California State University, San Bernardino
CSUSB ScholarWorks

Theses Digitization Project | John M. Pfau Library

1997

Differential geometry of surfaces and minimal surfaces

James Joseph Duran

Follow this and additional works at: https://scholarworks.lib.csusb.edu/etd-project

Part of the Algebraic Geometry Commons

Recommended Citation

Duran, James Joseph, "Differential geometry of surfaces and minimal surfaces" (1997). *Theses Digitization Project*. 1542.
https://scholarworks.lib.csusb.edu/etd-project/1542

This Project is brought to you for free and open access by the John M. Pfau Library at CSUSB ScholarWorks. It has been accepted for inclusion in Theses Digitization Project by an authorized administrator of CSUSB ScholarWorks. For more information, please contact scholarworks@csusb.edu.

DIFFERENTIAL GEOMETRY OF SURFACES AND MINIMAL SURFACES

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

James Joseph Duran

June 1997

DIFFERENTIAL GEOMETRY OF SURFACES AND MINIMAL SURFACES

A Project

Presented to the

Faculty of

California State University,

San Bernardino

---

by

James Joseph Duran

June 1997

Approved by:

[redacted] 5-29-97

Wenxiang Wang, Committee Chair — Date

Mathematics

[redacted]

John Sarli, Committee Member

[redacted]

Chetan Prakash, Committee Member

[redacted]

Paul Vicknair, Department Chair

[redacted]

Terry Hallett, Graduate Coordinator

ABSTRACT

Differential Geometry is the study of geometric figures in space using the methods of Analysis and Linear Algebra. Concepts involving the basis of a vector space, the differential of mappings and perhaps, most importantly, curvatures, are employed extensively when examining surfaces in three dimensions. These surfaces are given by the images of a function f: U—>$\mathbb{R}^3$, where U is understood to be an open set in $\mathbb{R}^2$. The first portion of my project will be devoted to examining the fundamental ideas underlying Differential Geometry, focusing on the features of a specific mapping. Secondly, I will concentrate on a special class of surfaces, referred to as minimal surfaces, defined to be any surface having a zero mean curvature. Minimal surfaces can be characterized by the quality of having the least surface area of all surfaces bounded by the same Jordan curve. This analysis will establish a framework to understand the Classical Plateau Problem asserting the existence of a minimal surface bounded by a given Jordan curve. This background will lead me to my third goal: to investigate which surface of revolution is a minimal surface. My fourth chapter will highlight the differences between Euclidian Geometry and the geometry on the surfaces by examining the properties of Parallel Translation and Geodesics. I will then conclude my project by summarizing a practical application of a minimal surface.

TABLE OF CONTENTS

CHAPTER ONE: THE FOUNDATIONS OF DIFFERENTIAL GEOMETRY

To understand Differential Geometry, I will begin by investigating the following properties of the continuous mapping $f: U \longrightarrow \mathbb{R}^3$ defined by the formula

$$f(u,v) = (u, v, 1 - u^2 + v^2):$$

(a) Differential mappings

(b) First and Second Fundamental Forms

(c) Gauss and Mean Curvatures

Generally, U refers to an infinite open set in $\mathbb{R}^2$; however, since we want to examine a surface with boundary for the purpose of finding its area, we will parametrize our surface by the compact unit disk $\overline{B}: = \{ w \in \mathbb{R}^2: |w| \leq 1 \}$ centered at the origin. That is, $\overline{B}$ is the domain of f. The image of this function, $f(\overline{B}) = S$, is the set $S = \{ (u,v, 1 - u^2 + v^2): (u,v) \in \mathbb{R}^2 \}$ in $\mathbb{R}^3$. This set S is defined to be a hyperbolic paraboloid, a closed surface with boundary. This function is illustrated in the first figure of the appendix.

It will be understood that

$f_1 = u$

$f_2 = v$

$f_3 = 1-u^2 + v^2.$

The tangent plane of this surface is the infinite set of all vectors tangent to the surface at a specific point. From this tangent point emanates infinitely many tangent vectors, which form a set. This set, denoted by $T_u f$, where u is the point of tangency on the surface, is the tangent plane to S at the tangent point u. To find the basis of this tangent plane, we will explore the differential mapping of f given by the matrix

$$df_u = \begin{bmatrix} \frac{df_1}{du} & \frac{df_1}{dv} \\ \frac{df_2}{du} & \frac{df_2}{dv} \\ \frac{df_3}{du} & \frac{df_3}{dv} \end{bmatrix}$$

In our example, this equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & 2v \end{bmatrix}$$

Composing this matrix df_u with $e_1 = (1,0)$, we will obtain the first element in the basis of the tangent plane: $f_u = (1, 0, -2u)$; similarly, the second element in our basis is found by composing df_u with $e_2 = (0,1)$: $f_v = (0, 1, 2v)$. It is clear that the natural basis of $\mathbb{R}^2$ is mapped by df_u into a basis of $T_u f$. Since the set $\{ f_u, f_v \}$ is the basis of our tangent plane, we can represent the tangent plane $T_u f$ by $\{ af_u + bf_v \mid (a,b) \in \mathbb{R}^2 \}$. The surface accompanied by its tangent plane at the point (0, 0, 1) is illustrated in figure 2 of the appendix. Notice that the point (0,0,1) on the surface was mapped from the origin in the domain $\overrightarrow{B}$; that is, f(0,0) = (0,0,1). At (0,0,1), we can find a line ℓ which is perpendicular to all the tangent vectors; this is referred to as the normal line orthogonal to the tangent plane. Observe that all the infinitely many vectors parallel to ℓ are also orthogonal to the tangent plane, also shown in figure 2 in the appendix.

However, we are only interested in the two unit normal vectors parallel to ℓ given by

$$\mathbf{N}(u,v) = \pm \frac{(f_u X f_v)}{| f_u X f_v |}$$

This definition arises from the fact that the cross product, a X b, of two nonzero vectors a and b is orthogonal to both a and b lying in a plane. For the mapping

$$f(u, v) = (u, v, 1\text{-}u^2 + v^2),$$

the two unit normal vectors with respect to the tangent point (0,0,1) are

$$\mathbf{N}(u,v) = \pm\, (2u/[4u^2 + 2v^2 + 1]^{1/2}, -2v/[4u^2 + 4v^2 + 1]^{1/2}, [4u^2 + 4v^2 + 1]^{-1/2}).$$
$$= \pm(0,0,1).$$

For the remainder of this analysis, let $D = [4u^2 + 4v^2 + 1]^{1/2}$. For our purposes, we will focus only on the unit normal vector (0,0,1). We can recognize from figure 2 that when we translate the unit normal vector (0,0,1) along the z-axis such that its initial point is at (0,0,1) and its terminal point is at (0,0,2), this is the unit normal vector of the tangent plane at the point (0,0,1).

The unit normal vector at an indicated point serves as an introduction to the Gauss mapping $N: \vec{B} \longrightarrow S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}$ defined by a similar formula given earlier:

$$N(u,v) = \frac{(f_u X f_v)}{| f_u X f_v |}$$

We can perceive the Gauss map as a composition of two functions:

$$N = h \circ f : \vec{B} \longrightarrow S^2$$

First, f is our mapping which will correspond a point in $\vec{B}$ to a point on the surface S: for $(u,v) \in \vec{B}$, $f(u,v) = P \in S$. Secondly, h maps this point f(u,v) = P on the surface to its unit normal vector in $\mathbb{R}^3$; that is,

$$N(u,v) = h(f(u,v)) = \frac{(f_u X f_v)}{| f_u X f_v |}$$

This unit normal vector in S is then translated to the unit sphere S^2 such that the initial point of the unit normal vector coincides with the origin and its terminal point is a point on the sphere itself. The image of the Gauss map, therefore, is the set of unit normal vectors of the various tangent planes $T_u f$ translated to the sphere S^2; that is, the Gauss map translates all the unit normal vectors of the infinitely many tangent planes at all points on the surface to S^2. For instance, if our surface were a plane parallel to the xy-plane, as illustrated in figure 3, the tangent plane of our surface is the surface itself having a single unit normal vector protruding from it; when we translate this unit normal vector to S^2, we discover that the image of the Gauss mapping is a unit vector parallel to the z-axis, having as its initial point (0,0,0) and its terminal point (0,0,1). The image is just a point having zero area. This preceding illustration demonstrates that the area of the image of the Gauss map, a subset of S^2, is a useful indication of the flatness of our surface: since our surface is a flat plane, the area of the image of the Gauss map is zero.

The unit normal vector, which was used to understand the mechanics of the Gauss map, also plays a key role in determining the fundamental forms. For the vectors $\mathbf{X} = (x_1, x_2)$ and $\mathbf{Y} = (y_1, y_2)$ in $\mathbb{R}^2$, the first fundamental form is defined as follows:

$$I(\mathbf{X}, \mathbf{Y}) = (x_1, x_2) \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$, where

$$E := f_u \cdot f_u$$

$$F := f_u \cdot f_v$$

$$G := f_v \cdot f_v$$

For our mapping $f(u,v) = (u, v, 1- u^2 + v^2)$,

$$E = 1 + 4u^2$$

$$F = - 4uv$$

$$G = 1 + 4v^2.$$

Therefore,

$$I(\mathbf{X}, \mathbf{Y}) = (x_1, x_2) \begin{bmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

This bilinear form equals

$$x_1y_1 + 4u^2x_1y_1 - 4uvx_2y_1 - 4uvx_1y_2 + x_2y_2 + 4v^2x_2y_2.$$

Important local geometric properties of surfaces, including length of curves and surface area can be developed from the first and second fundamental forms. It is important to note that since f is continuous and our domain is the compact set $\overline{B}$, our surface, $S = f(\overline{B})$, is also compact, which implies that although S is an infinite set in $\mathbb{R}^3$, it is bounded; consequently, we can speak of the surface area of S parameterized by the unit disk $\overline{B}$. For example, by a theorem, the area of a surface in three dimensions is given by

$$A = \int\int [1 + (g_u)^2 + (g_v)^2]^{1/2} dA.$$

To relate the first fundamental form with the surface area of the map under consideration, we will evaluate the determinant of the matrix whose entries are the coefficients of the first fundamental form:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

This determinant equals $EG - F^2 = (1 + 4u^2)(1 + 4v^2) - (-4uv)(-4uv) = 1 + 4v^2 + 4u^2$.

We can relate this expression with the square of the integrand in the above surface area formula by expressing the surface in parametric form:

$$x = u$$

$$y = v$$

$$z = g(u,v) = 1 - u^2 + v^2$$

Therefore, we have following partial derivatives of g(u,v):

$$g_u = -2u$$

$$g_v = 2v.$$

This implies that $1 + (g_u)^2 + (g_v)^2 = 1 + 4u^2 + 4v^2 = EG - F^2$; consequently, our area is

$$A = \int\int (1 + 4u^2 + 4v^2)^{1/2}\, dA.$$

Using methods developed in multivariable calculus, we can compute the area of our surface:

$$A = \int\int (1 + 4u^2 + 4v^2)^{1/2}\, dA.$$

$$= \pi/6[5^{3/2} - 1].$$

For vectors **X** and **Y** in $\mathbb{R}^2$, the second fundamental form is defined to be

$$\mathrm{II}(\mathbf{X}, \mathbf{Y}) = (x_1, x_2) \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The entries of the matrix are defined as follows:

$$L := f_{uu} \cdot \mathbf{N}(u,v)$$

$$M := f_{uv} \cdot \mathbf{N}(u,v)$$

$$N := f_{vv} \cdot \mathbf{N}(u,v)$$

With respect to the mapping under consideration,

$$f_{uu} = (0,0,-2)$$

$$f_{uv} = (0,0,0)$$

$$f_{vv} = (0,0,2).$$

Therefore, recalling $D = [4u^2 + 4v^2 + 1]^{1/2}$,

$$L = -2/D$$

$$M = 0$$

$$N = 2/D, \text{ and}$$

$$II(\mathbf{X}, \mathbf{Y}) = (x_1, x_2) \begin{bmatrix} -2/D & 0 \\ 0 & 2/D \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= (-2x_1y_1 + 2x_2y_2)/[4u^2 + 4v^2 + 1]^{1/2}$$

We will now be using the coefficients L, M, and N of the second fundamental form to calculate the minimum and maximum principal curvatures at a given point on our surface; this will serve as a basis to find the Gauss and Mean Curvatures of our surface.

Before examining these curvatures, however, we need to understand the concept of a principal direction. Let S_u^1f denote the unit circle in the tangent plane having as its center the tangent point u, where the tangent plane touches the surface; that is,

$$S_u^1 f := \{ \mathbf{X} \in T_u f : |\mathbf{X}| = 1 \},$$

which is the set comprised of unit tangent vectors. By a proposition given in Wilhelm Klingenberg's A Course in Differential Geometry, $\mathbf{X} \in S_u^1 f$ is a principal direction if and only if $\mathbf{X}$ is an eigenvector of the Weingarten map L_u having as entries coefficients of the second fundamental form[1]:

$$L_u = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} -2/D & 0 \\ 0 & 2/D \end{bmatrix}$$

There will be two unit principal directions, denoted by $\mathbf{X}_1$ and $\mathbf{X}_2$, which are found as follows:

(a) Solve $(L_u - \lambda \mathbf{I}_2) = \mathbf{0}$ for λ. This equation is equivalent to

$$\det\left(\begin{bmatrix} -2/D & 0 \\ 0 & 2/D \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to

$$[\lambda + 2/(4u^2 + 4v^2 + 1)^{1/2}][\lambda - 2/(4u^2 + 4v^2 + 1)^{1/2}] = 0,$$

giving us two eigenvalues:

$$\lambda_1 = 2/(4u^2 + 4v^2 + 1)^{1/2} \text{ and } \lambda_2 = -2/(4u^2 + 4v^2 + 1)^{1/2}$$

(b) To find the eigenvector $\mathbf{X}_1$ corresponding to $\lambda_1 = 2/(4u^2 + 4v^2 + 1)^{1/2}$, we need to substitute this value of λ into the expression

$$(L_u - \lambda \mathbf{I}_2)\mathbf{X}_1 = \mathbf{0}, \text{ obtaining}$$

$$\begin{bmatrix} -4/D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system yields

$$x_1 = 0$$

$$x_2 \in \mathbb{R}$$

Since $\mathbf{X}_1$ is unit vector, let $\mathbf{X}_1 = (0,1)$. Similarly, to find the eigenvector $\mathbf{X}_2$ corresponding to $\lambda_2 = -2/(4u^2 + 4v^2 + 1)^{1/2}$, we will substitute this value of λ into the expression

$$(L_u - \lambda I_2)\mathbf{X}_2 = \mathbf{0}.$$

Solving this system of equations yield

$$x_1 \in \mathbb{R}$$

$$x_2 = 0.$$

Again, since $\mathbf{X}_2$ is a unit vector, $\mathbf{X}_2 = (1,0)$.

The next step is to find $\kappa_1(\mathbf{X}_1)$ and $\kappa_2(\mathbf{X}_2)$, representing the minimum and maximum principal curvatures at a given point on the surface. It is important to understand that we do not speak of the curvature of a surface but rather the curvature of a curve on the surface with respect to a given point on this curve. This curve is generated by the intersection of the surface and the plane formed by the differential of a principal direction vector and its unit normal vector at a given point. To find these principal curvatures, we again utilize the coefficients L, M, and N of the second fundamental form; for our mapping, the minimum principal curvature, corresponding to the principal direction vector $\mathbf{X}_2 = (1,0)$, is defined to be $\kappa_2(\mathbf{X}_2) = II(\mathbf{X}_2, \mathbf{X}_2)$, which is equivalent to

$$(1, 0)\begin{bmatrix} -2/D & 0 \\ 0 & 2/D \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we evaluate this product, we will have our minimum principal curvature:

$$\kappa_2(\mathbf{X}_2) = -2/[4u^2 + 2v^2 + 1]^{1/2}.$$

Similarly, our maximum principal curvature corresponding to the principal direction $\mathbf{X}_1$ is defined to be $\kappa_1(\mathbf{X}_1) = II(\mathbf{X}_1, \mathbf{X}_1) = 2/[4u^2 + 2v^2 + 1]^{1/2}$. Observe that if $\kappa_1(\mathbf{X}_1)$ had been less than $\kappa_2(\mathbf{X}_2)$, then $\kappa_1(\mathbf{X}_1)$ would have been the minimum principal curvature. In other words, only when we evaluate $\kappa_1(\mathbf{X}_1)$ and $\kappa_2(\mathbf{X}_2)$ and determine their magnitudes can we specify which one denotes the minimum principal curvature and and which one denotes the maximum principal curvature .

We will now examine in detail these two curvatures, beginning with the minimum principal curvature, $\kappa_2(\mathbf{X}_2) = -2/[4u^2 + 2v^2 + 1]^{1/2}$, in the direction of $\mathbf{X}_2 = (1,0)$, at the following point which is mapped from the origin in $\vec{B}$ to the surface:

$$f(0,0) = (0,0,1).$$

To locate this curve on the surface having curvature $\kappa_2(\mathbf{X}_2)$ at the point (0,0,1), we will slice the surface in the direction of the principal direction vector $\mathbf{X}_2 = (1,0)$. To achieve this aim we will proceed as follows:

(a) By regarding $\mathbf{X}_2$ as the tangent vector, we mean that $df_u(\mathbf{X}_2) = (1, 0, -2u)$ is the tangent vector of the curve which we will produce at the point (0,0,1) determined by $\mathbf{X}_2$. Since we are mapping this point from the origin in $\vec{B}$, where u = v = 0, it is clear that

$$df_u(\mathbf{X}_2) = (1, 0, 0) \text{ and}$$

$$\kappa_2(\mathbf{X}_2) = -2/[4u^2 + 2v^2 + 1]^{1/2} = -2.$$

(b) Find the unit normal vector corresponding to the point (0,0,1). Again, the unit normal vector is

$$\mathbf{N}(u,v) = (2u/[4u^2 + 2v^2 + 1]^{1/2}, -2v/[4u^2 + 4v^2 + 1]^{1/2}, [4u^2 + 4v^2 + 1]^{-1/2})$$

$= (0,0,1).$

Our curve will be produced by the normal intersection of the surface and the plane formed by the vectors $df_u(\mathbf{X}_2) = (1, 0, 0)$ and $N(0,0) = (0, 0, 1)$. Notice that these two vectors are the basis of the xz-plane; since they span the plane, they generate the plane. Notice from figure 4, $df_u(\mathbf{X}_2)$ is the tangent vector of the curve at the point (0,0,1); to make this more evident, translate this tangent vector to the point (0,0,1). The normal intersection is a curve resembling an inverted parabola in the xz-plane having (0,0,1) as the vertex. From the perspective of the unit normal vector (0, 0, 1), translated up on the z-axis, the curve is concave down, justifying why our minimum principal curvature is negative.

The maximum principal curvature, $\kappa_1(\mathbf{X}_1) = 2/[4u^2 + 2v^2 + 1]^{1/2}$, in the direction of $\mathbf{X}_1 = (0, 1)$ at the point (0,0,1), is the curvature of the curve which is created when we slice the surface in the direction of the principal direction vector $\mathbf{X}_1 = (0, 1)$. Patterning our work from the previous exercise, we proceed accordingly:

(a) By regarding $\mathbf{X}_1$ as the tangent vector, we mean that $df_u(\mathbf{X}_1) = (0,1, 2v)$ is the tangent vector determined by $\mathbf{X}_1$. Again, since we are mapping our tangent point from the origin in $\vec{B}$, where $u = v = 0$, it is clear that

$$df_u(\mathbf{X}_1) = (0,1,0) \text{ and}$$

$$\kappa_1(\mathbf{X}_1) = 2/[4u^2 + 2v^2 + 1]^{1/2} = 2.$$

(b) The unit normal vector corresponding to the point (0,0,1) is the same one found in the prior case: $\mathbf{N}(0,0) = (0,0,1)$. Our curve, therefore, will be produced by the normal intersection of the surface and the plane generated by the vectors $df_u(\mathbf{X}_2) = (0,1,0)$ and $\mathbf{N}(0,0) = (0,0,1)$: the yz-plane. Notice from figure 5 that when we translate $df_u(\mathbf{X}_2) = (0,1,0)$ to the point (0,0,1), it becomes clear that

$df_u(\mathbf{X}_2)$ is tangent to the curve at the point (0,0,1).This normal intersection is a curve resembling an elongated parabola in the yz-plane. From the perspective of the unit normal vector, (0, 0, 1), translated up on the z-axis, this curve is concave up, justifying why our maximum principal curvature is positive.

The intermediate curvatures of the curves generated by different principal direction vectors at the point (0,0,1) will vary between the minimum principal curvature, -2, and the maximum principal curvature, 2. These curves are formed by slicing the surface along a principal direction vector other than $\mathbf{X}_1$ or $\mathbf{X}_2$. For instance, to find the principal direction vector $\mathbf{X}' \in S_u^1 f$ which would produce a curve with 0 curvature, we would solve the following equation for $\mathbf{X}'$:

$$II(\mathbf{X}',\mathbf{X}') = \kappa(\mathbf{X}') = 0.$$

The coordinates of $\mathbf{X}'$ assume the form $|x_1| = |x_2|$; without loss of generality, we will let (1,1), (-1,-1), (1,-1) and (-1,1) be our four possible solutions to $\mathbf{X}'$. Regarding each $\mathbf{X}'$ as the tangent vector, we are able to find the corresponding $df_u(\mathbf{X}')$, the tangent vector determined by each $\mathbf{X}'$. In all cases below, $u = v = 0$, and the corresponding unit normal vector is (0,0,1):

$$\text{(i)}df_u(1,1) = (1,1,-2u+2v) = (1,1,0)$$

$$\text{(ii)}df_u(-1,-1) = (-1,-1,2u-2v) = (-1,-1,0)$$

$$\text{(iii)}df_u(1,-1) = (1,-1,2u-2v) = (1,-1,0)$$

$$\text{(iv)}df_u(-1,1) = (-1,1,2u+2v) = (-1,1,0).$$

As illustrated in figure 6, when we cut the surface by the plane generated by $df_u(1,1) = (1,1,0)$ and the unit normal vector $\mathbf{N}(0,0) = (0,0,1)$, a line, which can be considered as a curve with 0 curvature, results; notice that this line coincides with the line produced by the intersection of the surface and the plane which is formed by $df_u(-1,-1) = (-1,-1,0)$ and $\mathbf{N}(0,0) = (0,0,1)$. It is clear that this plane, which we can refer to as the diagonal plane, cuts the xy plane at a 45 degree angle. The two other tangent vectors, $df_u(1,-1) = (1,-1,0)$ and $df_u(-1,1) = (-1,1,0)$ along with $\mathbf{N}(0,0) = (0,0,1)$ also generates a diagonal plane that, when intersected by the surface, forms a line. Observe from the figure, as we rotate either diagonal plane closer to the xz-plane, we will produce curves on the surface whose curvatures approach -2 at the point (0,0,1); similarly, as we rotate either diagonal plane closer to the yz-plane, we will produce curves on the surface whose curvatures approach 2 at the point (0,0,1).

The Gauss and mean curvatures at a given point on our surface are defined to be the product and the average of the maximum principal curvature and the minimum principal curvature, respectively:

$$K = \kappa_1(\mathbf{X}_1)\kappa_2(\mathbf{X}_2)$$

$$H = 1/2[\kappa_1(\mathbf{X}_1) + \kappa_2(\mathbf{X}_2)]$$

In our example, with respect to the point (0,0,1):

$$K = -4/[4u^2 + 2v^2 + 1]^{1/2} = -4$$

$$H = 1/2\{2/[4u^2 + 2v^2 + 1]^{1/2} - 2/[4u^2 + 2v^2 + 1]^{1/2}\} = 0$$

By definition, a point on a surface is elliptic if and only if $K > 0$; hyperbolic if and only if $K < 0$; parabolic or planar if and only if $K = 0$. At the point (0,0,1), the Gauss curvature of our surface is less than 0; consequently, we know that the tangent point (0,0,1) is hyperbolic. This means that we can always find points in the neighborhood of (0,0,1) on S which lie on either side of the tangent plane. Figure 2 illustrates this property of our surface. Consider a surface S', the hemisphere, shown in figure 7, with a positive Gauss curvature, $K > 0$, at a given point P. In this situation, we could find points, in the neighborhood of P, on S', all lying on one side of the tangent plane at P. Here, P, by definition, is an elliptic point.

For the case $K = 0$, let us consider the surface given by the image of

$j(u,v) = (u,v,1 - u^2)$, delineated in figure 8, accompanied by its tangent plane at the point (0,0,1). Since at least one of our principal curvatures must be zero, we can infer that one of our curves generated by a normal intersection must be a line. Similar to the manner in which we we found the maximum and minimum curvatures of f with respect to the point P = (0,0,1), we can recognize from the figure that when we intersect this surface with the xz-plane, we will produce a curve with a negative curvature at P; however, intersecting this surface with the yz-plane will generate a straight line having zero curvature. Such a point P is called parabolic, and is characterized by the property that near P, there are points on the surface which lie on the same side of the tangent plane.[2]

Since the mean curvature of our original surface given by the image of f at the point (0,0,1) is zero, our surface, by definition, is a minimal surface. Minimal surfaces have the least surface area of all surfaces bounded by the same Jordan curve. Another example, more trivial, of a minimal surface is a disk whose boundary is a circle. Any other ring bounded surface, whether barely wrinkled or strongly bulged, would have a larger area.

CHAPTER TWO: THE PLATEAU PROBLEM

Now that we understand the concept of a minimal surface, we can focus on the Classical Plateau Problem asserting the existence of a minimal surface bounded by a closed Jordan curve. By definition:

> Given a closed Jordan curve Γ in $\mathbb{R}^3$, we say that $X: \overline{B} \rightarrow \mathbb{R}^3$ is a solution of Plateau's problem for the boundary contour Γ (or : a minimal surface spanned by Γ) if it fulfills the following three conditions:
>
> (i) $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$
> This means that X is continuous on $\overline{B} := \{w \in \mathbb{R}^2 : |w| \leq 1\}$ and twice differentiable on $B := \{ w \in \mathbb{R}^2 : |w| < 1\}$.
>
> (ii) The surface X satisfies in B the equations
>
> (a) $\Delta X = \frac{\delta^2 x}{\delta u^2} + \frac{\delta^2 x}{\delta v^2} = 0$
>
> (b) $|X_u|^2 = |X_v|^2$
>
> (c) $\langle X_u, X_v \rangle = 0$
>
> (iii) The restriction $X|_C$ to the boundary C of the parameter domain B is a homeomorphism of C onto Γ.[3]

For two sets to be homeomorphic, there must exist a bijective and continuous mapping between the two sets whose inverse is also continuous.

Before exploring this problem, given in the form of a theorem in U. Dierkes' Minimal Surfaces I, we need to understand the variational problem, denoted by $P(\Gamma)$, that will enable us to solve the Plateau problem. First, we will define terms and notation which will be used in understanding this problem:

(1) Jordan curve: A closed curve, denoted by Γ, in $\mathbb{R}^3$ which is not self

intersecting. In topological terms, this curve can be described as a subset of $\mathbb{R}^3$ homoemorphic to a circle.

(2) $B := \{w \in \mathbb{R}^2 : |w| < 1\}$

(3) $\bar{B} := \{w \in \mathbb{R}^2 : |w| \leq 1\}$

(4) $C := \{w \in \mathbb{R}^2 : |w| = 1\} = \delta B$

(5) $H^1_2(B, \mathbb{R}^3)$: the set of surfaces which are continuous and differentiable from B to $\mathbb{R}^3$.

(6) X: this will denote, depending on the context, either a continuous and differentiable mapping from B to $\mathbb{R}^3$ or the image of this mapping, namely a surface in $\mathbb{R}^3$.

(7) Weakly monotonic : a mapping $\psi: C \rightarrow \Gamma$ is weakly monotonic if the image points ψ (w) traverse Γ in a constant direction when w moves along C in a constant direction. The image points may remain stationary, but never move backwards if w moves monotonically on C and ψ (w) moves once around Γ if w travels once around C.

(8) A mapping $X: B \rightarrow \mathbb{R}^3$ is said to be of class $\mathcal{C}(\Gamma)$ if $X \in H^1_2(B, \mathbb{R}^3)$ and if its trace $X|_C$ (that is, restricting the domain of X to $C = \delta B$) can be represented by a weakly, continuous mapping $\Phi : C \rightarrow \Gamma$ of C onto Γ.

(9) The Dirichlet integral

$D(X) = D_B(X) := \frac{1}{2}\int (|X_u|^2 + |X_v|^2)\, dudv.$

This integral furnishes a majorant for the area function given by

$A_B(X) := \int_B |X_u \times X_v|\, dudv$

Consequently, we will find that specific X which minimizes the Dirichlet integral, and thereby minimizes the area function. This is the crux of the variational problem.

The variational problem $P(\Gamma)$ associated with the Plateau problem for the Jordan curve Γ is defined as the following task: "minimize Dirichlet's integral $D(X)$ in the class $\mathcal{C}(\Gamma)$."[4]

This means the following:

First, define $e(\Gamma) := \inf\{D(X): X \in \mathcal{C}(\Gamma)\}$, where the expression on the right denotes the smallest $D(X)$ (some real number) given by a specific surface, say X_m. To clarify, $\{D(X) : X \in \mathcal{C}(\Gamma)\}$ denotes the set of all Dirichlet integrals $D(X)$; thus, $\inf\{D(X) : X \in \mathcal{C}(\Gamma)\}$ will give us the smallest value of $D(X)$ among all the $D(X)$ values in $\{D(X) : X \in \mathcal{C}(\Gamma)\}$. Note that the smallest value of $D(X)$, which we can denote by $D(X_m) = \inf\{D(X) : X \in \mathcal{C}(\Gamma)\}$, may or may not be an element of the set $\{D(X) : X \in \mathcal{C}(\Gamma)\}$. Secondly, we are to find that specific surface $X_m \in \mathcal{C}(\Gamma)$ such that $D(X_m) = e(\Gamma)$ is satisfied. This latter portion is the essence of the variational problem $P(\Gamma)$: finding the minimal surface X_m.

The Classical Plateau Problem asserting the existence of a minimal surface bounded by a given Jordan Curve is given by the following theorem from <u>Minimal Surfaces I</u> by U. Dierkes:

> If $\mathcal{C}(\Gamma)$ is nonempty, then the minimum problem $P(\Gamma)$ has at least one solution which is continuous on $\overline{B}$ and harmonic in B. In particular, $P(\Gamma)$ has such a solution for every rectifiable curve Γ.[5]

Here, it is my aim to clarify the proof given in the above mentioned textbook.

The proof proceeds as follows: We need to find a solution to $P(\Gamma)$, which is the following problem:

$$(1)\ D(X) \longrightarrow \min \text{ in the class } \mathcal{C}^*(\Gamma),$$

which means to minimize the Dirichlet integral D(X) in the class $\mathcal{C}^*(\Gamma)$. $\mathcal{C}^*(\Gamma)$ represents the set of surfaces in $\mathcal{C}(\Gamma)$ satisfying a fixed three-point condition:

$$(2)\ X(w_k) = Q_k,\ k = 1, 2, 3.$$

Here w_1, w_2, w_3 denote three distinct points on $C = \delta B$ and Q_1, Q_2, Q_3 are three different points on the Jordan Curve Γ. That is, X maps 3 different points on C to three different points on Γ.

We will denote sequence of mappings X_n, where $X_n \in \mathcal{C}^*(\Gamma)$, by $\{X_n\}$. That is, $\{X_n\}$, represents $X_1, X_2, \ldots X_n$. We choose a sequence $\{X_n\}$ of mappings such that

$$(3)\ \lim_{n \to \infty} D(X_n) = e^*(\Gamma) \text{ holds.}$$

Here we are taking the limit of this sequence of mappings (surfaces). As n increases, consecutive surfaces cluster toward the minimal surface, X_m, which may or may not be included in $\{X_n\}$. Consequently, $\lim_{n \to \infty} D(X_n) = D(X_m) = e^*(\Gamma)$.

Assume, without loss of generality, that X_n is a surface of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$. This means that X_n is continuous on $\bar{B}$ and twice differentiable on B. We will also assume that X_n is harmonic; that is,

$$(4)\ \Delta X_n = \frac{\delta^2 X_n}{\delta u^2} + \frac{\delta^2 X_n}{\delta v^2} = 0 \text{ in } B$$

We claim that the boundary values $X_n|_c$ of the terms of any minimizing sequence $\{X_n\}$ for $P^*(\Gamma)$ are equicontinuous on C. This means that for all $\varepsilon > 0$, the Euclidian distance between two image points of $X_n|_c$, the boundary values, which lie on the Jordan curve Γ, will always be less than ε provided that the Euclidian distance between the two preimage points, which lie on C, is less than a given $\delta > 0$. In symbols, we have for all $\varepsilon > 0$, $|X_m(w) - X_m(w')| < \varepsilon$ if $|w - w'| < \delta$. To prove this, we need to make use of the Courant-Lebesque lemma:

> Suppose that X is of class $C^0(\overline{B}, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$ (that is, X is continuous on $\overline{B}$ and once differentiable on B) and satisfies $D(X) \leq M$, where $0 \leq M < \infty$. Then for every $z_0 \in C$ and for every $\delta \in (0, 1)$, there exists a number $\rho \in (\delta, \delta^{\frac{1}{2}})$ such that the distance between X(z) and X(z'), the respective images of z and z', where $\{z, z'\} = C \cap \delta B_\rho(z_0)$, can be estimated by[6]

$$|X(z) - X(z')| \leq \left\{\frac{4M\pi}{\log 1/\delta}\right\}^{\frac{1}{2}}.$$

See figure 9. Note that $\delta B_\rho(z_0)$ is the circle with center z_0 and radius ρ. We will apply this lemma as follows:

Since Γ is the topological image of C (that is Γ and C are homeomorphic to each other), there exists for every $\varepsilon > 0$, another number, $\lambda(\varepsilon)$, a function of ε, which is also greater than 0 with the following property:

Any pair of points $P, Q \in \Gamma$ with

$$(6)\quad 0 < |P - Q| < \lambda(\varepsilon)$$

decomposes Γ into two arcs, $\Gamma_1(P, Q)$ and $\Gamma_2(P, Q)$, as shown in figure 10. That is, $\Gamma = \Gamma_1(P, Q) \cup \Gamma_2(P, Q)$, such that

$$(7)\quad \operatorname{diam}\Gamma_1(P, Q) < \varepsilon,$$

where $\operatorname{diam}\Gamma_1(P, Q)$ denotes the length of the line segment between P and Q. Thus, if

$$(8)\quad 0 < \varepsilon < \varepsilon_0 := \min_{j \neq k} |Q_j - Q_k|,$$

where $Q_j, Q_k \in \Gamma$, then $\Gamma_1(P, Q)$ contains at most one of the points Q_j appearing in the three point condition: $X(w_i) = Q_i$, as shown in figure 11.

Note that $\min_{j \neq k} |Q_j - Q_k|$ denotes the shortest distance between Q_j and Q_k on Γ.

By examining $0 < \varepsilon < \varepsilon_0$ closer, it is evident from prior definitions that $0 < \operatorname{diam} \Gamma_1(P, Q) < \varepsilon < \min_{j \neq k} |Q_j - Q_k|$, implying that $\operatorname{diam} \Gamma_1(P, Q) < \min_{j \neq k} |Q_j - Q_k|$. If $\Gamma_1(P, Q)$ contained both points Q_j and Q_k, we would have the situation illustrated in figure 12; clearly, in this case $\operatorname{diam} \Gamma_1(P,Q) > \min_{j \neq k} |Q_j - Q_k|$.

Let X be an arbitrary mapping in $\mathcal{C}^*(\Gamma)$ fulfilling the hypothesis of the Courant - Lebesque lemma and let $\delta_0 \in (0, 1)$ be a fixed number with

$$(9) \quad 2\delta_0^{\frac{1}{2}} < \min_{j \neq k} |w_j - w_k|, \text{ where } w_1, w_2, w_3 \in C.$$

See figure 13. For an arbitrary $\varepsilon \in (0, \varepsilon_0)$, we select some number $\delta = \delta(\varepsilon) > 0$ such that

$$(10) \quad |X(z) - X(z')| \leq \left\{\frac{4 M \pi}{\log 1/\delta}\right\}^{\frac{1}{2}} < \lambda(\varepsilon), \text{ and}$$

$$(11) \quad \delta < \delta_0$$

Now consider an arbitrary point $z_0 \in C$ and let $\rho \in (\delta, \delta^{\frac{1}{2}})$ be some number such that the images $P := X(z)$ and $Q := x(z')$ of the two intersection points, z and z', of C and $\delta B_\rho(z_0)$ satisfy

$$(12) \quad |X(z) - X(z')| = |P - Q| \leq \left\{\frac{4 M \pi}{\log 1/\delta}\right\}^{\frac{1}{2}}$$

Equations (12) and (10) imply

$$(13) \quad |P - Q| < \lambda(\varepsilon)$$

where from equation (7), $\operatorname{diam} \Gamma_1(P, Q) < \varepsilon$. From equations (7) and (8), we know $\operatorname{diam}\Gamma_1(P, Q) < \varepsilon < \min_{j \neq k} |Q_j - Q_k|$. This suggests the arc $\Gamma_1(P, Q)$ contains at most one of the points Q_j (As before, if the arc contained both points Q_j, the prior inequality would not hold). In addition, it follows from $X \in \mathscr{C}(\Gamma)$, and from the facts listed below:

(a) $X(w_k) = Q_k$, $k = 1,2,3$

(b) $2\,\delta_0^{\frac{1}{2}} < \min_{j \neq k} |w_j - w_k|$

(c) $\delta < \delta_0$

that $X(C \cap \overline{B_\rho(z_0)})$, the image of the intersection of the circle C and the disk with center z_0, and radius ρ, along with the boundary of this disk, contains at most one of the points Q_j. Thus, X maps this intersection to the arc $\Gamma_1(P, Q)$; that is, $X(C \cap \overline{B_\rho(z_0)}) = \Gamma_1(P, Q)$.

To understand why $X(C \cap \overline{B_\rho(z_0)}) = \Gamma_1(P, Q)$, we will carefully analyze the above facts along with figure 14.

From (b) and (c), we know that $\delta^{\frac{1}{2}} < \delta_0^{\frac{1}{2}}$ and $\delta_0^{\frac{1}{2}} < \frac{1}{2} \min_{j \neq k} |w_j - w_k|$. So by transitivity,

$\delta^{\frac{1}{2}} < \frac{1}{2} \min_{j \neq k} |w_j - w_k|$; consequently, $\delta^{\frac{1}{2}} < \min_{j \neq k} |w_j - w_k|$.

In the Courant-Lebesque lemma, we were give that $\rho \in (\delta, \delta^{\frac{1}{2}})$; hence, $\rho < \delta^{\frac{1}{2}}$. So again by transitivity:

$\rho < \delta^{\frac{1}{2}}$

$\delta^{\frac{1}{2}} < \min_{j \neq k} |w_j - w_k|$

$\rho < \min_{j \neq k} |w_j - w_k|$

This tells us that the radius of the disk $\overline{B_\rho(z_o)}$ will always be less than the minimum distance between two arbitrary points, w_j and w_k, on the circle C. So for three arbitrary points, $w_1, w_2, w_3, \in C$, we know that, one of these points, say w_1, lies in the set $C \cap \overline{B_\rho(z_o)}$.

From the fact that $X(w_k) = Q_k$, k = 1, 2, 3, we know that $X(w_1) = Q_1$, which, without loss of generality, we will assume to be a point in $\Gamma_1(P, Q)$ since we concluded earlier that $\Gamma_1(P, Q)$ contains at most one of the points Q_j appearing in the three point condition. Conclusively, the following equalities

$X(z) = P$

$X(z') = Q$

$X(w_1) = Q_1$

coupled with the fact that X is a continuous function, mapping closed sets to closed sets, we can infer that the closed set $C \cap \overline{B_\rho(z_o)}$ is mapped to the closed set $\Gamma_1(P, Q)$. Hence, $X(C \cap \overline{B_\rho(z_o)}) = \Gamma_1(P, Q)$.

Consequently, we have $|X(w) - X(w')| < \text{diam}\,\Gamma_1(P, Q)$ for all $w, w' \in C \cap \overline{B_\rho(z_o)}$. This is clear since X maps points from $C \cap \overline{B_\rho(z_o)}$ to points in $\Gamma_1(P, Q)$. The image points X(w) and X(w'), therefore, lie somewhere on $\Gamma_1(P, Q)$, ensuring that the Euclidian distance between them, denoted by $|X(w) - X(w')|$ is less than the distance between the endpoints of $\Gamma_1(P, Q)$, denoted by $\text{diam}\,\Gamma_1(P, Q)$. See figure 15. And since $\text{diam}\,\Gamma_1(P, Q) < \varepsilon$, we know, by transitivity,

$$(14)\quad |X(w) - X(w')| < \varepsilon \text{ for all } w, w' \in C \cap B_\rho(z_o).$$

This implies $|X(w) - X(w')| < \varepsilon$ for all $w, w' \in C$ with $|w - w'| < \delta/2$. That is, when we restrict the distance of any two points on C to less than $\delta/2$, where $\delta < \rho < \delta^{\frac{1}{2}}$, we can be sure that the distance between X(w) and X(w'), which may or may not lie in $\Gamma_1(P, Q)$ since w and w' are now any two points on

C (not necessarily lying on $C \cap B_\rho(z_o)$) is less than ε. This is true because X is a continuous mapping.

Now consider the minimizing sequence $\{X_n\} = X_1, X_2, \ldots X_n$.

Since $\lim_{n \to \infty} D(X_n) = e^*(\Gamma)$, there is some number $M > 0$ such that $D(X_n) \leq M$ is true for all $n \in \mathbb{N}$. That is, the Dirichlet integral, when applied to any X_i in $\{X_n\}$, will, when evaluated, produce a different number depending on the chosen X_i. Each one of these numbers is less than or equal to some number M which is greater than zero. Therefore, we can apply the fact that $| X(w) - X(w') | < \varepsilon$ for all $w, w' \in C$ with $| w - w' | < \delta/2$ to $X = X_n$, $n \in \mathbb{N}$, and conclude that the functions X_n, when the domain is restricted to C, is equicontinuous. By definition of equicontinuous, applied to this case, for all $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for all $X_i \in \{X_n\}$ and for all $w, w' \in C$ the following inequalities are true

$$(15) \quad | X_1(w) - X_1(w') | < \varepsilon$$

$$| X_2(w) - X_2(w') | < \varepsilon$$

$$\vdots$$

$$| X_n(w) - X_n(w') | < \varepsilon$$

whenever $| w - w' | < \delta/2$. That is, only one number $\delta > 0$ allows the previous inequalities to hold true. In short, this definition essentially asserts that the Euclidian distance between $X_i(w)$ and $X_i(w')$ for $i = 1, 2, \ldots, n$ will always be less than ε as long as we ensure that the Euclidian distance between w and w', for all $w, w' \in C$, is less than $\delta/2$. See figure 16. Note that all the points $w, w' \in C$ are "especially chosen" so that $| w - w' | < \delta/2$. Further, although every $X_i|_C$ has the same domain, C, and image, Γ, each specific $X_i|_C$ maps the points on Γ in a different order.

In addition, we can conclude from $X_n(C) = \Gamma$ that the functions $X_i|_c$ are uniformly bounded for n = 1, 2, 3... To understand why this is true, we need to understand what it means for a family of functions to be uniformly bounded; first, consider the definition of what it means for a given function of one variable to be bounded. A set of real numbers is said to be a bounded set if the set has both an upper and a lower bound. Consider a function $f : I \longrightarrow \mathbb{R}$ defined on a given interval I of real numbers. The function is bounded on the interval if the set of all values of the function is a bounded set; this means that for all $x \in I$, there is some real number A such that $| f(x) | \leq A$. Similarly, for a family of functions $X_n|_c : C \longrightarrow \Gamma$, i = 1, 2, ...n to be uniformly bounded, means that we can enclose Γ containing all the image points of all the mappings $X_i|_c : C \longrightarrow \Gamma$ in a sphere of radius r > 0; we can denote this sphere, centered at the origin, by $S^2 = \{x \in \mathbb{R}^3 | \ |x| = r\}$. Clearly, since Γ is enclosed in S^2 the image points $X_n(w)$ on Γ will always be less than r units from the origin. This is true regardless of where the Jordan curve is located inside S^2 in figure 17.
This case, as well as all cases where Γ is enclosed by S^2, we know $|X_n(w)| \leq r$ for all $w \in C$, n = 1, 2, ... That is, the norm of all $X_i(w)$ is always less than or equal to the radius r of the sphere, so the functions $X_n|_c$ are uniformly bounded for n = 1, 2, ...

Hence, our sequence of uniformly bounded mappings $\{X_n|_c\}$ satisfy the hypothesis of a version of the theorem of Arzelá - Ascoti :

> If $\{X_n|_c\}$ is a sequence of harmonic functions on C that is uniformly bounded on C, then all these mappings in $\{X_n|_c\}$ converge to a specific map $\varphi : C \longrightarrow \Gamma$[7]

To understand this theorem, let us examine the image points of each $X_1|_c$, $X_2|_c$, ... as $n \longrightarrow \infty$. As shown in figure 18, as n increases infinitely, each consecutive $X_n(w)$, for n = 1, 2, ... converges to a fixed point P; that is, $\lim_{n \to \infty} X_n(w) = P$.

Therefore, by the theorem of Arzelá-Ascoli, all the mappings in the sequence $\{X_n|_c\}$ converge uniformly to a specific continuous weakly monontonic map as $n \to \infty$, denoted by $\varphi : C \longrightarrow \Gamma$, where $\varphi(w) = P$ for a given $w \in C$. So we have

$\lim_{n \to \infty} X_n|_c = \varphi$. Moreover, from a result of harmonic function theory, we know

that the uniform limit of functions (1) continuous on a closed disk B, (2) harmonic in the interior B, and (3) satisfying the three point condition is a function having these three properties. This function $X : B \longrightarrow \mathbb{R}^3$ is similar to φ in the sense that just as each consecutive $X_i|_c(w)$ converges to a specific point P on the Jordan curve, each consecutive $X_i(w)$ converges to a specific point Q on the surface. Again, just as before, all the mappings in the sequence $\{X_n\}$ converge to a specific map X sharing the same properties of maps in the sequence. Finally, restricting the domain of X to C will produce image points on the boundary Γ of the surface; thus, Moreover, from a result of harmonic function theory, we know $X|_c = \varphi$. Consequently, X meets the requirements for being of class $C^*(\Gamma)$, defined earlier, and therefore

$$(16) \quad e^*(\Gamma) \leq D(X)$$

This is true given the definition of $e^*(\Gamma) := \inf\{D(X) \mid X \in C^*(\Gamma)\}$, which will produce the smallest possible value of D(X); this number will always be less than or equal to a given D(X).

Furthermore, a classical result for harmonic functions (recall that a function X is harmonic if

$$\Delta X = \frac{\delta^2 X}{\delta u^2} + \frac{\delta^2 X}{\delta v^2} = 0) \quad \text{implies that}$$

$$\text{grad } X_n = \frac{\delta X_n}{\delta u}\mathbf{i} + \frac{\delta X_n}{\delta v}\mathbf{j} \quad \text{tends to}$$

$$\text{grad } X = \frac{\delta X}{\delta u}\mathbf{i} + \frac{\delta X}{dv}\mathbf{j}$$

as $n \to \infty$ uniformly on every $B' \subset\subset B$. As grad X_n approaches grad X, by the definition that taking the limit of vectors entails taking the limit of each component, we know

$$\frac{\delta X_n}{\delta u} \quad \text{approaches} \quad \frac{\delta X}{\delta u} \quad \text{and}$$

$$\frac{\delta X_n}{\delta v} \quad \text{approaches} \quad \frac{\delta X}{\delta v}, \text{ as } n \to \infty, \text{ where}$$

(17)
$$\frac{\delta X_n}{\delta u} = X{n_u}$$

$$\frac{\delta X_n}{\delta v} = X{n_v}$$

$$\frac{\delta X}{\delta u} = X_u$$

$$\frac{\delta X}{\delta v} = X_v$$

Consequently, by substitution and squaring

$|Xn_u|^2$ approaches $|Xu|^2$ and

$|Xn_v|^2$ approaches $|Xv|^2$.

So we can infer that since, by definition,

$$D_{B'}(X_n) := \tfrac{1}{2}\int_{B'} (|Xn_u|^2 + |Xn_v|^2)\, dudv \qquad \text{and}$$

$$D_{B'}(X) := \tfrac{1}{2}\int_{B'} (|X_u|^2 + |X_v|^2)\, dudv,$$

$$\lim_{n \to \infty} D_{B'}(X_n) = D_{B'}(X) \text{ is true.}$$

When we restrict the Dirichlet integral to B', a strict subset of B, the result of the integration will produce a number less than or equal to the smallest $D_B(X)$; this number is denoted by $\liminf_{n \to \infty} D_B(X_n)$. As a result,

$$\liminf_{n \to \infty} D_B(X_n) \geq \lim_{n \to \infty} D_{B'}(X_n) \text{ if}$$

$B' \subset\subset B$. When we replace $\lim_{n \to \infty} D_{B'}(X_n)$ by $D_{B'}(X)$, we will obtain

$$(18) \quad \liminf_{n \to \infty} D_B(X_n) \geq D_{B'}(X).$$

Without loss of generality, $\liminf_{n \to \infty} D_B(X_n) = \lim D_B(X_n) = \lim_{n \to \infty} D(X_n)$.

Thus, we finally obtain $e^*(\Gamma) = \lim_{n \to \infty} D(X_n) \geq D(X) \geq e^*(\Gamma)$.

The first portion of this expression, $e^*(\Gamma) = \lim_{n \to \infty} D(X_n)$,

was an assumption we established earlier, and the last portion of the expression

$$(19) \; D(X) \geq e^*(\Gamma)$$

is true by the definition of $e^*(\Gamma)$. We can now explain why

$$(20) \quad \lim_{n \to \infty} D(X_n) \geq D(X) \text{ is true, where}$$

$$(21) \lim_{n \to \infty} D(X_n) = e^*(\Gamma).$$

By equations (18) and (21), we know $e^*(\Gamma) \geq D_{B'}(X)$, where $e^*(\Gamma)$ is a constant. Since B' is a strict subset of B, as $B' \to B$, $D_{B'}(X)$ approaches $D_B(X)$. By analysis, $e^*(\Gamma)$ will still be greater than $D_{B'}(X)$, as we take the limit of $D_{B'}(X)$, letting B' —> B. Consequently,

$$(22) \quad e^*(\Gamma) \geq \lim_{B' \to B} D_{B'}(X) = D_B(X)$$

From our previous work, we also know

$$(23) \quad e^*(\Gamma) \leq D_B(X)$$

Equations (22) and (23) suggest $D(X) = e^*(\Gamma)$.

Recall that $D(X) = D_B(X)$ by the definition of the Dirichlet integral. Therefore, $X \in \mathcal{C}^*(\Gamma)$ is a minimizer of the Dirichlet integral $D(X)$ within the class $\mathcal{C}(\Gamma)$.

CHAPTER THREE: THE MINIMAL SURFACE PROBLEM

Now that we have established that minimal surfaces do exist, we will now prove that the catenoid is the only surface of revolution which is also a minimal surface. The problem is stated as follows:

"Consider the surface of revolution f generated by the catenary (h(u), 0, k(u)), where

$$h(u) = a\cosh([k(u) - b]/a)$$

This surface is known as the catenoid. Prove that the catenoid is the only surface of revolution which is also a minimal surface."[8]

Solution:

First, I will prove that given $h(u) = a\cosh([k(u) - b]/a)$, then this surface of revolution, which is of the form

$$f(u,v) = (\, h(u)\cos v,\, h(u)\sin v,\, k(u)\,),$$

is a minimal surface. To prove this, I will show that the mean curvature of this surface equals zero. That is,

$$H = 1/2[\,\kappa_1(\mathbf{X}_1) + \kappa_2(\mathbf{X}_2)] = (EN + GL - 2FM)/2(EG - F^2) = 0.$$

To do this, I will show that the numerator, EN + GL - 2FM, equals zero. I will begin by computing the first and second partial derivatives with respect to u and v:

$$f_u = (h'(u)\cos v,\, h'(u)\sin v,\, k'(u))$$

$$f_{uu} = (h''(u)\cos v,\, h''(u)\sin v,\, k''(u))$$

$$f_v = (-h(u)\sin v,\, h(u)\cos v,\, 0)$$

$$f_{vv} = (-h(u)\cos v,\, -h(u)\sin v,\, 0)$$

$$f_{uv} = (-h'(u)\sin v,\, h'(u)\cos v,\, 0)$$

Next, I will find the unit normal vector **N**, defined to be

$$\frac{(f_u X f_v)}{| f_u X f_v |}$$

The numerator, $f_u X f_v$, equals

$= e_1(-h(u)k'(u)\cos v) - e_2(h(u)k'(u)\sin v) + e_3(h(u)h'(u)\cos^2 v + h(u)h'(u)\sin^2 v)$

$= (-h(u)k'(u)\cos v, -h(u)k'(u)\sin v, h(u)h'(u))$, and the denominator, $| f_u X f_v |$, is

$\{h^2(u)[k'(u)]^2\cos^2 v + h^2(u)[k'(u)]^2\sin^2 v + h^2(u)[h'(u)]^2\}^{1/2}$

$= \{(h^2(u)[k'(u)]^2 + h^2(u)[h'(u)]^2\}^{1/2}$

$= h(u)\{k'(u)^2 + h'(u)^2\}^{1/2}$.

Therefore,

$$\mathbf{N} = (-h(u)k'(u)\cos v, -h(u)k'(u)\sin v, h(u)h'(u))/h(u)([k'(u)]^2 + [h'(u)]^2)^{1/2}.$$

$$= (-k'(u)\cos v, -k'(u)\sin v, h'(u))/([k'(u)]^2 + [h'(u)]^2)^{1/2}.$$

Using the above information, we can find E,F,G,L,M and N:

$E := f_u \cdot f_u = [h'(u)]^2\cos^2 v + [h'(u)]^2\sin^2 v + [k'(u)]^2 = [k'(u)]^2 + [h'(u)]^2$.

$F := f_u \cdot f_v = -h(u)h'(u)\cos v\sin v + h(u)h'(u)\cos v\sin v + 0 = 0$.

$G := f_v \cdot f_v = h^2(u)\sin^2 v + h^2(u)\cos^2 v + 0 = h^2(u)$.

$L := f_{uu} \cdot \mathbf{N} = (k''(u)h'(u) - k'(u)h''(u))/([k'(u)]^2 + [h'(u)]^2)^{1/2}$.

$M := f_{uv} \cdot \mathbf{N} = h'(u)k'(u)\sin v\cos v/([k'(u)]^2 + [h'(u)]^2)^{1/2}$

$- h'(u)k'(u)\sin v\cos v/([k'(u)]^2 + [h'(u)]^2)^{1/2} + 0 = 0$.

$N := f_{vv} \cdot \mathbf{N} =$ $h(u)k'(u)\cos^2 v/(k'(u)^2 + h'(u)^2)^{1/2} + h(u)k'(u)\sin^2 v/([k'(u)]^2 + [h'(u)]^2)^{1/2}$

$= h(u)k'(u)/(k'(u)^2 + h'(u)^2)^{1/2}$.

Consequently, EN + GL -2FM equals

$\{[k'(u)]^2 + [h'(u)]^2\}\{h(u)k'(u)/(k'(u)^2 + h'(u)^2)^{1/2}\}$

$+ h^2(u)[(k''(u)h'(u) - k'(u)h''(u))/([k'(u)]^2 + [h'(u)]^2)^{1/2}]$, which equals

$\{h(u)[h'(u)]^2k'(u) + h(u)[k'(u)]^3 + h^2(u)h'(u)k''(u) - h^2(u)h''(u)k'(u)\}/([k'(u)]^2 + [h'(u)]^2)^{1/2}$

Again, to show that the above expression equals zero, I will show that the numerator

$$h(u)[h'(u)]^2k'(u) + h(u)[k'(u)]^3 + h^2(u)h'(u)k''(u) - h^2(u)h''(u)k'(u)$$

equals zero. First, I will take the first and second derivatives of

$$h(u) = a\cosh([k(u) - b]/a):$$

$$h'(u) = k'(u)\sinh([k(u) - b]/a)$$

$$h''(u) = k''(u)\sinh([k(u) - b]/a) + [k'(u)]^2/a\cosh([k(u) - b]/a)$$

Also,

$$h^2(u) = a^2\cosh^2([k(u) - b]/a), \text{ and}$$

$$[h'(u)]^2 = [k'(u)]^2\sinh^2([k(u) - b]/a)$$

Consequently, letting $q = [k(u) - b]/a$ and substituting the expressions for h(u) and h'(u), we will see that

$$h(u)[h'(u)]^2k'(u) + h(u)[k'(u)]^3 + h^2(u)h'(u)k''(u) - h^2(u)h''(u)k'(u)$$

$= a\cosh q k'(u)[k'(u)]^2\sinh^2 q + a\cosh q[k'(u)]^3 + a^2\cosh^2 q k''(u)k'(u)\sinh q$

$- a^2\cosh^2 q k'(u)\{k''(u)\sinh q + [k'(u)]^2/a\cosh q\}$

$= a\cosh q[k'(u)]^3\sinh^2 q + a\cosh q[k'(u)]^3 + a^2\cosh^2 q k''(u)k'(u)\sinh q$

$- k'(u)k''(u)a^2\cosh^2 q\sinh q - a[k'(u)]^3\cosh^3 q$

$= a\cosh q[k'(u)]^3\sinh^2 q + a\cosh q[k'(u)]^3 - a[k'(u)]^3\cosh^3 q$

$= a\cosh q[k'(u)]^3\{\sinh^2 q + 1 - \cosh^2 q\}$

$= a\cosh q[k'(u)]^3\{\cosh^2 q - \cosh^2 q\}$

$= a\cosh q[k'(u)]^3\{0\} = 0.$

Our surface of revolution, therefore, defined by

$$f(u,v) = (\, h(u)\cos v, h(u)\sin v, k(u)\,)$$

$$= (a\cos v\cosh [k(u) - b]/a, a\sin v\cosh [k(u) - b]/a, k(u))$$

is, by definition, a minimal surface since its mean curvature equals zero. Figure 19 provides an illustration of a catenoid.

Given that the mean curvature equals zero; that is EN + GL - 2FM = 0, which is equivalent to

$$[h'(u)]^2k'(u) + [k'(u)]^3 + h(u)h'(u)k''(u) - h(u)h''(u)k'(u) = 0,$$

we will solve this differential equation for h(u) and k(u). We will begin by making the following substitutions:

$$h(u) = t \text{ and } k = k(u) = k(t)$$

Consequently, using the chain rule and the product rule for differentiation, we have the following:

$$\frac{dh}{dt} = 1$$

$$h'(u) = \frac{dh}{du} = \frac{dh}{dt} \cdot \frac{dt}{du} = \frac{dt}{du}$$

$$h''(u) = \frac{d^2h}{du^2} = \frac{d}{du}\left(\frac{dh}{du}\right) = \frac{d}{du}\left(\frac{dt}{du}\right) = \frac{d^2t}{du^2}$$

$$k'(u) = \frac{dk}{du} = \frac{dk}{dt} \cdot \frac{dt}{du}$$

$$\begin{aligned} k''(u) &= \frac{d^2k}{du^2} = \frac{d}{du}\left(\frac{dk}{dt} \cdot \frac{dt}{du}\right) \\ &= \frac{d}{du}\left(\frac{dk}{dt}\right) \cdot \frac{dt}{du} + \frac{dk}{dt} \cdot \frac{d}{du}\left(\frac{dt}{du}\right) \\ &= \frac{d^2k}{dt^2} \cdot \frac{dt}{du} \cdot \frac{dt}{du} + \frac{dk}{dt} \cdot \frac{d^2t}{du^2} \\ &= \frac{d^2k}{dt^2}\left(\frac{dt}{du}\right)^2 + \frac{dk}{dt} \cdot \frac{d^2t}{du^2} \end{aligned}$$

Substituting these expressions into

$[h'(u)]^2k'(u) + [k'(u)]^3 + h(u)h'(u)k''(u) - h(u)h''(u)k'(u) = 0$, we have:

$$\left(\frac{dt}{du}\right)^2\left(\frac{dk}{dt}\cdot\frac{dt}{du}\right)+\left(\frac{dk}{dt}\cdot\frac{dt}{du}\right)^3+\frac{tdt}{du}\left[\left(\frac{d^2k}{dt^2}\cdot\left(\frac{dt}{du}\right)^2\right)+\frac{dk}{dt}\cdot\frac{d^2t}{du^2}\right]-\left(\frac{td^2t}{du^2}\right)\left[\frac{dk}{dt}\cdot\frac{dt}{du}\right]=0$$

$$\rightarrow\left(\frac{dt}{du}\right)^2\left(\frac{dk}{dt}\cdot\frac{dt}{du}\right)+\left(\frac{dk}{dt}\cdot\frac{dt}{du}\right)^3+\frac{tdt}{du}\cdot\frac{d^2k}{dt^2}\left(\frac{dt}{du}\right)^2+$$

$$\frac{tdt}{du}\cdot\frac{dk}{dt}\cdot\frac{d^2t}{du^2}-\frac{td^2t}{du^2}\cdot\frac{dk}{dt}\cdot\frac{dt}{du}=0$$

$$\rightarrow\left(\frac{dt}{du}\right)^3\left[\frac{dk}{dt}+\left(\frac{dk}{dt}\right)^3+\frac{td^2k}{dt^2}\right]=0$$

$$\rightarrow\quad\frac{dk}{dt}+\left(\frac{dk}{dt}\right)^3+\frac{td^2k}{dt^2}=0$$

$$\rightarrow\left(1+\left(\frac{dk}{dt}\right)^2\right)\frac{dk}{dt}+\frac{td^2k}{dt^2}=0$$

Letting $k' = k'(t) = \frac{dk}{dt}$ and $k'' = k''(t) = \frac{d^2k}{dt^2}$, we have

$$(1+(k')^2)\,k'+tk''=0$$

Now when we multiply this equation by k', we obtain

$$(k')^2+(k')^4+tk'k''=0$$

Letting $y = (k')^2$ and $\frac{dy}{dt} = 2k'k''$ our equation becomes

$$y+y^2+\frac{tdy}{2dt}=0,$$

which is equivalent to

$$\frac{dy}{y(1+y)}+\frac{2dt}{t}=0$$

When we integrate this equation, we will have

$$\ln|y| - \ln|1 + y| + 2\ln|t| = c,$$

for some constant c. Thus,

$$\ln|y/(1+y)| + \ln|t^2| = c, \text{ or } \ln\left(\frac{yt^2}{1 + y}\right) = c$$

Since $e^c > 0$,, we can let $e^c = a^2$, where a is another constant. Consequently,

$$yt^2/(1 + y) = a^2$$

This equation is equivalent to

$$yt^2 = a^2 (1 + y).$$

$$\text{Since } y = (k')^2 = [k'(u)]^2,$$

$$(k')^2 t^2 = a^2 (1 + (k')^2).$$

Solving for k', we have

$$k' = a/(t^2 - a^2)^{1/2}.$$

Integrating will produce the following

$$k = a\cosh^{-1} \frac{t}{a} + b,$$

where b is another constant. When we solve for t we will have

$$t = a\cosh[(k -b)/a].$$

Recalling that t = h(u) and k = k(u), we can rewrite this equation as

$$h(u) = a\cosh[(k(u) - b/a)].$$

CHAPTER FOUR: PARALLEL TRANSLATION AND GEODESICS

We will now explore two other properties of Differential Geometry, Parallel Translation and Geodesics, found in Wilhelm Klingenberg's A Course in Differential Geometry, to highlight the differences between Euclidian Geometry and the geometry on the surfaces. First, we will need to develop some background on the following topics:

(1) Tangential Vector Fields

(2) Orthogonal Projections

(3) Covariant Derivatives

By definition, a vector field is a mapping which corresponds points in $\mathbb{R}^2$, the domain, to vectors in $\mathbb{R}^3$. Likewise, a tangential vector field X: I —> $\mathbb{R}^3$, given by the vector function $X(t) = au(t)f_u + bv(t)f_v = (x(t), y(t), z(t))$ is a mapping from the real numbers in the interval I, which determines points on a surface, to tangent vectors, tangent to the surface at points given as a function of real numbers in the interval I. Selecting different points $t_i \in I$ will produce different points on the surface, and the tangential vector field X: I —> $\mathbb{R}^3$, will map these points to their respective tangent vectors, tangent to the surface at that point. Observe that there exists an infinite number of tangent vectors to a surface at a given point. Similarly, the function dX/dt: I—> $\mathbb{R}^3$ defined by $dX(t)/dt = X'(t) = (x'(t), y'(t), z'(t))$ is a mapping assigning vectors to real numbers in the interval I which determines some point on the surface. The function $pr_u: T_{f(u)}\mathbb{R}^3 \to T_uf$ is defined as an orthogonal projection in the direction of the normal vector n(u) mapping arbitrary vectors of the form $af_1 + bf_2 + cn$ in $T_{f(u)}\mathbb{R}^3 = \mathbb{R}^3$ to their corresponding tangent vectors of the form $af_1 + bf_2$ lying on the tangent space T_uf. Notice that

$T_{f(u)}\mathbb{R}^3$ represents the tangent plane of $\mathbb{R}^3$ at the point f(u), which is $\mathbb{R}^3$ itself; this is why $T_{f(u)}\mathbb{R}^3 = \mathbb{R}^3$. In short, pr_u projects a vector in $\mathbb{R}^3$ to the tangent plane T_uf, as shown in figure 20, where our surface is a hemisphere. Observe that if the vector we are projecting is parallel to the normal vector at that point, the orthogonal projection will be zero.

We are now ready to define the covariant derivative, denoted by $\Delta X(t)/dt$, as the composition of pr_u and dX/dt; that is, $\Delta X(t)/dt = pr_u \circ (dX/dt)(t): I \rightarrow T_uf$, where $t \in I$, as illustrated in figure 21. In this example, the tangent plane T_uf is parallel to the xy-plane. Note that $\Delta X(t)/dt: I \rightarrow \mathbb{R}^3$, is, by definition, a tangential vector field and $\Delta X(t)/dt = 0$ if $dX(t)/dt$ is parallel to the normal vector at a given point; in figure 21, the normal (unit) vector is the vector (0,0,1), parallel to the z-axis. To clarify, since the orthogonal projection pr_u projects the vector $dX(t)/dt$ onto the tangent plane, when $dX(t)/dt$ is parallel to the z-axis, the orthogonal projection will be zero. When the orthogonal projection equals zero, $\Delta X(t)/dt$ also equals zero.

Equipped with this background, we can begin our discussion of parallel translation. Covariant differentiation will be used to define what it means for vectors to be parallel along a curve on a surface. We will first consider two cases where our surface is the Euclidian plane $\mathbb{R}^2$. Let $X_1: I \rightarrow \mathbb{R}^3$ be a tangential vector field. In a plane, the image of this vector field $X_1(t)$, a set of vectors, along a curve, c(t), is defined to be constant, or parallel, if its value is constant: for all $t \in I$, $X_1(t) = X_0 = (a,b)$, where $a,b \in \mathbb{R}$. That is, all points on the plane are sent to a single vector X_0, and since $X_1(t)$ is a constant, $dX_1(t)/dt = (a,b)' = (0,0) = \mathbf{0}$. For example, suppose X_1 maps all the points on an interval I to the vector X_0, along our "curve" (a line) lying in the surface $\mathbb{R}^2$, as depicted in figure 22.

Here, all these vectors along the line l are the same vector X_0 being translated along l and are all tangent to $\mathbb{R}^2$. In the context of Euclidian Geometry, these vectors in the above figure resemble parallel vectors on the plane. As another example, suppose X_2 were a tangential vector field mapping points on a curve lying in a plane to their respective tangent vectors on the surface $\mathbb{R}^2$; similar to X_1, for all $t \in I$, $X_2(t) = X_0$, a constant vector being translated along the curve on the plane, as shown in figure 23. Again, these vectors are all tangent to the plane and in the context of Euclidian Geometry, these vectors in the above figure resemble parallel vectors on the plane.

We will now consider what it means for tangent vectors of a curve lying on a nonplanar surface to be parallel: by definition given a curve $c = f \circ u: I \longrightarrow \mathbb{R}^3$ on a surface $f: U \longrightarrow \mathbb{R}^3$, a vector field X along c is parallel along c provided $\Delta X(t)/dt = pr_u(dX(t)/dt) = 0$; this occurs when $dX(t)/dt$ is parallel to its normal vector at a given point. As indicated earlier, a surface in space has an infinite number of tangent vectors at a particular point; however, a curve on a surface has only one tangent vector at a particular point. To illustrate such a vector field X, let us examine the unit sphere S^2 given by the equation

$$f(u,v) = (\cos u \cos v, \cos u \sin v, \sin u).$$

As before, the unit normal vector at a given point on the surface is defined to be

$$\mathbf{N}(u,v) = \frac{(f_u X f_v)}{| f_u X f_v |}$$

For the unit sphere, we can find $\mathbf{N}(u,v)$ as follows:

Find the partial derivatives of f with respect to u and v.

$f_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$ and $f_v = (-\sin v \cos u, \cos u \cos v, 0)$.

Consequently,

$f_u X f_v = (-\cos^2 u \cos v)e_1 - (0-(-\cos^2 u \sin v)e_2 + (-\sin u \cos u \cos^2 v - \sin^2 v \cos u \sin u)e_3$

$= -\cos^2 u \cos v e_1 - \cos^2 u \sin v e_2 - \sin u \cos u e_3$

$= (-\cos^2 u \cos v, -\cos^2 u \sin v, -\sin u \cos u)$ and

$|f_u X f_v| = [\cos^4 u \cos^2 v + \cos^4 u \sin^2 v + \sin^2 u \cos^2 u]^{1/2}$

$= [\cos^4 u(\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u]^{1/2}$

$= [\cos^4 u + \sin^2 u \cos^2 u]^{1/2}$

$= [\cos^2 u(\cos^2 u + \sin^2 u)]^{1/2}$

$= \cos u.$

Therefore, $\dfrac{(f_u X f_v)}{|f_u X f_v|}$ is equal to (-cosucosv, -cosusinv, -sinu) = -f(u,v).

This tells us that for a tangent vector of our surface at a given point, we can find its corresponding unit normal vector merely by taking the additive inverse of the vector associated with that given point. To illustrate this, we will focus on a specific set of points lying on S^2: points on the the great circle c parametrically defined by c(t) = (0, cost, sint); this great circle is the intersection of S^2 and the yz-plane. Clearly, since c is a subset of S^2, all the points on c given by c(t) = (0, cost, sint) also lie on the sphere. So by our previous computation, where we found that **N**(u,v) = -f(u,v), we know that the normal vector of a tangent vector on the great circle c at the point c(t) on c is the additive inverse of c(t): -c(t). This surface together with the set of its distinct tangent vectors, the image of the tangential vector field X: I —> $\mathbb{R}^3$, and its normal vectors along the great circle is illustrated figure 24. We will define this tangential vector field by X(t) = c'(t) = (0, -sint, cost). For example, the point c(π/2) = (0,0,1) on the curve has as a (unit) tangent vector c'(π/2) = (0,-1,0) and a unit normal vector -c(π/2) = (0,0,-1). Similarly, at the point c(0) = (0,1,0), we have the tangent vector c'(0) = (0,0,1) and the unit normal vector

$-c(0) = (0, -1, 0)$. These tangent vectors are elements of the image of the tangential vector field. Notice that $dX(t)/dt = X'(t) = c''(t) = (0, -\cos t, -\sin t) = -c(t)$, which implies that $dX(t)/dt$ is parallel to the unit normal vector $-c(t)$. Therefore, $\Delta X(t)/dt = pr_u(dX(t)/dt) = 0$, and so, by definition, the tangential vector field X along the great circle c is parallel along c.

Another property of differential geometry, related to parallel translation, is geodesics. To understand the concept of geodesics, we need to comprehend the notion of geodesic curvature on a surface. Before proceeding, however, let us begin by exploring curvature on the Euclidian plane. In this context, curvature is the measure of how fast a curve on a plane turns at a given point as we move along it. For example, consider the curve $c:[a,b] \longrightarrow \mathbb{R}^2$ in figure 25. Loosely speaking, since there is more of a bend at D than at point F, we claim that the curvature is greater at point D, compared to point F. To make the notion of curvature more precise, we will now examine the previous figure accompanied by its unit tangent vectors **T** at various points, as shown in figure 26. Observe that as we move along the curve on the plane, the unit tangent vectors **T** turns as the curve bends. On the plane, the mathematical definition of curvature at a given point on the curve is given by $\kappa = |dt/ds|$, where s denotes the arc length and t the angle formed by the unit tangent vector **T** and $\mathbf{i} = (1,0)$; therefore dt is the change in the angle t and ds the change in arc length s. Since $\kappa = |dt/ds|$, the curvature is greatest where t changes the most rapidly. It is, therefore, greatest where the tangent vector changes the most rapidly. For instance, there is a major difference between the tangent vector at point D, and a point near D, say point E. Thus, the curvature at point D is large. Conversely, at point F and at a point near F, say point G, there is little difference between the corresponding tangent vectors, suggesting that the curvature at point F is small.

On a straight line, the angle t remains the same on each point on the line. Consequently, since there is no change in angle, the curvature of the line at all points is zero. In addition, the curvature at each point on a circle of radius r is a constant 1/r since the unit tangent vectors **T** turn at a constant rate. These two figures are illustrated in figure 27.

Similar to curvature on a plane, geodesic curvature is the measure of how fast a curve on a surface turns at a given point as we move along the curve. Geodesic curvature, therefore, is a local quality of a curve at a given point. Let us examine the curve shown in figure 28. At the point c(t) on the curve, the geodesic curvature is defined to be $\kappa_g(t) = e_2(t) \cdot ((\Delta e_1(t)/dt)/|c'(t)|$. Let us examine each component of the equation above:

(a) $e_1(t) = c'(t)/|c'(t)|$. This is the unit tangent vector of the curve at the point c(t)

(b) $e_2(t) = e_1'(t)$

(c) The covariant derivative, $\Delta e_1(t)/dt$, is formed by projecting $e_2(t)$ onto the tangent plane, obtaining a tangent vector.

(d) To find the geodesic curvature, we divide the dot product of $\Delta e_1(t)/dt$ and $e_2(t)$ and by $|c'(t)|$. Again, this quantity represents the amount of bend at the given point c(t).

By definition, a curve is geodesic when its geodesic curvature ,$\kappa_g(t)$, equals zero at all points on the curve. This occurs when $e_2(t)$ equals, or is parallel to, the normal vector of the tangent vector at a given point. To illustrate such a curve, let us again consider the unit sphere S^2 defined by the equation

$$f(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

and the great circle lying on the xy-plane defined by

$$c(t) = (\cos t, \sin t, 0).$$

We will now calculate the geodesic curvature at the point c(t) on this curve, shown in figure 29, using the steps below:

(a) $e_1(t) = c'(t)/|c'(t)|$

$c'(t) = (-\sin t, \cos t, 0)$ and $|c'(t)| = [(-\sin t)^2 + (\cos t)^2 + 0^2]^{1/2} = 1$.

Thus, $e_1(t) = c'(t) = (-\sin t, \cos t, 0)$.

(b) $e_2(t) = e_1'(t) = (-\cos t, -\sin t, 0)$.

(c) To find the covariant derivative, $\Delta e_1(t)/dt$, we project $e_2(t)$ onto the tangent plane at the point c(t). Before doing so, however, notice that since $\mathbf{N}(u,v) = -f(u,v)$, at the point c(t), the normal vector equals $-c(t) = (-\cos t, -\sin t, 0) = e_2(t)$. Since $e_2(t)$ equals the normal vector, the projection of $e_2(t)$ on the tangent plane will yield zero, ensuring that the geodesic curvature at that point is zero. All the points on this circle have this property; consequently, this curve, the great circle, is, by definition, a geodesic. In fact, all the great circles on the sphere are geodesics.

However, the non great circles on the sphere, the latitude circles, are not geodesics. For instance, consider the latitude circle of radius r and a units from the origin, shown in figure 30, given by the equation

$$d(t) = (r\cos t, r\sin t, a).$$

The unit tangent vector $e_1(t)$ equals $(-r\sin t, r\cos t, 0)/(r^2 + a^2)^{1/2}$ and $e_2(t)$ equals $(-r\cos t, -r\sin t, 0)/(r^2 + a^2)^{1/2}$. Since the normal vector, $-d(t) = (-r\cos t, -r\sin t, -a)$, does not equal $e_2(t)$, the geodesic curvature is not equal to 0; thus, this circle is not a geodesic.

CHAPTER FIVE: AN APPLICATION OF A MINIMAL SURFACE

The concept of minimal surfaces is an outgrowth of experiments involving soap films stretched across closed twisted wire frames. In the analysis of the Plateau Problem given in the second chapter, the closed Jordan curve Γ played the role of the twisted wire frame. The architect Frei Otto, inspired by the elegance and economy displayed by these soap films, designed exhibition halls, arenas and stadiums. His goal was to utilize the least amount of construction material to create strong lightweight structures that were easily erected, dismantled and moved, as well as be able to withstand the destructive forces of nature.

To design such a lightweight structure, for example, a roof, Otto would begin his task by constructing a plexiglass plate studded with thin rods of varying heights. These rods would have drooping threads defining the edges and ridges of the roof. This model would then be immersed into a soap solution, and, when withdrawn, would reveal a tent like shape. The resulting soap film, stretching out only as far as it must, pulls the threads taut to create a spectacular scalloped roof. Similar to the surface defined by the image of the mapping $f(u,v) = (u, v, 1 - u^2 + v^2)$ which we examined in chapter one, every section of this roof is shaped like a horse's saddle. Next, the soap film model is carefully photographed and measured in order to build a solid representative of the structure; this miniature is tested in wind tunnels to determine the potential impact of wind, rain and snow. If the model pasts these tests, Otto would use it as a basis to design his roof. In the actual construction of the roof, sheets of synthetic material serve the role of the soap film and steel cables play the part of the threads. This model also enables Otto and his colleagues to study new ways of utilizing minimal surfaces to design structures of optimal shape for a given contour or boundary.[9]

APPENDIX A: FIGURES

Figure 1

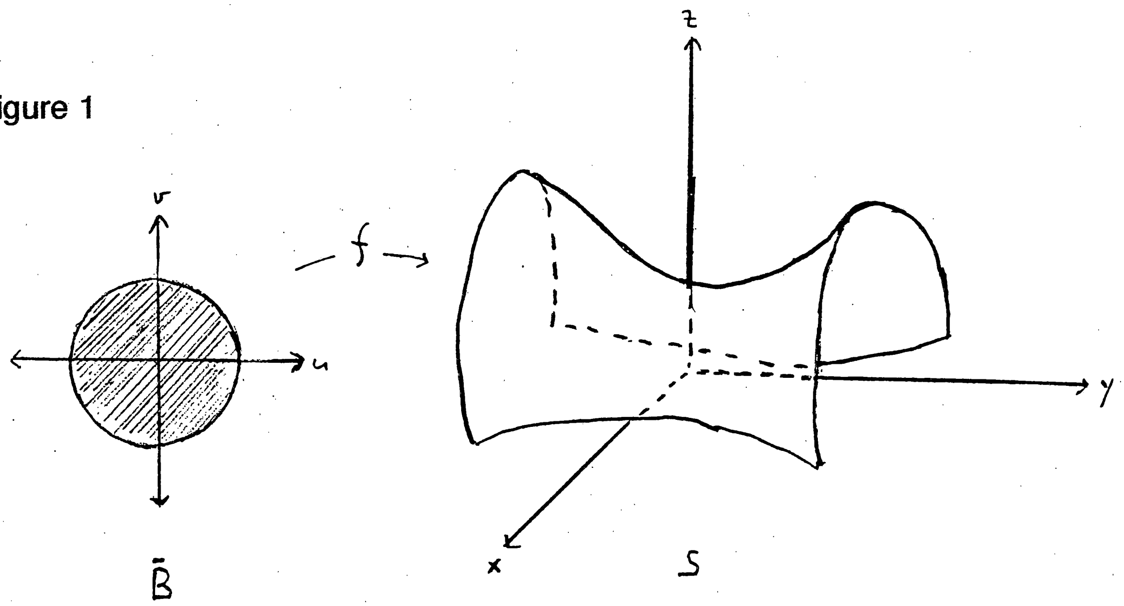


Figure 2

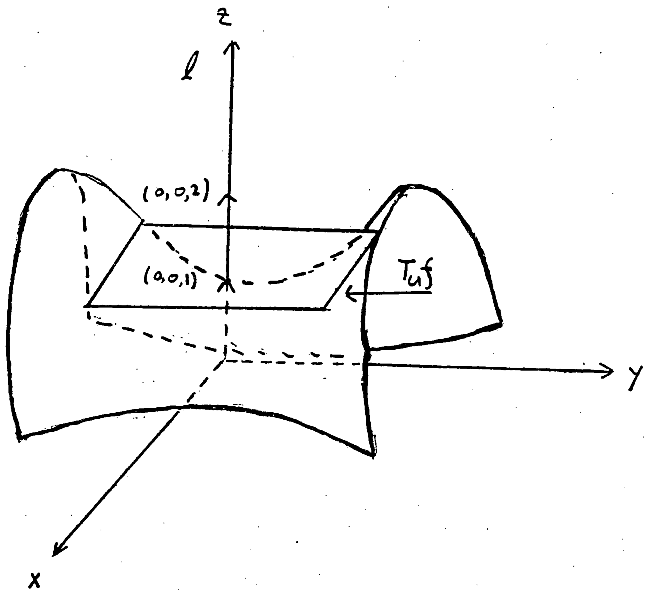


Figure 3

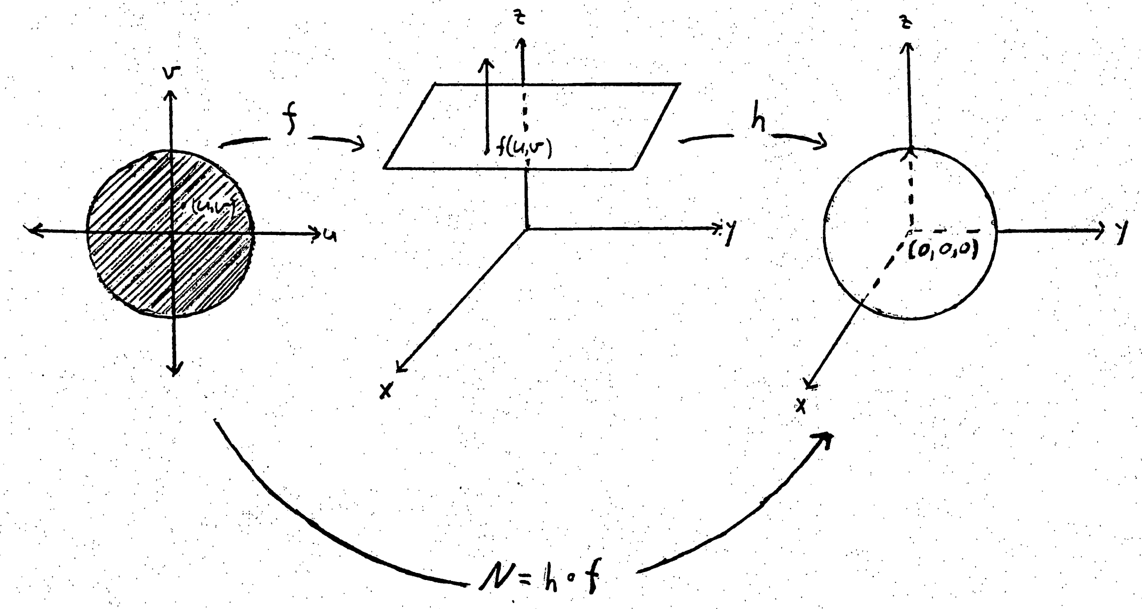


Figure 4

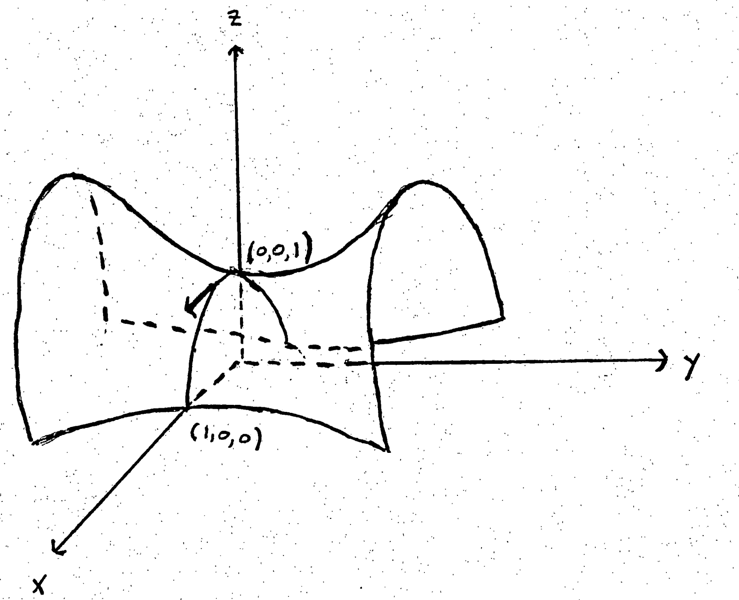


Figure 5

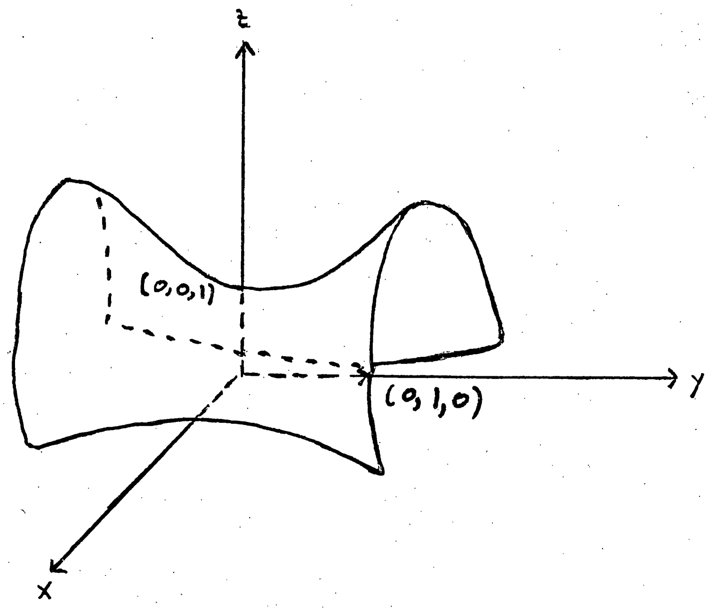


Figure 6

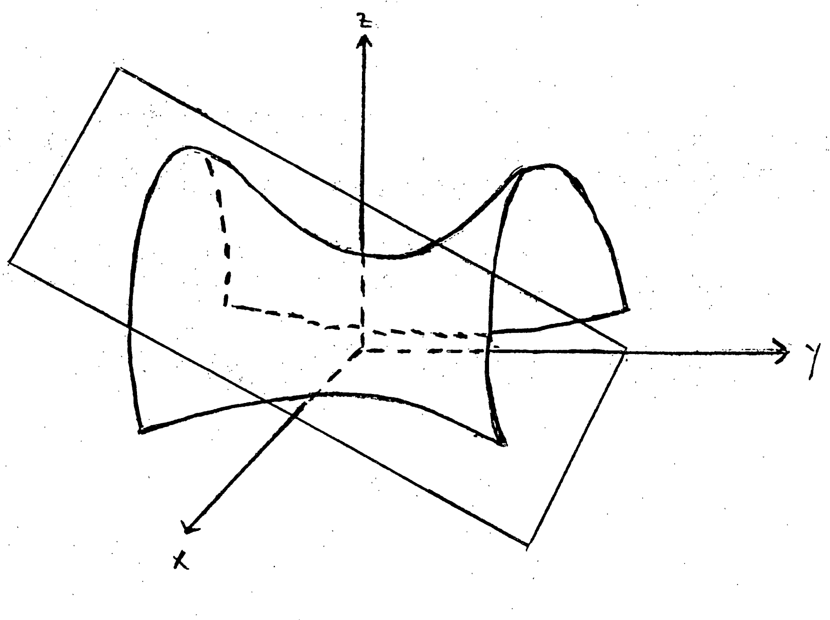


Figure 7

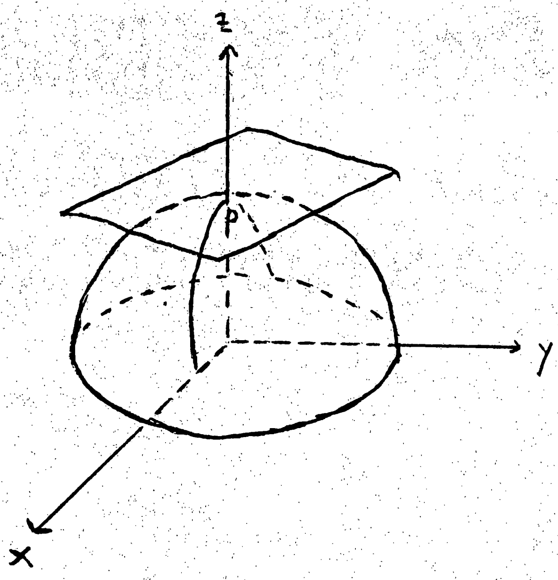


Figure 8

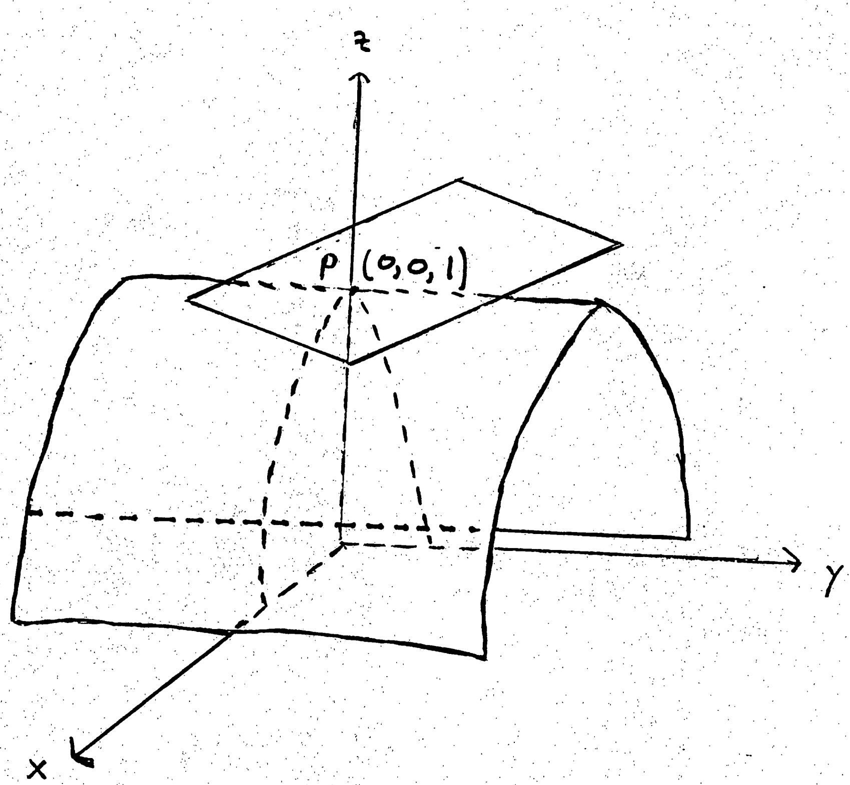


Figure 9

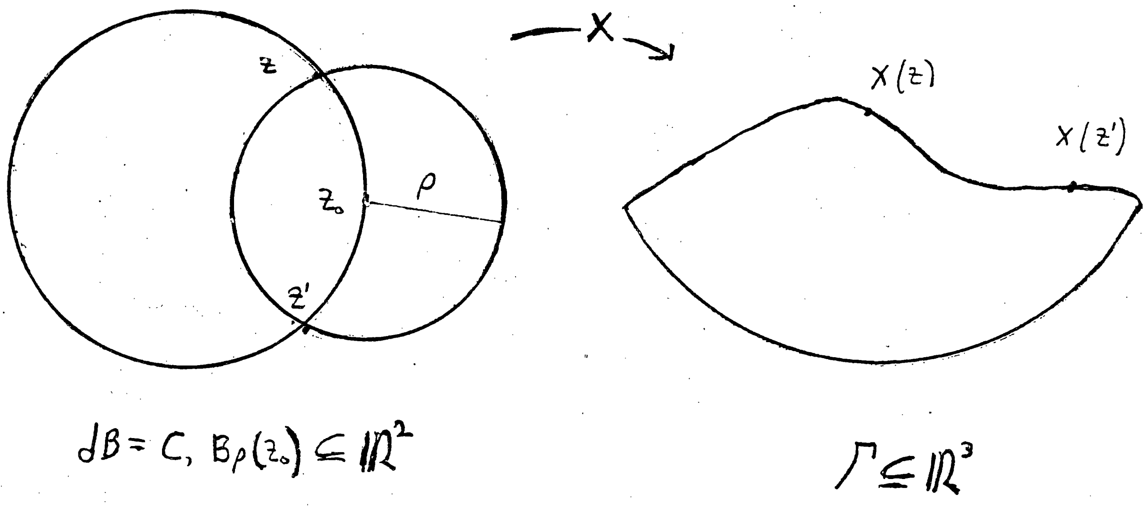


Figure 10

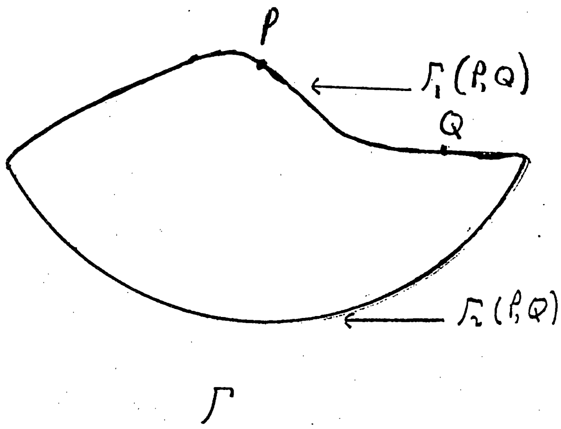


Figure 11

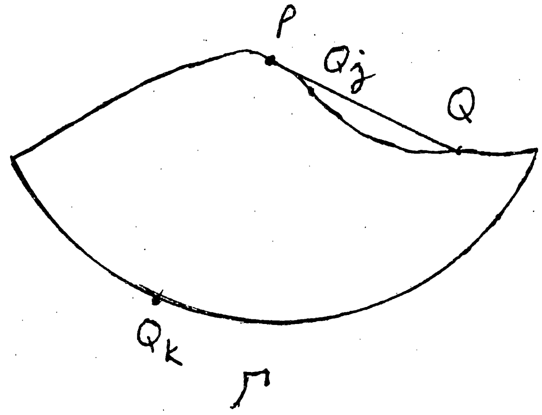


Figure 12

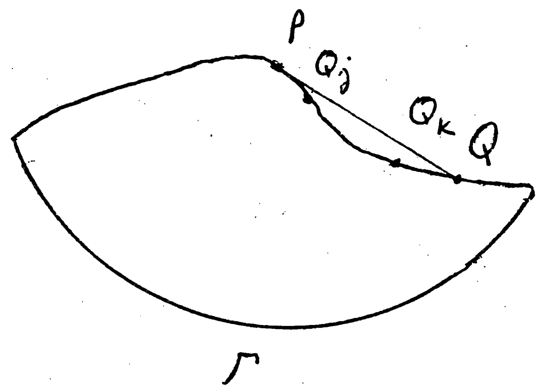


Figure 13

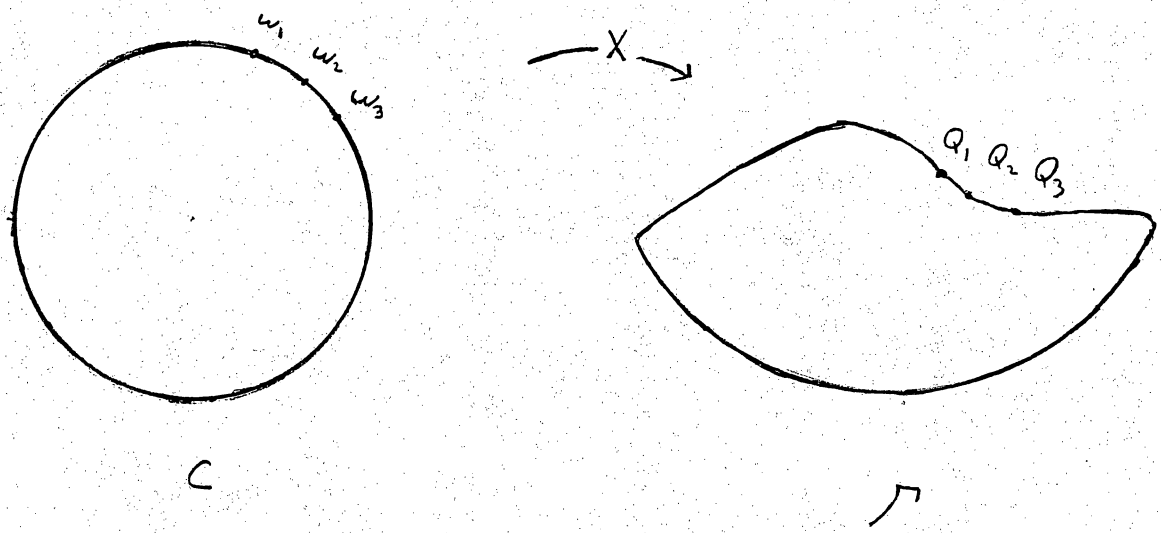


Figure 14

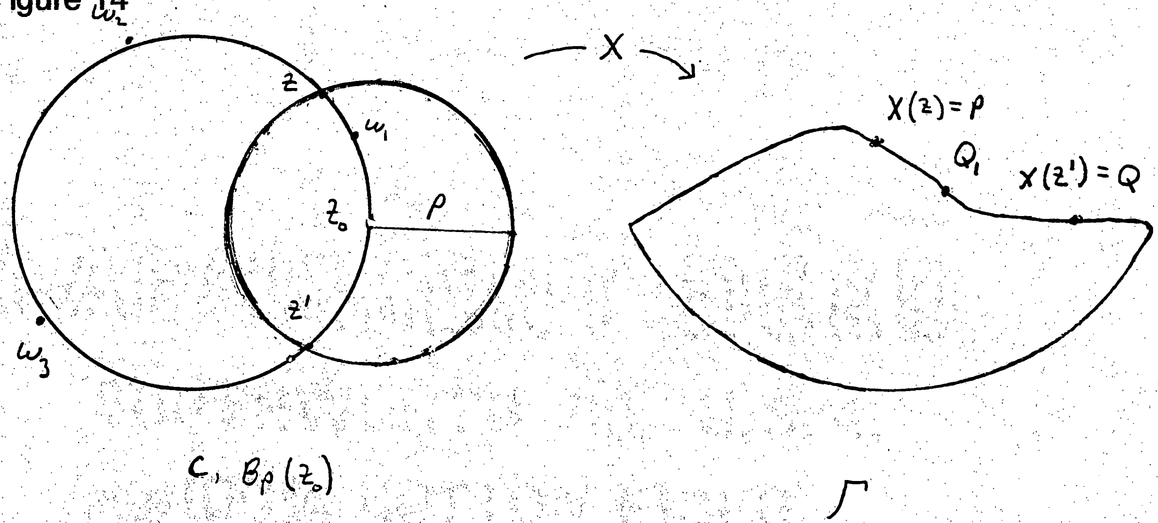


Figure 15

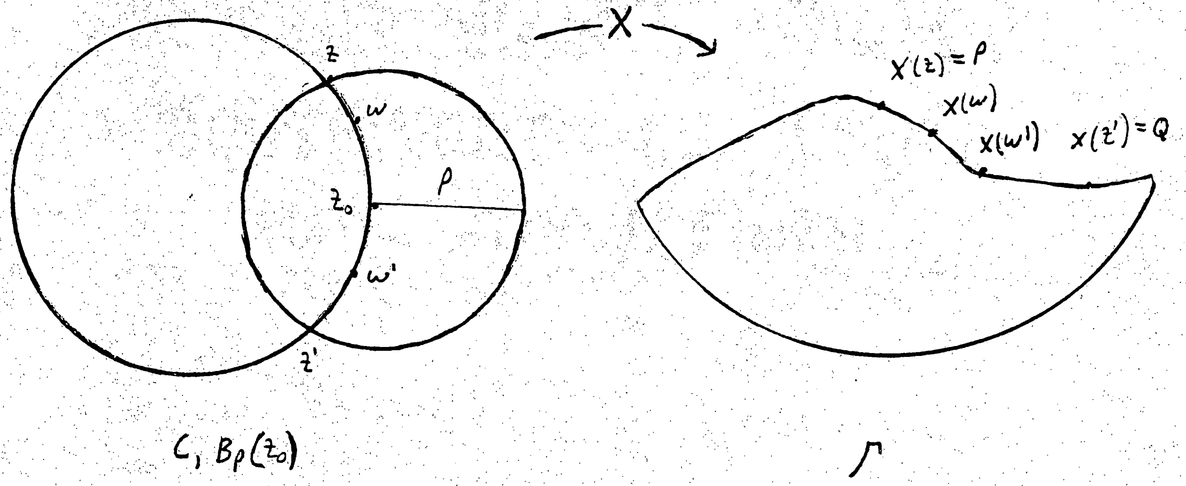


Figure 16

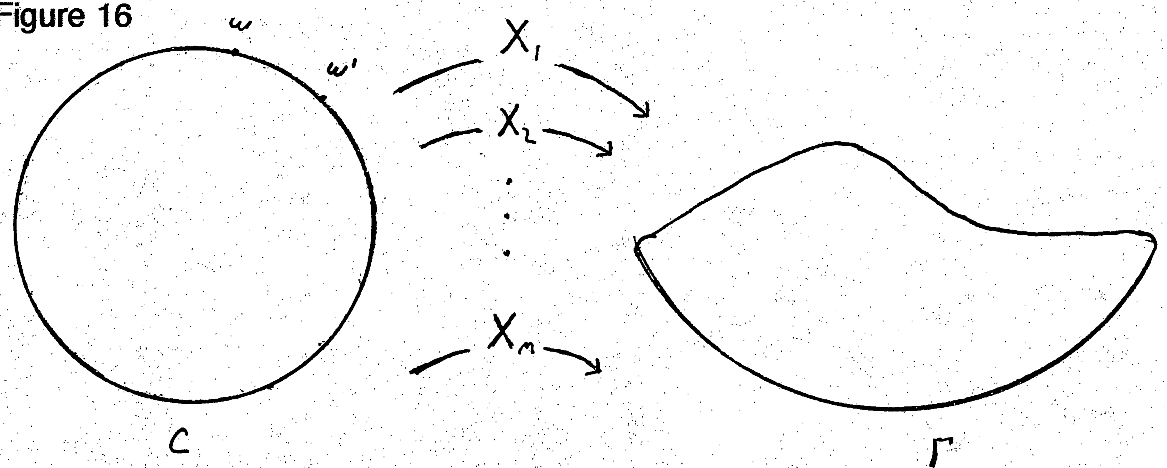


Figure 17

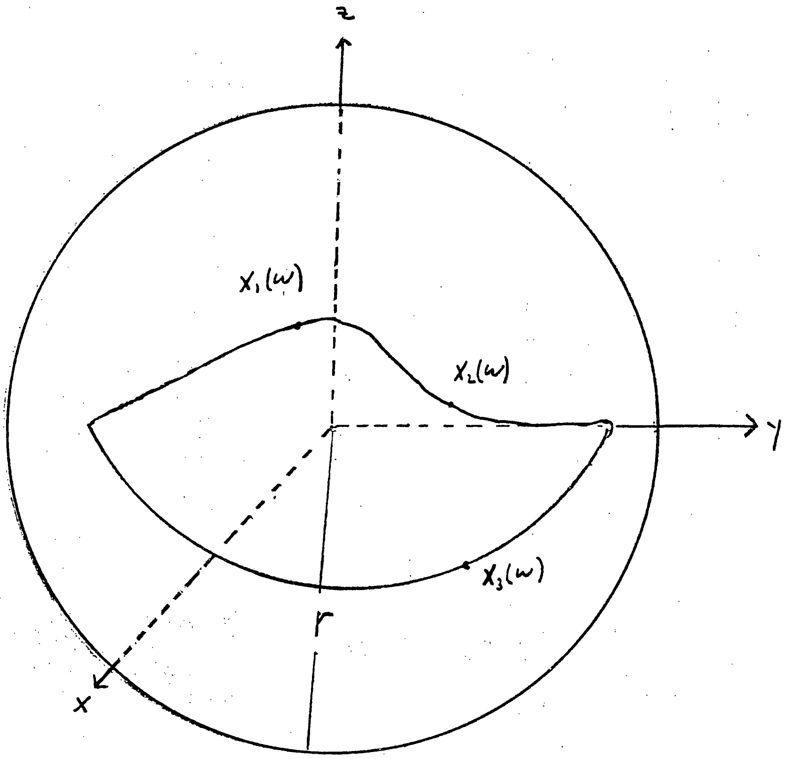


Figure 18

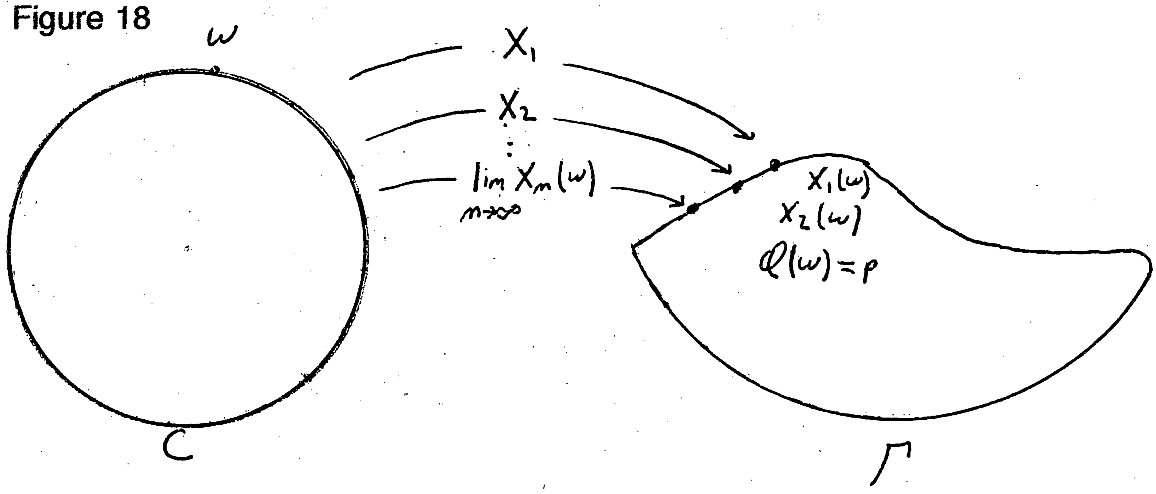


Figure 19

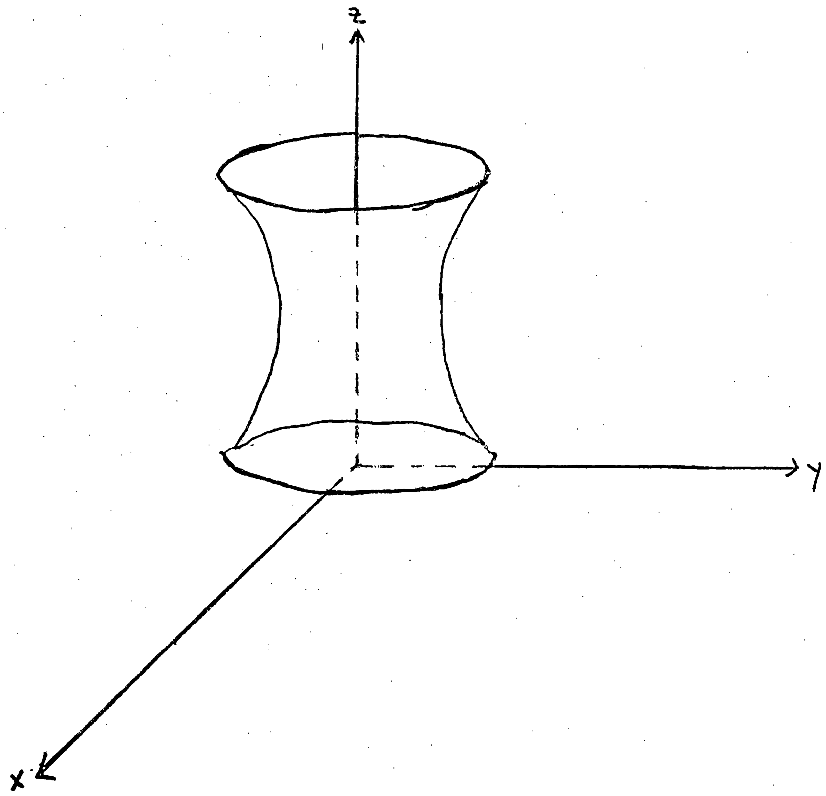


Figure 20

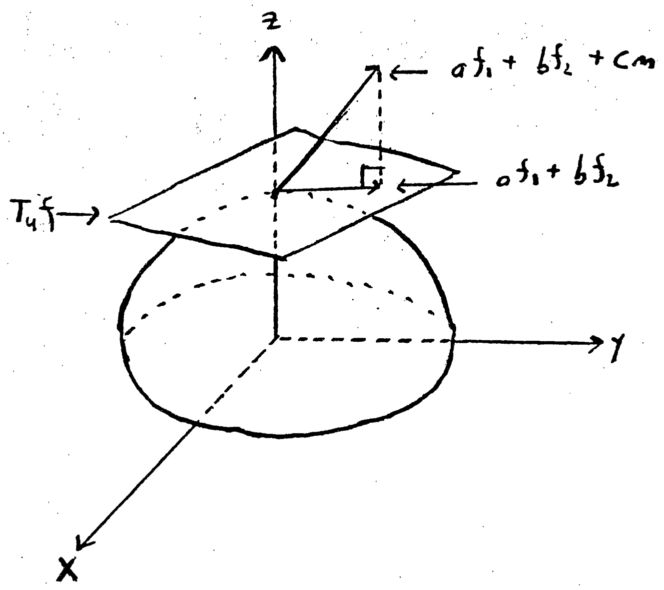


Figure 21

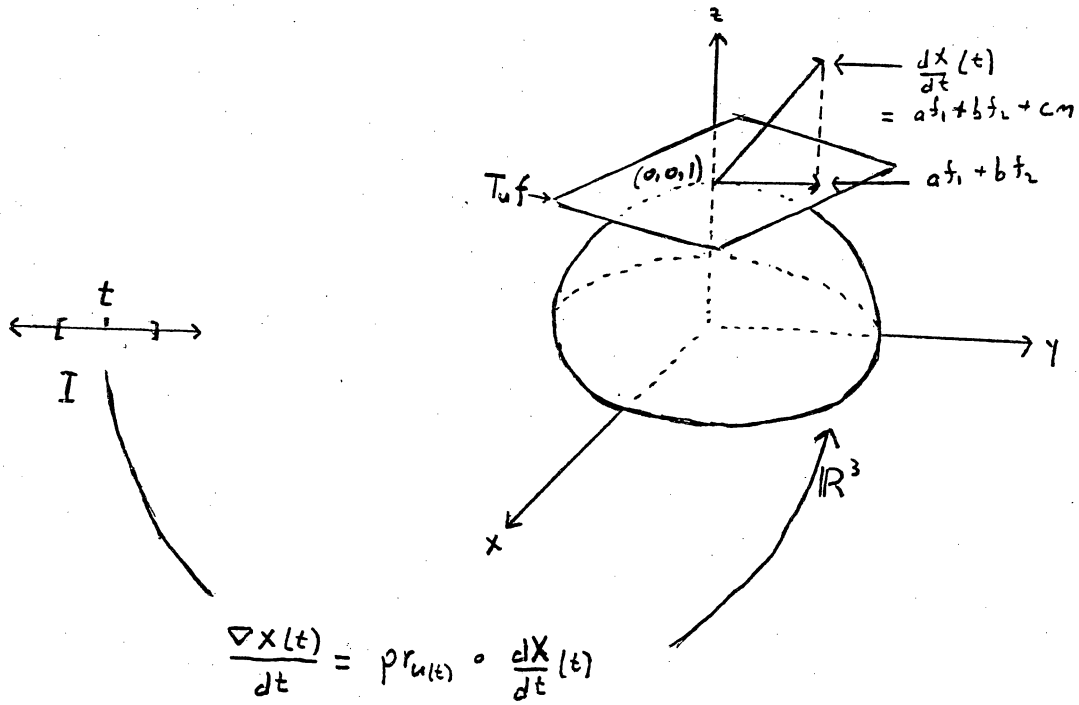


Figure 22

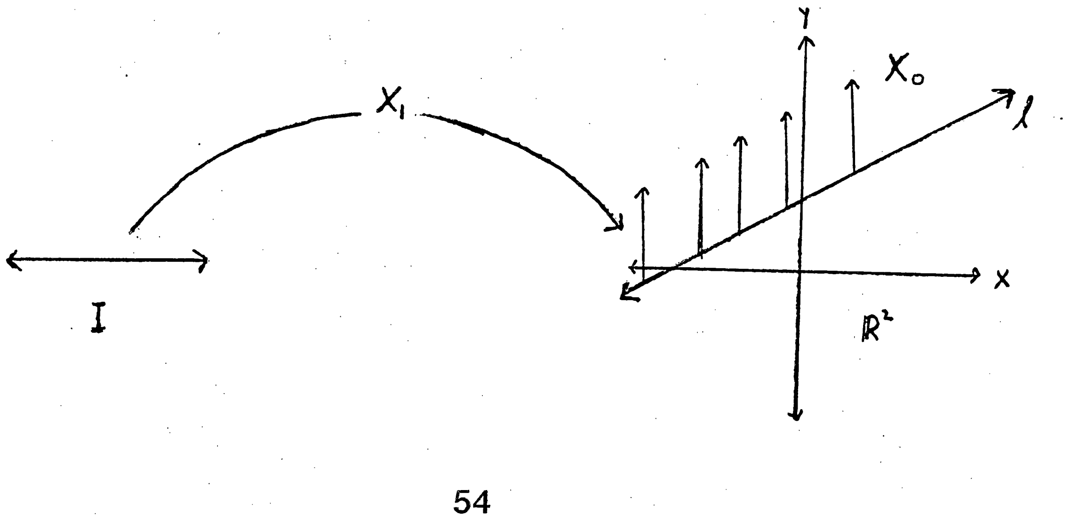


Figure 23

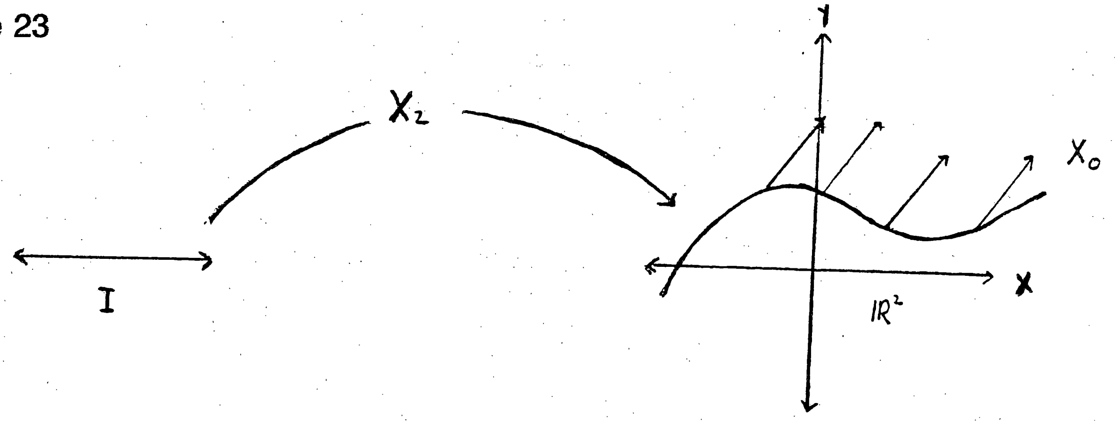


Figure 24

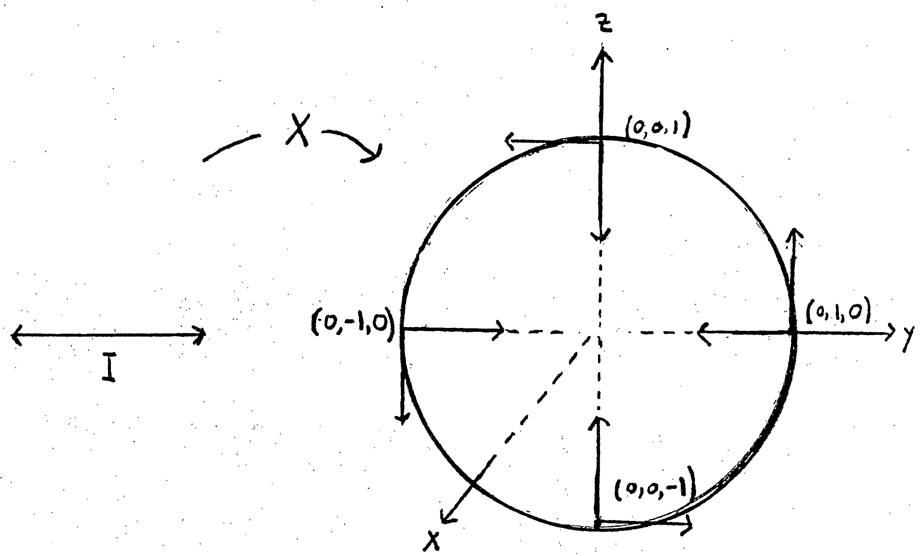


Figure 25

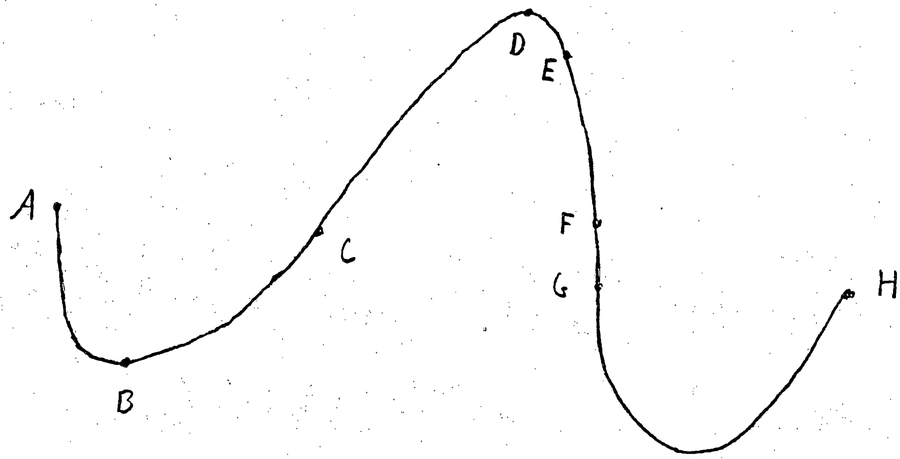


Figure 26

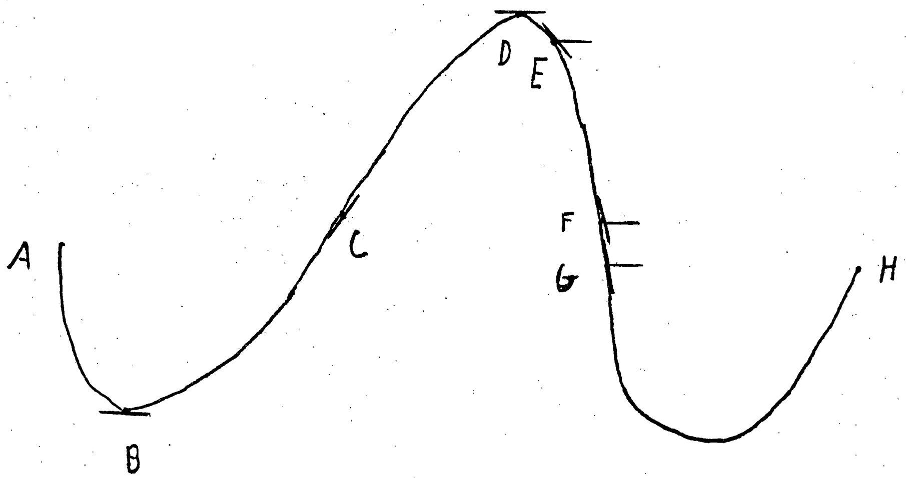


Figure 27

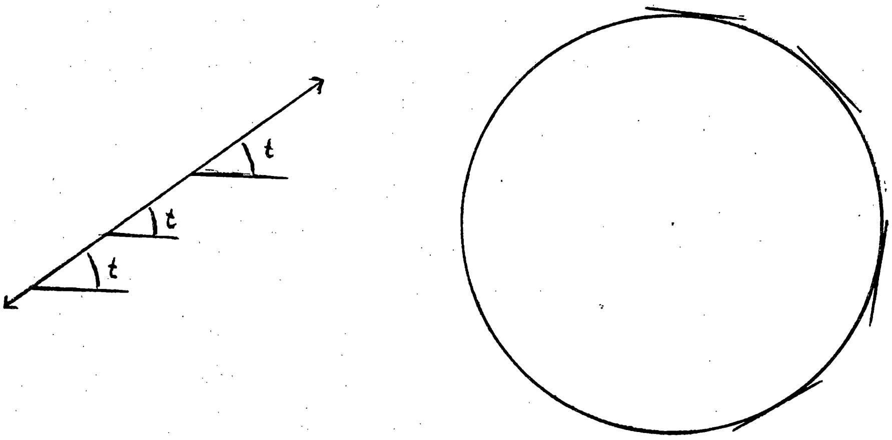


Figure 28

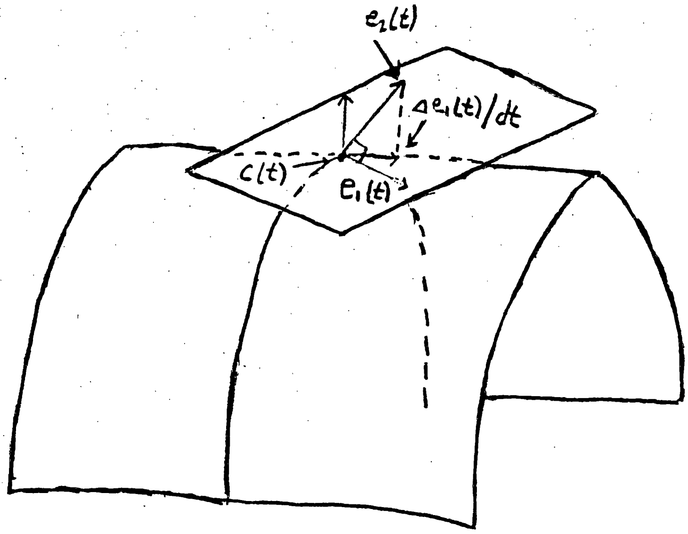


Figure 29

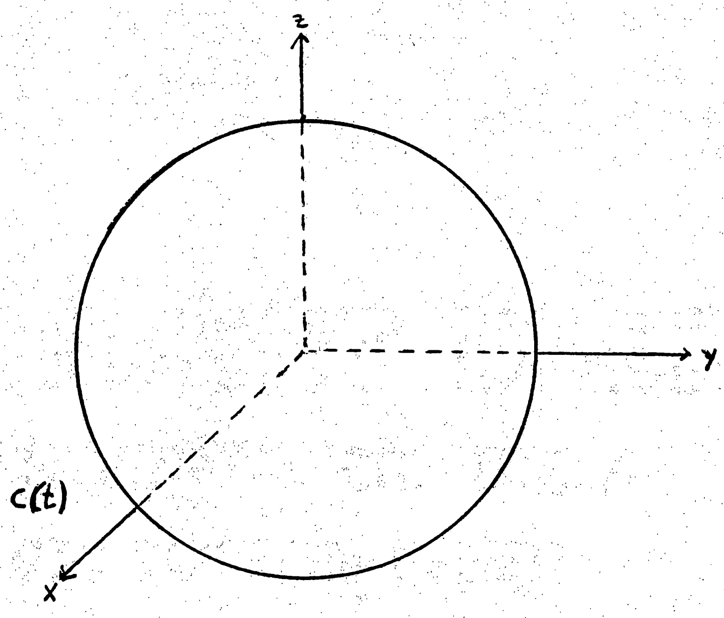
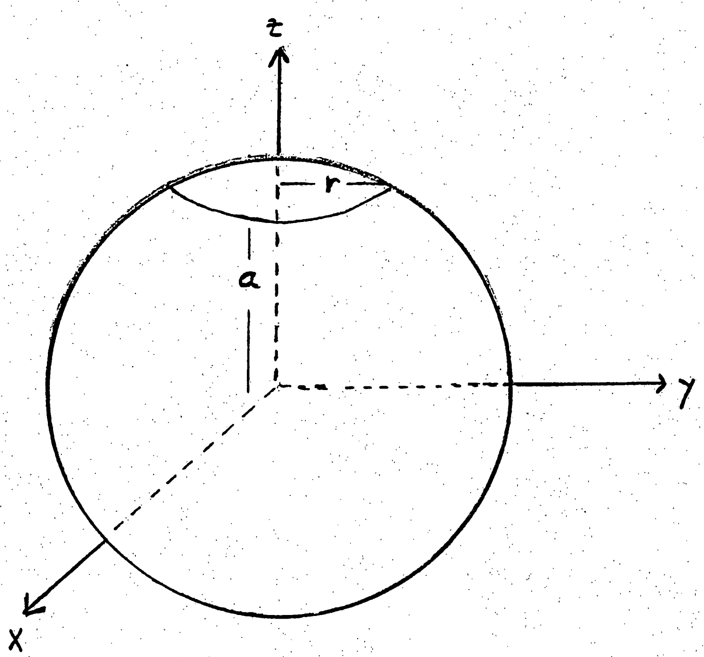


Figure 30



ENDNOTES

[1]Wilhelm Klingenberg, A Course in Differential Geometry (New York: Springer-Vertag, 1978) , 42.

[2]John McCleary, Geometry From a Differentiable Viewpoint (New York: Cambridge University Press, 1994) , 133-134.

[3]U. Dierkes, S. Hildebrandt, A. Kuster, and O. Wohlrab, Minimal Surfaces I (Berlin: Springer-Verlag, 1992) , 361.

[4]U.Dierkes, S. Hildebrandt, A. Kuster, and O. Wohlrab, Minimal Surfaces I (Berlin: Springer-Verlag, 1992) , 362.

[5]U. Dierkes, S. Hildebrandt, A. Kuster, and O. Wohlrab, Minimal Surfaces I (Berlin:Springer-Verlag, 1992) , 363.

[6]U. Dierkes, S. Hildebrandt, A. Kuster, and O. Wohlrab, Minimal Surfaces I (Berlin: Springer-Verlag, 1992) , 363.

[7] Sheldon Axler, Harmonic Function Theory (New York: Springer-Verlag, 1992) , 35.

[8]Wilhelm Klingenberg, A Course in Differential Geometry (New York: Springer-Vertag, 1978) , 42.

[9]Ivars Peterson, The Mathematical Tourist (New York: W.H. Freeman and Company, 1988) , 46-47.

BIBLIOGRAPHY

Axler, Sheldon, Paul Bourdon, and Wade Ramey. Harmonic Function Theory. New York: Springer-Verlag, 1992.

Chaun-Chi Hsiung. A First Course in Differential Geometry. New York: John Wiley & Sons, Inc., 1981.

Dierkes, U., S. Hildebrandt, A. Kuster, and O. Wohlrab. Minimal Surfaces I. Berlin: Springer-Verlag, 1992.

Douglas, Jesse. "Solution of the Problem of Plateau." American Mathematical Society, 12 (January 1931): 263-280.

Klingenberg, Wilhelm. A Course in Differential Geometry. New York: Springer-Verlag, 1978.

McCleary, John. Geometry from a Differential Viewpoint. New York: Cambridge University Press, 1994.

Meeks III, W.H. and S. T.Yau. "The Existence of Embedded Minimal Surfaces and the Problem of Uniqueness." Math Z, 179 (May1982): 151-168.

Osserman, Robert. A Survey of Minimum Surfaces. New York: Dover Publications, Inc., 1986.

Peterson, Ivars.The Mathematical Tourist. New York: W.H. Freeman and Company, 1988.