# Worldsheet methods for perturbative quantum field theory

**Eduardo Casali**

Department of Applied Mathematics and Theoretical Physics

University of Cambridge

This dissertation is submitted for the degree of

*Doctor of Philosophy*

Trinity College

October 2015

In loving memory of my mother.

# Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Eduardo Casali
October 2015

# Acknowledgements

I would like first and foremost to thank David for everything I've learned from (and with) him in the past four years, and for taking me up as a student at a time when his own academic situation was uncertain. I'm very happy with how that gamble panned out. It has been a joy to work with Dave and, even if I have yet to master his method of just guessing the right answer, I learned from his sheer enthusiasm to just keep trying.

I would like to thank Freddy for all his support while I was at the Perimeter Institute, without which I wouldn't have been able to keep working with Dave. I also thank Debbie and Diana for all the help they gave me in navigating Canadian bureaucracy and Montes with whom I shared an office and quite a few interesting discussions.

I would also like to thank Tim for his patience during the many times I complained about my knee, and perhaps more importantly, for all the discussions and the fruitful collaborations that resulted from them. I also thank Arnab and Piotr for their excellent company and insightful ideas, my time at DAMTP wouldn't have been as fun without them.

Lastly I would like to thank my father and sister for their unconditional support during my studies.

# Abstract

This thesis is divided into two parts. The first part concerns the study of the ambitwistor string and the scattering equations, while the second concerns the interplay of the symmetries of the asymptotic null boundary of Minkowski space, called $\mathscr{I}$, and scattering amplitudes.

The first part begins with a review of the CHY formulas for scattering amplitudes, the scattering equations and the ambitwistor string including its pure spinor version. Next are the results of this thesis concerning these topics, they are: generalizing the ambitwistor model to higher genus surfaces; calculating the one-loop NS-NS scattering amplitudes and studying their modular and factorization properties; deriving the one-loop scattering equations and analyzing their factorization; showing that, in the case of the four graviton amplitude, the ambitwistor amplitude gives the expected kinematical prefactor; matching this amplitude to the field theory expectation in a particular kinematical regime; solving the one loop scattering equations in this kinematical regime; a conjecture for the IR behaviour of the one-loop ambitwistor integrand; computing the four graviton, two-loop amplitude using pure spinors; showing that this two-loop amplitude has the correct kinematical prefactor and factorizes as expected for a field theory amplitude; generalizing the ambitwistor string to curved backgrounds; obtaining the field equations for type II supergravity as anomaly cancellation on the worldsheet; generalizing the scattering equations for curved backgrounds.

The second part begins with a review of the definition of the null asymptotic boundary of four dimensional Minkowski space, its symmetry algebra, and their relation to soft particles in the S-matrix. Next are the results of this thesis concerning these topics, they are: constructing two models consisting of maps from a worldsheet to $\mathscr{I}$, one containing the spectrum of $\mathcal{N} = 8$ supergravity, and the other the spectrum of $\mathcal{N} = 4$ super Yang-Mills; showing how certain correlators in these theories calculate the tree-level S-matrix of $\mathcal{N} = 8$ sugra and $\mathcal{N} = 4$ sYM respectively; defining worldsheet charges which encode the action of the appropriate asymptotic symmetry algebra and showing that their Ward-identities recover the soft graviton, and soft gluon factors; defining worldsheet charges for

proposed extensions of these symmetry algebras and showing that their Ward-identities give the subleading soft graviton and subleading soft gluon factors.

# Table of contents

# Chapter 1

# Introduction

The framework of quantum field theory (QFT) started, as the name indicates, by applying the principles of the then new quantum mechanics to fields, notably the electromagnetic field. At first particles were seen as distinct entities from fields, but after a few years it became clear that they were better described by quantized fields, thus fields came to describe both matter and interactions. As the decades of the last century passed we learned more and more about QFT's; the infinities that initially plagued the calculations were tamed by the idea of renormalization, observables were calculated to astonishing precision, QFTs found use beyond its initial application to high energy physics, etc. Among these, the concepts of renormalization group flow and effective field theory stand out as critical to our modern understanding of QFTs and why they are ubiquitous at low energy. But progress was not always smooth, describing the strong interaction using QFTs proved to be a challenge at first, and the approach almost fell out of favour until the advent of Yang-Mills theory and asymptotic freedom. Through many transformations and revolutions QFTs have shown to be immensely useful and natural to describe most diverse physical phenomena, be in the area of particle physics, cosmology or condensed matter. Even in mathematics QFTs have recently found beautiful applications, specially in certain areas of topology and geometry.

It is fair to say that QFTs are one of the most successful frameworks in the history of physics, and given its age it might be expected that by know we know pretty well what QFTs are, and front a certain point of view this is indeed true. Given a Lagrangian we can set up a perturbation theory around some classical vacuum and calculate observables order by order, any textbook on QFT will tell you how to do it. But consider that a quick glance at QFTs textbooks shows that there's not much agreement even on how these basics should should be taught. Canonical quantization first then path-integrals, or perhaps start directly with path-integrals. Do all free fields first then introduce interactions, or go through each helicity in detail first. Do all the maths of classical field theory and functional calculus or go straight into cross sections. Or even start with Feynman diagrams and fill in all the details later. This shows that although we have pretty good operational knowledge of the subject there's no agreed upon streamlined derivation of the framework from basic principles. This is a sign that we're missing some deep principles or idea on QFTs, and the way we are used to approach them might not be the best suited one to understand these issues.

Case in point, the central role that Lagrangians play on QFT. The usual story goes that a Lagrangian defines a QFT through some quantization procedure, and all one needs to know is the Lagrangian. But consider the cases when a QFT has several equivalent Lagrangian descriptions, so there is no uniqueness linking a classical theory to a QFT. Or even more extreme are the cases where no Lagrangian description is available, such as the mysterious six dimensional $(0,2)$ superconformal theory. The existence of these cases signals that the usual spacetime Lagrangian formulation does not capture the full richness of QFTs. But even when a unique Lagrangian is available it doesn't mean that's the better approach to study the QFT. For example, there are cases when a QFT can be solved exactly (through integrability or otherwise) and in these cases the Lagrangian description is not really useful or needed in the final answer. Lastly take the striking duality between a certain class of QFTs with conformal symmetry in $d$ dimensions, and quantum gravity in asymptotically Anti-de-Sitter (AdS) spacetimes. Here, calculations in a non gravitational strongly coupled field theory which would be inaccessible through usual techniques can be carried out in the AdS side using usual perturbative expansion in semiclassical supergravity.

Another strong evidence that there's more to QFTs than what we presently know is the recent progress in the area of scattering amplitudes. Though these observables have been studied since the beginnings of the subject, it is only recently that we started to uncover an enormous amount of structure encoded in them [1]. These properties of scattering amplitudes are not accessible from the usual spacetime Lagrangian formulation of QFT. To see them we have to think about QFTs from a different angle, using a mixture of tools and ideas borrowed from S-matrix theory, string theory and even pure maths.

One particularly useful tool came from twistor theory. Originally used to study classical gravitation, twistors have been crucial in unravelling the structure of scattering amplitudes, most notably in the context of $\mathcal{N}=4$ supersymmetric Yang-Mills (sYM) theory. Of particular interest for this thesis (and the origin of the use of twistors in scattering amplitudes) is the work of Witten and Berkovits on the twistor string [2–4].The idea of

[1] A comprehensive though not exhaustive review is [1]

these works was to construct a string theory whose target space was not spacetime but an associated space called twistor space, which has a non-local relationship to spacetime. Heuristically, a line in twistor space corresponds to a point in spacetime while a point in twistor space corresponds to a null ray in spacetime[2]. This mild non-locality brought forth the simplicity of scattering amplitudes of $\mathcal{N}=4$ sYM, which could be written very compactly as an integral over the moduli space of maps to twistor space. Very interestingly this integral was completely fixed by delta-functions insertions which impose a set of equations now know as the *scattering equations.* The scattering equations seem to be the backbone of massless scattering in QFTs, as they are a crucial part of the original twistor string, and other twistor string like models describing massless field in spacetime. Their study is and generalization is one of the main parts of this thesis and much more about them will be discussed in the next chapters. While the original twistor string is not a complete description of $\mathcal{N}$=4 sYM it nevertheless led to many new techniques and insights into the study of scattering amplitudes. And perhaps more importantly its existence gave more evidence that, even at the perturbative level, there might be better ways to think about QFTs.

This thesis is then a humble step in the program to push forwards our understanding of these new structures and formulations of QFT. Specifically, I focus on using worldsheet methods originating from the twistor string to study effectively new perturbative formulations of QFTs. In this regard this thesis accomplishes a few small but significant goals, in particular: QFT loop amplitudes are described using a worldsheet theory without the need to take any low energy limit; non-linear equations of motion for the target space QFT are obtained from worldsheet consistency conditions; and holographic descriptions of tree-level scattering are given by means of a worldsheet theory.

The above results form the bulk of this thesis which is structured in two parts: The first part, consisting of chapters 2, 3 and 4, deals with a worldsheet theory called the ambitwistor string. Chapter 2 is a review of the relevant material for the next two chapters, that includes the Cachazo-He-Yuan formulas for massless scattering, the scattering equations and the RNS and pure spinor formulations of the ambitwistor string. The definition of the ambitwistor string at loop level is given on chapter 3, where one and two loop amplitudes are discussed as well as the generalization of the scattering equations. Chapter 4 deals with the definition of the ambitwistor string for curved target spaces. The second part of the thesis is comprised of the last chapter 5 which introduces two worldsheet models, one for supergravity and the other for super Yang-Mills. What is notable about these models is that their target space is the null boundary of Minkowski

[2]More precisely to an $\alpha$-plane in complexified Minkwoski spacetime.

spacetime. Correlation functions of these models reproduce the tree-level S-matrices of $\mathcal{N} = 4$ sYM and $\mathcal{N} = 8$ supergravity, and thus gives an effective holographic descriptions of these theories in a certain parameter space. The beginning of every chapter starts with a brief introduction and review of the relevant literature, and more details can be found in the original papers [5–10].

# Chapter 2

# Review

## 2.1 CHY formulas

The Cachazo-He-Yuan (CHY) formulas, introduced in a series of papers [11–13], present tree-level scattering amplitudes for massless particles in a very interesting form. Amplitudes are given by evaluating a rational function on solutions to a set of equations, called the scattering equations, and summing over all the solutions. That is, an $n$-particle amplitude can be written as

$$\mathscr{A}_n^{(0)} = \sum_{\text{sols}} \frac{\mathrm{I}_n(k,\epsilon)}{\mathrm{J}(k)} \tag{2.1}$$

where $\mathrm{I}_n(k,\epsilon)$ is a function of the external momenta and polarization vectors, and $\mathrm{J}(k)$ is the Jacobian obtained from solving the scattering equations. This formula can be recast in a form that is easier to manipulate by writing it as an integral over the moduli space of a $n$-punctured sphere. Let $z_i$ label the position of these punctures, then the amplitude can be written as

$$\mathscr{A}_n^{(0)} = \int \frac{\prod_{i=1}^n \mathrm{d}z_i}{\text{Vol SL}(2|\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(z_i))\mathrm{I}_n(k,\epsilon,z). \tag{2.2}$$

Here the measure is written in a permutation invariant way by dividing by Vol Sl(2|$\mathbb{C}$), fixing this redundancy introduces a Jacobian. For example fixing $\{z_1, z_2, z_3\}$ to some arbitrary values gives

$$\frac{1}{\text{Vol Sl}(2|\mathbb{C})} \to \frac{z_{12}z_{23}z_{31}}{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}. \tag{2.3}$$

The scattering equations are imposed by

$$\prod_i{}' \bar{\delta}(k_i \cdot P(z_i)) = \frac{z_{12}z_{23}z_{31}}{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3} \prod_{i=4}^{n} \bar{\delta}(k_i \cdot P(z_i)) \tag{2.4}$$

where the holomorphic delta functions are to be interpreted as

$$\bar{\delta}(z) = \bar{\partial}\frac{1}{z} = \mathrm{d}\bar{z}\frac{\partial}{\partial \bar{z}}\frac{1}{z}. \tag{2.5}$$

The scattering equations themselves are the set of equations

$$k_i \cdot P(z_i) = \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \quad i \in \{1, \cdots, n\}. \tag{2.6}$$

They are invariant under SL(2|ℂ) transformation so only $n-3$ of them are independent. This is the same as the (complex) dimension of moduli space of a $n$-punctured sphere. Thus, for generic external kinematics, the SL(2|ℂ) invariant combination of delta functions (2.4) is enough to completely localize the integral (2.2).

The above prescription is generic, what differs from one set of massless amplitudes from another is the integrand $\mathrm{I}_n$. This is in general a function of external momenta, polarization vectors and the location of the punctures. Originally, CHY found integrands for three classes of tree-level amplitudes: Einstein gravity coupled to a dilaton and B-field; Yang-Mills; and massless coloured cubic scalar. But the CHY formulas are not limited to describing just these three cases, in fact the number of theories whose amplitudes can be expressed in this form is quite large, for and extensive list see [14–16]. These include theories with very complicated Lagrangian description such as Einstein-Yang-Mills, Dirac-Born-Infeld and non-linear sigma models. The existence of CHY formulas for such theories is compelling evidence that not only the CHY formula is in a sense universal for tree-level massless scattering, but also that there should be a better description of these theories, at least at the perturbative level.

It is quite remarkable that the integrands for the three original theories can be constructed from two building blocks. One of them are Parke-Taylor colour factors

$$\mathscr{C}_{i_1,1_2,\ldots,i_n} = \sum_{w\in S_n/\mathbb{Z}_n} \frac{\mathrm{Tr}(T^{I_{w(i_1)}}T^{I_{w(i_2)}}\ldots T^{I_{w(i_n)}})}{z_{w(i_1),w(i_2)}z_{w(i_2),w(i_3)}\cdots z_{w(i_n),w(i_1)}} \tag{2.7}$$

which encode all the possible tree level colour orderings of pure YM. And the other is the Pfaffian of a $2n \times 2n$ antisymmetric matrix defined as

$$\Psi = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}. \tag{2.8}$$

Its building blocks are the matrices $A, B$ and $C$ defined as

$$A_{ij} = \begin{cases} \dfrac{k_i \cdot k_j}{z_i - z_j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \qquad B_{ij} = \begin{cases} \dfrac{\epsilon_i \cdot \epsilon_j}{z_i - z_j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \tag{2.9}$$

and

$$C_{ij} = \begin{cases} \dfrac{\epsilon_i \cdot k_j}{z_i - z_j} & \text{if } i \neq j \\ -\sum_{l \neq i} \dfrac{\epsilon_i \cdot k_l}{z_i - z_l} & \text{if } i = j \end{cases} \tag{2.10}$$

The matrix $\Psi$ has two zero modes, so in order to get a non-zero result two lines and two columns must be removed from it before calculating the Pfaffian. Denoting the matrix with columns and lines $\{i, j\}$ removed as $\Psi^{ij}$, the pseudo-Pfaffian is defined as

$$\mathrm{Pf}'\,\Psi = \frac{(-1)^{i+j}}{z_i - z_j}\,\mathrm{Pf}\,\Psi^{ij}. \tag{2.11}$$

For example, a gravitational amplitude in the CHY form is written as

$$\mathscr{A}_n^{(0)} = \int \frac{\prod_{i=1}^n \mathrm{d}z_i}{\mathrm{Vol}\,\mathrm{SL}(2|\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(z_i))\,\mathrm{Pf}'\,\Psi\,\mathrm{Pf}'\,\tilde{\Psi} \tag{2.12}$$

where $\tilde{\Psi}$ is define analogously to $\Psi$ but with tilded polarization vectors. While a colour-ordered Yang-Mills amplitude is written as

$$\mathscr{A}_n^{(0)} = \int \frac{\prod_{i=1}^n \mathrm{d}z_i}{\mathrm{Vol}\,\mathrm{SL}(2|\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(z_i))\,\mathrm{Pf}'\,\Psi \frac{1}{z_{12} \dots z_{n1}}. \tag{2.13}$$

It can be shown that these formulas don't depend on which columns and rows have been removed, nor in how one fixes the SL(2|ℂ) redundancy. Consistency checks on these formulas were done in the original works and a proof was given not much later by Dolan and Goddard in [17]. Comparing (2.12) and (2.13) the only difference is the replacement of the Parke-Taylor factor by a Pfaffian

$$\frac{1}{z_{12}\dots z_{n1}} \to \text{Pf}'\,\Psi. \tag{2.14}$$

This is reminiscent of colour-kinematics duality and this connection was studied in the original CHY papers [12, 13]. Colour-kinematic duality is a fascinating subject but it is not the subject of this thesis, the interested reader is referred to the comprehensive review [1].

## 2.2 Scattering equations

The backbone of the CHY formulas are the scattering equations:

$$k_i \cdot P(z_i) = \sum_{j\neq i}^{n} \frac{k_i \cdot k_j}{z_i - z_j} = 0. \tag{2.15}$$

Invariance under SL(2, $\mathbb{C}$) transformations require conservation of external momenta, and implies that only $n-3$ of the scattering equations are linearly independent. For generic external kinematics the number of solutions to the scattering equations is $(n-3)!$. At low points explicit solutions are known, and numerical solutions can be found to higher points but it quickly becomes too time consuming and other methods have to be used[18–23]. One way of thinking about the scattering equations is as vanishing conditions on the square of a map $P : \Sigma_n \to \mathbb{C}^n$, from a $n$-punctured Riemann surface to the space of null external momenta:

$$P_\mu(z) = \sum_{i=1}^{n} \frac{k_{i\mu}}{z - z_i}. \tag{2.16}$$

It'll become clear later when the ambitwistor string is reviewed that $P$ is better thought as a meromorphic section of the worldsheet canonical bundle. An external momenta is associated to each marked point through the residues of $P$. The field $P^2$ is then a meromorphic quadratic differential, with possible double and simple poles. For on-shell external momenta the coefficient of the double poles vanish and the residue of its simple poles are

$$\text{Res}_{z_i} P^2(z) = \text{Res}_{z_i} \sum_{l\neq j} \sum_{j} = \frac{k_l \cdot k_j}{(z - z_l)(z - z_j)} = \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j}. \tag{2.17}$$

These are the same as the scattering equations, setting $n-3$ of them to zero kills $n-3$ of the poles of $P^2$. Since a meromorphic quadratic differential on a sphere must have at least four poles (Counted with multiplicity), these are enough conditions to set $P^2 = 0$ everywhere.

One of the most important features of the scattering equations is how they relate boundaries of the moduli space of Riemann surfaces to vanishing of kinematic invariants [11]. The moduli space of Riemann surfaces of genus $g$ and with $n$ marked points, denoted by $\mathcal{M}_{g,n}$, admits a compactification $\overline{\mathcal{M}}_{g,n}$, which is the Deligne-Mumford moduli space of marked curves. Heuristically, the Deligne-Mumford compactification adds to $\mathcal{M}_{g,n}$ nodal curves at infinity when either marked points collide, or the surface develops a long thin neck. At genus zero there is only one kind of degeneration, when one or more marked points approach each other. In this case the the boundary looks like the product of two genus zero Riemann surfaces connected by a nodal point; the colliding marked points are spread out in one of the surfaces, while the remaining points stay in the other sphere[1]. Given a subset $I$ of the external momenta there is an obvious kinematic invariant

$$(\sum_{i\in I} k_{i\mu})^2 \tag{2.18}$$

which is uniquely defined up to momentum conservation. At tree level, a kinematic invariant vanishing means that an internal propagator is going on-shell, that is, a factorization channel of the amplitude is being approached. Consider a subset of external momenta $I = \{1, 2, \cdots, m\}$, and do a change of coordinates on the sphere:

$$z_i = z_n + q w_i \quad i \in \{m+1, \cdots, n-1\}. \tag{2.19}$$

The parameter $q$ controls the degeneration of the surface when the points $\{m+1, \cdots, n\}$ collide, at the point $q = 0$ the original sphere pinches into two spheres joined at a nodal point [25]. Multiplying the $i$-th equation by $z_{in}$ and summing them up gives, after some manipulations:

$$(k_1 + \cdots k_m)^2 = q(\frac{w_{m+1}}{z_{1m+1}} k_1 \cdot k_{m+1} + \cdots + \frac{w_{n-1}}{z_{mn-1}} k_m \cdot k_{n-1}). \tag{2.20}$$

This explicitly links the boundary of the moduli space approached when $q \to 0$ to the factorization channel when the sum of external momenta $(\sum_{i\in I} k_i)^2$ becomes null. In this limit the scattering equations also factorize into two sets, one for each sphere with and added extra point (the nodal point) carrying null momenta $k_I = \sum_{i\in I} k_i$. This factorization is most easily seen from the CFT perspective given by the ambitwistor. In latter sections

[1] In the context of superstring theory and superRiemann surfaces this is reviewed in[24].

I'll show that these factorization properties carry over to higher genus surfaces with a few caveats.

Interestingly the same set of equations had already appeared in the study of dual models in the work of Farlie and Roberts [26–28] where the idea of localization to the solution of the scattering equations was already present. They also appeared later in the works of Gross and Mende on the high energy limit of string scattering [29, 30] and in a different guise in the Berkovits-Witten twistor-string [2, 3, 31].

## 2.3 Ambitwistor string

The ambitwistor string was introduced by Mason and Skinner in [32] and its geometry was further studied in [33]. The ambitwistor string is a 2D CFT whose correlation functions reproduce the CHY formulas, and effectively explains their origin. A pleasant feature of the ambitwistor string is its similarity with the conventional RNS string, this allows for worldsheet string theoretic techniques to be easily adapted to the study of the ambitwistor string. The model discussed here is the type II version of the ambitwistor string, so called because it has two real fermions on the worldsheet and reproduces the scattering amplitudes of type II supergravity. Given a Riemann surface $\Sigma$ the matter action of the model in conformal gauge is:

$$S_m = \frac{1}{2\pi}\int_\Sigma P_\mu \bar\partial X^\mu + \frac{1}{2}\psi_\mu\bar\partial\psi^\mu + \frac{1}{2}\tilde\psi_\mu\bar\partial\tilde\psi^\mu - \frac{1}{2}eP^2 - \chi\psi^\mu P_\mu - \tilde\chi\tilde\psi^\mu P_\mu. \tag{2.21}$$

The fields $X^\mu$ have zero conformal weight and represent coordinates on complexified Minkowski space $\mathbb{M}$. Their conjugated fields $P_\mu$ have conformal weight $(1,0)$ and *both* fermions have conformal weight $(\frac{1}{2},0)$[2]. This is a chiral action, all the fields are left-moving and the kinetic terms are given only in terms of $\bar\partial$. The Lagrange multiplier field $e$ has conformal weight $(-1,1)$ and imposes the mass-shell constraint $P^2 = 0$, while the both "gravitino" fields $\chi, \tilde\chi$ have conformal weight $(-\frac{1}{2},1)$ and impose the supersymmetric partners of the mass-shell constrain $\psi\cdot P = \tilde\psi\cdot P = 0$. This action is invariant under a gauge symmetry

$$\delta X^\mu = \alpha P^\mu \quad \delta P_\mu = 0 \quad \delta e = \bar\partial\alpha \tag{2.22}$$

[2]As usual, I say that fields taking values on the line bundle $K^h \otimes \bar K^{\bar h}$ have conformal weight $(h,\bar h)$, where $(\bar K)$ $K$ is the (anti)-canonical bundle of $\Sigma$.

as well as supersymmetries

$$\begin{aligned} \delta X^\mu &= \epsilon\psi^\mu \quad \delta\psi^\mu = \epsilon P^\mu \quad \delta P_\mu = 0 \\ \delta\chi &= \partial\epsilon \qquad \delta e = \epsilon\chi \end{aligned} \tag{2.23}$$

and their tilded version. The matter currents that generate the symmetries (2.22) and (2.23) are easy to obtain using the Noether procedure:

$$\begin{aligned} T_m &= -P_\mu\partial X^\mu - \frac{1}{2}\psi_\mu\partial\psi^\mu - \frac{1}{2}\tilde{\psi}_\mu\partial\tilde{\psi}^\mu \\ H_m &= -\frac{1}{2}P^2 \\ G_m &= -\psi^\mu P_\mu \\ \tilde{G}_m &= -\tilde{\psi}^\mu P_\mu. \end{aligned} \tag{2.24}$$

Alternatively, the fields $(P, X)$ can be considered as coordinates on the complexified cotangent bundle of Minkowski space, $T^*\mathbb{M}$, the gauge symmetry (2.22) corresponds to translations along complex null geodesics on target space. The simplectic reduction of $T^*\mathbb{M}$ by the constraint $P^2 = 0$ is the space of complex null geodesics $\mathbb{A} = T^*\mathbb{M}//\{P^2 = 0\}$, also called ambitwistor space. Although the action above lives on a two-dimensional space its form resembles that of the worldline action of a massless particle. Heuristically, the time derivatives $\partial_\tau$ are replaced by antiholomorphic ones $\bar{\partial}$ and the worldline is complexified, conformal weights are assigned in such a way that the action has conformal symmetry. From this perspective it might be expected that the ambitwistor strings describes field theory and not string theory, even though the calculational techniques are lifted from 2D CFTs. Notice also that there is no dimensional parameter in action, so there is no analogue of an $\alpha'$ expansion. This is consistent with the fact that there is no $XX$ OPE, the worldsheet is rigid and only the massless modes are present. The ambitwistor string seems to implement the $\alpha' \to 0$ limit of the string from the very beginning.

Gauge-fixing by the usual BRST procedure we find that it is necessary to add contributions to the currents coming from the ghosts:

$$\begin{aligned} T_{gh} &= c\partial b - 2b\partial c + \tilde{c}\partial\tilde{b} - 2\tilde{b}\partial\tilde{c} - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\tilde{\beta}\partial\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}\partial\tilde{\beta} \\ H_{gh} &= c\partial\tilde{b} - 2\tilde{b}\partial c \\ G_{gh} &= c\partial\beta + \frac{3}{2}\beta\partial c - 2\tilde{b}\gamma \\ \tilde{G}_{gh} &= c\partial\tilde{\beta} + \frac{3}{2}\tilde{\beta}\partial c - 2\tilde{b}\tilde{\gamma} \end{aligned} \tag{2.25}$$

These are defined so that the action of the BSRT operator

$$Q = \oint c(T_m + \frac{1}{2}T_{gh}) + \tilde{c}(H_m + \frac{1}{2}H_{gh}) + \gamma(G_m + \frac{1}{2}G_{gh}) + \tilde{\gamma}(\tilde{G}_m + \frac{1}{2}\tilde{G}_{gh}) \\ - \tilde{b}\gamma^2 - \tilde{b}\tilde{\gamma}^2. \tag{2.26}$$

on the antighost fields is the usual one:

$$\begin{aligned} Q\cdot b = T_m + T_{gh} = T \quad Q\cdot\tilde{b} = H_m + H_{gh} = H \\ Q\cdot\beta = G_m + G_{gh} = G \quad Q\cdot\tilde{\beta} = \tilde{G}_m + \tilde{G}_{gh} = \tilde{G} \end{aligned} \tag{2.27}$$

Then the gauge fixed ghost action is chiral and free:

$$S_g = \frac{1}{2\pi}\int_\Sigma b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} + \gamma\bar{\partial}\beta + \tilde{\gamma}\bar{\partial}\tilde{\beta} \tag{2.28}$$

$$\tag{2.29}$$

and ghost fields have the analogous conformal weight as in the RNS string. That is the fermionic ghosts $c, \tilde{c}$ have conformal weight $(-1,0)$ and $b, \tilde{b}$ have conformal weight $(2,0)$, while the bosonic ghosts $\gamma, \tilde{\gamma}$ have conformal weight $(-\frac{1}{2},0)$ and $\beta, \tilde{\beta}$ have conformal weight $(\frac{3}{2},0)$. Note that both sets of bosonic ghosts are left-moving which is expected given the chiral nature of the model, but is distinct from the type II superstring where one set is left-moving and the other right-moving. In practical calculations the ghost currents $H_{gh}, G_{gh}, \tilde{G}_{gh}$ can be ignored as long as only standard vertex operators are used, that is operators that don't contain derivatives of the ghost fields. In particular the algebra of constraints

$$G_m(z)G_m(w) \sim -2\frac{H_m}{z-w} \quad \tilde{G}_m(z)\tilde{G}_m(w) \sim -2\frac{H_m}{z-w} \quad G_m(z)\tilde{G}_m(w) \sim 0 \tag{2.30}$$

also holds for the currents $H, G, \tilde{G}$. Nilpotency of the BRST operator $Q$ The central charge of this theory is $c = 3(d-10)$, where $d$ is the complex dimension of spacetime. The ambitwistor is critical and $Q^2 = 0$ when $d = 10$ like in the superstring.

Standard vertex operators in the NS-NS sector of the theory are of the form:

$$\mathcal{V} = c\tilde{c}\delta(\gamma)\delta(\tilde{\gamma})V \tag{2.31}$$

where $V$ has conformal weight $(1,0)$ and depends only on the matter fields $(P, X, \psi, \tilde{\psi})$. Imposing a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry[3] on the fermions

$$\psi \to \pm\psi \quad \tilde{\psi} \to \pm\tilde{\psi} \tag{2.32}$$

restrict the possible contents of $V$ to

$$V = \epsilon_\mu \tilde{\epsilon}_\nu \psi^\mu \tilde{\psi}^\nu e^{\mathrm{i}k\cdot X} \tag{2.33}$$

where $\epsilon_\mu$ and $\tilde{\epsilon}_\mu$ are polarization tensors and $k_\mu$ is the external momentum. The form of this vertex operator is thus practically identical to that of fixed NS vertex operators in type II string theory; the only difference is that all the fields in the ambitwistor string are chiral and have only holomorphic conformal weight. It is easily checked that $Q \cdot \mathcal{V} = 0$ requires that $\epsilon \cdot k = \tilde{\epsilon} \cdot k = 0$ and $k^2 = 0$; this last condition comes from the OPE with the constraint $H$. Since the action is first order the exponential $e^{\mathrm{i}k\cdot X}$ does not carry conformal weight, so there are no vertex operators that could correspond to the tower of massive states of string theory. Considering the form of (2.33) there are only three states in the spectrum in this sector, a symmetric traceless tensor, an antisymmetric tensor, and a trace part. These will correspond to the graviton, B-field and dilaton of type II supergravity. The fact that on-shellness of the vertex operators are imposed by the constraint $H$ will be important later on when the ambitwistor string will be defined for curved target spaces. The operator (2.31) will be referred as the fixed vertex operator; from it the integrated form of the vertex operator can be derived using the usual descent procedure [24, 34]:

$$\mathcal{U} = \int \bar{\delta}(k\cdot P) U = \int \bar{\delta}(k\cdot P)(\epsilon \cdot P + k\cdot\psi\epsilon\cdot\psi)(\tilde{\epsilon}\cdot P + k\cdot\tilde{\psi}\tilde{\epsilon}\cdot\tilde{\psi}) e^{\mathrm{i}k\cdot X}. \tag{2.34}$$

This operator also resembles the integrated vertex operator of the RNS string except for the appearance of the $\delta$-function. While its presence might seem odd at first sight it comes naturally from the descent procedure as will be shown in the next chapter. It is also quite natural from the target space perspective; using the ambitwistor version of the Penrose transform [35, 36] it can be shown that the vertex operators (2.33)-(2.34) represent the NS-NS sector of supergravity in ten dimensions [32]. Note that the argument of the $\delta$-function are the scattering equations (2.15), here they appear from the proper gauge fixing of the worldsheet symmetries in the presence of vertex operators.

[3] This symmetry is analogous to the GSO projection of the usual RNS superstring.

At tree-level the ghosts $c, \tilde{c}$ have three zero modes and the ghosts $\gamma, \tilde{\gamma}$ have two zero modes each. A well defined correlation function is therefore given by:

$$\mathcal{A}_n^{(0)} = \langle \mathcal{V}_1 \mathcal{V}_2 c_3 \tilde{c}_3 U_3 \prod_{i=4}^{n} \mathcal{U}_i \rangle \,. \tag{2.35}$$

Evaluating this correlator is straightforward, the exponentials are absorbed in the action

$$\int_\Sigma P_\mu \bar{\partial} X^\mu + \sum_{i=1}^{n} k_{i\mu} X^\mu \bar{\delta}(z - z_i) \tag{2.36}$$

and the integration over the non-zero modes of $X$ fixes the field $P$ to obey

$$\bar{\partial} P^\mu(z) = \sum_{i=1}^{n} k_{i\mu} \bar{\delta}(z - z_i). \tag{2.37}$$

On the sphere the solution to this equation is

$$P_\mu(z) = \sum_{i=1}^{n} \frac{k_{i\mu}}{z - z_i}. \tag{2.38}$$

Since $P$ has no zero modes on the sphere its path integral is completely constrained, so the solution (2.38) can be substituted in the correlator (2.35). The only path-integral left to do is over the real fermions $\psi, \tilde{\psi}$; since the action is free each gives a Pfaffian of a matrix with columns and rows $1, 2$ removed. This is a consequence of the two $\gamma, \tilde{\gamma}$ zero modes which need to be fixed through the insertion of $\mathcal{V}_1$ and $\mathcal{V}_2$. From the worldsheet perspective it is obvious that the pseudo-Pfaffian won't depend on which rows and columns have been removed. The contribution from the $P$ part of the vertex operators can be neatly encoded into these Pfaffians as was shown in [32] and the correlation function (2.35) reproduces the CHY formula (2.12) for gravitons.

Although the type II ambitwistor string will be the focus of most of this thesis, it should be mentioned that there is a heterotic version of the ambitwistor string. This is nearly identical to the type II model described above, except that the $\tilde{\Psi}$ system is exchanged for a worldsheet current algebra for some gauge group. At genus zero and leading trace the heterotic model reproduces the tree-level amplitudes of Yang-Mills in the CHY form [32]. Multi-trace contributions are mediated by poorly understood gravitational degrees of freedom which don't correspond to Einstein gravity. There is also a bosonic version of the ambitwistor string [32] which also contains some kind of gravitational degrees of freedom which don't correspond to Einstein gravity. Thus the most interesting model to study is the type II ambitwistor string which does describe the interactions of usual type II A/B supergravity. There are modifications of the 10 dimensional ambitwistor string [37] which reproduces the other CHY formulas though the quantum consistency of these models is not as well understood as the type II.

There is also a four dimensional version of the ambitwistor string [38–41], which makes use of spinor helicity variables to present compact formulas for tree-level scattering in four dimensions. In spinor helicity variables the scattering equations have different guises depending on how parity is manifest, or not, in the amplitude formulas. This gives a variety of ways of writing down field theory amplitudes [31, 38, 42–45].

## 2.4 $\infty$-tension limit of the pure spinor

Although it is possible to describe target space fermions using the ambitwistor string [5], it involves bosonizing the worldsheet fermions and the bosonic ghosts, as is done in the RNS string [46, 47]. This introduces non-polynomial vertex operators for states in the R sector and computations of their correlation functions quickly become cumbersome. For the usual superstring this issue is overcome by working with a model with manifest target space supersymmetry, that is, either the Green-Schwarz or the pure spinor superstring. Of these two only the pure spinor superstring has a covariant quantization [48] which makes Lorentz symmetry manifest. Not longer after the ambitwistor string was introduced Berkovits described a pure spinor version of it [49] which has all the usual advantages of this formalism, like manifest spacetime supersymmetry and absence of worldsheet fermions. This is the model reviewed in this chapter which is known as the minimal version. Later a non-minimal version will be introduced to calculate higher-loop amplitudes. More about the pure spinor superstring can be found in the reviews [50, 51]

When discussing the pure spinor I'll adopt different conventions from the previous sections for spacetime indices, following the conventions in the pure spinor literature. In this section, $m, n, \ldots = 0, \ldots, 9$ denote ten-dimensional space-time indices, while $\alpha, \beta, \ldots = 1, \ldots, 16$ are spinor indices. The action of the model is chiral and first order, like in the ambitwistor string. As in that case, it can also be viewed as a chiral complexification of the pure spinor superparticle action [52]:

$$S = \frac{1}{2\pi}\int_\Sigma P_m \bar{\partial} X^m + p_\alpha \bar{\partial}\theta^\alpha + \tilde{p}_{\tilde{\alpha}} \bar{\partial}\tilde{\theta}^{\tilde{\alpha}} + w_\alpha \bar{\partial}\lambda^\alpha + \tilde{w}_{\tilde{\alpha}} \bar{\partial}\tilde{\lambda}^{\tilde{\alpha}} \tag{2.39}$$

where $X^m$ and $\theta^\alpha, \tilde{\theta}^{\tilde{\alpha}}$ are the conformal weight zero coordinates on target superspace. The fields $P_m$ and $p_\alpha, \tilde{p}_{\tilde{\alpha}}$ are their conjugate momenta and have conformal weight $(1,0)$.

The fields $\lambda^\alpha, \tilde{\lambda}^{\tilde{\alpha}}$ are bosonic spinors which satisfy the pure spinor constraint:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 = \tilde{\lambda}^{\tilde{\alpha}} \gamma^m_{\tilde{\alpha}\tilde{\beta}} \tilde{\lambda}^{\tilde{\beta}} ,$$

The fields $w_\alpha, \tilde{w}_{\tilde{\alpha}}$ are their conjugate momenta with weight $(1,0)$. Due to the pure spinor constraint not all the components of $\lambda, \tilde{\lambda}$ are independent. Using a parametrization which breaks the manifest SO(10) symmetry down to U(5) the pure spinor constraints can be solved explicitly and the number of independent components of $\lambda, \tilde{\lambda}$ is seen to be eleven. This constraint also generates a gauge transformation of their conjugates fields $w, \tilde{w}$. These can only ever appear in gauge-invariant combinations

$$N^{nm} = \frac{1}{2}(w\gamma^{nm}\lambda) , \qquad J = \lambda \cdot w , \qquad T_\lambda = -w_\alpha \partial \lambda^\alpha$$

The OPEs between the matter variables are free

$$X^m(z)\, P_n(w) \sim \frac{\delta^m_n}{z-w} , \qquad \theta^\alpha(z)\, p_\beta(w) \sim \frac{\delta^\alpha_\beta}{z-w} , \tag{2.40}$$

and likewise for the tilded variables. The OPEs of the operators built out of the pure spinor variables and their conjugate can be computed in the same way as the pure spinor superstring by using a U(5)-covariant parametrization of the space of pure spinors [48]. These are collected in the appendix for reference.

The BRST operator is a holomorphic generalization of the BRST operator in the type II pure spinor superparticle. The Green-Schwarz constraint is

$$d_\alpha = p_\alpha - \frac{1}{2} P_m \gamma^m_{\alpha\beta} \theta^\beta , \tag{2.41}$$

and the BRST charge is defined to be

$$Q = \oint \lambda^\alpha\, d_\alpha + \tilde{\lambda}^{\tilde{\alpha}}\, \tilde{d}_{\tilde{\alpha}} . \tag{2.42}$$

Nilpotency of $Q$ is straightforward to see using the OPEs (2.40) and the pure spinor constraints. Vertex operators are given by non-trivial cohomology classes with respect to the BRST operator. The fixed vertex operator is:

$$V = \lambda^\alpha\, \tilde{\lambda}^{\tilde{\alpha}}\, A_\alpha(\theta)\, \tilde{A}_{\tilde{\alpha}}(\tilde{\theta})\, \mathrm{e}^{ik\cdot X} , \tag{2.43}$$

where $A_\alpha, \tilde{A}_{\tilde{\alpha}}$ are the standard $\mathcal{N}=1$ superfields, which can be expanded in terms of vector and spinor polarizations. $Q$-closedness of the vertex operators impose the linearised equations of motion

$$k^2 = 0\,, \qquad (\gamma_{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0 = (\gamma_{mnpqr})^{\tilde{\alpha}\tilde{\beta}} \tilde{D}_{\tilde{\alpha}} \tilde{A}_{\tilde{\beta}}\,,$$

where the supersymmetric derivative is

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2} k^m (\gamma_m \theta)_\alpha\,.$$

Integrated vertex operators resemble those of the type II pure spinor superstring, but terms proportional to $\partial\theta$ and $\partial\tilde{\theta}$ are absent and the chirality of the model leads to the presence of holomorphic delta functions to balance the conformal weight:

$$\begin{aligned}&\int_\Sigma \bar{\delta}(k\cdot P)\, U \\ &\quad = \int_\Sigma \bar{\delta}(k\cdot P) \left(A\cdot P + d_\alpha W^\alpha + \frac{1}{2} N_{mn} \mathscr{F}^{mn}\right)\left(\tilde{A}\cdot P + \tilde{d}_{\tilde{\alpha}} \tilde{W}^{\tilde{\alpha}} + \frac{1}{2} \tilde{N}_{mn} \tilde{\mathscr{F}}^{mn}\right) \mathrm{e}^{ik\cdot X}\,, \qquad (2.44)\end{aligned}$$

where $\{A^m, W^\alpha, \mathscr{F}^{mn}, \dots\}$ are the standard superfields of $\mathscr{N} = 1$ super-Yang-Mills in ten dimensions [53, 54]. The integrated vertex operator is $Q$-closed, $[Q, U] = 0$, on the support of the delta function.

The vertex operators (2.43), (2.44) give the full spectrum of type II supergravity in ten dimensions. Individual fields can be picked out by expanding the various superfields in powers of $\theta$ (or $\tilde{\theta}$), and isolating those components proportional to the desired polarizations.

The genus zero worldsheet correlation function prescription given in [49] mimics the prescription for the superstring:

$$\mathscr{M}_n^{(0)} = \left\langle \prod_{i=1}^{3} V(z_i) \prod_{j=4}^{n} \int_\Sigma \bar{\delta}(k_i \cdot P(z_i))\, U(z_i) \right\rangle\,, \qquad (2.45)$$

with the usual zero-mode normalization for $\theta, \tilde{\theta}, \lambda, \tilde{\lambda}$ inherited from the superstring (*i.e.*, $\langle \lambda^3 \theta^5 \rangle = 1$) [48]. By restricting the vertex operators to the NS-NS sector, it is straightforward to see that this prescription reproduces the worldsheet correlators of the RNS-like model in [32]. These in turn are equal to the scattering equation representations for the tree-level S-matrix of gravitons, $B$-fields, and dilatons given by Cachazo, He, and Yuan [12].

In the case of generic external states, performing explicit amplitude calculations for an arbitrary number of external particles is difficult. However, by utilizing genus zero results from the pure spinor superstring [55–57] and KLT orthogonality, it can nevertheless be shown that the prescription (2.45) *does* reproduce the full tree-level S-matrix of type II supergravity, in a representation that is supported on the scattering equations [58]. The distinction between type IIA and IIB supergravity is built into the identification of the tilded spinor indices: for IIA tilded indices denote spinors of the opposite chirality as untilded-tilded indices, while for IIB they denote spinors of the same chirality.

# Chapter 3

# Ambitwistor string at loop level

The main objective of this chapter, which comprises the majority of this thesis, is the study of the ambitwistor string at loop level. A first objection to such generalization might be that the ambitwistor string describes type II supergravities in ten dimensions which, without a regularization scheme, have divergent loop amplitudes. In later sections I'll show that the origin of these possible divergences is well understood and they can be factored out easily, leaving finite *integrands*. From the worldsheet perspective there is no obstruction to extend the ambitwistor string to higher genus surfaces, its defining CFT has zero central charge in $d = 10$. Therefore the main objects of study in this chapter are the integrands given by the ambitwistor string.

Most of this chapter deals with the one loop generalization of the ambitwistor string. First, in section 3.1, I present the generalization of the scattering equations one loop, where evidence for their validity is given by studying their degenerations at the boundary of the moduli space. In section 3.2 the modular properties of the partition functions for the type II A/B ambitwistor string are analysed. The one loop scattering amplitudes for external NS-NS states are given in section 3.3, where they shown to be modular invariant. Factorization properties of these amplitudes on the boundaries of the moduli space is given in section 3.4 where manifestly permutation invariant formulas for the amplitudes are also given.

Next I move to the study of the one loop integrand and its comparison to the expected integrand from the field theory. This is done in section 3.5 where several results are obtained: the ambitwistor integrand is shown to have the correct IR behaviour expected for a field theory amplitude; analytic solutions for the one loop scattering equations are found at four points near the boundary of the moduli space; the ambitwistor integrand is evaluated on top of the solutions and shown to match the field theory integrand; and a conjecture to the behaviour of these integrands at n points is given.

The last section 3.6 extends the formalism to two loops by introducing the non-minimal version of the $\infty$-tension limit of the pure spinor. This formalism is used since it makes the calculation of loop amplitudes more straightforward, much like in the superstring case. In that section the non-minimal version of Berkovits model [49] is constructed, one and two loop four point amplitudes are computed and shown to have the correct kinematical factor, and the two loop amplitude is shown to have the correct factorization properties for a field theory amplitude.

## 3.1 The scattering equations at genus one

The ambitwistor string gives a natural derivation of the scattering equations for higher genus surfaces. They arise from the usual BRST quantization of the constrained action (2.21), so it is worthwhile here to go over this procedure more carefully. Consider the matter action (2.21).The chiral worldsheet gravity and gravitinos of this theory are gauge-fixed by the usual BRST procedure, that is introducing a $bc$-ghost system and two copies of the superconformal ghost system denoted as $\beta\gamma$ and $\tilde{\beta}\tilde{\gamma}$. But the ambitwistor string has an additional gauge symmetry compared to the string (2.22) which needs to be fixed. To do this, introduce in the action the gauge-fixing term:

$$\left\{Q, \int_{\Sigma} \tilde{b}\, F(e)\right\}, \tag{3.1}$$

where $\tilde{b}$ has conformal weight $(2,0)$ and $F(e)$ is a gauge-fixing functional. The natural choice for $F$ is to set $e=0$; but there might be obstructions to achieving this gauge globally, these make up part of the moduli of the problem. In particular, in a punctured Riemann surface the gauge parameter is required to vanish at the marked points, so naively the gauge $e=0$ can't be achieved when punctures are present. More generally, the BRST transformations of the gauge fields only allow $e$ to be varied within a fixed Dolbeault cohomology class. If $\Sigma$ is a genus $g$ Riemann surface with $n$ marked points $\{z_i\}$ the field $e$ can be fixed up to elements of $H^{0,1}(\Sigma, T_{\Sigma}(-z_1-\cdots-z_n))$ which is non-trivial. For $r=1,\ldots,3g-3+n$ let $\{\mu_r\}$ be a basis of $H^{0,1}(\Sigma, T_{\Sigma}(-z_1-\cdots-z_n))$, the gauge-fixing functional is chosen to be:

$$F(e) = e - \sum_{r=1}^{3g-3+n} s_r\, \mu_r, \tag{3.2}$$

where $s_r \in \mathbb{C}$ are coefficients of the basis.

The action of the BRST operator $Q$ on the various fields in the gauge-fixing term is

$$\delta\tilde{b} = m, \qquad \delta e = \bar{\partial}\tilde{c}, \qquad \delta s_r = q_r, \qquad \delta m = 0, \qquad \delta q_r = 0,$$

so after integrating out the Nakanishi-Lautrup $m$, the relevant part of the action (2.21) becomes

$$\frac{1}{2\pi}\int_\Sigma \tilde{b}\,\bar{\partial}\tilde{c} - \sum_r s_r \int_\Sigma \mu_r P^2 - \sum_{r=1}^{3g-3+n} q_r \int_\Sigma \tilde{b}\,\mu_r\,. \tag{3.3}$$

Integrating out the bosonic and fermionic parameters $s_r$ and $q_r$ leaves us with an insertion of

$$\prod_{r=1}^{3g-3+n} \bar{\delta}\left(\int_\Sigma \mu_r P^2\right) \int_\Sigma \tilde{b}\,\mu_r \tag{3.4}$$

inside the path integral. At genus zero choose a basis of $n-3$ Beltrami differentials so that $\int \tilde{b}\,\mu_r$ simply extracts the residue of $\tilde{b}$ at the location of the $r^{\text{th}}$ vertex operator [59]. This then strips off the $\tilde{c}$ ghost associated with a fixed vertex operator insertion. Similarly, the integral $\int \mu_r P^2$ in (3.4) extracts the residue of the quadratic differential $P^2$ at the location of the vertex operator, leaving a $\delta$-function that forces this residue to vanish. At genus zero, a quadratic differential must have at least four poles (counted with multiplicity). Since $P^2$ has at most simple poles, enforcing the vanishing of all but three of its residues ensures that in fact $P^2 = 0$ globally over the genus zero Riemann surface. This is exactly the content of the scattering equations; as mentioned before, they emerge as a natural consequence of the gauge redundancy enforcing that the target space is ambitwistor space in the presence of vertex operator.

A $n$-punctured genus one Riemann surface has $n$ moduli. These are the position of $n-1$ punctures[1] and the complex structure parameter $\tau$. Just like at genus zero, choose a basis of Beltrami differentials so that $n-1$ of the fixed vertex operators become integrated vertex operators. The remaining $\delta$-function should be seen as providing part of the measure on the moduli space:

$$\int_\Sigma \tilde{b}\,\mu \;\times\; \bar{\delta}\left(\int_\Sigma P^2 \mu\right) = \tilde{b}_0\,\bar{\delta}(P^2(z_0|\tau))\,, \tag{3.5}$$

where $\mu$ is the Beltrami differential associated to changes in the complex structure of the torus. The insertion of $\tilde{b}_0$ serves to absorb the single zero mode of $\tilde{b}$ at genus 1, and its insertion point is arbitrary. The remaining $\delta$-function is part of the genus 1 scattering

[1] Tori have a translation symmetry which can be used to fix the position of one of the punctures.

equations. It should be interpreted as

$$\bar{\delta}(P^2(z_0|\tau)) = \mathrm{d}\bar{\tau}\frac{\delta}{\delta\bar{\tau}}\left(\frac{1}{P^2(z_0|\tau)}\right), \tag{3.6}$$

so it fixes the integral over the modular parameter $\tau$. There are now two kinds of scattering equations; $\mathrm{Res}_i P^2(z) = 0$ which fixes the moduli corresponding to the position of the punctures on the curve, and $P^2(z_0) = 0$ for some arbitrary point $z_0$ which fixes the value of the modular parameters determining the shape of the curve. Note that the geometrical content of these equations is the same as at tree-level. At genus one the only dependence on $X$ is on the exponentials in the vertex operators, these can be absorbed in the action (2.36) and integrated out, constraining $P$ to obey

$$\bar{\partial}P^\mu(z) = \sum_{i=1}^{n} k_i^\mu \bar{\delta}(z - z_i)\mathrm{d}z \tag{3.7}$$

which sets $P(z)$ to to be a meromorphic differential on the torus with residue $k_i^\mu$ at the simple pole located at $z_i$. Being a section of the canonical bundle $P$ has one zero mode at genus one, so the solution to (3.7) is given by

$$P_\mu(z) = p_\mu \mathrm{d}z + \sum_{i=1}^{n} k_{i\mu}\tilde{S}(z, z_i|\tau), \tag{3.8}$$

where $\mathrm{d}z$ is the global holomorphic differential on the torus and $\tilde{S}(z, z_i|\tau)$ is the $PX$ propagator defined by

$$\tilde{S}(z, z_i|\tau) = \mathrm{d}z\frac{\partial}{\partial z}G(z, z_i|\tau) \tag{3.9}$$

where

$$G(z, z_i|\tau) = -\ln|E(z, z_i|\tau)|^2 + 2\pi\frac{(\mathrm{Im}(z - z_i))^2}{\mathrm{Im}(\tau)} \tag{3.10}$$

is the usual genus one propagator for a non-chiral scalar, written in terms of the prime form $E(z, w|\tau)$. Just like at tree level, the field $P^2$ is a quadratic differential with $n$ simple poles, higher order poles have zero coefficients when the external momenta are on-shell. Imposing that $n-1$ residues of $P^2$ vanish automatically kills the last pole, since there are no elliptic functions with a single pole. Thus, $P^2$ must be globally holomorphic over $\Sigma$. The role of the "new" scattering equation $P^2(z_0) = 0$ is to kill this last holomorphic piece, which on top of the "old" scattering equations enforces that $P^2 = 0$ everywhere on

$\Sigma$. Explicitly, the one loop scattering equations are:

$$\mathrm{Res}_{z_i} P^2(z) = k_i \cdot p + \sum_{j \neq i} k_i \cdot k_j \, \tilde{S}_1(z_i, z_j|\tau) = 0 \tag{3.11}$$

at all but one of the marked points, and:

$$\begin{aligned} P^2(z_0) = & p^2 (\mathrm{d}z)^2 + (\mathrm{d}z) \sum_{i=1}^{n} p \cdot k_i \, \tilde{S}_1(z_0, z_i|\tau) \\ & + \sum_{i \neq j} k_i \cdot k_j \, \tilde{S}_1(z_0, z_i|\tau) \tilde{S}_1(z_0, z_j|\tau) = 0, \end{aligned} \tag{3.12}$$

where the second sum runs over both $i$ and $j$.

The $n$ scattering equations completely fix the integral over the $n$-dimensional moduli space $\mathscr{M}_{1,n}$ of $n$-pointed genus 1 curves in terms of the external momentum $k_i$ and the zero mode coefficients $p$. These coefficients are unconstrained and must be integrated over to recover the full amplitude, so it is natural to interpret them as the loop momentum. The integral over $\mathscr{M}_{1,n}$ give a loop *integrand*, to recover the amplitude the integral over the $p$'s must be performed. In general there's no canonical way to define a loop integrand starting from Feynman diagrams, the ambitwistor string seems to give one such prescription. In section 3.5 I'll match the ambitwistor integrand to the corresponding Feynman graphs in a specific kinematical regime, and give a conjecture to what is meant by loop integrand in the ambitwistor string.

At genus $g$ the expected number of scattering equations is $n+3g-3$, of those (for $g \geq 2$) $n$ would be of the type $k_i \cdot P(z_i) = 0$ which are the constraints on the residues of $P^2(z)$, while $3g-3$ would be of the type $P^2(z_r) = 0$ constraining the contributions of the holomorphic quadratic differentials to $P^2$ to vanish at $3g-3$ points. Since $h^0(\Sigma, K^2(z_1 + \cdots + z_n)) = n + 3g - 3$ these scattering equations suffice to impose $P^2(z) = 0$ globally over the marked Riemann surface, ensuring as in [32] that the true target space of the string is ambitwistor space $P\mathbb{A}$. At genus $g$ there are $g$ holomorphic Abelian differentials $\omega_a$ (with $a = 1, \ldots, g$) which contribute to the zero modes of $P$. Higher genus amplitudes will involve an integral over these zero modes $\prod_a \mathrm{d}^{10} p_a$ of $P(z)$, which should to the loop momenta at $g$ loops in field theory.

### 3.1.1 Factorization on boundaries of $\overline{\mathscr{M}}_{1,n}$

The one loop scattering equations, like their tree-level counterpart, also have the property of tying up the boundaries of $\overline{\mathscr{M}}_{1,n}$ and factorization channels. Since the loop momentum appears explicitly in the one loop scattering equations the factorization properties here are those of the loop integrand. Tori can degenerate in two distinct ways: either by pinching a non-trivial cycle, reducing the torus to a Riemann sphere, or by pinching a trivial cycle which factors the worldsheet into a sphere and another torus. These are referred to as *non-separating* or *separating* degenerations, respectively, and both can be understood as contributions from the boundary in the moduli space of curves $\overline{\mathcal{M}}_{1,n}$.

Recall that $P_\mu$ is constrained by the equation:

$$\bar{\partial} P_\mu(z) = 2\pi \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \sum_{i=1}^{n} k_{imu}\, \delta^2(z - z_i).$$

Which holds regardless of the form of the surface. At genus one the solution is

$$P_\mu(z) = p_\mu \mathrm{d}z + \sum_{i=1}^{n} k_{i\mu} \tilde{S}_1(z, z_i|\tau).$$

To understand how the scattering equations behave under degenerations it is enough to study how the field $P_\mu(z)$ behaves. The behaviour of the abelian differentials $p_\mu \mathrm{d}z$ and the Szëgo kernels $\tilde{S}_1(z_i, z_j|\tau)$ under degenerations of the torus is well known [25]. The non-separating degeneration is approached when the modular parameter $q = \mathrm{e}^{2\mathrm{i}\pi\tau} \to 0$, this corresponds to the $a$-cycle pinching. At $q = 0$ the resulting surface is a sphere with two extra marked points identified. In this case the abelian differential $\mathrm{d}z$ develops poles at these two new marked points with residues 1 and $-1$. That is, denoting coordinates on the nodal sphere by $x$'s, the abelian differential behave as

$$p_\mu \mathrm{d}z \to p_\mu \mathrm{d}x \frac{x_a - x_b}{(x - x_a)(x - x_b)} \tag{3.13}$$

where $x_a$ and $x_b$ are the positions of the nodes. The behaviour of the Szëgo kernels is also straightforward, it simply goes over to the Szëgo kernel on the sphere

$$\tilde{S}_1(z, z_j|\tau) \to \frac{\mathrm{d}x}{x - x_j}. \tag{3.14}$$

At the degeneration point the scattering equations (3.11) become

$$k_i \cdot P(z_i) \to \frac{k_i \cdot k}{x_i - x_a} - \frac{k_i \cdot k}{x_i - x_b} + \sum_{j \neq i} \frac{k_i \cdot k_j}{x_i - x_j}. \tag{3.15}$$

The interpretation is that two new particles were created at points $x_a, x_b$ with equal and opposite momentum $k$. Taking this factorization limit corresponds to a $(n+2)$-point tree amplitude which should come with $n-1$ scattering equations. This is precisely the number of equations given for each choice of $i$ in (3.15)[2]. On the support of (3.15) the remaining scattering equation becomes

$$P^2(z_0|q=0) = p^2 \mathrm{d}z_0^2 \to k^2 \mathrm{d}x_0^2 \left( \frac{x_a - x_b}{(x_0 - x_a)(x_0 - x_b)} \right)^2 = 0\,, \tag{3.16}$$

which forces the momentum at the nodal points (which is the momentum running through the cut) to be on-shell. The points $\{x_0, x_a, x_b\}$ are fixed by the $\mathrm{SL}(2,\mathbb{C})$ freedom on the degenerated worldsheet.

I should note that for generic values of the modular parameter $\tau$, $\bar{\delta}(P^2)$ does *not* constrain $p_\mu$ to be null. If this were true, then the loop momentum would always be constrained to be on-shell. For a generic value of $\tau$, the remaining $n-1$ scattering equations and momentum conservation can be used to write (3.12) as

$$P^2(z_1) = p^2\, \mathrm{d}z^2 + \sum_{j\neq i} k_j \cdot k_i\, f(z_i, z_j, \tau)\, \mathrm{d}z^2\,, \tag{3.17}$$

where the function $f(z_i, z_j, \tau)$ is smooth and has no singularity when $z_i \to z_j$. Furthermore, when $q = 0$, $f$ approaches a constant independent of the worldsheet coordinates. By momentum conservation, this means that $P^2(z) \to p^2$ as we pinch the non-separating cycle. Hence, the degeneration parameter $q$ is directly related to the off-shellness of the internal loop momentum, this behaviour will be studied more explicitly in section 3.5.

This implies that in general the scattering equation (3.12) can be seen as fixing the integration over $\tau$, leaving a loop integral over the non-compact space of $P$ zero modes. Integrating over this space might introduce divergences which are absent from string theory amplitudes but are expected from a theory which gives field theory amplitudes. Alternatively this equation can be interpreted as reducing the integral over the $P$ zero modes to some hypersurface parametrized by $\tau$. The moduli of the Riemann surface then can be seen as an off-shellness parameter for the loop momentum and we retain the interpretation that the target space is ambitwistor space.

The other boundaries are approached when two or more marked points collide. This case is fairly similar to factorization at tree-level [11]. The degenerate surface has two components, a torus $\Sigma_R$ and a sphere $\Sigma_L$, connected through a nodal point. The colliding points are distributed on the sphere while the other points remain on the torus. In this case the abelian differential is simply zero on the sphere component, while its component on the torus remains the same. The behaviour of the Szëgo kernels depend on which

[2] The usual CFT interpretation is that at the factorization limit $c$ and $\tilde{c}$ operators were inserted which create the punctures. The states inserted at these points are fixed; hence there are no scattering equations for the particles inserted at $z_a, z_b$.

components its two points are;

$$\tilde{S}_1(z_{ij}|\tau) \to \begin{cases} \tilde{S}_1(z_{ij}|\tau) & \text{if } z_i, z_j \in \Sigma_R, \\ \dfrac{1}{z_i - z_j} & \text{if } z_i, z_j z \in \Sigma_L, \\ \tilde{S}_1(z_i, z_a|\tau) + \dfrac{1}{z_b - z_j} & \text{if } z_i \in \Sigma_R \text{and } z_j \in \Sigma_L, \end{cases}$$

where $z_a, z_b$ are the coordinates for the nodal points on $\Sigma_R$ and $\Sigma_L$ respectively. Thus the field $P$ in each component is:

$$P_\mu(z)|_{\Sigma_L} \to -\frac{k_{R\,\mu}}{z-w}\mathrm{d}z + \sum_{i\in L}\frac{k_{i\,\mu}}{z-z_i}\mathrm{d}z, \tag{3.18}$$

$$P_\mu(z)|_{\Sigma_R} \to p_\mu \mathrm{d}z + k_{R\,\mu}\tilde{S}_1(z,y|\tau) + \sum_{j\in R} k_{j\,\mu}\tilde{S}_1(z,z_j|\tau). \tag{3.19}$$

As at tree level, the scattering equations separate into two sets; one corresponding to the sphere component, and the other corresponding to the torus component. The remaining scattering equation enforces that the momentum flowing through the cut is on-shell. Later on, when discussing factorization properties of the one-loop amplitude this computation will be done explicitly.

### 3.1.2 Relation to Gross and Mende's equations

Before moving on to the next section I wish to discuss the relation of one loop scattering equations presented here, (3.11) and (3.12), to the equations found in the high energy scattering of strings by Gross and Mende [29, 30]. First, a rewriting of the field $P$ makes the comparison more transparent[3]. The field $P(z)$ can also be written as

$$P_\mu(z) = \ell_\mu \mathrm{d}z + \sum_{i=1}^{n} k_{i\mu} S_1(z, z_i|\tau) \tag{3.20}$$

where $S_1$ is the Szëgo kernel in the odd spin structure,

$$S_1(z|\tau) = \frac{\partial\theta_1(z|\tau)}{\theta_1(z|\tau)}. \tag{3.21}$$

[3] A more compelling argument for using this parametrization of $P$ will be given later when the IR behaviour of the scattering equations and loop integrand is discussed.

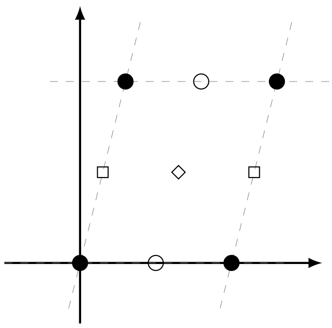

Fig. 3.1 Gross and Mende equilibrium; the charges should be placed at half-periods of the lattice.

which is related to the full propagator by

$$\partial G = -S_1(z|\tau) - 2i\pi \frac{\mathrm{Im z}}{\mathrm{Im}\tau}. \tag{3.22}$$

This is one of several ways of defining $P$, depending on how the representation of the bosonic propagator is chosen and on how to divide it into zero and nonzero modes. What constrains the possible representations is that the field $P(z)$ has to obey the differential equation (3.7), which (3.20) does. This manifestly holomorphic representation of the propagator obscures the modular properties of the scattering equations. So when discussing the modular transformations of partition functions and amplitudes I'll use (3.8) while (3.20) is more useful when studying the IR behaviour of the scattering amplitudes and their solutions. The relation between these two representation is simple, just redefine the loop momentum

$$\ell^\mu \to \ell^\mu + 2i\pi \sum_{i=1}^{n} k_i^\mu \frac{\mathrm{Im(z - z_i)}}{\mathrm{Im}(\tau)}. \tag{3.23}$$

to go between them. Gross and Mende studied the high energy limit of closed string amplitudes. The type II 4-graviton string theory amplitude in 10 dimensions is:

$$\int_{\mathscr{F}} \frac{\mathrm{d}^2\tau}{\mathrm{Im}\tau^2} \int \prod_{i=2}^{4} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\tau} \left| e^{2\alpha' \sum_{i,j} k_i \cdot k_j G(z_{ij}|\tau)} \right|^2 . \tag{3.24}$$

To study the high energy behaviour of this integral Gross and Mende used a saddle point approximation around the extremals of the energy functional $\mathscr{E} = \alpha' \sum_{i,j} k_i \cdot k_j G_{ij}$ with respect to variations of the moduli $z_i$ and $\tau$. The leading contribution is claimed to come from the saddle corresponding to the most symmetric way to arrange four charges on the torus; this is achieved when the charges sit at half-periods of the lattice, such that $\{z_1, z_2, z_3, z_4\} = \{1/2, \tau/2, (\tau+1)/2, 0\}$, (up to permutations), as pictured in fig.3.1.

With this choice the $\partial_z\mathscr{E}$ scattering equation vanish and every single term in the sum is actually zero. The last saddle point equation, $\partial_\tau\mathscr{E}$ is solved by the condition:

$$\frac{\theta_2(0,\tau)^4}{\theta_3(0,\tau)^4} = -\frac{u}{s}\,. \tag{3.25}$$

To connect this saddle point to the one-loop scattering equations one crucial ingredient is missing; there is no loop momentum. This can be cured by reverse engineering a string amplitude with explicit loop momentum, this is done when proving the chiral splitting of the superstring integrand at higher genus [59]. Starting from (3.24), one has to undo the $\partial X$ zero mode integral, the amplitude is

$$\int \frac{d^{10}\ell}{(2\pi)^{10}} \int_{\mathscr{F}} \frac{d^2\tau}{(\text{Im}\tau)^{2-5}} \int \prod_{i=2}^{4} \frac{d^2 z_i}{\text{Im}\tau} \left| e^{i\pi\tau\ell^2 + 2i\pi\sum_{i=1}^4 \ell\cdot k_i z_i} \right|^2 \left| e^{2\alpha' \sum_{i,j} k_i\cdot k_j S_{ij}} \right|^2 , \tag{3.26}$$

where the $-5$ in the exponent of $\text{Im}\tau$ comes from the reintroduction of the loop momentum Gaussian integral. It is easily checked that integrating out the loop momentum provides the expected non-holomorphic part of the propagator.

This introduces an explicit loop momentum dependence in the argument of the exponential, this alters the "energy" functional that has to be extremized $\tilde{\mathscr{E}}(\ell)$. In doing the saddle point analysis this amplitude there are two options; either extremize with relation to the $\ell$ directions, that is, add the $\partial_\ell\tilde{\mathscr{E}}(\ell) = 0$ equation, or leave unfixed the integration over the loop momentum and solve the saddle point for each value of $\ell^\mu$. The former gives

$$\ell_*^\mu = \sum_{i=1}^{n} k_i^\mu \frac{\text{Imz}_\text{i}}{\text{Im}\tau}\,, \tag{3.27}$$

which, once inserted in the $\partial_{z/\tau}\tilde{\mathscr{E}}(\ell) = 0$ saddle conditions, gives back the Gross and Mende saddle point equations. The latter option gives the one loop scattering equations (3.11) and (3.12).

Interestingly, the Gross and Mende saddle point gives a preferred value at a threshold for the loop momentum;

$$\ell_*^\mu = k_2^\mu + k_3^\mu\,, \tag{3.28}$$

while the ambitwistor string doesn't fix $\ell$ to a particular value, but requires that the scattering equations be solved for any value of $\ell$. Modular transformations act by permuting which scattered particles sit on the half periods, changing the loop momentum (3.28) to a different threshold.

## 3.2 Modular invariance and the partition function

On surfaces of higher genus the path integrals over the non-zero modes of the fields are non-trivial, even in the absence of any vertex operator. Genus one surfaces have one complex moduli $\tau$ and the functional determinants obtained from the path-integral are functions[4] of this modular parameter. Furthermore, there is a choice of spin structure for the fermions, that is a choice of the which square root of the canonical bundle the fermions take values in. At genus one there are four choices of spin structures; three even which have no zero modes and one odd which has a zero mode. For the odd spin structure, the fields $\psi^\mu$ and $\tilde{\psi}^\mu$ each have zero modes that, in the absence of vertex operator insertions, kill the contribution of the odd spin structure to the partition function, while for an even spin structure, neither the fermionic fields $\Psi$, $\tilde{\Psi}$ nor the associated $\beta\gamma$ and $\tilde{\beta}\tilde{\gamma}$ ghost systems have any zero modes. Therefore the partition function is

$$Z_{\boldsymbol{\alpha}}(\tau)\tilde{Z}_{\boldsymbol{\beta}}(\tau) = \frac{\det'(\bar{\partial}_{T_\Sigma})^2}{\det'(\bar{\partial}_{\mathcal{O}})^{10}}\frac{\mathrm{Pf}(\bar{\partial}_{K_\Sigma^{1/2}(\boldsymbol{\alpha})})^{10}}{\det(\bar{\partial}_{T_\Sigma^{1/2}(\boldsymbol{\alpha})})}\frac{\mathrm{Pf}(\bar{\partial}_{K_\Sigma^{1/2}(\boldsymbol{\beta})})^{10}}{\det(\bar{\partial}_{T_\Sigma^{1/2}(\boldsymbol{\beta})})} = \frac{1}{\eta(\tau)^{16}}\frac{\theta_{\boldsymbol{\alpha}}(0|\tau)^4}{\eta(\tau)^4}\frac{\theta_{\boldsymbol{\beta}}(0|\tau)^4}{\eta(\tau)^4}, \tag{3.29}$$

Primes mean that the zero modes are removed prior to evaluating the determinant of Pfaffian. The labels $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ refer to the spin structures associated to $\{\psi,\gamma,\beta\}$ and $\{\tilde{\psi},\tilde{\gamma},\tilde{\beta}\}$ respectively, and $\eta(\tau)$ is the Dedekind eta function.

By themselves these partition functions are not modular invariant. In general, the spin structures are swapped by modular transformations so it is possible to combine the partition functions for different spin structures into objects with better modular properties. In the usual superstring these combinations are given by the GSO projection, in the case of the ambitwistor string these correspond to the partition functions

$$Z_{\mathrm{IIA}}(\tau) = \left(Z_1 + \sum_{\boldsymbol{\alpha}=2,3,4}(-1)^{\boldsymbol{\alpha}}Z_{\boldsymbol{\alpha}}\right)\left(\tilde{Z}_1 - \sum_{\boldsymbol{\alpha}=2,3,4}(-1)^{\boldsymbol{\alpha}}\tilde{Z}_{\boldsymbol{\alpha}}\right) \tag{3.30}$$

$$Z_{\mathrm{IIB}}(\tau) = \left(Z_1 + \sum_{\boldsymbol{\alpha}=2,3,4}(-1)^{\boldsymbol{\alpha}}Z_{\boldsymbol{\alpha}}\right)\left(\tilde{Z}_1 + \sum_{\boldsymbol{\alpha}=2,3,4}(-1)^{\boldsymbol{\alpha}}\tilde{Z}_{\boldsymbol{\alpha}}\right), \tag{3.31}$$

for type IIA and type IIB ambitwistor strings [5]. Here $Z_1$ and $\tilde{Z}_1$ are the (vanishing) partition functions of the $\Psi$ and $\tilde{\Psi}$ systems in the odd spin structure. As usual, both these partition

[4]In general the determinants take values on line bundles over the moduli space. These line bundles are usually non-trivial, but combinations of them might be and thus can be identified with functions over the moduli space and integrated.

[5]There is also a type 0 ambitwistor string by requiring the $\Psi$ and $\tilde{\Psi}$ systems to have the same spin structures. This choice breaks spacetime supersymmetry. However, unlike the real partition function $\propto |\theta_2(\tau)|^N + |\theta_3(\tau)|^N + |\theta_4(\tau)|^N$ of non–chiral Type 0 strings which is modular for any value of $N$, the chiral

functions vanish as a consequence of the Jacobi 'abstruse identity' $\theta_2(\tau)^4 - \theta_3(\tau)^4 + \theta_4(\tau)^4 = 0$ which is a consequence of spacetime supersymmetry and imposes the one-loop vanishing of the spacetime cosmological constant.

These partition functions, (3.31), have modular weight $-8$. The full ambitwistor string partition function includes the integral over the zero modes of $x^\mu$ of the fields $X^\mu$ and $p_\mu$ of $P_\mu$, together with the zero modes of the $bc$ and $\tilde{b}\tilde{c}$ ghost systems, and the measure on the moduli space. The full genus one partition function of the type II string is formally

$$\mathcal{Z}_{\mathrm{IIA/B}} = \int \frac{\mathrm{d}^{10}x\,\mathrm{d}^{10}p}{(\mathrm{vol}\,\mathbb{C}^*)^2}\,\bar{\delta}\big(p^2(\mathrm{d}z)^2\big)\,Z_{\mathrm{IIA/B}}(\tau)\,\mathrm{d}\tau\,, \tag{3.32}$$

where $P_\mu(z) = p_\mu \mathrm{d}z$ in the absence of any vertex operators. The zero mode $p_\mu$ is the coefficient of the abelian differential $\mathrm{d}z$ which under the modular transformation $\tau \to -1/\tau$ behaves as $\mathrm{d}z \to \mathrm{d}z/\tau$. To keep the field invariant the zero mode coefficient should transform as

$$p_\mu \to \tau\, p_\mu \tag{3.33}$$

With this definition, the loop integral measure $\mathrm{d}^{10}p$ acquires a factor of $\tau^{10}$ under this modular transformation. This compensates the weight of the modular function $Z_{\mathrm{IIA/B}}(\tau)\,\mathrm{d}\tau$ so that (3.32) is invariant.

The factor of $1/(\mathrm{vol}\,\mathbb{C}^*)^2$ arises from fixing the zero modes of the $c$ and $\tilde{c}$ ghosts. The $c$ ghost zero mode may be used to fix the insertion point of $\bar{\delta}(p^2(\mathrm{d}z)^2)$ to any point on the torus. Recall that the $\tilde{c}$ ghost is associated to the transformation $\delta X^\mu = \tilde{c} P^\mu$ which translates $X$ along the null geodesic in the $P$ direction. So the remaining $\mathrm{vol}\,\mathbb{C}^*$ can be used to fix one of the $x$ integrals, picking a representative point on each null geodesic. Combining this with the constraint $p^2 = 0$ the integral over zero modes of $X$ and $P$ becomes an integral over the target space $P\mathbb{A}$. This once again emphasizes the fact that the target space of this chiral model is best thought of as ambitwistor space, rather than spacetime.

partition function $\propto \theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8$ of the type 0 ambitwistor string can be modular only in $8k+2$ space-time dimensions.

## 3.3 NS-NS scattering amplitudes at genus one

At genus one, both ghosts $c$ and $\tilde{c}$ have one zero mode corresponding to translations around the torus and along spacetime null geodesics respectively. These are taken care of by inserting a fixed vertex operator, all other vertex operators are integrated. The picture number of these vertex operators depends on the spin structure of the fermions, and therefore of the ghosts. So even and odd spin structure contributions to the amplitudes are considered separately. Only vertex operators in the NS-NS sector will be considered, in principle there's no obstruction to using vertex operators in the R sector, but as at tree-level their correlation functions quickly become cumbersome.

### 3.3.1 Even spin structure

In any of the even spin structures, neither the worldsheet fermions $\Psi^\mu$, $\tilde{\Psi}^\mu$ nor the ghosts $\gamma$, $\tilde{\gamma}$ have zero modes, so no insertions of $\delta(\gamma)$ or $\delta(\tilde{\gamma})$ are necessary. Only vertex operators $U$ descended in the fermionic directions are necessary. The relevant correlator is:

$$\mathcal{M}_n^{1;\text{even}} = \left\langle b_0 \tilde{b}_0 \bar{\delta}(P^2)\, c\tilde{c} U_1(z_1) \prod_{i=2}^{n} \int \bar{\delta}(k_i \cdot P(z_i)) U_i(z_i) \right\rangle , \tag{3.34}$$

where the factor of $\bar{\delta}(P^2)$ in the measure was explained in section 3.1.

Since none of the vertex operators involve $\delta(\gamma)$ or $\delta(\tilde{\gamma})$, the correlator of the $\psi$ fields and of the $\tilde{\psi}$ fields each lead to Pfaffians of $2n \times 2n$ matrices $M'_{\boldsymbol{\alpha}}$ and $\widetilde{M}'_{\boldsymbol{\beta}}$. In other words, unlike at genus zero [12, 13, 32], no rows or columns need to be removed from these matrices. The matrix $M'_{\boldsymbol{\alpha}}$ has elements

$$M'_{\boldsymbol{\alpha}} = \begin{pmatrix} A & -C'^{\mathrm{T}} \\ C' & B \end{pmatrix} \tag{3.35}$$

where

$$A_{ij} = k_i \cdot k_j \, S_{\boldsymbol{\alpha}}(z_{ij}|\tau) \qquad B_{ij} = \epsilon_i \cdot \epsilon_j \, S_{\boldsymbol{\alpha}}(z_{ij}|\tau) \qquad C'_{ij} = \epsilon_i \cdot k_j \, S_{\boldsymbol{\alpha}}(z_{ij}|\tau) \tag{3.36}$$

and $A_{ii} = B_{ii} = C'_{ii} = 0$. In this matrix,

$$S_{\boldsymbol{\alpha}}(z_{ij}, \tau) = \frac{\theta'_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \frac{\theta_{\boldsymbol{\alpha}}(z_{ij}|\tau)}{\theta_{\boldsymbol{\alpha}}(0|\tau)} \sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j} \tag{3.37}$$

is the $g = 1$ free fermion propagator, or Szëgo kernel, in the even spin structure $\boldsymbol{\alpha}$. This is defined to be a half-form in both $z_i$ and $z_j$ (like $\psi(z_i)\psi(z_j)$) so that its modular properties are simple. Under a modular transformation the Szëgo kernel simply changes to a Szëgo kernel in a different even spin structure, that is, it does not acquire any factors of $\sqrt{\tau}$.

The elements of $M'_{\boldsymbol{\alpha}}$ come from a calculation similar to the tree-level one. The $\psi$ insertions at distinct points $z_i$ and $z_j$ on the worldsheet contract with each other to form a Pfaffian. As at genus zero [32], the contributions from the $\epsilon_i \cdot P(z_i)$ in the vertex operators are incorporated by modifying the matrix $C' \to C$, by adding to its diagonal the terms

$$C_{ii} = \epsilon_i \cdot p \,\mathrm{d}z_i + \sum_{j\neq i} \epsilon_i \cdot k_j \, \tilde{S}_1(z_i, z_j|\tau)\,, \tag{3.38}$$

which are independent of the spin structure of the fermions. So the contribution from the vertex operators in the spin structures $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are Pfaffians $\mathrm{Pf}(M_{\boldsymbol{\alpha}})\,\mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}})$, where

$$M_{\boldsymbol{\alpha}} = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix} \tag{3.39}$$

and $\tilde{M}_{\boldsymbol{\beta}}$ is similar but with tilded polarization tensors and a possible different spin structure $\bar{\delta}\beta$. On the support of the scattering equations, these Pfaffians are invariant under gauge transformations $\epsilon_i \to \epsilon_i + k_i$, as follows from BRST invariance.

The result of the correlator is given by combining all the ingredients above. Adding the Pfaffians from the vertex operators, the partition functions and the GSO projection gives

$$\begin{aligned} \mathscr{M}_n^{1;\mathrm{even}} = \delta^{10}\left(\sum_{i=1}^n k_i\right) \int \mathrm{d}^{10}p \wedge \mathrm{d}\tau \; \bar{\delta}\big(P^2(z_1|\tau)\big) \prod_{j=2}^n \bar{\delta}(k_j \cdot P(z_j)) \\ \times \sum_{\boldsymbol{\alpha};\boldsymbol{\beta}} (-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau)\; \mathrm{Pf}(M_{\boldsymbol{\alpha}})\, \mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}}) \end{aligned} \tag{3.40}$$

as the contribution to 1-loop scattering amplitudes from even spin structures. Note that the integrand in (3.40) is a (top,top) form on $\mathscr{M}_{n,1}$; the product of the two Pfaffians transforms as a quadratic differential at each marked point $z_i$ for $i \in \{1,\ldots,n\}$, while the constraints $\prod_{j=2}^n \bar{\delta}(k_j \cdot P(z_j))$ provide holomorphic conformal weight $-1$ at all the marked points except $z_1$, whereas the constraint $\bar{\delta}\big(P^2(z_1|\tau)\big)$ provides holomorphic weight $-2$ at $z_1$.

As mentioned above, the one loop scattering equations fix the positions $\{z_i\}$ of the vertex operators and the complex structure moduli $\tau$ in terms of the external and loop momenta $\{k_i\}$ and $p$. The integral over the loop momentum $p$ must be treated as a contour integral and is expected to diverge on the physical contour $\mathbb{R}^{9,1} \subset \mathbb{C}^{10}$. The loop momentum appears in the Pfaffians, through the diagonal elements (3.38) of $C$, as well as in the scattering equations. Modular invariance of the right hand side of (3.40) follows trivially from the modular invariance of the partition function; indeed, the form weights in the elements of $M_{\bar{\delta}\alpha}$ and $\widetilde{M}_{\bar{\delta}\beta}$ were included to ensure that their Pfaffians are invariant under modular transformations, up to a change in spin structure.

### 3.3.2 Odd spin structure

At genus one, there is a single odd spin structure corresponding to periodic boundary conditions around each of the two non-trivial cycles on the torus. In this spin structure the the ghosts and antighost have one, constant zero mode each. The zero modes of the antighosts correspond to fermionic moduli, which as in the RNS string are fixed by inserting two picture changing operators

$$\Upsilon_0 = \delta(\beta)(P\cdot\psi + \tilde{b}\gamma) \qquad\qquad \widetilde{\Upsilon}_0 = \delta(\tilde{\beta})(P\cdot\tilde{\psi} + \tilde{b}\tilde{\gamma})\,. \tag{3.41}$$

At least at genus one, there are no spurious singularities and BRST invariance ensures the amplitude is independent of the choice of insertion point of these operators.

Each component of the fermionic fields $\psi^\mu$ and $\tilde{\psi}^\mu$ also has a zero mode. So, amplitudes involving fewer than 5 particles don't receive any contribution from this spin structure. For $n \geq 5$ the amplitude receives a contribution from the correlator

$$\mathcal{M}_n^{1;\,\mathrm{odd}} = \left\langle b_0\tilde{b}_0\,\bar{\delta}(P^2(z_0))\,\Upsilon_0\widetilde{\Upsilon}_0\;c_1\tilde{c}_1\delta(\gamma_1)\delta(\tilde{\gamma}_1)V(z_1)\prod_{i=2}^{n}\int\bar{\delta}(k_i\cdot P(z_i))U(z_i)\right\rangle. \tag{3.42}$$

The correlator gives again Pfaffians of $2n\times 2n$ matrices. The $\psi$ system gives the matrix

$$M = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}, \tag{3.43}$$

where the entries now depend on the $\psi$ zero modes $\psi_0$. For $i\neq j$ and $i\neq 1$ these entries are

$$A_{ij} = k_i\cdot k_j\,\tilde{S}_1^F(z_{ij}|\tau) + k_i\!\cdot\!\psi_0\,k_j\!\cdot\!\psi_0 \qquad\qquad i,j\neq 1 \tag{3.44}$$

$$B_{ij} = \epsilon_i\cdot\epsilon_j\,\tilde{S}_1^F(z_{ij}|\tau) + \epsilon_i\!\cdot\!\psi_0\,\epsilon_j\!\cdot\!\psi_0 \tag{3.45}$$

$$C_{ij} = \epsilon_i\cdot k_j\,\tilde{S}_1^F(z_{ij}|\tau) + \epsilon_i\!\cdot\!\psi_0\,k_j\!\cdot\!\psi_0\,. \tag{3.46}$$

The diagonal entries of $C$, when $i \neq 1$, are

$$C_{ii} = -\epsilon_i \cdot P(z_0) \mathrm{d}z_i - \sum_{j \neq i}^{n} \epsilon_i \cdot k_j \ \tilde{S}_1^F(z_{ij}|\tau) \,. \tag{3.47}$$

while the diagonals of $A$ and $B$ are zero. When $i = 1$, the entries of $A$ and $C$ are modified to

$$A_{1j} = P(z_0) \cdot k_j \ \tilde{S}_1^F(z_{0j}) + P(z_0) \cdot \psi_0 \ k_j \cdot \psi_0 \tag{3.48}$$

$$C_{11} = \epsilon_i \cdot P(z_0) \ \tilde{S}_1^F(z_{10}) + \epsilon_i \cdot \psi_0 \ P(z_0) \cdot \psi_0 \,, \tag{3.49}$$

since they come from contractions involving the picture changing operators. In these expressions, $\tilde{S}_1^F(z_{ij}|\tau)$ is the free fermion propagator in the odd spin structure

$$\tilde{S}_1^F(z_{ij}|\tau) := \left( \frac{\theta_1'(z_i - z_j|\tau)}{\theta_1(z_i - z_j|\tau)} + 4\pi \frac{\mathrm{Im}(z_i - z_j)}{\mathrm{Im}(\tau)} \right) \sqrt{\mathrm{d}z_i} \sqrt{\mathrm{d}z_j}. \tag{3.50}$$

Note that it is a half-form in each of $z_i$ and $z_j$[6], which makes it invariant under modular transformations. Here the zero mode $\psi_0^\mu = \psi_{0z}^\mu \sqrt{\mathrm{d}z}$, where $\psi_{0z}^\mu$ are anticommuting constants. Keeping the coefficients and the form degree of the zero modes together makes it easier to examine these expressions under the worldsheet degenerations.

After performing all contractions to obtain the Pfaffian of $M$ (and similarly a Pfaffian of a matrix $\widetilde{M}$ from the other fermion), there are still contributions from the worldsheet partition functions to the path-integral. These end up cancelling among themselves

$$\frac{\det'(\bar{\partial}_{T_\Sigma})^2}{\det'(\bar{\partial}_{\mathscr{O}})^{10}} \frac{\mathrm{Pf}(\bar{\partial}_{K_\Sigma^{1/2}})^{10}}{\det(\bar{\partial}_{T_\Sigma^{1/2}})} \frac{\mathrm{Pf}(\bar{\partial}_{K_\Sigma^{1/2}})^{10}}{\det(\bar{\partial}_{T_\Sigma^{1/2}})} = 1 \,, \tag{3.51}$$

due to the isomorphisms $K_\Sigma^{1/2} \simeq T_\Sigma^{1/2} \simeq \mathscr{O}$ for the odd spin structure at genus one.

Leaving explicit the integration over zero modes, the contribution of the odd spin structure to the $n \geq 5$ particle amplitudes is

$$\mathscr{M}_n^{1;\,\mathrm{odd}} = \delta^{10}\left(\sum k_i\right) \int \mathrm{d}^{10}p \, \mathrm{d}^{10}\psi_0 \, \mathrm{d}^{10}\tilde{\psi}_0 \, \mathrm{d}\tau \, \bar{\delta}(P^2(z_1)) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \times \mathrm{Pf}(M) \, \mathrm{Pf}(\tilde{M}) \, \frac{\mathrm{d}z_1}{(\mathrm{d}z_0)^3}, \tag{3.52}$$

[6]The fermionic propagator looks almost the same as the bosonic propagator $\tilde{S}_1(ij)$, but notice that the latter is a one-form on its first entry and a zero form on its second entry.

where $\mathrm{d}^{10}\psi_0$ and $\mathrm{d}^{10}\tilde{\psi}_0$ are the integrals over the $\psi$ and $\tilde{\psi}$ zero modes, while the ratio $\mathrm{d}z_1/(\mathrm{d}z_0)^3$ comes from the zero modes of the ghost and antighosts in the picture changing operators. It is easy to see that (3.52) is invariant under $\tau \to \tau+1$. Under $\tau \to -1/\tau$, invariance of $p\mathrm{d}z$ again implies that $\mathrm{d}^{10}p \to \tau^{10}\,\mathrm{d}^{10}p$. Likewise, invariance of $\psi_0\sqrt{\mathrm{d}z}$ implies that the fermionic measure changes as $\mathrm{d}^{10}\psi_0 \to \tau^{-5}\,\mathrm{d}^{10}\psi_0$, and similarly for the $\tilde{\psi}$ zero modes. Therefore, under $\tau \to -1/\tau$,

$$\mathrm{d}^{10}p\,\mathrm{d}^{10}\psi_0\,\mathrm{d}^{10}\tilde{\psi}_0\,\mathrm{d}\tau \to \frac{1}{\tau^2}\mathrm{d}^{10}p\,\mathrm{d}^{10}\psi_0\,\mathrm{d}^{10}\tilde{\psi}_0\,\mathrm{d}\tau\,. \tag{3.53}$$

Since the Pfaffians and $\delta$-functions are modular invariant, the only remaining factor comes from the ghost zero mode contribution $\mathrm{d}z_1/(\mathrm{d}z_0)^3$. This produces the missing $\tau^2$ and renders the result modular invariant.

## 3.4 Factorization at genus one

Recall from section 3.1.1 that at genus one, there are two distinct factorization limits. These correspond to the two ways in which the torus can degenerate: either by pinching a non-trivial cycle which reduces the torus to a Riemann sphere, or by pinching a trivial cycle which factors the worldsheet into a sphere and another torus. These were called *non-separating* and *separating* degeneration, respectively, and both can be understood as contributions from the boundary in the moduli space of curves $\overline{\mathcal{M}}_{1,n}$.

In the non-separating case, the boundary divisor being approach is denoted by $\mathfrak{D}^{\text{ns}}$. This looks like the moduli space of genus zero Riemann surfaces with two additional punctures:

$$\mathfrak{D}^{\text{ns}} \cong \overline{\mathcal{M}}_{0,n+2}.$$

The separating degeneration corresponds to a divisor $\mathfrak{D}^{\text{sep}}$ where the worldsheet pinches off a genus zero component $\Sigma_L \cong \mathbb{CP}^1$. The $n$ marked points corresponding to the vertex operators distribute themselves between the two factors, with $n_L$ on $\Sigma_L$ and $n_R$ on $\Sigma_R$ such that $n_L + n_R = n$. This boundary divisor looks like the product

$$\mathfrak{D}^{\text{sep}} \cong \overline{\mathcal{M}}_{0,n_L+1} \times \overline{\mathcal{M}}_{1,n_R+1}.$$

The behaviour near the boundaries of the moduli space can be studied from the worldsheet perspective using CFT methods just like in the superstring [24, 60], or twistor-string theory [61]. However, it is instructive to check that the calculated formula, obtained after evaluating all the correlators, has the correct behaviour near the factorization channels.

In both of these factorization limits, the expression for the genus one scattering amplitude develops a simple pole in the modulus transverse to the boundary divisor. This confirms that the IR behaviour of the amplitude is in accordance with unitarity: the amplitude develops simple poles in the internal momenta as the boundary divisor is approached. In the non-separating case, the residue of this pole is a surprisingly simple expression living on the resulting genus zero worldsheet, which cannot be identified with a CHY formula due to Ramond sector vertex operators which now contribute to the external states. In the separating case, a tree-level amplitude in CHY form factors off from the one-loop result when the residue is extracted.

### 3.4.1 Pinching a non–separating cycle

Pinching a non-separating cycle corresponds to approaching the non-separating boundary divisor $\mathfrak{D}^{\text{ns}} \subset \overline{\mathcal{M}}_{1,n}$, which is described by a degenerate limit of the complex structure $\tau$

of the torus, the relevant limit being $\mathrm{Im}\tau \to \infty$. It is convenient to work with the alternative parametrization $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, where the boundary divisor sits at $q \to 0$.

As this boundary is approached, it is necessary to know how the various ingredients appearing in the amplitude behave. Some were already discussed in section 3.1.1 where the factorization properties of the scattering equations were studied. The others are the Dedekin eta function and theta constants:

$$\eta(\tau) \sim q^{1/24}, \qquad \theta_3(0|\tau),\, \theta_4(0|\tau) \sim 1, \qquad \theta_2(0|\tau) \sim q^{1/8}, \tag{3.54}$$

to leading order in the limit $q \to 0$. The factorization properties of the Szëgo kernels for the different spin structures can obtained from (A.4) or can be rigorously derived using the sewing formalism for Riemann surfaces [62, 63]:

$$S_{\boldsymbol{\alpha}}(z_{ij},\tau) \sim \begin{cases} \frac{\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j}}{z_i - z_j} & \text{if } \boldsymbol{\alpha} = 2 \\ \kappa \times \sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j} & \text{otherwise} \end{cases}, \tag{3.55}$$

where $\kappa$ is some constant. And as already noted before

$$\tilde{S}_1(z_i, z_j|\tau) \sim \frac{\mathrm{d}z_i}{z_i - z_j}, \tag{3.56}$$

as $q \to 0$.

At the degeneration point the contribution to the amplitude coming from the odd spin structure vanishes since there are no odd spin structures on the sphere. So in this limit only the contribution coming from the even spin structures needs to be considered. Start with the behaviour of the Pfaffians $\mathrm{Pf}(M_{\boldsymbol{\alpha}})$, $\mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}})$ in (3.40). Using (3.55) and (3.56), it is clear that when $\boldsymbol{\alpha} = 2$, the block entries of $M_{\boldsymbol{\alpha}}$ become:

$$A_{ij} = k_i \cdot k_j \frac{\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j}}{z_i - z_j}, \qquad B_{ij} = \epsilon_i \cdot \epsilon_j \frac{\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j}}{z_i - z_j}, \qquad C_{ij} = \epsilon_i \cdot k_j \frac{\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j}}{z_i - z_j},$$

which are the expected entries at genus zero [12, 32]. The only subtlety is in the diagonal entries of the $C$-block:

$$C_{ii}|_{q\to 0} = -\sum_{j\neq i} \frac{\epsilon_i \cdot k_j}{z_i - z_j} \mathrm{d}z_i + \epsilon_i \cdot p|_{q\to 0}\, \mathrm{d}z_i,$$

where $p_\mu \mathrm{d}z_i$ is the zero mode of $P_\mu(z_i)$ on the torus. As seen above, on the boundary divisor $\mathfrak{D}^{\mathrm{ns}}$, a holomorphic differential degenerates into a meromorphic differential on the sphere with simple poles at the two new marked points, with equal and opposite residues at those points. Calling this residue $k_\mu$, and denoting the two new marked points as $z_a, z_b \in \mathbb{CP}^1$, the diagonal entries in $C$ become:

$$C_{ii}|_{q\to 0} = \left( -\sum_{j\neq i} \frac{\epsilon_i \cdot k_j}{z_i - z_j} + \frac{\epsilon_i \cdot k}{z_i - z_a} - \frac{\epsilon_i \cdot k}{z_i - z_b} \right) \mathrm{d}z_i = C_{ii}^{n+2}.$$

This is the diagonal entry for the $C$-block with $n+2$ particles, two of which have equal and opposite momentum. The same calculation holds for the matrix $\widetilde{M}_{\boldsymbol{\alpha}}$.

Hence:

$$\mathrm{Pf}(M_2),\ \mathrm{Pf}(\widetilde{M}_2) \xrightarrow{q\to 0} \mathrm{Pf}(M_{ab}^{ab}),\ \mathrm{Pf}(\widetilde{M}_{ab}^{ab}), \tag{3.57}$$

where $M_{ab}^{ab}$ is the matrix whose entries are the same as in the genus zero case for $n+2$ particles, with rows and columns corresponding to the new external states at $z_a, z_b$ removed. Note that unlike boson scattering amplitudes at genus zero, the rank of the Pfaffian is uncharged-changed. For the other two even spin structures, the matrices $M_{\boldsymbol{\alpha}}$, $\widetilde{M}_{\boldsymbol{\alpha}}$ do not approach recognizable structures, however, their contributions cancel due to the GSO projection.

Now the only factors in $\mathcal{M}_n^{1;\mathrm{even}}$ which encode the spin structure and potential $q$-dependence are

$$\begin{aligned} &\mathrm{d}\tau \sum_{\boldsymbol{\alpha};\boldsymbol{\beta}} (-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau) \mathrm{Pf}(M_{\boldsymbol{\alpha}})\,\mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}}) \\ &\qquad = \frac{1}{2\pi i}\frac{\mathrm{d}q}{q} \sum_{\boldsymbol{\alpha};\boldsymbol{\beta}} (-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \frac{\theta_{\boldsymbol{\alpha}}(0|\tau)^4\,\theta_{\boldsymbol{\beta}}(0|\tau)^4}{\eta(\tau)^{24}} \mathrm{Pf}(M_{\boldsymbol{\alpha}})\,\mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}}). \end{aligned} \tag{3.58}$$

Using the leading behaviour given by (3.54), this sum looks like

$$\frac{\mathrm{d}q}{q^2} \sum_{\boldsymbol{\beta}} (-1)^{\boldsymbol{\beta}} \theta_{\boldsymbol{\beta}}(0|\tau)^4\, \mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}}) \left[ q^{1/2}\mathrm{Pf}(M_2) - \mathrm{Pf}(M_3) + \mathrm{Pf}(M_4) \right], \tag{3.59}$$

as $q \to 0$, which appears to have a tachyonic double pole in $q$. But in this limit, $\mathrm{Pf}(M_3) = \mathrm{Pf}(M_4)$, so the last two terms in (3.59) cancel with each other due to the GSO projection. The same argument holds for the sum over $\boldsymbol{\beta}$, leading to the single power of $q$ in the numerator from the only surviving terms where $\boldsymbol{\alpha} = \boldsymbol{\beta} = 2$. Hence, close to the boundary divisor $\mathfrak{D}^{\text{ns}}$ the contribution to the measure from (3.58) is given by:

$$\mathrm{d}\tau \sum_{\boldsymbol{\alpha};\boldsymbol{\beta}} (-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau) \mathrm{Pf}(M_{\boldsymbol{\alpha}}) \, \mathrm{Pf}(\widetilde{M}_{\boldsymbol{\beta}}) \sim \frac{\mathrm{d}q}{q} \mathrm{Pf}(M^{ab}_{ab}) \, \mathrm{Pf}(\widetilde{M}^{ab}_{ab}). \tag{3.60}$$

This is in direct analogy with the role of the GSO projection in superstring theory: a generic term in $\mathscr{M}_n^{1;\text{even}}$ has a tachyonic double pole in the modulus $q$ as the boundary divisor is approached, but the sum over spin structures, with appropriate signs, cancels these double poles and leaves only the simple pole consistent with unitarity.

To summarize; when a non-separating cycle is pinched a pole of order one appears and the amplitude factorizes in terms of an expression on a genus zero worldsheet with two additional particles of equal and opposite null momenta. This null momentum is being integrated over the phase space of the on-shell loop momentum, and there is an implicit sum over all possible intermediate states flowing through the node. In this limit, the integrand depends only on algebraic functions of kinematic invariants, as in the tree-level case. It is expected that a rational function of the external kinematics is recovered after summing over all the solutions to the scattering equations. Note that because of the scattering equations the various elliptic functions only contribute to the simple pole rather than adding higher mode dependence as in ordinary superstring theory, so there is no tower of massive modes.

In this factorized amplitude, the intermediate states could be any state in the $\mathcal{N}=2$ sugra massless multiplet. While there is a compact expression for $n$-graviton scattering that could be used to check the above formula, there is no similarly simple expression for 2-gravitino and $(n-2)$-graviton scattering written in terms of Pfaffians as above. Nevertheless the result of this factorization limit seems to imply that a simple expression for such amplitudes might exist.

### 3.4.2 Pinching a separating cycle

Pinching a separating cycle on the genus one worldsheet factors off a Riemann sphere $\Sigma_L \cong \mathbb{CP}^1$ as the boundary divisor $\mathfrak{D}^{\text{sep}}$ is approached. In this case, the degeneration of the worldsheet is not controlled by the modular parameter $\tau$; instead, it corresponds to a set of $n_L$ of the marked points coming very close to each other. A conformally equivalent statement is that these $n_L$ points lie on a sphere $\Sigma_L$ which is connected to the torus $\Sigma_R$ by a long tube.

A local model for this degeneration can be given near the divisor. In this case the worldsheet is modelled by

$$(z_L - w)(z_R - y) = s, \tag{3.61}$$

where $z_L$ is a local coordinate on $\Sigma_L$ and $z_R$ is a local coordinate on $\Sigma_R$[7]. The parameter $s$ acts as a modulus for the length of the tube connecting the two branches, and as $s \to 0$ the worldsheet separates into $\Sigma_L \cup \Sigma_R$, joined at the points $z_L = w$ and $z_R = y$. The modulus $s$ is actually the natural transverse modulus to the boundary divisor $\mathfrak{D}^{\text{sep}} \subset \overline{\mathcal{M}}_{1,n}$.

Unfortunately, the expression for the $g = 1$ amplitude computed in 3.3 is not optimal for studying the separating degeneration. This is because the amplitude was calculated in a picture with no insertions of $\delta(\gamma)$ or $\delta(\tilde{\gamma})$; this is a natural because there are no zero modes of the superconformal ghosts which need to be fixed at genus one. However, upon pinching the separating cycle, the branch $\Sigma_L$ is a sphere on which $\gamma$ and $\tilde{\gamma}$ have two zero modes each. In other words, the two worldsheets produced by the separating degeneration have different numbers of fermionic moduli. The new states that appear at the nodes of $w \in \Sigma_L$ and $y \in \Sigma_R$ should be represented by fixed vertex operators with picture number $-1$, which is unnatural-natural from the perspective of the picture used in section 3.3. In other words, the use of integrated vertex operators corresponds to a choice of gauge which makes studying this boundary behaviour difficult.

This issue is familiar from the conventional RNS superstring: at arbitrary genus, amplitudes are easiest to compute using a mixture of fixed and integrated vertex operators appropriate to the number of zero modes in the superconformal ghost system. At the level of the moduli space integrand, this expression minimizes the number of picture changing operator insertions and behaves appropriately under all non-separating factorizations and all separating factorizations for which the resulting worldsheets have the same number of fermionic zero modes.[8] However, this choice of picture is unnatural-natural for generic worldsheet degenerations where new states will appear in the fixed picture, making it cumbersome to isolate the boundary behaviour of the amplitude.

One solution to this issue is to represent all external states by fixed vertex operators at the expense of introducing an appropriate number of picture changing operators. The resulting amplitude appears to be different from an expression obtained with integrated vertex operators, but it will be independent of the PCO insertions and equal to the alternative expression. The amplitude in this all-fixed picture is naturally suited to studying the

[7] The choice of a coordinate system on $\Sigma_L$ or $\Sigma_R$ is left implicit from now on.

[8] For example, at genus two the expression factorizes correctly for a non-separating degeneration as well as the separating degeneration that results in two tori, see [64].

behaviour near any boundary divisors in the moduli space since all external states are on the same footing as internal states appearing in the factorization channel. Another way of seeing this is by considering the worldsheet perspective on factorization, where it is essential to work in the all-fixed picture [24, 61].

At genus one, in an even spin structure, this means that the NS-NS sector scattering amplitude should be computed from the worldsheet correlation function:

$$\mathcal{M}_n^{1;\,\text{even}} = \left\langle \prod_{i=1}^{n} \mathcal{V}_i \prod_{a=1}^{n} \Upsilon_a \widetilde{\Upsilon}_a \prod_{r=1}^{n-1} (b_r|\mu_r)\, (\tilde{b}_r|\mu_r) \bar{\delta}\left(\int_\Sigma \mu_r\, P^2\right) \right\rangle , \tag{3.62}$$

where

$$(b_r|\mu_r) = \int_\Sigma b_r \wedge \mu_r \,,$$

is used as a shorthand for the measure on the moduli space.

The resulting amplitude can be computed in much the same way as the previous expression. In an even spin structure the amplitude is:

$$\mathcal{M}_n^{1;\,\text{even}} = \delta^{10}\left(\sum_i k_i\right) \int \mathrm{d}^{10} p \wedge \mathrm{d}\tau \wedge \bar{\delta}\left(P^2(z_1)\right) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \times \sum_{\boldsymbol{\alpha};\boldsymbol{\beta}} (-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau) \frac{\mathrm{Pf}(\mathsf{M}_{\boldsymbol{\alpha}})}{|\mathsf{R}_{\boldsymbol{\alpha}}|} \frac{\mathrm{Pf}(\widetilde{\mathsf{M}}_{\boldsymbol{\beta}})}{|\widetilde{\mathsf{R}}_{\boldsymbol{\beta}}|} , \tag{3.63}$$

where the partition function $Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau)$ is as in (3.29). The skew-symmetric $2n \times 2n$ matrix $\mathsf{M}_{\boldsymbol{\alpha}}$ arises from the matter systems, and is analogous to the matrix $M_{\boldsymbol{\alpha}}$ appearing in (3.40). It can be written in a block decomposition

$$\mathsf{M}_{\boldsymbol{\alpha}} = \begin{pmatrix} \mathsf{A} & -\mathsf{C}^{\mathrm{T}} \\ \mathsf{C} & \mathsf{B} \end{pmatrix} .$$

Entries of the A-block are indexed by the locations of the PCOs, which are denoted by $x_a, x_b \in \Sigma$, for $a, b = 1, \ldots, n$:

$$\mathsf{A}_{ab} = S_{\boldsymbol{\alpha}}(x_{ab}|\tau) \left( \sum_{i,j=1}^{n} k_i \cdot k_j\, \tilde{S}_1(x_a, z_i|\tau)\, \tilde{S}_1(x_b, z_j|\tau) + \sum_{i=1}^{n} k_i \cdot p\, \mathrm{d}x_b\, \tilde{S}_1(x_a, z_i|\tau) + \sum_{j=1}^{n} p \cdot k_j\, \mathrm{d}x_a\, \tilde{S}_1(x_b, z_j|\tau) + p^2\, \mathrm{d}x_a\, \mathrm{d}x_b \right) , \tag{3.64}$$

with $\mathsf{A}_{aa} = 0$. The entries of the $\mathsf{B}$-block are indexed by the vertex operator locations, and are identical to those in (3.36):

$$\mathsf{B}_{ij} = \epsilon_i \cdot \epsilon_j \, S_{\boldsymbol{\alpha}}(z_{ij}|\tau), \qquad \mathsf{B}_{ii} = 0. \tag{3.65}$$

Finally, the rows of the $\mathsf{C}$-block are indexed by the vertex operators, while its columns are indexed by the PCOs:

$$\mathsf{C}_{ia} = S_{\boldsymbol{\alpha}}(x_a - z_i|\tau) \left( \sum_{j=1}^{n} \epsilon_i \cdot k_j \, \tilde{S}_1(x_a, z_j|\tau) + \epsilon_i \cdot p \, \mathrm{d}x_a \right). \tag{3.66}$$

A determinant of the $n \times n$ matrix $\mathrm{R}_{\boldsymbol{\alpha}}$ arises in the denominator due to the correlator of the $\beta\gamma$-system. This is the bosonic 'Slater determinant' [24] whose entries are composed of the propagators between the $\gamma$ insertions for vertex operators and the $\beta$ insertions for the PCOs:

$$\mathsf{R}_{ia} = S_{\boldsymbol{\alpha}}(z_i - x_a|\tau) \frac{\mathrm{d}x_a}{\mathrm{d}z_i}. \tag{3.67}$$

The entries of $\widetilde{\mathsf{M}}_{\boldsymbol{\beta}}$ and $\widetilde{\mathsf{R}}_{\boldsymbol{\beta}}$ are exactly the same, except for the spin structure and polarization vectors.

At first, it may appear that (3.63) cannot be equivalent to the earlier expression (3.40): not only are the various Pfaffians different, but there are also Slater determinants as well with apparent dependence on the locations of the PCOs. By usual BRST arguments this expression must be independent of the PCOs locations $x_a$, but there appear to be various poles in $\mathsf{M}_{\boldsymbol{\alpha}}$ and $\widetilde{\mathsf{M}}_{\boldsymbol{\beta}}$ when these points coincide with the locations of the vertex operators $z_i$. However, by carefully considering the limit where $x_i \to z_i$, it can be shown that all these apparent singularities vanish, and the resulting expression is in fact *equal to* (3.40). By Liouville's theorem, this means that (3.63) and (3.40) are equivalent representations of the even spin structure contribution to the amplitude. A similar story holds for the odd spin structure, although this will not be present explicitly here.

With the expression (3.63) for the amplitude it is now easy to study the behaviour of the amplitude near the non-separating degeneration using the local model (3.61). All the ingredients in the amplitude which are associated uniquely with the torus simply remain on the $\Sigma_R$ factor without contributing any dependence on the parameter $s$. In particular, the integrals over $\mathrm{d}^{10}p$ and $\mathrm{d}\tau$, as well as $Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}$ simply move onto $\Sigma_R$ as $s \to 0$. The odd spin structure also contributes nothing to the $\Sigma_L$ branch since there is no odd spin structure on the sphere.

As the separating cycle is pinched, $n_L$ of the vertex operators move onto $\Sigma_L$, while the remaining $n_R = n - n_L$ remain on $\Sigma_R$. The PCOs locations also divide themselves between the two factors; in order for the result to be non-vanishing, there must be $n_L - 1$ of the $x_a$ on $\Sigma_L$ and $n_R + 1$ on $\Sigma_R$. Near the boundary divisor, there is a natural identification of three of the moduli in play: the modulus $s$, and the locations of the two new fixed points $w, y$. These will contribute to the overall measure as [24]

$$\mathrm{d}w\,\mathrm{d}y\,\frac{\mathrm{d}s}{s^2}, \tag{3.68}$$

by the scaling properties of (3.61). The form degrees in $w, y$ will be absorbed by the various Pfaffians and scattering equations.

Turning to the behaviour of the Pfaffians as $s \to 0$, every entry in $\mathsf{M}_{\boldsymbol{\alpha}}$ falls into one of two classes: either both of its indices are on the same side of the separating cycle, or they are on different sides. If $z, z' \in \Sigma_L$, then as $s \to 0$ the Szëgo kernel $S_{\boldsymbol{\alpha}}(z - z'|\tau)$ simply reduces to the Szëgo kernel on $\Sigma_L$, and similarly for $z, z' \in \Sigma_R$.

On the other hand, when $z \in \Sigma_L$ and $z' \in \Sigma_R$, homogeneity and conformal invariance dictate that the Szëgo kernel behaves like

$$S_{\boldsymbol{\alpha}}(z - z'|\tau) = \frac{\sqrt{s}}{\sqrt{\mathrm{d}w}\sqrt{\mathrm{d}y}}\frac{\sqrt{\mathrm{d}z}\sqrt{\mathrm{d}w}}{z - w}\,S_{\boldsymbol{\alpha}}(y - z'|\tau) + O(s^{3/2}), \tag{3.69}$$

as $s \to 0$. Similar reasoning dictates that the propagator $\tilde{S}_1$ behaves as

$$\tilde{S}_1(z, z'|\tau) = \frac{s}{\mathrm{d}y}\frac{\mathrm{d}z}{z - w}\,\tilde{S}_1(y, z'|\tau) + O(s^2), \tag{3.70}$$

in this situation.

This gives the behaviour of the entries in $\mathsf{M}_{\boldsymbol{\alpha}}$ in the $s \to 0$ limit. For instance, if $x_a, x_b \in \Sigma_L$ then

$$\mathsf{A}_{ab} = \frac{\sqrt{\mathrm{d}x_a}\sqrt{\mathrm{d}x_b}}{x_a - x_b}\sum_{i,j\in L\cup\{w\}} k_i\cdot k_j\,\frac{\mathrm{d}x_a\,\mathrm{d}x_b}{(x_a - z_i)(x_b - z_j)} + O(s). \tag{3.71}$$

Using (3.18)–(3.19) in conjunction with (3.69)–(3.70) it is easy to see that for a general entry in $\mathsf{M}_{\boldsymbol{\alpha}}$:

$$(\mathsf{M}_{\boldsymbol{\alpha}})_{i_L j_L} \to (\mathsf{M}^L)_{i_L j_L}, \qquad (\mathsf{M}_{\boldsymbol{\alpha}})_{i_R j_R} \to (\mathsf{M}^R_{\boldsymbol{\alpha}})_{i_R j_R}, \tag{3.72}$$

where $\mathsf{M}^L$ is the matrix for the genus zero amplitude on $\Sigma_L$ with external particles in $L \cup \{a\}$ and $\mathsf{M}^R_{\boldsymbol{\alpha}}$ is the matrix for the genus one amplitude on $\Sigma_R$ with external particles in $R \cup \{b\}$.

There are also entries in $\mathsf{M}_{\boldsymbol{\alpha}}$ which tie together locations on opposite sides of the separating cycle. A simple calculation reveals that for $x_a \in \Sigma_L$, $x_b \in \Sigma_R$,

$$\begin{aligned}\mathsf{A}_{ab} = &\frac{\sqrt{s}}{\sqrt{\mathrm{d}w}\sqrt{\mathrm{d}y}} \frac{\sqrt{\mathrm{d}x_a}\sqrt{\mathrm{d}w}}{x_a - w} S_{\boldsymbol{\alpha}}(y - x_b|\tau) \\ &\times \left( \sum_{i \in L\cup\{w\}} \sum_{j \in R\cup\{y\}} k_i \cdot k_j \frac{\mathrm{d}x_a}{x_a - z_i} \tilde{S}_1(x_b, z_j|\tau) + \sum_{i \in L\cup\{w\}} k_i \cdot p \, \mathrm{d}x_b \frac{\mathrm{d}x_a}{x_a - z_i} \right) + O(s^{3/2}) \end{aligned} \tag{3.73}$$

as $s \to 0$. Likewise, for $x_a \in \Sigma_L$ and $z_i \in \Sigma_R$,

$$\mathsf{C}_{ia} = \frac{\sqrt{s}}{\sqrt{\mathrm{d}w}\sqrt{\mathrm{d}y}} \frac{\sqrt{\mathrm{d}x_a}\sqrt{\mathrm{d}w}}{x_a - w} S_{\boldsymbol{\alpha}}(y - z_i|\tau) \sum_{j \in L\cup\{w\}} \epsilon_i \cdot k_j \frac{\mathrm{d}x_a}{x_a - z_j} + O(s^{3/2}), \tag{3.74}$$

and for $z_i \in \Sigma_L$, $z_j \in \Sigma_R$,

$$\mathsf{B}_{ij} = \frac{\sqrt{s}}{\sqrt{\mathrm{d}w}\sqrt{\mathrm{d}y}} \frac{\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}w}}{z_i - w} S_{\boldsymbol{\alpha}}(y - z_i|\tau) \, \epsilon_i \cdot \epsilon_j + O(s^{3/2}). \tag{3.75}$$

In each of these entries, there is a product $e_i \cdot e_j$, where $e^\mu$ is either a momentum or polarization vector. Using the completeness relation these contractions can be written in terms of polarization vectors:

$$e_i \cdot e_j = e_i^\mu \, e_j^\nu \left( \sum_{\epsilon_I} \epsilon_{a\,\mu} \epsilon_{b\,\nu} - \frac{k_{R\,\mu} k_{R\,\nu}}{k_R^2} \right),$$

where the sum runs over the possible polarizations of the internal particle. The second term in this expression is actually just a gauge transformation so it can be neglected. Upon inspecting (3.73)-(3.75), the completeness relation actually generates all the entries in the $(2w)^{\text{th}}$ row and column of $\mathsf{M}^L$ as well as the $(2y)^{\text{th}}$ row and column of $\mathsf{M}^R_{\boldsymbol{\alpha}}$, up to an overall factor proportional to $\sqrt{s}$.

Using the basic properties of Pfaffians the behaviour of $\mathrm{Pf}(\mathsf{M}_{\boldsymbol{\alpha}})$ as the separating cycle is pinched is:

$$\mathrm{Pf}(\mathsf{M}_{\boldsymbol{\alpha}}) \to \frac{\sqrt{s}}{\sqrt{\mathrm{d}w}\sqrt{\mathrm{d}y}} \mathrm{Pf}(\mathsf{M}^L)\, \mathrm{Pf}(\mathsf{M}^R_{\boldsymbol{\alpha}}), \tag{3.76}$$

where $\mathsf{M}^L$ is the $2n_L \times 2n_L$ matrix at genus zero and $\mathsf{M}^R_{\boldsymbol{\alpha}}$ is the $2(n_R+1)\times 2(n_R+1)$ matrix at genus one. The final ingredient is given by the factorization of the determinant $|\mathsf{R}_{\boldsymbol{\alpha}}|$, which is guaranteed by the properties of the $\beta\gamma$-system.[9] In particular:

$$|\mathsf{R}_{\boldsymbol{\alpha}}| \to \frac{1}{\sqrt{s}}|\mathsf{R}^L|\,|\mathsf{R}^R_{\boldsymbol{\alpha}}|, \tag{3.77}$$

for the appropriate $(n_L+1)\times(n_L+1)$ Slater determinant on $\Sigma_L$ and $(n_R+1)\times(n_R+1)$ determinant on $\Sigma_R$. The factor of $s^{-1/2}$ ensures the appropriate homogeneity, since there is now a row corresponding to $w$ in $\mathsf{R}^L$ and a row corresponding to $y$ in $\mathsf{R}^R_{\boldsymbol{\alpha}}$.

Pulling all the pieces together, the genus one amplitude near the separating boundary divisor looks like:

$$\int \frac{z_{12}z_{2w}z_{w1}}{\mathrm{d}z_1\,\mathrm{d}z_2\mathrm{d}w}\prod_{i\in L\setminus\{1,2\}}\bar{\delta}\left(k_i\cdot P(z_i)\right)\frac{\mathrm{Pf}(\mathsf{M}^L)}{|\mathsf{R}^L|}\frac{\mathrm{Pf}(\widetilde{\mathsf{M}}^L)}{|\widetilde{\mathsf{R}}^L|}\frac{\mathrm{d}s}{s}\,\bar{\delta}\left(s\mathscr{F}+k_R^2\right)$$
$$\mathrm{d}^{10}p\,\mathrm{d}\tau\,\bar{\delta}\left(P^2(y)\right)\prod_{j\in R}\bar{\delta}\left(k_j\cdot P(z_j)\right)\sum_{\boldsymbol{\alpha};\boldsymbol{\beta}}(-1)^{\boldsymbol{\alpha}+\boldsymbol{\beta}}Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\tau)\,\frac{\mathrm{Pf}(\mathsf{M}^R_{\boldsymbol{\alpha}})}{|\mathsf{R}^R_{\boldsymbol{\alpha}}|}\frac{\mathrm{Pf}(\widetilde{\mathsf{M}}^R_{\boldsymbol{\beta}})}{|\widetilde{\mathsf{R}}^R_{\boldsymbol{\beta}}|}. \tag{3.78}$$

As expected, there is only a simple pole in the degeneration modulus $s$; taking the residue of this pole sets the momentum flowing across the cut to be null ($k_R^2=0$), and it is easy to show that the resulting on-shell amplitudes for $\Sigma_L$ and $\Sigma_R$ are equivalent to the genus zero NS-NS formula and (3.63) respectively.

Hence, the genus one amplitude of the ambitwistor string factorizes correctly in the separating channel. Note that in this case the resulting amplitudes were identified as the tree-level and one-loop with bosonic external states. This is because the Ramond sector cannot contribute to the separating degeneration, since the resulting amplitudes would have only one external fermion and therefore vanish.

[9] This behaviour is universal for the superconformal ghost system, or for any general Slater determinant, in superstring as well as the ambitwistor string.

## 3.5 The IR behaviour of the one-loop amplitude

It is far from obvious that the prescription given by the ambitwistor string reproduces the one-loop amplitudes of type II sugra. The fact that the amplitude is presented in terms of elliptic functions and as an integral over the moduli space of a marked torus makes it look much more like a string theory amplitude than a field theory one. Compelling evidence was given at the end of the last section where the ambitwistor string amplitude was shown to factorize as expected from a field theory, *not* as a string theory. In particular no tower of massive modes is observed running in the loop. Still it would be better to have a proof of the equivalence between the ambitwistor prescription and usual field theory. Even at low points this is not a trivial task, it entails finding all solutions to the one-loop scattering equations, evaluating the integrand on top of them and summing over the whole set of solutions. The integrand itself is not very friendly, depending on rational functions of elliptic functions summed over the different spin structures on the torus.

The aim of this chapter is to provide more evidence that (3.40) and (3.42) indeed reproduces the amplitudes of type II sugra. In order to do so I'll study the simplest amplitude with four external NS-NS states in a particular kinematical regime, the deep IR. In this region the scattering equations simplify enough so that, with the help of some numerics, explicit solutions can be found. Since the sugra loop amplitudes are in general divergent in 10 dimensions I stripping out the integration over the zero modes of $P$, which gives the ambitwistor integrand. This integrand is evaluated on the solutions of the scattering equations, summed over them, and matched explicitly with the integrand obtained from field theory, including non-trivial kinematic dependence. I'll also give some conjectures about the contribution of the scattering equations to the integrand for any number of external particles.

In this section the loop momenta will be denoted by $\ell$, the reason for this change of notation will become clear later. The usual Mandelstam variables will be denoted by $s=(k_1+k_2)^2$, $t=(k_1+k_4)^2$, $u=(k_1+k_3)^2$. The holomorphic derivative of the propagators with respect to the worldsheet coordinate is denoted by a prime, that is $\frac{\partial}{\partial z}S(z|\tau)=S'(z|\tau)$. Finally $S(z_{ij})=S_{ij}$ without any explicit label for spin structure will stand for the propagator in the odd spin structure.

### 3.5.1 Boundary behaviour of the ambitwistor amplitude

The factorisation of the amplitude on the boundaries of the moduli space was already studied in section 3.4. Here I'll start by describing a subtlety of this limit when $q$ is small but finite. Consider the kinematic regime where the loop momenta $\ell^2 \to 0$. In this region a factor of the ambitwistor integrand should produce a factor of $1/\ell^2$. This behaviour is insensitive to the number of external particles so it should come from an universal feature of the amplitudes. Indeed, it is the Jacobian coming from solving the scattering equations that produces this term. This Jacobian has to contain all the information about the scalar propagators of the amplitude, as it does in the CHY formulas at tree level. The difference is that at one-loop there is an extra loop momentum which is not localised. The structure of the Jacobian is:

$$J = \left(\begin{array}{c|c} A_{ij} & B_i \\ \hline C_j & D \end{array}\right) \tag{3.79}$$

where

$$A_{ij} = \begin{cases} k_i \cdot k_j S'_{ij}, & \text{if } i \neq j\,, \\ \sum_l k_l \cdot k_i S'_{il}, & \text{if } i = j\,, \end{cases} \tag{3.80}$$

with

$$B_i = \ell \cdot k_i S'_{0i} + \sum_j k_i \cdot k_j S_{j0} S'_{i0}\,, \tag{3.81}$$

$$C_i = \sum_j k_i \cdot k_j \partial_\tau S_{ij}\,, \tag{3.82}$$

$$D = \sum_i \ell \cdot k_i \partial_\tau S_{i0} + \sum_{j \neq i} k_i \cdot k_j S_{i0} \partial_\tau S_{j0}\,. \tag{3.83}$$

After solving the scattering equations, the integrand for the amplitudes is computed by evaluating the Pfaffians and the Jacobian on these solutions and summing over all of them[10]. Schematically:

$$\sum_{\text{solutions}} \frac{\text{Pf}(M)\,\text{Pf}(\tilde{M})}{J} = \text{“generalized integrand"}\,, \tag{3.84}$$

where the right hand side stands for the result of bringing under the same integral sign the field theory integrands corresponding to the the various Feynman graphs.

In section 3.1.1 it was shown that when $\ell \to 0$, the parameter $q$ can be consistently considered to vanish as well for certain solutions of the scattering equations. The converse is not necessarily true; in principle, there could be solutions for which $\ell^2 \to 0$ but $q$ stays finite and the following analysis won't be sensitive to those solutions. By general worldsheet factorisation arguments even is such solutions exist they shouldn't contribute to IR divergences.

[10]Here a sum over the spin structures has been omitted for clarity.

At $q=0$ and $\ell^2=0$, the $n$ scattering equations reduce $n-1$ independent ones

$$P\cdot k_i(z_i)=\ell\cdot k_i+\sum_{j\neq i}\frac{\pi k_i\cdot k_j}{\tan(\pi z_{ij})}=0. \tag{3.85}$$

where only leading terms in the expansion of the propagator[11] were kept. The last equation (3.12), that is $P^2=0$, is automatically satisfied at $q=0$; the finite piece cancels due to a trigonometric identity, somewhat analogous to a partial fraction decomposition

$$\frac{1}{\tan(\pi z_{ij})\tan(\pi z_{jk})}+\frac{1}{\tan(\pi z_{jk})\tan(\pi z_{ki})}+\frac{1}{\tan(\pi z_{ki})\tan(\pi z_{ij})}=-1\,, \tag{3.86}$$

valid for any set of three complex numbers $z_i, z_j, z_k$.

At this stage, the choice of which propagator to use is immaterial since $q=0$ is equivalent to $1/\mathrm{Im}\tau=0$, so that both propagators coincide. But at finite $q$ there is a possible difference and using the full propagator obscures the correct $1/\ell^2$ behaviour, thereby motivating the choice of a holomorphic representation in this section.

Consider the case of a large but not infinite $\mathrm{Im}\tau$, or small but nonzero $q$. Working with the full propagator (3.9), that is, the one with the non-holomorphic term, the $\epsilon=1/\mathrm{Im}\tau$ correction is much bigger than corrections of order $q$. So it makes sense to consider corrections of order $\epsilon$, such that $z_i=z_i^0+\epsilon z_i^\epsilon$ is a new solution to the scattering equations.

The first $P(z_i)\cdot k_i$, $i=1,\cdots n-1$ equations are still satisfied at order zero while the $O(\epsilon)$ terms give a system of linear equations for the $z_i^\epsilon$. Once plugged back in the last equation $P^2(z_0)=0$, the zeroth order cancels again but the $O(\epsilon)$ seems to undergo no further obvious cancellations, indicating that $\epsilon$ is of the order of the zero mode part $\ell^2$. This,*a priori*, is a possibility. Knowing that the expected leading infrared behaviour of the integrand is $1/\ell^2$, it means that the Jacobian should be of order $1/\epsilon$, that is, $\mathrm{Im}\tau$. As the analysis below will demonstrate, the presence of $\tau$ derivatives in the Jacobian always produces order $O(\epsilon^2)$ terms due to the fact that $\partial_\tau(1/\mathrm{Im}\tau)=(2\mathrm{i})^{-1}(\mathrm{Im}\tau)^{-2}$. This second order contributions to the Jacobian in turn seems to give an incorrect IR behaviour, of the form $\frac{\mathrm{d}\ell}{\epsilon^2}\sim\frac{\mathrm{d}\ell}{\ell^4}$ instead of the expected $1/\ell^2$.

On the other hand, dropping the non-holomorphic part of the propagator, the first small correction to be turned on is of order $q$. The same analysis as above holds, but with $\epsilon=q\sim\ell^2$. This is easily seen to produce the correct qualitative IR behaviour since the $\tau$ derivatives do not change anymore the overall degree of $\epsilon$; $\partial_\tau q=2\mathrm{i}\pi q$. This motivates the choice to adopt the purely holomorphic propagators from now on.

[11] See appendix A for one loop formulas and identities

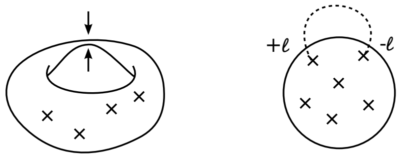


Fig. 3.2 4-point pinched torus creates a 6-point sphere with two back-to-back momenta.

Now, back to studying the behaviour of the Jacobian (3.79) on the support of solutions for which $\ell^2 \to 0$ implies $q \to 0$. The propagators themselves reduce to $1/\tan$ trigonometric functions, as in (3.85). The derivatives of the propagator with respect to the coordinates $z_i$ are finite

$$S'_{ij} \to -\frac{\pi^2}{\sin^2(\pi z_{ij})} + O(q)\,, \tag{3.87}$$

but the $\tau$ derivatives are of order $q$

$$\partial_\tau S_{ij} = 8\mathrm{i}\pi^2 q \sin(2\pi z_{ij}) + O(q^2)\,. \tag{3.88}$$

Therefore, the last line of the Jacobian (3.79) is proportional to $q$, which means that $|J| \to q|M|$ where $M$ has no other dependence on $q$ at leading order. Since $\ell^2 \propto q$ for small $q$ this explains how the Jacobian produces the scalar propagator that is going on shell. Schematically;

$$\frac{1}{\text{Jacobian}} \propto \frac{1}{\ell^2}\,. \tag{3.89}$$

Before moving on it is good to recall once again the geometry of pinching a non-trivial cycle in the torus. The factorisation properties of the ambitwistor string in the $q \to 0$ limit are very reminiscent of the traditional picture in string theory. In particular, the fact that the torus pinches in the limit is completely compatible with factorisation of the amplitude in the $\ell^2 = 0$ channel. The resulting geometry can be interpreted as the forward limit of an $(n+2)$-point tree-level amplitude, where the two new punctures have back to back momentum $\ell^\mu$ and $-\ell^\mu$, see figure 3.2. Since the external kinematics are not generic the number of independent solutions is smaller in this limit.

Numerically (using the simple `NSolve` routine of `Mathematica`), there are at 6,7 and 8 points, 2,12 and 72 solutions respectively. A reasonable conjecture for the generic pattern of the number of solutions is $(n-3)! - 2(n-4)!$;

$$N_{\text{sols}}^{\text{forward}-\text{tree}} = (n-3)! - 2(n-4)!\,. \tag{3.90}$$

This is just a conjecture, so far there is no satisfactory proof of this. Table 3.1 displays the known number of solutions for generic kinematics, the number of solutions in the forward

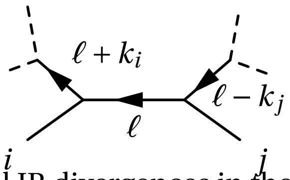


Fig. 3.3 Typical IR divergences in theories of gravity.

limit at low number of points and the number of trivalent diagrams at $n$ points. This emphasises that the number of solutions is much smaller than the number of diagrams at tree level.

| $n$ | $N_{\text{sols}}^{\text{tree}}$ | $N_{\text{sols}}^{\text{forward-tree}}$ | Number of cubic graphs |
|---|---|---|---|
| 4 | 1 | $\emptyset$ | 3 |
| 5 | 2 | $\emptyset$ | 15 |
| 6 | 6 | 2 | 105 |
| 7 | 24 | 12 | 945 |
| 8 | 120 | 72 | 10395 |

Table 3.1 Number of solutions to the tree-level scattering equations (known to be $(n-3)!$), number of solutions in the forward kinematics, number of cubic graphs; $(2n-5)!!$.

Since the geometry is similar to tree-level, it is expected that the number of boundary solutions to the one-loop scattering equations is equal to the number of solutions in the tree level forward kinematics, making it equal to $(n-3)!-2(n-4)!$. Numerical agreement with this claim was observed at 4 and 5 points, at 5 points the one-loop system was solved for vanishing $q$. If there exists additional solutions which are not sent to the boundary of the moduli space in this limit, then the analysis done so far is insensitive to it. Therefore the total number of solutions is bounded by the number of conjectured tree-level forward solutions;

$$N_{\text{sols}}^{1-\text{loop}} \geq (n-3)!-2(n-4)!. \tag{3.91}$$

### 3.5.2 Three propagators on-shell

The kinematic regime in which analytic results will be obtained is characterised by the fact that three adjacent propagators are going on shell, $\ell^2, (\ell+k_i)^2, (\ell-k_j)^2 \to 0$. From the point of view of the pinched worldsheet described before, this can be seen as a sort of a double collinear limit, where the loop momentum $\ell^\mu$ is tuned to be collinear with two external particles $k_i^\mu$ and $k_j^\mu$. The leading infrared divergence originates from the configuration where the legs $i$ and $j$ are adjacent, as pictured in 3.3, which gives the following behaviour

$$\text{leading IR} \sim \frac{1}{(\ell\cdot k_i)\ell^2(\ell\cdot k_j)} \tag{3.92}$$

up to an overall product of propagators corresponding to the ordering of the graph. In gauge theory, these would be dressed with appropriate colour factors selecting possible divergences. In gravity or QED [65] this is not the case, since all orderings contribute equally. Therefore these divergent terms can be grouped under the same integration sign. As will be shown later, the explicit solutions of the scattering equations in this IR regime modify the scaling of $q$ to

$$q \propto \ell^2(\ell\cdot k_i)(\ell\cdot k_j)\,. \tag{3.93}$$

From this the qualitative IR behaviour of the ambitwistor Jacobian can be obtained. It follows from the fact that the Jacobian is of order $q$ in this limit and the leftover determinant is finite and nonzero, as in (3.89). At four points, this can be made very precise. Consider taking $\ell^2 \to 0$ as well as taking the loop momenta to be collinear with particles 2 and 3. The boxes which contribute to the the leading IR divergence are given in figure 3.4.

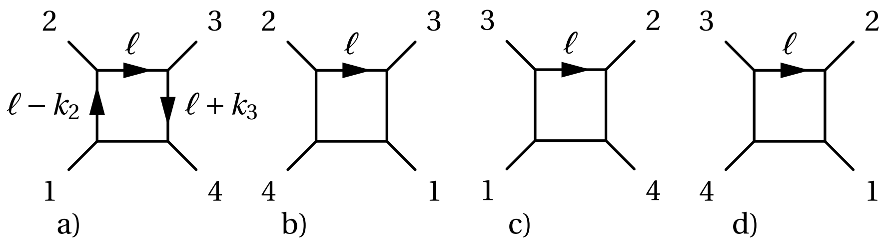


Fig. 3.4 The four boxes that contribute to the IR divergence

Their contribution is

$$\begin{aligned} \text{box}_a &= \frac{1}{2\ell\cdot k_4 + s} & \text{box}_b &= \frac{1}{-2\ell\cdot k_4 + u} \\ \text{box}_c &= \frac{1}{2\ell\cdot k_4 + u} & \text{box}_d &= \frac{1}{-2\ell\cdot k_4 + s} \end{aligned} \tag{3.94}$$

up to a global divergent factor

$$\frac{-1}{4(\ell\cdot k_2)\ell^2(\ell\cdot k_3)}\,. \tag{3.95}$$

Bringing all these divergent integrands under the same integral sign, gives the leading IR divergence

$$\frac{-1}{2(\ell\cdot k_2)\ell^2(\ell\cdot k_3)}\left(\frac{-stu + t(2\ell\cdot k_4)^2}{(s^2-(4\ell\cdot k_4)^2)(u^2-(4\ell\cdot k_4)^2)}\right). \tag{3.96}$$

It is this non-trivial factor, including its functional dependence on the last propagator $\ell\cdot k_4$, that will be matched against the ambitwistor integrand in the following sections.

### 3.5.3 Numerator structure

The Pfaffians entering (3.40) may seem to be extremely complicated objects, as they depend on various theta functions and derivatives thereof. It is far from obvious that these objects not only give rational functions of the kinematic invariants but also reproduce the very simple integrands of maximal supergravity. However, these type of spin structure sums are well known in RNS string amplitudes, for which simplifications arise due to Riemann's theta-function identities (see for example [66]). The identity needed here is

$$\sum_{\alpha=1,2,3,4}(-1)^{\alpha-1}\prod_{i=1}^{4}\theta_\alpha(v_i)=-2\prod_{i=1}^{4}\theta_1(v_i'),\tag{3.97}$$

with $v_1'=\frac{1}{2}(-v_1+v_2+v_3+v_4)$, $v_2'=\frac{1}{2}(v_1-v_2+v_3+v_4)$, $v_3'=\frac{1}{2}(v_1+v_2-v_3+v_4)$, $v_4'=\frac{1}{2}(v_1+v_2+v_3-v_4)$.

This identity gives rise to four vanishing identities

$$\begin{aligned}&\sum_{\alpha=2,3,4}(-1)^{\alpha-1}\frac{\theta_\alpha(0|\tau)^4}{\eta(\tau)^{12}}(\tau)=0\,,\\&\sum_{\alpha=2,3,4}(-1)^{\alpha-1}\frac{\theta_\alpha(0|\tau)^4}{\eta(\tau)^{12}}\prod_{r=1}^{n}S_\alpha(z_r)=0\,,\end{aligned}\tag{3.98}$$

for $n=1,2,3$, where the $z_r$'s are arbitrary. The first non-vanishing identity is

$$\sum_{\alpha=2,3,4}(-1)^{\alpha-1}\frac{\theta_\alpha(0|\tau)^4}{\eta(\tau)^{12}}\prod_{i=1}^{4}S_\alpha(z_i|\tau)=-(2\pi)^4\,,\tag{3.99}$$

for $z_1+\cdots+z_4=0$. In order to write (3.99), the identity

$$\partial_z\theta_1(0|\tau)=\pi\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)=2\pi\eta^3(\tau)\tag{3.100}$$

was used. The Dedekind $\eta$ functions was introduced in order to have the partition functions $Z_{\boldsymbol{\alpha}}$ defined in (3.29) explicit in the left hand side of (3.98), and (3.99). These identities imply, as in string theory, that the 0, 1, 2 and 3-point amplitudes vanish due to target space supersymmetry. This is a statement about the numerator of the integrand, the scattering equations should still be valid for $n\leq 4$. The 4-point amplitude simplifies considerably and the whole ambitwistor numerator boils down to a single kinematical term, the $t_8F^4t_8\tilde{F}^4 = t_8t_8R^4$ tensor. This is the only kinematic invariant at four points allowed by maximal supersymmetry of the form $R^4$.[12] In In the ends the four point amplitude is simplifies to

$$I_4 = t_8t_8R^4\int \mathrm{d}\tau\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4\bar{\delta}(P^2(z_0))\bar{\delta}(k_2\cdot P(z_2))\bar{\delta}(k_3\cdot P(z_3))\bar{\delta}(k_4\cdot P(z_4))\,. \qquad (3.101)$$

The leftover physics of the integrand is captured solely by the Jacobian. Its evaluation on top of the solutions of the scattering equations should reproduce the the one-loop four-graviton integrand, which is a simple sum of scalar box integrands [68]. This also gives a tempting interpretation of integrals of the type of $I_4$ for generic $n$ as a representation of scalar $n$-gons integrals.

### 3.5.4 IR solution to the four-point one-loop scattering equations

The manifestly holomorphic scattering equations are [13]

$$\ell\cdot k_i+\sum_{j\neq i}k_i\cdot k_jS_{ij}=0\,,\qquad i=2,\dots,n-1 \qquad (3.102a)$$

$$\ell^2+2\sum_{i=1}^{n}\ell\cdot k_i\,S_{0i}+\sum_{i\neq j}^{n}k_i\cdot k_j\,S_{0i}S_{0j}=0\,. \qquad (3.102b)$$

The last equation may be rewritten as

$$0=\ell^2-2\sum_{1\le i<j\le 4}k_i\cdot k_j\left(S_{0i}S_{ij}+S_{j0}S_{0i}+S_{ij}S_{j0}\right). \qquad (3.103)$$

on the support of the other equations. It is now easy to check that this equation has no poles in $z_0$ and since it is a holomorphic elliptic function on $z_0$ without any poles, by Liouville's theorem it has to be a constant.

The kinematical regime to be studied is given by $\ell\cdot k_1$ and $\ell\cdot k_4$ and sending $\ell\cdot k_2\to 0$ and $\ell\cdot k_3\to 0$, with $\ell\cdot k_2<\ell\cdot k_3$. In this regime, the equations that need to be solved are similar to the 6-point tree-level equations, which are easy to solve numerically. The first

[12] The field strength $F^{\mu\nu}$ is the linearized field strength defined by $F^{\mu\nu}=\varepsilon^\mu k^\nu-k^\mu\varepsilon^\nu$ and $R^{\mu\nu\rho\sigma}=F^{\mu\nu}F^{\rho\sigma}$. The $t_8$ tensor is defined in [67, Appendix 9.A], where it is given by $t_8F^4=4\,\mathrm{Tr}(F^{(1)}F^{(2)}F^{(3)}F^{(4)})-\mathrm{Tr}(F^{(1)}F^{(2)})\,\mathrm{Tr}(F^{(3)}F^{(4)})+\text{perms}(2,3,4)$, traces are taken over the Lorentz indices. In the spinor-helicity formalism one has $2t_8F^4=\langle 12\rangle^2[34]^2$ and $4t_8t_8R^4=\langle 12\rangle^4[34]^4$. Note also that $\langle 12\rangle^2[34]^2=\mathrm{i}stA^{tree}(1^-,2^-,3^+,4^+)$ where $A^{tree}$ is the tree level four graviton amplitude.

[13] Note that the $(n+1)$-th equation $\ell\cdot k_1+\sum_{j\neq 1}k_1\cdot k_jS_{1j}=0$ holds automatically by momentum conservation

outcome of the numerics is that for $q=0$ and $\ell^2=0$ there are only two solutions, complex conjugate to one another. This still holds after turning on a small but finite $q$.

The second one is that the leading part of the positions of the vertex operators scale as;

$$\begin{aligned} \mathrm{i}\pi z_2 &= \log(\sqrt{\ell\cdot k_2 c_2}) \\ \mathrm{i}\pi z_3 &= -\log(\sqrt{\ell\cdot k_3 c_3}) \end{aligned} \tag{3.104}$$

where $c_2$ and $c_3$ are complex constants of mass dimension $(-2)$, to be determined. Finally, it should be noted that the signs are obtained for a given kinematic configuration, that is $\ell\cdot k_2 < \ell\cdot k_3$. For consistency, in other kinematical configurations the signs might change.

Now, declare that (3.104) an *ansatz*, in which $c_2$, $c_3$ and $z_4$, or rather

$$c_4 := \exp(-2\mathrm{i}\pi z_4)\,, \tag{3.105}$$

are unknowns to be determined to first order in $q, \ell\cdot k_2, \ell\cdot k_3$. In this manner, the scattering equations can be simplified by Taylor expanding the propagators

$$\begin{aligned} \mathrm{i}\cot(\pi z_{21}) &= 1+2\ell\cdot k_2 c_2 \\ \mathrm{i}\cot(\pi z_{23}) &= 1+2\ell\cdot k_2\ell\cdot k_3 c_2 c_3 \\ \mathrm{i}\cot(\pi z_{24}) &= 1+2\ell\cdot c_2 c_4 \\ -\mathrm{i}\cot(\pi z_{31}) &= 1+2\ell\cdot k_3 c_3 \\ -\mathrm{i}\cot(\pi z_{34}) &= 1+2\ell\cdot k_3 c_3/c_4 \end{aligned} \tag{3.106}$$

where $O(q)$ terms on the right hand side were omitted for clarity. It is easy to derive similar rules for any trigonometric function of the same arguments that is required to explicit evaluate the Jacobian. With these, the $k_4\cdot P(z_4)$ scattering equation simplifies drastically

$$\pi\cot(\pi z_4) = \frac{\ell\cdot k_4}{k_1\cdot k_4} + \mathrm{i}\pi\frac{s-u}{t} \tag{3.107}$$

from which $c_4$ can be extracted. The scattering equations $k_2\cdot P(z_2)$ and $k_3\cdot P(z_3)$ can be rewritten, at leading order,

$$\begin{aligned} 2\ell\cdot k_2 - \mathrm{i}s(1+2\ell\cdot k_2 c_2) - \mathrm{i}t(1+2\ell\cdot k_2\ell\cdot k_3 c_2 c_3) - \mathrm{i}u(1+2\ell\cdot k_2 c_2 c_4) = 0\,, \\ 2\ell\cdot k_3 + \mathrm{i}u(1+2\ell\cdot k_3 c_3) + \mathrm{i}t(1+2\ell\cdot k_2\ell\cdot k_3 c_2 c_3) + \mathrm{i}s(1+2\ell\cdot k_3 c_3/c_4) = 0\,. \end{aligned} \tag{3.108}$$

After using momentum conservation, these reduce to a degenerate system of quadratic equations with unique solution given by

$$
\begin{aligned}
c_2 &= \frac{\mathrm{i}\ell\cdot k_4 - \pi u}{\pi t \ell\cdot k_4}\,,\\
c_3 &= -\frac{\mathrm{i}\ell\cdot k_4 + \pi s}{\pi t \ell\cdot k_4}\,,\\
c_4 &= -\frac{\pi s + \mathrm{i}\ell\cdot k_4}{\pi u - \mathrm{i}\ell\cdot k_4}\,.
\end{aligned}
\tag{3.109}
$$

The last scattering equation, $P^2(z_0)$, can now be used to determine $q$ to first order, considering the new scaling (3.104) in this limit. The Fourier-Jacobi expansion includes sine functions as coefficients of $q$. These produce divergent terms when its arguments involve momenta becoming collinear to $\ell^\mu$. In particular, it is not hard to see in (3.103) that the most divergent term will come from $\sin(2\pi z_{23})$, so that

$$
0 = \ell^2 + 4\pi^2 q k_2\cdot k_3\,(S_{23}S_{30} + S_{32}S_{20})\big|_{(q)}\,,
\tag{3.110}
$$

at leading order. To extract the exact value of this term, use the independence of $P^2(z_0)$ with respect to $z_0$ and set $z_0 = 1/2$. In this case, the $\cot(\pi z_{20})$ and $\cot(\pi z_{30})$ terms become tan's which are readily evaluated to $\pm i$, as in (3.106) (recall that $z_1 = 0$). Finally the modular parameter is fixed to:

$$
q = -\frac{c_2 c_3}{8\pi^2 k_2\cdot k_3}\ell^2(\ell\cdot k_2)(\ell\cdot k_3)\,.
\tag{3.111}
$$

This equation indicates that the scaling of $q$ is not only dictated by the $\ell^2 \to 0$ but also by the collinear $\ell\cdot k_2 \to 0$ and $\ell\cdot k_3 \to 0$ and other kinematic invariants, as claimed in section 3.5.2.

The final part of the computation is the determination of the Jacobian. This will verify that there are no further divergences that could change this IR behaviour, and will match the ambitwistor prescription to the field theory result (3.96).

### 3.5.5 Computation of the Jacobian

First, observe that since $q$ was stripped off from the Jacobian, no more factors of $\ell\cdot k_2$ or $\ell\cdot k_3$ can contribute at first order.[14] Thus, this stripped determinant depends only on $c_4$, $s, t, u$ and $\ell\cdot k_4$.

[14] There are possible divergences inside the Jacobian. It is not hard to see that they multiply terms of order $\ell\cdot k_2 \ell\cdot k_3$ inside the Jacobian, thus rendering them finite. This pattern extends to higher points.

Analytically evaluating it gives a remarkable simplification of the determinant, which reduces to a single term:

$$J = -64q\mathrm{i}\pi^7 t^2 (\ell\cdot k_4)^2 \,. \tag{3.112}$$

Replacing $q$ (3.111) as well as $c_2$ and $c_3$, gives:

$$J = -16\mathrm{i}\pi^3 \frac{\ell^2(\ell\cdot k_2)(\ell\cdot k_3)}{t}(\pi u - \mathrm{i}\ell\cdot k_4)(\pi s + \mathrm{i}\ell\cdot k_4)\,. \tag{3.113}$$

At this point, there is already an interesting combination appearing on the right side of the last expression. This is highly reminiscent of a combination of two IR boxes in fig. 3.4, up to a rescaling of $\ell \to 2\mathrm{i}\pi\ell$.

The last step of the prescription is to sum over the solutions of the scattering equations. At four-point two solutions contribute to this IR limit, the one just described and its complex conjugate. Before performing this sum a last subtlety must be addressed; the Jacobian contains a $\partial_\tau$ derivative, which is not a holomorphic operation on $q$. Therefore, the evaluation of the Jacobian for the second solution, denoted $\tilde{J}$, is obtained by exchanging the $z_i$'s and $q$ for their complex conjugate, while not complex conjugating the i coming from $\partial_\tau = 2\mathrm{i}\pi q\partial_q$. The final result is:

$$\frac{1}{J} + \frac{1}{\tilde{J}} = \frac{-1}{(16\mathrm{i}\pi^3)\ell^2(\ell\cdot k_2)(\ell\cdot k_3)}\frac{2\pi^2 stu + 2(\ell\cdot k_4)^2}{((\pi u)^2 + (\ell\cdot k_4)^2)((\pi s)^2 + (\ell\cdot k_4)^2)}\,, \tag{3.114}$$

which is exactly the sum of symmetrized boxes (3.96), after rescaling $\ell \to 2\mathrm{i}\pi\ell$.

Note that nowhere in this computation the spacetime dimension was used explicitly. This is evidence that the result is actually independent of the spacetime dimension, and that the integral eq. (3.101) is well defined in any dimension.

### 3.5.6 Extension to $n$ points

Remarkably, the solution presented above extends straightforwardly to $n$ points, at least qualitatively. Going again to the limit where three adjacent propagators go on shell, use the ansatz of eq. (3.104). The qualitative behaviour follows from the fact that the arguments given for factoring $q$ out of the Jacobian still hold, and so does the scaling obtained in eq. (3.111). Therefore, it is immediate that *the Jacobian possess terms with the qualitative IR behaviour expected from scalar $n$-gons.* This strengthens the interpretation of the scalar integrals of the type of (3.101) as scalar $n$-gons, which can be defined in any dimension.

It is even possible to actually extract information on the behaviour of $z_2$ and $z_3$ in this limit. The scattering equations for $z_2$ and $z_3$ are solved exactly in the same manner as above in (3.108), these are:

$$2\ell\cdot k_2 - \mathrm{i}k_1\cdot k_2(1+2\ell\cdot k_2c_2) - \mathrm{i}k_2\cdot k_3(1+2\ell\cdot k_2\ell\cdot k_3c_2c_3) \\ -\mathrm{i}\sum_{j=4}^{n} k_2\cdot k_j(1+2\ell\cdot k_2c_2c_j) = 0\,, \tag{3.115}$$

and

$$2\ell\cdot k_3 - \mathrm{i}k_1\cdot k_3(1+2\ell\cdot k_3c_3) + \mathrm{i}k_2\cdot k_3(1+2\ell\cdot k_2\ell\cdot k_3c_2c_3) \\ +\mathrm{i}\sum_{j=4}^{n} k_3\cdot k_j(1+2\ell\cdot k_3c_3c_j) = 0\,, \tag{3.116}$$

where the $c_j$ for $j \geq 4$ are defined similarly to $c_4$ in (3.105).

These equations can be solved as in (3.109), using momentum conservation and replacing $k_{2/3}\cdot k_4c_4$ by the sum $\sum_{j=1} k_{2/3}\cdot k_jc_j$. The unknowns $c_2$ and $c_3$ can be expressed in terms of $c_4$ as:

$$c_2 = \frac{1}{\mathrm{i}\pi(k_1\cdot k_2 + k_2\cdot k_4c_4)}\,, \qquad c_3 = \frac{-c_4}{\mathrm{i}\pi(k_1\cdot k_2 + k_2\cdot k_4c_4)}\,. \tag{3.117}$$

It is now straightforward to replace $c_4$ by its $n$-point value. A more precise statement would require solving for the remaining $c_j$, which quickly becomes difficult for large values of $n$.

## 3.6 Non-minimal $\infty$-tension limit of the pure spinor

In the context of superstring theory, the pure spinor formalism gives a manifestly super-Poincaré-invariant quantization which avoids the difficulties of dealing with space-time supersymmetry or light-cone gauge in the RNS and Green-Schwarz formalisms, respectively [48, 50, 51]. By now the pure spinor formalism has been used extensively in the study of perturbative scattering amplitudes, enabling explicit calculations at higher-genus which have so far been beyond the reach of other methods, see for example the three loop calculation in [69].

A pure spinor version of the chiral, 'infinite tension' worldsheet model has also been proposed [49], and shown to give the correct tree-level S-matrix of fully supersymmetric type II supergravity [58]. Given the efficacy of the pure spinor approach to superstring amplitudes at higher genus, it seems natural to ask if there is a prescription for the calculation of loop integrands in supergravity using this formalism.

In the superstring context, higher genus prescriptions can be made using the 'minimal' worldsheet variables; unfortunately, it entails the use of complicated picture changing operators to define the functional integrals [70]. Furthermore, the prescription for integrating over the worldsheet modular parameters requires an effective $b$-antighost which is not manifestly covariant, its the definition depends on the choice of a patch of pure spinor space [71]. While explicit calculations at genus one [70, 72] and two [73, 74] can be made with this formalism, the picture changing operators complicate the functional integration and break manifest Lorentz covariance at intermediate stages, although the final amplitudes are covariant [75].

A more elegant prescription is provided by adding *non-minimal* worldsheet variables to the model and modifying the BRST charge [76, 77]. This eliminates the need for picture changing operators and allows one to define a covariant effective $b$-ghost to perform moduli integrals. In this section definitions of non-minimal versions of the *supergravity* worldsheet model worldsheet action, BRST charge, effective $b$-ghost, and regulator prescriptions will be given. In many aspects, these objects closely resemble (or are even identical to) their string theoretic counterparts, while also inheriting much of the structure of pure spinor world*line* formalisms for supergravity [52, 78].

Following the non-minimal pure spinor superstring, define the non-minimal version of the model (2.39) by adding two sets of worldsheet fields: bosonic spinors $\bar{\lambda}_\alpha, \bar{\tilde{\lambda}}_{\tilde{\alpha}}$ and fermionic spinors $r_\alpha, \tilde{r}_{\tilde{\alpha}}$, along with their respective conjugate fields $\bar{w}^\alpha, \bar{\tilde{w}}_{\tilde{\alpha}}$ and $s^\alpha, \tilde{s}^{\tilde{\alpha}}$. These variables obey the constraints

$$\bar{\lambda}_\alpha \gamma_m^{\alpha\beta} \bar{\lambda}_\beta = 0 = \bar{\tilde{\lambda}}_{\tilde{\alpha}} \gamma_m^{\tilde{\alpha}\tilde{\beta}} \bar{\tilde{\lambda}}_{\tilde{\beta}}, \qquad \bar{\lambda}_\alpha \gamma_m^{\alpha\beta} r_\beta = 0 = \bar{\tilde{\lambda}}_{\tilde{\alpha}} \gamma_m^{\tilde{\alpha}\tilde{\beta}} \tilde{r}_{\tilde{\beta}}. \tag{3.118}$$

This means that $\bar{\lambda}, \bar{\tilde{\lambda}}$ are pure spinors of opposite chirality to $\lambda, \tilde{\lambda}$; if the space-time signature is taken to be Euclidean, then they can be interpreted as complex conjugates of the original variables. The constraints also restrict the fermions $r, \tilde{r}$ to having eleven independent components.

The modified action is

$$S = \frac{1}{2\pi}\int_{\Sigma} P_m \bar{\partial} X^m + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha + \bar{w}^\alpha \bar{\partial}\bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha + \text{tilded}. \tag{3.119}$$

The constraints ensure that the $\bar{w}\bar{\lambda}$-system contributes +22 units of central charge, which is balanced by the −22 contributions from the $rs$-system. Hence, the condition for the conformal anomaly cancellation, is unchanged, that is, this non-minimal worldsheet model has critical dimension $d = 10$, just like the minimal version. The action for the non-minimal fields is free, but the various constraints require a careful treatment of their OPEs. In particular, the variables of conformal weight $(1,0)$ can only appear in currents that are invariant under the gauge transformations induced by the pure spinor constraints. These are precisely the same as those used in the superstring [76]:

$$\bar{N}_{mn} = \frac{1}{2}\left(\bar{w}\gamma_{mn}\bar{\lambda} + s\gamma_{mn} r\right), \qquad \bar{J} = \bar{w}\cdot\bar{\lambda} + s\cdot r, \qquad T_{\bar{\lambda},r} = -\bar{w}^\alpha \partial\bar{\lambda}_\alpha - s^\alpha \partial r_\alpha,$$

$$S_{mn} = \frac{1}{2}(s\gamma_{mn}\bar{\lambda}), \qquad S = s\cdot\bar{\lambda}. \tag{3.120}$$

The currents for the tilded variables are identical, and have the *same* conformal weight as the untilded-tilded currents. The various OPEs between these currents are collected in appendix B for reference.

Define the non-minimal BRST operator to be

$$Q = \oint \lambda^\alpha d_\alpha + \tilde{\lambda}^{\tilde{\alpha}} \tilde{d}_{\tilde{\alpha}} + \bar{w}^\alpha r_\alpha + \tilde{\bar{w}}^{\tilde{\alpha}} \tilde{r}_{\tilde{\alpha}}, \tag{3.121}$$

which is nilpotent due to the pure spinor constraint. Since the 'quartet' of non-minimal variables do not affect $Q^2 = 0$, standard arguments [79, 80] ensure that they have no impact on the BRST cohomology. In particular, external supergravity states can be represented in the non-minimal worldsheet model by the same fixed (2.43) and integrated (2.44) vertex operators used in the minimal model.

Note that just as the minimal model and BRST charge (2.39), (2.42) resemble a holomorphic complexification of the pure spinor superparticle, the non-minimal action and BRST charge (3.119), (3.121) are a holomorphic complexification of the non-minimal superparticle developed in [78]. This worldline formalism has been used to check the UV divergence structure of maximally supersymmetric supergravity loop amplitudes [81], suggesting that the worldsheet model should be related to field theory beyond tree-level.

### 3.6.1 Effective $b$-ghost

In the RNS formalism for superstring theory, the prescription for integrating over worldsheet moduli at arbitrary genus is provided by the functional integral over the conformal $bc$-ghost system. In the RNS-like worldsheet formulation of supergravity, there are *two* conformal ghost systems: one corresponds to gauging the worldsheet stress tensor as in string theory, while the other corresponds to gauging the Hamiltonian constraint $P^2 = 0$ [32]. This latter constraint ensures that the resulting worldsheet correlation functions are supported on the scattering equations – indeed, in the presence of vertex operator insertions, $P^2 = 0$ is *equivalent* to the scattering equations at any genus [5].

Of course, there is no $bc$-ghost system in either the pure spinor superstring or the worldsheet model discussed here. In the superstring, a prescription for integrating over moduli is nonetheless available by defining a composite operator $b \in \Pi\Omega^0(\Sigma, K_\Sigma^2)$, called an effective $b$-ghost, which obeys $\{Q, b\} = T$. In the worldsheet model, it is also possible to construct an effective $b$-ghost, but instead of being related to the stress tensor, this composite operator obeys $\{Q, b\} = P^2$. The effective $b$-ghost of the pure spinor superparticle is also related to the Hamiltonian constraint (albeit a real function on the worldline rather than a quadratic differential on the worldsheet), and ensures the gauge invariance of the propagator [78]. Viewing the worldsheet model as a complexification of the worldline theory, this choice of ghost will likewise ensure gauge invariance, as well as modular invariance and the appropriate scattering equations at arbitrary genus.

While the lack of an explicit relationship with the stress tensor is slightly mysterious, it seems to be related to the fact that both the Virasoro and Hamiltonian constraints are implied by a single twistor-like constraint in conjunction with a $\lambda^\alpha$ constraint[15]. In the superstring, the twistor-like constraint implies the Virasoro constraint only [82, 83]. Of course, the ultimate test of this choice will be the resulting amplitude prescription.

The construction of the effective $b$-ghost proceeds in direct analogy to the superstring calculation [76, 84]). So, what is needed is an operator $G^\alpha \in \Pi\Omega^0(\Sigma, K_\Sigma^2)$ which obeys $\{Q, G^\alpha\} = \lambda^\alpha P^2$. Using the various OPEs between currents and fields in the worldsheet model, it is easy to see that

$$G^\alpha = -P_m (\gamma_m d)^\alpha \,, \tag{3.122}$$

[15] See the appendix of [6].

has the desired property. Now, since

$$\{Q, \lambda^\alpha G^\beta\} = \lambda^\alpha \lambda^\beta P^2 , \tag{3.123}$$

the operator $(\lambda^\alpha G^\beta - \lambda^{((\alpha} G^{\beta))})$ is BRST-closed, where $((\cdots))$ denotes the symmetric, gamma-matrix-traceless part. As the $Q$-cohomology at ghost number one with non-zero conformal weight is trivial, there must exist some $H^{\alpha\beta}$ of conformal weight $(2,0)$ such that

$$\left[Q, H^{\alpha\beta} - H^{((\alpha\beta))}\right] = \lambda^\alpha G^\beta - \lambda^{((\alpha} G^{\beta))} . \tag{3.124}$$

A calculation identical to the analogous step in the superstring reveals that

$$H^{\alpha\beta} = \frac{(\gamma^{mnp})^{\alpha\beta}}{96} \left[ (d\gamma_{mnp} d) + 24 N_{mn} P_p \right] . \tag{3.125}$$

Cohomological arguments allow for the continued construction of a chain of operators, each related to the previous operator in the chain by the action of $Q$, until the chain terminates by virtue of the pure spinor constraint. These operators can then be arranged into a single composite operator by making use of the non-minimal pure spinor variables:

$$b = -\frac{(\bar\lambda \gamma^m d) P_m}{\bar\lambda \cdot \lambda} - \frac{(\bar\lambda \gamma^{mnp} r)}{96 (\bar\lambda \cdot \lambda)^2} \left[ (d\gamma_{mnp} d) + 24 N_{mn} P_p \right] \\ + \frac{(r\gamma_{mnp} r)(\bar\lambda \gamma^m d)}{8 (\bar\lambda \cdot \lambda)^3} N^{np} - \frac{(r\gamma_{mnp} r)(\bar\lambda \gamma^{pqr} r)}{64 (\bar\lambda \cdot \lambda)^4} N^{mn} N_{qr} , \tag{3.126}$$

which obeys $\{Q, b\} = P^2$. The effective $b$-ghost for the tilded worldsheet fields takes an identical form. This composite operator is identical to the holomorphic complexification of the $b$-ghost appearing in the non-minimal pure spinor superparticle [78, 81], up to an overall constant factor.

### 3.6.2 Zero modes measure

In any path integral calculation, regardless of the details of the amplitude prescription, zero modes of the various worldsheet fields must be integrated over. Remarkably, the only variable in the model (3.119) which does not appear in the pure spinor superstring is $P_m \in \Omega^0(\Sigma, K_\Sigma)$; all other worldsheet fields appear as left-movers in the superstring. Hence, the subtleties associated with their functional integrations can be dealt with in *exactly* the same manner as they are handled in the superstring context. Crucially, the tilded sector of the worldsheet model is just a second (left-moving) copy of the untilded-tilded sector.

The conformal weight zero matter fields $\theta^\alpha, \tilde{\theta}^{\tilde{\alpha}}$ have the usual zero mode integration measures, which will be denoted by $\mathrm{d}^{16}\theta$, $\mathrm{d}^{16}\tilde{\theta}$ at arbitrary worldsheet genus. Likewise, at any genus the bosonic and fermionic pure spinor variables $\lambda^\alpha$, $\bar{\lambda}_\alpha$, $r_\alpha$ and their tilded counterparts have eleven zero modes. Since these are identical to the pure spinor variables of the superstring, we can use the same integration measures that were developed in that context for both the tilded and untilded-tilded variables. The precise definition of the zero mode measures can be found in [70, 76, 85]; here they will simply be denoted as $[\mathrm{d}\lambda]$, $[\mathrm{d}\bar{\lambda}]$, $[\mathrm{d}r]$, etc.

All of the conjugate fields in this model are left-moving, with conformal weight $(1,0)$. So on a genus $g$ worldsheet, they acquire $g$ zero modes which must be integrated over. Let $f$ be any such worldsheet field; at genus $g$ it can be expanded as

$$f \to \widehat{f} + \sum_{I=1}^{g} f^I_{\text{z.m.}}\, \omega_I\,, \tag{3.127}$$

where $\widehat{f}$ is the quantum (non-zero mode) field, $\{\omega_I\}$ form a basis of $H^0(\Sigma, K_\Sigma)$, and $f^I_{\text{z.m.}}$ are the functions (bosonic or fermionic) which parametrize the zero modes. Choosing a canonical basis $\{A_1, \dots, A_g, B_1 \dots, B_g\}$ for $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ and the $\{\omega_I\}$ such that

$$\int_{A_I} \omega_J = \delta_{IJ}\,, \qquad \int_{B_I} \omega_J = \Omega_{IJ}\,, \tag{3.128}$$

where $\Omega_{IJ}$ is the period matrix of $\Sigma$, the zero mode of a field can be extracted unambiguously as

$$f^I_{\text{z.m.}} = \int_{A_I} f\,.$$

The various conformal weight one fields which have zero mode structure of this form are $p_\alpha, w_\alpha, \bar{w}^\alpha, s^\alpha$ and their tilded counterparts (the field $P_m$ will be treated later). Once again, their zero mode integrals can be performed in an identical manner to the left-movers of the superstring. Hence, the integral over $p^I_{\text{z.m.}\,\alpha}$ is exchanged for an integral over $d^I_{\text{z.m.}\,\alpha}$, and the various integral measures are denoted by

$$[\mathrm{d}d] = \prod_{I=1}^{g} [\mathrm{d}^{16} d^I_{\text{z.m.}}]\,, \qquad [\mathrm{d}w] = \prod_{I=1}^{g} [\mathrm{d}^{11} w^I_{\text{z.m.}}]\,, \dots \tag{3.129}$$

The technical definitions of these measures can be found throughout the literature on the pure spinor formalism [70, 76, 85].

Just as in the pure spinor superstring, there are two important subtleties associated with these zero mode integrations. Firstly, there are non-compact integrals which can introduce potential divergences. If the non-minimal formalism is to be equivalent to the minimal prescription at genus zero, it cannot have new divergences, so these integrals require regularization. But since the pure spinor variables of the worldsheet model are identical to the left-moving pure spinor variables of the superstring, the *same* regulator can be used. In particular, taking $\mathcal{N} = \exp(\{Q, \chi\})$ will not affect worldsheet correlation functions of BRST-closed vertex operators, so on a genus $g$ worldsheet set [76]

$$\chi = -\bar{\lambda}\cdot\theta - \sum_{I=1}^{g} \left( N^{I}_{\text{z.m.}\,mn}\, S^{mn\,I}_{\text{z.m.}} + J^{I}_{\text{z.m.}}\, S^{I}_{\text{z.m.}} \right) . \tag{3.130}$$

The exponential suppression then provides a regulator for the large $\lambda, \bar{\lambda}$ region.

The second subtlety arises from the zero mode integration near the tip of the pure spinor cone, where $\bar{\lambda}\cdot\lambda \to 0$. It can be shown that the zero mode measures are convergent in this region [76, 85]:

$$[\mathrm{d}\lambda]\,[\mathrm{d}\bar{\lambda}]\,[\mathrm{d}r] \sim \lambda^8 \bar{\lambda}^{11} .$$

However, the effective $b$-ghost (3.126) contains a term which diverges like $(\bar{\lambda}\cdot\lambda)^{-3}$ near the tip of the pure spinor cone. There are $3g-3$ insertions of such $b$-ghosts for any correlator on a genus $g \geq 2$ worldsheet, so potential divergences can arise for $g > 2$.

Once more, the pure spinor superstring provides a resolution for this problem. There a solution has be proposed in the form of a BRST-invariant regularization of the effective $b$-ghost. While the functional form of (3.126) differs slightly from the effective $b$-ghost of the superstring, its dependence on the pure spinor variables is the same, so the same the pure spinor regularization for the $b$-ghost given by Berkovits and Nekrasov [77] can be adopted for the worldsheet model. The precise details of this regularization will not be needed for the calculations below.

The regularized $b$-ghost will be denoted by $b_\epsilon$; accounts of its use in several calculations can be found in [77, 86, 87]. This prescription has yet to be used in a full, non-trivial superstring amplitude computation (the divergences do not arise for the four-point function until $g = 5$ due to fermionic zero mode saturation). However, any potential issues which could arise from practical computations in the superstring will be identical in the worldsheet model.

### 3.6.3 Amplitude prescription

Given the similarities between the model (3.119) and the superstring, the higher genus amplitude prescription follows closely [76]. In particular, on a genus $g \geq 2$ worldsheet the

$n$-point amplitude is defined by the worldsheet correlation function:

$$\mathcal{M}_n^{(g)} = \lim_{\epsilon\to 0}\int \prod_{a=1}^{3g-3} \mathrm{d}\tau_a \left\langle \mathcal{N}\widetilde{\mathcal{N}} \prod_{j=1}^{3g-3} \bar{\delta}\left(P^2(z_j)\right) (b_\epsilon|\mu)_j (\tilde{b}_\epsilon|\tilde{\mu})_j \prod_{i=1}^{n} \int_\Sigma \bar{\delta}\left(k_i\cdot P(z_i)\right) U(z_i) \right\rangle . \tag{3.131}$$

The complex parameters $\{\tau_a\}$ are the complex structure moduli of the genus $g$ Riemann surface $\Sigma$ integrated over the fundamental domain of the modular group[16]; $\mathcal{N}, \widetilde{\mathcal{N}}$ are the regulators defined by (3.130); $b, \tilde{b}$ are the effective $b$-ghosts of (3.126); $\epsilon$ is the regulation parameter of [77]; and $U(z)$ is the integrated vertex operator (2.44). The Beltrami differentials $\mu_j$ form a basis of $H^{0,1}(\Sigma, T_\Sigma)$, with

$$(b|\mu) := \frac{1}{2\pi}\int_\Sigma \mu \lrcorner b, \tag{3.132}$$

and likewise for the tilded variables. The brackets $\langle\cdots\rangle$ indicate the correlator in the worldsheet CFT; that is, integrating over zero modes and eliminating non-zero modes via worldsheet OPEs. Note that for $g=2$, the regulator $\epsilon$ can be dropped from this prescription.

As usual, the amplitude prescription for a genus one worldsheet should include a single fixed vertex operator in accordance with the ghost number anomaly. Thus, the $g=1$ amplitudes are defined by

$$\mathcal{M}_n^{(1)} = \int \mathrm{d}\tau \left\langle \mathcal{N}\widetilde{\mathcal{N}}\bar{\delta}\left(P^2(z_1)\right)(b|\mu)(\tilde{b}|\tilde{\mu})\, V(z_1) \prod_{i=2}^{n}\int_\Sigma \bar{\delta}\left(k_i\cdot P(z_i)\right) U(z_i)\right\rangle . \tag{3.133}$$

On the Riemann sphere, we have three fixed vertex operators in accordance with $\mathrm{SL}(2,\mathbb{C})$ invariance, leading to

$$\mathcal{M}_n^{(0)} = \left\langle \mathcal{N}\widetilde{\mathcal{N}} \prod_{i=1}^{3} V(z_i) \prod_{j=4}^{n}\int_\Sigma \bar{\delta}(k_i\cdot P(z_i))\, U(z_i)\right\rangle . \tag{3.134}$$

Despite the apparent complexity of the general amplitude prescription, there are nevertheless some important universal properties which can be easily observed. Note that with the momentum eigenstates of (2.43), (2.44), the worldsheet field $X^m$ enters the correlator only via the plane wave exponentials $\mathrm{e}^{ik\cdot X}$. Following the strategy adopted for the RNS-like model [32], the $X$ path integral can be performed explicitly, enforcing

[16]This is consistent with modular invariance. Modular invariance at the level of the correlation function is obscured due to the regulator, it only becomes manifest in the final amplitude.

ten-dimensional momentum conservation and the equation of motion

$$\bar{\partial} P_m(z) = 2\pi i \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \sum_{i=1}^{n} k_{i\,m} \delta^2(z - z_i) \,. \tag{3.135}$$

This indicates that $P_m$ is a meromorphic differential on $\Sigma$, with singularities only at the vertex operator insertions $\{z_i\} \subset \Sigma$.

On a genus $g$ Riemann surface, the kernel of $\bar{\partial} : \mathscr{O} \to K_\Sigma$, denoted by $\tilde{S}_g(z, w|\Omega)$, serves as the propagator for the $PX$-system. This is a $(1,0)$-form with respect to $z$ and a scalar with respect to $w$, and can be defined as

$$\tilde{S}_g(z, w|\Omega) = \partial_z G_g(z, w|\Omega) \,, \tag{3.136}$$

$$G_g(z, w|\Omega) = -\ln|E_g(z, w)|^2 + 2\pi \sum_{I,J=1}^{g} (\mathrm{Im}\,\Omega)^{-1}_{IJ} \left( \mathrm{Im} \int_z^w \omega_I \right) \left( \mathrm{Im} \int_z^w \omega_J \right) , \tag{3.137}$$

where $E_g(z, w)$ is the prime form [25, 59]. In the limit where $z \to w$, this propagator has the expected simple pole

$$\lim_{z \to w} \tilde{S}_g(z, w|\Omega) \sim \frac{\mathrm{d}z}{z - w} \,, \tag{3.138}$$

in appropriately chosen inhomogeneous coordinates on $\Sigma$.

The equation (3.135) can be integrated using (3.137), this gives

$$P_m(z) = \sum_{I=1}^{g} \ell^I_m \omega_I(z) + \sum_{i=1}^{n} k_{i\,m} \tilde{S}_g(z, z_i|\Omega) \,. \tag{3.139}$$

Combined with the on-shellness of the $\{k_i\}$, this indicates that $P^2$ is a meromorphic quadratic differential with only simple poles at the vertex operator insertion points:

$$\begin{aligned} P^2(z) = \sum_{I,J=1}^{g} \ell^I \cdot \ell^J \omega_I(z) \omega_J(z) + 2 \sum_{I=1}^{g} \sum_{i=1}^{n} \ell^I \cdot k_i \, \omega_I(z) \tilde{S}_g(z, z_i|\Omega) \\ + \sum_{i \neq j} k_i \cdot k_j \, \tilde{S}_g(z, z_i|\Omega) \tilde{S}_g(z, z_j|\Omega) \,. \end{aligned} \tag{3.140}$$

The vectors $\{\ell^I_m\}$ are the zero modes of $P_m$, associated with homogeneous solutions of (3.135), whereas the residue of $P^2$ at $z_i$ is easily seen to be

$$\mathrm{Res}_{z=z_i} P^2(z) = \sum_{I=1}^{g} k_i \cdot \ell^I \omega_I(z_i) + \sum_{j \neq i} k_i \cdot k_j \, \tilde{S}_g(z_i, z_j|\Omega) \,. \tag{3.141}$$

In light of (3.140), the delta functions appearing in the correlators (3.131), (5.45), (3.134) have a natural interpretation: they enforce the condition that $P^2=0$ globally on the worldsheet $\Sigma$. As noted in [5], this is the geometric content of the *scattering equations* at generic genus. Indeed the amplitude prescription ensures that there are $3g-3+n$ delta function constraints for $g\geq 2$: $n$ of them to set the residues (3.141) to zero, and $3g-3$ to kill the remaining globally-defined moduli. At $g=0,1$ this counting is modified in the obvious way in accordance with $h^0(\Sigma, K_\Sigma^2(z_1+\cdots+z_n))$.

Hence, it is clear that the amplitude prescription will give the expected scattering equations at a given genus, along with a non-compact zero-mode integral over the $\{\ell^I_m\}$. These scattering equations completely fix all the moduli integrals (over $\{\tau_a\}$ and $\{z_i\}$) in terms of the kinematics (the external and loop momenta $\{k_i,\ell^I\}$). Therefore a general amplitude takes the form:

$$\begin{aligned}\mathcal{M}_n^{(g)} &= \delta^{10}\left(\sum_{i=1}^n k_i^m\right)\int\prod_{I=1}^g \mathrm{d}^{10}\ell^I\prod_{a=1}^{3g-3}\mathrm{d}\tau_a\bar{\delta}\left(P^2(z_a)\right)\prod_{j=1}^n\bar{\delta}\left(k_j\cdot P(z_j)\right)\left\langle\mathcal{N}\widetilde{\mathcal{N}}\cdots\right\rangle \\ &:= \delta^{10}\left(\sum_{i=1}^n k_i^m\right)\int\prod_{I=1}^g\mathrm{d}^{10}\ell^I\,\mathfrak{M}_n^{(g)}, \end{aligned} \tag{3.142}$$

where the integrand $\mathfrak{M}_n^{(g)}$ represents the full correlator, localized on the support of the scattering equations with all OPEs and zero mode integrations performed, except for the loop integrals $\mathrm{d}^{10}\ell$.

The quantity $\mathfrak{M}_n^{(g)}$ is conjectured to be equal to the $g$-loop *integrand* of type II supergravity, before any loop integrals have been performed. What is meant by the 'integrand' is a sum over the complete symmetrization of all $g$-loop Feynman diagrams in the field theory without performing the loop integrations. Although type II supergravity is UV divergent in ten-dimensions, these divergences are expected to emerge only after performing the $\mathrm{d}^{10}\ell$ integrals, so the integrand $\mathfrak{M}_n^{(g)}$ itself is a well-defined object.

It is far from obvious that the worldsheet correlators will have even the most rudimentary properties of field theory amplitudes, such as being rational functions of the kinematic data, producing the correct kinematic prefactors, or factorizing correctly. However, it will be shown that in the special case of the four-point amplitudes, the correlators do indeed pass several non-trivial tests in favour of the conjecture. In particular, the correct kinematic prefactor are recovered and the IR behaviour is consistent with supergravity amplitudes. These tests are enabled by a combination of similar results for the higher-genus amplitudes of the RNS-like formalism [5, 7], as well as the similarities between this worldsheet theory and the non-minimal formalism of the pure spinor superstring, where extensive calculations have been performed explicitly.

At genus zero, there are no zero modes of $P_m$ to integrate over and the conjecture reduces to the claim that $\mathcal{M}_n^{(0)}$ gives the full tree-level S-matrix of type II supergravity. On the genus zero worldsheet, the regulator is simply

$$\mathcal{N} = \mathrm{e}^{-\lambda\cdot\bar{\lambda}-r\cdot\theta}\,, \tag{3.143}$$

since none of the conformal weight $(1,0)$-fields have any zero modes. Performing the $X$ path integral fixes $P_m$ via (3.139) to be

$$P_m(z) = \mathrm{d}z\sum_{i=1}^{n}\frac{k_{i\,m}}{z-z_i}\,, \tag{3.144}$$

so all the remaining fields in the correlator (3.134) are the *same* as left-moving variables of the superstring. After contracting all the conformal weight $(1,0)$ fields via their OPEs, the same strategy as the superstring [76] reveals that after integrating out the non-minimal variables,

$$\mathcal{M}_n^{(0)} = \int [\mathrm{d}\lambda][\mathrm{d}\tilde{\lambda}]\,\mathrm{d}^{16}\theta\,\mathrm{d}^{16}\tilde{\theta}\prod_{i=4}^{n}\bar{\delta}\left(k_i\cdot P(z_i)\right)\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}\tilde{\lambda}^{\tilde{\alpha}}\tilde{\lambda}^{\tilde{\beta}}\tilde{\lambda}^{\tilde{\gamma}}f_{\alpha\tilde{\alpha}\beta\tilde{\beta}\gamma\tilde{\gamma}}(\theta,\tilde{\theta})\,, \tag{3.145}$$

where $f$ is a function of the kinematic data, the insertion points, and takes values in $\otimes_{i=4}^{n}K^2_{\Sigma\,i}$.

It is easy to see that this is equivalent to the minimal prescription (2.45) given by Berkovits [49], and in turn proven to give the full tree-level S-matrix of supergravity [58]. So at genus zero, the non-minimal formalism reduces to the minimal formalism in exactly the same way as for superstring theory, and gives the desired classical scattering amplitudes of type II supergravity.[17]

### 3.6.4 Four-point function: Genus one

On a genus one surface the fields of conformal weight $(1,0)$ acquire zero modes. In particular the fermionic fields $s^{\alpha}$ and $\tilde{s}^{\tilde{\alpha}}$ have 11 zero modes each, which must be soaked up by operator insertions in the path-integral to give a non-vanishing result. The only operators which can provide these zero modes are the regulators $\mathcal{N}$ and $\widetilde{\mathcal{N}}$, given at

[17] In principle, one could define a higher genus prescription for the minimal model analogous to the superstring. While avoiding this for the reasons mentioned above, it is expected that an abstract equivalence between the two formalisms holds beyond tree-level, again in analogy with superstring theory [88].

genus one by [76]

$$\mathcal{N} = \exp\left(-\lambda\cdot\bar{\lambda} - r\cdot\theta - w_{\text{z.m.}}\cdot\bar{w}_{\text{z.m.}} + s_{\text{z.m.}}\cdot d_{\text{z.m.}}\right), \tag{3.146}$$

where $f_{\text{z.m.}}$ denotes the zero mode of the conformal weight $(1,0)$ field $f$. The 11 zero modes of $s^\alpha$ are thus accompanied by 11 zero modes of the $d_\alpha$ field, the latter of which has 16 unconstrained components. So there are 5 remaining zero modes of $d_\alpha$ left to be soaked up by contributions coming either from vertex operators or the $b$-ghost insertion in (5.45).

Fixed vertex operators cannot contribute $d$ zero modes, so they must come either from integrated vertex operators, which can contribute at most one $d$ zero mode each, or from the effective $b$-ghost, which can contribute at most 2 zero modes. The counting is exactly the same for the tilded variables. Using this zero mode counting, it is clear that the first non-vanishing amplitude at genus one is the four-point amplitude; $\mathcal{M}^{(1)}_{n<4} = 0$ since the fermionic zero mode integrals cannot be saturated. This vanishing is a consequence of spacetime supersymmetry, which is manifest in the pure spinor approach.

At four points there is only one way to pick terms from the vertex operators and $b$-ghost in order to saturate the $d$ zero mode path integral, just as in superstring theory [70, 72, 89]. Each of the three integrated vertex operators (2.44) contributes a zero mode from the term $d_\alpha W^\alpha$ and the $b$-ghost (3.126) contributes

$$(b|\mu) \propto \frac{(\bar{\lambda}\gamma_{mnp}r)(d_{\text{z.m.}}\gamma^{mnp}d_{\text{z.m.}})}{(\bar{\lambda}\cdot\lambda)^2}. \tag{3.147}$$

After performing the $d$ zero mode integral the expression becomes

$$\int \mathrm{d}^{16}\theta \int [\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}r]\, \frac{(\bar{\lambda}\gamma_{mnp}D)}{(\bar{\lambda}\cdot\lambda)^2}(\lambda\cdot A_1)(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)\,\mathrm{e}^{-\lambda\cdot\bar{\lambda}-r\cdot\theta}. \tag{3.148}$$

This has been shown [75, 89] to be proportional to the pure spinor superspace expression

$$K = \left\langle (\lambda\cdot A_1)(\lambda\gamma_m W_2)(\lambda\gamma_n W_3)\mathcal{F}_4^{mn}\right\rangle, \tag{3.149}$$

where these angle brackets stand for the pure spinor and theta zero mode integrations. The calculation in the tilded variables is identical. Thus, the amplitude can be written as

$$\mathcal{M}_4^{(1)} \propto K\widetilde{K}\int \mathrm{d}^{10}\ell \int \mathrm{d}\tau\,(\mathrm{d}z_0)^2\,\bar{\delta}\left(P^2(z_0)\right)\prod_{i=2}^{4}\bar{\delta}(k_i\cdot P(z_i))\,(\mathrm{d}z_i)^2, \tag{3.150}$$

omitting the overall momentum conserving delta function.

The integrand of the amplitude (3.150) is equal to the integrand given by the ambitwistor string formalism after summing over spin structures as shown in section 3.5.3. The prefactor $K\widetilde{K}$ is the correct supersymmetric prefactor for supergravity, which reduces to the $t_8 t_8 R^4$ tensor when all external states are gravitons. As expected, the integral over the moduli space of a four-punctured torus is completely localized by the scattering equations. The integrand is seen to be modular invariant by adopting the prescription of section 3.2 for the transformation of the zero modes of $P_m$.

### 3.6.5 Four-point function: Genus two

By now it should be clear that computations involving only zero mode counting in this model will be almost the same as in the usual pure spinor superstring. In particular, the computation of the genus two four-point amplitude can be carried out in much the same way as in the pure spinor superstring. In this case there are now 22 zero modes of the field $s$; these, again, can only come from the regulators and thus are accompanied by 22 zero modes of the $d$ field. At genus two the field $d$ has 32 zero modes, so 10 other zero modes must be provided by the integrated vertex operators and $b$-ghosts. Each integrated vertex operator can contribute at most one zero mode, so each $b$-ghost contributes two zero modes.

This completely fixes the contributions from each operator, which are the same as in the one-loop case. After doing the path integral over the zero modes of $d$, $s$, $w$, and $\bar{w}$, the remaining superspace expression can be written as [73, 75, 89, 90]

$$\int \mathrm{d}^{16}\theta \int [\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}r]\, \frac{(\lambda\gamma_{mnpqr}\lambda)}{(\bar{\lambda}\cdot\lambda)^3} \mathscr{F}^{mn}\mathscr{F}^{pq}\mathscr{F}^{rs}(\lambda\gamma_s W)\, \mathrm{e}^{-\lambda\cdot\bar{\lambda}-r\cdot\theta}, \tag{3.151}$$

where the various numerical factors and the distribution of particle labels on the superfields are suppressed. Upon summing over permutations of particle labels, this superspace expression vanishes unless it is dressed with holomorphic differentials arising from a combination of the moduli integrals and the $b$-ghost insertions. The result can be identified with the kinematic prefactor of supergravity [73, 74] by comparison with the computation in the RNS formalism [91, 92], or via BRST cohomology arguments [93]. The counting and calculation for the tilded variables follows identically.

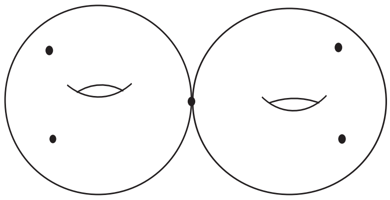

Fig. 3.5 *The genus two worldsheet at the boundary of the moduli space.*

The BRST cohomology techniques of [93] relate the two-loop kinematic prefactor to the one-loop prefactor (3.149), see also [90]. Applying this relationship gives the expression

$$\mathscr{M}_4^{(2)} \propto K\widetilde{K} \int \mathrm{d}^{10}\ell_1 \mathrm{d}^{10}\ell_2 \int \mathrm{d}^3\Omega\, \mathscr{Y}^2 \prod_{j=1}^{3} \bar{\delta}\left(P^2(x_j)\right) (\mathrm{d}x_j)^2 \prod_{i=1}^{4} \bar{\delta}\left(k_i \cdot P(z_i)\right), \tag{3.152}$$

where $\mathrm{d}^3\Omega$ stands for the integrals over the complex structure moduli of the genus two Riemann surface and $\mathscr{Y}$ is the quadri-holomorphic form [92]

$$\mathscr{Y} = (t-u)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(4,2) + (u-s)\Delta(1,4)\Delta(2,3)\,. \tag{3.153}$$

Here, $\{s,t,u\}$ are the standard Mandelstam parameters (*e.g.*, $s = 2k_1 \cdot k_2$) and

$$\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$$

for $\omega_I$ the Abelian differentials on the genus two worldsheet.

The conjecture is that the integrand of (3.152) is a representation for the two-loop integrand of type IIA/B supergravity. In particular, the massive modes that usually run through the loops of string theoretic amplitudes at genus two should be absent. There is an easy test that can be done in this amplitude to show that no massive modes are propagating by looking at the boundary of the moduli space where the genus two surface degenerates into two tori glued at a nodal point, see Figure 3.5. In the superstring the only poles at this boundary come from the propagation of massive modes through the node [94]. In terms of the field theory integrand, this boundary corresponds to a non-existent cut of a double box. Therefore if (3.152) represents a field theory amplitude, it must vanish at this separating boundary.

Using the period matrix

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$$

to parametrize the genus two surface, the separating boundary divisor of the moduli space sits at $\tau_{12} \to 0$; $\tau_{11}, \tau_{22}$ are the modular parameters of the two resulting tori. Near this boundary the external states $\{1,2\}$ move to one of the tori, call it $\Sigma_1$ with modular parameter $\tau_{11}$, while states $\{3,4\}$ move to the other torus, call it $\Sigma_2$ with modular parameter $\tau_{22}$. With this choice, the quadri-holomorphic form (3.153) becomes simply [94]

$$\mathscr{Y} \xrightarrow{\tau_{12}\to 0} -s = -2k_1 \cdot k_2\,, \tag{3.154}$$

with no pole arising from the measure factors. At this stage it is not obvious why this should be zero. The crucial fact is that the scattering equations in (3.152) enforce the momentum flowing through the node to be on-shell (*i.e.*, $s = 0$), so the amplitude vanishes.

To see this it is convenient to make use of an explicit parametrization of the moduli space near this boundary, which has been deployed often in the study of factorization in string theory (the so-called 'plumbing fixture' [25, 59, 60, 95]). On the two tori $\Sigma_1, \Sigma_2$ pick local coordinates $z_I$ around one point on each surface $p_I \in \Sigma_I$ such that $p_I = \{z_I = 0\}$. Remove an open neighbourhood around these points $U_I = \{|z_I| < |t|^{1/2}\}$ where $t$ is a coordinate on the unit disk $D = \{t \in \mathbb{C}|\,|t| < 1\}$ (not to be confused with the Mandelstam variable). Now glue them together using the annulus $A_t = \{w \in \mathbb{C}|\,|t|^{1/2} < |w| < |t|^{-1/2}\}$ via

$$w = \begin{cases} \dfrac{t^{1/2}}{z_1} & \text{if } |t|^{1/2} < |w| < 1 \\ t^{-1/2}\, z_2 & \text{if } 1 < |w| < |t|^{1/2} \end{cases}. \tag{3.155}$$

This gives a family of genus two Riemann surfaces fibered over the unit disk which can be seen as the union of three distinct components, $(\Sigma_1 \setminus U_1) \cup A_t \cup (\Sigma_2 \setminus U_2)$. The singular fiber over $t = 0$ corresponds to the boundary of interest, and one can show that $t \propto \tau_{12}$. We now distribute the scattering equations among these components. The four scattering equations of the form $k_i \cdot P(z_i)$ accompany the punctures, so the $i = 1,2$ equations go to $\Sigma_1 \setminus U_1$, while $i = 3,4$ go to $\Sigma_2 \setminus U_2$. There are also three $P^2(x)$ scattering equations, corresponding to each of the three moduli of the genus two surface. The natural choice is to place one of these equations on each component of the family of surfaces (see Figure 3.6).

The form of these equations as the boundary is approached is dictated by the field $P_m(z)$, whose behaviour under the degeneration depends on which component it is being evaluated at. Using standard degeneration formulas for the Abelian differentials and propagators, it is easy to see what happens to $P$. The Abelian differentials behave as [25]

$$\omega_I(z) = \begin{cases} \varpi_I(z) + O(t) & \text{if } z \in \Sigma_I \\ O(t) & \text{otherwise} \end{cases}, \qquad I = 1,2\,, \tag{3.156}$$

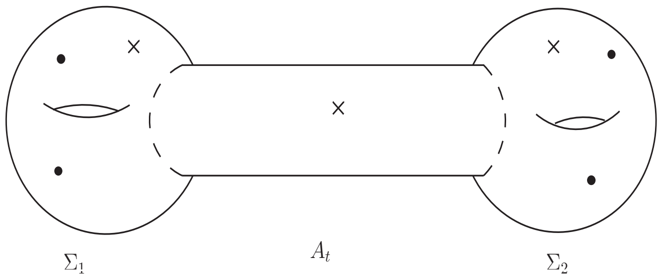


Fig. 3.6 *The degenerating worldsheet modelled on two tori* $\Sigma_1, \Sigma_2$ *connected by the annulus* $A_t$. *Solid dots denote scattering equations of the form* $k \cdot P$, *while crosses denote scattering equations of the form* $P^2$.

where $\varpi_I$ are the global holomorphic differentials on the respective tori. The behaviour of the propagator $\tilde{S}_2(z,w|\Omega)$ can be deduced from that of the prime form

$$E_2(z,w|\Omega) = \begin{cases} -E_1(z,p_1|\tau_{11})\, w\, t^{-1/4} & \text{if } z\in\Sigma_1,\ w\in A_t \\ E_1(z,p_2|\tau_{22})\, t^{-1/4} & \text{if } z\in\Sigma_2,\ w\in A_t \\ E_1(z,p_1|\tau_{11})E_1(p_2,w|\tau_{22})\, t^{-1/2} & \text{if } z\in\Sigma_1,\ w\in\Sigma_2 \end{cases} . \tag{3.157}$$

Using (3.156)–(3.157) with (3.139) it is straightforward to see that as $t\to 0$, the scattering equations on each component $\Sigma_I \setminus U_I$ go to the one-loop scattering equations with an extra puncture at $p_I$ of momentum $\pm(k_1+k_2)$. This is a consequence of

$$\begin{aligned} \lim_{t\to 0} P_m(z)|_{\Sigma_1} &= \ell_{1\,m}\,\varpi_1(z) + \sum_{i=1,2} k_{i\,m}\tilde{S}_1(z,z_i|\tau_{11}) - (k_1+k_2)_m \tilde{S}_1(z,p_1|\tau_{11})\,, \\ \lim_{t\to 0} P_m(z)|_{\Sigma_2} &= \ell_{2\,m}\,\varpi_2(z) + \sum_{i=3,4} k_{i\,m}\tilde{S}_1(z,z_i|\tau_{22}) + (k_1+k_2)_m \tilde{S}_1(z,p_2|\tau_{22})\,. \end{aligned}$$

The remaining scattering equation on the annulus enforces the momentum flowing through the node to be on-shell, since

$$P_m(w) = (k_1+k_2)_m \frac{\mathrm{d}w}{w} + O(t) \ \text{ if } w\in A_t\,, \tag{3.158}$$

where $\frac{\mathrm{d}w}{w}$ is the holomorphic differential on the annulus.

Therefore $P^2(w) \propto s + O(t) = 0$, enforcing $k_1\cdot k_2 = 0$ in the $t\to 0$ limit. Since the amplitude in this limit is multiplied by a factor of $s$ from (3.154), it vanishes on top of the scattering equations. This gives further evidence that the model describes only field theory amplitudes in type IIA/B supergravity. For more general external kinematics, it should also be possible to extract scalar integrals from (3.152) by probing the deep IR behaviour of the integrand (*i.e.*, considering multiple adjacent propagators going on-shell) and using the techniques of section 3.5. The squared Mandelstam invariants, which accompany each planar or non-planar double box in four-point two-loop supergravity amplitudes [96], are supplied by $\mathscr{Y}^2$.

Note that the factorization arguments for the scattering equations generalize in the obvious way to $n$-points and arbitrary genus. Combined with the non-separating degenerations (*i.e.*, pinching a cycle of the worldsheet non-homologous to zero) and the separating degenerations that pinch off a sphere from the worldsheet, both of which were studied in section 3.4, this encompasses *all* possible degenerations of the scattering equations near a boundary of the moduli space for any genus and any number of external states.

# Chapter 4

# Ambitwistor string on curved backgrounds

It is a highly non-trivial (if well-known) fact that General Relativity emerges as the low energy limit of closed string theory. This equivalence was first observed via the tree-level S-matrices of the two theories: the $\alpha' \to 0$ limit of a sphere amplitude in string theory gives the corresponding tree-level scattering amplitude of gravity [97–99]. The relationship can also be captured at the non-linear level by considering the worldsheet sigma model on an arbitrary curved background, composed of a metric $g$, $B$-field, and dilaton $\Phi$. Maintaining worldsheet conformal invariance requires the vanishing of the worldsheet $\beta$-functionals, which imply the target space fields obey certain equations of motion that at low energies are the Einstein equation together with equations of motion for $B$ and $\Phi$ [100–103].

The two ways of obtaining target space field equations are of course different aspects of the same thing. Perturbatively, vertex operators in the worldsheet CFT are infinitesimal deformations of the worldsheet action, and correspond to infinitesimal fluctuations of the background geometry (at least for massless states). In order for a vertex operator to be admissible, the fluctuation it describes must obey the target space field equations, linearised around the background. The linearised field equations arise from the requirement that the vertex operators have the correct anomalous conformal weight, reflecting the fact that the non-linear field equations are the condition for vanishing worldsheet Weyl anomaly.

In either approach, for a generic target space it is prohibitively difficult to write down the exact string equations of motion. Rather, one typically works perturbatively in the string length $\sqrt{\alpha'}$, which governs a derivative expansion in the target space geometry, or equivalently a loop expansion parameter in the worldsheet non-linear sigma model. Higher curvature corrections were first seen from the point of view of the $\alpha'$ expansion of amplitudes in [104], and emerge from the four-loop $\beta$-function of the superstring [105, 106]. This infinite series of higher-order corrections play an important role in guaranteeing the excellent high energy behaviour of strings.

The theory ambitwistor string describes maps into flat space-time and computes amplitudes perturbatively around flat space. It is natural to ask if there is a formulation describing maps into curved space-time. Since the theory produces pure supergravity amplitudes when linearised around flat space, the supergravity field equations — with no $\alpha'$ corrections — should be the *exact* conditions for quantum consistency of such a model.

The aim of this chapter is to provide such a description. Section 4.1 presents, at the classical level, a generalization of the ambitwistor string describing maps into a curved target space. The key is to generalize the worldsheet current algebra that in the flat space model was responsible for localization on the scattering equations. The appropriate generalization is closely related to the Hamiltonian framework of worldline supersymmetry in supersymmetric quantum mechanics (for example [107, 108]). These currents are gauged and, as in flat space, at genus zero it is possible to choose a gauge in which the gauge fields vanish so that the currents disappear from the action. The remaining action is free, opening the possibility of making exact statements about its quantum behaviour. In fact, the action of the model is a type of supersymmetric curved $\beta\gamma$-system. The quantum properties of curved $\beta\gamma$-systems have been extensively investigated [109–118], and are rather subtle. In section 4.2 the behaviour of the currents under diffeomorphisms of both the target space and worldsheet is studied. The classical curved space currents of section 4.1 receive quantum corrections in order to remain covariant under diffeomorphisms at the quantum level. Finally, in section 4.3 the algebra generated by the quantum-corrected currents is shown to be anomaly free if and only if the target space satisfies the nonlinear supergravity equations of motion, with no higher curvature corrections.

## 4.1 Curved target space: classical aspects

This section deals with the generalization of the ambitwistor string to curved target space at the classical level. Modifications due to quantum effects will be studied in the next section. Let $(M_{\mathbb{R}}, g_{\mathbb{R}})$ be a pseudo-Riemannian space-time and $(M, g)$ its complexification with holomorphic metric $g$. That is, $g : \mathrm{Sym}^2\, T_M \to \mathbb{C}$ where $T_M$ is the holomorphic tangent bundle of $M$. Temporarily ignoring the gauge fields $(\chi, \bar{\chi}, e)$, the natural generalization of the matter action for the case of a curved $(M, g)$ is

$$S_{\mathrm{cl}} = \frac{1}{2\pi}\int_\Sigma P_\mu \bar{\partial} X^\mu + \bar{\psi}_\mu \bar{D}\psi^\mu\,, \tag{4.1}$$

where the fermions $\psi$ and $\bar{\psi}$ are now understood to take values in the pullbacks $X^* T_M$ and $X^* T_M^*$, respectively, while $\bar{D}\psi^\mu = \bar{\partial}\psi^\mu + \Gamma^\mu{}_{\nu\rho}\psi^\nu\bar{\partial}X^\rho$ is the $(0,1)$-part of the pullback to $\Sigma$ of the Levi-Civita connection on $M$. Notice that, unlike the standard string, here it is not possible to include a four-fermion interaction in $S_{\mathrm{cl}}$, since all the fermions are left-moving. Here the real fermions from the previous chapters are grouped into one *complex* fermion, that is, in this chapter $\psi$ denotes a complex fermion and $\bar{\psi}$ its complex conjugate. This definition is quite natural from the curved space perspective.

The action may be further simplified by introducing the field $\Pi$ as

$$\Pi_\mu := P_\mu + \Gamma^\lambda{}_{\mu\nu}\bar{\psi}_\lambda\psi^\nu\,, \tag{4.2}$$

whereupon the matter portion of the worldsheet action becomes

$$S_{\mathrm{cl}} = \frac{1}{2\pi}\int_\Sigma \Pi_\mu \bar{\partial} X^\mu + \bar{\psi}_\mu \bar{\partial}\psi^\mu\,, \tag{4.3}$$

and does not depend on the choice of target metric $g$. The presence of the Levi-Civita connection in the definition of $\Pi$ is reflected by its non-tensorial transformation

$$\Pi_\mu \;\mapsto\; \tilde{\Pi}_\mu = \frac{\partial X^\nu}{\partial \tilde{X}^\mu}\Pi_\nu + \frac{\partial^2 X^\lambda}{\partial\tilde{X}^\mu\partial\tilde{X}^\nu}\frac{\partial\tilde{X}^\nu}{\partial X^\sigma}\bar{\psi}_\lambda\psi^\sigma \tag{4.4}$$

under the diffeomorphism $X^\mu \mapsto \tilde{X}^\mu(X)$ of $M$, so that classically (4.3) remains invariant.

The target space metric does play a role in the curved space generalization of the currents $\mathscr{G}^0$, $\bar{\mathscr{G}}^0$ and $\mathscr{H}^0$. The action (4.3) is invariant under the supersymmetry transformations

$$
\begin{aligned}
\delta X^\mu &= -\bar{\epsilon}\,\psi^\mu - \epsilon\, g^{\mu\nu}\bar{\psi}_\nu \\
\delta\psi^\mu &= \epsilon\, g^{\mu\nu}(\Pi_\nu - \Gamma^\kappa{}_{\nu\lambda}\bar{\psi}_\kappa\psi^\lambda) + \epsilon\, g^{\kappa\nu}\Gamma^\mu{}_{\nu\lambda}\bar{\psi}_\kappa\psi^\lambda \\
\delta\bar{\psi}_\mu &= \bar{\epsilon}\,\Pi_\mu - \epsilon\, g^{\kappa\nu}\Gamma^\lambda{}_{\mu\nu}\bar{\psi}_\kappa\bar{\psi}_\lambda \\
\delta\Pi_\mu &= \epsilon\, g^{\rho\sigma}\Gamma^\nu{}_{\rho\mu}\left(\bar{\psi}_\sigma\Pi_\nu + \bar{\psi}_\nu\Pi_\sigma\right) - \frac{\epsilon}{2}\bar{\psi}_\nu\bar{\psi}_\rho\psi^\sigma R^{\nu\rho}{}_{\mu\sigma}
\end{aligned}
\tag{4.5}
$$

with parameters $\epsilon, \bar{\epsilon} \in \Pi\Omega^0(\Sigma, T_\Sigma^{1/2})$, where $R^\mu{}_{\nu\kappa\lambda}$ is the Riemann curvature of the Levi-Civita connection. At the classical level, these transformations are generated by the Noether currents

$$
\begin{aligned}
\mathcal{G}^{\text{cl}} &:= \psi^\mu\left(\Pi_\mu - \Gamma^\kappa{}_{\mu\lambda}\bar{\psi}_\kappa\psi^\lambda\right) = \psi^\mu\Pi_\mu \\
\bar{\mathcal{G}}^{\text{cl}} &:= g^{\mu\nu}\bar{\psi}_\nu\left(\Pi_\mu - \Gamma^\kappa{}_{\mu\lambda}\bar{\psi}_\kappa\psi^\lambda\right)
\end{aligned}
\tag{4.6}
$$

where the equality in the first line follows by the symmetry of the Levi-Civita connection. The Poisson brackets of these curved space currents obey the same algebra

$$
\left\{\mathcal{G}^{\text{cl}}, \bar{\mathcal{G}}^{\text{cl}}\right\} = \mathcal{H}^{\text{cl}}\,, \qquad \left\{\mathcal{G}^{\text{cl}}, \mathcal{G}^{\text{cl}}\right\} = 0\,, \qquad \left\{\bar{\mathcal{G}}^{\text{cl}}, \bar{\mathcal{G}}^{\text{cl}}\right\} = 0 \tag{4.7}
$$

as in flat space, where now

$$
\mathcal{H}^{\text{cl}} := g^{\mu\nu}(\Pi_\mu - \Gamma^\kappa{}_{\mu\lambda}\bar{\psi}_\kappa\psi^\lambda)(\Pi_\nu - \Gamma^\rho{}_{\nu\sigma}\bar{\psi}_\rho\psi^\sigma) - \frac{1}{2}R^{\kappa\lambda}{}_{\mu\nu}\bar{\psi}_\kappa\bar{\psi}_\lambda\psi^\mu\psi^\nu\,. \tag{4.8}
$$

The currents (4.6) & (4.8) generalize the flat space currents $\mathcal{G}^0$, $\bar{\mathcal{G}}^0$ and $\mathcal{H}^0$. They take a similar form to the worldline supersymmetry currents and Hamiltonian in supersymmetric quantum mechanics. In particular, since $\Pi_\mu$ is canonically conjugate to $X^\mu$ while $J_\mu{}^\nu = \bar{\psi}_\mu\psi^\nu$ generates target space Lorentz transformations, after quantization $\mathcal{H}^{\text{cl}}$ is a Lichnerowicz Laplacian acting on forms on the infinite dimensional space of maps from $\Sigma$ to $M$.

If there is a $B$-field on $M$, with 3-form field strength $H = \mathrm{d}B$, then the currents are further modified to

$$
\begin{aligned}
\mathcal{G}^{\text{cl}} &= \psi^\mu\Pi_\mu + \frac{1}{3!}\psi^\mu\psi^\nu\psi^\kappa H_{\mu\nu\kappa} \\
\bar{\mathcal{G}}^{\text{cl}} &= g^{\mu\nu}\bar{\psi}_\nu\left(\Pi_\mu - \Gamma^\kappa{}_{\mu\lambda}\bar{\psi}_\kappa\psi^\lambda\right) + \frac{1}{3!}\bar{\psi}_\mu\bar{\psi}_\nu\bar{\psi}_\kappa H^{\mu\nu\kappa} \\
\mathcal{H}^{\text{cl}} &= g^{\mu\nu}\left(\Pi_\mu - \Gamma^\kappa{}_{\mu\lambda}\bar{\psi}_\kappa\psi^\lambda + \frac{1}{2}H_{\mu\kappa\lambda}\psi^\kappa\psi^\lambda\right)\left(\Pi_\nu - \Gamma^\rho{}_{\nu\sigma}\bar{\psi}_\rho\psi^\sigma + \frac{1}{2}H_{\nu\rho\sigma}\bar{\psi}^\rho\bar{\psi}^\sigma\right) \\
&\quad - \frac{1}{2}R^{\kappa\lambda}{}_{\mu\nu}\bar{\psi}_\kappa\bar{\psi}_\lambda\psi^\mu\psi^\nu - \frac{1}{3!}\psi^\mu\bar{\psi}_\nu\bar{\psi}_\kappa\bar{\psi}_\lambda\nabla_\mu H^{\nu\kappa\lambda} - \frac{1}{3!}\bar{\psi}_\mu\psi^\nu\psi^\kappa\psi^\lambda\nabla^\mu H_{\nu\kappa\lambda}\,,
\end{aligned}
\tag{4.9}
$$

without changing the action. The Poisson brackets of these currents still obey (4.7). Note that the $B$-field here does *not* appear simply as the torsion of the connection, but rather breaks the $\mathbb{C}^*$-symmetry of the fermion system to $\mathbb{Z}_2$. As in the classical string, including a target space dilaton is best done in the context of the quantum theory.

At the classical level, the transformations generated by these currents, with local parameters $\{\bar{\epsilon}, \epsilon, \alpha\}$ respectively, are gauge symmetries of the action

$$S = \frac{1}{2\pi}\int_\Sigma \Pi_\mu \bar{\partial} X^\mu + \bar{\psi}_\mu \bar{\partial}\psi^\mu + \bar{\chi}\mathscr{G}^{\text{cl}} + \chi\bar{\mathscr{G}}^{\text{cl}} + \frac{e}{2}\mathscr{H}^{\text{cl}} \tag{4.10}$$

provided the gauge fields transform as $\delta\bar{\chi} = -\bar{\partial}\bar{\epsilon}$, $\delta\chi = -\bar{\partial}\epsilon$ and $\delta e = -\bar{\partial}\alpha$. As in the flat space model, at genus zero, in the absence of vertex operators it is possible to choose the parameters so that the gauge fields vanish and the currents disappear from the action.

## 4.2 Quantum corrections

The generalization of the flat space model to a curved target involves only changing the currents, after a field redefinition the kinetic terms in the action are the same as in flat space. In the gauge where $e$, $\chi$ and $\bar{\chi}$ vanish the worldsheet action is free and the theory knows about the target space fields $(g, B, \Phi)$ only through the BRST operator. The resulting action is an example of a *curved $\beta\gamma$-system*.

In this section the properties of this theory at the quantum level are studied. The quantum behaviour of curved $\beta\gamma$-systems is known to be subtle [109–113], though the supersymmetric case is much more straightforward than the purely bosonic one [114–118]. The first piece of good news is that since the action is free, correlation functions may be computed using the free OPEs

$$X^\mu(z)\,\Pi_\nu(w) \sim \frac{\delta^\mu{}_\nu}{z-w}, \qquad \psi^\mu(z)\,\bar{\psi}_\nu(w) \sim \frac{\delta^\mu{}_\nu}{z-w}. \tag{4.11}$$

This is one of the main advantages of curved $\beta\gamma$-systems in general. It also corresponds to what is expected from the curved space version of a worldsheet theory describing pure supergravity with no higher curvature corrections. Since higher order corrections come from the loop expansion on the worldsheet, a theory that only gives the supergravity equations of motion should at least be solvable. Here the situation is even better since the gauge-fixed action turns out to be locally free. Therefore calculations in this theory can be carried out using only free OPEs. In the next two sections these are used to examine the transformation properties of the currents (4.9) under diffeomorphisms of both the target and worldsheet. In order to be covariant at the quantum level these currents will receive corrections quantum corrections.

### 4.2.1 Target space diffeomorphisms

Infinitesimally, target space diffeomorphisms are generated by the Lie derivative $\mathscr{L}_V$ along some vector field $V$. In the quantum theory this is realized by an operator $\mathscr{O}_V$, whose OPE with the currents should generates diffeomorphisms. In order for $\mathscr{O}_V$ to represent the diffeomorphism algebra, given two vectors $V$ and $W$, the operators $\mathscr{O}_V$ and $\mathscr{O}_W$ must have the OPE

$$\mathscr{O}_V(z)\,\mathscr{O}_W(w) \sim \frac{\mathscr{O}_{[V,W]}(w)}{z-w}, \tag{4.12}$$

where $[V, W]$ is the Lie bracket of the two vector fields. A naive guess is

$$\mathcal{O}_V^{\text{naive}}(z) := -:V^\mu(X)\Pi_\mu: \equiv \lim_{\epsilon\to 0}\left(V^\mu(X(z+\epsilon))\,\Pi_\mu(z) - \frac{1}{\epsilon}\partial_\mu V^\mu(z)\right), \tag{4.13}$$

but this fails for two reasons. Firstly, the OPE $\mathcal{O}_V^{\text{naive}}(z)\,\mathcal{O}_W^{\text{naive}}(w)$ does not agree with (4.12) because of double contractions. This is a common feature of curved $\beta\gamma$-systems [109–113] whose resolution usually requires replacing the Lie bracket on $T_M$ by the Courant bracket on $T_M \oplus T_M^*$. In the supersymmetric context a further problem with $\mathcal{O}_V^{\text{naive}}$ is that it does not act on the fermions, whereas these transform non-trivially under Diff($M$) since they take values in the pullbacks of the target space tangent and cotangent bundles.

Remarkably, these two problems cure one another. The operator

$$\mathcal{O}_V := -\left(:V^\mu\Pi_\mu: + \partial_\nu V^\mu :\bar\psi_\mu\psi^\nu:\right) \tag{4.14}$$

both obeys the desired OPE (4.12) and generates the correct Diff($M$) transformations of all fields. That is, its OPE with the fields are

$$\begin{aligned}
\mathcal{O}_V(z)\,X^\mu(w) &\sim \frac{V^\mu(w)}{z-w}\,,\\
\mathcal{O}_V(z)\,\psi^\mu(w) &\sim \frac{\partial_\nu V^\mu\,\psi^\nu(w)}{z-w}\,,\\
\mathcal{O}_V(z)\,\bar\psi_\mu(w) &\sim \frac{-\partial_\mu V^\nu\,\bar\psi_\nu(w)}{z-w}\\
\mathcal{O}_V(z)\,\Pi_\mu(w) &\sim -\frac{1}{z-w}\left(:\partial_\mu V^\nu\,\Pi_\nu: + \partial_\mu\partial_\nu V^\kappa :\bar\psi_\kappa\psi^\lambda:\right)(w)\,,
\end{aligned} \tag{4.15}$$

where the second term in the transformation of $\Pi$ is the expected non-tensorial behaviour of the Levi-Civita connection. The fact that supersymmetric curved $\beta\gamma$-systems behave more straightforwardly under Diff($M$) than their bosonic counterparts has been noted before, see *e.g.* [115–118].

Although the choice (4.14) ensures that the fundamental fields $\{X, \Pi, \psi, \bar\psi\}$ transform as expected under target space diffeomorphisms, this does not guarantee that the same is true of composite operators because of the potential for double (or higher) contractions between $\mathcal{O}_V$ and the composite operator. In particular, while at the classical level the currents $\mathcal{G}^{\text{cl}}$, $\bar{\mathcal{G}}^{\text{cl}}$ and $\mathcal{H}^{\text{cl}}$ introduced in (4.9) transform geometrically under Diff($M$), this is not true in the quantum theory. For example, the OPE of $\mathcal{G}^{\text{cl}}$ with $\mathcal{O}_V$ contains a non-vanishing first-order pole

$$\mathcal{O}_V(z)\,\mathcal{G}^{\text{cl}}(w) \sim \cdots + \frac{\partial\left(\partial_\mu\partial_\nu V^\mu\,\psi^\nu\right)}{z-w} + \cdots\,, \tag{4.16}$$

which does not combine with other terms to form any sort of Lie derivative along $V$. As with all quantum anomalies, the origin of this term is a double contraction between $\mathscr{G}^{\text{cl}}$ and $\mathscr{O}_V$.

To correct this anomalous behaviour, the currents (4.9) must be modified in the quantum theory. The required modification is to add new terms that involve (holomorphic) worldsheet derivatives. Such terms generate both new contributions to the higher-order pole terms in the OPE with $\mathscr{O}_V$, and also modify the coefficients of the simple poles by terms involving worldsheet derivatives. After some experimentation, one finds that the modifications should be

$$\begin{aligned}\mathscr{G} &= :\mathscr{G}^{\text{cl}}: + \partial\left(\psi^\mu \Gamma^\kappa{}_{\mu\kappa}\right)\\ \bar{\mathscr{G}} &= :\bar{\mathscr{G}}^{\text{cl}}: + g^{\mu\nu}\partial\left(\bar{\psi}_\kappa \Gamma^\kappa{}_{\mu\nu}\right).\end{aligned} \tag{4.17}$$

These quantum currents do indeed behave appropriately under target space diffeomorphisms, having the OPEs

$$\mathscr{O}_V(z)\,\mathscr{G}(w) \sim \cdots + \frac{\mathscr{L}_V\mathscr{G}}{z-w}, \qquad \mathscr{O}_V(z)\,\bar{\mathscr{G}}(w) \sim \cdots + \frac{\mathscr{L}_V\bar{\mathscr{G}}}{z-w}, \tag{4.18}$$

and so are covariant under target space diffeomorphisms at the quantum level.

In order to include a dilaton it is convenient to rewrite these currents as

$$\begin{aligned}\mathscr{G} &= :\mathscr{G}^{\text{cl}}: + \partial\left(\mathscr{L}_{\psi^\mu\partial_\mu}\log\Omega\right),\\ \bar{\mathscr{G}} &= :\bar{\mathscr{G}}^{\text{cl}}: + \partial\left(\mathscr{L}_{g^{\mu\nu}\bar{\psi}_\mu\partial_\nu}\log\Omega\right) + \bar{\psi}_\mu\Gamma^\mu{}_{\nu\rho}\,\partial g^{\nu\rho},\end{aligned} \tag{4.19}$$

where $\Omega = X^*(\sqrt{g}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d)$ is the pullback to $\Sigma$ of a top holomorphic form on the (complex) target space $M$.[1] To incorporate a dilaton field $\Phi$ on $M$, simply choose $\Omega$ to be the pullback of $\mathrm{e}^{-2\Phi}\sqrt{g}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d$ instead.

### 4.2.2 Worldsheet diffeomorphisms

While the quantum corrections ensure that the currents (4.17) transform covariantly under Diff($M$) transformations, they also affect their behaviour under *worldsheet* diffeomorphisms. This can be seen by considering their OPEs with the worldsheet stress tensor

$$T^{\text{cl}} := -:\Pi_\mu\partial X^\mu: - \frac{1}{2}\left(:\bar{\psi}_\mu\partial\psi^\mu: + :\psi^\mu\partial\bar{\psi}_\mu:\right) \tag{4.20}$$

[1]Thus, for any vector field $V$, $\mathscr{L}_V\log\Omega = \Omega^{-1}\mathscr{L}_V\Omega = \nabla_\mu V^\mu$, where $\nabla_\mu$ is the Levi-Civita covariant derivative. The existence of $\Omega$ is not restrictive on an affine complex space, but may be expected to lead to interesting constraints on possible compactifications.

that follows from the free action (4.3). For example, there is now a triple pole in the OPE between the stress tensor and $\mathcal{G}$

$$T^{\mathrm{cl}}(z)\,\mathcal{G}(w) \sim -\frac{1}{2}\frac{\mathcal{L}_{\psi^\mu\partial_\mu}\log\Omega}{(z-w)^3} + \cdots\,,$$

showing that $\mathcal{G}$ is no longer primary. The resolution is to modify the stress tensor by a total derivative term; that is, the actual stress tensor of the quantum theory is

$$T := T^{\mathrm{cl}} - \frac{1}{2}\partial^2 \log\left(\mathrm{e}^{-2\Phi}\sqrt{g}\right)\,. \tag{4.21}$$

It is straightforward to check that using this stress tensor, the currents $\mathcal{G}$ and $\bar{\mathcal{G}}$ of (4.17) are primary operators, transforming as sections of $K_\Sigma^{3/2}$ under worldsheet diffeomorphisms.

Note that unlike in string theory, this modification does not affect the condition for worldsheet conformal invariance, because here the $X(z)\,X(w)$ OPE is trivial, so there are no new contributions to the fourth order pole in $T(z)\,T(w)$. Thus, despite the presence of a non-trivial metric, $B$-field and dilaton on the target, the only restriction on the model to emerges from the $T(z)\,T(w)$ OPE (including ghosts) is the critical dimension $\dim_{\mathbb{C}}(M) = 10$, as in flat space. In particular, unlike in usual string theory [103, 119], the target space field equations do not appear in $T(z)\,T(w)$, and so are not related to worldsheet $\beta$-functions. This is as expected from the flat space theory [5, 32] reviewed in chapter 2: the requirement that the vertex operators had to obey linearised field equations came not from any anomalous conformal weight, but rather from their potentially anomalous behaviour under transformations generated by the gauged currents.

The choice (4.21) of stress tensor implies that the worldsheet action should likewise be modified to

$$S \to S + \frac{1}{8\pi}\int_\Sigma R_\Sigma \log\left(\mathrm{e}^{-2\Phi}\sqrt{g}\right)\,, \tag{4.22}$$

where $R_\Sigma$ is the worldsheet curvature. In two dimensions $R_\Sigma$ can always be chosen to vanish locally, so the addition of this term does not affect the short distance OPE, and the above calculations are self-consistent. Actually, the dilaton coupling (4.22) is well-known in first-order formulations of string theory [112, 120], in particular the fact that the dilaton is effectively shifted $\Phi \to \Phi - \frac{1}{2}\log\sqrt{g}$ compared to the usual dilaton coupling in string theory. This shifted coupling also plays an important role in T-duality, see *e.g.* [121], and analogous shifts also appear when studying $\alpha'$-corrections to string theory using doubled geometry see*e.g.* [122].

## 4.3 Supergravity equations of motion as an anomaly

In the previous section currents (4.17) that behave correctly under both target space and worldsheet diffeomorphisms at the quantum level were constructed. Contrary to usual string theory, the requirement of quantum worldsheet conformal invariance places no restrictions on the target space fields.

Instead, the target space field equations come from quantum consistency of the current algebra. At the quantum level, the Poisson bracket relations

$$\left\{\mathcal{G}^{\mathrm{cl}},\bar{\mathcal{G}}^{\mathrm{cl}}\right\}=\mathcal{H}^{\mathrm{cl}}\,,\qquad\left\{\mathcal{G}^{\mathrm{cl}},\mathcal{G}^{\mathrm{cl}}\right\}=0\,,\qquad\left\{\bar{\mathcal{G}}^{\mathrm{cl}},\bar{\mathcal{G}}^{\mathrm{cl}}\right\}=0 \tag{4.23}$$

between the classical currents should be replaced by OPEs of the quantum currents (4.17), so that the $\mathcal{G}(z)\,\mathcal{G}(w)$ and $\bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w)$ OPEs are non-singular, while the $\mathcal{G}(z)\,\bar{\mathcal{G}}(w)$ OPE has only a simple pole. Only if this is true, so that the algebra of currents is non-anomalous, will the BRST operator (5.31) obey $Q^2=0$. It is a remarkable fact that because the worldsheet action is (locally) free, these OPEs can be computed *exactly*, and so the exact quantum consistency conditions can be obtained. This is quite distinct from the usual case in string theory, where for generic backgrounds, one is faced with an interacting worldsheet CFT and so must work perturbatively around some fixed background, treating $\alpha'$ as a loop expansion on the worldsheet, or derivative expansion in the target. The ambitwistor string ha no $\alpha'$ parameter.

The simplest OPE is $\mathcal{G}(z)\,\mathcal{G}(w)$. Performing all possible contractions and expanding the coefficients of higher order poles around the mid-point gives

$$\mathcal{G}(z)\,\mathcal{G}(w)\sim-\frac{1}{3}\frac{\psi^\kappa\psi^\lambda\psi^\mu\psi^\nu}{z-w}\,\partial_\kappa H_{\lambda\mu\nu}-\frac{\partial\left(\psi^\mu\psi^\nu\,\partial_\mu\Gamma^\kappa{}_{\nu\kappa}\right)}{z-w}+2\,\frac{\partial\left(\psi^\mu\psi^\nu\partial_\mu\partial_\nu\Phi\right)}{z-w}\,. \tag{4.24}$$

The second and third terms in this expression vanish by the antisymmetry of fermions contracted into partial derivatives. (Recall that $\partial_\mu\Gamma^\kappa{}_{\nu\kappa}=\partial_\mu\partial_\nu\log\sqrt{g}$.) Hence the only non-trivial anomaly cancellation condition in (4.24) is given by the first term. This is simply the requirement that the 3-form $H$ is closed so that $H=\mathrm{d}B$ at least locally on $M$. Thus $H$ is indeed the field strength of a $B$-field.

Next is the $\bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w)$ OPE. Again, performing all possible contractions and expanding around the mid-point gives

$$\begin{aligned}\bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w)\;\sim\;&\frac{1}{2}\frac{:\!\bar\psi_\kappa\bar\psi_\lambda\bar\psi_\mu\psi^\nu\!:}{z-w}\,R_\nu{}^{\kappa\lambda\mu}+\frac{\partial\left(\bar\psi_\mu\bar\psi_\nu\,R^{\mu\nu}\right)}{z-w}-\frac{1}{3}\frac{\bar\psi_\kappa\bar\psi_\lambda\bar\psi_\mu\bar\psi_\nu}{z-w}\,\partial^\kappa H^{\lambda\mu\nu}\\&+2\,\frac{\bar\psi_\mu\bar\psi_\nu\,\partial X^\kappa}{z-w}\left[\Gamma^\nu{}_{\rho\sigma}\,R^{\sigma\rho\mu}{}_\kappa+\Gamma^\rho{}_{\kappa\sigma}(R^{\mu\sigma\nu}{}_\rho+R^{\nu\sigma\mu}{}_\rho)\right]\,.\end{aligned} \tag{4.25}$$

These anomalies vanish provided that again $\mathrm{d}H = 0$ and the Riemann and Ricci tensors obey the identities

$$R_\nu{}^{[\kappa\lambda\mu]} = 0\,, \quad R^{[\mu\nu]} = 0 \quad \text{and} \quad R^{(\mu\nu)}{}_{\rho\sigma} = 0\,. \tag{4.26}$$

These are of course the first Bianchi identity and basic symmetries of the Riemann and Ricci tensors that hold provided the connection $\Gamma$ is indeed Levi-Civita. So neither of these two OPEs impose any dynamical restrictions on the target space fields.

The only remaining OPE to be checked is that of $\mathscr{G}(z)$ and $\bar{\mathscr{G}}(w)$. This OPE has first, second and third order poles. The coefficient of the first order pole defines the quantum corrected current $\mathscr{H}$, but the coefficients of the higher order poles must be made to vanish. Proceeding as above, a straightforward, if somewhat lengthy, calculation yields

$$\begin{aligned}\mathscr{G}(z)\,\bar{\mathscr{G}}(w) \;\sim\; & \frac{2}{(z-w)^3}\left(R + 4\nabla_\mu\nabla^\mu\Phi - 4\nabla_\mu\Phi\,\nabla^\mu\Phi - \frac{1}{12}H^2\right) \\ & + 2\,\frac{(\Gamma^\mu{}_{\kappa\nu}\partial X^\kappa + \psi^\mu\bar{\psi}_\nu)}{(z-w)^2}\,g^{\nu\lambda}\left(R_{\mu\lambda} + 2\nabla_\mu\nabla_\lambda\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\lambda{}^{\rho\sigma}\right) \\ & + \frac{(\psi^\mu\psi^\nu - \bar{\psi}^\mu\bar{\psi}^\nu)}{(z-w)^2}\left(\nabla_\kappa H^\kappa{}_{\mu\nu} - 2H^\kappa{}_{\mu\nu}\nabla_\kappa\Phi\right) + \frac{\mathscr{H}}{z-w}\,.\end{aligned} \tag{4.27}$$

The quantum corrected current $\mathscr{H}$ takes the somewhat unenlightening form

$$\begin{aligned}\mathscr{H} = \mathscr{H}^{\mathrm{cl}} &+ \partial\left(\mathscr{L}_{g^{\mu\nu}\Pi_\mu\partial_\nu}\log\Omega\right) - \frac{1}{2}\partial^2(g^{\mu\nu})\,\partial_\mu\partial_\nu\log\left(\sqrt{g}\mathrm{e}^{-2\Phi}\right) - \bar{\psi}_\kappa\partial\psi^\lambda\,g^{\mu\nu}\partial_\lambda\Gamma^\kappa{}_{\mu\nu} \\ -\frac{1}{4}\partial(g^{\mu\nu})\,\partial\left[\partial_\mu\partial_\nu\log\left(\sqrt{g}\mathrm{e}^{-2\Phi}\right)\right] &+ \frac{1}{2}H^{\mu\nu\kappa}\bar{\psi}_\kappa\,\partial\left(H_{\mu\nu\lambda}\psi^\lambda\right) + \partial\left(H_{\kappa\lambda\nu}\psi^\nu\right)\,g^{\kappa\sigma}\Gamma^\lambda{}_{\sigma\rho}\psi^\rho \\ -\frac{1}{2}\partial_\sigma H_{\mu\nu\rho}\psi^\nu\psi^\rho\,\partial(g^{\sigma\mu}) &- \frac{1}{12}H^{\mu\nu\rho}\partial^2 H_{\mu\nu\rho} + \frac{1}{2}\partial(g^{\mu\nu})\Gamma^\rho{}_{\mu\nu}\left(2\Pi_\rho + H_{\sigma\lambda\rho}\psi^\sigma\psi^\lambda\right) \\ -\partial\Big[\partial(g^{\mu\nu})&\left(\partial_\sigma\Phi\Gamma^\sigma{}_{\mu\nu} + \frac{1}{2}\Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\sigma\rho} - \frac{1}{2}\partial_\sigma\Gamma^\sigma{}_{\mu\nu}\right) + g^{\mu\nu}\Gamma^\rho{}_{\mu\sigma}\partial(\Gamma^\sigma{}_{\nu\rho})\Big] \\ &-\partial\left[\bar{\psi}_\kappa\psi^\lambda\left(\nabla^\kappa\nabla_\lambda\Phi - 2g^{\mu\nu}\Gamma^\kappa{}_{\mu\lambda}\partial_\nu\Phi\right)\right]\,.\end{aligned} \tag{4.28}$$

Equation (4.27) shows that the algebra of currents is anomaly free if and only if the space-time fields $(g, B, \Phi)$ obey the equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4}H_{\mu\kappa\lambda}H_\nu{}^{\kappa\lambda} + 2\nabla_\mu\nabla_\nu\Phi &= 0\,, \\ \nabla_\kappa H^\kappa{}_{\mu\nu} - 2H^\kappa{}_{\mu\nu}\nabla_\kappa\Phi &= 0\,, \\ R + 4\nabla_\mu\nabla^\mu\Phi - 4\nabla_\mu\Phi\,\nabla^\mu\Phi - \frac{1}{12}H^2 &= 0\,. \end{aligned} \tag{4.29}$$

These are precisely the field equations of general relativity with a $B$-field and dilaton. Hence, the *exact* condition for the worldsheet theory to be consistent at the quantum level is that the target space $(M; g, B, \Phi)$ obeys the non-linear $d = 10$ supergravity field equations, in the Neveu-Schwarz sector.

The BRST operator constrains physical field configurations to obey $\mathcal{H} = 0$, which in flat space is the condition $\eta^{\mu\nu}\Pi_\mu\Pi_\nu = 0$ at every point of the worldsheet. Recall that this the content of the scattering equations. The $\mathcal{G}(z)\bar{\mathcal{G}}(w)$ OPE has $\mathcal{H}$ as its classical contribution, while the field equations (4.29) appear as the coefficients of higher poles. In this sense, the Einstein equations emerge as quantum corrections to the curved space generalization of the scattering equations.

A couple of remarks before moving to the next chapter. Firstly, the curved space worldsheet theory encodes the vertex operators for perturbations of the metric, $B$-field, and dilaton around flat space. In the non-linear sigma model of string theory, the flat space vertex operators are found by considering linearised perturbations of the action; here the vertex operators arise by perturbing the *currents*. For example, expanding the metric in $\mathcal{H}$ to linear order around the Minkowski metric one finds

$$\mathcal{H} - \mathcal{H}^0 = \delta g^{\mu\nu}\,\Pi_\mu\Pi_\nu - 2\eta^{\mu\nu}\,\Pi_\mu\delta\Gamma^\kappa_{\nu\lambda}\,\bar{\psi}_\kappa\psi^\lambda - \delta_\mu(\delta\Gamma^\kappa_{\nu\lambda})\,\bar{\psi}_\kappa\bar{\psi}^\lambda\psi^\mu\psi^\nu, \tag{4.30}$$

up to terms which vanish on the support of the flat space scattering equations $\mathcal{H}^0 = \eta^{\mu\nu}\Pi_\mu\Pi_\nu = 0$. This quadratic differential is essentially the vertex operator describing fluctuations $\delta g$ around flat space. When the fluctuations are plane waves with target space momentum $k_\mu$, the remaining factor of the integrated vertex operator is $\bar{\delta}(k\cdot\Pi) \in H^{0,1}(\Sigma, T_\Sigma)$, which is best interpreted as a modulus of the gauge field $e$ on the marked worldsheet. The integrated vertex operators describing fluctuations $\delta B$ or $\delta\Phi$ around flat space are obtained similarly. Expanding the currents $\mathcal{G}$ and $\bar{\mathcal{G}}$ around flat space (and re-expressing them in terms of real fermions) likewise gives the vertex operators in different pictures. See [32] for details.

Secondly, note that the dilaton equation of motion enters in the $\mathcal{G}(z)\bar{\mathcal{G}}(w)$ OPE (4.27) at order $(z-w)^{-3}$, whereas the Einstein and $B$-field equations enter at order $(z-w)^{-2}$. This is analogous to the way the dilaton equation of motion appears at higher loop order in the worldsheet $\beta$-functionals in usual string theory. Of course, the dilaton equation of motion is implied by the Einstein and $B$-field equations, so that the triple pole in $\mathcal{G}(z)\bar{\mathcal{G}}(w)$ is guaranteed to vanish if the double poles do. In this sense, the exact target space field equations indeed arise from a 1-loop anomaly of the currents.

# Chapter 5

# CFTs, soft theorems and the geometry of $\mathscr{I}$

It is a fair question to ask if the methods of the past chapters are useful to understand more than just scattering amplitudes. In the previous chapter I showed that indeed, the ambitwistor model knows non-linear information about the target space and can, in principle, be used to calculate interesting quantities in curved space. In this chapter I'll show another application of similar methods to the ambitwistor string. Here I'll be interested in studying the properties of asymptotically flat spacetimes viewed from the perspective of its boundary $\mathscr{I}$. The aim will be to understand the interplay between asymptotic symmetries of $\mathscr{I}$ and bulk observables, in particular the S-matrix. Worldsheet methods similar to the ambitwistor string, and including it, have been very successful in giving alternative formulations of S-matrices that don't rely on Lagrangians or even ordinary spacetime. Therefore one might wonder if they can provide insight into the structure of a putative holographic dual to gravity in asymptotically flat space-time, at least in some limiting regime.

This chapter starts with a brief introduction and literature review relevant for this work. Next, the gravitational model is introduced and it is shown that the Ward identities for a certain generator of the symmetry group of $\mathscr{I}$ are equivalent to Weinberg's soft theorem. In the same section it is also shown that the Ward identities for an extension of the symmetry group of $\mathscr{I}$ are equivalent to the recently introduced gravitational subleading soft factor.

## 5.1 Introduction

The conformal boundary of a four dimensional asymptotically flat space-time is a null hypersurface $\mathscr{I}$, whose past and future components $\mathscr{I}^{\pm}$ are topologically $\mathbb{R}\times S^2$ [123, 124]. The symmetry group of each of $\mathscr{I}^{\pm}$ is a copy of the infinite dimensional BMS group [125, 126]. The BMS group is the asymptotic symmetry group of the bulk space-time and, as in AdS/CFT, it is expected to play an important role in any candidate holographic description of gravity. Indeed, this perspective was taken well before the advent of AdS/CFT, for instance in Ashtekar's asymptotic quantization programme [127, 128] which encodes bulk gravitational degrees of freedom in terms of geometric data defined intrinsically on $\mathscr{I}$.

Much subsequent research has followed this general line of thought (often with the language of holography), seeking to determine the symmetry properties required for a boundary theory in asymptotically flat space-time (*e.g.* [129–133]). Most recently, Strominger [134] has shown that in space-times where space-like infinity is sufficiently well-behaved [135, 136], one can identify a diagonal action of the BMS groups on $\mathscr{I}^{+}$ and $\mathscr{I}^{-}$; this diagonal action is a symmetry of the gravitational S-matrix. In particular, the Ward identity associated with certain carefully chosen BMS generators is equivalent [137] to Weinberg's soft graviton theorem [138] in the bulk. It has further been suggested that the subleading behaviour of soft gravitons [139–141] are be due to a Ward identity for an extension of the BMS group proposed by Barnich and Troessaert [142, 143].

Yang-Mills theory has also been studied from the perspective of asymptotic symmetries of $\mathscr{I}$ [144, 145]. The Ward identity for these extended gauge transformations is associated with a Kac-Moody symmetry on the sphere of null generators of $\mathscr{I}$, and has been shown to encode the soft gluon theorem for both Abelian and non-Abelian gauge groups [146–148]. There is also a subleading soft gluon factor [149], first found in the Abelian setting of QED [150, 151], which can be obtained an asymptotic symmetry perspective [152].

This kinematic work is important because of its universality: the soft graviton theorem holds irrespective of the matter content of the theory, and receives no quantum corrections to all orders in perturbation theory. The subleading behaviour of soft gravitons does receive corrections but only at one loop, the soft gluon theorem does receive loop corrections [153–156]. Both soft behaviour of soft gravitons and soft gluons are tightly constrained and can be obtained, with minimal assumptions, from Poincaré and gauge invariance. They are universal features of the gravitational and gauge theory S-matrix [157–160].

In this chapter I'll go beyond these purely kinematic considerations. The most obvious, diffeomorphism invariant observable in an asymptotically flat space-time is the S-matrix. Indeed, the S-matrix is almost tautologically holographic, being defined in terms of how states look in the distant past and future. For massless particles, the relevant asymptotic region is $\mathscr{I}$, and one might hope that correlation functions in a boundary theory on $\mathscr{I}$ can compute scattering amplitudes in the bulk. In fact, the usual on-shell momentum eigenstates considered in scattering amplitudes are extremely closely related to local insertions on $\mathscr{I}$. The precise form of the scattering amplitudes of course depends on the details of the quantum gravity in the bulk, but it seems reasonable to expect that there should exist a regime where classical (super)gravity is a good bulk description.

Traditionally, amplitudes have been computed using Feynman diagrams to evolve fields through the bulk, or else by considering a string theory whose worldsheet is mapped to a minimal surface in the bulk space-time. In recent years however, powerful techniques have been developed that compute amplitudes purely using on-shell quantities: notions such as a space-time Lagrangian or off-shell propagator do not arise. Furthermore, in these methods the building blocks from which amplitudes are constructed do not have a straightforward bulk space-time interpretation.

The aim of this chapter is to construct a worldsheet models that lives entirely on (complexified) $\mathscr{I}$, and whose states encode the asymptotic radiative modes of gravity and gauge theory in the bulk. These models have an action of the relevant asymptotic symmetry groups, either BMS or 'large' gauge transformations. The Ward identities for the charges that generate these symmetries recover the soft graviton and soft gluon theorems. The theories also accommodate charges for the proposed extensions of the asymptotic symmetry algebras [142], and when acting on correlation functions, these produce the sub-leading gravitational and gauge soft factors found in [140, 149]. Lastly, the simplest correlation functions of these theories reproduce the tree-level S-matrix of $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ sYM.

Of course these theories are not a full realization of a boundary theories dual to gravity in asymptotically flat space. Rather, they may provide a perturbative description of such a theory in a regime where classical supergravity is valid in the bulk. Nevertheless, it still provides a dynamical realization of a theory defined entirely on $\mathscr{I}$ which produces bulk observables and carries a natural action of the BMS group.

## 5.2 The geometry of $\mathscr{I}$

In four dimensions, the conformal boundary $\mathscr{I}$ of an asymptotically flat space-time is a null hypersurface in the conformally re-scaled metric, composed of two disjoint factors $\mathscr{I} = \mathscr{I}^- \cup \mathscr{I}^+$. Each of $\mathscr{I}^\pm$ has the topology of a light cone $\mathscr{I}^\pm \cong \mathbb{R} \times S^2$ [123, 124]. Null infinity is the natural holographic screen on which the S-matrix of massless states may be defined. In Lorentzian signature, the scattering process evolves initial data on $\mathscr{I}^-$ to final data on $\mathscr{I}^+$.

From now the space being considered won't be $\mathscr{I}$ itself, but rather its *complexification* $\mathscr{I}_{\mathbb{C}}$ [161]. There are many physically interesting situations in classical relativity where one *must* complexify $\mathscr{I}$ in order to obtain non-trivial information (*e.g.* [162–164]), reality conditions being imposed only subsequently. In the context of the S-matrix, crossing symmetry implies that amplitudes extend analytically to $\mathscr{I}_{\mathbb{C}}$. More generally, the possibility of working on $\mathscr{I}_{\mathbb{C}}$ without reference to a future or past boundary should be closely tied to the 'Christodoulou-Klainerman' property of real space-times [135, 136]. This, in particular, allows one to make an identification between the generators of $\mathscr{I}^-$ and $\mathscr{I}^+$, thereby selecting a single copy of the BMS group (or gauge group) to act on all asymptotic data [134].

Complexified null infinity $\mathscr{I}_{\mathbb{C}}$ is a complex three-manifold which can be charted with coordinates $(u,\zeta,\tilde{\zeta})$, where $u$ is a complex coordinate along the null generators of $\mathscr{I}_{\mathbb{C}}$ and $(\zeta,\tilde{\zeta})$ are complex stereographic coordinates related to the usual $(\theta,\phi)$ by $\zeta = \mathrm{e}^{\mathrm{i}\phi}\cot(\theta/2)$ and $\tilde{\zeta} = \mathrm{e}^{-\mathrm{i}\phi}\cot(\theta/2)$. (Note that $(\zeta,\tilde{\zeta})$ are not necessarily complex conjugates if $(\theta,\phi)$ are not assumed real.) Equivalently, the complexified space of generators can be viewed as the product $\mathbb{CP}^1\times\mathbb{CP}^1$ of two Riemann spheres, described by homogeneous coordinates $\lambda_\alpha = (\lambda_0,\lambda_1)$ and $\tilde{\lambda}_{\dot\alpha} = (\tilde{\lambda}_{\dot 0},\tilde{\lambda}_{\dot 1})$, respectively. Hence, $\mathscr{I}_{\mathbb{C}}$ can be charted with 'projective' coordinates $(u,\lambda,\tilde{\lambda})$, defined up to the equivalence [165]

$$(u,\lambda,\tilde{\lambda}) \sim (r\tilde{r}u, r\lambda, \tilde{r}\tilde{\lambda}), \qquad r,\tilde{r}\in\mathbb{C}^*.$$

Denoting the line bundle of complex functions on $\mathbb{CP}^1\times\mathbb{CP}^1$ which are homogeneous of degree $m$ in $\lambda$ and degree $n$ in $\tilde{\lambda}$ by $\mathscr{O}(m,n)$, this means that $\mathscr{I}_{\mathbb{C}}$ is realized as the total space of the line bundle

$$\mathscr{O}(1,1)\to\mathbb{CP}^1\times\mathbb{CP}^1. \tag{5.1}$$

To recover the Lorentzian real slice one simply imposes $\tilde{\lambda}_{\dot\alpha} = \overline{\lambda_\alpha}$ and $u=\bar{u}$. Thus, the real Lorentzian cones $\mathscr{I}^\pm$ may each be viewed as the total space of the bundle $\mathscr{O}_{\mathbb{R}}(1,1)\to\mathbb{CP}^1$, where $\mathscr{O}_{\mathbb{R}}(1,1)$ is the restriction of $\mathscr{O}(1,1)$ to real-valued functions.

The BMS group (the asymptotic symmetry group of asymptotically flat space-times [125, 126]) acts naturally on $\mathscr{I}$, and hence on $\mathscr{I}_{\mathbb{C}}$ by analytic continuation. This group is the semi-direct product

$$\mathrm{BMS} = \mathrm{ST}\ltimes\mathrm{SL}(2,\mathbb{C}). \tag{5.2}$$

of an infinite dimensional Abelian group ST of *supertranslations* that moves one up and down a generator of the null cone, with *rotations* that are the global diffeomorphisms of the space $S^2$ of generators. In terms of the coordinates $(u, \lambda, \tilde{\lambda})$, the supertranslations act as

$$u \to u + \alpha(\lambda, \tilde{\lambda}), \qquad \lambda \to \lambda, \qquad \tilde{\lambda} \to \tilde{\lambda}, \tag{5.3}$$

where $\alpha$ transforms in the same way as $u$ under a rescaling of the homogeneous coordinates, and where $\tilde{\lambda} = \bar{\lambda}$ and $\alpha$ is real in Lorentzian signature. Expanding $\alpha$ in spherical harmonics, the $\ell = 0, 1$ terms correspond to Poincaré translations. These Poincaré translations are a symmetry of any asymptotically flat space-time, while a generic *super*translation transforms one asymptotically flat solution of general relativity to another [128, 166].

An asymptotically flat Lorentzian space-time carries two copies of this BMS group, acting at $\mathscr{I}^{\pm}$ separately. As explained in [134], only the diagonal subgroup can act on the S-matrix. $\mathscr{I}_{\mathbb{C}}$ carries an action of (one copy of) the complexified BMS group that admits independent SL$(2, \mathbb{C})$ transformations of $\lambda$ and $\tilde{\lambda}$ and allows $\alpha(\lambda, \tilde{\lambda})$ to be complex.

It has been suggested that the BMS group can be extended by supplementing globally well-defined SL$(2, \mathbb{C})$ rotations with *any* local conformal transformations of the sphere [130, 142, 167]. This leads to an enhanced set of (singular) rotations, known as *superrotations*, on the space of null generators of $\mathscr{I}_{\mathbb{C}}$, which contains two copies of the Virasoro algebra at the infinitesimal level [143].

In the case where massless Yang-Mills (with gauge group $G$) states propagate in the bulk spacetime, one must give boundary conditions at $\mathscr{I}$ in order to have a well-defined asymptotic symmetry group. This can be done, following [144], by imposing boundary conditions on the gauge field such that the charge and energy flux through any subset of $\mathscr{I}$ is finite. With these boundary conditions, it is easy to show that the asymptotic radiative degrees of freedom of the gauge field are controlled by a single function on $\mathscr{I}$, taking values in the Lie algebra $\mathfrak{g}$ of the gauge group.

On $\mathscr{I}_{\mathbb{C}}$, the gravitational radiative information from the interior of the space-time is controlled by a single complex function taking values in $\mathcal{O}(-3, 1)$, denoted here by $\sigma^0(u, \lambda, \tilde{\lambda})$. In the Newman-Penrose formalism, this is known as the 'asymptotic shear' [168]. The energy flux from the interior of the space-time is encoded in the *Bondi news function* [125],

$$N(u, \lambda, \tilde{\lambda}) = \frac{\partial \sigma^0}{\partial u} \equiv \dot{\sigma}^0, \tag{5.4}$$

taking values in $\mathcal{O}(-4, 0)$. The news function has long been regarded as fundamental to studying quantum gravity on $\mathscr{I}$, since it encodes the asymptotic 'radiative modes' of the gravitational field [127, 128].

In the Yang-Mills case, analytically continuing the gauge field to complexified Minkowski space, the function controlling the asymptotic radiative information becomes a $\mathfrak{g}$-valued function on $\mathscr{I}_{\mathbb{C}}$, taking values in $\mathcal{O}(-1,1)$. This function is denoted here by $\mathscr{A}^0(u,\lambda,\tilde{\lambda})$, suppressing gauge indices. The energy flux of the gauge field from the interior of Minkowski space through $\mathscr{I}_{\mathbb{C}}$ is encoded by the *broadcasting function* [127, 169, 170]

$$\mathscr{F}(u,\lambda,\tilde{\lambda}) = \frac{\partial \mathscr{A}^0}{\partial u}, \tag{5.5}$$

taking values in $\mathcal{O}(-2,0)\otimes\mathfrak{g}$.[1] This broadcasting function is the gauge theoretic version of the Bondi news function [125].

Hence, a description of gravitational scattering states at $\mathscr{I}_{\mathbb{C}}$ should encode scattering data in terms of 'insertions' of news functions, while a description gauge theoretic scattering states should encode the date in terms of 'insertions' of the broadcasting function.

It is worth noting that the coordinate $u$ is naturally conjugate to the 'frequency' of on-shell momentum eigenstates of massless particles. To define this, in place of the standard spinor helicity variables $p_{\alpha\dot{\alpha}} = \Lambda_\alpha\tilde{\Lambda}_{\dot{\alpha}}$, where $\Lambda_\alpha$ and $\tilde{\Lambda}_{\dot{\alpha}}$ are defined up to $(\Lambda,\tilde{\Lambda}) \sim (r\Lambda, r^{-1}\tilde{\Lambda})$, the null momentum is taken to be

$$p_{\alpha\dot{\alpha}} = \omega\,\lambda_\alpha\,\tilde{\lambda}_{\dot{\alpha}}$$

with the equivalence $(\omega,\lambda,\tilde{\lambda}) \sim (r^{-1}\tilde{r}^{-1}\omega, r\lambda, \tilde{r}\tilde{\lambda})$. Thus, on a (complex) Minkowski background, massless momentum eigenstates appear on $\mathscr{I}_{\mathbb{C}}$ as plane waves $\mathrm{e}^{\mathrm{i}\omega u}$ of frequency $\omega$, localized along the generator of $\mathscr{I}_{\mathbb{C}}$ at fixed angular location $(\lambda,\tilde{\lambda}) \in \mathbb{CP}^1\times\mathbb{CP}^1$.

Finally, the extension of $\mathscr{I}$ required to incorporate $\mathcal{N} = 2p$ extended supersymmetry is straightforward: one replaces the complexified space of null generators by $\mathbb{CP}^{1|p}\times\mathbb{CP}^{1|p}$, where each factor may now be described by homogeneous coordinates $\lambda_A = (\lambda_\alpha, \eta_a)$ and $\tilde{\lambda}_{\dot{A}} = (\tilde{\lambda}_{\dot{\alpha}}, \tilde{\eta}_{\dot{a}})$, respectively. The $\eta_a, \tilde{\eta}_{\dot{a}}$ are Grassmann (anti-commuting) coordinates, with $a, \dot{a} = 1,\ldots,p$. In the case of the gravity model $p = 4$, corresponding to a parity symmetric treatment of $\mathcal{N} = 8$ supergravity in which only $\mathcal{N} = 4$ supersymmetry is manifest. Latter, when discussing the YM model $p = 2$, which gives a parity symmetric treatment of $\mathcal{N} = 4$ super Yang-Mills where only $\mathcal{N} = 2$ supersymmetry is manifest.

In the supergravity context, that is $p = 4$, the news function (5.4) is replaced by a *news supermultiplet* $\Phi$ that takes values in $\mathcal{O}(0,0)$. The first component $\phi = \Phi|_{\eta=\tilde{\eta}=0}$ represents a scalar field at null infinity, while the usual news tensor and its conjugate are encoded by the coefficients of $(\eta)^4$ and $(\tilde{\eta})^4$. The multiplet terminates with a further scalar at order $(\eta\tilde{\eta})^4$. As for Yang-Mills ($p = 2$), the broadcasting function is replaced by a *broadcasting*

[1] In the case $G = \mathrm{U}(1)$, the broadcasting function is the Newman-Penrose coefficient for the asymptotic Maxwell field $\phi_2^0$ [168].

*supermultiplet* 0 which also takes values in $\mathcal{O}(0,0)$. The broadcasting function and its conjugate are the coefficients of the $(\eta)^2$ and $(\tilde{\eta})^2$ respectively.

From here on the $\mathscr{I}_{\mathbb{C}}$ will also be used to denote the total space of $\mathcal{O}(1,1) \to \mathbb{CP}^{1|p} \times \mathbb{CP}^{1|p}$, it should be clear from the context which manifold is being referred to.

## 5.3 The gravity model

The model which describes gravitational degrees of freedom is given by a chiral CFT describing holomorphic maps $(u,\lambda,\tilde{\lambda}) : \Sigma \to \mathscr{I}_{\mathbb{C}}$ from a Riemann sphere $\Sigma$ to the supersymmetric extension of complexified null infinity, taken to be the total space of $\mathcal{O}(1,1) \to \mathbb{CP}^{1|4} \times \mathbb{CP}^{1|4}$ as above. This model is expected to serve as a description for some effective theory on $\mathscr{I}_{\mathbb{C}}$, analogous to a worldline formalism for a field theory. $\Sigma$ serves as the chiral complexification of the usual worldline.

In order to implement the $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ scaling on $\mathscr{I}_{\mathbb{C}}$ associated with (5.1) at the level of $\Sigma$, introduce two line bundles $\mathscr{L}, \tilde{\mathscr{L}} \to \Sigma$ of degree $d, \tilde{d} \geq 0$, respectively. The basic fields of the model are then

$$u \in \Omega^0(\Sigma, \mathscr{L} \otimes \tilde{\mathscr{L}}), \qquad \lambda_A \in \Omega^0(\Sigma, \mathbb{C}^{2|4} \otimes \mathscr{L}), \qquad \tilde{\lambda}_{\dot{A}} \in \Omega^0(\Sigma, \mathbb{C}^{2|4} \otimes \tilde{\mathscr{L}}),$$

which describe the pullbacks to $\Sigma$ of homogeneous coordinates on $\mathscr{I}_{\mathbb{C}}$. These fields have a chiral action

$$S_1 = \frac{1}{2\pi} \int_\Sigma w\,\bar{\partial} u + v^A \bar{\partial} \lambda_A + \tilde{v}^{\dot{A}} \bar{\partial} \tilde{\lambda}_{\dot{A}} \tag{5.6}$$

where $\{w, v^A, \tilde{v}^{\dot{A}}\}$ are each $(1,0)$-forms on the worldsheet, with gauge charges opposite those of $\{u, \lambda_A, \tilde{\lambda}_{\dot{A}}\}$, respectively. These are Lagrange multipliers that ensure the map to $\mathscr{I}_{\mathbb{C}}$ is holomorphic. Introduce also the fields $\psi_A$ and $\tilde{\psi}_{\dot{A}}$ of opposite statistics to $\lambda_A$ and $\tilde{\lambda}_{\dot{A}}$, together with their conjugates $\bar{\psi}^A$ and $\bar{\tilde{\psi}}^{\dot{A}}$, respectively. Each of these fields is a worldsheet spinor, neutral under both $\mathrm{GL}(1,\mathbb{C})$ scalings. Their action is

$$S_2 = \frac{1}{2\pi} \int_\Sigma \bar{\psi}^A \bar{\partial} \psi_A + \bar{\tilde{\psi}}^{\dot{A}} \bar{\partial} \tilde{\psi}_{\dot{A}} \tag{5.7}$$

and the combined action $S_1 + S_2$ is invariant under the fermionic transformations

$$\delta\psi_A = \varepsilon_1 \lambda_A\,, \qquad \delta\bar{\psi}^\alpha = \varepsilon_2 \epsilon^{\alpha\beta} \lambda_\beta\,, \qquad \delta v^A = \varepsilon_1 \bar{\psi}^A - \varepsilon_2 \delta^A_{\ \alpha} \epsilon^{\alpha\beta} \psi_{\dot{\alpha}}\,, \tag{5.8}$$

with similar transformations for the tilded fields. All other fields remain invariant.

To gauge these fermionic symmetries include bosonic ghosts $s^{1,2} \in \Omega^0(\Sigma, K^{1/2} \otimes \mathscr{L}^{-1})$ and $\tilde{s}^{1,2} \in \Omega^0(\Sigma, K^{1/2} \otimes \tilde{\mathscr{L}}^{-1})$ together with their antighosts $r_{1,2}$ and $\tilde{r}_{1,2}$. Fermionic ghosts are also needed, these are given by the fields $\mathrm{n}, \tilde{\mathrm{n}} \in \Omega^0(\Sigma)$ and the antighosts are $\mathrm{m}, \tilde{\mathrm{m}} \in \Omega^0(\Sigma, K)$, these are associated to gauging the $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ transformations. The ghost action is

$$S_3 = \frac{1}{2\pi} \int_\Sigma r_a \bar{\partial} s^a + \tilde{r}_a \bar{\partial} \tilde{s}^a + \mathrm{m}\bar{\partial}\mathrm{n} + \tilde{\mathrm{m}}\bar{\partial}\tilde{\mathrm{n}}. \tag{5.9}$$

The final ingredient is a conjugate pair of fermionic fields $\xi \in \Omega^0(\Sigma, (\mathscr{L} \otimes \tilde{\mathscr{L}})^{-1})$ and $\chi \in \Omega^0(\Sigma, K \otimes \mathscr{L} \otimes \tilde{\mathscr{L}})$ with action

$$S_4 = \frac{1}{2\pi} \int_\Sigma \chi \bar{\partial} \xi \,. \tag{5.10}$$

The role of these fields will be explained below.

The BRST operator is taken to be[2]

$$\begin{aligned} Q_{\mathrm{BRST}} = \oint -\mathrm{n}(w\,u + v^A \lambda_A + r_a s^a + \chi\,\xi) - \tilde{\mathrm{n}}(w\,u + \tilde{v}^{\dot{A}} \tilde{\lambda}_{\dot{A}} + \tilde{r}_a \tilde{s}^a + \chi\,\xi) \\ + s^1 \lambda_A \bar{\psi}^A + s^2 \langle \lambda \psi \rangle + \tilde{s}^1 \tilde{\lambda}_{\dot{A}} \bar{\tilde{\psi}}^{\dot{A}} + \tilde{s}^2 [\tilde{\lambda}\tilde{\psi}] \,. \end{aligned} \tag{5.11}$$

This includes gaugings of the fermionic symmetries above as well as the gaugings associated to $\mathscr{L}$ and $\tilde{\mathscr{L}}$. It is straightforward to check that $Q_{\mathrm{BRST}}$ is nilpotent and anomaly free. For example, there is a potential $\mathrm{GL}(1,\mathbb{C})$ anomaly $\mathfrak{a}_{\mathrm{GL}(1)}$ associated with the line bundle $\mathscr{L}$. This is given by the sum of squares of the fields' $\mathrm{GL}(1,\mathbb{C})$-charges, weighted by a sign for their respective statistics:

$$\mathfrak{a}_{\mathrm{GL}(1)} = \sum_i (-1)^{F_i} q_i^2 = 1_{wu} + (2-4)_{v\lambda} + 2_{rs} - 1_{\chi\xi} = 0 \,.$$

The anomalies associated with $\tilde{\mathscr{L}}$ and $\mathscr{L} \times \tilde{\mathscr{L}}$ vanish by identical calculations. Note that the central charge of this chiral CFT is given by[3]:

$$\mathfrak{c} = 2_{wu} + 3(2-4)_{v\lambda,\bar{\psi}\psi} + 3(2-4)_{\tilde{v}\tilde{\lambda},\bar{\tilde{\psi}}\tilde{\psi}} - 4_{rs,\tilde{r}\tilde{s}} - 4_{\mathrm{mn},\tilde{\mathrm{m}}\tilde{\mathrm{n}}} - 2_{\chi\xi} = -20 \,.$$

BRST closed vertex operators are built out of the worldsheet fields in such a way as to have vanishing charge under $\mathscr{L}$ and $\tilde{\mathscr{L}}$, and be invariant under the fermionic transformations (5.8). The simplest such operators are gauge-invariant functions $\Phi =$

[2]Here the brackets denote the usual invariants $\langle ab \rangle = \epsilon_{\alpha\beta} a^\alpha b^\beta, [\tilde{a}\tilde{b}] = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{a}^{\dot{\alpha}} \tilde{b}^{\dot{\beta}}$.

[3]Non-vanishing $\mathfrak{c}$ corresponds to a non-vanishing Virasoro anomaly, but as there is no gauging of gravity on $\Sigma$ (*i.e.*, no $bc$-ghost system) the role of such an anomaly is unclear.

$\Phi(u, \lambda, \tilde{\lambda})$ of the target space coordinate fields that have vanishing worldsheet conformal weight. Crucially, these operators encode the energy flux through $\mathscr{I}_{\mathbb{C}}$, so their correlation functions should contain information about the bulk space-time. Expanding $\Phi$ in the fermionic coordinates on $\mathscr{I}_{\mathbb{C}}$ gives

$$\Phi_{(0,0)}(u, \lambda, \tilde{\lambda}) = \phi_{(0,0)} + \cdots + (\eta)^4 N_{(-4,0)} + \cdots + (\tilde{\eta})^4 \tilde{N}_{(0,-4)} + \cdots + (\eta)^4 (\tilde{\eta})^4 \tilde{\phi}_{(-4,-4)}\,, \quad (5.12)$$

where subscripts denote weights with respect to $(\lambda, \tilde{\lambda})$ and the component fields are functions only of the bosonic coordinates. In particular, $N_{(-4,0)}$ represents the Bondi news function (5.4), encoding the radiative data of a negative helicity graviton, while $\tilde{N}_{(0,-4)}$ is the news function for the positive helicity graviton. The other components represent analogous 'news functions' for the other particle content of $\mathcal{N} = 8$ supergravity; for instance the 28 components with two more $\eta$s than $\tilde{\eta}$s represent negative helicity photons, while the 70 components with equal numbers of $\eta$ and $\tilde{\eta}$s are scalars[4]. Note that the vertex operators of the worldsheet CFT are not constrained to have vanishing conformal weight, so there will be an infinite tower of states beyond these simplest ones. It will be shown below that correlation functions of arbitrarily many $\Phi(u, \lambda, \tilde{\lambda})$ operators do not excite these other states. These states won't be discussed further since they don't affect the relevant correlation functions.

The bosonic antighost fields $r, \tilde{r}$ have zero modes when $d, \tilde{d} > 0$. To make sense of the path-integral measure picture changing operators (PCOs) must be inserted to absorb these zero modes:

$$\Upsilon = \delta(r_1)\delta(r_2)\, \lambda_A \bar{\psi}^A \langle \lambda\, \psi \rangle, \qquad \tilde{\Upsilon} = \delta(\tilde{r}_1)\delta(\tilde{r}_2)\, \tilde{\lambda}_{\dot{A}} \bar{\tilde{\psi}}^{\dot{A}}\, [\tilde{\lambda}\, \tilde{\psi}]. \quad (5.13)$$

Insertions of $\Upsilon$ ($\tilde{\Upsilon}$) absorb $2d$ ($2\tilde{d}$) zero modes of the $r_1, r_2$ ($\tilde{r}_1, \tilde{r}_2$) antighosts. As usual, the correlation function doesn't depend on the location of the PCOs.

The next step is picking measure with which to integrate over the moduli space of vertex operator locations. This will reveal the role of the $\xi\chi$ system. Consider the composite operator $w\chi$. This is an uncharged fermionic quadratic differential on the worldsheet, and is BRST closed[5]. In the presence of vertex operators at points $\{x_1, \cdots, x_n\} \in \Sigma$, it has $n-3$ zero modes. As usual in string theory, if $\{\mu_j\}$ form a basis of Beltrami differentials on the punctured worldsheet, a top holomorphic form on the moduli space of these punctures can be constructed with $\prod_{j=1}^{n-3} (w\chi|\mu_j)$, where the bracket denotes integration over $\Sigma$. This

[4]Encoding the $\mathcal{N} = 8$ gravitational multiplet in this way breaks the SU(8) R-symmetry group to a SU(4) × SU(4) subsector (in Lorentzian signature) where the two factors are related by parity symmetry.

[5]In particular $\{Q_{\text{BRST}}, w\chi\} \neq T$, so $w\chi$ cannot be interpreted as a composite $b$ ghost.

choice of measure places an important constraint on the possible degrees of the line bundles $\mathscr{L}$ and $\tilde{\mathscr{L}}$. Since $\chi$ has $d+\tilde{d}-1$ zero modes, the correlation function vanishes unless

$$d+\tilde{d}=n-2, \tag{5.14}$$

As in [31], this amounts to the requirement that $\mathscr{L}\otimes\tilde{\mathscr{L}}\cong K(x_1+\cdots+x_n)$.

The simplest correlation function in this model is thus

$$\mathscr{M}_{n,d}=\left\langle\prod_{i=1}^{n}\Phi(\sigma_i)\prod_{j=4}^{n}\big(w\,\chi|\mu_j\big)\prod_{k=1}^{d}\Upsilon_k\prod_{l=1}^{\tilde{d}}\tilde{\Upsilon}_l\right\rangle=\left\langle\!\!\left\langle\prod_{i=1}^{3}\Phi(\sigma_i)\prod_{j=4}^{n}\int_{\Sigma}\chi(\sigma_j)\dot{\Phi}(\sigma_j)\right\rangle\!\!\right\rangle, \tag{5.15}$$

where $\dot{\Phi}=\partial_u\Phi$ takes values in $\mathscr{O}(-1,-1)$ on $\mathscr{I}_{\mathbb{C}}$ and $\langle\!\langle\cdots\rangle\!\rangle$ denotes a correlator in the presence of the PCOs.

## 5.4 Symmetries and Ward identities

The symmetries of $\mathscr{I}_{\mathbb{C}}$ must have a realization in terms of charges on $\Sigma$, which act on correlators such as (5.15). The simplest such symmetry is the Poincaré group, which is generated by the charges

$$Q_{\mathrm{SL}(2,\mathbb{C})}=\oint a^{\alpha}{}_{\beta}\,\lambda_{\alpha}(\sigma)\nu^{\beta}(\sigma)+\text{c.c.}\qquad\text{and}\qquad Q_{\mathrm{T}}=\oint b^{\alpha\dot{\alpha}}\,\lambda_{\alpha}(\sigma)\tilde{\lambda}_{\dot{\alpha}}(\sigma)\,w(\sigma)\,, \tag{5.16}$$

where $a^{\alpha}{}_{\beta}$ and $b^{\alpha\dot{\alpha}}$ are constant and the former is traceless. It is easy to see that these charges commute with the action and are *bona fide* symmetries of the model, just as the Poincaré group is an asymptotic symmetry of every asymptotically flat space-time.

However, as discussed in section 5.2, there is a larger symmetry group on $\mathscr{I}_{\mathbb{C}}$: the infinite dimensional BMS group, built from Lorentz transformations and supertranslations. The latter are generated on $\Sigma$ by charges

$$Q_{\mathrm{ST}}=\oint f(\lambda,\tilde{\lambda})\,w(\sigma), \tag{5.17}$$

where $f$ is a function of weight one in both $\lambda$ and $\tilde{\lambda}$.[6] Unlike the Poincaré charges (5.16), the general supertranslation charge (5.17) will have poles, and its commutator with the action will be non-vanishing at these poles. This is expected, since the realization of $\mathscr{I}_{\mathbb{C}}$ as a vector bundle over $\mathbb{CP}^1\times\mathbb{CP}^1$ endows it with more structure than necessary. In particular,

[6] $Q_{\mathrm{ST}}$ generates 'complexified' supertranslations on $\mathscr{I}_{\mathbb{C}}$; to restrict to the real $\mathscr{I}$, set $\tilde{\lambda}=\bar{\lambda}$ so that $f$ becomes a smooth function on $\mathbb{CP}^1$, defining a real supertranslation.

the choice of an origin for this vector bundle is equivalent to a choice of classical vacuum from the perspective of asymptotic quantization [127, 128]; since supertranslations map one vacuum to an inequivalent vacuum, they shouldn't be exact symmetries of the model. Nonetheless, $Q_{\text{ST}}$ *does* give nontrivial information in the form of a Ward identity containing information about soft gravitons [134, 137].

In particular, consider the supertranslation given by the charge

$$Q^{(1)}_{\text{ST}} = \oint f^{(1)}(\lambda, \tilde{\lambda})\, w(\sigma), \qquad f^{(1)}(\lambda, \tilde{\lambda}) = \frac{a^\alpha b^\beta \tilde{\lambda}_s^{\dot{\alpha}}}{\langle a\, s\rangle \langle b\, s\rangle} \frac{\lambda_\alpha(\sigma)\lambda_\beta(\sigma)\tilde{\lambda}_{\dot{\alpha}}(\sigma)}{\langle s\, \lambda(\sigma)\rangle}, \tag{5.18}$$

where $(\lambda_s, \tilde{\lambda}_s)$ is a fixed point on the space of generators of $\mathscr{I}_{\mathbb{C}}$ associated with the insertion of a soft graviton. Note that $f^{(1)}$ has weight (1,1) in $(\lambda(\sigma), \tilde{\lambda}(\sigma))$ as required for a supertranslation, and weight $(-3, 1)$ in $(\lambda_s, \tilde{\lambda}_s)$ as for the asymptotic shear of a soft graviton. Inserting this charge into (5.15), its effect is to differentiate each vertex operator $\Phi_i$ in the $u$-direction. Assuming that these operators are momentum eigenstates of frequency (or energy) $\omega_i$, this results in a Ward identity

$$\left\langle\!\!\left\langle Q^{(1)}_{\text{ST}} \prod_{i=1}^{3} \Phi(\sigma_i) \prod_{j=4}^{n} \int_\Sigma \chi(\sigma_j)\dot{\Phi}(\sigma_j) \right\rangle\!\!\right\rangle = \sum_{i=1}^{n} \omega_i\, f^{(1)}(\lambda(\sigma_i), \tilde{\lambda}(\sigma_i))\, \mathscr{M}_{n,d}\,, \tag{5.19}$$

where on the left hand side, the contour in $Q^{(1)}_{\text{ST}}$ is taken along $|\langle \lambda(\sigma)\, s\rangle| = \epsilon$. This is equivalent to the Ward identity found in [134] for the supertranslations generated by (5.18).

More specifically, suppose that one-particle states are represented by the explicit momentum eigenstates

$$\Phi_i = \int \frac{\mathrm{d}t_i\, \mathrm{d}\tilde{t}_i}{t_i\, \tilde{t}_i\, \omega_i}\, \delta^{2|4}(\lambda_i - t_i \lambda(\sigma_i))\, \delta^{2|4}(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(\sigma_i))\, \mathrm{e}^{t_i \tilde{t}_i \omega_i\, u(\sigma_i)}. \tag{5.20}$$

Then on the support of the delta functions in these vertex operators, the action of $Q^{(1)}_{\text{ST}}$ on the correlator reads

$$\begin{aligned} \sum_{i=1}^{n} \frac{a^\alpha b^\beta \tilde{\lambda}_s^{\dot{\alpha}}}{\langle a\, s\rangle \langle b\, s\rangle} t_i \tilde{t}_i \omega_i \left\langle\!\!\left\langle \frac{\lambda_\alpha(\sigma_i)\lambda_\beta(\sigma_i)\tilde{\lambda}_{\dot{\alpha}}(\sigma_i)}{\langle s\, \lambda(\sigma_i)\rangle} \prod_{k=1}^{3} \Phi(\sigma_k) \prod_{j=4}^{n} \int_\Sigma \chi(\sigma_j)\dot{\Phi}(\sigma_j) \right\rangle\!\!\right\rangle \\ = \sum_{i=1}^{n} \omega_i \frac{[s\, i]}{\langle s\, i\rangle} \frac{\langle a\, i\rangle \langle b\, i\rangle}{\langle a\, s\rangle \langle b\, s\rangle} \mathscr{M}_{n,d}. \end{aligned} \tag{5.21}$$

This is precisely Weinberg's soft graviton theorem as re-derived in the context of supertranslations acting on the S-matrix in [137]. In this worldsheet model, the universal soft graviton factor arises from the action of a charge generating a supertranslation, which effectively creates the soft graviton at the position $(\lambda_s, \tilde{\lambda}_s) \in \mathbb{CP}^1 \times \mathbb{CP}^1$.

More general supertranslations (having additional or higher-order poles) are related to the creation of multiple soft gravitons. The worldsheet model also includes supersymmetric extensions of supertranslations, which correspond to other soft particles in the spectrum of $\mathcal{N} = 8$ supergravity. Hence, the supertranslations (5.17) combined with $Q_{\mathrm{SL}(2,\mathbb{C})}$ generate the action of the BMS group in the worldsheet model.

Interestingly, the superrotations of the extended BMS group can be easily incorporated. The relevant charge on $\Sigma$ is

$$Q_{\mathrm{SR}} = \oint R(\lambda,\tilde{\lambda})^{\alpha}{}_{\beta}\,\lambda_{\alpha}(\sigma)\,v^{\beta}(\sigma) + \tilde{R}(\lambda,\tilde{\lambda})^{\dot{\alpha}}{}_{\dot{\beta}}\,\tilde{\lambda}_{\dot{\alpha}}(\sigma)\,\tilde{v}^{\dot{\beta}}(\sigma)\,, \tag{5.22}$$

where $R(\lambda,\tilde{\lambda})^{\alpha}{}_{\beta}$, $R(\lambda,\tilde{\lambda})^{\dot{\alpha}}{}_{\dot{\beta}}$ are traceless, weightless holomorphic functions of $(\lambda,\tilde{\lambda})$. General operators of this form suffer from normal ordering ambiguities, but a large interesting class are free from such problems. For instance, consider (5.22) with

$$R^{\alpha}{}_{\beta} = 0, \qquad \tilde{R}^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{\langle a\,\lambda(\sigma)\rangle}{\langle s\,\lambda(\sigma)\rangle}\,\frac{\tilde{\lambda}_s^{\dot{\alpha}}\,\tilde{\lambda}_{s\dot{\beta}}}{\langle a\,s\rangle}.$$

A calculation similar to that which led to (5.21) gives the action of this charge on the correlator with momentum eigenstates:

$$\left\langle\!\!\left\langle Q_{\mathrm{SR}}\prod_{i=1}^{3}\Phi(\sigma_i)\prod_{j=4}^{n}\int_{\Sigma}\chi(\sigma_j)\dot{\Phi}(\sigma_j)\right\rangle\!\!\right\rangle = -\sum_{i=1}^{n}\frac{[s\,i]}{\langle s\,i\rangle}\frac{\langle a\,i\rangle}{\langle a\,s\rangle}\tilde{\lambda}_{s\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_{i\dot{\alpha}}}\mathcal{M}_{n,d}\,, \tag{5.23}$$

where the contour for $Q_{\mathrm{SR}}$ is as before. The last line is the holomorphic subleading soft graviton contribution recently discussed by Cachazo and Strominger [140]. It was conjectured that this subleading contribution is related to the action of superrotations; (5.23) demonstrates this explicitly at the level of charges acting on the correlator. This was latter verified using different methods in [].

## 5.5 Gravitational scattering amplitudes

Having shown that the model has vertex operators naturally encoding the asymptotic radiative degrees of freedom, and that the charges corresponding to the BMS group have a natural action on its correlators, the last thing to do is to evaluate the correlation functions (5.15) themselves. First, notice that the PCO insertions only have non-trivial Wick contractions with other PCOs of the same type. The resulting correlation function on the PCOs can then be computed using the arguments employed for PCOs in [43], resulting in

$$\left\langle \prod_{k=1}^{d} \Upsilon_k \prod_{l=1}^{\tilde{d}} \tilde{\Upsilon}_l \right\rangle = \mathrm{R}(\lambda)\, \mathrm{R}(\tilde{\lambda}), \tag{5.24}$$

where $\mathrm{R}(\lambda)$, $\mathrm{R}(\tilde{\lambda})$ are the *resultants* of the maps

$$\lambda_\alpha : \Sigma \to \mathbb{CP}^1, \qquad \tilde{\lambda}_{\dot{\alpha}} : \Sigma \to \mathbb{CP}^1, \tag{5.25}$$

respectively [44]. The resultant $\mathrm{R}(\lambda)$ vanishes iff both $\lambda_\alpha(\sigma_*)$ vanish simultaneously for some $\sigma_* \in \Sigma$. The factor (5.24) thus ensures that the amplitude receives contributions only when $(\lambda_\alpha(\sigma), \tilde{\lambda}_{\dot{\alpha}}(\sigma))$ is a well-defined map to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Evaluating the remainder of the correlator with the choice of momentum eigenstates (5.41) for $\Phi$ leads to

$$\begin{aligned}\mathcal{M}_{n,d} = \int \frac{\prod_{r=1}^{d+1} \mathrm{d}^{2|4}\lambda_r^0 \prod_{s=1}^{\tilde{d}+1} \mathrm{d}^{2|4}\tilde{\lambda}_s^0}{\mathrm{vol}\,(\mathbb{C}^* \times \mathbb{C}^*)} \; \frac{\mathrm{R}(\lambda)\,\mathrm{R}(\tilde{\lambda})\; |\sigma_4 \cdots \sigma_n|}{\prod_{j=1,2,3} t_j\, \tilde{t}_j\, \omega_j\, \mathrm{D}\sigma_j} \\ \prod_{a=1}^{d+\tilde{d}+1} \delta\!\left(\sum_{i=1}^{n} t_i \tilde{t}_i \omega_i\, \mathfrak{s}_a(\sigma_i)\right) \prod_{i=1}^{n} \mathrm{D}\sigma_i\, \mathrm{d}t_i\, \mathrm{d}\tilde{t}_i\; \delta^{2|4}(\lambda_i - t_i \lambda(\sigma_i))\; \delta^{2|4}(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(\sigma_i))\,.\end{aligned} \tag{5.26}$$

Here, the measure in the first line is over the zero modes of the maps (5.25), while the quotient by two $\mathbb{C}^*$-freedoms reflects the rescaling symmetry associated with $\mathcal{L}$ and $\tilde{\mathcal{L}}$. The $\sigma_i^{\underline{\alpha}} = (\sigma_i^{\underline{0}}, \sigma_i^{\underline{1}})$ are homogeneous coordinates on $\Sigma$, which have $\mathrm{SL}(2,\mathbb{C})$-invariant contraction $\epsilon_{\underline{\alpha}\underline{\beta}} \sigma_i^{\underline{\alpha}} \sigma_j^{\underline{\beta}} = (i\, j)$. The Vandermonde determinant

$$|\sigma_4 \cdots \sigma_n| := \prod_{4 \le i < j \le n} (i\, j)$$

is produced by the $n-3$ $\chi$-insertions, and $\mathrm{D}\sigma_i := (\sigma_i\, \mathrm{d}\sigma_i)$ is the natural weight +2 holomorphic measure on $\Sigma$. Finally, the first set of $\delta$-functions in the second line arises by performing the integral over zero modes for the map component $u : \Sigma \to \mathbb{C}$,

$$\int \mathrm{d}^{d+\tilde{d}+1} u^0 \, \exp\left[\mathrm{i} \sum_{i=1}^{n} t_i \tilde{t}_i \omega_i\, u(\sigma_i)\right] = \prod_{a=1}^{d+\tilde{d}+1} \delta\!\left(\sum_{i=1}^{n} t_i \tilde{t}_i \omega_i\, \mathfrak{s}_a(\sigma_i)\right),$$

for $\{\mathfrak{s}_a\}$ a basis of $H^0(\Sigma, \mathcal{L} \otimes \tilde{\mathcal{L}})$.

The expression (5.45) for $\mathcal{M}_{n,d}$ can be manipulated into a more recognizable form. Using the constraint (5.14), one can show ([31, 45]) that

$$\prod_{a=1}^{d+\tilde{d}+1} \delta\left(\sum_{i=1}^{n} t_i \tilde{t}_i \omega_i \,\mathfrak{s}_a(\sigma_i)\right) = \frac{1}{|\sigma_1 \cdots \sigma_n|} \int_{\mathbb{C}} \mathrm{d}r \prod_{i=1}^{n} \delta\left(t_i \tilde{t}_i \omega_i - \frac{r}{\prod_{j\neq i}(\sigma_i \sigma_j)}\right).$$

Inserting this identity into (5.45) and working on the support of the various delta-functions produces an equivalent expression for the correlator:

$$\mathcal{M}_{n,d} = \int \frac{\prod_{r=1}^{d+1} \mathrm{d}^{2|4}\lambda_r^0 \prod_{s=1}^{\tilde{d}+1} \mathrm{d}^{2|4}\tilde{\lambda}_s^0}{\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*)} \mathrm{R}(\lambda)\,\mathrm{R}(\tilde{\lambda})\, \frac{\mathrm{d}r}{r^3} \prod_{i=1}^{n} \delta\left(t_i \tilde{t}_i \omega_i - \frac{r}{\prod_{j\neq i}(\sigma_i \sigma_j)}\right)$$
$$\times\ \mathrm{D}\sigma_i\, \mathrm{d}t_i\, \mathrm{d}\tilde{t}_i\, \delta^{2|4}(\lambda_i - t_i \lambda(\sigma_i))\, \delta^{2|4}(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(\sigma_i)). \quad (5.27)$$

Using one of the $\mathbb{C}^*$-freedoms to fix the $r$-integral, this expression is equal to a representation derived in [44] of the Cachazo-Skinner formula for the tree-level S-matrix of $\mathcal{N} = 8$ supergravity [171]. Hence, the simplest correlation function of the model, with vertex operators represented by momentum eigenstates, produces the tree-level scattering amplitudes of gravity.

## 5.6 The Yang-Mills model

The set up of the Yang-Mills model is similar to the gravitational one. Matter fields are mostly the same, the difference lies on the symmetries being gauged. It's worth nothing from the beginning that if this model is to be interpreted as giving a holographic description of Yang-Mills theory in some regime of validity it should also include gravity. Indeed, it turns out that the YM model contains states of *conformal* gravity, quite similar to the original twistor string of Berkovits and Witten [2–4]. For tree-level single trace amplitudes there is no contribution from the conformal gravity states, so I'll focus on this case.

As in the gravitational case studied above, the worldsheet model for Yang-Mills is a CFT on the Riemann sphere $\Sigma$ governing holomorphic maps from $\Sigma$ to $\mathscr{I}_{\mathbb{C}}$. Now $\mathscr{I}_{\mathbb{C}}$ is the total space of $\mathcal{O}(1,1) \to \mathbb{CP}^{1|2} \times \mathbb{CP}^{1|2}$, which gives $\mathcal{N} = 4$ supersymmetry on the target space. As above the $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ scaling built into the projective description of $\mathscr{I}_{\mathbb{C}}$ is given on the Riemann sphere in terms of two line bundles $\mathscr{L}, \tilde{\mathscr{L}} \to \Sigma$, of degree $d, \tilde{d} \geq 0$. And the coordinates on $\mathscr{I}_{\mathbb{C}}$ are given by the same worldsheet fields as in the gravitational case

$$u \in \Omega^0(\Sigma, \mathscr{L} \otimes \tilde{\mathscr{L}}), \qquad \lambda_A \in \Omega^0(\Sigma, \mathbb{C}^{2|2} \otimes \mathscr{L}), \qquad \tilde{\lambda}_{\dot{A}} \in \Omega^0(\Sigma, \mathbb{C}^{2|2} \otimes \tilde{\mathscr{L}}),$$

with the chiral, first-order action:

$$S_1 = \frac{1}{2\pi}\int_\Sigma w\,\bar{\partial}u + v^A\,\bar{\partial}\lambda_A + \tilde{v}^{\dot{A}}\,\bar{\partial}\tilde{\lambda}_{\dot{A}}\,. \tag{5.28}$$

The conformal weight $(1,0)$ fields $\{w, v^A, \tilde{v}^{\dot{A}}\}$ are conjugate to the coordinates on $\mathscr{I}_{\mathbb{C}}$.

To describe gauge degrees of freedom add the action $S_C$ for a worldsheet current algebra corresponding to the space-time gauge group $G$. As usual, this can be realized explicitly in terms of free fermions on the worldsheet. The two $\mathrm{GL}(1,\mathbb{C})$ symmetries associated with the line bundles $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are being gauge as well as two-dimensional gravity on the worldsheet.[7]. The ghost action is then

$$S_2 = \frac{1}{2\pi}\int_\Sigma \mathrm{m}\,\bar{\partial}\mathrm{n} + \tilde{\mathrm{m}}\,\bar{\partial}\tilde{\mathrm{n}} + b\,\bar{\partial}c\,, \tag{5.29}$$

where all ghost fields have fermionic statistics and $\mathrm{n}, \tilde{\mathrm{n}} \in \Omega^0(\Sigma)$, $c \in \Omega^0(\Sigma, T_\Sigma)$. Finally, add an additional set of fermionic fields

$$\chi \in \Omega^0(\Sigma, \mathscr{L}\otimes\tilde{\mathscr{L}})\,, \qquad \xi \in \Omega^0(\Sigma, K_\Sigma \otimes (\mathscr{L}\otimes\tilde{\mathscr{L}})^{-1})\,,$$

with action

$$S_3 = \frac{1}{2\pi}\int_\Sigma \xi\,\bar{\partial}\chi\,. \tag{5.30}$$

Note that this pair of fields have *different* conformal weights from the ones in the gravitational model. They play a similar role in this model as in the gravitational one so they will be denoted by the same symbols. Since from now on only the gauge theory model will be discussed I trust that no confusion arise from it.

The full action is $S = S_1 + S_2 + S_3 + S_C$ and the BRST charge implementing scale and conformal invariance on $\Sigma$ is:

$$Q = \oint c\,T - \mathrm{n}\left(w\,u + v^A\,\lambda_A + \chi\,\xi\right) - \tilde{\mathrm{n}}\left(w\,u + \tilde{v}^{\dot{A}}\,\tilde{\lambda}_{\dot{A}} + \chi\,\xi\right), \tag{5.31}$$

where $T$ is the holomorphic stress tensor of the action. It is easy to see that the anomalies associated with $\mathscr{L}, \tilde{\mathscr{L}}$ vanish, and the theory has vanishing conformal anomaly provided the worldsheet current algebra contributes $+30$ to the central charge. When working on the Riemann sphere ($\Sigma \cong \mathbb{CP}^1$) the conformal anomaly is somewhat tame, so no explicit choice of gauge group that would make $c = 0$ will be made.

[7] This was not the case for the gravitational model, where the stress tensor $T$ wasn't part of the BRST operator.

Vertex operators of the model sit inside the cohomology of the BRST charge $Q$. The structure of (5.31) makes clear that these vertex operators should be functions of conformal weight zero, uncharged-charged under $\mathscr{L} \times \tilde{\mathscr{L}}$. Such vertex operators fall into two broad classes: gauge theoretic and gravitational. The former are given by

$$c\,\mathrm{tr}\big(O(u,\lambda,\tilde{\lambda})\; j\big)\,, \tag{5.32}$$

where $O(u,\lambda,\tilde{\lambda})$ is a homogeneous function on $\mathscr{I}_{\mathbb{C}}$ taking values in $\mathfrak{g}$ (the Lie algebra of the gauge group) and $j$ is the conformal weight $(1,0)$ Kac-Moody current provided by the worldsheet current algebra. Inside the trace there is a $c$-ghost, which absorbs the conformal weight of the current $j$ and creates a puncture on $\Sigma$. As usual, the current obeys

$$j^{\mathsf{a}}(z)\, j^{\mathsf{b}}(w) \sim \frac{k\,\delta^{\mathsf{ab}}}{(z-w)^2} + \frac{f^{\mathsf{ab}}_{\ \ \mathsf{c}}\, j^{\mathsf{c}}}{(z-w)}\,, \tag{5.33}$$

where $k$ is the level of the current algebra and $f^{\mathsf{abc}}$ are the structure constants of $\mathfrak{g}$.[8] The current algebra is chosen to have level $k=0$ as in [148], but this assumption can be relaxed for most calculations.

Expanding the function $O(u,\lambda,\tilde{\lambda})$ with respect to the fermionic coordinates reveals how the radiative degrees of freedom for the gauge field at $\mathscr{I}_{\mathbb{C}}$ are encoded. The supermultiplet is expanded as:

$$O(u,\lambda,\tilde{\lambda}) = \varphi_{(0,0)} + \cdots + (\eta)^2 \mathscr{F}_{(-2,0)} + \cdots + (\tilde{\eta})^2 \tilde{\mathscr{F}}_{(0,-2)} + \cdots + (\eta)^2 (\tilde{\eta})^2 \tilde{\varphi}_{(-2,-2)}\,,$$

with the components being $\mathfrak{g}$-valued functions on the bosonic body of $\mathscr{I}_{\mathbb{C}}$ whose weight is indicated by the subscripts. The component $\mathscr{F}_{(-2,0)}$ ($\tilde{\mathscr{F}}_{(0,-2)}$) is the broadcasting function for a positive (negative) helicity gluon defined by (5.5), while each of the other components corresponds to the radiative degrees of freedom of the full spectrum of $\mathcal{N}=4$ super-Yang-Mills theory. For example, the six components coming with an equal number of $\eta$s and $\tilde{\eta}$s represent the scalars.

However, there are additional vertex operators which do not involve the worldsheet current algebra. Roughly, these correspond to deformations of the complex and Hermitian structures of $\mathscr{I}_{\mathbb{C}}$, and are given by

$$c\,w\,v\,, \quad c\,v^A\,v_A\,, \quad c\,\tilde{v}^{\dot{A}}\,\tilde{v}_{\dot{A}}\,,$$

[8]Note that here the indices $\mathsf{a},\mathsf{b},\mathsf{c}$ run over the dimension of the Lie algebra, and are not to be confused with those of fermionic variables $\eta_a$.

$$c\partial u g\,,\quad c\partial\lambda_A g^A\,,\quad c\partial\tilde{\lambda}_{\dot{A}}\tilde{g}^{\dot{A}}\,, \tag{5.34}$$

where $v, v_A, \tilde{v}_{\dot{A}}, g, g^A$, and $\tilde{g}^{\dot{A}}$ are functions on $\mathscr{I}_{\mathbb{C}}$ taking values in appropriate powers of $\mathscr{L}$ and $\tilde{\mathscr{L}}$ to ensure overall homogeneity. Given their geometric action, it is natural to interpret these vertex operators as *gravitational* perturbations away from the Minkowski vacuum defining $\mathscr{I}_{\mathbb{C}}$. Indeed, from a holographic perspective it is expected that any model living at null infinity with gauge-theoretic degrees of freedom must be coupled to gravitation in some way. Below I'll argue that (5.34) actually correspond to a non-unitary theory of gravity. On the Riemann sphere and at single trace it is possible to consistently study only the gauge-theoretic vertex operators (5.32), conformal gravity states will only contribute to multi trace amplitudes and at loop level.

Several of the fields in this model have zero modes on the Riemann sphere, thus a prescription for computing correlation functions in this CFT is required. While the $bc$-ghost system provides a measure on the moduli space of $\Sigma$ in the presence of vertex operator insertions, the field $\chi$ from (5.30) has $d+\tilde{d}$ zero modes which must be absorbed. To do this, note that the composite operator $w\chi$ has conformal weight $(1,0)$ and is uncharged-charged with respect to $\mathscr{L}, \tilde{\mathscr{L}}$. In the presence of vertex operator insertions at points $\{z_1,\ldots,z_n\}\in\Sigma$, $w\chi$ acquires poles at these insertions. Thus, define

$$\oint_{z_i} w(z)\,\chi(z)\,,$$

to be the scalar given by the residue of $w\chi$ at $z_i \in \Sigma$.

As an uncharged-charged scalar, this quantity is clearly BRST-closed and has $n-1$ zero modes. In analogy to the gravitational model, demand that these zero modes also saturate the zero modes of $\chi$ itself. This results in the constraint

$$d+\tilde{d} = n-2\,. \tag{5.35}$$

With this prescription it is possible to eliminate the degree of $\tilde{\mathscr{L}}$ in terms of $d$ and $n$ in any correlation function, and reproduces the identification between line bundles $\mathscr{L}\otimes\tilde{\mathscr{L}} \cong K_{\Sigma}(z_1+\cdots+z_n)$ given by [31].

The correlation function involving $n$ gauge theory vertex operators (5.32) is given by

$$\mathscr{A}_{n,d} = \left\langle \prod_{i=1}^{n} c_i \,\mathrm{tr}\big(O\, j\big)_i \prod_{j=1}^{n-3} (b|\mu)_j \prod_{k=2}^{n} \oint_{\sigma_k} w(\sigma)\,\chi(\sigma) \right\rangle$$
$$= \left\langle c_1 \,\mathrm{tr}\big(O\, j\big)_1\, c_2\,\chi_2 \,\mathrm{tr}\big(\dot{O}\, j\big)_2\; c_3\,\chi_3 \,\mathrm{tr}\big(\dot{O}\, j\big)_3 \prod_{k=3}^{n} \int_{\Sigma} \chi_k \,\mathrm{tr}\big(\dot{O}\, j\big)_k \right\rangle, \quad (5.36)$$

where a subscript denotes dependence on an insertion point (*e.g.*, $c_1 = c(\sigma_1)$) and $\dot{O} = \partial_u O$ takes values in $\mathscr{O}(-1,-1)$ on $\mathscr{I}_{\mathbb{C}}$.

## 5.7 Kac-Moody symmetries and Ward identities

The correlator (5.36) involves insertions of the broadcasting function, which encodes the radiative degrees of freedom of (super-)Yang-Mills theory asymptotically. The asymptotic symmetries of the gauge field should have a realization in the context of this worldsheet model and a natural action on correlators.

The most obvious asymptotic symmetry of the gauge theory are global gauge transformations on $\mathscr{I}_{\mathbb{C}}$. These are implemented in this model by charges

$$\mathscr{J}_{\mathfrak{g}} = \oint \mathrm{tr}\big(\mathsf{T}\, j(\sigma)\big)\,, \quad (5.37)$$

where $\{\mathsf{T}^{\mathsf{a}}\}$ are the generators of the Lie algebra $\mathfrak{g}$ and $j^{\mathsf{a}}(\sigma)$ is the $\mathfrak{g}$-Kac-Moody current. Since they are global on $\mathscr{I}_{\mathbb{C}}$ and holomorphic on the worldsheet, it is obvious that these charges commute with the action and generate exact symmetries of the correlators.

The asymptotic symmetries of Yang-Mills theory are larger than global gauge transformations, though. Strominger showed that the possibility of non-zero colour flux through $\mathscr{I}$ leads to 'large' gauge transformations which are constant along the null generators [144]. Such gauge transformations are parametrized by a function on the sphere of null generators which is only locally holomorphic. These are implemented in the model by charges

$$\mathscr{J}_{\varepsilon} = \oint \varepsilon(\lambda,\tilde{\lambda})\,\mathrm{tr}\big(\mathsf{T}\, j(\sigma)\big)\,, \quad (5.38)$$

where the function $\varepsilon$ specifies the 'large' gauge transformation.[9]

Crucially, $\varepsilon$ may have poles, in which case the charge $\mathscr{J}_{\varepsilon}$ is not an exact symmetry due to the residue at those poles. In this case, the contour in (5.38) is taken around these

[9] This charge actually generates a *complexified* 'large' gauge transformation; the real transformation is obtained by restricting to the real slice $\mathscr{I}$.

poles; assuming only simple poles, inserting the charge into the correlator (5.36) results in a Ward identity:

$$\langle \mathcal{J}_\varepsilon\, c_1 \operatorname{tr}\big(O\, j\big)_1 \cdots \int_\Sigma \chi_n \operatorname{tr}\big(\dot{O}\, j\big)_n \rangle \\ = \sum_{w\in\mathcal{S}_n/\mathbb{Z}_n} \sum_{i=1}^{n} \varepsilon(\lambda(\sigma_{w_i}),\tilde{\lambda}(\sigma_{w_i}))\operatorname{tr}(\mathsf{T}_{w_1}\cdots[\mathsf{T},\mathsf{T}_{w_i}]\cdots\mathsf{T}_{w_n})\mathscr{A}_{n,d}\,. \quad (5.39)$$

Here $\mathsf{T}^{\mathsf{a}}_i$ is the generator of $\mathfrak{g}$ acting in the representation carried by the vertex operator at $\sigma_i \in \Sigma$. This is closely related to the Ward identity found in [144] for the action of the Kac-Moody current implementing the 'large' gauge transformation (5.38). Here, $\mathscr{A}^{a,i}_{n,d}$ is shorthand for the correlator

$$\mathscr{A}^{a,i}_{n,d} = \left\langle c_1 \operatorname{tr}\big(O\, j\big)_1 \cdots \mathsf{T}^a_i \int_\Sigma \chi_i \operatorname{tr}(\dot{O}\, j)_i \cdots \int_\Sigma \chi_n \operatorname{tr}\big(\dot{O}\, j\big)_n \right\rangle,$$

with $\mathsf{T}^a_i$ in the representation carried by the vertex operator at insertion $\sigma_i \in \Sigma$.

It has been shown that the content of the Ward identity (5.39) should be equivalent to Weinberg's soft gluon theorem [144, 148], and this is confirmed in the context of the YM model with an appropriate choice for the 'large' gauge transformation parameter. Indeed, the simplest non-trivial choice for the function $\varepsilon$ is

$$\varepsilon^{(1)}(\lambda,\tilde{\lambda}) = \frac{\langle a\,\lambda(\sigma)\rangle}{\langle a\,s\rangle\,\langle s\,\lambda(\sigma)\rangle}\,, \quad (5.40)$$

where $a_\alpha$ is an arbitrary point on $\mathbb{CP}^1$, $(\lambda_s,\tilde{\lambda}_s)$ labels the generator of $\mathscr{I}_\mathbb{C}$ associated with the insertion. Since $\varepsilon^{(1)}$ is homogeneous with respect to $a$ and $\lambda(\sigma)$, and is independent of $\tilde{\lambda}(\sigma)$, the associated charge corresponds to a 'holomorphic' Kac-Moody current. Furthermore, $\varepsilon^{(1)}$ has weight $(-2,0)$ with respect to $(\lambda_s,\tilde{\lambda}_s)$, the same as the broadcasting function for a positive helicity gluon.

The single particle states in (5.36) can be given in the momentum eigenstate representation:

$$O^{\mathsf{a}}_i = \mathsf{T}^{\mathsf{a}}_i \int \frac{\mathrm{d}t_i\,\mathrm{d}\tilde{t}_i}{t_i\,\tilde{t}_i\omega_i}\,\delta^{2|2}(\lambda_i - t_i\lambda(\sigma_i))\,\delta^{2|2}(\tilde{\lambda}_i - \tilde{t}_i\tilde{\lambda}(\sigma_i))\,\mathrm{e}^{t_i\tilde{t}_i\omega_i u(\sigma_i)}\,. \quad (5.41)$$

In a fixed colour-ordering, the Ward identity (5.39) then becomes

$$\frac{a^\alpha}{\langle a\,s\rangle}\left(\left\langle \frac{\lambda_\alpha(\sigma_1)}{\langle s\,\lambda(\sigma_1)\rangle} c_1 \operatorname{tr}(O\, j)_1\cdots\right\rangle - \left\langle \frac{\lambda_\alpha(\sigma_n)}{\langle s\,\lambda(\sigma_n)\rangle} c_1 \operatorname{tr}(O\, j)_1\cdots\right\rangle\right) \\ = \left(\frac{\langle a\,1\rangle}{\langle a\,s\rangle\,\langle s\,1\rangle} - \frac{\langle a\,n\rangle}{\langle a\,s\rangle\,\langle s\,n\rangle}\right)\mathscr{A}_{n,d} = \frac{\langle 1\,n\rangle}{\langle s\,1\rangle\,\langle s\,n\rangle}\mathscr{A}_{n,d}\,, \quad (5.42)$$

which is precisely the soft gluon theorem [138]. So the soft gluon theorem is realized (for a positive helicity soft gluon) by the action of a holomorphic Kac-Moody current generating a 'large' gauge transformation at $\mathscr{I}_{\mathbb{C}}$, which corresponds to the insertion of a soft gluon broadcasting function at $(\lambda_s, \tilde{\lambda}_s) \in \mathbb{CP}^1 \times \mathbb{CP}^1$. Analogous to the gravitational model introduced above, more general 'large' gauge transformations will be related to the insertion of other soft particles in the spectrum of $\mathcal{N} = 4$ super-Yang-Mills or to multiple such soft insertions.

Following the discovery of subleading soft factors for gravity [139, 140], a similar subleading soft factor for gauge theory amplitudes was found [149]. This subleading soft theorem has recently been derived from an asymptotic symmetry perspective for Abelian gauge group, where the associated symmetry is not simply a gauge transformation, but also acts as a vector field on the sphere of null generators at $\mathscr{I}$ [152]. It seems natural that the subleading soft theorem is generated by a similar symmetry in the non-Abelian setting.

It is straightforward to write charges generating such rotations on the (complexified) sphere of null generators:

$$\mathscr{J}_V = \oint V^{\dot{\alpha}}(\lambda, \tilde{\lambda})\tilde{v}_{\dot{\alpha}}(\sigma)\,\mathrm{tr}\big(\mathsf{T}\, j(\sigma)\big)\,,$$

where $V^{\dot{\alpha}}$ must take values in $\mathcal{O}(0,-1)$ on $\mathscr{I}_{\mathbb{C}}$ and have conformal weight $(-1,0)$. There is the possibility for a rotation in the $\lambda$-direction which has been dropped for simplicity. A particularly interesting choice for $V^{\dot{\alpha}}$ is

$$V^{\dot{\alpha}}(\lambda, \tilde{\lambda}) = \frac{\tilde{\lambda}_s^{\dot{\alpha}}}{w(\sigma)\,\langle s\,\lambda(\sigma)\rangle}\,. \tag{5.43}$$

As above there is a Ward identity associated with the action of $\mathscr{J}_V$ inside the correlator (5.36). Once more, the vector $V^{\dot{\alpha}}$ has poles which ensure that it is not an exact symmetry of the theory. In order to evaluate the Ward identity when all external states are in momentum eigenstate representations (5.41), note that the path integral over non-zero modes of $u(\sigma)$ can be performed explicitly, fixing [39]

$$w(\sigma) = \sum_{i=1}^{n} t_i\,\tilde{t}_i\,\omega_i\,\frac{\mathrm{D}\sigma_i}{(\sigma\,\sigma_i)}\prod_{a=0}^{d+\tilde{d}}\frac{(p_a\,\sigma_i)}{(p_a\,\sigma)}\,,$$

where $(\sigma_i\,\sigma_j)$ is the SL(2,$\mathbb{C}$)-invariant inner product $\epsilon_{\underline{\alpha\beta}}\sigma_i^{\underline{\alpha}}\sigma_j^{\underline{\beta}}$ for homogeneous coordinates on $\Sigma$, $\mathrm{D}\sigma_i = (\sigma_i\,\mathrm{d}\sigma_i)$ is the weight $+2$ holomorphic measure on $\Sigma$, and the $\{p_a\} \subset \Sigma$ are an arbitrary collection of $d + \tilde{d} + 1$ points.

Taking the usual contour to pick out the poles, and making a choice of colour-ordering as before, the Ward identity reads:

$$\left\langle \mathscr{J}_V\, c_1\, \mathrm{tr}(O\,j)_1 \cdots \int_\Sigma \chi_n\, \mathrm{tr}(\dot{O}\,j)_n \right\rangle = \left( \frac{\tilde{\lambda}_s^{\dot\alpha}}{\omega_1 \langle s\,1\rangle} \frac{\partial}{\partial \tilde{\lambda}_1^{\dot\alpha}} - \frac{\tilde{\lambda}_s^{\dot\alpha}}{\omega_n \langle s\,n\rangle} \frac{\partial}{\partial \tilde{\lambda}_n^{\dot\alpha}} \right) \mathscr{A}_{n,d}\,, \tag{5.44}$$

which is the subleading soft factor for a positive helicity gluon inserted between particles 1 and $n$ in the colour ordering [149]. In the Abelian case, this is equivalent to the Ward identity used to derive Low's subleading soft theorem [152]. For general gauge group, (5.44) explicitly confirms that the subleading gluon soft factor is related to the action of vector fields on the conformal two-sphere.

## 5.8 Scattering amplitudes in the YM model

At this point, only correlators involving gauge-theoretic vertex operators (5.32) encoding the broadcasting data of the gauge field were considered. On the Riemann sphere this restriction is consistent and corresponds to isolating single trace contributions in the colour structure. The Ward identity (5.39) establishes that charges implementing the action of asymptotic 'large' gauge transformations act on these correlators in a natural way, implying the soft gluon theorem. This extends to other charges, acting as rotations on the space of generators of $\mathscr{I}_{\mathbb{C}}$, which give the subleading soft theorem. What is left to do is to actually evaluate the correlator (5.36), and discuss the role of the other states in the theory given by vertex operators (5.34).

To evaluate (5.36), consider all vertex operator insertions to be given by the momentum eigenstates (5.41). The only non-trivial Wick contractions between the vertex operators are in the worldsheet current algebra; using (5.33) and choosing a specific colour-ordering this leads to the usual Parke-Taylor factor:

$$\left\langle \prod_{i=1}^{n} j^{\mathsf{a}_i}(\sigma_i) \right\rangle = \mathrm{tr}\left(\mathsf{T}^{\mathsf{a}_1} \cdots \mathsf{T}^{\mathsf{a}_n}\right) \prod_{i=1}^{n} \frac{\mathrm{D}\sigma_i}{(\sigma_i\, \sigma_{i+1})}\,.$$

The remainder of the correlator is given by the zero mode integrals for the various worldsheet fields, with the result:

$$\mathscr{A}_{n,d} = \mathrm{tr}\left(\mathsf{T}^{\mathsf{a}_1}\cdots\mathsf{T}^{\mathsf{a}_n}\right)\int \frac{\prod_{r=0}^{d}\mathrm{d}^{2|2}\lambda_r^0\prod_{s=0}^{\tilde d}\mathrm{d}^{2|2}\tilde\lambda_s^0}{\mathrm{vol}(\mathbb{C}^*\times\mathbb{C}^*)}\frac{|\sigma_1\sigma_2\sigma_3|}{\mathrm{D}\sigma_1\mathrm{D}\sigma_2\mathrm{D}\sigma_3}\frac{|\sigma_2\cdots\sigma_n|}{t_1\tilde t_1\omega_1}$$
$$\prod_{a=0}^{d+\tilde d}\delta\left(\sum_{i=1}^n t_i\tilde t_i\omega_i\,\mathfrak{s}_a(\sigma_i)\right)\prod_{i=1}^n\frac{\mathrm{D}\sigma_i}{(\sigma_i\sigma_{i+1})}\mathrm{d}t_i\,\mathrm{d}\tilde t_i\,\delta^{2|2}(\lambda_i - t_i\lambda(\sigma_i))\,\delta^{2|2}(\tilde\lambda_i-\tilde t_i\tilde\lambda(\sigma_i)) \quad (5.45)$$

In this expression, the measure in the first line is over the zero modes of the maps $\lambda_A(\sigma),\tilde\lambda_{\dot A}(\sigma):\Sigma\to\mathbb{CP}^{1|2}$, the two $\mathbb{C}^*$-freedoms are associated with the scalings of $\mathscr{L}$ and $\tilde{\mathscr{L}}$, and the Vandermonde determinants

$$\frac{|\sigma_1\sigma_2\sigma_3|}{\mathrm{D}\sigma_1\,\mathrm{D}\sigma_2\,\mathrm{D}\sigma_3}=\frac{(\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_3\sigma_1)}{\mathrm{D}\sigma_1\mathrm{D}\sigma_2\mathrm{D}\sigma_3},\qquad |\sigma_2\cdots\sigma_n|=\prod_{2\le i<j\le n}(\sigma_i\sigma_j),$$

are given by the 3 zero modes of $c$ and $n-1$ zero modes of $\chi$, respectively. The delta functions in the second line are from momentum eigenstate insertions and the integral over $u(\sigma)$ zero modes:

$$\int\prod_{a=0}^{d+\tilde d}\mathrm{d}u_a^0\,\exp\left[\sum_{i=1}^n t_i\tilde t_i\omega_i\,u(\sigma_i)\right]=\prod_{a=0}^{d+\tilde d}\delta\left(\sum_{i=1}^n t_i\tilde t_i\omega_i\,\mathfrak{s}_a(\sigma_i)\right),$$

where $\{\mathfrak{s}_a\}$ form a basis of $H^0(\Sigma,\mathscr{L}\otimes\tilde{\mathscr{L}})$.

This expression can be manipulated into a more recognizable form by using the identity (see [31, 44, 45])

$$\prod_{a=0}^{d+\tilde d}\delta\left(\sum_{i=1}^n t_i\tilde t_i\omega_i\,\mathfrak{s}_a(\sigma_i)\right)=\frac{1}{|\sigma_1\cdots\sigma_n|}\int\mathrm{d}r\prod_{i=1}^n\delta\left(t_i\tilde t_i\omega_i-\frac{r}{\prod_{j\neq i}(\sigma_i\sigma_j)}\right).$$

The result,

$$\mathscr{A}_{n,d} = \mathrm{tr}\left(\mathsf{T}^{\mathsf{a}_1}\cdots\mathsf{T}^{\mathsf{a}_n}\right)\int \frac{\prod_{r=0}^{d}\mathrm{d}^{2|2}\lambda_r^0\prod_{s=0}^{\tilde d}\mathrm{d}^{2|2}\tilde\lambda_s^0}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*\times\mathbb{C}^*)}\frac{\mathrm{d}r}{r}\prod_{i=1}^n\delta\left(t_i\tilde t_i\omega_i-\frac{r}{\prod_{j\neq i}(\sigma_i\sigma_j)}\right)$$
$$\times\frac{\mathrm{D}\sigma_i}{(\sigma_i\sigma_{i+1})}\mathrm{d}t_i\,\mathrm{d}\tilde t_i\,\delta^{2|2}(\lambda_i-t_i\lambda(\sigma_i))\,\delta^{2|2}(\tilde\lambda_i-\tilde t_i\tilde\lambda(\sigma_i)) \quad (5.46)$$

is equal to the parity-invariant form of the Roiban-Spradlin-Volovich [45] expression for the classical S-matrix of $\mathscr{N}=4$ super-Yang-Mills given by Witten [31]. Thus, the correlators (5.36) of the YM model reproduce all tree-level amplitudes of gauge theory, confirming the interpretation of the Ward identities in the previous section.

Now consider the gravitational degrees of freedom in this model, corresponding to the vertex operators (5.34). While consistently omitted from single trace gauge theory interactions at tree-level, these states can mediate multi-trace tree-level amplitudes in the gauge theory and would also run in any amplitudes computed to higher-order in perturbation theory. The claim is that these degrees of freedom correspond to a non-unitary theory of gravity with fourth-order equations of motion.

The non-unitary nature of the gravitational degrees of freedom is manifest by considering a multi-trace correlator where all external states are given by gauge-theoretic vertex operators (5.32). Since the only non-trivial Wick contractions between these operator insertions are in the worldsheet current algebra, the arguments of [61] ensure that this double trace is mediated by gravitational degrees of freedom with fourth-order equations of motion. As an explicit example, consider a double trace correlator of $n$ external states, with $n_1$ in one trace and $n_2$ in the other ($n_1 + n_2 = n$). It is straightforward to show that such a correlator can be written (for momentum eigenstates) as:

$$\mathrm{tr}\left(\mathsf{T}^{\mathsf{a}_1}\cdots\mathsf{T}^{\mathsf{a}_{n_1}}\right)\mathrm{tr}\left(\mathsf{T}^{\mathsf{b}_1}\cdots\mathsf{T}^{\mathsf{b}_{n_2}}\right)\int\frac{\prod_{r=0}^{d}\mathrm{d}^{2|2}\lambda_r^0\,\prod_{s=0}^{\tilde{d}}\mathrm{d}^{2|2}\tilde{\lambda}_s^0}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*\times\mathbb{C}^*)}\,\frac{\mathrm{d}r}{r}$$
$$\times\prod_{j=1}^{n_1}\frac{\mathrm{D}\sigma_j}{(\sigma_j\,\sigma_{j+1})}\,\mathrm{d}t_j\,\mathrm{d}\tilde{t}_j\,\delta\left(t_j\tilde{t}_j\omega_j-\frac{r}{\prod_{l\neq j}(\sigma_j\,\sigma_l)}\right)\delta^{2|2}(\lambda_j-t_j\lambda(\sigma_j))\,\delta^{2|2}(\tilde{\lambda}_j-\tilde{t}_j\tilde{\lambda}(\sigma_j))$$
$$\times\prod_{k=1}^{n_2}\frac{\mathrm{D}\sigma_k}{(\sigma_k\,\sigma_{k+1})}\,\mathrm{d}t_k\,\mathrm{d}\tilde{t}_k\,\delta\left(t_k\tilde{t}_k\omega_k-\frac{r}{\prod_{m\neq k}(\sigma_k\,\sigma_m)}\right)\delta^{2|2}(\lambda_k-t_k\lambda(\sigma_k))\,\delta^{2|2}(\tilde{\lambda}_k-\tilde{t}_k\tilde{\lambda}(\sigma_k))\,.$$

Consider the limit where this correlator factorizes without splitting the colour traces: the worldsheet $\Sigma$ degenerates into two Riemann spheres $\Sigma_1$ and $\Sigma_2$ attached at a node, with vertex operators $1,\ldots,n_1$ on $\Sigma_1$ and $1,\ldots,n_2$ on $\Sigma_2$. In this limit the only states flowing through the factorization channel are gravitational. The worldsheet can be modelled on a conic in $\mathbb{CP}^2$ with a complex parameter $q$ controlling the degeneration; in the $q\to 0$ limit, $\Sigma$ factorizes into $\Sigma_1\cup\Sigma_2$, with the marked points $\sigma_x\in\Sigma_1$ and $\sigma_y\in\Sigma_2$ identified at the node.

Standard arguments (see Appendix C of [61]) show that in the $q\to 0$ limit, the portion of the correlator encoding the trace structure factorizes as

$$\frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_{j=1}^{n_1}\frac{\mathrm{D}\sigma_j}{(\sigma_j\,\sigma_{j+1})}\prod_{k=1}^{n_2}\frac{\mathrm{D}\sigma_k}{(\sigma_k\,\sigma_{k+1})}$$
$$\to\frac{\mathrm{d}q}{q^2}\left(\frac{\mathrm{D}\sigma_x}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_{j=1}^{n_1}\frac{\mathrm{D}\sigma_j}{(\sigma_j\,\sigma_{j+1})}\right)\left(\frac{\mathrm{D}\sigma_y}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_{j=1}^{n_2}\frac{\mathrm{D}\sigma_k}{(\sigma_k\,\sigma_{k+1})}\right)+O(q^{-1})\,.$$

Due to the $\mathcal{N}=4$ supersymmetry in play, all other parts of the correlator are homogeneous, introducing no additional powers of $q$. The various delta functions can be used to show that in the factorization limit, $q$ scales as the square of the total momentum flowing through the channel. Thus, the presence of a double pole in $q$ indicates a momentum space propagator of the form $p^{-4}$, as expected for a theory with fourth-order equations of motion.

While this factorization argument confirms that the gravitational vertex operators of the YM model do not correspond to Einstein gravity, a more precise statement can be made by considering correlators of the operators (5.34) themselves. These give gravitational interactions which are consistent with a particular *non-minimal* $\mathcal{N}=4$ conformal supergravity [172] arising in the twistor-strings of Witten and Berkovits [2–4].[10] In the YM model conformal invariance is explicitly broken by the choice of target space: $\mathscr{I}_{\mathbb{C}}$ is topologically distinct from the conformal boundary of (anti-)de Sitter space, which is a conformally equivalent bulk space-time. Minkowski space *is* a vacuum solution to the conformal gravity equations of motion, though, so the vertex operators (5.34) can be thought of as linearised perturbations around this fixed background conformal structure.

Low-point amplitudes of non-minimal conformal supergravity have been calculated in the context of twistor-string theory [4, 175–177], and the structure of the vertex operators makes it clear that the YM model will reproduce those amplitudes in a parity-symmetric form, analogous to the gauge theory calculation above. For instance, the $n=3$, $d=0$ correlator

$$\left\langle \prod_{i=1}^{3} c_i\, v_i^A\, v_{i\,A} \prod_{j=2,3} \oint_{\sigma_j} w(\sigma)\,\chi(\sigma) \right\rangle \\ = \int \frac{\mathrm{d}^{2|2}\lambda^0 \prod_{s=0,1} \mathrm{d}^{2|2}\tilde{\lambda}^0_s \prod_{t=0}^{2} \mathrm{d}u^0_t}{\mathrm{vol}\,(\mathbb{C}^* \times \mathbb{C}^*)} \left( \frac{\partial v_{1\,B}}{\partial \lambda_A} \frac{\partial \dot{v}_{2\,C}}{\partial \lambda_B} \frac{\partial \dot{v}_{3\,A}}{\partial \lambda_C} - \frac{\partial v_{1\,B}}{\partial \lambda_C} \frac{\partial \dot{v}_{2\,C}}{\partial \lambda_A} \frac{\partial \dot{v}_{3\,A}}{\partial \lambda_B} \right),$$

is non-vanishing, and corresponds to a cubic interaction between two conformal gravitons and a conformal scalar in Minkowski space. This interaction is forbidden in minimal $\mathcal{N}=4$ conformal supergravity by a global $\mathrm{SU}(1,1)$ symmetry acting on the conformal scalar [172, 178].

Another test of non-minimality is given by embedding Einstein degrees of freedom inside the gravitational vertex operators. Fixing a conformal structure to perform this embedding corresponds to picking a cosmological constant, $\Lambda$; in minimal conformal super-

[10] 'Non-minimal' refers to the presence of interaction terms between scalars and two Weyl tensors in the space-time action [173]. The existence of this non-minimal theory at the quantum level is questionable due to $\mathrm{SU}(4)$ axial anomaly calculations [174], but here only tree-level observables were used.

gravity, correlators of the embedded Einstein operators will be $O(\Lambda)$ polynomials [179, 180]. In this model, such an embedding can be given by taking linear combinations of (5.34):

$$v^A\, v_A + \tilde{v}^{\dot{A}}\, \tilde{v}_{\dot{A}} \to \Lambda v^A \frac{\partial h}{\partial \lambda_A} + \tilde{v}^{\dot{A}} \frac{\partial \tilde{h}}{\partial \tilde{\lambda}_{\dot{A}}}, \qquad \partial \lambda_A\, g^A + \partial \tilde{\lambda}_{\dot{A}}\, \tilde{g}^{\dot{A}} \to \langle \partial \lambda\, \lambda \rangle\, G + \Lambda\, [\partial \tilde{\lambda}\, \tilde{\lambda}]\, \tilde{G},$$

where $h$, $\tilde{h}$, $G$, and $\tilde{G}$ encode Einstein degrees of freedom. For example, in the expansion

$$h = f_{(-2,0)} + \cdots + (\eta)^2 N_{(-4,0)} + \cdots + (\tilde{\eta})^2 \phi_{(-2,-2)} + \cdots + (\eta)^2 (\tilde{\eta})^2 \tilde{f}_{(-4,-2)}$$

the component $N_{(-4,0)}$ is the news function of a positive helicity Einstein graviton, while $f_{(-2,0)}$, $\tilde{f}_{(-4,-2)}$ and $\phi_{(-2,-2)}$ encode the radiative degrees of freedom for photons and scalars in $\mathcal{N} = 4$ supergravity. It is easy to see that correlators of these operators will have an $O(\Lambda^0)$ piece, indicating non-minimal structure.[11]

While this hardly suffices to establish that the gravitational interactions of the YM model are *equivalent* to non-minimal conformal supergravity, it does seem to indicate that this theory is at least a subsector of the model (presented in a conformally broken target space framework). Combined with other obvious similarities to the Berkovits-Witten twistor-string, this suggests that there could be a transcription (in some sense) of this model from $\mathscr{I}_{\mathbb{C}}$ to twistor space.

It is easy to see that correlators of these operators will have an $O(\Lambda^0)$ piece, indicating non-minimal structure. Furthermore, the correlators

$$\left\langle \prod_{i=1}^{d+1} c_i (\langle \partial \lambda\, \lambda \rangle\, G + \Lambda\, [\partial \tilde{\lambda}\, \tilde{\lambda}]\, \tilde{G})_i \prod_{j=d+2}^{n} c_j (\Lambda v^A \frac{\partial h}{\partial \lambda_A} + \tilde{v}^{\dot{A}} \frac{\partial \tilde{h}}{\partial \tilde{\lambda}_{\dot{A}}})_j \prod_{k=4}^{n} (b|\mu)_k \prod_{l=2}^{n} \oint_{\sigma_l} w(\sigma)\, \chi(\sigma) \right\rangle,$$

which would lead to $\mathrm{N}^{d-1}$MHV amplitudes in minimal conformal supergravity are structurally equivalent to those produced by the non-minimal Berkovits-Witten model in the $\Lambda \to 0$ limit [181].

---

[11] The $\Lambda \to 0$ limit of certain correlators also matches those produced by the non-minimal Berkovits-Witten model at arbitrary $n$ and $d$ [181], though there are other correlators whose interpretation is less clear.

# Chapter 6

# Conclusion

I have shown in this thesis that there's much to be gained from pushing forwards the twistor string approach to scattering amplitudes. There are now many variants of the twistor string, and many more twistor string inspired formulas, which share the as a basic ingredient the scattering equations. Among them the type II ambitwistor string, either in the RNS or pure spinor guise, stands out as giving a geometrical interpretation of the scattering equations as enforcing the worldsheet to be mapped into ambitwistor space. As shown in this thesis, this geometrical view is key in generalizing the scattering equations to loop level and to curved spacetimes, giving more heft to the claim that it gives an alternative description of type II supergravity.

I have also shown that these worldsheet methods have applications to the study of asymptotic symmetries of the S-matrix and flat space holography, at least in some specific cases. Given the connection between scri and asymptotic twistor space it might have been expected that twistor string methods might be applied in these cases, but it is still intriguing that a theory living completely on scri can reproduce the tree-level S-matrix of $\mathcal{N} = 4$ sYM and $\mathcal{N} = 8$ sugra. There's also a connection between ambitwistor space and the cotangent bundle to scri [39] which would be interesting to study further.

The work done here is a small step in furthering our understanding of QFTs and there are many, many things left to be done. Particularly, in the subject of twistor strings there has been many exciting recent developments. Among them are: the great number of theories described in the CHY formalism [14, 15], new and simpler loop formulas [41], a geometrical understanding of the ambitwistor string through localization [33], an elucidation of the role of twistors in superstring theory [83], and others. It seems that all of those should somehow come together under some unified description or framework, though at this point it is not clear exactly how. The next few years promise a lot of new surprises and insights, and I look forward to it.

# References


[1] H. Elvang and Y.-t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

[2] E. Witten, "Perturbative gauge theory as a string theory in twistor space," *Commun.Math.Phys.* **252** (2004) 189–258, arXiv:hep-th/0312171 [hep-th].

[3] N. Berkovits, "An Alternative string theory in twistor space for N=4 superYang-Mills," *Phys.Rev.Lett.* **93** (2004) 011601, arXiv:hep-th/0402045 [hep-th].

[4] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory," *JHEP* **0408** (2004) 009, arXiv:hep-th/0406051 [hep-th].

[5] T. Adamo, E. Casali, and D. Skinner, "Ambitwistor strings and the scattering equations at one loop," *JHEP* **1404** (2014) 104, arXiv:1312.3828 [hep-th].

[6] T. Adamo and E. Casali, "Scattering equations, supergravity integrands, and pure spinors," *JHEP* **05** (2015) 120, arXiv:1502.06826 [hep-th].

[7] E. Casali and P. Tourkine, "Infrared behaviour of the one-loop scattering equations and supergravity integrands," *JHEP* **1504** (2015) 013, arXiv:1412.3787 [hep-th].

[8] T. Adamo and E. Casali, "Perturbative gauge theory at null infinity," *Phys.Rev.* **D91** no. 12, (2015) 125022, arXiv:1504.02304 [hep-th].

[9] T. Adamo, E. Casali, and D. Skinner, "Perturbative gravity at null infinity," *Class.Quant.Grav.* **31** no. 22, (2014) 225008, arXiv:1405.5122 [hep-th].

[10] T. Adamo, E. Casali, and D. Skinner, "A Worldsheet Theory for Supergravity," *JHEP* **1502** (2015) 116, arXiv:1409.5656 [hep-th].

[11] F. Cachazo, S. He, and E. Y. Yuan, "Scattering equations and Kawai-Lewellen-Tye orthogonality," *Phys.Rev.* **D90** no. 6, (2014) 065001, arXiv:1306.6575 [hep-th].

[12] F. Cachazo, S. He, and E. Y. Yuan, "Scattering of Massless Particles in Arbitrary Dimensions," *Phys.Rev.Lett.* **113** no. 17, (2014) 171601, arXiv:1307.2199 [hep-th].

[13] F. Cachazo, S. He, and E. Y. Yuan, "Scattering of Massless Particles: Scalars, Gluons and Gravitons," *JHEP* **1407** (2014) 033, arXiv:1309.0885 [hep-th].

[14] F. Cachazo, S. He, and E. Y. Yuan, "Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM," arXiv:1412.3479 [hep-th].

[15] F. Cachazo, S. He, and E. Y. Yuan, "Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations," *JHEP* **1501** (2015) 121, arXiv:1409.8256 [hep-th].

[16] N. E. J. Bjerrum-Bohr, P. H. Damgaard, P. Tourkine, and P. Vanhove, "Scattering Equations and String Theory Amplitudes," *Phys.Rev.* **D90** no. 10, (2014) 106002, arXiv:1403.4553 [hep-th].

[17] L. Dolan and P. Goddard, "Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension," *JHEP* **1405** (2014) 010, arXiv:1311.5200 [hep-th].

[18] F. Cachazo and H. Gomez, "Computation of Contour Integrals on $\mathscr{M}_{0,n}$," arXiv:1505.03571 [hep-th].

[19] C. Baadsgaard, N. Bjerrum-Bohr, J. L. Bourjaily, and P. H. Damgaard, "Integration Rules for Scattering Equations," arXiv:1506.06137 [hep-th].

[20] C. Baadsgaard, N. Bjerrum-Bohr, J. L. Bourjaily, and P. H. Damgaard, "Scattering Equations and Feynman Diagrams," arXiv:1507.00997 [hep-th].

[21] C. Kalousios, "Scattering equations, generating functions and all massless five point tree amplitudes," *JHEP* **1505** (2015) 054, arXiv:1502.07711 [hep-th].

[22] C. Kalousios, "Massless scattering at special kinematics as Jacobi polynomials," *J.Phys.* **A47** (2014) 215402, arXiv:1312.7743 [hep-th].

[23] C. Lam, "Permutation Symmetry of the Scattering Equations," *Phys.Rev.* **D91** no. 4, (2015) 045019, arXiv:1410.8184 [hep-th].

[24] E. Witten, "Superstring Perturbation Theory Revisited," arXiv:1209.5461 [hep-th].

[25] J. Fay, *Theta Functions on Riemann Surfaces.* Lecture Notes in Mathematics. Springer, 1973.

[26] D. Fairlie and D. Roberts, "DUAL MODELS WITHOUT TACHYONS - A NEW APPROACH." unpublished, 1972.

[27] D. Roberts, *Mathematical Structure of Dual Amplitudes.* PhD thesis, 1972. http://etheses.dur.ac.uk/8662/.

[28] D. B. Fairlie, "A Coding of Real Null Four-Momenta into World-Sheet Co-ordinates," *Adv.Math.Phys.* **2009** (2009) 284689, arXiv:0805.2263 [hep-th].

[29] D. J. Gross and P. F. Mende, "The High-Energy Behavior of String Scattering Amplitudes," *Phys.Lett.* **B197** (1987) 129.

[30] D. J. Gross and P. F. Mende, "String Theory Beyond the Planck Scale," *Nucl.Phys.* **B303** (1988) 407.

[31] E. Witten, "Parity invariance for strings in twistor space," *Adv.Theor.Math.Phys.* **8** (2004) 779–796, arXiv:hep-th/0403199 [hep-th].

[32] L. Mason and D. Skinner, "Ambitwistor strings and the scattering equations," *JHEP* **1407** (2014) 048, arXiv:1311.2564 [hep-th].

[33] K. Ohmori, "Worldsheet Geometries of Ambitwistor String," *JHEP* **1506** (2015) 075, arXiv:1504.02675 [hep-th].

[34] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond.* Cambridge University Press, 1998.

[35] C. LeBrun, "Spaces of Complex Null Geodesics in Complex Riemannanian Geometry," *Trans. Amer. Math. Soc.* **278** (1983) 209.

[36] R. Baston and L. Mason, "Conformal Gravity, the Einstein Equations and Spaces of Complex Null Geodesics," *Class. Quant. Grav.* **4** (1987) 815–826.

[37] E. Casali, Y. Geyer, L. Mason, R. Monteiro, and K. A. Roehrig, "New Ambitwistor String Theories," arXiv:1506.08771 [hep-th].

[38] Y. Geyer, A. E. Lipstein, and L. J. Mason, "Ambitwistor Strings in Four Dimensions," *Phys.Rev.Lett.* **113** no. 8, (2014) 081602, arXiv:1404.6219 [hep-th].

[39] Y. Geyer, A. E. Lipstein, and L. Mason, "Ambitwistor strings at null infinity and (subleading) soft limits," *Class.Quant.Grav.* **32** no. 5, (2015) 055003, arXiv:1406.1462 [hep-th].

[40] A. E. Lipstein, "Soft Theorems from Conformal Field Theory," *JHEP* **1506** (2015) 166, arXiv:1504.01364 [hep-th].

[41] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, "Loop Integrands from the Riemann Sphere," arXiv:1507.00321 [hep-th].

[42] T. Adamo, E. Casali, K. A. Roehrig, and D. Skinner, "All tree amplitudes of supersymmetric Einstein-Yang-Mills theory," arXiv:1507.02207 [hep-th].

[43] D. Skinner, "Twistor Strings for N=8 Supergravity," arXiv:1301.0868 [hep-th].

[44] F. Cachazo, "Resultants and Gravity Amplitudes," arXiv:1301.3970 [hep-th].

[45] R. Roiban, M. Spradlin, and A. Volovich, "On the tree level S matrix of Yang-Mills theory," *Phys.Rev.* **D70** (2004) 026009, arXiv:hep-th/0403190 [hep-th].

[46] D. Friedan, E. J. Martinec, and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," *Nucl.Phys.* **B271** (1986) 93.

[47] D. Friedan, S. H. Shenker, and E. J. Martinec, "Covariant Quantization of Superstrings," *Phys.Lett.* **B160** (1985) 55.

[48] N. Berkovits, "Super Poincare covariant quantization of the superstring," *JHEP* **0004** (2000) 018, arXiv:hep-th/0001035 [hep-th].

[49] N. Berkovits, "Infinite Tension Limit of the Pure Spinor Superstring," *JHEP* **1403** (2014) 017, arXiv:1311.4156 [hep-th].

[50] N. Berkovits, “ICTP lectures on covariant quantization of the superstring,” arXiv:hep-th/0209059 [hep-th].

[51] O. A. Bedoya and N. Berkovits, “GGI Lectures on the Pure Spinor Formalism of the Superstring,” arXiv:0910.2254 [hep-th].

[52] N. Berkovits, “Covariant quantization of the superparticle using pure spinors,” *JHEP* **0109** (2001) 016, arXiv:hep-th/0105050 [hep-th].

[53] J. P. Harnad and S. Shnider, “Constraints and field equations for ten-dimensional super Yang-Mills theory,” *Commun.Math.Phys.* **106** (1986) 183.

[54] E. Witten, “Twistor - Like Transform in Ten-Dimensions,” *Nucl.Phys.* **B266** (1986) 245.

[55] C. R. Mafra, O. Schlotterer, S. Stieberger, and D. Tsimpis, “A recursive method for SYM n-point tree amplitudes,” *Phys.Rev.* **D83** (2011) 126012, arXiv:1012.3981 [hep-th].

[56] C. R. Mafra, O. Schlotterer, and S. Stieberger, “Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation,” *Nucl.Phys.* **B873** (2013) 419–460, arXiv:1106.2645 [hep-th].

[57] J. Broedel, O. Schlotterer, and S. Stieberger, “Polylogarithms, Multiple Zeta Values and Superstring Amplitudes,” *Fortsch.Phys.* **61** (2013) 812–870, arXiv:1304.7267 [hep-th].

[58] H. Gomez and E. Y. Yuan, “N-point tree-level scattering amplitude in the new Berkovits‘ string,” *JHEP* **1404** (2014) 046, arXiv:1312.5485 [hep-th].

[59] E. D’Hoker and D. Phong, “The Geometry of String Perturbation Theory,” *Rev.Mod.Phys.* **60** (1988) 917.

[60] J. Polchinski, “Factorization of Bosonic String Amplitudes,” *Nucl.Phys.* **B307** (1988) 61.

[61] T. Adamo, “Worldsheet factorization for twistor-strings,” *JHEP* **1404** (2014) 080, arXiv:1310.8602 [hep-th].

[62] A. Yamada, “Precise variational formulas for abelian differentials,” *Kodai Math. J.* **3** no. 1, (1980) 114–143. http://projecteuclid.org/euclid.kmj/1138036124.

[63] M. P. Tuite and A. Zuevsky, “The Szĕgo kernel on a sewn Riemann surface,” *Comm. Math. Phys.* **306** no. 3, (2011) 617–645. http://dx.doi.org/10.1007/s00220-011-1310-1.

[64] E. D’Hoker and D. Phong, “Lectures on two loop superstrings,” *Conf.Proc.* **C0208124** (2002) 85–123, arXiv:hep-th/0211111 [hep-th].

[65] S. Badger, N. Bjerrum-Bohr, and P. Vanhove, “Simplicity in the Structure of QED and Gravity Amplitudes,” *JHEP* **0902** (2009) 038, arXiv:0811.3405 [hep-th].

[66] D. Mumford, *Tata lectures on theta. I,*. Birkhäuser Boston Inc., Boston, MA, 1983.

[67] M. B. Green, J. Schwarz, and E. Witten, "SUPERSTRING THEORY. VOL. 1: INTRODUCTION," *Cambridge Monogr.Math.Phys.* (1987) .

[68] M. B. Green, J. H. Schwarz, and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," *Nucl.Phys.* **B198** (1982) 474–492.

[69] H. Gomez and C. R. Mafra, "The closed-string 3-loop amplitude and S-duality," *JHEP* **1310** (2013) 217, `arXiv:1308.6567 [hep-th]`.

[70] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," *JHEP* **0409** (2004) 047, `arXiv:hep-th/0406055 [hep-th]`.

[71] N. Berkovits, "Relating the RNS and pure spinor formalisms for the superstring," *JHEP* **0108** (2001) 026, `arXiv:hep-th/0104247 [hep-th]`.

[72] C. R. Mafra, "Four-point one-loop amplitude computation in the pure spinor formalism," *JHEP* **0601** (2006) 075, `arXiv:hep-th/0512052 [hep-th]`.

[73] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," *JHEP* **0601** (2006) 005, `arXiv:hep-th/0503197 [hep-th]`.

[74] N. Berkovits and C. R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," *Phys.Rev.Lett.* **96** (2006) 011602, `arXiv:hep-th/0509234 [hep-th]`.

[75] C. R. Mafra, "Superstring Scattering Amplitudes with the Pure Spinor Formalism," `arXiv:0902.1552 [hep-th]`.

[76] N. Berkovits, "Pure spinor formalism as an N=2 topological string," *JHEP* **0510** (2005) 089, `arXiv:hep-th/0509120 [hep-th]`.

[77] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," *JHEP* **0612** (2006) 029, `arXiv:hep-th/0609012 [hep-th]`.

[78] J. Bjornsson, "Multi-loop amplitudes in maximally supersymmetric pure spinor field theory," *JHEP* **1101** (2011) 002, `arXiv:1009.5906 [hep-th]`.

[79] T. Kugo and I. Ojima, "Local Covariant Operator Formalism of Nonabelian Gauge Theories and Quark Confinement Problem," *Prog.Theor.Phys.Suppl.* **66** (1979) 1–130.

[80] G. Rybkin, "STATE SPACE IN BRST QUANTIZATION AND KUGO-OJIMA QUARTETS," *Int.J.Mod.Phys.* **A6** (1991) 1675–1692.

[81] J. Bjornsson and M. B. Green, "5 loops in 24/5 dimensions," *JHEP* **1008** (2010) 132, `arXiv:1004.2692 [hep-th]`.

[82] N. Berkovits, "Twistor Origin of the Superstring," *JHEP* **1503** (2015) 122, `arXiv:1409.2510 [hep-th]`.

[83] N. Berkovits, "Origin of the Pure Spinor and Green-Schwarz Formalisms," arXiv:1503.03080 [hep-th].

[84] I. Oda and M. Tonin, "Y-formalism and b ghost in the non-minimal pure spinor formalism of superstrings," *Nucl.Phys.* **B779** (2007) 63–100, arXiv:0704.1219 [hep-th].

[85] N. Berkovits and S. A. Cherkis, "Higher-dimensional twistor transforms using pure spinors," *JHEP* **0412** (2004) 049, arXiv:hep-th/0409243 [hep-th].

[86] Y. Aisaka and N. Berkovits, "Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization," *JHEP* **0907** (2009) 062, arXiv:0903.3443 [hep-th].

[87] P. A. Grassi and P. Vanhove, "Higher-loop amplitudes in the non-minimal pure spinor formalism," *JHEP* **0905** (2009) 089, arXiv:0903.3903 [hep-th].

[88] J. Hoogeveen and K. Skenderis, "BRST quantization of the pure spinor superstring," *JHEP* **0711** (2007) 081, arXiv:0710.2598 [hep-th].

[89] N. Berkovits and C. R. Mafra, "Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism," *JHEP* **0611** (2006) 079, arXiv:hep-th/0607187 [hep-th].

[90] H. Gomez and C. R. Mafra, "The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors," *JHEP* **1005** (2010) 017, arXiv:1003.0678 [hep-th].

[91] E. D'Hoker and D. Phong, "Two loop superstrings. 1. Main formulas," *Phys.Lett.* **B529** (2002) 241–255, arXiv:hep-th/0110247 [hep-th].

[92] E. D'Hoker and D. Phong, "Two-loop superstrings VI: Non-renormalization theorems and the 4-point function," *Nucl.Phys.* **B715** (2005) 3–90, arXiv:hep-th/0501197 [hep-th].

[93] C. R. Mafra, "Pure Spinor Superspace Identities for Massless Four-point Kinematic Factors," *JHEP* **0804** (2008) 093, arXiv:0801.0580 [hep-th].

[94] E. D'Hoker, M. Gutperle, and D. Phong, "Two-loop superstrings and S-duality," *Nucl.Phys.* **B722** (2005) 81–118, arXiv:hep-th/0503180 [hep-th].

[95] E. P. Verlinde and H. L. Verlinde, "Chiral Bosonization, Determinants and the String Partition Function," *Nucl.Phys.* **B288** (1987) 357.

[96] Z. Bern, L. J. Dixon, D. Dunbar, M. Perelstein, and J. Rozowsky, "On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences," *Nucl.Phys.* **B530** (1998) 401–456, arXiv:hep-th/9802162 [hep-th].

[97] J. Scherk, "Zero-slope limit of the dual resonance model," *Nucl.Phys.* **B31** (1971) 222–234.

[98] T. Yoneya, "Quantum gravity and the zero slope limit of the generalized Virasoro model," *Lett.Nuovo Cim.* **8** (1973) 951–955.

[99] J. Scherk and J. H. Schwarz, “Dual Models for Nonhadrons,” *Nucl.Phys.* **B81** (1974) 118–144.

[100] L. Alvarez-Gaume and D. Z. Freedman, “Kahler Geometry and the Renormalization of Supersymmetric Sigma Models,” *Phys.Rev.* **D22** (1980) 846.

[101] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, “The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model,” *Annals Phys.* **134** (1981) 85.

[102] E. Braaten, T. L. Curtright, and C. K. Zachos, “Torsion and Geometrostasis in Nonlinear Sigma Models,” *Nucl.Phys.* **B260** (1985) 630.

[103] J. Callan, Curtis G., E. Martinec, M. Perry, and D. Friedan, “Strings in Background Fields,” *Nucl.Phys.* **B262** (1985) 593.

[104] D. J. Gross and E. Witten, “Superstring Modifications of Einstein’s Equations,” *Nucl.Phys.* **B277** (1986) 1.

[105] M. T. Grisaru, A. van de Ven, and D. Zanon, “Four Loop beta Function for the N=1 and N=2 Supersymmetric Nonlinear Sigma Model in Two-Dimensions,” *Phys.Lett.* **B173** (1986) 423.

[106] M. T. Grisaru, A. van de Ven, and D. Zanon, “Four Loop Divergences for the N=1 Supersymmetric Nonlinear Sigma Model in Two-Dimensions,” *Nucl.Phys.* **B277** (1986) 409.

[107] M. Blau, “The Mathai-Quillen formalism and topological field theory,” *J.Geom.Phys.* **11** (1993) 95–127, `arXiv:hep-th/9203026 [hep-th]`.

[108] J. Frohlich and K. Gawedzki, “Conformal field theory and geometry of strings,” `arXiv:hep-th/9310187 [hep-th]`.

[109] F. Malikov, V. Schechtman, and A. Vaintrob, “Chiral de Rham complex,” *Commun.Math.Phys.* **204** (1999) 439–473, `arXiv:math/9803041 [math-ag]`.

[110] F. Malikov and V. Schechtman, “Chiral de Rham complex. II,” in *Differential topology, infinite-dimensional Lie algebras, and applications*, vol. 194 of *Amer. Math. Soc. Transl. Ser. 2*, pp. 149–188. Amer. Math. Soc., Providence, RI, 1999.

[111] V. Gorbounov, F. Malikov, and V. Schechtman, “On chiral differential operators over homogeneous spaces,” *Int. J. Math. Math. Sci.* **26** no. 2, (2001) 83–106. http://dx.doi.org/10.1155/S0161171201020051.

[112] N. A. Nekrasov, “Lectures on curved beta-gamma systems, pure spinors, and anomalies,” `arXiv:hep-th/0511008 [hep-th]`.

[113] E. Witten, “Two-dimensional models with (0,2) supersymmetry: Perturbative aspects,” *Adv.Theor.Math.Phys.* **11** (2007) , `arXiv:hep-th/0504078 [hep-th]`.

[114] L. Baulieu, A. S. Losev, and N. A. Nekrasov, “Target space symmetries in topological theories. 1.,” *JHEP* **0202** (2002) 021, `arXiv:hep-th/0106042 [hep-th]`.

[115] D. Ben-Zvi, R. Heluani, and M. Szczesny, "Supersymmetry of the chiral de Rham complex," *Compos.Math.* **144** (2008) 503–521.

[116] E. Frenkel, A. Losev, and N. Nekrasov, "Instantons beyond topological theory. I," arXiv:hep-th/0610149 [hep-th].

[117] E. Frenkel, A. Losev, and N. Nekrasov, "Instantons beyond topological theory II," arXiv:0803.3302 [hep-th].

[118] J. Ekstrand, R. Heluani, J. Kallen, and M. Zabzine, "Non-linear sigma models via the chiral de Rham complex," *Adv.Theor.Math.Phys.* **13** (2009) 1221–1254, arXiv:0905.4447 [hep-th].

[119] T. Banks, D. Nemeschansky, and A. Sen, "Dilaton Coupling and BRST Quantization of Bosonic Strings," *Nucl.Phys.* **B277** (1986) 67.

[120] A. S. Schwarz and A. A. Tseytlin, "Dilaton shift under duality and torsion of elliptic complex," *Nucl.Phys.* **B399** (1993) 691–708, arXiv:hep-th/9210015 [hep-th].

[121] T. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," *Phys.Lett.* **B201** (1988) 466.

[122] O. Hohm, W. Siegel, and B. Zwiebach, "Doubled $\alpha'$-geometry," *JHEP* **1402** (2014) 065, arXiv:1306.2970 [hep-th].

[123] R. Penrose, "Asymptotic properties of fields and space-times," *Phys.Rev.Lett.* **10** (1963) 66–68.

[124] R. Penrose, "Zero rest mass fields including gravitation: Asymptotic behavior," *Proc.Roy.Soc.Lond.* **A284** (1965) 159.

[125] H. Bondi, M. van der Burg, and A. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc.Roy.Soc.Lond.* **A269** (1962) 21–52.

[126] R. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc.Roy.Soc.Lond.* **A270** (1962) 103–126.

[127] A. Ashtekar and M. Streubel, "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity," *Proc.Roy.Soc.Lond.* **A376** (1981) 585–607.

[128] A. Ashtekar, *Asymptotic Quantization: Based on 1984 Naples Lectures.* Bibliopolis, 1987.

[129] G. Arcioni and C. Dappiaggi, "Holography in asymptotically flat space-times and the BMS group," *Class.Quant.Grav.* **21** (2004) 5655, arXiv:hep-th/0312186 [hep-th].

[130] G. Barnich and C. Troessaert, "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited," *Phys.Rev.Lett.* **105** (2010) 111103, arXiv:0909.2617 [gr-qc].

[131] A. Bagchi, "The BMS/GCA correspondence," arXiv:1006.3354 [hep-th].

[132] A. Bagchi, "Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories," *Phys.Rev.Lett.* **105** (2010) 171601.

[133] C. Duval, G. Gibbons, and P. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class.Quant.Grav.* **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[134] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP* **1407** (2014) 152, arXiv:1312.2229 [hep-th].

[135] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," *Phys.Rev.Lett.* **67** (1991) 1486–1489.

[136] D. Christodoulou and S. Klainerman, "The Global nonlinear stability of the Minkowski space,".

[137] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," arXiv:1401.7026 [hep-th].

[138] S. Weinberg, "Infrared photons and gravitons," *Phys.Rev.* **140** (1965) B516–B524.

[139] C. D. White, "Factorization Properties of Soft Graviton Amplitudes," *JHEP* **1105** (2011) 060, arXiv:1103.2981 [hep-th].

[140] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," arXiv:1404.4091 [hep-th].

[141] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, "Semiclassical Virasoro symmetry of the quantum gravity $\mathscr{S}$-matrix," *JHEP* **1408** (2014) 058, arXiv:1406.3312 [hep-th].

[142] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* **1005** (2010) 062, arXiv:1001.1541 [hep-th].

[143] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **1112** (2011) 105, arXiv:1106.0213 [hep-th].

[144] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," *JHEP* **1407** (2014) 151, arXiv:1308.0589 [hep-th].

[145] G. Barnich and P.-H. Lambert, "Einstein-Yang-Mills theory: Asymptotic symmetries," *Phys.Rev.* **D88** (2013) 103006, arXiv:1310.2698 [hep-th].

[146] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, "New Symmetries of Massless QED," *JHEP* **1410** (2014) 112, arXiv:1407.3789 [hep-th].

[147] A. Mohd, "A note on asymptotic symmetries and soft-photon theorem," *JHEP* **1502** (2015) 060, arXiv:1412.5365 [hep-th].

[148] T. He, P. Mitra, and A. Strominger, "2D Kac-Moody Symmetry of 4D Yang-Mills Theory," arXiv:1503.02663 [hep-th].

[149] E. Casali, "Soft sub-leading divergences in Yang-Mills amplitudes," *JHEP* **1408** (2014) 077, arXiv:1404.5551 [hep-th].

[150] F. Low, "Scattering of light of very low frequency by systems of spin 1/2," *Phys.Rev.* **96** (1954) 1428–1432.

[151] F. Low, "Bremsstrahlung of very low-energy quanta in elementary particle collisions," *Phys.Rev.* **110** (1958) 974–977.

[152] V. Lysov, S. Pasterski, and A. Strominger, "Low's Subleading Soft Theorem as a Symmetry of QED," *Phys.Rev.Lett.* **113** no. 11, (2014) 111601, arXiv:1407.3814 [hep-th].

[153] Z. Bern, S. Davies, and J. Nohle, "On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons," *Phys.Rev.* **D90** no. 8, (2014) 085015, arXiv:1405.1015 [hep-th].

[154] S. He, Y.-t. Huang, and C. Wen, "Loop Corrections to Soft Theorems in Gauge Theories and Gravity," *JHEP* **1412** (2014) 115, arXiv:1405.1410 [hep-th].

[155] M. Bianchi, S. He, Y.-t. Huang, and C. Wen, "More on Soft Theorems: Trees, Loops and Strings," arXiv:1406.5155 [hep-th].

[156] J. Broedel, M. de Leeuw, J. Plefka, and M. Rosso, "Factorized soft graviton theorems at loop level," arXiv:1411.2230 [hep-th].

[157] A. J. Larkoski, "Conformal Invariance of the Subleading Soft Theorem in Gauge Theory," *Phys.Rev.* **D90** no. 8, (2014) 087701, arXiv:1405.2346 [hep-th].

[158] J. Broedel, M. de Leeuw, J. Plefka, and M. Rosso, "Constraining subleading soft gluon and graviton theorems," *Phys.Rev.* **D90** no. 6, (2014) 065024, arXiv:1406.6574 [hep-th].

[159] Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, "Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance," *Phys.Rev.* **D90** no. 8, (2014) 084035, arXiv:1406.6987 [hep-th].

[160] C. White, "Diagrammatic insights into next-to-soft corrections," *Phys.Lett.* **B737** (2014) 216–222, arXiv:1406.7184 [hep-th].

[161] R. Penrose and W. Rindler, *Spinors and Space-Time,* vols 1 & 2. Cambridge University Press, 1984 & 1986.

[162] E. Newman, "Heaven and Its Properties," *Gen.Rel.Grav.* **7** (1976) 107–111.

[163] E. Newman and K. Tod, "ASYMPTOTICALLY FLAT SPACE-TIMES,".

[164] T. Adamo, C. Kozameh, and E. Newman, "Null Geodesic Congruences, Asymptotically Flat Space-Times and Their Physical Interpretation," *Living Rev.Rel.* **12** (2009) 6, arXiv:0906.2155 [gr-qc].

[165] M. Eastwood and P. Tod, "Edth-a differential operator on the sphere," *Math.Proc.Camb.Phil.Soc.* **92** (1982) 317–330.

[166] E. Newman and R. Penrose, "Note on the Bondi-Metzner-Sachs group," *J.Math.Phys.* **7** (1966) 863–870.

[167] G. Barnich and C. Troessaert, "Supertranslations call for superrotations," *PoS* **21** (2010) 010, `arXiv:1102.4632 [gr-qc]`.

[168] E. Newman and R. Penrose, "An Approach to gravitational radiation by a method of spin coefficients," *J.Math.Phys.* **3** (1962) 566–578.

[169] M. van der Burg, "Gravitational waves in general relativity. 10. Asymptotic Expansions for the Einstein-Maxwell Field," *Proc.Roy.Soc.Lond.* **A310** (1969) 221–230.

[170] A. Exton, E. Newman, and R. Penrose, "Conserved quantities in the Einstein-Maxwell theory," *J.Math.Phys.* **10** (1969) 1566–1570.

[171] F. Cachazo and D. Skinner, "Gravity from Rational Curves in Twistor Space," *Phys.Rev.Lett.* **110** no. 16, (2013) 161301, `arXiv:1207.0741 [hep-th]`.

[172] E. Fradkin and A. A. Tseytlin, "CONFORMAL SUPERGRAVITY," *Phys.Rept.* **119** (1985) 233–362.

[173] E. Fradkin and A. A. Tseytlin, "ASYMPTOTIC FREEDOM IN EXTENDED CONFORMAL SUPERGRAVITIES," *Phys.Lett.* **B110** (1982) 117–122.

[174] H. Romer and P. van Nieuwenhuizen, "Axial Anomalies in $N = 4$ Conformal Supergravity," *Phys.Lett.* **B162** (1985) 290.

[175] C.-h. Ahn, "Comments on MHV tree amplitudes for conformal supergravitons from topological B-model," *JHEP* **0507** (2005) 004, `arXiv:hep-th/0504109 [hep-th]`.

[176] L. Dolan and J. N. Ihry, "Conformal Supergravity Tree Amplitudes from Open Twistor String Theory," *Nucl.Phys.* **B819** (2009) 375–399, `arXiv:0811.1341 [hep-th]`.

[177] T. Adamo and L. Mason, "Einstein supergravity amplitudes from twistor-string theory," *Class.Quant.Grav.* **29** (2012) 145010, `arXiv:1203.1026 [hep-th]`.

[178] E. Bergshoeff, M. de Roo, and B. de Wit, "Extended Conformal Supergravity," *Nucl.Phys.* **B182** (1981) 173.

[179] J. Maldacena, "Einstein Gravity from Conformal Gravity," `arXiv:1105.5632 [hep-th]`.

[180] T. Adamo and L. Mason, "Conformal and Einstein gravity from twistor actions," *Class.Quant.Grav.* **31** no. 4, (2014) 045014, `arXiv:1307.5043 [hep-th]`.

[181] T. Adamo and L. Mason, "Twistor-strings and gravity tree amplitudes," *Class.Quant.Grav.* **30** (2013) 075020, `arXiv:1207.3602 [hep-th]`.

# Appendix A

# One loop formulas

This appendix contains some useful expressions for the ingredients of the loop amplitudes appearing in various places in chapter 3.

Let $\tau \in \mathbb{C}$ define a torus by the quotient $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The modular parameter will also be denoted by $q$ such that

$$q = e^{2i\pi\tau}\,, \tag{A.1}$$

The Jacobi theta functions are defined by their Fourier-Jacobi $q$-expansions,

$$\theta_{\boldsymbol{\alpha}}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{(1/2)(n-a/2)^2} e^{2i\pi(z-b/2)(n-a/2)}\,. \tag{A.2}$$

Here $\boldsymbol{\alpha} := (a,b) = \{(0,0),(0,1),(1,0)\}$ are the even characteristics (spin structures) and $(1,1)$ is the odd one. In the $\boldsymbol{\alpha} = \{1,2,3,4\}$ notation used above, they correspond to $\boldsymbol{\alpha} = \{3,4,2\}$ and $\boldsymbol{\alpha} = 1$ respectively. These are used to define the propagators on the elliptic curve.

The function $G(z_{ij}|\tau)$ denotes the bosonic propagator on the torus defined by

$$G(z|\tau) = -\ln\left|\frac{\theta_1(z|\tau)}{\partial\theta_1(0|\tau)}\right|^2 + 2\pi\frac{(\mathrm{Imz})^2}{\mathrm{Im}\tau}\,. \tag{A.3}$$

where the notation $z_{ij} = z_i - z_j$ and $\partial \equiv (\partial/\partial z)$ (respectively for $\bar\partial$) is used. The functions

$$S_{\boldsymbol{\alpha}}(z_{ij}|\tau) = \frac{\partial\theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)}\frac{\theta_{\boldsymbol{\alpha}}(z_{ij}|\tau)}{\theta_{\boldsymbol{\alpha}}(0|\tau)}\sqrt{\mathrm{d}z_i}\sqrt{\mathrm{d}z_j} \tag{A.4}$$

are the torus free fermion propagators, or Szëgo kernels, in the even spin-structure $\boldsymbol{\alpha}$. For example the Fourier-Jacobi $q$-expansion of $S_1$ is

$$S_1(z|\tau) = \frac{\pi}{\tan(\pi z)} + 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin(2n\pi z)\,. \tag{A.5}$$

Finally the Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \,. \tag{A.6}$$

# Appendix B

# Worldsheet currents and OPEs

For convenience, this appendix lists the various composite currents appearing in the non-minimal formalism discussed in this thesis, and their various OPEs with each other. Only the untilded-tilded variables are listed, as the currents and OPEs for the tilded sector are identical. Note that the only difference between the list here and that for the superstring is the definition of the Green-Schwarz current $d_\alpha$ [51].

The pure spinor conditions on minimal and non-minimal variables imply a gauge invariance, meaning that the conformal weight $(1,0)$ pure spinor fields can only appear in the currents:

$$N^{nm} = \frac{1}{2}(w\gamma^{nm}\lambda)\,, \qquad J = \lambda\cdot w\,, \qquad T_\lambda = -w_\alpha\,\partial\lambda^\alpha\,,$$

$$\bar{N}^{nm} = \frac{1}{2}\left(\bar{w}\gamma^{nm}\bar{\lambda} + s\gamma^{nm}r\right), \qquad \bar{J} = \bar{w}\cdot\bar{\lambda} + s\cdot r\,, \qquad T_{\bar{\lambda},r} = -\bar{w}^\alpha\partial\bar{\lambda}_\alpha - s^\alpha\partial r_\alpha\,,$$

$$S_{mn} = \frac{1}{2}(s\gamma_{mn}\bar{\lambda})\,, \qquad S = s\cdot\bar{\lambda}\,.$$

The minimal currents have OPEs:

$$N^{nm}(z)\,\lambda^\alpha(w) \sim -\frac{1}{2}\frac{(\gamma^{nm}\lambda)^\alpha}{z-w}\,, \qquad J(z)\,\lambda^\alpha(w) \sim -\frac{\lambda^\alpha}{z-w}\,, \qquad J(z)\,N^{nm}(w) \sim 0\,,$$

$$J(z)\,J(w) \sim \frac{-4}{(z-w)^2}\,, \qquad N^{pq}(z)\,N^{nm}(z) \sim -3\frac{\eta^{m[p}\eta^{q]n}}{(z-w)^2} + \frac{\eta^{m[q}N^{p]n} - \eta^{n[q}N^{p]m}}{z-w}\,,$$

$$N^{nm}(z)\,T_\lambda(w) \sim \frac{N^{nm}(z)}{(z-w)^2}\,, \qquad J(z)\,T_\lambda(w) \sim \frac{8}{(z-w)^3} + \frac{J(z)}{(z-w)^2}\,,$$

$$T_\lambda(z)\,T_\lambda(w) \sim \frac{11}{(z-w)^4} + \frac{2\,T_\lambda(z)}{(z-w)^2} + \frac{\partial T_\lambda}{z-w}\,.$$

The last of these confirms the +22 central charge contribution of the pure spinor variables.

For the non-minimal variables:

$$\bar{N}^{nm}(z)\,\bar{\lambda}_\alpha(w) \sim -\frac{1}{2}\frac{(\gamma^{nm}\bar{\lambda})_\alpha}{z-w}\,, \qquad \bar{N}^{nm}(z)\,r_\alpha(w) \sim -\frac{1}{2}\frac{(\gamma^{nm}r)_\alpha}{z-w}\,, \qquad \bar{J}(z)\,\bar{N}^{nm}(w) \sim 0\,,$$

$$\bar{J}(z)\,\bar{\lambda}_\alpha(w) \sim -\frac{\bar{\lambda}_\alpha}{z-w}\,, \qquad \bar{J}(z)\,r_\alpha(w) \sim -\frac{r_\alpha}{z-w}\,, \qquad \bar{J}(z)\,\bar{J}(w) \sim 0\,,$$

$$\bar{N}^{pq}(z)\,\bar{N}^{nm}(w) \sim \frac{\eta^{m[q}\bar{N}^{p]n} - \eta^{n[q}\bar{N}^{p]m}}{z-w}\,,$$

$$\bar{N}^{nm}(z)\,T_{\bar{\lambda},r}(w) \sim \frac{\bar{N}^{nm}(z)}{(z-w)^2}\,, \qquad \bar{J}(z)\,T_{\bar{\lambda},r}(w) \sim \frac{\bar{J}(z)}{(z-w)^2}\,,$$

$$T_{\bar{\lambda},r}(z)\,T_{\bar{\lambda},r}(w) \sim \frac{2\,T_{\bar{\lambda},r}(z)}{(z-w)^2} + \frac{\partial T_{\bar{\lambda},r}}{z-w}\,,$$

Any additional OPEs (involving $S_{nm}$ or $S$) can be read off directly from the superstring [76]. Note that the OPE of the stress tensor $T_{\bar{\lambda},r}$ with itself confirms that the non-minimal variables do not modify the central charge of the model.

Finally, the BRST charge for both the minimal and non-minimal models is built upon the Green-Schwarz constraint $d_\alpha$ (2.41), which is the holomorphic generalization of the superparticle constraint:

$$d_\alpha = p_\alpha - \frac{1}{2}P_m\gamma^m_{\alpha\beta}\theta^\beta.$$

The OPEs of this constraint with the other matter variables are crucial in proving that the BRST charge is nilpotent, the closure of the vertex operators, as well as deriving the effective $b$-ghost (3.126), and can be derived using the free OPEs (2.40):

$$d_\alpha(z)\,f(X,\theta)\,(w) \sim \frac{D_\alpha f}{z-w}\,, \qquad d_\alpha(z)\,d_\beta(w) \sim -\frac{P_m\gamma^m_{\alpha\beta}}{z-w}\,,$$

where $D_\alpha$ is the supersymmetric derivative.