five reactive elements and at most seven passive elements in total, and let G be the graph of elements corresponding to N. Further let N have impedance $H(s)$ which is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$. Then either G contains one maximal ω_0-blocked subnetwork which connects to the rest of G at either two or three vertices, or G contains two maximal ω_0-blocked subnetworks each of which connect to the rest of G at two vertices.*

Proof. Since $H(s)$ is not lossless, then N must contain at least one resistor. Moreover, conditions 1 to 3 in Lemma 3.5.5 must hold.

Suppose G contains a maximal ω_0-blocked subnetwork which connects to the rest of G at exactly x vertices. From conditions 2 and 3 in Lemma 3.5.5, it follows that there are at least $2x$ edges in G which are not $\tilde{i}(j\omega_0)$-blocked. Furthermore, at most one of the edges is the source, and the rest correspond to reactive elements, by condition 1 of Lemma 3.5.5. Since N contains at most five reactive elements, we conclude that $x \leq 3$, and for G to be biconnected (see Remark 1.2.4) we require $x \geq 2$.

Suppose next that G contains y maximal ω_0-blocked subnetworks, with the kth such subnetwork connecting to the rest of G at x_k vertices ($k = 1, 2, \ldots, y$). In a similar manner to the proof of Lemma 3.5.5, let us sum the number of edges incident at each vertex where a maximal ω_0-blocked subnetwork is connected to the rest of G. In so doing, we will count each edge which is not $\tilde{i}(j\omega_0)$-blocked at most twice. We conclude that the number of edges in G which are not $\tilde{i}(j\omega_0)$-blocked is greater than or equal to $\sum_{k=1}^{y} x_k$ by conditions 2 and 3 of Lemma 3.5.5. Furthermore, at most one of these edges is the source, and the rest correspond to reactive elements, by condition 1 of Lemma 3.5.5. Since N contains at most five reactive elements, and $x_k \geq 2$ for $k = 1, \ldots, y$, we conclude that $y \leq 3$.

If $y = 3$ then there must be at least three edges in G within the maximal ω_0-blocked subnetworks, and there must be at least six edges which are not $\tilde{i}(j\omega_0)$-blocked (one of which is the source) as explained in the preceding paragraph. This contradicts the assumption that N contains at most seven passive elements in total. If $x_k = 3$ for some $k \in 1, \ldots, y$, then the corresponding maximal ω_0-blocked subnetwork must contain at least two edges (since it connects to the rest of G at three vertices). As explained in the second paragraph of this proof, there must also be at least six edges which are not $\tilde{i}(j\omega_0)$-blocked (again, one of these is the source). Since N contains at most seven passive elements in total, we conclude that there is only one maximal ω_0-blocked subnetwork in this case. Finally, if $y = 2$ then, from the preceding arguments, each of these two maximal ω_0-blocked subnetworks must connect to the rest of G at exactly two vertices. This completes the proof of the present lemma. □

Let N be a network whose impedance is a minimum function (with a minimum frequency at ω_0), and consider a $j\omega_0$-trajectory of N. Then the only one-ports N_k with non-zero current (resp. non-zero voltage) are those whose impedance (resp. admittance) has a zero real part at $s = j\omega_0$. In the following lemma, we place constraints on those networks whose impedance satisfies such a condition. Prior to stating this lemma, we recall the definition of the network class $\mathcal{N}_1$ from Section 3.3. This is defined as the set of all networks N_1 in Fig. 21 for $A, B > 0$. The corresponding network quartet is denoted by $\mathcal{Q}_1$.

Lemma 3.6.2. *Let N be a transformerless network with impedance (resp. admittance) $H(s)$ which is not lossless, does not have a pole at $s = j\omega_0$, and satisfies $\Re(H(j\omega_0)) = 0$ and $\Im(H(j\omega_0)) \neq 0$. Further, let N_u be a one-port in N which contains a resistor but is not an ω_0-blocked subnetwork. Then N_u contains at least one capacitor and at least* *one inductor. Moreover, if N_u contains exactly three passive elements, then either N_u belongs to the class $\mathcal{N}_1$ and is $\tilde{v}(j\omega_0)$-blocked, or N_u belongs to the class $\mathcal{N}_1^d$ and is $\tilde{i}(j\omega_0)$-blocked.*

Proof. Let $Z_u(s)$ be the impedance, and $Y_u(s) = 1/Z_u(s)$ the admittance, of N_u. Suppose all reactive elements in N_u are of the same kind. Then, from the partial fraction expansion for networks containing only one kind of reactive element (see, e.g., [36, Section 9.6]), it is straightforward to verify that neither $Z_u(s)$ nor $Y_u(s)$ have a pole at $s = j\omega_0$, $\Re(Z_u(j\omega_0)) > 0$, and $\Re(Y_u(j\omega_0)) > 0$. From Lemma 3.5.1, we conclude that N_u is both $\tilde{i}(j\omega_0)$-blocked and $\tilde{v}(j\omega_0)$-blocked, which implies that N_u is an ω_0-blocked subnetwork. Hence, N_u must contain at least one inductor and at least one capacitor.

Suppose next that N_u contains exactly three passive elements, so N_u must be SP by [54, p. 326]. Since either N_u is not $\tilde{i}(j\omega_0)$-blocked, or it is not $\tilde{v}(j\omega_0)$-blocked, then either $\Re(Z_u(j\omega_0)) = 0$, or $\Re(Y_u(j\omega_0)) = 0$, by Lemma 3.5.1. It then follows, from the proof of Lemma 3.3.3, that N_u is from the quartet $\mathcal{Q}_1$. Since the impedance (resp. admittance) of any network from $\mathcal{N}_1$ (resp. $\mathcal{N}_1^d$) has a zero at $j\omega_0$, then it is $\tilde{v}(j\omega_0)$-blocked (resp. $\tilde{i}(j\omega_0)$-blocked) by Lemma 3.5.1. □

Combining the preceding lemma with Lemma 3.5.1 allows us to conclude that if N is a network which realises a minimum function, then any one-port subnetwork of N which contains a resistor and at most one other passive element is an ω_0-blocked subnetwork of N (this need not be a maximal ω_0-blocked subnetwork). Moreover, any one-port subnetwork of N which contains a resistor and at most two other passive elements is either an ω_0-blocked subnetwork of N or is from the quartet $\mathcal{Q}_1$.

We now turn our attention to the enumeration of those networks which contain at most five reactive elements and at most seven passive elements in total and which realise a minimum function. In Section 3.3 we considered those networks which realised a minimum function and which contain fewer than five reactive elements. Therefore, in this section, it remains to consider networks containing exactly five reactive elements and at most two resistors.

Those SP networks which contain at most seven passive elements and which realise a minimum function are described in Section 3.3. The following lemma extends the results in that section by considering those minimum functions which may be realised by networks which contain at most seven passive elements and which are a series or parallel connection of two one-ports. First, we recall the definition of the network class $\mathcal{N}_3$ from Section 3.3. This was defined as the set of all networks N_3 in Fig. 23 for $A, B, C, D, E > 0$. The corresponding quartet $\mathcal{Q}_3$ was defined in the usual manner.

Lemma 3.6.3. *Let N be a transformerless network which contains at most seven passive elements. Further let N have impedance $H(s)$ which is a minimum function, and let N be either a series or a parallel connection of two one-ports. Then N is from the quartet $\mathcal{Q}_3$.*

Proof. Consider first the case where N is a series connection of two one-ports with $N = N_u + N_v$, and so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s), Z_v(s)$ are the impedances of N_u, N_v. Then neither $Z_u(s)$ nor $Z_v(s)$ have any poles on $j\mathbb{R} \cup \infty$, and $\Re\left(Z_u(j\omega_0)\right) = \Re\left(Z_v(j\omega_0)\right) = 0$, by Lemma 3.3.1. Furthermore, without loss of generality, $\Im\left(Z_u(j\omega_0)\right) \neq 0$. Then N_u contains at least three reactive elements by Lemma 3.5.5. Furthermore, N_v contains at least two reactive elements by Lemma 3.3.2, and both N_u and N_v contain at least one resistor since neither $Z_u(s)$ nor $Z_v(s)$ are lossless. It follows that N_u (resp. N_v) contains exactly three reactive elements (resp. two reactive elements) and one resistor. Since both N_u and N_v contain fewer than five passive elements, then they must both be SP by [54, p. 326]. It follows that N is from one of the classes $\mathcal{N}_3$ or $\mathcal{N}_3^i$ by Theorem 3.3.5.

The case where N is a parallel connection of two one-ports is similar. In this case, we conclude that N is from one of the classes $\mathcal{N}_3^d$ or $\mathcal{N}_3^{di}$. □

By combining Lemmas 3.6.1 to 3.6.3, we arrive at the following lemma.

Lemma 3.6.4. *Let N be a transformerless network which contains exactly five reactive elements and at most two resistors and whose impedance is a minimum function (with ω_0 a minimum frequency). Then either N is from the quartet $\mathcal{Q}_3$, or one of the conditions 1 to 4 holds.*

1. *N takes the form of Fig. 32, one of the one-ports $N_1, N_2, \ldots, N_5$ comprises two reactive elements, one comprises a resistor, and the remaining one-ports comprise a single reactive element.*

2. *N takes the form of Fig. 32, two of the one-ports $N_1, N_2, \ldots, N_5$ comprise two passive elements, the remaining one-ports comprise a single passive element, and either one of the following two subconditions 2a or 2b must hold.*

   (a) N_1 and N_3 each comprise a resistor, N_4 comprises a series connection of an inductor and a capacitor, N_5 comprises a parallel connection of an inductor and a capacitor, and N_2 comprises a single reactive element.

   (b) N_1 and N_2 each contain a resistor (and possibly other elements), N_3, N_4, and N_5 each contain only reactive elements.

3. *N takes the form of Fig. 32, one of the one-ports* $N_1, N_2, \ldots, N_5$ *comprises three passive elements, the remaining one-ports comprise a single passive element, and at least one of the following four subconditions 3a to 3d must hold.*
   (a) N_1 *and* N_2 *each contain a resistor (and possibly other elements) and are* ω_0*-blocked subnetworks,* N_3*,* N_4*, and* N_5 *each contain only reactive elements.*
   (b) N_1 *comprises a resistor,* N_2 *belongs to the quartet* $\mathcal{Q}_1$*,* N_3*,* N_4*, and* N_5 *each comprise a single reactive element.*
   (c) N_1 *comprises two resistors and a single reactive element,* N_2*,* N_3*,* N_4*, and* N_5 *each comprise a single reactive element.*
   (d) N_3 *comprises two resistors and a single reactive element,* N_1*,* N_2*,* N_4*, and* N_5 *each comprise a single reactive element.*
4. *N takes one of the forms in Fig. 45 where each of the one-ports* $N_1, N_2, \ldots, N_7$ *comprise a single passive element, which are reactive elements with the exception of the following two one-ports each of which comprises a single resistor:*
   (a) N_2 *and* N_5*.*
   (b) N_2 *and* N_4*.*
   (c) N_2 *and* N_7*.*
   (d) N_1 *and* N_7*.*
   (e) N_1 *and* N_5*.*

Proof. From Lemma 3.6.3, and the enumeration of biconnected graphs which contain eight or fewer edges [54, pp. 325 - 327], we conclude that either N is from $\mathcal{Q}_3$ or N takes the form of one of the networks in Figs. 32 or 45. Since the impedance of N is not lossless, then N contains at least one resistor.

Suppose initially that N contains exactly five reactive elements and one resistor, so N must take the form of the network in Fig. 32, one of the one-ports $N_1, \ldots, N_5$ comprises two passive elements, and the remainder comprise a single passive element. First consider the case in which the resistor is in the one-port containing two passive elements. Then, it may be verified that N must have either an L-path, C-path, L-cut-set, or C-cut-set, and so the impedance of N must possess either a pole or a zero at either 0 or ∞. This contradicts the assumption that the impedance of N is a minimum function. We thus conclude that one of the one-ports $N_1, \ldots, N_5$ comprises a single resistor, and so N satisfies condition 1.

If, on the other hand, N contains exactly two resistors, then N contains exactly seven passive elements. If N takes the form of the network in Fig. 32 then, since N contains exactly seven passive elements, either two of the one-ports $N_1, N_2, \ldots, N_5$ comprise two passive elements and the remaining one-ports comprise a single passive element, or one of the one-ports $N_1, N_2, \ldots, N_5$ comprises three passive elements and the remaining one-ports comprise a single passive element. If, instead, N takes the form of one of the networks in Fig. 45, then each of the one-ports $N_1, N_2, \ldots, N_7$ must comprise a single passive element, since N contains exactly seven passive elements.

Suppose initially that N takes the form of the network in Fig. 32, two of the one-ports $N_1, N_2, \ldots, N_5$ comprise two passive elements, and the remaining one-ports comprise a single passive element. For N to contain no series or parallel connected elements of the same kind, then the two resistors in N must be in separate one-ports. Moreover, both one-ports which contain a resistor must be ω_0-blocked subnetworks by Lemma 3.6.2. Furthermore, N satisfies the conditions of Lemma 3.6.1 and so the three conditions listed in that lemma statement must hold. Given the symmetry of the network, it may then be verified that N satisfies condition 2 in the present lemma statement.

Suppose next that N takes the form of the network in Fig. 32, one of the one-ports $N_1, N_2, \ldots, N_5$ comprises three passive elements, and the remaining one-ports comprise a single passive element. In this case, it is possible for both resistors to be in the one-port comprising three passive elements. Suppose initially that this is the case. Since this one-port can contain at most one reactive element then it must be an ω_0-blocked subnetwork by Lemma 3.6.2. Then, given the symmetry of the network, it follows that either condition 3c or 3d must hold.

Suppose, instead, that the resistors are in different one-ports. From Lemma 3.6.2, at least one of these one-ports must be an ω_0-blocked subnetwork, and the second one-port is either an ω_0-blocked subnetwork or it belongs to the quartet $\mathcal{Q}_1$. Suppose initially that this one-port belongs to the quartet $\mathcal{Q}_1$. Then each of the other one-ports comprise a single passive element, and, given the symmetry of the network, there are four cases to consider: (i) N_3 is from $\mathcal{N}_1$, (ii) N_2 is from $\mathcal{N}_1$, (iii) N_3 is from $\mathcal{N}_1^d$, (iv) N_2 is from $\mathcal{N}_1^d$. We consider here cases (i) and (ii). In these two cases, we consider a $j\omega_0$-trajectory of the form of (65) for the graph of one-ports G corresponding to the interconnection of $N_1, N_2, \ldots, N_5$ shown in Fig. 32, where $\tilde{i}_k$ and $\tilde{v}_k$ correspond to the one-port N_k, and we denote the impedance (resp. admittance) of N_k by $Z_k(s)$ (resp. $Y_k(s)$), for $k = 1, 2, \ldots, 5$. The analysis for the two other cases is similar, and allows us to conclude that N satisfies condition 3b when one of the one-ports is from $\mathcal{Q}_1$.

Consider first case (i). Since N_3 is from $\mathcal{N}_1$ then it is $\tilde{v}(j\omega_0)$-blocked (i.e. $\tilde{v}_3 = 0$) and, without loss of generality, let N_1 comprise a resistor (and hence $\tilde{i}_1 = \tilde{v}_1 = 0$). Then $\tilde{v}_4 = 0$ by Kirchhoff's voltage law. Moreover, since N_4 is a single reactive element, then $Y_4(s)$ does not have a pole at $s = j\omega_0$, and hence $\tilde{i}_4 = Y_4(j\omega_0)\tilde{v}_4 = 0$, and there is a cut-set in G which comprises the source together with the $\tilde{i}(j\omega_0)$-blocked one-ports N_1 and N_4. This contradicts the assumption that the impedance of N is a minimum function by Lemma 3.5.10.

Consider next case (ii). Then N_4 cannot be an ω_0-blocked subnetwork by Lemma 3.5.10, since this would result in a circuit in G comprising the source together with the $\tilde{v}(j\omega_0)$-blocked subnetworks N_2 and N_4. By Kirchhoff's voltage law, if either of the one-ports N_3 or N_5 is $\tilde{v}(j\omega_0)$-blocked then so too is the other one-port. Since $Y_k(s)$ does not have a pole at $s = j\omega_0$ then $\tilde{i}_k = Y_k(j\omega_0)\tilde{v}_k = 0$ for $k = 3$ and $k = 5$, and so both of these one-ports will then also be $\tilde{i}(j\omega_0)$-blocked. Then, by Kirchhoff's current law, we find that N_1 is also $\tilde{i}(j\omega_0)$-blocked, and hence $\tilde{v}(j\omega_0)$-blocked since $Z_1(s)$ does not have a pole at $s = j\omega_0$. Again, there is a circuit in G comprising the source together with $\tilde{v}(j\omega_0)$-blocked one ports, which is not possible by Lemma 3.5.10. Hence, the only possibility is for N_1 to be the ω_0-blocked subnetwork, this corresponding to condition 3b.

If neither of the preceding subconditions (3b, 3c, or 3d) holds then the two resistors are in separate one-ports from among $N_1, N_2, \ldots N_5$, and neither of these one-ports is from $\mathcal{Q}_1$. We then conclude that both of these one-ports are ω_0-blocked subnetworks by Lemma 3.6.2. In this case, from Lemma 3.6.1 and given the symmetry of the network, it follows that condition 3a must hold.

Suppose finally that N takes the form of one of the networks in Fig. 45, where each of the one-ports $N_1, N_2, \ldots, N_7$ comprises a single passive element. We consider only the network on the left of Fig. 45. The corresponding conclusions for the network on the right follow by a similar argument. Let us consider the graph of elements G corresponding to N. Then, by Lemma 3.6.1, there are two sub-cases to consider: (i) G contains one maximal ω_0-blocked subnetwork which connects to the rest of G at two or three vertices, (ii) G contains two maximal ω_0-blocked subnetworks, each of which connects to the rest of G at two vertices. Moreover, all edges in G which correspond to resistors in N are ω_0-blocked by Lemma 3.6.2. In subcase (i), the only possibility which satisfies conditions 2 and 3 of Lemma 3.5.5 (given the symmetry of the network) corresponds to case 4a in the present lemma statement. In subcase (ii), consider first the case where N_2 comprises a single resistor. The only possibilities in this case which satisfy conditions 2 and 3 of Lemma 3.5.5 correspond to cases 4b and 4c in the present lemma statement. The remaining possibilities, given the symmetry of the network, are

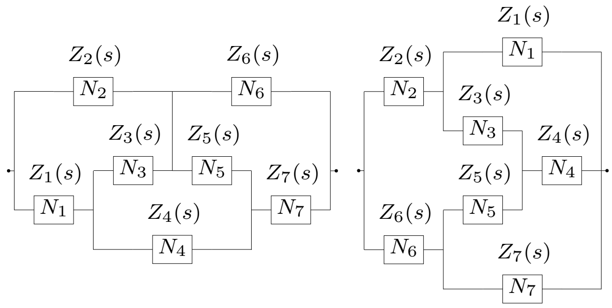


Figure 45: Networks comprised of the one-ports $N_1, N_2, \ldots, N_7$.

those for which neither N_2 nor N_6 contain a resistor. Without loss of generality, and again considering conditions 2 and 3 of Lemma 3.5.5, these correspond to cases 4d and 4e in the present lemma statement. □

We now combine Lemmas 3.5.10 and 3.6.4 to identify those network classes whose networks realise a minimum function and contain exactly five reactive elements and at most two resistors. These network classes are given in Theorem 3.6.5. As in Theorem 3.5.14, the networks comprising such classes all satisfy certain constraints among the impedances of their constituent one-ports. Prior to stating Theorem 3.6.5, we introduce the pertinent network classes together with these associated one-port impedance constraints.

Network classes $\mathcal{N}_{30}$ to $\mathcal{N}_{38}$ are the sets of all networks with the form of Fig. 32, where the one-ports $N_1 \ldots N_5$ are from the respective classes listed in Table 2, and their impedances satisfy the following constraints. For class $\mathcal{N}_{30}$, $Z_3(j\omega_0) = -Z_5(j\omega_0)$. For class $\mathcal{N}_{31}$, the admittance (resp. impedance) of the one-port N_4 (resp. N_5) must have a pole at $j\omega_0$. For classes $\mathcal{N}_{32}$ to $\mathcal{N}_{38}$, $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$.

Network classes $\mathcal{N}_{39}$ to $\mathcal{N}_{49}$ are the sets of all networks with the form of Fig. 32, where the one-ports $N_1 \ldots N_5$ are from the respective classes listed in Table 3, and their impedances satisfy the following constraints. For classes $\mathcal{N}_{39}$ to $\mathcal{N}_{44}$, $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$. For network classes $\mathcal{N}_{45}$ and $\mathcal{N}_{46}$ (resp. $\mathcal{N}_{47}$ and $\mathcal{N}_{48}$; $\mathcal{N}_{49}$), $Z_3(j\omega_0) = -Z_5(j\omega_0)$ (resp. $Z_2(j\omega_0)(Z_3(j\omega_0)+Z_4(j\omega_0))+Z_4(j\omega_0)(Z_3(j\omega_0)+Z_5(j\omega_0)) = 0$; $Z_1(j\omega_0)Z_2(j\omega_0) - Z_4(j\omega_0)Z_5(j\omega_0) = 0$).

Network classes $\mathcal{N}_{50}$ to $\mathcal{N}_{64}$ are the sets of all networks with the form of the network on the left of Fig. 45, where the one-ports $N_1 \ldots N_7$ are from the respective classes listed in Table 4, and their impedances satisfy the following constraints. For classes $\mathcal{N}_{50}$ and $\mathcal{N}_{51}$ (resp. $\mathcal{N}_{52}$ to $\mathcal{N}_{55}$; $\mathcal{N}_{56}$ to $\mathcal{N}_{61}$; $\mathcal{N}_{62}$; $\mathcal{N}_{63}$ and $\mathcal{N}_{64}$), $Z_3(j\omega_0)Z_7(j\omega_0) - Z_4(j\omega_0)Z_6(j\omega_0) = 0$ and $Z_1(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0)) + Z_3(j\omega_0)Z_4(j\omega_0) = 0$ (resp. $Z_3(j\omega_0)(Z_5(j\omega_0) +$

	N_1	N_2	N_3	N_4	N_5
$\mathcal{N}_{30}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}\cdot\mathcal{C}$
$\mathcal{N}_{31}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{L}+\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$
$\mathcal{N}_{32}$	$\mathcal{R}+\mathcal{C}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{33}$	$\mathcal{R}+\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{34}$	$\mathcal{R}\cdot\mathcal{C}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{35}$	$\mathcal{R}$	$\mathcal{R}+\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{36}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{L}+\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{37}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}\cdot\mathcal{C}$	$\mathcal{L}+\mathcal{C}$
$\mathcal{N}_{38}$	$\mathcal{R}\cdot\mathcal{C}$	$\mathcal{R}+\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$

Table 2: Realisations of minimum functions corresponding to cases 1 and 2 in Lemma 3.6.4.

	N_1	N_2	N_3	N_4	N_5
$\mathcal{N}_{39}$	$\mathcal{R}\cdot(\mathcal{L}+\mathcal{C})$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{40}$	$\mathcal{C}\cdot(\mathcal{R}+\mathcal{C})$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{41}$	$\mathcal{C}\cdot(\mathcal{R}+\mathcal{L})$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{42}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{C}\cdot(\mathcal{L}+\mathcal{C})$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{43}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}\cdot(\mathcal{L}+\mathcal{C})$	$\mathcal{L}$
$\mathcal{N}_{44}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{C}\cdot(\mathcal{L}+\mathcal{C})$	$\mathcal{L}$
$\mathcal{N}_{45}$	$\mathcal{R}$	$\mathcal{N}_1$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{46}$	$\mathcal{R}$	$\mathcal{N}_1$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$
$\mathcal{N}_{47}$	$\mathcal{R}\cdot(\mathcal{R}+\mathcal{C})$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{48}$	$\mathcal{R}\cdot(\mathcal{R}+\mathcal{L})$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{N}_{49}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{R}\cdot(\mathcal{R}+\mathcal{C})$	$\mathcal{L}$	$\mathcal{L}$

Table 3: Realisations of minimum functions corresponding to case 3 in Lemma 3.6.4.

$Z_6(j\omega_0)+Z_7(j\omega_0))+Z_5(j\omega_0)Z_6(j\omega_0) = 0$ and $Z_1(j\omega_0)+Z_3(j\omega_0) = 0$; $Z_1(j\omega_0)Z_5(j\omega_0)-Z_6(j\omega_0)(Z_4(j\omega_0)+Z_5(j\omega_0)) = 0$ and $Z_1(j\omega_0)Z_5(j\omega_0)+Z_3(j\omega_0)(Z_5(j\omega_0)+Z_6(j\omega_0)) = 0$; $Z_3(j\omega_0)+Z_4(j\omega_0)+Z_5(j\omega_0) = 0$ and $Z_2(j\omega_0)Z_5(j\omega_0)-Z_3(j\omega_0)Z_6(j\omega_0) = 0$; $Z_3(j\omega_0)+Z_4(j\omega_0) = 0$ and $Z_2(j\omega_0)Z_3(j\omega_0)Z_7(j\omega_0) - Z_4(j\omega_0)Z_6(j\omega_0)(Z_2(j\omega_0)+Z_3(j\omega_0)) = 0$).

The corresponding network quartets $\mathcal{Q}_{30}$ to $\mathcal{Q}_{64}$ are defined in the usual manner.

Theorem 3.6.5. *Let N be a transformerless network containing at most five reactive elements and at most seven passive elements in total and with impedance $H(s)$ which is a minimum function (with ω_0 a minimum frequency). Then N is from one of the quartets $\mathcal{Q}_3$, $\mathcal{Q}_{22}$ to $\mathcal{Q}_{27}$, or $\mathcal{Q}_{30}$ to $\mathcal{Q}_{64}$.*

Proof. If N contains fewer than five reactive elements then N satisfies the conditions of Theorem 3.5.14. It is straightforward to verify that the only networks which satisfy the conditions of Theorem 3.5.14 and which contain at most seven passive elements in

	N_1	N_2	N_3	N_4	N_5	N_6	N_7
$\mathcal{N}_{50}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$
$\mathcal{N}_{51}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{52}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$
$\mathcal{N}_{53}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{54}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$
$\mathcal{N}_{55}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{56}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{57}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{58}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{59}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{60}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{61}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}_{62}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{R}$
$\mathcal{N}_{63}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$
$\mathcal{N}_{64}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{L}$

Table 4: Realisations of minimum functions corresponding to case 4 in Lemma 3.6.4.

total, with no passive elements of the same kind in series or in parallel, are those in quartets $\mathcal{Q}_{22}$ to $\mathcal{Q}_{27}$. It remains to consider the case where N contains exactly five reactive elements and at most two resistors, and so N satisfies the conditions of Lemma 3.6.4.

Consider first the case where N satisfies condition 1 in Lemma 3.6.4. Given the symmetry of the network, we need consider only two cases. Firstly, where the one-port comprising the resistor is N_1, and, secondly, where the one-port comprising the resistor is N_3. We will consider here only the case where the one-port comprising two reactive elements is from $\mathcal{L} \cdot \mathcal{C}$, the remaining possibilities will then correspond to the duals of the identified networks (these duals are certain to exist since the network contains fewer than eight passive elements, see e.g. [22, Section III]). We will also arbitrarily assign the kind of one of the reactive elements in one of the remaining one-ports in the network, and the remaining possibilities will correspond to the frequency inverses of the identified networks. If either N_2 or N_4 are from $\mathcal{L} \cdot \mathcal{C}$ then the network must contain either an L-path or a C-path, so let N_4 be from $\mathcal{L}$. We conclude that N_2 must then be from $\mathcal{C}$ since the network must not contain an L-path. If N_3 belongs to $\mathcal{L} \cdot \mathcal{C}$ then the network contains either a C-cut-set if N_5 belongs to $\mathcal{C}$, or an L-path if N_5 belongs to $\mathcal{L}$. Moreover, if N_3 belongs to $\mathcal{R}$, then all possibilities for N_1 and N_5 result in networks containing either an L-path or a C-path. We therefore conclude that N_1 must be from $\mathcal{R}$ and N_5 from $\mathcal{L} \cdot \mathcal{C}$. Finally, N_3 must belong to $\mathcal{C}$ since N has no L-path. Since N_1 is an ω_0-blocked subnetwork by Lemma 3.6.2, then the $j\omega_0$-

impedances of the networks in Fig. 34 must be equal by Lemma 3.5.10, which implies $Z_2(j\omega_0)(Z_3(j\omega_0)+Z_4(j\omega_0))+Z_4(j\omega_0)(Z_3(j\omega_0)+Z_5(j\omega_0))=0$. We thus conclude that N must be from $\mathcal{Q}_{30}$ when condition 1 in Lemma 3.6.4 is satisfied.

When N satisfies condition 2a in Lemma 3.6.4 then, since N_1 and N_3 are both ω_0-blocked subnetworks by Lemma 3.6.2, it may be verified that the admittance (resp. impedance) of network N_4 (resp. N_5) must have a pole at $j\omega_0$, and so N must be from $\mathcal{Q}_{31}$ in this case.

Consider next the case where N satisfies condition 2b in Lemma 3.6.4. Since both N_1 and N_2 must be ω_0-blocked subnetworks by Lemma 3.6.2, then we require the $j\omega_0$-impedance of the four networks in Fig. 36 to be equal by Lemma 3.5.10. It may then be verified that $Z_3(j\omega_0)=-Z_4(j\omega_0)=-Z_5(j\omega_0)$. Given the symmetry of the network then, without loss of generality, either N_1 or N_4 must contain two passive elements. We consider only the case where at least one of the one-ports N_1 and N_4 is essentially parallel, the remaining networks will then be the duals of those identified here. Moreover, at least one of the one-ports N_3, N_4, N_5 will comprise a single reactive element. We will assign the kind of reactive element for one of those one-ports which comprise a single reactive element from among N_3, N_4, and N_5 to ensure that the impedance of the network at $s=j\omega_0$ is equal to αXj with $X>0$. The kind of reactive element for any other one-port which comprises a single reactive element from among the one-ports N_3, N_4, and N_5 may then be determined by the relationship $Z_3(j\omega_0)=-Z_4(j\omega_0)=-Z_5(j\omega_0)$. The remaining networks which satisfy condition 2b in Lemma 3.6.4 are then the frequency inverse networks to those identified here. There are two sub-cases to consider: (i) N_4 is from $\mathcal{L}\cdot\mathcal{C}$, (ii) N_1 is from either $\mathcal{R}\cdot\mathcal{L}$ or $\mathcal{R}\cdot\mathcal{C}$ and, in order to avoid duplicating networks which are covered by case (i) networks and their duals, both N_4 and N_5 comprise a single reactive element.

In case (i), suppose initially that N_1 is the second one-port which comprises two passive elements. As explained in the preceding paragraph, we consider only the case where N_5 is from $\mathcal{L}$ and N_3 from $\mathcal{C}$, so clearly N_2 must be from $\mathcal{R}$. If N_1 is from $\mathcal{R}\cdot\mathcal{L}$ then there is an L-path between the network terminals, hence we require N_1 to be from $\mathcal{R}+\mathcal{C}$, $\mathcal{R}+\mathcal{L}$, or $\mathcal{R}\cdot\mathcal{C}$, and so N must be from $\mathcal{N}_{32}$, $\mathcal{N}_{33}$, or $\mathcal{N}_{34}$. Suppose next that N_2 is the second one-port comprising two passive elements, and so N_1 must belong to $\mathcal{R}$. Again, we consider just the case where N_5 is from $\mathcal{L}$ and N_3 from $\mathcal{C}$. In this case, the only possibility which does not contain an L-cut-set, L-path or C-path is when N_2 is from $\mathcal{R}+\mathcal{C}$, and so N belongs to $\mathcal{N}_{35}$. Next, suppose N_3 is the second one-port comprising two passive elements. In this case, both N_1 and N_2 are from $\mathcal{R}$, and we consider just the case where N_5 is from $\mathcal{L}$. The only possibility which does not contain an L-path is when N_3 belongs to $\mathcal{L}+\mathcal{C}$, in which case N belongs to $\mathcal{N}_{36}$. Suppose finally that N_5 is the second one-port comprising two passive elements, so again N_1 and N_2 belong to $\mathcal{R}$. As explained previously, we will consider only the case where N_3 belongs to $\mathcal{C}$. In this case, the only possibility which does not contain an L-path is when N_5 belongs to $\mathcal{L}+\mathcal{C}$, and then N belongs to $\mathcal{N}_{37}$.

In case (ii) then, as explained earlier, we will consider only the case where both N_4 and N_5 are from $\mathcal{L}$. If N_3 contains two reactive elements then N necessarily contains either an L-path or an L-cut-set, and hence N_3 must be from $\mathcal{C}$ for $Z_3(j\omega_0) = -Z_4(j\omega_0)$. Given the symmetry of the network, the only possibility which does not contain an L-path, L-cut-set, or C-path, is when N_1 belongs to $\mathcal{R}\cdot\mathcal{C}$ and N_2 belongs to $\mathcal{R}+\mathcal{C}$, in which case N belongs to $\mathcal{N}_{38}$. We have shown that N is from one of the quartets $\mathcal{Q}_{31}, \mathcal{Q}_{32}, \ldots, \mathcal{Q}_{38}$ when N satisfies condition 2 of Lemma 3.6.4.

Consider next the case where N satisfies condition 3a in Lemma 3.6.4. Once again, we require the $j\omega_0$-impedance of the four networks in Fig. 36 to be equal, and so $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$. As before, we consider only those cases in which the one-port comprising three passive elements is essentially parallel, and we assign one of those one-ports comprising a single reactive element from amongst N_3, N_4, N_5 to ensure the impedance of the network at $s = j\omega_0$ is equal to $\alpha X j$ with $X > 0$. Again, the kind of reactive element for any other one-port which comprises a single reactive element from among the one-ports N_3, N_4, and N_5 may then be determined by the relationship $Z_3(j\omega_0) = -Z_4(j\omega_0) = -Z_5(j\omega_0)$. The remaining networks will then be the duals and frequency inverses of the networks identified here. We then have three sub-cases to consider: (i) N_1 comprises three passive elements, (ii) N_3 comprises three passive elements, (iii) N_4 comprises three passive elements.

In case (i), we consider just the case where N_3 is from $\mathcal{C}$ and N_4 and N_5 are both from $\mathcal{L}$, and clearly N_2 must be from $\mathcal{R}$. The only possibilities which do not contain an L-path are when N belongs to $\mathcal{N}_{39}$, $\mathcal{N}_{40}$, or $\mathcal{N}_{41}$. In case (ii), we consider just the case where N_4 and N_5 are both from $\mathcal{L}$, and in this case both N_1 and N_2 must be from $\mathcal{R}$. The only possibilities which do not contain an L-path, and do not contain two elements of the same kind in series or parallel, are when N_3 is from $\mathcal{C}\cdot(\mathcal{L}+\mathcal{C})$, in which case N belongs to $\mathcal{N}_{42}$. In case (iii), we again require both N_1 and N_2 to belong to $\mathcal{R}$, and we consider the case where N_3 is from $\mathcal{C}$ and N_5 from $\mathcal{L}$. Since we consider only the case where N_4 is EP, and N_4 has no two elements of the same kind in series or parallel, then N must be from either $\mathcal{N}_{43}$ or $\mathcal{N}_{44}$.

Next, suppose N satisfies condition 3b in Lemma 3.6.4. We consider here only the case where N_2 is from $\mathcal{N}_1$. The remaining cases will correspond to the dual networks to those identified here. Since N_1 is an ω_0-blocked subnetwork, and N_2 is $\tilde{v}(j\omega_0)$-blocked by Lemma 3.6.2, then we require the $j\omega_0$-impedance of the network second to the left in Fig. 34 to be equal to the $j\omega_0$-impedance of the rightmost network in that figure by Lemma 3.5.10. It may then be verified that $Z_3(j\omega_0) = -Z_5(j\omega_0)$, and so N_3 and N_5 must comprise reactive elements of different kind. As in previous cases, we consider the case where N_4 is from $\mathcal{L}$, since this implies that the impedance of the network at $s = j\omega_0$ is equal to $\alpha X j$ with $X > 0$. The remaining cases will correspond to the frequency inverse networks to those identified here. It is then clear that N_1 must belong to $\mathcal{R}$, and since N_3 and N_5 comprise reactive elements of different kind then N belongs to either $\mathcal{N}_{45}$ or $\mathcal{N}_{46}$.

Suppose now that N satisfies condition 3c in Lemma 3.6.4. Since N_1 is an ω_0-blocked subnetwork by Lemma 3.6.2, then the $j\omega_0$-impedances of the networks in Fig. 34 must be equal by Lemma 3.5.10, which implies $Z_2(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0)) + Z_4(j\omega_0)(Z_3(j\omega_0) + Z_5(j\omega_0)) = 0$. Similarly to before, we consider only the case where N_1 is EP and N_4 is from $\mathcal{L}$, the remaining cases will then be the dual and frequency inverse networks to those identified here. The only possibilities which do not contain an L-path or C-cut-set are those for which N_2 and N_3 are from $\mathcal{C}$, and N_5 is from $\mathcal{L}$. Since N_1 is EP and comprises two resistors and a single reactive element, then N must be from either $\mathcal{N}_{47}$ or $\mathcal{N}_{48}$.

Consider next the case where N satisfies condition 3d in Lemma 3.6.4. In this case, the only possibilities which contain no L-paths, C-paths, L-cut-sets or C-cut-sets are those where N_1 and N_2 are from $\mathcal{C}$, and N_4 and N_5 are from $\mathcal{L}$. Since N_3 comprises two resistors and a single reactive element, then we need consider only the case where N_3 is from $\mathcal{R} \cdot (\mathcal{R} + \mathcal{C})$, and so N is from $\mathcal{N}_{49}$. The remaining networks are then the dual and frequency inverses of networks from this class. We have thus shown that N is from one of the quartets $\mathcal{Q}_{39}, \mathcal{Q}_{40}, \ldots, \mathcal{Q}_{49}$ when N satisfies condition 3 of Lemma 3.6.4.

Suppose now that N satisfies condition 4a in Lemma 3.6.4. For the $j\omega_0$-impedance of the network with N_2 opened (resp. shorted) and N_5 opened to be equal to that of the network with N_2 opened (resp. shorted) and N_5 shorted, then $Z_3(j\omega_0)Z_7(j\omega_0) - Z_4(j\omega_0)Z_6(j\omega_0) = 0$ (resp. $Z_1(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0)) + Z_3(j\omega_0)Z_4(j\omega_0) = 0$). This implies the following two conditions: (i) N_3 and N_7 comprise reactive elements of the same kind if and only if N_4 and N_6 do also, and (ii) N_1, N_3 and N_4 don't all comprise reactive elements of the same kind. Also, for N to contain no L-path, C-path, L-cut-set or C-cut-set, the following three conditions must also hold: (iii) N_6 and N_7 comprise reactive elements of different kind, (iv) N_1, N_3 and N_6 don't all comprise reactive elements of the same kind, and (v) N_1, N_4 and N_7 don't all comprise reactive elements of the same kind. We consider only the case where N_1 belongs to $\mathcal{L}$, and the remaining cases will correspond to the frequency inverse networks to those identified here. Suppose initially that N_3 belongs to $\mathcal{C}$. If N_4 also belongs to $\mathcal{C}$ then either N_6 belongs to $\mathcal{C}$ and N_7 to $\mathcal{L}$, or vice-versa, by condition (iii). However, both of these possibilities contradict condition (i). Hence, if N_3 belongs to $\mathcal{C}$, then N_4 belongs to $\mathcal{L}$, and so N_7 belongs to $\mathcal{C}$ by condition (v), and finally N_6 belongs to $\mathcal{L}$ by condition (iii). Otherwise, N_3 belongs to $\mathcal{L}$, so N_4 and N_6 belong to $\mathcal{C}$ by conditions (ii) and (iv) respectively, and N_7 belongs to $\mathcal{L}$ by condition (iii). We thus conclude that if N satisfies condition 4a in Lemma 3.6.4, then N is from either $\mathcal{N}_{50}$ or $\mathcal{N}_{51}$.

Suppose next that N satisfies condition 4b in Lemma 3.6.4. For the $j\omega_0$-impedance of the network with N_2 opened (resp. shorted) and N_4 opened to be equal to that of the network with N_2 opened (resp. shorted) and N_4 shorted, then $Z_3(j\omega_0)(Z_5(j\omega_0) + Z_6(j\omega_0) + Z_7(j\omega_0)) + Z_5(j\omega_0)Z_6(j\omega_0) = 0$ (resp. $Z_1(j\omega_0) + Z_3(j\omega_0) = 0$). This implies that N_1 and N_3 comprise reactive elements of different kind. In this case, for the network to contain no L-cut-set or C-cut-set, we additionally require the reactive element in N_6 to be of different kind to the reactive element in N_7. We consider only the case where N_5 belongs to $\mathcal{L}$, the remaining networks being the frequency inverse networks to those identified here. There are then four possibilities: (i) N_1 and N_6 belong to $\mathcal{L}$, and N_3 and N_7 belong to $\mathcal{C}$, (ii) N_1 and N_7 are from $\mathcal{L}$, and N_3 and N_6 are from $\mathcal{C}$, (iii) N_1 and N_7 are from $\mathcal{C}$, and N_3 and N_6 are from $\mathcal{L}$, and (iv) N_1 and N_6 are from $\mathcal{C}$, and N_3 and N_7 are from $\mathcal{L}$. These correspond to the cases where N belongs to $\mathcal{N}_{52}$ to $\mathcal{N}_{55}$ respectively.

Consider now the case where N satisfies condition 4c in Lemma 3.6.4. For the $j\omega_0$-impedance of the network with N_2 opened (resp. shorted) and N_7 opened to be equal to that of the network with N_2 opened (resp. shorted) and N_7 shorted, then we require

$$Z_6(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0) + Z_5(j\omega_0)) + Z_3(j\omega_0)Z_5(j\omega_0) = 0,$$
$$\text{and} \quad Z_1(j\omega_0)(Z_3(j\omega_0) + Z_4(j\omega_0) + Z_5(j\omega_0)) + Z_3(j\omega_0)(Z_4(j\omega_0) + Z_5(j\omega_0)) = 0.$$

Pre-multiplying the first of the above equations by $Z_1(j\omega_0)$ (resp. $Z_1(j\omega_0) + Z_3(j\omega_0)$), and the second by $Z_6(j\omega_0)$, and subtracting, we find $Z_1(j\omega_0)Z_5(j\omega_0) - Z_6(j\omega_0)(Z_4(j\omega_0) + Z_5(j\omega_0)) = 0$ (resp. $Z_1(j\omega_0)Z_5(j\omega_0) + Z_3(j\omega_0)(Z_5(j\omega_0) + Z_6(j\omega_0)) = 0$). This implies the following two conditions: (i) if N_4 and N_5 comprise reactive elements of the same kind then so too do N_1 and N_6, and (ii) if N_5 and N_6 comprise reactive elements of the same kind then N_1 and N_3 must comprise reactive elements of different kind. Moreover, for N to contain no L-path, C-path, L-cut-set or C-cut-set, we require the following two conditions to hold: (iii) N_1, N_3 and N_6 don't all comprise reactive elements of the same kind, and (iv) N_4, N_5 and N_6 don't all comprise reactive elements of the same kind. We consider only the case where N_6 belongs to $\mathcal{L}$, and again the remaining cases are the frequency inverse networks of those identified here. Suppose initially that N_1 is from $\mathcal{L}$, so N_3 belongs to $\mathcal{C}$ by condition (iii). If, in addition, N_4 belongs to $\mathcal{L}$ then N_5 belongs to $\mathcal{C}$ by condition (iv), and so N belongs to $\mathcal{N}_{56}$. If, on the other hand, N_4 belongs to $\mathcal{C}$, then N_5 belongs to either $\mathcal{L}$ or $\mathcal{C}$, and so N belongs to either $\mathcal{N}_{57}$ or $\mathcal{N}_{58}$. Suppose, instead, that N_1 belongs to $\mathcal{C}$, and so N_4 and N_5 must comprise reactive elements of different kind by condition (i). If, in addition, N_3 belongs to $\mathcal{L}$, then N must belong to either $\mathcal{N}_{59}$ or $\mathcal{N}_{60}$. If, on the other hand, N_3 belongs to $\mathcal{C}$, then the only possibility which does not violate condition (ii) is when N belongs to $\mathcal{N}_{61}$.

Next, suppose that N satisfies condition 4d in Lemma 3.6.4. For the $j\omega_0$-impedance of the network with N_1 opened (resp. shorted) and N_7 opened to be equal to that of the network with N_1 opened (resp. shorted) and N_7 shorted, then we require

$$Z_3(j\omega_0) + Z_4(j\omega_0) + Z_5(j\omega_0) = 0,$$

$$\text{and} \quad Z_6(j\omega_0)(Z_2(j\omega_0) + Z_3(j\omega_0))(Z_3(j\omega_0) + Z_4(j\omega_0) + Z_5(j\omega_0)) + Z_3(j\omega_0)(Z_2(j\omega_0)Z_5(j\omega_0) - Z_3(j\omega_0)Z_6(j\omega_0)) = 0.$$

Pre-multiplying the first of the above equations by $Z_6(j\omega_0)(Z_2(j\omega_0)+Z_3(j\omega_0))$ and subtracting the second, we see that $Z_3(j\omega_0)+Z_4(j\omega_0)+Z_5(j\omega_0) = 0$ and $Z_2(j\omega_0)Z_5(j\omega_0) - Z_3(j\omega_0)Z_6(j\omega_0) = 0$. This implies the following two conditions: (i) N_3, N_4 and N_5 don't all comprise reactive elements of the same kind, and (ii) N_2 and N_5 comprise reactive elements of the same kind if and only if N_3 and N_6 do also. Also, for N to contain no L-path, C-path, L-cut-set or C-cut-set, we additionally require the following three conditions to hold: (iii) N_2 and N_6 comprise reactive elements of different kind, (iv) N_2, N_3 and N_4 don't all comprise reactive elements of the same kind, and (v) N_4, N_5 and N_6 don't all comprise reactive elements of the same kind. Given the symmetry of the network together with condition (iii), we conclude that, without loss of generality, N_2 must belong to $\mathcal{L}$ and N_6 to $\mathcal{C}$. We consider only the case where N_4 belongs to $\mathcal{L}$, and the remaining cases will be the frequency inverse networks to those identified here. Then, N_3 belongs to $\mathcal{C}$ by condition (iv), and N_5 belongs to $\mathcal{L}$ by condition (ii). We conclude that N must belong to $\mathcal{N}_{62}$ in this case.

Consider finally the case where N satisfies condition 4e in Lemma 3.6.4. For the $j\omega_0$-impedance of the network with N_1 opened (resp. shorted) and N_5 opened to be equal to that of the network with N_1 opened (resp. shorted) and N_5 shorted, then $Z_3(j\omega_0) + Z_4(j\omega_0) = 0$ (resp. $Z_2(j\omega_0)Z_3(j\omega_0)Z_7(j\omega_0) - Z_4(j\omega_0)Z_6(j\omega_0)(Z_2(j\omega_0) + Z_3(j\omega_0)) = 0$).

We thus conclude that N_3 and N_4 must comprise reactive elements of different kind. Moreover, for N to contain no L-path, C-path, L-cut-set, or C-cut-set then N_2 and N_6 must comprise reactive elements of different kind, as must N_6 and N_7. We consider only the case where N_2 belongs to $\mathcal{L}$, the remaining cases will be the frequency inverse networks to those identified here. Then N_6 must belong to $\mathcal{C}$, N_7 to $\mathcal{L}$, and we require N to belong to either $\mathcal{N}_{63}$ or $\mathcal{N}_{64}$.

We have thus shown that N must be from one of the 42 quartets $\mathcal{Q}_3$, $\mathcal{Q}_{22}$ to $\mathcal{Q}_{27}$, or $\mathcal{Q}_{30}$ to $\mathcal{Q}_{64}$. □

It remains to determine which networks from the quartets described in Theorem 3.6.5 realise a biquadratic minimum function. These networks are stated in Theorem 3.6.6, which describes those quartets which realise biquadratic minimum functions and whose networks contain exactly seven elements. Before stating this theorem, we introduce the network quartets $\mathcal{Q}_{65}$ to $\mathcal{Q}_{73}$.

We first introduce networks N_{65} and N_{66} in Figs. 46 and 47. We remark that the impedance of the network N_{65}, and the impedance of the network N_{66}, is equal to $H_p(s)$ in (145) with $W = 1/2$ and $\alpha, X > 0$. We define the network class $\mathcal{N}_{65}$ as the set of all such networks N_{65} for $1 \leq g_3 \leq 4$, $g_3 \neq 2$, $\alpha, X > 0$, and we denote the corresponding quartet by $\mathcal{Q}_{65}$. Similarly, we define the network class $\mathcal{N}_{66}$ as the set of all such networks N_{66} for $\alpha, X > 0$, and we denote the corresponding quartet by $\mathcal{Q}_{66}$. From the proof of Theorem 3.5.18, the network class $\mathcal{N}_{65}$ contains those networks from the class $\mathcal{N}_{26}$ which realise a biquadratic minimum function.

We now introduce network N_{67} (resp. N_{68}; N_{69}) in Fig. 48 (resp. Fig. 49; 50). We remark that the impedance of the network N_{67} is equal to $H_p(s)$ in (145) with $\alpha > 0$,

$$1/2 < W < 1, \tag{165}$$

$$\text{and} \quad X = W\sqrt{(2W-1)/(1-W)}. \tag{166}$$

Also, the impedance of the network N_{68} is equal to $H_p(s)$ in (145) with $\alpha > 0$,

$$0 < W < 1/2, \tag{167}$$

$$\text{and} \quad X = \frac{W\sqrt{(1-2W)(1-W)((1-2W)(1-2W^2)+2W^3(2-3W)-\sqrt{(1-2W)(1+2W)}(1-2W)(1-2W^2))}}{\sqrt{2}((1-W)(1-2W)(1+2W)+W^3)}. \tag{168}$$

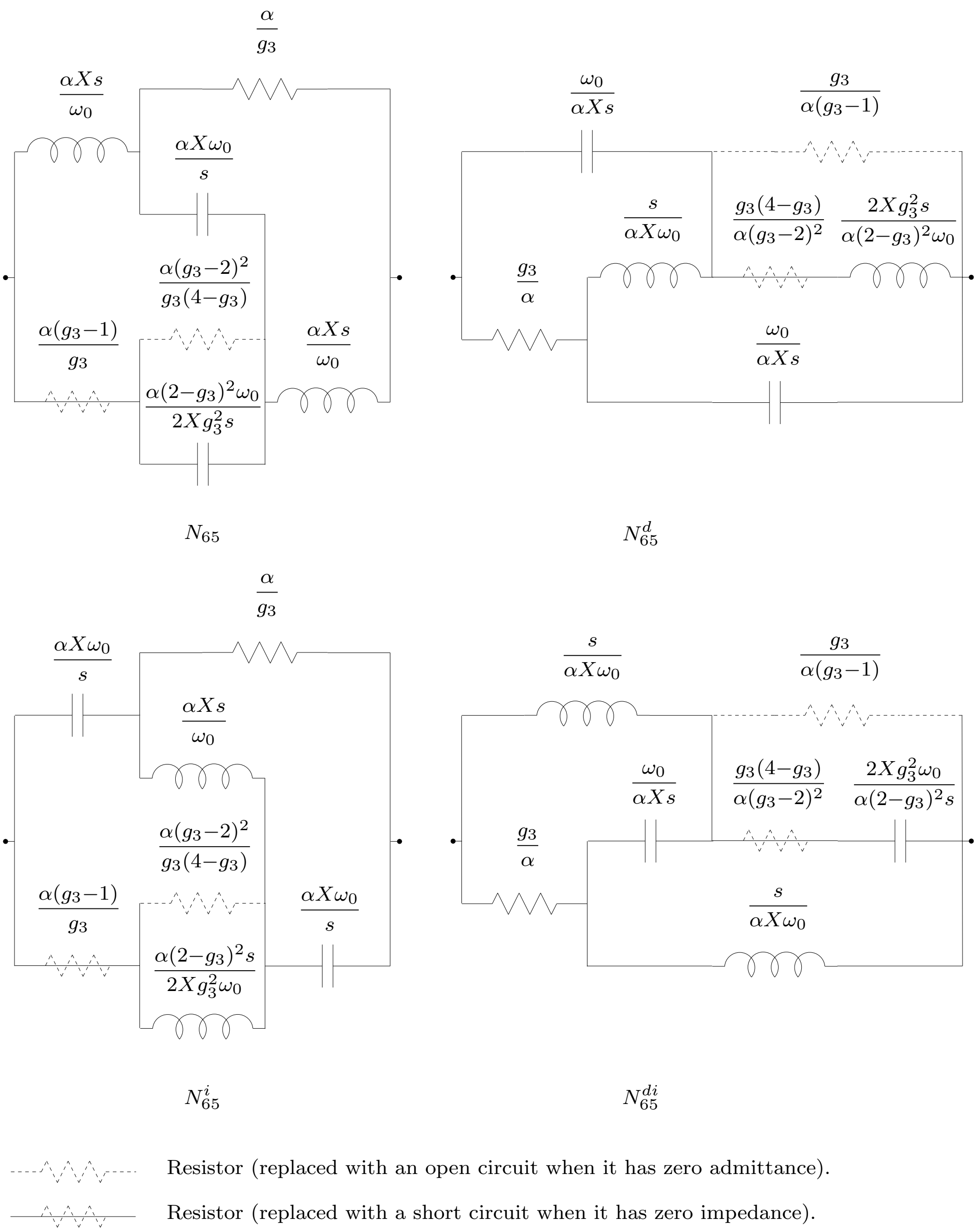


Figure 46: Quartet $\mathcal{Q}_{65}$, $1 \leq g_3 \leq 4$, $g_3 \neq 2$, $X, \alpha > 0$.

In addition, the impedance of the network N_{69} is equal to $H_p(s)$ in (145) with $\alpha > 0$,

$$0 < W < 1, \tag{169}$$

$$\text{and} \quad X = (1+W)(1-W)^2\sqrt{1-W+W^2}/(1-W+W^3). \tag{170}$$

We define the network class $\mathcal{N}_{67}$ (resp. $\mathcal{N}_{68}$; $\mathcal{N}_{69}$) as the set of all such networks

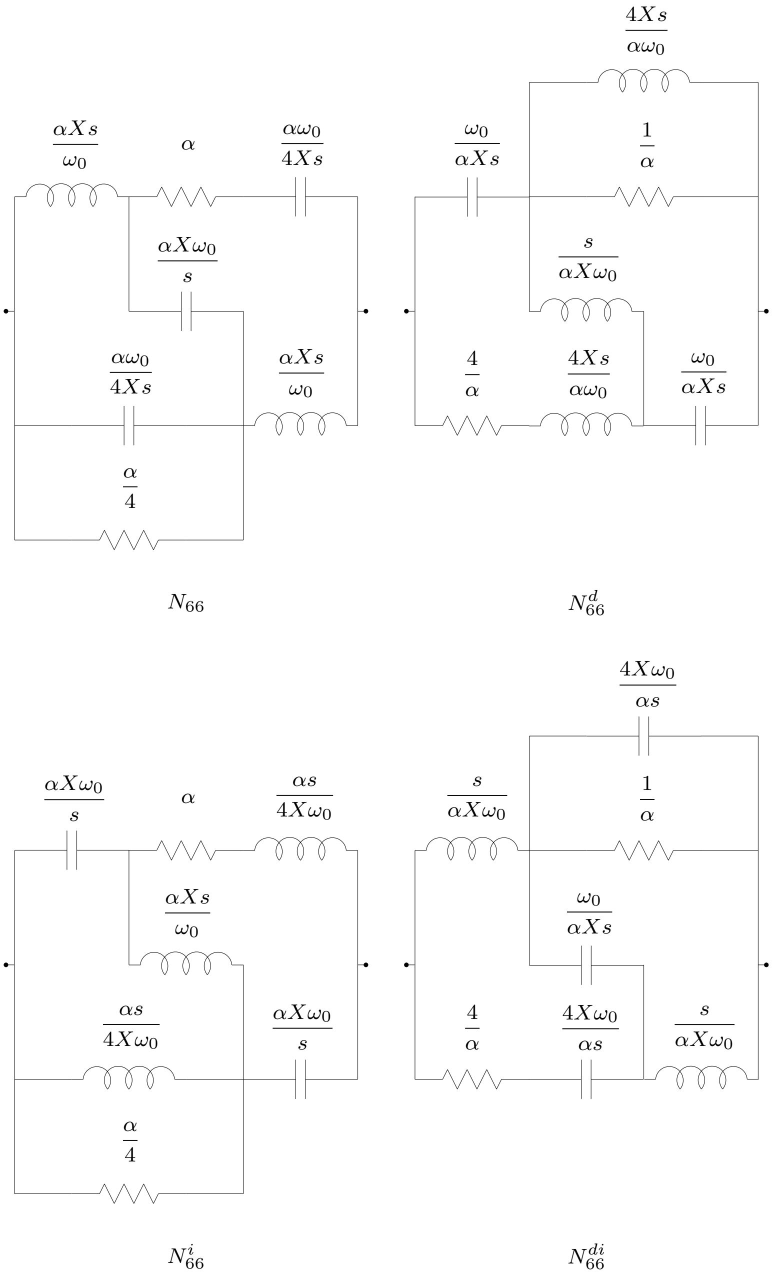


Figure 47: Quartet $\mathcal{Q}_{66}$, $X\alpha > 0$.

N_{67} (resp. N_{68}; N_{69}) for $\alpha > 0$, and for which W satisfies the constraint (165) (resp. constraint (167); constraint (169)), and X satisfies equation (166) (resp. equation (168); equation (170)). We denote the corresponding quartets $\mathcal{Q}_{67}$, $\mathcal{Q}_{68}$, and $\mathcal{Q}_{69}$.

Next, we introduce network N_{70} (resp. N_{71}) in Fig. 51 (resp. Fig. 52). We remark that the impedance of the network N_{70} (resp. N_{71}) is equal to $H_p(s)$ in (145) with $\alpha > 0$, $1/2 < W < 1$, and $X > W\sqrt{2W-1}/(1-W)$ (resp. $0 < X < W(1-W)/\sqrt{2W-1}$).

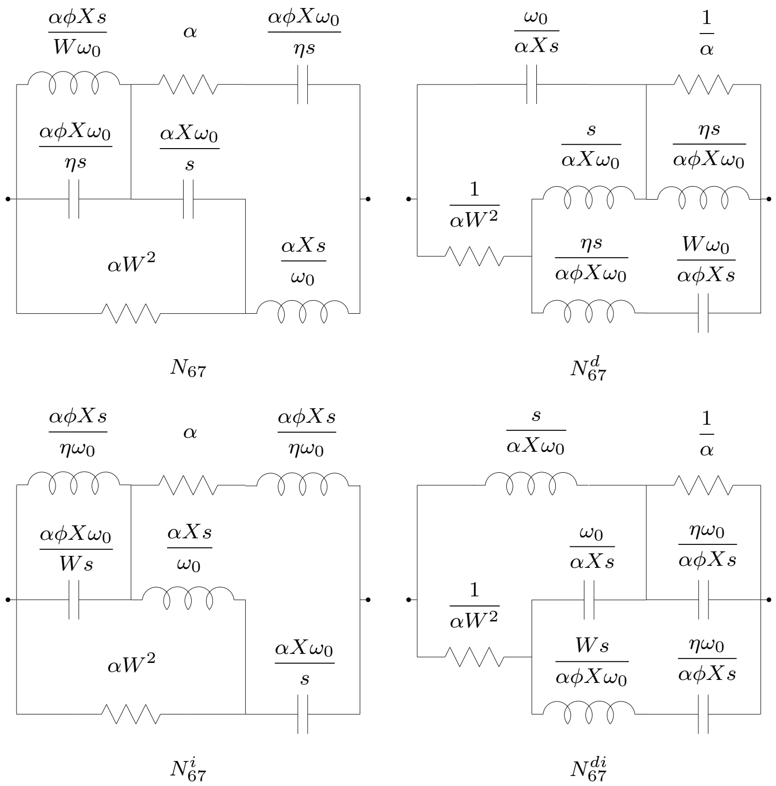


Figure 48: Quartet $\mathcal{Q}_{67}$, $\phi = 1 - W$, $\eta = 2W - 1$, $X = W\sqrt{(2W-1)/(1-W)}$, $1/2 < W < 1$, $\alpha > 0$.

We define the network class $\mathcal{N}_{70}$ (resp. $\mathcal{N}_{71}$) as the set of all such networks N_{70} (resp. N_{71}) for $\alpha > 0$, $1/2 < W < 1$, and $X > W\sqrt{2W-1}/(1-W)$ (resp. $0 < X < W(1-W)/\sqrt{2W-1}$). We denote the corresponding quartets $\mathcal{Q}_{70}$ and $\mathcal{Q}_{71}$. We remark that any network from one of the quartets $\mathcal{Q}_{70}$ and $\mathcal{Q}_{71}$ may be obtained from a network in the other quartet by the application of a star-delta transformation.

Finally, we introduce network N_{72} (resp. N_{73}) in Fig. 53 (resp. Fig. 54). We remark that the impedances of the networks N_{72} and N_{73} are equal to $H_p(s)$ in (145) when $0 < W < 1$ and $\alpha, X > 0$. Moreover, these are the networks obtained by applying the Reza-Pantell-Fialkow-Gerst simplification and our alternative simplification (described in Section 3.1.5) to the function $H_p(s)$ in (145), respectively, for the case when $\Im(H_p(j\omega_0)) > 0$. We define the network class $\mathcal{N}_{72}$ (resp. $\mathcal{N}_{73}$) as the set of all such networks N_{72} (resp. N_{73}) for $\alpha, X > 0$ and $0 < W < 1$. It follows that the two network classes $\mathcal{N}_{72}$ and $\mathcal{N}_{72}^d$ (or, equivalently, $\mathcal{N}_{72}^i$ and $\mathcal{N}_{72}^{di}$; $\mathcal{N}_{73}$ and $\mathcal{N}_{73}^d$: $\mathcal{N}_{73}^i$ and $\mathcal{N}_{73}^{di}$) collectively realise all of the biquadratic minimum functions. We denote the corresponding quartets $\mathcal{Q}_{72}$ and $\mathcal{Q}_{73}$.

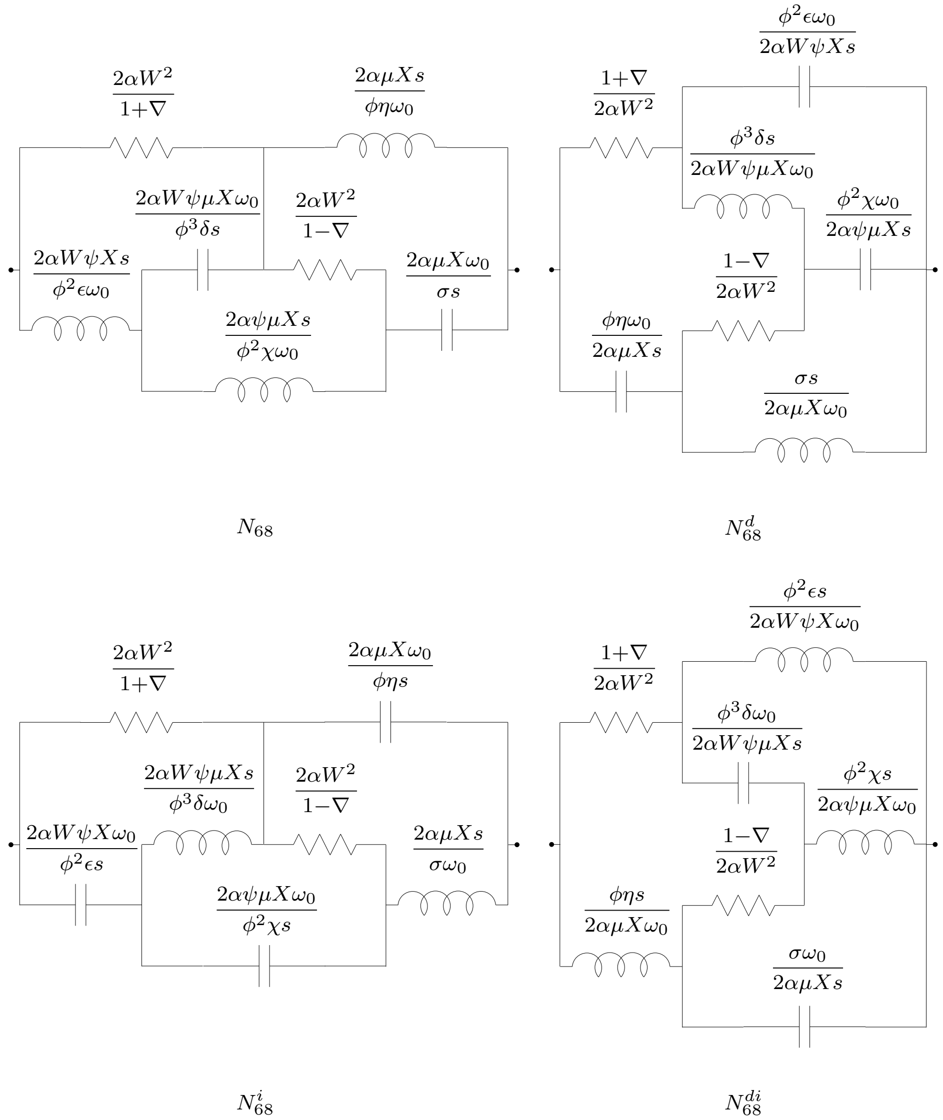


Figure 49: Quartet $\mathcal{Q}_{68}$, $\phi = 1 - W$, $\nabla = \sqrt{(1-2W)(1+2W)}$, $\psi = (1-W)(1-2W)(1+2W) + W^3$, $\mu = 2(1-2W) + W^3(1+2W)$, $\eta = 3(1-W) - 2W^2 + \nabla(1-W)$, $\chi = \nabla(1 - 3W + 4W^2 - 4W^3 + 2W^4) - (1 - 3W - 2W^2 + 10W^3 - 8W^5)$, $\delta = 2 - W - 10W^2 + 9W^3 + 2W^4 + \nabla(2 - 3W - 2W^2 + 3W^3)$, $\sigma = \nabla(1 - 2W + W^2) - (1 - 2W - W^2 + 4W^4)$, $\epsilon = ((1-2W)(1+2W) + W) + \nabla(1-W)$, $0 < W < 1/2$, $X = \frac{W\sqrt{(1-2W)(1-W)((1-2W)(1-2W^2)+2W^3(2-3W)-\sqrt{(1-2W)(1+2W)}(1-2W)(1-2W^2))}}{\sqrt{2}((1-W)(1-2W)(1+2W)+W^3)}$, $\alpha > 0$.

Theorem 3.6.6. *Let N be a transformerless network containing at most five reactive elements and at most seven elements in total, and let the impedance of N be a biquadratic minimum function (with minimum frequency ω_0). Then N is from one of the eleven quartets $\mathcal{Q}_{28}$, $\mathcal{Q}_{29}$, $\mathcal{Q}_{65}$, $\mathcal{Q}_{66}$, $\mathcal{Q}_{67}$, $\mathcal{Q}_{68}$, $\mathcal{Q}_{69}$, $\mathcal{Q}_{70}$, $\mathcal{Q}_{71}$, $\mathcal{Q}_{72}$ and $\mathcal{Q}_{73}$. Of these, the quartets $\mathcal{Q}_{28}$, $\mathcal{Q}_{29}$, and $\mathcal{Q}_{65}$ to $\mathcal{Q}_{69}$, realise sets of biquadratic minimum*

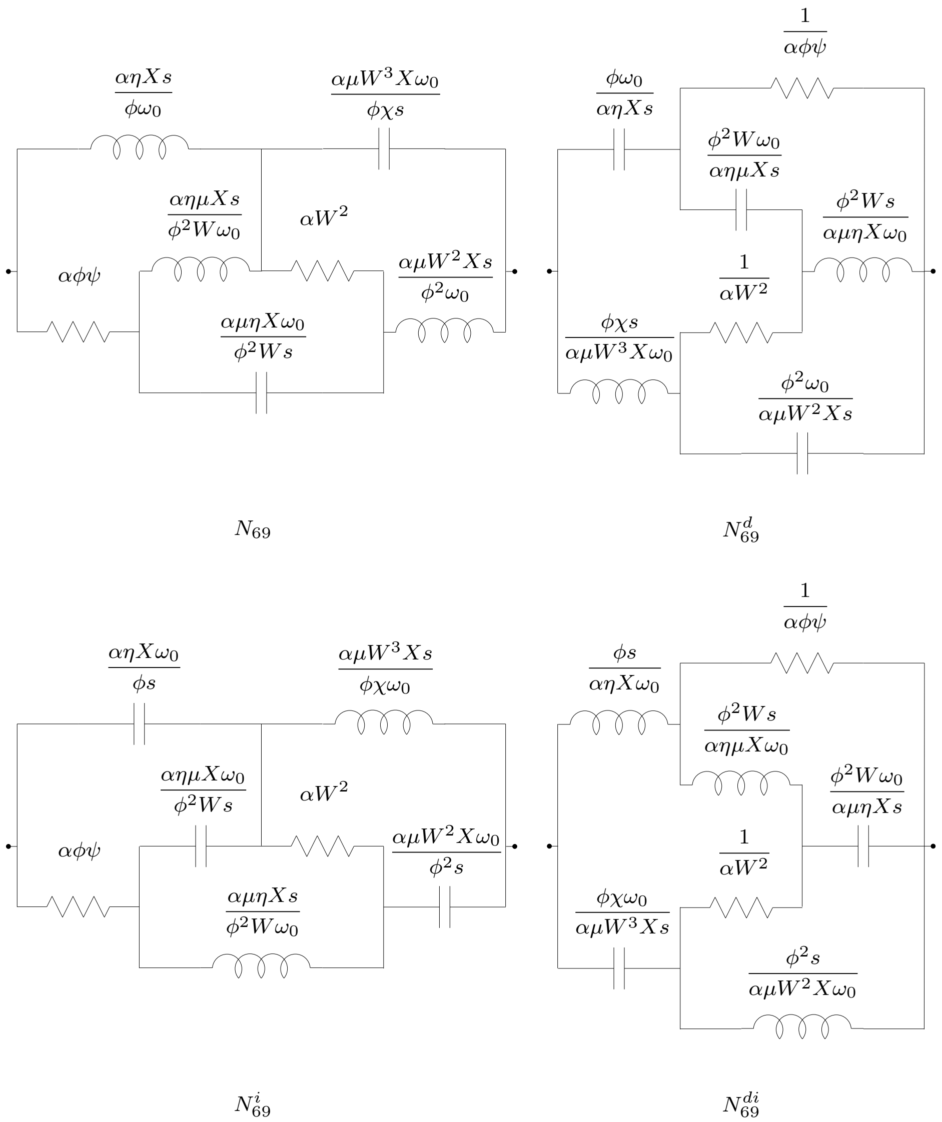


Figure 50: Quartet $\mathcal{Q}_{69}$, $\phi = 1 - W$, $\psi = 1 + W$, $\eta = 1 - W + W^3$, $\mu = 1 - 2W + 2W^2$, $\chi = 1 - W + W^2$, $X = (1 + W)(1 - W)^2\sqrt{1 - W + W^2}/(1 - W + W^3)$, $0 < W < 1$, $\alpha > 0$.

functions of codimension one in the parameters α, X, ω_0, and W of the parametrisation $H_p(s)$ in (145). Moreover, only the quartets $\mathcal{Q}_{72}$ and $\mathcal{Q}_{73}$ can realise all of the biquadratic minimum functions.

Proof. We denote the impedance of N by $H(s)$. From Theorem 3.6.5, N must be from one of the quartets $\mathcal{Q}_3$, $\mathcal{Q}_{22}$ to $\mathcal{Q}_{27}$, or $\mathcal{Q}_{30}$ to $\mathcal{Q}_{64}$. Since $H(s)$ is biquadratic then N cannot be from either of the quartets $\mathcal{Q}_{24}$, $\mathcal{Q}_{25}$, or $\mathcal{Q}_{30}$ by Corollary 3.5.12. From the proof of Theorem 3.5.17, if N belongs to the quartet $\mathcal{Q}_{22}$ then N must also belong

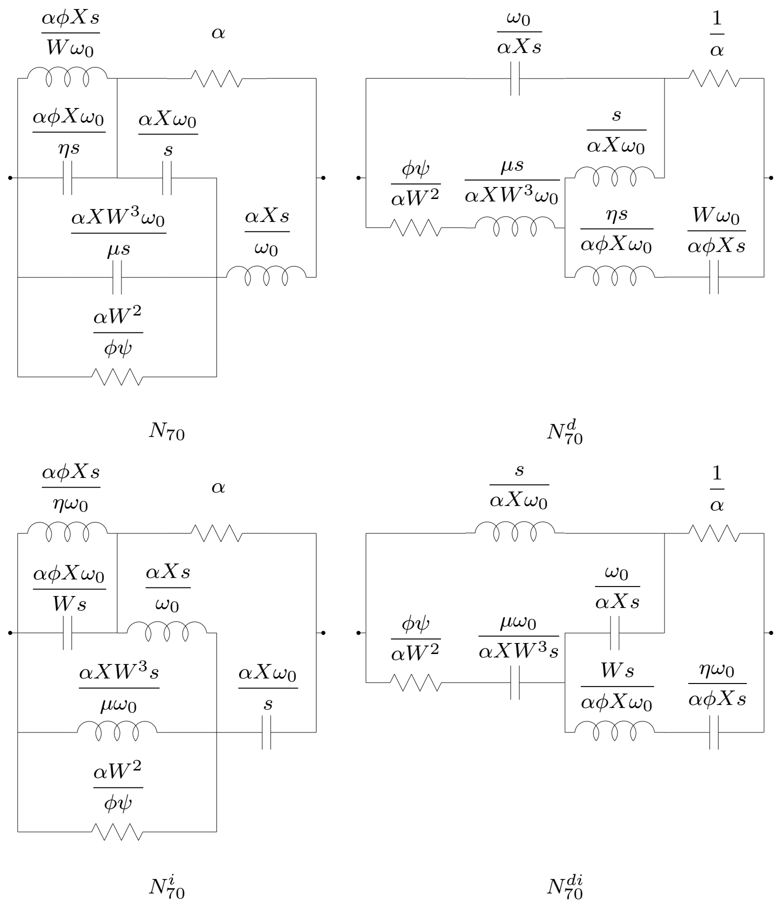


Figure 51: Quartet $\mathcal{Q}_{70}$, $\phi = 1 - W$, $\psi = 1 + W$, $\eta = 2W - 1$, $\mu = X^2\phi^2 - W^2\eta$, $X > W\sqrt{2W-1}/(1-W)$, $1/2 < W < 1$, $\alpha > 0$.

to the quartet $\mathcal{Q}_{28}$ since $H(s)$ is biquadratic. Similarly, from the proof of Theorem 3.5.18, if N belongs to the quartet $\mathcal{Q}_{23}$ (resp. $\mathcal{Q}_{26}$) then N must also belong to the quartet $\mathcal{Q}_{29}$ (resp. $\mathcal{Q}_{65}$) since $H(s)$ is biquadratic, and no network in $\mathcal{Q}_{27}$ realises a biquadratic minimum function. Furthermore, from Theorem 3.3.5, no network in $\mathcal{Q}_3$ realises a biquadratic minimum function. It remains to determine those networks in the quartets $\mathcal{Q}_{31}$ to $\mathcal{Q}_{64}$ which realise a biquadratic minimum function. As in the proof of Theorem 3.5.17, we will find those networks in the class $\mathcal{N}_k$ whose impedance $H_k(s)$ takes the form of $H_p(s)$ in (145), for $k = 31, 32, \ldots, 64$. The remaining networks in the quartets $\mathcal{Q}_{31}$ to $\mathcal{Q}_{64}$ which realise a biquadratic minimum function will then be the duals and frequency inverted networks to those identified here.

From Corollary 3.5.11, we may obtain expressions for $H_k(0)$ and $H_k(\infty)$ as functions of the impedances of the resistors in a network from the class $\mathcal{N}_k$ ($k = 31, 32, \ldots, 64$). Moreover, we may obtain an expression for $\Im(H_k(j\omega_0))$ as a function of reactive element

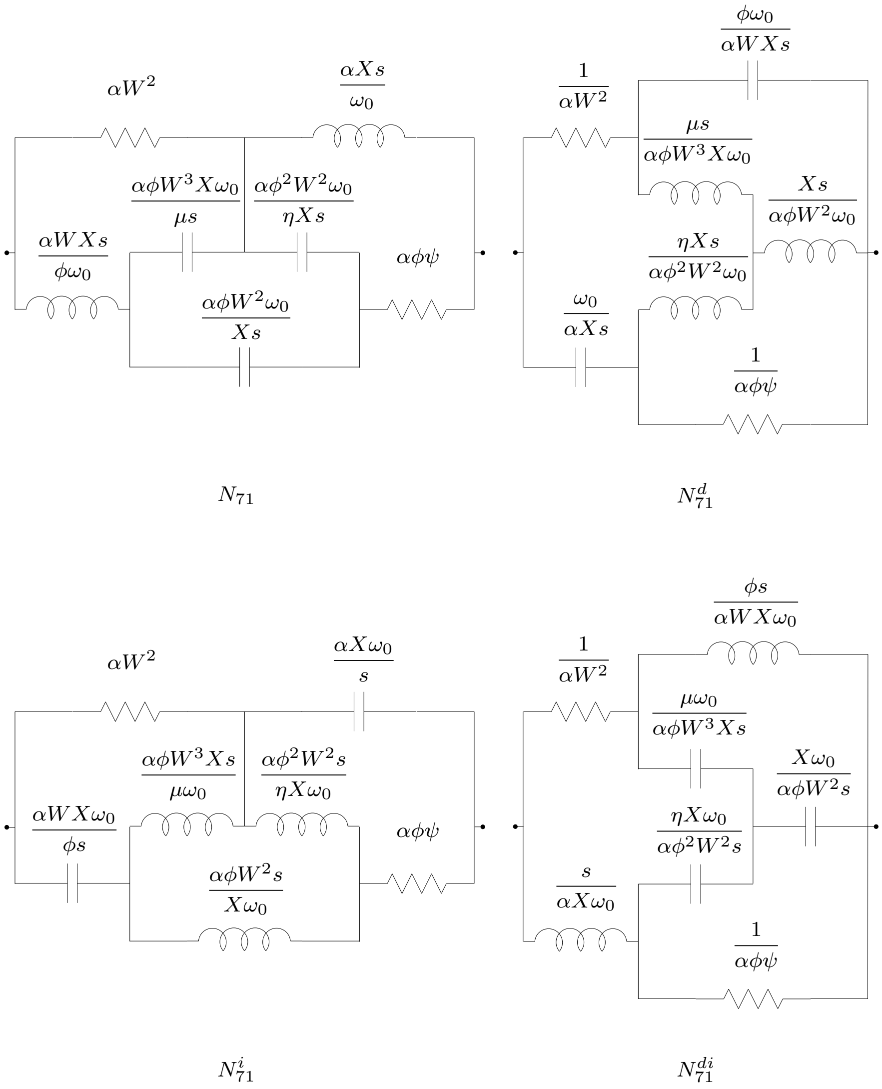


Figure 52: Quartet $\mathcal{Q}_{71}$, $\phi = 1 - W$, $\psi = 1 + W$, $\eta = 2W - 1$, $\mu = W^2\phi^2 - X^2\eta$, $0 < X < W(1-W)/\sqrt{2W-1}$, $1/2 < W < 1$, $\alpha > 0$.

impedances by using Lemma 3.5.10. These functions are shown in Table 5 on p. 212. In that table, we denote the impedance of the ith resistor in a network N_k from the class $\mathcal{N}_k$ by R_i, the capacitance of the ith capacitor in N_k by C_i, and the inductance of the ith inductor in N_k by L_i, in the order in which these elements appear in Tables 2, 3 and 4. As explained in Section 3.2.2, since $H(s)$ is a biquadratic minimum function (with minimum frequency ω_0), then $H(0) \neq H(\infty)$, and $H(0) < H(\infty)$ if and only if $\Im(H(j\omega_0)) > 0$. It is then evident from Table 5 that there are no networks from the quartets $\mathcal{Q}_{31}$, $\mathcal{Q}_{32}$, $\mathcal{Q}_{37}$, $\mathcal{Q}_{40}$, $\mathcal{Q}_{47}$, or $\mathcal{Q}_{55}$ which realise a biquadratic minimum function, and that any network N_k which realises a biquadratic minimum function has impedance $H_k(s)$ which satisfies $H_k(0) < H_k(\infty)$ and $\Im(H_k(j\omega_0)) > 0$

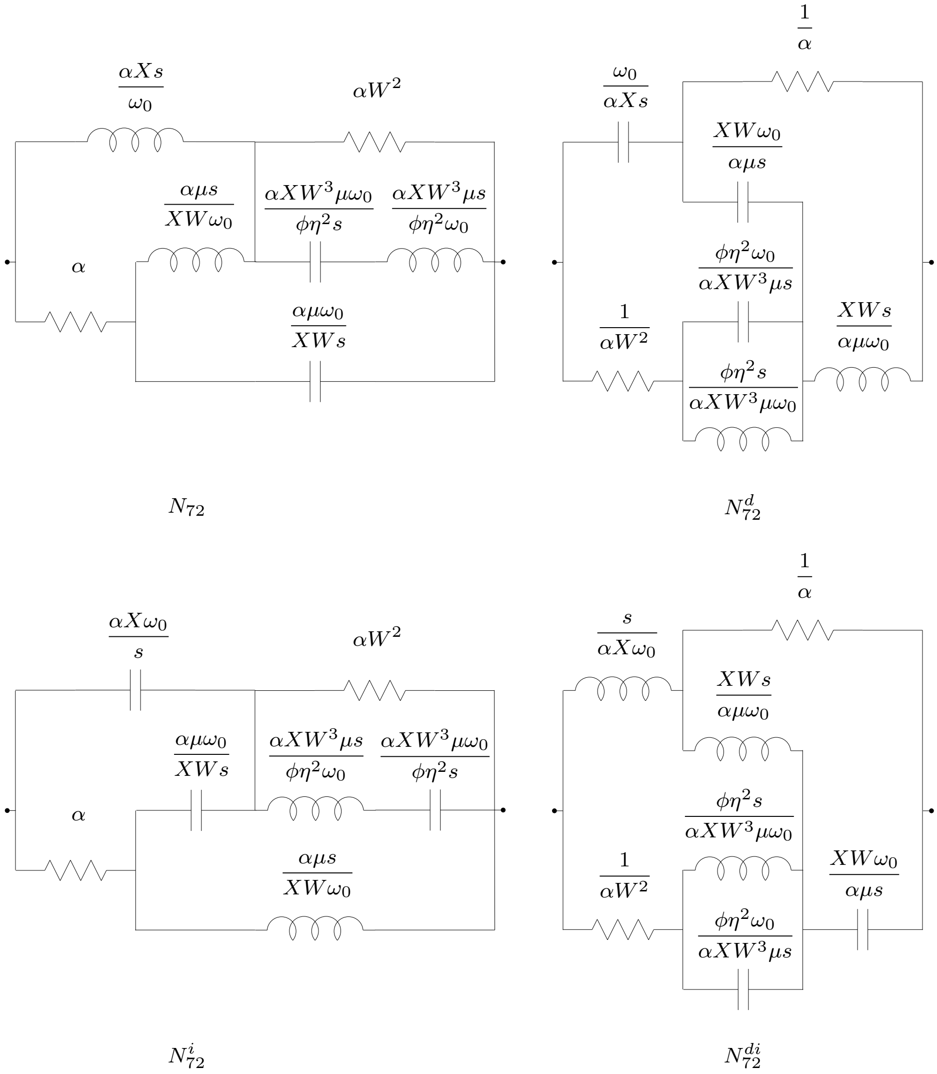


Figure 53: Quartet $\mathcal{Q}_{72}$, $\phi = 1 - W$, $\eta = W^2 + X^2$, $\mu = W^2 + X^2\phi$, $X > 0$, $0 < W < 1$, $\alpha > 0$.

$(k = 33, \ldots, 36, 38, 39, 41, \ldots, 46, 48, \ldots, 54, 56, \ldots, 64)$. Consequently, we examine the conditions under which $H_k(s)$ takes the form of $H_p(s)$ in (145) with $\alpha = H_k(\infty) > 0$, $X = \Im(H_k(j\omega_0))/H_k(\infty) > 0$, and $0 < W = \sqrt{H_k(0)/H_k(\infty)} < 1$. Accordingly, for each element in N_k, we let $R_i = \alpha/g_i$, $C_i = c_i/(\alpha X \omega_0)$ and $L_i = \alpha X/(x_i \omega_0)$, and, since $\alpha, X, R_i, C_i, L_i > 0$, we require $g_i > 0$, $c_i > 0$ and $x_i > 0$.

From equation (38), we find in each case that $H_k(s) = \alpha X n_k(s)/d_k(s)$, where $n_k(s)$ and $d_k(s)$ are both polynomials in the parameters c_i, x_i, g_i, X, and s. For $H_k(s)$ to be biquadratic, we then require $R_l(n_k(s), d_k(s)) = 0$ for $l = 0, 1, \ldots, (\deg(n_k(s)) - 3)$. Here, we adopt the notation for the Sylvester determinants introduced in Subsection 3.2.3, and in each case the determinants $R_l(n_k(s), d_k(s))$ are polynomials in the

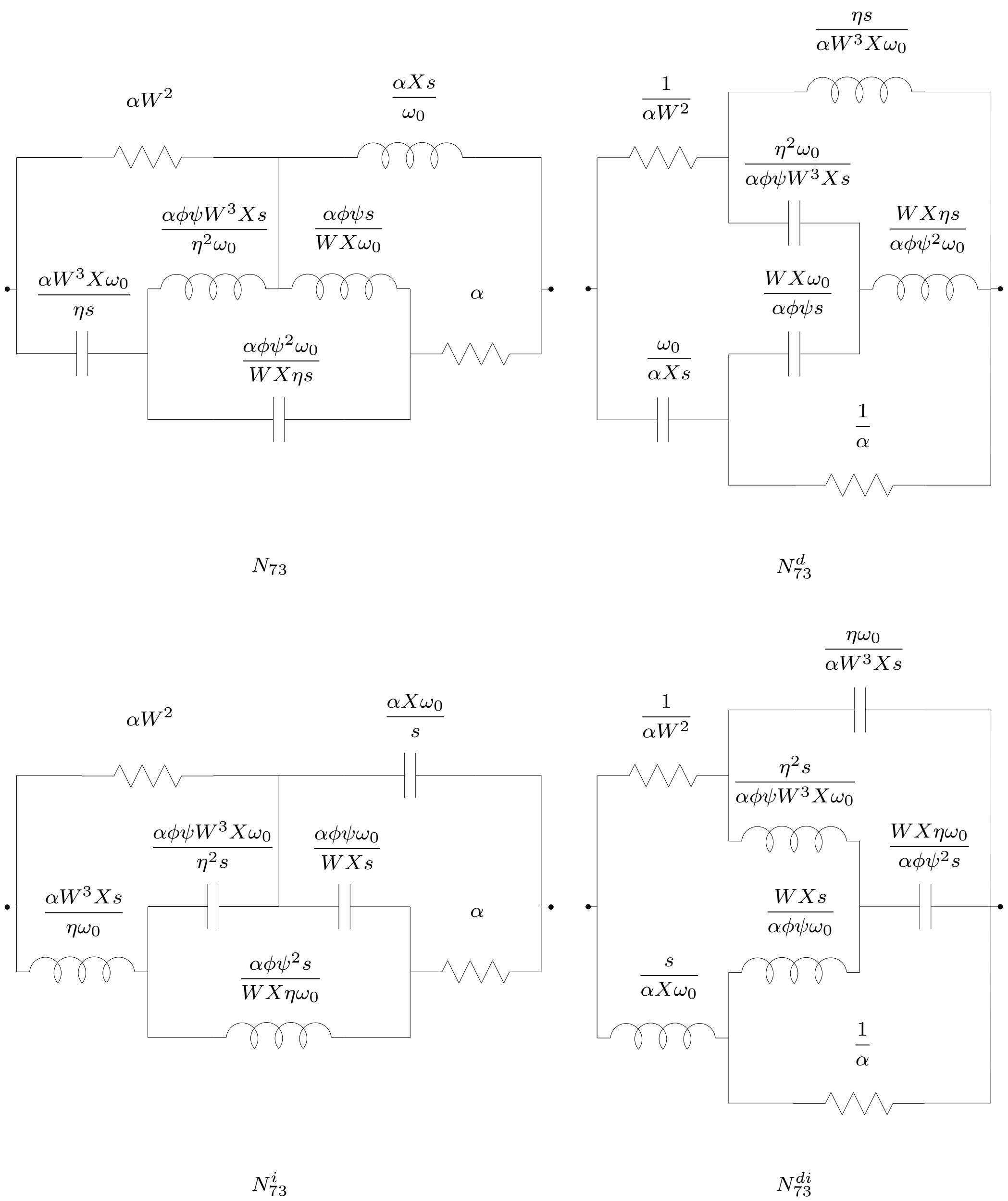


Figure 54: Quartet $\mathcal{Q}_{73}$, $\phi = 1 - W$, $\psi = W^2 + X^2$, $\eta = W^2\phi + X^2$, $X > 0$, $0 < W < 1$, $\alpha > 0$.

parameters c_i, x_i, g_i, ω_0 and X. These polynomials are of high degree and contain many terms, and determining whether these polynomials can all be equal to zero given the constraints $c_i, x_i, g_i, X > 0$ is complicated. In certain cases, it is preferable to equate $H_k(s)$ with $H_p(s)$ in equation (145) directly. In other words, we let $f(s)(s^2 + \omega_0(1-W)X/Ws + \omega_0^2 W) = n_k(s)$ and $f(s)(Xs^2 + \omega_0(1-W)s + X\omega_0^2/W) = d_k(s)$, for some polynomial $f(s)$ whose degree is equal to $\deg(n_k(s)) - 2$. The algebra is further simplified if we make the substitutions $z = s/(\omega_0 X)$ and $X^2 = F$. Then, $H_p(s)$ in (145) takes the form $\alpha n_p(z)/d_p(z)$ where

$$n_p(z) = FWz^2 + F(1-W)z + W^2, \tag{171}$$

$$\text{and} \quad d_p(z) = FWz^2 + W(1-W)z + 1. \tag{172}$$

Moreover, in each case, we find that both $n_k(z)$ and $d_k(z)$ are polynomials in c_i, x_i, g_i, and F. Then, for $H_k(s)$ to be biquadratic, we require the existence of a polynomial $f(z)$ such that

$$n_k(z) = n_p(z)f(z), \tag{173}$$

$$\text{and} \quad d_k(z) = d_p(z)f(z). \tag{174}$$

By equating coefficients of z in these two equations we arrive at a series of conditions which must all be simultaneously satisfied in order for $H_k(s)$ to be biquadratic.

The computation detailed in this proof was facilitated by use of the symbolic algebra package *Maple 17*.

Consider first the impedance $H_{33}(s)$ of a network from the class $\mathcal{N}_{33}$. For $H_{33}(j\omega_0) = \alpha X j$, we require $c_1 = x_3 = 1$, and $x_2 = 1 + c_2$. In this case, we find

$$\begin{aligned} R_0(n_{33}(s), d_{33}(s)) &= X^6\omega_0^{25}c_2g_1^4x_1(1+c_2)(x_1^2(1+g_1g_2X^2)^2 + g_1^2X^2)^2\gamma_1^2, \\ R_1(n_{33}(s), d_{33}(s)) &= X^6\omega_0^{16}g_1^4c_2\gamma_2, \\ \text{and} \quad R_2(n_{33}(s), d_{33}(s)) &= -X^6\omega_0^{9}g_1^4c_2\gamma_3, \end{aligned}$$

where γ_1, γ_2, and γ_3 are all polynomials in g_1, g_2, x_1, c_2, and X. For $H_{33}(s)$ to be biquadratic we thus require $\gamma_1 = \gamma_2 = \gamma_3 = 0$, so, in particular, $R_0(\gamma_1(x_1), \gamma_2(x_1)) = R_0(\gamma_1(x_1), \gamma_3(x_1)) = 0$. Here,

$$\begin{aligned} R_0(\gamma_1(x_1), \gamma_2(x_1)) = &-X^2g_1^4g_2^5(1+c_2)(c_2(1+c_2) + X^2g_2^2) \\ &\times (g_1^2g_2^4(1+c_2)^2X^6 + (g_2(g_2+g_1c_2(1+c_2))X^2 + c_2(1+c_2))^2)^2\mu_1, \end{aligned}$$

and $R_0(\gamma_1(x_1), \gamma_3(x_1)) = g_1^2g_2^3\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}[g_1, g_2, c_2, X]$. We thus require

$$R_0(\mu_1(g_1), \mu_2(g_1)) = -X^{10}g_2^7(c_2(1+c_2) + X^2g_2^2)^3\chi^2 = 0,$$

and so $\chi = 0$, where $\chi \in \mathbb{R}[g_2, c_2, X]$. However, χ cannot be zero, since

$$4\chi(1+c_2)^2 = (2(1+c_2)^2g_2^2X^2 + c_2(2c_2^3 + 4c_2^2 - c_2 - 4))^2 + c_2^3(2c_2+3)^2(4c_2^2 + 11c_2 + 8).$$

It follows that there are no networks from $\mathcal{Q}_{33}$ which realise a biquadratic minimum function.

Consider next the impedance $H_{34}(s)$ of a network from $\mathcal{N}_{34}$. For $H_{34}(j\omega_0) = \alpha X j$, we require $c_2 = x_2 = 1$ and $x_1 = 1 + c_3$. In this case, we obtain

$$R_0(n_{34}(s), d_{34}(s)) = \omega_0^{16}(1+c_3)(c_1(1+c_3)+c_3)((1+g_1g_2X^2)^2 + c_1^2g_2^2X^2)^2\gamma_1^2,$$

and $$R_1(n_{34}(s), d_{34}(s)) = -\omega_0^9(c_1(1+c_3)+c_3)\gamma_2,$$

where $\gamma_1, \gamma_2 \in \mathbb{R}[g_1, g_2, c_1, c_3, X]$. Since n_{34} and d_{34} are both of degree four in s, then $H_{34}(s)$ is biquadratic if and only if $\gamma_1 = \gamma_2 = 0$. Here,

$$\gamma_1 = X^2g_2(g_1(1+c_3) - g_2) - c_3(1+c_3) - c_1(1+c_3)^2.$$

In particular, we require $R_0(\gamma_1(c_1), \gamma_2(c_1)) = 0$. Here,

$$R_0(\gamma_1(c_1), \gamma_2(c_1)) = X^6g_2^5(1+c_3)^2((X^2(1+c_3)g_1g_2+1+c_3-X^2g_2^2)^2+X^2g_2^2(1+c_3)^2)^2\mu,$$

where $g_1\mu = (g_1(1+c_3) - g_2)^2 - g_2(g_1+g_2)$. Moreover, for $H_{34}(\infty) = H_p(\infty)$ and $H_{34}(0) = H_p(0)$, we require $g_2 = 1$ and $g_1 + g_2 = 1/W^2$. Since $g_1, g_2, c_1, c_3, X > 0$, we find that $H_{34}(s)$ is biquadratic if and only if $g_1 = (1-W^2)/W^2$, $g_2 = 1$, $c_3 = (2W-1)/(1-W)$, $c_1 = (X^2(1-W)^2 - W^2(2W-1))/W^3$, $1/2 < W < 1$, and $X > W\sqrt{2W-1}/(1-W)$. We thus conclude that if N is from $\mathcal{Q}_{34}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{70}$ in Fig. 51.

Consider now the impedance $H_{35}(s)$ of a network from the class $\mathcal{N}_{35}$. In this case, for $H_{35}(j\omega_0) = H_p(j\omega_0)$, we require $c_2 = x_2 = 1$ and $x_1 = 1 + c_3$. Moreover, $H(\infty) = H_p(\infty)$ and $H(0) = H_p(0)$ imply $g_2 = 1$ and $g_1 = 1/W^2$. We define $n_p(z)$, $d_p(z)$, $n_{35}(z)$ and $d_{35}(z)$ as described earlier in the proof of this theorem, and (for the case $k = 35$) we equate coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174). Since $c_1, c_3, F, W > 0$, we then obtain

$$Wf(z) = (Fz+W)(c_1c_3W^2z^2 + c_1(1+c_3)z + (1+c_3)).$$

Then, equating coefficients of z^2 in (174), and given $W < 1$, we require $c_3 = (2W-1)/(1-W)$, which implies $W > 1/2$. Equating coefficients of z in (174) and z^2 in (173), we then require $c_1 = (2W-1)/(1-W)$ and $F = W^2(2W-1)/(1-W)$. It may then be verified that $H_{35}(s)$ takes the form of $H_p(s)$ in (145) with $1/2 < W < 1$ and $X = W\sqrt{2W-1}/\sqrt{1-W}$. We thus conclude that if N is from $\mathcal{Q}_{35}$ and has a biquadratic impedance, then N must also belong to the quartet $\mathcal{Q}_{67}$ in Fig. 48.

Consider now the impedance $H_{36}(s)$ of a network from $\mathcal{N}_{36}$. In this case, we require $c_1 = x_1/(1+x_1)$, $c_2 = x_2 - 1$, and $x_3 = 1$ for $H_{36}(j\omega_0) = H_p(j\omega_0)$, and hence $c_2 > 0$ implies $x_2 > 1$. Moreover, we have $g_1 = (1-W^2)/W^2$ and $g_2 = 1$ for $H_{36}(0) = H_p(0)$ and $H_{36}(\infty) = H_p(\infty)$. Equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$Wf(z) = F^2(1-W^2)(x_2-1)z^3 + F(x_2-1)(x_1W + 1 + W(1-W))z^2 \\ + F(x_1+1)(x_2(1-W)+W)z + Wx_2(x_1+1).$$

From equating coefficients of z^4 in (173), and given $0 < W < 1$ and $F > 0$, we require $x_2 = (F+W^2)/(F(1-W)+W^2)$, and it may be verified that $x_2 > 1$. Then, equating coefficients of z^3 in (174), we require either $F = 0$ or $W = 0$, which contradicts the requirements $0 < W < 1$ and $F > 0$. We conclude that there are no networks in $\mathcal{Q}_{36}$ which realise a biquadratic minimum function.

Consider next the impedance $H_{38}(s)$ of a network from $\mathcal{N}_{38}$. In this case, we have $x_1 = x_2 = c_3 = 1$, $g_1 = 1/W^2$, and $g_2 = 1$ for $H_{38}(j\omega_0) = H_p(j\omega_0)$, $H_{38}(0) = H_p(0)$, and $H_{38}(\infty) = H_p(\infty)$. As in previous cases, we equate coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), and we obtain

$$Wf(z) = (Fz + W)(c_1c_2W^2z^2 + c_2z + 1).$$

Equating coefficients of z^2 in (174), and given $W, F, c_2 > 0$, we require $W = 1/2$. Then, equating coefficients of z in (174) and z^2 in (173), we obtain $c_1 = c_2 = 4F$. It may then be verified that $H_{38}(s)$ takes the form of $H_p(s)$ in (145) with $W = 1/2$. We thus conclude that if N is from $\mathcal{Q}_{38}$ and has a biquadratic impedance, then N must also belong to the quartet $\mathcal{Q}_{66}$ in Fig. 47.

Now, consider the impedance $H_{39}(s)$ of a network from $\mathcal{N}_{39}$. In this case, we require $c_2 = x_2 = x_3 = 1$ for $H_{39}(j\omega_0) = H_p(j\omega_0)$. Furthermore, for $H_{39}(s)$ to be biquadratic, we require $R_0(n_{39}(s), d_{39}(s)) = R_1(n_{39}(s), d_{39}(s)) = R_2(n_{39}(s), d_{39}(s)) = 0$. Here,

$$R_0(n_{39}(s), d_{39}(s)) = X^2\omega_0^{25}c_1^3x_1^3((c_1-x_1)^2(1+g_1g_2X^2)^2 + X^2g_2^2c_1^2x_1^2)^2\gamma_1^2,$$
$$R_1(n_{39}(s), d_{39}(s)) = X^2\omega_0^{16}c_1^3x_1\gamma_2,$$
and
$$R_2(n_{39}(s), d_{39}(s)) = X^2\omega_0^{9}c_1^3\gamma_3,$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}[g_1, g_2, c_1, x_1, X]$. We thus require $\gamma_1 = 0$, $R_0(\gamma_1(c_1), \gamma_2(c_1)) = 0$, and $R_0(\gamma_1(c_1), \gamma_3(c_1)) = 0$. Here,

$$\gamma_1 = (g_1 - g_2(1 + x_1))c_1 + X^2 g_2^2 x_1(g_1 - g_2),$$
$$R_0(\gamma_1(c_1), \gamma_2(c_1)) = X^6 g_2^6 x_1^5 (g_1 - g_2(1 + x_1))\mu_1^2 \mu_2,$$
$$\text{and} \quad R_0(\gamma_1(c_1), \gamma_3(c_1)) = -X^4 g_2^4 x_1^3 \mu_3,$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}[g_1, g_2, x_1, X]$. Since

$$\mu_1 = (X^2 g_2(g_1 - g_2)(g_1 - g_2 x_1) + g_1 - g_2(1 + x_1))^2 + g_2^2 X^2 (X^2 g_1 g_2(g_1 - g_2) + g_1 - g_2(1 + x_1))^2,$$

which cannot be zero for $g_2, x_1, X > 0$, then we require $\mu_2 = \mu_3 = 0$. In particular, we require

$$R_0(\mu_2(x_1), \mu_3(x_1)) = -X^2 g_2^7 (g_1 + g_2)(g_1 - g_2)^5 \times (((g_1 - 3g_2)g_1 g_2 X^2)^2 + 2g_2(g_1 + g_2)(g_1 - g_2)^2 X^2 + (g_1 + g_2)^2(1 + 2g_2^2 X^2))^2 = 0.$$

It may be verified that there are no solutions which satisfy both the above equation and the equation $\gamma_1 = 0$ for $g_1, g_2, x_1, c_1, X > 0$. We conclude that there are no networks in $\mathcal{Q}_{39}$ which realise a biquadratic minimum function.

Consider next the impedance $H_{41}(s)$ of a network from $\mathcal{N}_{41}$. In this case, for $H_{41}(j\omega_0) = H_p(j\omega_0)$, we require $c_2 = x_2 = x_3 = 1$. For $H_{41}(s)$ to be biquadratic, we require $R_0(n_{41}(s), d_{41}(s)) = R_1(n_{41}(s), d_{41}(s)) = R_2(n_{41}(s), d_{41}(s)) = 0$. Here,

$$R_0(n_{41}(s), d_{41}(s)) = X^6 \omega_0^{25} c_1 g_1^4 x_1 (g_1^2 g_2^2 (c_1 - x_1)^2 X^4 + x_1^2 + X^2(g_1^2 + 2g_1 g_2 x_1^2 + g_2^2 x_1^2 c_1^2))^2 \gamma_1^2,$$
$$R_1(n_{41}(s), d_{41}(s)) = X^6 \omega_0^{16} c_1 g_1^4 \gamma_2,$$

and

$$R_2(n_{41}(s), d_{41}(s)) = -X^6 \omega_0^9 c_1 g_1^4 \gamma_3,$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}[g_1, g_2, c_1, x_1, X]$. In this case, we require $\gamma_1 = R_0(\gamma_1(x_1), \gamma_2(x_1)) = R_0(\gamma_1(x_1), \gamma_3(x_1)) = 0$, where

$$\gamma_1 = (X^2 g_1 g_2 - (c_1 + g_2^2 X^2))g_2 x_1 + (c_1 + g_2^2 X^2)g_1,$$
$$R_0(\gamma_1(x_1), \gamma_2(x_1)) = -X^2 g_1^4 g_2^5 (c_1 + g_2^2 X^2) \times (X^2 g_2^2 (X^2 g_1 g_2 (1 + c_1) - c_1(c_1 + g_2^2 X^2))^2 + (c_1 + X^2 g_2(g_2 + g_1 c_1))^2)^2 \mu_1,$$

and

$$R_0(\gamma_1(x_1), \gamma_3(x_1)) = g_1^2 g_2^3 \mu_2,$$

where $\mu_1, \mu_2 \in \mathbb{R}[g_1, g_2, c_1, X]$. In particular, we require

$$R_0(\mu_1(g_1), \mu_2(g_1)) = -X^{10}g_2^7(c_1 + g_2^2X^2)^3((X^2(2c_1{+}1)g_2^2{-}c_1(c_1{+}2))^2{+}9X^2c_1^2g_2^2) = 0,$$

which is not possible since $c_1, g_2, X > 0$. We conclude that there are no networks in $\mathcal{Q}_{41}$ which realise a biquadratic minimum function.

Now, consider the impedance $H_{42}(s)$ of a network from $\mathcal{N}_{42}$. In this case, we require $x_2 = x_3 = 1$ and $c_1 = (x_1c_2{+}c_2{-}x_1)/(c_2{-}x_1)$ for $H_{42}(j\omega_0) = H_p(j\omega_0)$, so in particular we require $c_2 - x_1 \neq 0$ and $x_1c_2 + c_2 - x_1 \neq 0$, and both $c_2 - x_1$ and $x_1c_2 + c_2 - x_1$ must have the same sign. In this case, we obtain

$$\begin{aligned}
R_0(n_{42}(s), d_{42}(s)) &= X^2c_2^5x_1^3\omega_0^{25}(x_1c_2{+}c_2{-}x_1)(c_2{-}x_1)^5(1{+}X^2g_1g_2)^4 \\
&\quad \times (c_2(x_1c_2{+}c_2{-}x_1){+}X^2g_1g_2x_1(c_2{-}x_1))^4\gamma_1^2, \\
R_1(n_{42}(s), d_{42}(s)) &= X^2c_2^5x_1\omega_0^{16}(x_1c_2{+}c_2{-}x_1)(c_2{-}x_1)^3(1{+}X^2g_1g_2)^2 \\
&\quad \times (c_2(x_1c_2{+}c_2{-}x_1){+}X^2g_1g_2x_1(c_2{-}x_1))^2\gamma_2, \\
\text{and } R_2(n_{42}(s), d_{42}(s)) &= -X^2c_2^5\omega_0^9(x_1c_2{+}c_2{-}x_1)(c_2{-}x_1)\gamma_3,
\end{aligned}$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}[g_1, g_2, x_1, c_2, X]$. Since $c_2 - x_1 \neq 0$ and $x_1c_2 + c_2 - x_1 \neq 0$, both $c_2 - x_1$ and $x_1c_2 + c_2 - x_1$ have the same sign, and $g_1, g_2, x_1, c_2, X > 0$, we require $\gamma_1 = \gamma_2 = \gamma_3 = 0$. In particular, we require $R_0(\gamma_1(g_1), \gamma_2(g_1)) = 2X^6g_2^8x_1^2(c_2{-}x_1)^2 = 0$, which is not possible since $c_2 - x_1 \neq 0$ and $g_2, x_1, X > 0$. We conclude that there are no networks in $\mathcal{Q}_{42}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{43}(s)$ of a network from $\mathcal{N}_{43}$. In this case, we have $c_1 = x_3 = 1$ and $x_1 = (c_2 - x_2(1 + c_2))/(c_2 - x_2)$ for $H_{43}(j\omega_0) = H_p(j\omega_0)$. Moreover, for $H_{43}(j\omega_0) = H_p(j\omega_0)$ and $H_{43}(j\omega_0) = H_p(j\omega_0)$, we require

$$g_1 = \frac{1+\nabla}{2W^2}, \tag{175}$$

$$\text{and} \quad g_2 = \frac{1-\nabla}{2W^2}, \tag{176}$$

where $\nabla \in \mathbb{R}$ is one of the solutions to the equation

$$\nabla^2 - (1-2W)(1+2W) = 0. \tag{177}$$

We thus require $W \leq 1/2$. As in previous cases, we equate coefficients of z^0, z^1, and

z^5 in equation (173), and coefficients of z^4 in equation (174), and we obtain

$$f(z) = 4WF^2c_2(c_2 - x_2)z^3 + 2WFc_2(x_2^2(\nabla - 1) + 2W(c_2 - x_2))z^2 + 2Fx_2(x_2c_2(1 - \nabla) + 2W(c_2 - x_2(1 + c_2)))z + 4W^2x_2^2(c_2 - x_2(1 + c_2)).$$

Equating coefficients of z^4 in (173), we find

$$x_2(2W(W(1 - 2W^2 - \nabla)x_2 + 2(1 - 2W)(W^2 + F)) + F(\nabla^2 - (1 - 2W)(1 + 2W))) = c_2(4W(1 - 2W)(W^2 + F) + F(\nabla^2 - (1 - 2W)(1 + 2W))).$$

From equation (177), and since $0 < W \leq 1/2$ and $c_2, x_2, F > 0$, we may solve the above equation for c_2, and we require $W \neq 1/2$ and so $\nabla \neq 0$. Then, equating coefficients of z^2 in (173), we obtain

$$2((1 - 2W^2) - \nabla)W^2(\nabla(W^3 - F(1 - W)) + (1 - 2W)(1 - W)F - W^3(3 - 4W))x_2 - (1-W)(\nabla-(1-2W))(W^2+F)(F(\nabla^2-(1-2W)(1+2W))+4W(1-2W)(W^2+F)) = 0.$$

Since $(1 - 2W^2)^2 > (1 - 2W)(1 + 2W) > (1 - 2W)^2$ when $0 < W < 1$, then $1 - 2W^2 > |\nabla| > 1 - 2W$ by equation (177), and so we may solve the above equation for x_2. Then, given $x_2 > 0$, we may equate coefficients of z in (174) to obtain

$$F(4W(F+W^2)-F(1-\nabla))(\nabla^2-(1-2W)(1+2W))+8(1-2W)W^2(W^2+F)^2 = 0,$$

which from equation (177), and since $W > 0$, implies $W = 1/2$, which contradicts the preceding requirement for $W < 1/2$. We conclude that there are no networks from $\mathcal{Q}_{43}$ which realise a biquadratic minimum function.

Consider next the impedance $H_{44}(s)$ of a network from $\mathcal{N}_{44}$. In this case, $c_1 = x_2 = 1$ and $c_2 = (x_1c_3 - (c_3 - x_1))/(c_3 - x_1)$ for $H_{44}(j\omega_0) = H_p(j\omega_0)$. In particular, for $c_2 > 0$, we require $x_1c_3 > c_3 - x_1 > 0$. Moreover, we require $g_1 = 1/W^2$ and $g_2 = 1$ for $H_{44}(0) = H_p(0)$ and $H_{44}(\infty) = H_p(\infty)$. Then, equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$Wf(z) = (Fz+W)(W^2c_3(x_1c_3-(c_3-x_1))z^2+c_3^2x_1z+x_1(c_3-x_1)).$$

By equating coefficients of z^4 in (173), we then find

$$x_1 = \frac{c_3(F + W^2)}{c_3(F(1 - W) + W^2) + F + W^2},$$

and then, equating coefficients of z^2 in (173), we obtain

$$c_3 = \frac{(1-W)(W^2+F)^2}{W^3(F(1-W)+W^2)}.$$

We then equate coefficients of z^2 in equation (174), and the resulting equation has no solutions for $0 < W < 1$ and $F > 0$. We conclude that there are no networks in $\mathcal{Q}_{44}$ which realise a biquadratic minimum function.

Now, consider the impedance $H_{45}(s)$ of a network from $\mathcal{N}_{45}$. Here, $c_1 = x_1$, $c_2 = x_3$, and $x_2 = 1$ for $H_{45}(j\omega_0) = H_p(j\omega_0)$. Moreover, for $H_{45}(0) = H_p(0)$ and $H_{45}(\infty) = H_p(\infty)$, we require g_1 and g_2 to satisfy equations (175) and (176) respectively, where ∇ is a solution to equation (177). In particular, we require $W \leq 1/2$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} f(z) = 4WF^2x_3z^3 + 2WFx_3((\nabla+1)(x_1+x_3-1)+2W)z^2 \\ + 2F((\nabla+1)(1-x_3)+2Wx_3)z + 4W^2x_3. \end{aligned}$$

Then, equating coefficients of z^2 in (174), we obtain

$$x_1 = \frac{(x_3-1)^2(1-W)}{W - x_3(1-W)},$$

so, given $x_1 > 0$ and $0 < W \leq 1/2$, we require $W - x_3(1-W) > 0$ and $x_3 \neq 1$. Equating coefficients of z^2 and z^4 in (173), and given $W - x_3(1-W) > 0$ and $F > 0$, we obtain two polynomials in x_3, ∇, F, W, both of which must be zero, and from taking the difference of these two polynomials we obtain

$$2(1-W)(x_3-1)(1+\nabla)(FW - x_3(F(1-W)+W^2(1-2W))) = 0.$$

Since $x_3 \neq 1$, $0 < W \leq 1/2$, $\nabla^2 = 1 - 4W^2 < 1$, and $F > 0$, then we require

$$x_3 = \frac{FW}{F(1-W)+W^2(1-2W)},$$

and then

$$x_1 = \frac{(F+W^2)^2(1-W)(1-2W)}{W^3(F(1-W)+W^2(1-2W))}$$

so, in particular, $W < 1/2$. Then, equating coefficients of z^2 in (173), and given $x_1, F, W > 0$, we require

$$(\nabla^2-(1-2W)(1+2W))(F(1-W)+W^2(1-2W))+2(1-2W)(F+W^2)(\nabla+1-2W^2) = 0.$$

As $W > 0$, then $(1-2W)(1+2W) < (1-2W^2)^2$, and hence $|\nabla| < 1-2W^2$ by equation (177). Since, in addition, $F > 0$ and $W < 1/2$, then we see that the above equation cannot be satisfied. We conclude that there are no networks from $\mathcal{Q}_{45}$ which realise a biquadratic minimum function.

Next, consider the impedance $H_{46}(s)$ of a network from $\mathcal{N}_{46}$. In this case, we require $c_1 = x_1$, $c_2 = x_2$, and $x_3 = 1$ for $H_{46}(j\omega_0) = H_p(j\omega_0)$. For $H_{46}(s)$ to be biquadratic, we require $R_0(n_{46}(s), d_{46}(s)) = R_1(n_{46}(s), d_{46}(s)) = R_2(n_{46}(s), d_{46}(s)) = 0$. Here, $R_0(n_{46}(s), d_{46}(s)) = X^6\omega_0^{25}x_1^2x_2^4g_1^4g_2^2\gamma_1^4$, $R_1(n_{46}(s), d_{46}(s)) = X^4\omega_0^{16}x_1x_2g_1^2g_2^2\gamma_1^2\gamma_2$, and $R_2(n_{46}(s), d_{46}(s)) = X^2\omega_0^9x_2g_2^2\gamma_3$, for which $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}[x_1, x_2, g_1, g_2, X]$. In particular, we require $R_0(\gamma_1(x_1), \gamma_3(x_1)) = x_2^4(X^2g_1g_2 - x_2)^2\mu = 0$, where

$$\mu = X^4g_1^3g_2(x_2+1)^2 - (X^2g_1g_2+1)^2x_2^2.$$

For $H_{46}(0) = H_p(0)$ and $H_{46}(\infty) = H_p(\infty)$ we require $g_1 = 1$ and $g_2 = 1/W^2$, and we obtain

$$\mu W^4 = -(x_2(X^2(1+W)+W^2)+X^2W)(x_2(X^2(1-W)+W^2)-X^2W),$$

which, since $0 < W < 1$, $X > 0$, $x_2 > 0$, and $\mu = 0$, implies

$$x_2 = \frac{X^2W}{W^2+X^2(1-W)},$$

and then $\gamma_1 = 0$ implies

$$x_1 = \frac{(1-W)(W^2+X^2)^2}{W^3(X^2(1-W)+W^2)}.$$

It may then be verified that $H_{46}(s)$ takes the form of $H_p(s)$ in (145) with $0 < W < 1$ and $X > 0$. We thus conclude that if N is from $\mathcal{Q}_{46}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{72}$ in Fig. 53.

Consider next the impedance $H_{48}(s)$ of a network from $\mathcal{N}_{48}$. In this case, $x_1 = (c_1 + c_2)/(1+c_1)$ and $x_3 = c_2(1+c_1)/(c_1+c_2)$ for $H_{48}(j\omega_0) = H_p(j\omega_0)$, and we find

$$\begin{aligned} R_0(n_{48}(s), d_{48}(s)) = X^2c_1c_2^4g_2^2x_1\omega_0^{25}(c_1+1)^6(c_1+c_2)^4 \\ \times (g_1^2g_2^2X^4+c_1(g_2^2(c_2+x_1)^2+x_1^2g_1(g_1+2g_2))X^2+c_1^2c_2^2x_1^2)^2 \\ \times (c_1^2g_1^2g_2^2X^4+(g_2^2(c_1x_1+c_2)^2+x_1^2c_1^2g_1(g_1+2g_2))X^2+c_2^2x_1^2)^2 \end{aligned}$$

which cannot be zero since $c_1, c_2, x_1, g_1, g_2, X > 0$. We conclude that there are no networks from $\mathcal{Q}_{48}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{49}(s)$ of a network from $\mathcal{N}_{49}$. In this case, for $H_{49}(j\omega_0) = H_p(j\omega_0)$, we require $c_2 = (x_3(1+c_1)-c_1)/c_1$, $x_2 = (x_3(1+c_1)-c_1)/x_3$, and $x_3 \neq c_1$, and for $x_2 > 0$ we require $x_3(1+c_1)-c_1 > 0$. For $H_{49}(s)$ to be biquadratic, we require $R_0(n_{49}(s), d_{49}(s)) = R_1(n_{49}(s), d_{49}(s)) = R_2(n_{49}(s), d_{49}(s)) = 0$. In this case,

$$R_0(n_{49}(s), d_{49}(s)) = X^2\omega_0^{25}x_1x_3^6c_1^6g_2^2(x_3(1+c_1)-c_1)^4(x_3-c_1)^4\gamma_1^2\gamma_2^2\gamma_3^2,$$

and
$$R_1(n_{49}(s), d_{49}(s)) = X^2\omega_0^{16}x_3^4c_1^4g_2^2(x_3(1+c_1)-c_1)^2(x_3-c_1)^2\gamma_4,$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}[c_1, x_1, x_3, g_1, g_2, X]$. Here,

$$\gamma_1 = (x_3(1+c_1)-c_1)(X^2g_1g_2+X(g_2(c_1+x_3)+x_1(g_1+g_2))+x_1(c_1+x_3)) \\ +Xc_1x_3(Xg_1g_2+x_1(g_1+g_2)),$$

which is necessarily positive given $x_3(1+c_1)-c_1 > 0$ and $c_1, x_1, x_3, g_1, g_2, X > 0$. Also,

$$\gamma_2 = X^4g_1^2g_2^2+X^2((x_1+x_3(1+c_1)-c_1)^2g_2^2+x_1^2g_1(g_1+2g_2))+x_1^2(x_3(1+c_1)-c_1)^2),$$

which is necessarily positive since $g_1, g_2, X > 0$. Since, in addition, $x_3 \neq c_1$ and $x_3(1+c_1)-c_1 > 0$, we require $\gamma_3 = \gamma_4 = 0$. Here,

$$\gamma_3 = Xg_2\mu_1 - x_1(\mu_1 + Xg_2(x_3(1+2c_1)-c_1)), \tag{178}$$

with

$$\mu_1 = g_1X(x_3(1+2c_1)-c_1) - (c_1+x_3)(x_3(1+c_1)-c_1),$$

where $x_3(1+2c_1)-c_1 = x_3(1+c_1)-c_1+c_1x_3 > 0$ given $x_3(1+c_1)-c_1 > 0$ and $c_1, x_3 > 0$. We also require $\mu_1 \neq 0$ for $\gamma_3 = 0$ to have a solution with $x_1, g_2, X > 0$, and $x_3(1+2c_1)-c_1 > 0$. Moreover, we require $R_0(\gamma_3(x_1), \gamma_4(x_1)) = -X^6g_2^6(x_3-c_1)^2\mu_1\mu_2\mu_3^2\mu_4^2 = 0$, where $\mu_2, \mu_3, \mu_4 \in \mathbb{R}[c_1, x_3, g_1, g_2, X]$.

Suppose initially that $\mu_2 = 0$. Here,

$$\mu_2 = X(g_1(x_3(1+2c_1)-c_1)^2X - 2c_1(x_3^2+x_3(1+c_1)-c_1)(x_3(1+c_1)-c_1))g_2 + \mu_1^2.$$

Since $\mu_1 \neq 0$, we thus require

$$g_2 = \frac{\mu_1^2}{X(2c_1(x_3^2+x_3(1+c_1)-c_1)(x_3(1+c_1)-c_1) - g_1(x_3(1+2c_1)-c_1)^2X)},$$

and then $\gamma_3 = 0$ implies

$$\mu_1((x_3 - c_1)^2(x_3(1 + c_1) - c_1)x_1 + \mu_1^2) = 0,$$

which is not possible since $\mu_1 \neq 0$, $c_1 \neq x_3$, $x_1 > 0$, and $x_3(1 + c_1) - c_1 > 0$. We conclude that μ_2 and γ_3 cannot be simultaneously zero.

Suppose next that $\mu_3 = 0$. Since

$$\mu_3 = X^2 g_1(x_3(1+2c_1) - c_1)^2 g_2 + \mu_1(g_1 X(x_3(1+2c_1) - c_1) + (c_1 + x_3)(x_3(1+c_1) - c_1)),$$

and $X > 0$, $g_2 > 0$, and $x_3(1 + 2c_1) - c_1 > 0$, then

$$g_2 = \frac{-\mu_1(g_1(x_3(1 + 2c_1) - c_1)X + (c_1 + x_3)(x_3(1 + c_1) - c_1))}{X^2 g_1(x_3(2c_1 + 1) - c_1)^2},$$

and hence $\mu_1 < 0$ since $x_3(1 + 2c_1) - c_1 > x_3(1 + c_1) - c_1 > 0$ and $c_1, x_3, g_1, X > 0$. Then,

$$\begin{aligned}\gamma_3 X g_1(x_3(1+2c_1)-c_1)^2/\mu_1 = {} & (c_1+x_3)(x_3(1+c_1)-c_1)(x_3(1+2c_1)-c_1)x_1 \\ & - \mu_1(g_1(x_3(1+2c_1)-c_1)X+(c_1+x_3)(x_3(1+c_1)-c_1)),\end{aligned}$$

which, given the constraints obtained previously, implies that $\gamma_3 \neq 0$. Hence, μ_3 and γ_3 cannot be simultaneously zero.

Finally, suppose that $\mu_4 = 0$. Since

$$\mu_4 = X^2\eta_1^2 + (x_3(1 + c_1) - c_1)^2\eta_2^2,$$

with $\eta_1, \eta_2 \in \mathbb{R}[g_1, g_2, c_1, x_3, X]$, and since $x_3(1 + c_1) - c_1 > 0$, we require $\eta_1 = \eta_2 = 0$. Here,

$$\begin{aligned} & \eta_1 = g_1(g_1+g_2)(x_3(1+2c_1)-c_1)X-(x_3(1+c_1)-c_1)(g_1(c_1+x_3)+g_2c_1(1-x_3)), \\ \text{and} \quad & \eta_2 = X(x_3c_1(g_2+2g_1)-g_1(c_1-x_3)-g_2c_1)-(c_1+x_3)(x_3(1+c_1)-c_1),\end{aligned}$$

and, in particular, we require

$$R_0(\eta_1(X), \eta_2(X)) = (x_3 - c_1(1-x_3))g_2(g_1(c_1^2(x_3-1)^2 + x_3^2(1+c_1)^2) + g_2c_1^2(x_3-1)^2) = 0,$$

which has no solutions for $g_1, g_2, c_1, x_3 > 0$ and $x_3(1 + c_1) - c_1 > 0$. We conclude that there are no networks from $\mathcal{Q}_{49}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{50}(s)$ of a network from $\mathcal{N}_{50}$. In this case, we require $c_1 = x_1 + x_2$, $c_2 = x_2/x_1$, and $x_3 = (x_1 + x_2)/x_1$ for $H_{50}(j\omega_0) = H_p(j\omega_0)$. Furthermore, for $H_{50}(0) = H_p(0)$ and $H_{50}(\infty) = H_p(\infty)$, we require g_1 and g_2 to satisfy equations (175) and (176) respectively, where $\nabla \in \mathbb{R}$ is a solution to equation (177), so in particular we require $W \leq 1/2$ and $|\nabla| < 1$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned}f(z) = 4WF^2x_1x_2(x_1{+}x_2)z^3 \\ +2WFx_2(x_1{+}x_2)(x_2(1{+}x_1)(1{+}\nabla){+}x_1(x_1{-}1)(1{-}\nabla){+}2Wx_1)z^2 \\ +2Fx_1((x_1{+}x_2)^2(1{-}\nabla){+}2x_1x_2(W(x_1{+}x_2){-}x_2))z{+}4W^2x_1^2x_2(x_1{+}x_2).\end{aligned}$$

Now, equating coefficients of z^2 in (174), we obtain

$$-2FW(\nabla(x_1^2 - x_2^2) - (x_1^2 + x_2^2))(x_1(Wx_1 - (1 - W)) - (1 + x_1)(1 - W)x_2) = 0,$$

which, since $|\nabla| < 1$, $0 < W \leq 1/2$, and $x_1, x_2, F > 0$, implies

$$x_2 = \frac{x_1(Wx_1 - (1 - W))}{(1 + x_1)(1 - W)},$$

and hence $Wx_1{-}(1{-}W) > 0$. Then, equating coefficients of z^2 in (173), and coefficients of z in (174), we obtain

$$\begin{aligned}\gamma_1 = -(1{-}W)^2(4W(W(1{+}x_1){-}x_1{-}2){-}x_1{+}5{+}2\nabla x_1{-}(1{+}x_1)\nabla^2)F \\ +2W^3(Wx_1{-}(1{-}W))(2W(x_1^2(W{-}1){+}2{-}W){-}x_1(4W{-}3){-}2{+}x_1(2W{-}1)\nabla) = 0,\end{aligned}$$

in addition to

$$\begin{aligned}\gamma_2 = -(1{-}W)(4W(W(1{+}x_1){-}x_1{-}2){-}x_1{+}4{+}2\nabla x_1{-}x_1\nabla^2)F \\ -4W^3x_1(Wx_1{-}(1{-}W))^2 = 0.\end{aligned}$$

In particular, we require

$$\begin{aligned}\gamma_3 = \gamma_1 \times (4W(W(1{+}x_1){-}x_1{-}2){-}x_1{+}4{+}2\nabla x_1{-}x_1\nabla^2) \\ - \gamma_2 \times (1{-}W)(4W(W(1{+}x_1){-}x_1{-}2){-}x_1{+}5{+}2\nabla x_1{-}(1{+}x_1)\nabla^2) = 0,\end{aligned}$$

where

$$\gamma_3 = 2W^3(1 - W)(Wx_1 - (1 - W))\gamma_4,$$

with γ_4 a quadratic in x_1 whose discriminant is zero given equation (177).

Now, let

$$\gamma_5 = 2W((1-W)(1-2W)(1+2W)+W^3)x_1-(1-W)^2((1-2W)(1+2W)+W+\nabla(1-W)).$$

Then it may be verified that $R_0(\gamma_4(x_1), \gamma_5(x_1))$ is also zero given equation (177). Since γ_4 has exactly two roots for x_1, which are coincident, and γ_5 has exactly one root for x_1, then we may solve $\gamma_5 = 0$ for x_1 to give

$$x_1 = \frac{(1-W)^2((1-2W)(1+2W)+W+\nabla(1-W))}{2W((1-W)(1-2W)(1+2W)+W^3)}.$$

Then, equating coefficients of z^4 in (173), we obtain

$$(\nabla^2 - (1-2W)(1+2W))\mu_1 + 32W^5((1-W)(1-2W)(1+2W)+W^3)^3\mu_2 = 0$$

where $\mu_1, \mu_2 \in \mathbb{R}[W, F, \nabla]$. By equation (177), and since $0 < W \leq 1/2$, this implies

$$\begin{aligned}\mu_2 &= 2((1-W)(1-2W)(1+2W)+W^3)^2F \\ &-W^2(1-2W)(1-W)((1-2W)(1-2W^2)+2W^3(2-3W)-\nabla(1-2W)(1-2W^2)) = 0,\end{aligned}$$

which may be solved for F to give

$$F = \frac{W^2(1-2W)(1-W)((1-2W)(1-2W^2)+2W^3(2-3W)-\nabla(1-2W)(1-2W^2))}{2((1-W)(1-2W)(1+2W)+W^3)^2}.$$

That $(1-2W)(1-2W^2)+2W^3(2-3W)-\nabla(1-2W)(1-2W^2) > 0$ may be seen since from equation (177) and the preceding constraints, since

$$\begin{aligned}((1-2W)(1-2W^2)+2W^3(2-3W))^2 &= \nabla^2(1-2W)^2(1-2W^2)^2 \\ &+ 4W^2((1-W)(1-2W)(1+2W)+W^3)^2,\end{aligned}$$

and so $(1-2W)(1-2W^2)+2W^3(2-3W) > \nabla(1-2W)(1-2W^2)$. Hence, for $F > 0$, we require $W < 1/2$. Furthermore, from $Wx_1 - (1-W) > 0$, we require $\nabla > 0$, and so $\nabla = \sqrt{(1+2W)(1-2W)}$. It may then be verified that $H_{50}(s)$ takes the form of $H_p(s)$ in (145) with $0 < W < 1/2$ and

$$X = \frac{W\sqrt{(1-2W)(1-W)((1-2W)(1-2W^2)+2W^3(2-3W)-\sqrt{(1-2W)(1+2W)}(1-2W)(1-2W^2))}}{\sqrt{2}((1-W)(1-2W)(1+2W)+W^3)}.$$

We thus conclude that if N is from $\mathcal{Q}_{50}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{68}$ in Fig. 49.

Consider now the impedance $H_{51}(s)$ of a network from class $\mathcal{N}_{51}$. In this case, we have $c_1 = x_1 + x_2$, $c_2 = x_2/x_1$, and $x_3 = (x_1 + x_2)/x_1$ for $H_{51}(j\omega_0) = H_p(j\omega_0)$. Moreover, for $H_{51}(0) = H_p(0)$ and $H_{51}(\infty) = H_p(\infty)$, we require $g_1 = 1$ and $g_2 = 1/W^2$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} W^2 f(z) = WF^2 x_1 x_2 (x_1 + x_2) z^3 \\ + WFx_2(x_1+x_2)(W^2(1+x_1)(x_1+x_2)+x_1(x_1-(1-W)))z^2 \\ + Fx_1((W^2+1)(x_1+x_2)^2 + x_1x_2(W(x_1+x_2)-x_2))z + W^2x_1^2x_2(x_1+x_2). \end{aligned}$$

Then, equating coefficients of z^3 in (173), and given $x_1, x_2, F > 0$, we require

$$(Wx_1 - (1-W))x_1 - (1-W)(1+x_1)x_2 = 0,$$

and since $W < 1$ we may solve the above equation for x_2, to give

$$x_2 = \frac{x_1(Wx_1 - (1-W))}{(1-W)(1+x_1)},$$

so in particular we require $Wx_1 - (1-W) > 0$. Then, equating coefficients of z^4 in (173) and coefficients of z in (174), we require

$$\begin{aligned} \gamma_1 &= (1-W)F - W(x_1 - (1-W))(Wx_1 - (1-W)) = 0 \\ \text{and} \quad \gamma_2 &= (1-W)((1-W)^2 - x_1(1 + W(1-W)))F + x_1W^3(Wx_1 - (1-W))^2 = 0, \end{aligned}$$

so in particular we require

$$R_0(\gamma_1(F), \gamma_2(F)) = -W(1-W)^2(Wx_1 - (1-W))(Wx_1 - (1-W) + x_1)^2 = 0$$

which is not possible since $x_1 > 0$, $0 < W < 1$, and $Wx_1 - (1-W) > 0$. We conclude that there are no networks from $\mathcal{Q}_{51}$ which realise a biquadratic minimum function.

Next, consider the impedance $H_{52}(s)$ of a network from class $\mathcal{N}_{52}$. Here, we require $c_1 = x_1$, $c_2 = x_2/(1-x_1)$, and $x_3 = (x_1 - x_2)/x_1$ for $H_{52}(j\omega_0) = H_p(j\omega_0)$, so for $c_2, x_3 > 0$ we require $x_2 < x_1 < 1$. Furthermore, for $H_{52}(0) = H_p(0)$ and $H_{52}(\infty) = H_p(\infty)$, we find that g_1 and g_2 must satisfy equations (175) and (176) respectively, where $\nabla \in \mathbb{R}$ is a solution to equation (177). In particular, we require $W \leq 1/2$. Equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we find

$$f(z) = 4WF^2x_1^2x_2z^3 + 2FWx_1x_2(2x_1W + (1-x_1)(\nabla(x_1-x_2) - x_1 - x_2))z^2$$
$$+2F(1-x_1)(2x_1x_2^2(1-W) + x_1^2(2Wx_2+1) - x_2^2 + \nabla(x_2^2 - x_1^2))z + 4x_1x_2W^2(1-x_1)(x_1-x_2).$$

Then, by equating coefficients of z^2 in (174), we find

$$x_2(1-x_1)(1-W) - x_1(x_1W - (1-W)) = 0,$$

Since $x_1, x_2 > 0$ and $0 < W < 1$, we may solve the above equation for x_2 to give

$$x_2 = \frac{x_1(x_1W - (1-W))}{(1-x_1)(1-W)}.$$

Hence, $x_1W - (1-W) > 0$, which together with $x_1 < 1$ implies $2W - 1 > 0$, which contradicts the preceding requirement for $W \leq 1/2$. We conclude that there are no networks from $\mathcal{Q}_{52}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{53}(s)$ of a network from $\mathcal{Q}_{53}$. In this case, we have $x_1 = c_1$, $x_2 = c_1(1+c_2)$, and $x_3 = c_1(1+c_2)/(c_1-1)$ for $H_{53}(j\omega_0) = H_p(j\omega_0)$, so for $x_3 > 0$ we require $c_1 > 1$. Furthermore, for $H_{53}(0) = H_p(0)$ and $H_{53}(\infty) = H_p(\infty)$, we require $g_1 = 1$ and $g_1 = (1-W^2)/W^2$. From equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$W^2f(z) = WF^2c_1(1-W)(1+W)(c_1-1)z^3$$
$$+Fc_2W((c_1-1)^2 + W(c_1-1) + W^2(c_1^2c_2 + 2c_1 - 1) + W^3(1-c_1))z^2$$
$$+Fc_1(1+c_2)(c_2(1-c_1) + Wc_1(1+c_2) - W^2c_2)z + Wc_1^2(1+c_2)^2.$$

Then, by equating coefficients of z^2 in (174), and given $c_2, F > 0$, $0 < W < 1$, and $c_1 > 1$, we require $c_2c_1(1-W) - (Wc_1 - (1-W)) = 0$, which may be solved for c_2 to give

$$c_2 = \frac{Wc_1 - (1-W)}{c_1(1-W)}.$$

In particular, we require $Wc_1 - (1-W) > 0$. Then, equating coefficients of z^4 in (173), and coefficients of z in equation (174), and given $c_1 > 1$, $0 < W < 1$, and $Wc_1 - (1-W) > 0$, we require

$$\gamma_1 = (1-W)^2F - W(c_1 - (1-W))(Wc_1 - (1-W)) = 0,$$

and $\gamma_2 = (1-W)^2((c_1-1)(W^3 - W - 1) - W^2)F + c_1W^3(c_1 - (1-W))(Wc_1 - (1-W)) = 0.$

Hence, we require

$$R_0(\gamma_1(F), \gamma_2(F)) = (1-c_1)W(1+W)^2(1-W)^3(c_1-(1-W))(Wc_1-(1-W)) = 0,$$

which has no solutions for $c_1 > 1$, $Wc_1 - (1 - W) > 0$, and $0 < W < 1$. We conclude that there are no networks from $\mathcal{Q}_{53}$ which realise a biquadratic minimum function.

Consider next the impedance $H_{54}(s)$ of a network from $\mathcal{N}_{54}$. In this case, we find $c_1 = x_1$, $c_2 = x_2/(1 + x_1)$, and $x_3 = (x_1 + x_2)/x_1$ for $H_{54}(j\omega_0) = H_p(j\omega_0)$, and for $H_{54}(0) = H_p(0)$ and $H_{54}(\infty) = H_p(\infty)$ we require $g_1 = 1/W^2$ and $g_2 = 1$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we find

$$\begin{aligned}W^2 f(z) = WF^2x_1^2x_2z^3&+WFx_1x_2((1+x_1)(W^2(x_1+x_2)+x_2)+Wx_1)z^2\\ &+F(1+x_1)(W(x_1+x_2)(W(x_1+x_2)+x_1x_2)-x_1x_2^2)z+W^2x_1x_2(1+x_1)(x_1+x_2).\end{aligned}$$

By equating coefficients of z^2 in (174), and given $x_1, x_2, F > 0$ and $0 < W < 1$, we find $x_2(1 - W)(1 + x_1) - x_1(Wx_1 - (1 - W)) = 0$, which may be solved for x_2 to give

$$x_2 = \frac{x_1(Wx_1 - (1 - W))}{(1 - W)(1 + x_1)},$$

and hence $Wx_1-(1-W) > 0$. Then, equating coefficients of z^4 in (173), and coefficients of z in equation (174), and given $Wx_1 - (1 - W) > 0$, $0 < W < 1$, and $x_1, F > 0$, we obtain

$$\begin{aligned}&\gamma_1 = W(Wx_1 - (1 - W))(1 - W(x_1 + 2 - W)) + (1 - W)F = 0,\\ \text{and}\quad &\gamma_2 = W^3x_1(Wx_1 - (1 - W)) - (1 - W)F = 0.\end{aligned}$$

In particular, we require $R_0(\gamma_1(F), \gamma_2(F)) = -W(1-W)^2(Wx_1-(1-W))^2 = 0$ which cannot be satisfied given the constraints identified earlier. We conclude that there are no networks from $\mathcal{Q}_{54}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{56}(s)$ of a network from $\mathcal{N}_{56}$. Here, we have $x_1 = c_1/(1 - c_2)$, $x_2 = c_1c_2/(c_2 - (1 - c_1))$, and $x_3 = 1$ for $H_{56}(j\omega_0) = H_p(j\omega_0)$. In particular, for $x_1, x_2 > 0$, we require $1 > c_2 > 1 - c_1$. Moreover, for $H_{56}(0) = H_p(0)$ and $H_{56}(\infty) = H_p(\infty)$, we require g_1 and g_2 to satisfy equations (175) and (176), where $\nabla \in \mathbb{R}$ is a solution to equation (177), and hence we require $W \leq 1/2$ and $|\nabla| < 1$. Equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of

z^4 in equation (174), we obtain

$$f(z) = 4c_2F^2W(c_2-(1-c_1))(1-c_2)z^3 \\ +2Wc_2F(c_2-(1-c_1))(c_1-(1-c_2)(1-2W)-\nabla(c_2-(1-c_1)))z^2 \\ +2F(c_1(1-c_2(1-2W))-(1-c_2)^2-\nabla(1-c_2)(c_2-(1-c_1)))z+4c_1c_2W^2.$$

Then, equating coefficients of z^2 in (174), and given $1 > c_2 > 1 - c_1$, $|\nabla| < 1$, and $W \leq 1/2$ together with $c_2, F > 0$, we require $c_1(W - c_2(1-W)) - (1-c_2)^2(1-W) = 0$, which we may solve for c_1 to give

$$c_1 = \frac{(1-c_2)^2(1-W)}{W - c_2(1-W)}.$$

In particular, we require $W - c_2(1-W) > 0$. By equating coefficients of z^2 and z^4 in (173), and given the preceding constraints, we require

$$\gamma_1 = (W-c_2(1-W))(\nabla-(1-2W))(\nabla(1+c_2-W(2+c_2))+(1-2W)(c_2(1-W)-1))F \\ -2W^3c_2(1-2W)^2(\nabla+2c_2(1-W)-1) = 0,$$

together with either $W = 1/2$ or

$$\gamma_2 = (W-c_2(1-W))(\nabla^2+4Wc_2(1-W)-1)F \\ -2W^2c_2(1-2W)(\nabla+2Wc_2(1-W)-1) = 0.$$

In particular, we require

$$\gamma_1 \times (\nabla^2+4Wc_2(1-W)-1) \\ - \gamma_2 \times (\nabla-(1-2W))(\nabla(1+c_2-W(2+c_2))+(1-2W)(c_2(1-W)-1)) = 0,$$

which, given equation (177) together with the constraints $0 < W \leq 1/2$, $c_2 > 0$, and $W - c_2(1-W) > 0$, implies

$$(1-2W)(1-3W^2-\nabla(1-W^2)) = 0.$$

Moreover, given equation (177), then $1 - 3W^2 - \nabla(1-W^2) = 0$ implies $W = 0$ which contradicts the preceding requirement for $W > 0$. Hence, the above equation implies $W = 1/2$, and hence $\nabla = 0$. Then, $\gamma_2 = -F(1-c_2)^2/2$ which has no solutions for $1 > c_2$ and $F > 0$. We conclude that there are no networks from $\mathcal{Q}_{56}$ which realise a biquadratic minimum function.

Now consider the impedance $H_{57}(s)$ of a network from $\mathcal{N}_{57}$. In this case, we require $x_1 = c_2/(c_2+c_3)$, $c_1 = c_2(1-c_3)/(c_2+c_3)$, and $x_2 = 1$ for $H_{57}(j\omega_0) = H_p(j\omega_0)$. In particular, since $c_1 > 0$, we require $1 - c_3 > 0$. In this case, for $H_{57}(s)$ biquadratic, we require $R_0(n_{57}(s), d_{57}(s)) = R_1(n_{57}(s), d_{57}(s)) = 0$. Here,

$$R_0(n_{57}(s), d_{57}(s)) = X^4 c_2^2 g_2^2 \omega_0^{16}(c_2+c_3)^5(X^2 g_2^2(1-c_3)^2 + (c_2+c_3)^2(1+X^2 g_1 g_2)^2)^2\gamma_1^2,$$

and

$$R_1(n_{57}(s), d_{57}(s)) = -X^2 c_2 \omega_0^9 (c_2+c_3)^4 \gamma_2,$$

where $\gamma_1, \gamma_2 \in \mathbb{R}[g_1, g_2, c_2, c_3, X]$, with

$$\gamma_1 = g_1(X^2 g_1 g_2 - c_2)c_3 + c_2(g_1(X^2 g_1 g_2 - c_2) + c_2 g_2).$$

In particular, we require $R_0(\gamma_1(c_3), \gamma_2(c_3)) = X^6 c_2^6 g_1^3 g_2^{10}(X^2 g_1^2(1+c_2)^2 + c_2^2)^2\mu = 0$, where

$$\mu = (g_1+g_2)c_2^2 - 2g_1 X^2 g_2(g_1+g_2)c_2 + X^4 g_1^3 g_2^2.$$

Moreover, for $H_{59}(0) = H_p(0)$ and $H_{59}(\infty) = H_p(\infty)$, we require $g_1 = 1/W^2$ and $g_2 = 1/(1-W^2)$. Then, since $c_2, c_3, X > 0$ and $0 < W < 1$, we may solve for c_3 from $\gamma_1 = 0$, which gives

$$c_3 = \frac{c_2(X^2 - c_2 W^2(1-2W^2))}{W^2 c_2(1-W^2) - X^2}.$$

Here, $W^2 c_2(1-W^2) - X^2 > 0$ and $X^2 - c_2 W^2(1-2W^2) > 0$, since if $W^2 c_2(1-W^2) - X^2 \leq 0$ then $X^2 - c_2 W^2(1-2W^2) = c_2 W^4 - (W^2 c_2(1-W^2) - X^2) > 0$ which implies $c_3 \leq 0$. Then, $\mu = 0$ implies that either $c_2 W^2(1+W) - X^2 = 0$ or $c_2 W^2(1-W) - X^2 = 0$. Since $W^2 c_2(1-W^2) - X^2 > 0$ then $c_2 > X^2/(W^2(1-W^2))$, and hence $c_2 = X^2/(W^2(1-W))$. This implies that $c_1 = (W^2(1-W)^2 - X^2(2W-1))/(W^3(1-W))$ and $c_3 = X^2(2W-1)/W^2(1-W)^2$, and hence $1/2 < W < 1$ and $X < W(1-W)/\sqrt{2W-1}$. We thus conclude that if N is from $\mathcal{Q}_{57}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{71}$ in Fig. 52.

Next, consider the impedance $H_{58}(s)$ of a network from $\mathcal{N}_{58}$. Here, for $H_{58}(j\omega_0) = H_p(j\omega_0)$, we require $x_1 = (c_1+c_2)/(1+c_2)$, $x_2 = c_2(c_1-1)/(c_1+c_2)$, and $x_3 = 1$, so in particular $c_1 - 1 > 0$ for $x_2 > 0$. Furthermore, for $H_{58}(0) = H_p(0)$ and $H_{58}(\infty) = H_p(\infty)$, we have $g_1 = 1/W^2$ and $g_2 = 1/(1-W^2)$. From equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} Wf(z) = {} & F^2 c_1 c_2(c_2+1)z^3 + Fc_1 c_2 W(c_2 + 1 + W(c_1-1))z^2 \\ & + F((1-W^2)c_2(c_1-1) + Wc_1(1+c_2))z + Wc_2(c_1-1)(1-W)(1+W). \end{aligned}$$

Then, equating coefficients of z^2 in (174), and given $c_1, W, F > 0$, we have

$$(W - c_1(1 - W))c_2 - ((1 - W) - c_1 W) = 0.$$

It may be verified that $(1 - W) - c_1 W = W - c_1(1 - W) = 0$ has no solutions for $c_1 > 1$, and hence the above equation may be solved for c_2 to give

$$c_2 = \frac{(1 - W) - c_1 W}{W - c_1(1 - W)}.$$

In particular, we require $(1 - W) - c_1 W \neq 0$ and $W - c_1(1 - W) \neq 0$. By equating coefficients of z^4 in (173), and coefficients of z in equation (174), and given $c_1 - 1 > 0$, $(1 - W) - c_1 W \neq 0$, and $W - c_1(1 - W) \neq 0$, together with $0 < W < 1$ and $F > 0$, we require

$$\gamma_1 = (c_1(1-W)-W)F - W^2 c_1 (W - c_1(1-W))(W^3(1+c_1) - W^2 - W + 1) = 0,$$

and $$\gamma_2 = (W - c_1(1-W))F - W^2 c_1 (1-W)((1-W) - c_1 W) = 0.$$

In particular, we require $R_0(\gamma_1(F), \gamma_2(F)) = W^4 c_1 (W - c_1(1-W))((1-W) - c_1 W)^2 = 0$ which has no solutions given $0 < W < 1$, $c_1 > 0$, $(1 - W) - c_1 W \neq 0$, and $W - c_1(1 - W) \neq 0$. We conclude that there are no networks from $\mathcal{Q}_{58}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{59}(s)$ of a network from $\mathcal{N}_{59}$. In this case, for $H_{59}(j\omega_0) = H_p(j\omega_0)$, we have $x_1 = c_1(1 - c_2)$, $x_2 = c_1 c_2/(1 + c_1)$, and $x_3 = 1$, so in particular we require $1 - c_2 > 0$. Furthermore, for $H_{59}(0) = H_p(0)$ and $H_{59}(\infty) = H_p(\infty)$, we require $g_1 = 1/W^2$ and $g_2 = 1/(1 - W^2)$. Then, equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} W f(z) &= F^2 c_2 (1 + c_1) z^3 + F c_2 W (1 + c_1 + W(c_1^2(1 - c_2) - 1)) z^2 \\ &+ F((1 - W^2) c_1 c_2 (1 - c_2) + W(c_1(1 - c_2^2) + 1)) z + W c_1 c_2 (1 - W)(1 + W)(1 - c_2). \end{aligned}$$

Moreover, equating coefficients of z^2 in (174), and given $0 < c_2 < 1$, $0 < W < 1$ and $c_1, F > 0$, we require $c_1(W - c_2(1 - W)) - (1 - W) = 0$ which may be solved for c_1 to give

$$c_1 = \frac{1 - W}{W - c_2(1 - W)}.$$

In particular, we require $W - c_2(1 - W) > 0$ for $c_1 > 0$. Then, equating coefficients of z^4 in equation (173), and coefficients of z in equation (174), and given the constraints already established, we require

$$\begin{aligned}
&\gamma_1 = (1-c_2(1-W))(W-c_2(1-W))^2F \\
&\qquad -c_2W^2(1-W)^2(W(c_2W+1)^2-Wc_2(W^2+2c_2)+c_2(c_2-1)) = 0, \\
\text{and} \quad &\gamma_2 = (c_2(1-W)-W)F+c_2(1-W)^2W^2 = 0.
\end{aligned}$$

Hence, we require $R_0(\gamma_1(F), \gamma_2(F)) = W^4c_2^2(c_2-1)(1-W)^3(W-c_2(1-W)) = 0$, which has no solutions for $0 < W < 1$, $1-c_2 > 0$, $W-c_2(1-W) > 0$, and $c_2 > 0$. We thus conclude that there are no networks from $\mathcal{Q}_{59}$ which realise a biquadratic minimum function.

Consider now the impedance $H_{60}(s)$ of a network from $\mathcal{N}_{60}$. In this case, we find $x_1 = c_2(1+c_1)+c_1$, $x_2 = c_2(1+c_1)/c_1$, and $x_3 = 1$ for $H_{60}(j\omega_0) = H_p(j\omega_0)$. Then, for $H_{60}(s)$ to be biquadratic, we require $R_0(n_{60}(s), d_{60}(s)) = R_1(n_{60}(s), d_{60}(s)) = R_2(n_{60}(s), d_{60}(s)) = 0$, where

$$\begin{aligned}
&R_0(n_{60}(s), d_{60}(s)) = X^2\omega_0^{25}c_1^8c_2^4g_1^2(1+c_1)^3(c_2(1+c_1)+c_1)^3 \\
&\qquad\qquad \times (c_2^2(1+X^2g_1g_2)^2+X^2g_2^2(c_2(1+c_1)+c_1)^2)^2\gamma_1^4, \\
&R_1(n_{60}(s), d_{60}(s)) = X^2\omega_0^{16}c_1^6c_2g_1(c_2(1+c_1)+c_1)\gamma_1^2\gamma_2, \\
\text{and} \quad &R_2(n_{60}(s), d_{60}(s)) = -X^2\omega_0^9c_1^4c_2g_1^2\gamma_3,
\end{aligned}$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}[g_1, g_2, c_1, c_2, X]$. Since $c_1, c_2, g_1, g_2, X > 0$, we require $\gamma_1 = \gamma_3 = 0$, so in particular $R_0(\gamma_1(c_2), \gamma_3(c_2)) = X^{10}c_1^{13}g_1^4g_2^5(1+c_1)^3(X^2g_1g_2-c_1)^5\mu = 0$, where $\mu \in \mathbb{R}[g_1, g_2, c_1, X]$. Since $g_1, g_2, c_1, X > 0$, we require $\gamma_1 = 0$ together with either $\mu = 0$ or $X^2g_1g_2 - c_1 = 0$. Here,

$$\begin{aligned}
&\gamma_1 = X^2c_1g_1g_2 - c_2(1+c_1)(c_1 - X^2g_1g_2), \\
\text{and} \quad &\mu = g_1(1+X^2g_1g_2)^2 - g_2(c_1+1)^2.
\end{aligned}$$

Then, since $X^2g_1g_2 - c_1 = \gamma_1 = 0$ has no solutions for $c_1, c_2, g_1, g_2, X > 0$, we find $\mu = 0$. Moreover, for $H_{60}(0) = H_p(0)$ and $H_{60}(\infty) = H_p(\infty)$, we require $g_1 = 1/W^2$ and $g_2 = 1$. We may then solve the equations $\mu = \gamma_1 = 0$ for c_1 and c_2 to give

$$\begin{aligned}
&c_1 = \frac{W^2(1-W)+X^2}{W^3}, \\
\text{and} \quad &c_2 = \frac{WX^2(X^2+W^2(1-W))}{(1-W)(W^2+X^2)^2}.
\end{aligned}$$

We thus conclude that if N is from $\mathcal{Q}_{60}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{73}$ in Fig. 54.

Consider now the impedance $H_{61}(s)$ of a network from $\mathcal{N}_{61}$. In this case, we have $x_1 = (c_1 + c_2)/(1 + c_1)$, $x_2 = 1$, and $c_3 = (c_1 + c_2)/c_1$ for $H_{61}(j\omega_0) = H_p(j\omega_0)$. Moreover, for $H_{61}(0) = H_p(0)$ and $H_{61}(\infty) = H_p(\infty)$, we require $g_1 = 1/W^2$ and $g_2 = 1$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we find

$$Wf(z) = (c_1 + c_2)(c_1 c_2 W^2(1 + c_1)z^2 + (1 + c_1)(c_1 + c_2)z + c_1)(Fz + W).$$

Then, equating coefficients of z^2 in (174), and given $c_1, c_2, F > 0$ and $0 < W < 1$, we require $c_2(1 + c_1)(1 - W) - c_1((2W - 1)c_1 - (1 - W)) = 0$, which we may solve for c_2 to give

$$c_2 = \frac{c_1((2W - 1)c_1 - (1 - W))}{(1 + c_1)(1 - W)}.$$

In particular, we require $(2W - 1)c_1 - (1 - W) > 0$ for $c_2 > 0$, which implies $W > 1/2$ and $c_1 > (1 - W)/(2W - 1)$. Then, equating coefficients of z^4 in (173), and coefficients of z in (174), we obtain

$$\gamma_1 = (W(2 + 3c_1) - (1 + W^2)(1 + c_1))F - W^2 c_1((2W - 1)c_1 - (1 - W)) = 0,$$

and $$\gamma_2 = -F + W^2(Wc_1 - (1 - W)) = 0.$$

In particular, we require $R_0(\gamma_1(F), \gamma_2(F)) = W^2(1 + c_1)^2(1 - W)^3 = 0$, which has no solutions for $c_1 > 0$ and $0 < W < 1$. We conclude that there are no networks from $\mathcal{Q}_{61}$ which realise a biquadratic minimum function.

Next, consider the impedance $H_{62}(s)$ of a network from $\mathcal{N}_{62}$. In this case, $c_1 = x_2x_3/(x_2 + x_3)$, $c_2 = x_3/x_2$, and $x_1 = x_3/(x_2 + x_3)$ for $H_{62}(j\omega_0) = H_p(j\omega_0)$. Moreover, for $H_{62}(0) = H_p(0)$ and $H_{62}(\infty) = H_p(\infty)$, we require $g_1 = 1$ and $g_2 = 1/W^2$. Equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} W^2 f(z) &= WF^2 x_2 x_3(x_2 + x_3)z^3 + WFx_2x_3(W^2(x_2 + x_3)^2 + W(x_2 + x_3) - x_2)z^2 \\ &\quad + Fx_2(x_2 + x_3)(W^2(x_2 + x_3)(x_2 + 1) + x_2(Wx_3 + x_2 + 1))z + W^2x_2^2x_3(x_2 + x_3). \end{aligned}$$

Then, equating coefficients of z^2 in (174), and given $x_2, x_3, F > 0$ and $0 < W < 1$, we require $(W - x_2(1 - W))x_3 - x_2(1 + x_2)(1 - W) = 0$ which may be solved for x_3 to give

$$x_3 = \frac{x_2(1 + x_2)(1 - W)}{W - x_2(1 - W)}.$$

In particular, we require $W - x_2(1 - W) > 0$. By equating coefficients of z^4 in equation (173), and coefficients of z in equation (174), and given the constraints already established, we then obtain

$$\gamma_1 = (W - x_2(1-W))F - Wx_2^2(1-W)^2 = 0,$$
$$\text{and} \quad \gamma_2 = (W - x_2(1-W))(x_2(1-W)^2 - W)F + W^3x_2^2(1-W)^2 = 0.$$

We thus require $R_0(\gamma_1(F), \gamma_2(F)) = -x_2^2W(1-W)^3(W-x_2(1-W)) = 0$, which is not possible given $x_2 > 0$, $0 < W < 1$, and $W - x_2(1-W) > 0$. We conclude that there are no networks from $\mathcal{Q}_{62}$ which realise a biquadratic minimum function.

Now consider the impedance $H_{63}(s)$ of a network from $\mathcal{Q}_{63}$. In this case, we obtain $c_1 = x_2$, $c_2 = (x_2(x_3-1) - x_3)/x_2$, and $x_1 = (x_2 + x_3)/x_3$ for $H_{63}(j\omega_0) = H_p(j\omega_0)$. In particular, we require $x_2(x_3-1) - x_3 > 0$ for $c_2 > 0$. Moreover, for $H_{63}(0) = H_p(0)$ and $H_{63}(\infty) = H_p(\infty)$, we require $g_1 = 1$ and $g_2 = (1-W^2)/W^2$. By equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we find

$$\begin{aligned} f(z)W^2/x_2 = {} & F^2Wx_3(1-W)(1+W)(x_2(x_3-1)-x_3)z^3 \\ & + FW(x_2(x_3-1)-x_3)(W^2(x_3(x_3-W)+x_2(x_3-1))+Wx_3+x_2)z^2 \\ & + F(x_2(1-x_3)(W^2x_3+x_2)+Wx_3^2(W+x_2)+x_2x_3(1+Wx_2))z \\ & + W^2x_2x_3(x_2+x_3). \end{aligned}$$

Then, equating coefficients of z^2 in (174), and given $x_2(x_3-1)-x_3 > 0$ and $x_2, x_3, F, W > 0$, we require $x_2 - x_3(x_2(1-W) - W) = 0$, which may be solved for x_3 to give

$$x_3 = \frac{x_2}{x_2(1-W) - W},$$

and then

$$c_2 = \frac{x_2W - (1-W)}{x_2(1-W) - W}.$$

Hence, we require $x_2(1-W) - W > 0$ and $x_2W - (1-W) > 0$. Then, equating coefficients of z^4 in (173), and coefficients of z in equation (174), and given the constraints already established, we obtain

$$\gamma_1 = (x_2(1-W)-W)F - Wx_2^2(x_2W-(1-W)) = 0,$$
$$\begin{aligned} \text{and} \quad \gamma_2 = {} & ((W^2-x_2)(1+W)+W(W^2x_2+1))(x_2(1-W)-W)F \\ & +W^3x_2^2(1+x_2)(x_2W-(1-W)) = 0. \end{aligned}$$

This implies that $R_0(\gamma_1(F), \gamma_2(F)) = -Wx_2^2(1+W)^2(x_2W-(1-W))(x_2(1-W)-W)^2 = 0$, which is not possible since $x_2, W > 0$, $x_2(1-W) - W > 0$, and $x_2W - (1-

$W) > 0$. We conclude that there are no networks from $\mathcal{Q}_{63}$ which realise a biquadratic minimum function.

Consider finally the impedance $H_{64}(s)$ of a network from $\mathcal{N}_{64}$. In this case, $c_1 = x_2$, $c_2 = (x_1 + x_2)/(1 - x_1)$, and $x_3 = x_2/(1 - x_1)$ for $H_{64}(j\omega_0) = H_p(j\omega_0)$. Since $x_3 > 0$ we then require $1 - x_1 > 0$. Moreover, for $H_{64}(0) = H_p(0)$ and $H_{64}(\infty) = H_p(\infty)$ we require $g_1 = 1/(1 - W^2)$ and $g_2 = 1/W^2$. From equating coefficients of z^0, z^1, and z^5 in equation (173), and coefficients of z^4 in equation (174), we obtain

$$\begin{aligned} f(z)W^2/x_2 = {} & F^2W(1-x_1)(x_1+x_2)z^3 \\ & +FW(x_1+x_2)(W^2x_2(2-x_1)+W(1-x_1)-(1-x_1)^2)z^2 \\ & +F(1-x_1)((x_1+x_2)(1-x_1(1-W^2))+Wx_1x_2(1-W^2))z \\ & +W^2x_1x_2(1-W)(1+W)(1-x_1). \end{aligned}$$

Then, equating coefficients of z^2 in (174), and given $x_1, x_2, F > 0$, $0 < W < 1$, and $1 - x_1 > 0$, we require $(2 - x_1)(1 - W)x_2 - x_1(x_1 - (1 - W)) = 0$, which we may solve for x_2 to give

$$x_2 = \frac{x_1(x_1 - (1 - W))}{(2 - x_1)(1 - W)}.$$

Since $2 - x_1 > 1 - x_1 > 0$, $x_1 > 0$, and $W < 1$, we require $x_1 - (1 - W) > 0$. Then, by equating coefficients of z^4 in (173), and coefficients of z in (174), and given the constraints already established, we find

$$\begin{aligned} \gamma_1 = {} & (1-x_1)^2(1-W)^2(1+W)^2F \\ & -W(x_1W^4+(1-x_1)(1-W))(Wx_1+(1-W))(x_1-(1-W))^2 = 0, \end{aligned}$$

and

$$\begin{aligned} \gamma_2 = {} & (1-x_1)(1-W)(x_1W^2+(1-x_1)(1-W))F \\ & -W^3x_1(Wx_1+(1-W))(x_1-(1-W))^2 = 0. \end{aligned}$$

In particular, we require $R_0(\gamma_1(F), \gamma_2(F)) = -W(1-x_1)(1-W)(Wx_1+(1-W))(x_1-(1-W))^2(x_1W^3 - (1 - W)(1 - x_1))^2 = 0$. Since $0 < W < 1$, $1 - x_1 > 0$ and $x_1 - (1 - W) > 0$, this implies

$$x_1 = \frac{1 - W}{1 - W + W^3},$$

and then $\gamma_1 = 0$ implies

$$F = \frac{(1 - W + W^2)(1 + W)^2(1 - W)^4}{(1 - W + W^3)^2}.$$

$\mathcal{N}_k$	$H_k(0)$	$H_k(\infty)$	$\Im(H_k(j\omega_0))$
$\mathcal{N}_{31}$	R_1	R_1	$L_1\omega_0$
$\mathcal{N}_{32}$	R_2	R_2	$L_2\omega_0$
$\mathcal{N}_{33}$	$1/(1/R_1+1/R_2)$	R_2	$L_3\omega_0$
$\mathcal{N}_{34}$	$1/(1/R_1+1/R_2)$	R_2	$L_2\omega_0$
$\mathcal{N}_{35}$	R_1	R_2	$L_2\omega_0$
$\mathcal{N}_{36}$	$1/(1/R_1+1/R_2)$	R_2	$L_3\omega_0$
$\mathcal{N}_{37}$	R_2	R_2	C_1/ω_0
$\mathcal{N}_{38}$	R_1	R_2	$L_1\omega_0$
$\mathcal{N}_{39}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{40}$	R_2	R_2	$L_2\omega_0$
$\mathcal{N}_{41}$	$1/(1/R_1+1/R_2)$	R_2	$L_3\omega_0$
$\mathcal{N}_{42}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{43}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{44}$	R_1	R_2	$L_2\omega_0$
$\mathcal{N}_{45}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_2\omega_0$
$\mathcal{N}_{46}$	R_2	R_1	$L_3\omega_0$
$\mathcal{N}_{47}$	R_1	$1/(1/R_1+1/R_2)$	$L_1L_2C_3\omega_0^3$
$\mathcal{N}_{48}$	$1/(1/R_1+1/R_2)$	R_1	$L_2L_3C_2\omega_0^3$
$\mathcal{N}_{49}$	$1/(1/R_1+1/R_2)$	R_1	$(L_3C_1\omega_0^2-1)/(C_1(1-L_3C_2\omega_0^2)\omega_0)$
$\mathcal{N}_{50}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3(L_1+L_2)\omega_0/L_2$
$\mathcal{N}_{51}$	R_2	R_1	$L_3(L_1+L_2)\omega_0/L_2$
$\mathcal{N}_{52}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3(L_2-L_1)\omega_0/L_2$
$\mathcal{N}_{53}$	$1/(1/R_1+1/R_2)$	R_1	$(L_3C_1\omega_0^2-1)/(C_1(1-L_3C_2\omega_0^2)\omega_0)$
$\mathcal{N}_{54}$	R_1	R_2	$L_3(L_1+L_2)\omega_0/L_2$
$\mathcal{N}_{55}$	R_1	R_1	$-L_1(L_2+L_3)\omega_0/L_2$
$\mathcal{N}_{56}$	$1/(1/R_1+1/R_2)$	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{57}$	R_1	R_1+R_2	$L_2\omega_0$
$\mathcal{N}_{58}$	R_1	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{59}$	R_1	R_1+R_2	$L_3\omega_0$
$\mathcal{N}_{60}$	R_1	R_2	$L_3\omega_0$
$\mathcal{N}_{61}$	R_1	R_2	$L_2\omega_0$
$\mathcal{N}_{62}$	R_2	R_1	$L_1L_2\omega_0/(L_2+L_3)$
$\mathcal{N}_{63}$	$1/(1/R_1+1/R_2)$	R_1	$L_1(L_2+L_3)\omega_0/L_2$
$\mathcal{N}_{64}$	R_2	R_1+R_2	$L_1(L_2-L_3)\omega_0/L_2$

Table 5: The values of $H_k(0)$, $H_k(j\omega_0)$, and $H_k(\infty)$ for the impedance $H_k(s)$ of a network from the class $\mathcal{N}_k$ $(k = 31, 32, \ldots, 64)$.

We thus conclude that if N is from $\mathcal{Q}_{64}$ and has a biquadratic impedance, then N is also from the quartet $\mathcal{Q}_{69}$ in Fig. 50. □

We remark that the quartets $\mathcal{Q}_{72}$ and $\mathcal{Q}_{73}$, which are the only quartets which can realise all biquadratic minimum functions and comprise transformerless networks which contain exactly seven elements, comprise those networks obtained by the Reza-Pantell-Fialkow-Gerst simplification and our alternative simplification (described in Section 3.1.5), respectively.

3.7 On numbers of reactive elements in transformerless network realisations of other positive-real functions

In this section, we show how the techniques employed in the present paper may be extended to investigate the realisation of PR functions whose McMillan degree exceeds two.

In the discussion which follows Theorem 3.3.5, it is shown that there are minimum functions of McMillan degrees three, four, and five which can be realised by a SP network containing five reactive elements. Consequently, the Bott-Duffin procedure does not obtain SP networks which contain the minimum number of reactive elements for the realisation of all minimum functions. This does not exclude the possibility that the Bott-Duffin procedure is still minimal in the number of reactive elements used for *some* PR functions of higher McMillan degree than two. We are not able to settle this question completely. However, some parts of the present argument can be generalised to provide partial results for functions of higher McMillan degree. In particular, we demonstrate that there are PR functions of McMillan degree $2r$ which cannot be realised by SP networks containing fewer than $4r$ reactive elements, this result holding for any integer r.

We focus on PR functions whose real part is zero at r points on the positive imaginary axis. In Lemma 3.7.1, we show that if such a PR function is realised by a network containing fewer than $2r$ reactive elements then it must be lossless. In Lemma 3.7.2, we show that if such a function is a minimum function then any network realisation contains at least $4r$ reactive elements. In Lemma 3.7.3, we then demonstrate the existence of such a minimum function whose McMillan degree is equal to $2r$.

Lemma 3.7.1. *Let N be a network containing at most $2r-1$ reactive elements. Suppose N has impedance (admittance) $H(s)$ satisfying $\Re(H(j\omega_i)) = 0$ for $\omega_i > 0$, $i = 1, 2, \ldots, r$, with $\omega_i \neq \omega_j$ for $i \neq j$. Then $H(s)$ is lossless.*

Proof. Consider the function $P(s) = H(s)+H(-s)$. Since $H(s)$ is PR and $\Re(H(j\omega_i)) = 0$, then $P(s)$ has zeros of even multiplicity at $s = \pm j\omega_i$ for $i = 1, 2, \ldots, r$ (see, e.g. [36, Section 9.4]). It follows that either $P(s)$ has $4r$ or more zeros on $j\mathbb{R} \cup \infty$, or $P(s)$ is identically zero there. Hence, either the McMillan degree of $P(s)$ is greater than or equal to $4r$, or $P(s) \equiv 0$. Since N contains at most $2r-1$ reactive elements, then the McMillan degree of $P(s)$ is at most $4r-2$. Hence $P(s) \equiv 0$ which implies $H(s)$ is lossless. $\square$

Lemma 3.7.2. *Let N be a SP network and let $H(s)$ be the impedance of N where $H(s)$ is a minimum function with minimum frequencies $\omega_i > 0$ for $i = 1, 2, \ldots, r$ with $\omega_i \neq \omega_j$ for $i \neq j$. Then N contains at least $4r$ reactive elements and at least two resistors.*

Proof. By definition $\Re(H(j\omega_i)) = 0$ and $\Im(H(j\omega_i)) \neq 0$ for $i = 1, 2, \ldots, r$, and $H(s)$ has no poles or zeros on $j\mathbb{R}\cup\infty$. It is straightforward to show that $1/H(s)$ also satisfies these three conditions.

Suppose initially N is ES with $N = N_u + N_v$ so $H(s) = Z_u(s) + Z_v(s)$ where $Z_u(s)$, $Z_v(s)$ are the impedances of N_u, N_v. Then $\Re(Z_u(j\omega_i)) = \Re(Z_v(j\omega_i)) = 0$ for $i = 1, 2, \ldots, r$ and neither $Z_u(s)$ nor $Z_v(s)$ has a pole on $j\mathbb{R}\cup\infty$ by Lemma 3.3.1. In particular neither $Z_u(s)$ nor $Z_v(s)$ are lossless hence both N_u and N_v contain a resistor. Furthermore, by Lemma 3.7.1, both N_u and N_v must contain at least $2r$ reactive elements. The case where N is EP is similar. Combining the ES and EP case shows that N contains at least $4r$ reactive elements and at least two resistors. $\square$

Lemma 3.7.3. *Let $H(s) := p(s)/q(s)$, where*

$$p(s) := m(1)(n(s) + sm(s)) + m(s)(n(1) + m(1)), \tag{179}$$

$$\text{and} \quad q(s) := sm(s)(m(1) + n(1)) + m(1)(m(s) + sn(s)), \tag{180}$$

$n(s)$ and $m(s)$ are monic coprime polynomials in s with degree $2r-1$ and $2r$ respectively, all the roots of $n(s)$ and $m(s)$ are on $j\mathbb{R}$ and have multiplicity one, and roots of $n(s)$ interlace those of $m(s)$. Then $H(s)$ is a minimum function with McMillan degree $2r$ and with minimum frequencies $\omega_i > 0$ for some ω_i, $i = 1, 2, \ldots, r$, with $\omega_i \neq \omega_j$ for $i \neq j$.

Proof. From the properties of $n(s)$ and $m(s)$, we have $n(s) = s\prod_{k=1}^{r-1}(s^2 + \tilde{\omega}_k^2)$, and $m(s) = \prod_{k=1}^{r}(s^2 + \omega_k^2)$, for some $0 < \omega_1 < \tilde{\omega}_1 < \ldots < \omega_{r-1} < \tilde{\omega}_{r-1} < \omega_r$. Let $X(s) := n(s)/m(s)$, so $X(s)$ is lossless and $X(1) > 0$. It follows that $H(s) = p(s)/q(s)$ is PR, since

$$H(s) = \frac{1}{\frac{1}{1+X(1)+X(s)} + s} + \frac{1}{\frac{1}{s} + 1 + X(1) + X(s)}, \tag{181}$$

which is a composition of sums and inverses of PR functions (see footnote 24).

First, note that $p(s)$ and $q(s)$ are polynomials of degree $2r+1$, so in particular $H(s)$ has neither a pole nor a zero at $s = \infty$. Now, suppose $j\alpha$ is a root of $p(s)$ (resp. $q(s)$) for some $\alpha \in \mathbb{R}$. Then, $(s^2+\alpha^2)$ must be a factor of $p(s)$ (resp. $q(s)$), and so $(s^2+\alpha^2)$ must be a factor of both the even and odd parts of $p(s)$ (resp. $q(s)$). It may be seen that this is not possible, noting that (179) and (180) express $p(s)$ and $q(s)$ in terms of even and odd parts, and that $m(s)$ and $n(s)$ are coprime. Hence, $H(s)$ has no poles or zeros on $j\mathbb{R} \cup \infty$. Moreover, since $m(j\omega_k) = 0$, then $H(j\omega_k) = -j/\omega_k$ for $k = 1, 2, \ldots, r$. Hence, $H(s)$ is a minimum function with minimum frequencies $\omega_k > 0$ for $k = 1, 2, \ldots, r$. It follows that the McMillan degree of $H(s)$ is at least 2r, and at most $2r+1$ since $p(s)$ and $q(s)$ have degree $2r+1$. To see that the McMillan degree is exactly $2r$, note that $p(-1) = m(1)n(-1) + m(-1)n(1) = -q(-1)$. Since $m(-1) = m(1)$ and $n(-1) = -n(1)$, this implies $p(-1) = q(-1) = 0$, and hence $(s+1)$ is a factor of both $p(s)$ and $q(s)$. □

Remark 3.7.4.
The function $H(s)$ in Lemma 3.7.3 is a particular construction of a minimum function with r minimum frequencies $\omega_1, \omega_2, \ldots, \omega_r$ ($\omega_i > 0, i = 1, 2, \ldots, r$). The intuition behind this construction is based on the Bott-Duffin procedure. In particular, it may be verified that $H(s)$ in (181) satisfies $H(1) = 1$, and so $1 \times H(1) = j\omega_k \times H(j\omega_k)$ for $k = 1, 2, \ldots r$. It follows that $R(s)$ in equation (142) has poles at $s = \pm j\omega_k$ ($k = 1, 2, \ldots, r$). Since, in addition the McMillan degree of $R(s)$ is no greater than that of $H(s)$ (see Section 3.1.3), then $R(s)$ can be written as a sum of a positive constant and a lossless function with poles at $\pm j\omega_k$ ($k = 1, 2, \ldots, r$) in the manner of Theorem 3.1.4. Indeed, $X(s)$ is a lossless function with poles at $s = \pm j\omega_k$ ($k = 1, 2, \ldots, r$), and setting $R(s) = 1 + X(1) + X(s)$ and rearranging we arrive at equation (181).

Lemmas 3.7.2 and 3.7.3 can now be combined to give the following theorem.

Theorem 3.7.5. *Let $H(s)$ be the PR function of McMillan degree $2r$ which is defined in Lemma 3.7.3. If $H(s)$ is the impedance of a SP network N then N contains at least $4r$ reactive elements and at least two resistors.*

Part 4

Conclusions

This thesis represents a further contribution towards the understanding of passive network synthesis, with a particular focus on the number of reactive elements required for the realisation of a given positive-real function. The thesis was divided into three parts, and the principal results for each part were presented as theorem statements. In this final section, we provide a summary of these results, referring to the corresponding theorems in the main text.

4.1 Conclusions from Part 1

In **Theorem 1.4.1**, we presented a kernel description of the driving-point behaviour of a general transformerless network. In particular, we showed that the autonomous part of any driving-point trajectory of a transformerless network must decay to zero as $t \to \infty$.

Theorem 1.6.1 described a parametrisation of the behaviour of a general transformerless network. In particular, it was shown that the autonomous part of any trajectory of a transformerless network is bounded into the future. However, unlike the autonomous part of the driving-point trajectory, the autonomous part of the trajectory need not decay to zero as $t \to \infty$.

In **Definition 1.7.1** and **Definition 1.7.3**, we formalised the phasor approach to the analysis of transformerless networks through the notions of an s_0-trajectory, and s_0-driving-point trajectory, and the s_0-impedance and s_0-admittance. Then, **Theorem 1.7.4** proved that, for a given transformerless network N and a given s_0 in the closed right half plane, the s_0-impedance (resp. s_0-admittance) exists if and only if the impedance $Z(s)$ (resp. admittance $Y(s)$) of the network does not have a pole at $s = s_0$, in which case it is equal to the value $Z(s_0)$ (resp. $Y(s_0)$). This allows the value of the impedance at a particular point in the closed right half plane to be determined by finding an s_0-trajectory for the network. However, in Section 1.7, we showed that the s_0-impedance may differ from the value of the impedance of a network at s_0 for certain

s_0 in the open left half plane.

In **Definition 1.8.1**, we extended the notion of an s_0-trajectory and the s_0-impedance and s_0-admittance to cover the point at ∞. Then, in **Theorem 1.8.3**, we showed how Theorem 1.7.4 can be extended to include the point at ∞.

4.2 Conclusions from Part 2

In **Theorem 2.1.1**, we showed that the number of capacitors (resp. inductors) in a minimally-reactive reciprocal network is equal to the number of positive (resp. negative) eigenvalues of the Hankel matrix for the network's impedance. **Theorem 2.1.2** showed how these figures may be obtained algebraically by evaluating the leading principal minors of the Hankel matrix. Equivalent results in terms of the Cauchy index for the impedance between $-\infty$ and $+\infty$, and in terms of a Sylvester matrix corresponding to the impedance, were then stated in **Theorem 2.2.1** and **Theorem 2.2.2** respectively.

Definition 2.3.1 introduced the notion of the extended Cauchy index for a real-rational function. In **Theorem 2.3.3**, we showed how the difference between the number of capacitors and the number of inductors in any minimally-reactive reciprocal network (including those whose impedance is non-proper) is equal to the extended Cauchy index of the network's impedance. Then, in **Theorem 2.3.4**, we showed how these numbers may be calculated algebraically by considering the leading principal minors of a Sylvester matrix corresponding to the impedance.

In Section 2.4, we showed how equivalent results to the preceding sections may be stated in terms of a Bezoutian matrix corresponding to the network's impedance. Specifically, in **Theorem 2.4.1**, it is shown that the number of capacitors (resp. inductors) in a minimally-reactive network is equal to the number of positive (resp. negative) eigenvalues of a Bezoutian matrix corresponding to the network's impedance. Moreover, it is shown how these numbers may be determined algebraically by considering the leading principal minors of the Bezoutian matrix.

The results are extended to the case of networks which need not by minimally reactive in **Theorem 2.5.1**. That theorem states that the number of capacitors (resp. inductors) in any reciprocal network is greater than or equal to the number of positive (resp. negative) eigenvalues of a Bezoutian matrix corresponding to the network's impedance. Alternative equivalent characterisations are also given. Further extensions to multi-port reciprocal networks are then made in **Theorem 2.6.1** and **Theorem 2.6.5**. In the latter of these two theorems, lower bounds are stated on the numbers of capacitors and inductors in any multi-port reciprocal network in terms of a Bezoutian matrix corresponding to a hybrid matrix for the network.

In **Theorem 2.7.4**, it is shown that the impedance of a network which contains only resistors, capacitors, and transformers is also realised by a Cauer form network. Moreover, explicit expressions for the element parameters in the Cauer form network are provided in terms of the overall network impedance function.

4.3 Conclusions from Part 3

Theorem 3.1.5 provided four procedures for the realisation of a general minimum function as the impedance of a transformerless network. These include the Bott-Duffin procedure, and the one know simplification to that procedure (which we call the Reza-Pantell-Fialkow-Gerst simplification). In addition, two new alternative procedures are provided. One of these procedures produces series-parallel networks which contain the same number of reactive elements and the same number of resistors as the networks obtained by the Bott-Duffin procedure. The second provides a simplification to Bott-Duffin which produces transformerless networks which contain the same number of reactive elements and the same number of resistors as the networks obtained by the Reza-Pantell-Fialkow-Gerst simplification to Bott-Duffin's procedure. It is further shown how the networks obtained by each of these procedures may be obtained from the networks obtained by Bott and Duffin through a sequence of network transformations.

In **Theorem 3.3.5**, we described those minimum functions which can be realised by series-parallel networks which contain at most five reactive elements, and we showed that all such minimum functions are realised by a single network quartet. Then, in **Theorem 3.3.6**, we proved that there are no biquadratic minimum functions which can be realised by series-parallel networks which contain fewer than six reactive elements. We conclude that the networks produced by the Bott-Duffin procedure contain the least possible number of reactive elements for the realisation of a biquadratic minimum function among all series-parallel networks.

In **Theorem 3.4.3**, it is shown that those series-parallel network which contain exactly six reactive elements and exactly two resistors and which realise a minimum function belong to one of ten quartets. Then, in **Theorem 3.4.4**, it is shown that only two of those quartets contain networks which realise biquadratic minimum functions, these quartets comprising those networks obtained by the Bott-Duffin procedure, and those obtained by our series-parallel alternative to Bott-Duffin which was described in

Theorem 3.1.5.

In **Theorem 3.5.14**, we described those *transformerless* networks which contain fewer than five reactive elements and which realise a minimum function. In **Corollary 3.5.6**, it was shown that such networks necessarily contain at least three reactive elements. Then, in **Corollary 3.5.15**, we showed that those minimum functions which are realised by transformerless networks which contain three reactive elements are also realised by a single network quartet. Next, in **Corollary 3.5.16**, we showed that those additional minimum functions which are realised by transformerless networks containing four reactive elements are also realised by a further five network quartets.

Theorem 3.5.17 described those biquadratic minimum functions which are realised by transformerless networks which contain exactly three reactive elements. It is shown that such functions are realised by a single network quartet. Then, in **Theorem 3.5.18**, those additional biquadratic minimum functions which can be realised by transformerless networks which contain exactly four reactive elements are described, and it is shown that such additional functions are also realised by a single network quartet. From these two theorems. we concluded that the networks produced by the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure, and our alternative simplification described in Theorem 3.1.5, contain the least possible number of reactive elements for the realisation of almost all biquadratic minimum functions among the class of transformerless networks.

In **Theorem 3.6.5**, we showed that if N is a transformerless network which contains at most five reactive elements and at most seven passive elements in total and which realises a minimum function then N belongs to one of 42 quartets. Then, in **Theorem 3.6.6**, we showed that only eleven of these quartets contain networks which can realise a biquadratic minimum function. Of these, only four quartets realise sets of biquadratic minimum functions of non-zero measure. Moreover, only two such quartets can realise all of the biquadratic minimum functions, these quartets comprising the networks obtained by the Reza-Pantell-Fialkow-Gerst simplification to the Bott-Duffin procedure, and our alternative simplification described in Theorem 3.1.5.

Finally, **Theorem 3.7.5** demonstrated the existence of positive-real functions of McMillan degree $2r$ which cannot be realised by series-parallel networks which contain fewer than $4r$ reactive elements.

4.4 Directions for future research

This thesis has placed lower bounds on the numbers of inductors, capacitors and resistors required for the realisation of certain types of positive-real functions by transformerless networks. The question of the minimum numbers of inductors, capacitors and resistors required for the realisation of a general positive-real function remains open. One avenue for future research is to establish a complete description of those positive-real functions which can be realised by transformerless networks which contain certain specific numbers of inductors and capacitors. Coupled with this, it is instructive to find sets of networks which contain those exact numbers of inductors and capacitors and which collectively realise all such positive-real functions. In addition, it is preferable for the networks in these sets to contain the least possible number of resistors.

Reichert [25] provides an answer to these questions in the case of transformerless networks which contain exactly one capacitor and exactly one inductor. In [25], those positive-real functions which can be realised by transformerless networks containing exactly one inductor and one capacitor are described. Furthermore, a set of networks is provided which collectively realise all such functions, with each network containing one capacitor, one inductor, and three resistors. The proof provided in that paper is complex, and it is not presently clear how the arguments may be extended to networks containing a greater number of reactive elements. This provides motivation for developing a new approach to the Reichert problem which is more readily generalised to networks with greater numbers of elements.

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