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The Mandelbrot set

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THE MANDELBROT SET

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by

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ABSTRACT

The field of chaos has been hailed as one of the most important breakthroughs in this century. The study of nonlinear dynamical systems and fractals has important applications in a variety of sciences. Benot Mandelbrot and others have shown that cloud formations, radio static, the motion of molecules, the holes in swiss cheese,and even the shape of galaxies can be modeled by fractals.

Fractal boundaries, called Julia sets, can be formed by an algebraic iterated process in the complex plane. Some Julia sets look like circles that have been pinched and deformed. While others are unconnected regions that resemble particles of dust. A Julia set is either connected or it is a totally disconnected set of points.

In 1979, Mandelbrot looked in the complex parameter plane for quadratic polynomials of the form $z^2 + c$. He plotted computer pictures of those c-values for which the orbit of 0 remained bounded. A solution set of c-values representing these connected Julia sets is constructed in the complex plane. The set **M** that results is called proved that the **M**-set is connected.

Based on the work done by Robert Devaney, John Hubbard, and Adrien Douady, it will be shown that studying the **M**-set is equivalent to analyizing the iteration of all quadratic polynomials at once. The fact that the **M**-set has an infinite boundary, gives one the impression that the mathematics behind the set must be immense and abundant.

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INTRODUCTION

There are many dynamical systems that have long term behaviors that are chaotic. Since the work of Poincare, the study of dynamical systems has made use of the results of iterating functions. One such functional operator is called the logistic equation, $f(x)= ax(1-x)$, which is used in biology and ecology to model growth. In ecology, the initial seed x_0 for the iteration is a percentage of some limiting population, so x_0 is chosen between 0 and 1.

As the parameter changes, the quadratic function $f(x)=ax(1-x)$ exhibits fairly complicated behavior. A great deal of interest is attached to invariant points, that is, initial seeds for which $f(x)=x$.

In addition to fixed points there can be cycles, defined by

$$f(x_i)= x_{i+1} \text{ for } i=1,2,3,\dots,k, x_{k+1}= x_1.$$

The orbit of x_0 is the sequence of points such that $x_n=f^n(x_0)$. Depending on the parameter, the iteration of these initial seeds could tend to infinity, have a finite cycle, be completely chaotic within an interval.

Graphical iteration on the logistic function will exhibit one of two forms. If a is greater than four, then the iteration of zero gives a sequence of values that escapes to infinity. If we restrict a between one and four, then the sequence of values will be contained within the invariant interval [0,1]. Even without reference to the value of the parameter, it is possible to determine which of these two outcomes will occur [3].

The Mandelbrot set and its Julia sets are generated in the complex plane from the functional operator $f_c(z)=z^2+c$. Where z is a complex variable and c is a complex constant. The logistic equation, $f(x)=ax(x-1)$, can be transformed into z^2+c form by a sequence of three geometric transformations. The image parabola has the equation

$$f(x)= x^2 + (a/2 -a^2/4), \text{ where } c=(a/2-a^2/4).$$

Each setting of the parameter c defines a single function of the form $w=z^2+c$. For certain values of c the iteration of the function remains finite and bounded within a circle of radius two, while other values of c will produce functions that iterate to infinity.

If the parameter value c and the initial point are restricted to the real numbers, then the collection of orbit points, P_c, is either a connected invariant interval or a totally disconnected set of points called Cantorian "dust" [7].

If the parameter value c and the initial point are complex numbers, then the collection of orbit points, P_c, is either a connected region or a totally disconnected set of points. Suppose A_0 is an arbitrary set of points in the complex plane and A_1 is the set of points set of points z such that z^2+c belongs to A_0. This is written as $A_1=f_c^{-1}(A_0)$, and in general $A_n=f_c^{-1}(A_{n-1})$. Thus, A_n is a decreasing sequence of points, each contained in its predecessor, and the limit of this sequence is a bounded orbit, P_c. The boundary of P_c is called the Julia set, J_c. The Julia sets that are generated by iteration have very different forms depending on the initial parameter value of c. These different forms of J_c fall into two different classes: they are either connected or disconnected.

The Julia set of a complex function is named for the French mathematician Gaston Julia, who discovered many of the basic properties of this set in the early twentieth century. A more precise definition of the Julia set of a polynomial, is that it is the boundary of the set of points that escape to infinity. In other words, a point in the Julia set has an orbit that does not escape to infinity, but arbitrarily nearby there are points whose orbits do escape to infinity.

The Mandelbrot set, denoted M-set, identifies those parameter values c, of the function $f_c=z^2+c$ for which iteration of the critical point zero yields an orbit that fails to escape to infinity. Such parameter values are associated with connected Julia sets. Points outside of the Mandelbrot set relate to parameter values that iterate to infinity. These c-values correspond to disconnected Julia sets. The irregular boundary of the M-set forms a barrier between these two behaviors. The M-set is a topologically connected set whose complicated boundary is of infinite length and contained within a circle of radius 2 in the c-plane [1].

The M-set corresponds to the collection of all connected Julia sets for f_c. The Julia sets of the quadratic family z^2+c have the fractal property of being self-similar. If the orbit of zero escapes to infinity then the associated Julia set is fractal dust and totally disconnected. Zero is called the critical point because the derivative of f_c vanishes only at zero. The orbit of zero is called the critical orbit. Please note the difference between these two sets: the M-set is a picture in parameter space that records the fate of the orbit of zero, while the Julia set is a picture in the dynamical plane that records the fate of all orbits.

In other words, the M-set is the set of c-values for which the critical orbit of f_c does not tend to infinity. Therefore the M-set is a large collection of all connected Julia sets. The M-set contains descriptions of all the different dynamics that occur for the quadratic family [9].

ITERATION OF THE LOGISTIC EQUATION ax(1-x)

Iteration can be viewed as a recursive process that is used as an important mathematical tool to model natural phenomena in the real world. Iteration is often refered to as a feedback loop. Let's closely study the iteration process for the real number case of x^2+c, by examining the logistic equation for different parameter values.

Web diagrams are used to visually display the iteration of a function. The web diagram is obtained by a simple process of drawing vertical and horizontal line segments starting from an initial seed to the graph of a function and then to y=x which reflects back to the function. This process repeats and generates a continuous path of alternating vertical and horizontal line segments.

Fixed points of a function are found by locating intersections of the graph with the diagonal line, y=x. By examining the web diagram in the neighborhood of a fixed point we are able to classify the fixed point as attracting, repelling, or indifferent. These classifications are determined by the slope of the function at the fixed points.

	Slope,m	Behavior
If:	m is less than -1	repelling, spiral out
	m is between -1 and 0	attracting, spiral in
	m is between 0 and 1	attracting, staircase in
	m is greater than 1	repelling, staircase out [7].

If our initial input is a small subinterval ,then the web diagram would exhibit either an interval expansion or an interval compression. Interval compression shows that small errors contract and converge to zero, while interval expansion show that small errors expand into large errors. Hence, the iteration process could exhibit sensitive dependence on initial seeds [6].

Slope, m	Behavior
$|m|$ is greater than 1	interval expansion
$|m|$ is between 0 and 1	interval compression

The iteration behaviors of f(x)=ax(1-x) changes as the parameter value changes from 1 to 4.

Example 1: Consider f(x)=2.8x(1-x) for values in the interval (0,1). Each x_0 converges on the point 0.6429.... This point is called an attractor. This behavior of convergence on some single value occurs for all parameter values between 1 and 3.

Example 2: Iterating f(x)=3.2x(1-x) has the long term behavior of oscillating between two values, 0.799455... , and 0.513044... This behavior is called a period two attractor and occurs for parameter values between 3.449... and 3. In this case, iteration paths spiral towards a box that is defined by these values. The fixed point of the function serves as a repellor, and the attractor consists of the two points. These examples have predictable behaviors that always lead to the same stable orbits [7].

Example 3: Iterating f(x)=4x(1-x) produces a chaotic orbit for initial values taken from the interval [0,1]. Two points, very close to one another, will generate very different unstable orbits. This is a consequence of the sensitive dependence condition of chaos (related to small errors of our input). In fact, as the parameter value approaches 4, a larger portion of the input interval has a slope less than -1 or greater than 1. Hence, much of the interval exhibits an interval expansion and very small errors magnify greatly making any kind of output prediction impossible.

The quadratic function $f(x)=ax(1-x)$ exhibits a variety of behaviors as the parameter increases in value from 1 through 4. For low values of a, the behavior is a stable predictable orbit in the form of a fixed point attractor. As a increases the orbits experience a period doubling route to chaos. The sequence of period-doubling bifurcations are plotted on a graph of parameter values [1,4] to the attractors within the interval (0,1). The first six bifurcation points (parameter values) are listed below.

Bifurcation points	Periodic orbit
$a_1=3$	2-cycle
$a_2=3.449490...$	4-cycle
$a_3=3.544090...$	8-cycle
$a_4=3.564407...$	16-cycle
$a_5=3.568759...$	32-cycle
$a_6=3.569692...$	64-cycle

The ratio $(a_k-a_{k-1})/(a_{k+1}-a_k)$ converges to a number **d** called the Feigenbaum constant. In 1975, the number, **d**= 4.669202... , was shown to be a universal constant in the field study of chaos by Micheal

Feigeenbaum. For example, the constant is also associated with the iteration of the transcedental function $g(x)= ax^2\sin(\pi x)$ [7]. The Feigenbaum constant can be found in the bifurcation graph of the differential equations that are associated with the Rossler system.

When performing graphical iteration on the logistic equation, the prisoner set is either an invariant interval or a totally disconnected set of points. The prisoner set consists of the invariant interval [0,1], when the parameter has a value within [1,4]. A totally disconnected set of points occurs when the parameter value is greater than 4. Even without reference to the value of the parameter, it is possible to determine which of these two outcomes will occur. If the iteration of x=0 gives a sequence of points that escape to infinity then the prisoner set will be disconnected, otherwise the set is connected.

The corresponding parameter values that represent the invariant interval for the real valued function $f(x)=x^2+ c$ is [-2, 1/4] [8].

FIXED POINTS AND ORBITS FOR $x^2 + c$

Chaos is a condition of extreme unpredictability which occurs in a dynamical system, just as fractality is a condition of extreme irregularity in a geometric configuration. Chaotic behavior is found in discrete dynamical systems, such as the Julia sets that are associated with the M-set.

Iterated mappings are the simplest form of a discrete dynamical system. There are many different types of orbits that are associated with an iterated mapping. The most important kind of orbit is called a fixed point ($f(x)=x$). A fixed point, x, is called attracting provided that the orbits of all points in some small neighborhood close to x converge to x. A fixed point, x, is called repelling if the orbits of al sufficiently nearby points move away from x.

If the magnitude of $f'(x)$ is less than one, then x is attracting. If the magnitude of $f'(x)$ is greater than one, then x is repelling. If $f'(x)= 1$, then x is called neutral and no conclusion can be drawn, since x could be attracting, repelling, or neither. If $f'(x)=0$, then x is called super attracting since nearby orbits are attracted to these fixed points quickly [5].

Another important orbit is called a periodic orbit. The point x is called periodic with prime period n if $f^n(x)=x$ (where n is the least such value that is greater than zero). Notice that if x has prime period k, then $f^{nk}(x)=x$ (where nk is the period of the orbit). A point x is called eventually periodic if x itself is not fixed or periodic, but some point on the orbit is fixed or periodic.

Example 1: $f(x)=x^2-1$ is eventually periodic for f(1), $x_0=1$, $x_1=0$, $x_2=-1$, $x_3=0$, $x_4=-1$...

Example 2: $f(x)=x^2$ is eventually fixed for f(-1), $x_0=1$, $x_1=1$....

Most orbits are neither fixed nor periodic, but rather tend to a specific limit.

Example 3: Given $f(x)=x^2$ one finds that if the initial seed has $|x_0|$ is less than 1, then the orbit of x_0 converges to zero. If the $|x_0|$ is greater than 1, then the orbit converges to infinity.

Lastly, some orbits are called chaotic since their sequence appears to have a random order.

Example 4: $f(x)= x^2-2$ has an eventually fixed orbit for $x_0=0$ since we have 0,-2, 2,2,2,.... However, if we look at the orbits of points near zero, x= 0.1, x= 0.01, or x= 0.001, we'll get orbits that are different and randomly jump within the interval [2,-2]. This function exhibits chaotic behavior.

Definiton of Chaos Let (**X**,d) be a metric space, and let f:**X**to**X** be a function. The map f is chaotic provided that the following hold:.

1. f has sensitive dependence on initial conditions.
2. f is transitive
3. Periodic points in f are dense in X.

However, it was shown in 1992 by J. Banks that the sensitive dependence condition is redundant since the second and third conditions imply the first. It can also be shown by using the Baire category theorem from metric space theory, that if the metric space is complete, then the transitivity condition implies the existence of a dense orbit [11].

Theorem 3.1 Let (**X**,d) be a complete metric space such that T:**X**to**X** is a contraction mapping. Then there exists a fixed point x_k that satisfies $T(x_k)=x_k$. The fixed point is unique. Moreover the iterative mapping of an arbitrary point x_0 in **X** converges to the fixed point x_k.

Web diagrams are convenient way to graphically display the orbit dynamics of the real-valued function $x^2+ c$. For any real value of c, the fixed points can be found by solving $x^2+ c = x$. The fixed points are real numbers if and only if 1-4c is greater than or equal to 0. If we consider the two roots r_1, r_2, [where r_1= (1+discriminant)/2 and r_2=(1-discriminant)/2] we'll find that if c is less than or equal to .25, then r_2 is between $-r_1$ and r_1, where $f(-r_1)=r_1$. The orbits of the x_0-values that are greater than r_1 and less than $-r_1$ tend to positive infinity. Therefore, consider the values of x_0 between $-r_1$ & r_1 where c is less than or equal to 0.25.

Let **I** denote the closed interval $[-r_1,r_1]$. If c is between -2 & .25 and x_0 is chosen from **I**, then $f(x_0)$ is also in **I**. Hence the whole orbit is trapped in **I**. If c is less than -2 and x_0 is taken from **I**, then either the orbit is trapped in **I** or eventually some x_n drops below **I x I** and the orbit tends to positive infinity.

When c is between -0.75 and 0.25, then the fixed point r_2 is attracting since the $|f'(r_2)|$ is less than 1, and all orbits beginning in **I** converge to r_2. As c decreases through -3/4, the absolute value of $f'(r_2)$ increases through 1 and r_2 becomes repelling. At the same time the second iterated function acquires a pair of new attracting fixed points that exhibit a period two cycle for f. In other words, the system experiences a period-doubling bifurication [7].

Example 5: Consider $f(x)= x^2-1$, the two fixed points, r_1, r_2 are (1+ sqrt 5)/2 and (1-sqrt 5)/2. Both roots are repelling since $|f'(r_1)|$, $|f'(r_2)|$ are greater than one. However, when we consider the dynamics of the second iteration $f^2(x)= (x^2-1)^2 -1$ one discovers a periodic point of period two. The two new fixed points are 0 and -1. The iteration yields an orbit of 0, -1, 0, -1, ...

As c decreases through -5/4, another period bifurication occurs and the orbits are attracted to a period four cycle. As c continues to decrease, attracting period orbits of 8, 16, 32,..., 2^n,.... are found. This process is referred to as a period route to chaos.

Example 6: Consider $f(x)= x^2-2$, then the closed interval **I** is [-2,2]. The graph of the web for f(x) when x is taken from **I** appears to fill up the space **I x I**. The line y=x intersects the graph of f(x) exactly 2^n times within the space **I x I**. Each of the crossings represents a fixed point of $f^n(x)$ and hence a periodic point of f(x) having a period of n (not necessarily a least period). This implies that there exists infinitely many orbits of f(x) having periods of length 1, 2, 3, 4, ... [9].

Theorem 3.2 Suppose c is less than -2. Then the set of points,

P, whose orbits under $f_c = x^2 + c$ do not tend to infinity is a nonempty closed set in **I** that contains no intervals.

A disconnected set that contains no intervals is called a <u>Cantor dust</u>. The Cantor set is closed, totally disconnected uncountable set. In fact, it can be shown that the defined set **P** that was generated from our quadratic map of the theorem is homeomorphic to a Cantor middle third set. The set is self-similar and has a fractal dimension of log2/log3. The total length of the remaining line segments approaches zero as n goes to infinity, which implies that the topological dimension of the set is zero [8].

JULIA SETS

The birth of Julia sets has origins that begin in the nineteenth century. Mathematicians such as Leau, Schroder, Koenigs, and Bottcher became interested in iterating complex functions. They studied the behavior of iterations near a fixed point.

In 1918, Gaston Julia and Pierre Fatou expanded on the earlier work by considering the behavior of the iterations of complex functions outside the neighborhood of fixed points. They discovered dynamical behavior that was sometimes stable and at other times chaotic. The dynamics of these two radically different sets were further investigated by Julia and Fatou and they discovered many new properties. But, they were unable to classify all of the possible dynamics that were related to the stable set. They could not exclude the possibility of wandering domains from the stable set, nor could they prove that connected Siegel disks exists for certain c-values.

In 1940, C.L. Siegel showed that Siegel disks exist within a complex dynamical system. In addition, I.N. Baker extended much of the earlier work by showing other types of stable behavior could occur for entire and meromorphic functions.

In 1980, Benot Mandelbrot discovered the M-set. Dennis Sullivan introduced the use of quasi-conformal mappings which allowed him to prove the No Wandering Domain Theorem. This completed the classification of stable dynamics for rational maps begun by Fatou and Julia. The stable region of a complex dynamical system is called the

Fatou set, and the chaotic region is called the Julia set [4].

Definition: Suppose z is a fixed point. Then z is

1. attratting . . . . if the modulus of $f_c'(z)$ is between 0 and 1.
2. superattracting . . . . . . if $f_c'(z) = 0$
3. repelling . . . . if the modulus of $f_c'(z)$ is greater than 1
4. neutral . . . . if modulus of $f_c'(z) = 1$

Recall that the orbit of zero is called our critical orbit since $f_c'(0)=0$. In fact, 0 is the only critical point of $f_c=z^2+c$ in the complex plane. If z_0 is either superattracting or an attracting fixed point, then there is an open neighborhood **U** of z_0 having the property that $f^n(z)$ tends to z_0 as n goes to infinity for each z in **U**. The set of all points that converge to z_0 is called the basin of attraction.

If z_0 is a repelling fixed point, then there is a neighborhood **U** of z_0 in which there exists an analytic branch of the inverse of f_c. Since $f(z_0)$ is an attracting fixed point for this inverse, it follows then that the orbits in **U** are attracted to $f(z_0)$ under f_c^{-1}. Therefore, all of the orbits for z in **U** are repelled from z_0 by f_c.

If one studies the dynamics of f_c on the Reiman sphere, then the point at infinity becomes a superattracting fixed point for f_c (since $f_c'(\text{infinity})=0$), this can be shown by changing the coordinates to $h(z)=1/z$). Therefore, the second basin of attraction is associated with the point at infinity [10].

A complex analytic map always decomposes the plane into two disjoint subsets: the stable set on which the dynamics are calm, and the Julia set where the mapping is chaotic.

Definition of Julia set: The Julia set of f_c, denoted J_c, is the set of all points at which f_c exhibits sensitive dependence. The complement of this chaotic set J_c, is called the stable set, denoted S_c [7].

Let's recall a few facts from complex analysis, a collection of functions [($\mathbf{T}_i$): i=1,2,3,..., where ($\mathbf{T}_i$) are the iterated functions for the different values of c] is said to be normal on an open set, **U** in the complex plane if the functions, ($\mathbf{T}_i$), form a equicontinuous family on **U**. If the functions, ($\mathbf{T}_i$), are holomorphic functions then normality of the family ($\mathbf{T}_i$) on **U** is equivalent to the following: Every sequence of the ($\mathbf{T}_i$) has a subsequence that converges uniformly on compact sets either to an analytic function on **U** or to infinity.

Montel's Theorem: Suppose the family of analytic functions, ($\mathbf{T}_i$), is not normal on **U**. Then the family ($\mathbf{T}_i$) assumes all values in the complex plane except at most two.

This theorem tells us that if the family of iterated functions, does not form a normal family in any neighborhood of z_0, then z_0 is within the Julia set, J_c.

Theorem 4.1: The Julia set, J_c = [z_0 in **C**: $(f_c)^n$ is not a normal family in any neighborhood of z_0] [10].

Theorem 4.2: The Julia set, J_c, is the closure of the set of repelling periodic points of f_c.

Definition: The filled Julia set of f_c, denoted K_c, is the set of points whose orbits are bounded under the iterationof f_c.

Theorem 4.3: The Julia set, J_c is the boundary of K_c.

Two mappings, F, G: **C** to **C** are said to be conjugate if there exist a homemorphism, h:**C** to **C** such that $h \circ F = G \circ h$.
Note that h maps orbits of F to G since $h \circ F^n = G^n \circ h$. In addition, h^{-1} takes orbits of G to those of F and we have a one-to-one correspondence between the orbits of F and G.

Theorem 4.4: Suppose $g(z)=az^2+bz+c'$ with a not equal to zero and b,c' in **C**. Then g(z) is conjugate to $f_c(z)=z^2+c$ for some c in **C**. The affine conjugacy, $h(z)=az+b/2$, satisfies

$h(z) \circ g(z)=f_c(z) \circ h(z)$, where $c= ac'+ b/2 - b^2/4$.

Therefore, the general quadratic equation is dynamically equivalent to an equation in the family of f_c [10].

The dynamics on the Julia set are chaotic. All Julia sets exhibit the three properties of sensitive dependence, transitivity, and dense periodic points. The basin of attraction for the critical point at zero is a stable set, and all of the points with orbits that escape to the superattractive point at infinity are points in the other stable set. The points on the Julia set are a sequence of points that lie on the boundary between the two basins of attraction. There are two values for c of $f_c(z)=z^2+c$ that are not fractals. They are when c=0 or c=-2. The first case $f_0(z)=z^2$ has dynamics on the Julia set that are chaotic. Let **R/Z** be the unit circle, so that the points on the unit circle are defined mod 1. Then the orbit of any point on J_0 is obtained by iterating the doubling function, **D**(theta)=2·(theta) mod 1. If (theta) = p/q (where p/q is in lowest terms with q odd), then (theta) lies on a periodic orbit, and we have dense periodic points on the unit circle. If (theta) = $p/(2^n q)$, then (theta) is eventually periodic under **D**(theta) and these points will also be dense in J_0.

The function operator, **D**(theta), exhibits the property of sensitive dependence. If one chooses any point in an open neighborhood of a point on the unit circle and repeatedly apply $f_0= z^2$, one would find that the iteration of the open neighborhood would grow in size and eventually contain any point in **C** except 0 and infinity.

The doubling function expands arclengths on the unit circle by a factor of two. Thus, any segment of the unit circle is eventually mapped onto the entire circle. This gives the third condition of transitivity. Therefore, $f_0(z) = z^2$ has dynamics on the Julia set (the unit circle) that are chaotic [6].

The second case, $f_{-2}(z)=z^2-2$ also has dynamics on its Julia set that are chaotic. The critical orbit is eventually fixed since the second iteration gives $f_{-2}(0)=2$, which is a fixed point. The map

f_{-2} is similar to f_0. Consider $h(z) = z + 1/z$ (for all z such that $|z|$ is greater than or equal to 1). The function, h(z), maps the exterior of the open disk onto the entire plane.

The unit circle has a 2-to-1 mapping onto a closed interval [-2,2] (where the straight rays of the circle are mapped onto pieces of hyperbolas of the interval [-2,2]) [10].

The Julia set that is generated by $z^2 - 2$ is a line, [-2,2].

The two previous cases represent the simplest of the Julia sets to compute. The computer algorithms that generate pictures of the Julia sets for the remaining c-values will require numerical techniques.

The Julia set that is generated for c = -1 has a fractal boundary. The critical orbit is a two cycle: 0, -1, 0, -1, 0,.... The following two theorems show a dichotomy that is related to the critical orbit of f_c.

<u>Theorem 4.5</u>: If the orbit of 0 escapes under the iteration of f_c, then $K_c = J_c$ is a Cantor set.

<u>Thereom 4.6</u>: If the orbit of 0 does not escape under the iteration of f_c, then both K_c and J_c are connected sets [1].

THE M-SET AND MISIUREWICZ POINTS

The dynamics of complex analytical functions has made considerable progress in recent years. A complex analytic map always decomposes the plane into two disjoint subsets: the stable set on which the dynamics are calm, and the Julia set where the mapping is chaotic. Quadratic maps in the complex plane share many of the features of one-dimensional systems. But the dynamics of quadratic maps in the form $f_c = z^2+c$ (where c is a complex parameter) will be more complicated than the previous real-valued map $f_c = x^2 + c$.

In 1979, Mandelbrot looked in the complex plane for those values of c for which the orbit of zero remained bounded. One could say that the M-set is a solution set within the parameter plane where the iterations of z^2+c remain bounded and represent connected Julia sets. Studying the M-set is like studying the iterating dynamics of all quadratic polynomials at once.

The M-set is contained within a circle of radius 2. It is a very complicated fractal, that is not a self-similar set. The computer generated pictures show regions of bounded orbits, colored black, and regions of unbounded orbits, colored white. The closer we zoom in on the M-set's boundary, the more detail appears. As we blow-up the boundary of the Mandelbrot set, dark islands appear that resemble baby mandelbrot sets [2].

In 1982, Adrien Douady and John Hubbard proved that the M-set is connected (those islands off the edges are actually connected to the whole set by thin connected filaments). Currently, there is work being done on a conjecture concerning the possible property of the M-set being locally connected. In 1990, J. C. Yoccoz gave an almost complete answer by showing that for each point c in M which is not infinitely normalized, there is a neighborhood base of closed connected sets.

Definition of the M-set:

The Mandelbrot set is [c: $f^n{}_c(0)$ does not go to infinity]. Equivalently, The M-set is the set of c-values for which $K(f_c)$ is a connected set [2].

The Julia sets are either connected or not connected. The disconnected Julia sets are all homeomorphic to a Cantors set. The connected Julia sets have one of two forms either the set has a connected stable interior, or the Julia set has no interior where $K(f_c)=J(f_c)$ and is called a dendrite.

The M-set is a computer generated picture of the parameter plane. In order to give a pictorial description of the M-set, we need to analyze the critical orbits: if the critical orbit does not escape to infinity then the Julia set is connected and c is part of M. If the orbit of zero goes to infinity then c is not in M.

Theorem 5.1: If f_c has an attracting periodic orbit, then the orbit of zero must be attracted to this orbit .

Therefore, If one is able to locate a c-value such that f_c has an attracting periodic orbit, then we are guaranteed that the critical orbit will be attached to this orbit. Our function, f_c, can have at most one attracting cycle and the critical orbit can be is attached to one of these attracting cycles. Some iterating functions have infinitely many periodic orbits.

Example 1: Consider $f(x)= x^2 -2$, then our function has 2^n distinct periodic points that are fixed. At most one of these orbits can be attracting.

Critical orbits will determine much of the dynamics and structure of the stable set, $S(f_c)$. Every quadratic polynomial that has infinitely many periodic points, can have at most one point that is an attracting cycle. Consider the following two cases:

1. If f_c that has a rationally neutral cycle, then the critical orbit must again be attracted to it.

2. If f_c has a critical orbit that is pre-periodic (see next definition), then there is no attracting or neutral cycles($f_c'=1$) associated with f_c and we have a Julia set where $K(f_c)=J(f_c)$. This form of connected Julia set has an empty interior and is called a dendrite. The example above, when c=-2 gives have a connected Julia set with no interior, where $K(f_{-2})=J(f_{-2})$ [10].

The similarity between Julia sets and the M-set is well-understood in the neighborhood of certain points called Misiurewicz points.

Definition of pre-periodic: An initial point, z_0, is called pre-periodic if there exist **m** and **p** that are greater than or equal to 1, such that $z_{\mathbf{m}}=z_{\mathbf{m}+\mathbf{p}}$

<u>Definition of Misiurewicz point</u>: A value of c for which the orbit of 0 under f_c is pre-periodic (but not periodic) is called a Misiurewicz point.

If c is a Misiurewicz point then the orbit does not escape. Therefore all Misiurewicz points are in the M-set. Douady and Hubbard proved the following properties:

<u>1.</u> If c is a Misiurewicz point, then the periodic cycle $z_{\mathbf{m}}, z_{\mathbf{m}+1}, \ldots, z_{\mathbf{m}+\mathbf{p}-1}$ is a repelling orbit.

<u>2.</u> If c is a Misiurewicz point then $K(f_c)=J(f_c)$ and we have a dendrite.

<u>3.</u> Misiurewicz points are dense at the boundary of the M-set. (this means if we take any arbitrary point on the boundary of the M-set and construct a small disc around that point, then there exist a Misiurewicz point in the disc) [1].

<u>Example 2</u>: Consider $f_i(z)=z^2+i$. The critical orbit is pre-periodic: $z_0=0$, $z_1=i$, $z_2=-1+i$, $z_3=-i$, $z_4=-1+i$,.... (where $\mathbf{m}=2$ and $\mathbf{p}=2$, we have $z_2=z_{2+2}$). The Julia set, J_i, is an infinitely branched connected fractal called a <u>dendrite</u>.

<u>Example 3</u>: Consider $f(z)=z^2-2$. Again the critical orbit is pre-periodic: $z_0=0$, $z_1=-2$, $z_2=2$, $z_3=2$,.... (where $\mathbf{m}=2$ and $\mathbf{p}=1$, we have $z_2=z_{2+1}$).

In 1989, Tan Lei proved that if c is a Misiurewicz point, then:

1. The Julia set and the M-set are both asymptotically self-similar in the point z=c using the multiplier p.
2. Then the associated (shapes) limit objects L_j and L_m are essentially the same. They differ only by a scaling and a rotation. ($L_m = aL_j$, where a is in C) [1].

Since the Misiurewicz points are dense in the M-set we have a powerful method for visually seeing the shape of the corresponding Julia sets.

EXTERNAL RAYS

It is hard to find sufficient superlatives to describe the geometric complexity and beauty of the Julia sets that are generated as c varies along the boundary of the Mandelbrot set. Recall that the shape of the Julia set is the boundary between attractor points: zero and infinty. All Julia sets are symmetric about the origin and fall within a circle of radius two. Except for the attractor at infinity, the number and location of the attractors depends on c. In some cases there are no attractors except infinity and no "inside" to the Julia set. When we choose a value for c that is outside the M-set we get a disconnected Julia set. If we choose our starting point for iteration as one of the points of the Julia set, the iteration of the point will hop around locally from point to point in the set. But if we don't start on a point in the Julia set, then our point will escape to infinity.

Each of the various parts of the M-set represents dynamically different Julia sets. The large cardioid-shaped region of M. The equation for the curve that bounds the large cardiod region of M can be determined by solving two equations simultaneously: $z^2 + c = z$ and the derivative $|2z|$ is greater than or equal to 1, which yields $c = p(y) = \frac{1}{2}e^{iy} - \frac{1}{4}e^{2iy}$ (where y is an angle in [0, 2pi]). If c=p(y), then the derivative at the fixed point is e^{iy}. Notice that the derivative wraps once around the unit circle as c travels the boundary of the cardioid [11].

In 1983, Dennis Sullivan showed that the following four cases describe some of the possible connected Julia sets that are associated with the M-set (the only one remaining is called a herman ring).

CASE #1: The c-values that are in the interior of the M-set are called hyperbolic points. The cardioid main body contains all values of c for which f_c has one attracting fixed point and the associated J_c is a deformed circle (a Jordan curve). Each bud on the M-set corresponds to a cyclical attractor of a particular period.

CASE #2: When c is a Misurewicz point we have a dendrite.

CASE #3: When c is a point of M where the bud is connected (called a root point) we acquire a Julia set with a germination point of a bud for an n-cycle. The n-points of these cycles split off from the thick dot when c moves into the bud. The boundary has tendrils that reach up to the marginally stable attractor. At the branch point, the J_c becomes a stable attractor called the parabolic case.

CASE #4: If c is any other boundary point of the cardioid or a bud, then the J_c is associated with a Seigel disk. Unlike the parabolic case, where the boundary extends to the fixed point, the Seigel disk has invariant concentric circles in the J_c that surround the fixed point. There are some technical conditions regarding the irrationality of the fixed point for the Seigel disk [1].

The theory of external rays that was developed by Douady and Hubbard supply a method for understanding the structure of both the M-set and related Julia sets. A complex analytic map with a superattracting fixed point (infinity) is locally conjugate in a neighborhood, $\mathbf{U}_c$, of this fixed point of z to z^n, for some n greater than or equal to 2. The point at infinity is superattracting. Since $f_c(z)=z^2+c$ has degree two our map is conjugate of z to z^2 near infinity. Let $\mathbf{V}_R$=[z in C: |z| is greater than or equal to R, where R is greater than 1].

<u>Theorem 6.1</u> Let c be in **C**. Then there exist a neighborhood $\mathbf{U}_c$ of infinity in **C**, R is greater than 1, and an analytic isomorphism $\mathbf{P}_c$: $\mathbf{U}_c$ to $\mathbf{V}_R$ such that $\mathbf{P}_c(f_c(z))=(\mathbf{P}_c(z))^2$

<u>Definition of external rays</u>: The external ray of argument y is the curve $$g_y(t)= \mathbf{P}_c^{-1}(te^{2(pi)iy})$$

Recall that the eventually fixed orbit of z^2-2 had a map h(z)= z + 1/z, and this gave a uniformization,

$$H: [z: |z| \text{ is greater than } 1] \text{ to } \mathbf{C} - [-2,2]$$

which conjugates the map (z to z^2) to the map (z to $z^2 - 2$). This mapping determines the external rays, which turn out to be pieces of hyperbolas.

If our Julia set is locally connected, then each external ray will have a limit that lands on a unique point of the J_c. We will get information about the topology and the dynamics on the J_c by studying the landing points of the external rays. However, not all J_c are locally connected, so our external rays are not always guaranteed to land [10].

<u>Theorem 6.2</u> If the critical orbit of zero under $f_c= z^2+c$ is

eventually periodic or f_c has an attracting cycle, then J_c is locally connected.

Consequently, for a large number of cases, the external rays do land on J_c. Julia sets that have attracting cycles are part of the M-set. If $f_c=z^2+c$ has an attracting cycle of period n, then there exists a neighborhood, $\mathbf{U}_c$, for which the following holds: if c' is in $\mathbf{U}_c$, then $f_{c'}$ also has an attracting cycle of period n which remains close to the cycle of f_c. Therefore any such c-value lies in the interior of the M-set. The interior of M consists of c-values for which the corresponding f_c has an attracting cycle of some period. These components are called <u>hyperbolic</u>.

The region of the M-set where f_c has an attracting cycle of period two is the boundary of the circle that is given by $|c+1| = 1/4$. (which is derived by solving two equations simultaneously: $f_c^2(z)=z$, and $|(f_c^2)'|$ is less than 1). This region meets the main body cardioid at a unique point, c=-3/4. This bulb represents an area of M where the atractive cycle is period two [6].

All of the decorative bulbs of the M-set have a rational number associatred with them called the <u>rotation number</u> (**p**/q'). Each of the **p**/q'-bulbs is decorated with a antenna. Each antenna is unique in its topological structure and features a "junction point" from which different spokes emanate. The degree of this vertex (junction point) has a value of q' for each **p**/q'-bulb. The numerator, **p**, gives valuable information about the dynamics that are related to the c-value of f_c in the **p**/q'-bulb. The filled Julia set, K_c, for f_c

contains a fixed point p at which exactly q' components of $K(f_c)$-[**p**] meet. The iteration of the mapping, $f_c=z^2+c$ rotates about these components centered at **p** by an angle 2(pi)(**p**/q').

The local structure of the M-set around hyperbolic points is complicated but predictable. Recall, that the set of Misiurewicz points is dense in the boundary of M. In addition, the accumulation points of hyperbolic components will give the entire boundary of the Mandelbrot set. In other words, the boundary of M is contained in the closure of the set of centers of hyperbolic components, (where the <u>center</u> is definned to be the unique c-value for which the attracting cycle is a super-attracting cycle). This is a consequence of Montel's theorem [10].

Douady and Hubbard showed that the M-set has infinitely many small copies of the M-set embedded within the connected M-set [11].

RATIONAL ANGLES

Definition of external ray of M: The external ray of M with angle y is the curve $R_y(t) = \mathbf{Q}^{-1}(te^{(2(pi)iy)})$.

(see definition of mapping in next section)

When we refer to the external rays of the M-set, let's use the term field line. It is known that all field lines with rational angles do, in fact, land on the boundary of the M-set. Furthermore, the dynamics of $f_c(z)=z^2+c$ at the landing point is effectivelly determined by the argument of the field line. However, it is not known whether all field lines with irrational angles will land on the boundary of M. If it were proved that the M-set is locally connected, we would be allowed to say that all field lines land on bd(M).

Theorem 7.2: Suppose the angle y = **p**/q', in lowest terms. Then the field line of M with angle y lands at a point c_y in the bd(M):

1. If q' is odd, then c_y is a root point of a hyperbolic component.
2. If q' is even, then c_y is a Misiurewicz point.

When q' is odd, **p**/q' is an infinitely repeating binary expansion, B(**p**/q')=$b_1 \ldots b_n$, where b_j=0 or 1 for each j. Then the root point at which the **p**/q' ray lands separates a hyperbolic component of period n from a lower period component (usually a period dividing n).

For example, B(**1**/3)=010101.... and B(**2**/3)=101010.... These rays land at the root point (c=-3/4) of the period 2 bulb. The field lines land at the root point (c=-3/4) of the period 2 bulb. The field lines of angle y=**p**/7 give Binary expansions B(**1**/7)=001001...., B(**2**/7)=010010...., B(**3**/7)=011011...., etc. So the field lines of angle **p**/7 land at root points for period three components. Two of these are bulbs that are attached to the main cardioid where trifurcation takes place. The other must be a root point of a baby M-set with period 3 that is within the main M-set. Period three bulbs can only be attached to the main cardioid by direct bifurcation, and not to bulbs with other periods. Hence the field lines of **3**/7 and **4**/7 must terminate at the baby M-set of period 3 that is located on the negative real axis.

Schleicher's algorithm provides a procedure that is used to determine the binary expansions of field lines that land at root points of **p**/q' bulbs attached to the cardioid.

<u>Step 1</u> Using Farey addition determine the largest **p**/q' bulb. **p**/q' + r/s =(**p**+r)/(q'+s)

(Note that the ray 0 or 1 land at the cusp ofthe cardioid and the rays $\overline{01}$ and $\overline{10}$ land at the root point of 1/2 bulb. i.e. 0/1 + 1/2 = 1/3)

<u>Step 2</u> Find the rays closest to the (**p**+r)/(q'+s) bulb.

Let $\overline{s_1 \ldots s_n}$ be attached to the **p**/q' bulb, and let $\overline{t_1 \ldots t_m}$ be attached to the root point of the r/s bulb.

<u>Step 3</u> The ray attached to the (**p**+r)/(q'+s) bulb closest to the **p**/q' bulb is $\overline{s_1 \ldots s_n t_1 \ldots t_m}$. The ray attached to the (**p**+r)/(q'+s) bulb closest to the r/s bulb is $\overline{t_1 \ldots t_m s_1 \ldots s_n}$.

(the ray that is attached to the 1/3 bulb, closest to the $\overline{0}$ ray, can be found in the following way: use the nearest ray $\overline{0}$ then glue the farthest ray $\overline{01}$ to get $\overline{001}$).

Using this inductive procedure one could find all of the rational rays that land on the root points of the decorative bulbs of **M**. There is an extension to this algorithm that allows one to determine all of the rational rays that land on the smaller bulbs attached to the decorative bulbs of **M** [10].

THE M-SET IS CONNECTED

The Mandelbrot set is the set of c such that the filled Julia set is connected. Every such polynomial from the family z^2+c has a filled Julia set that is formed of iterated points with bounded orbits. Recall, that if the critical point 0 is in the filled Julia set, K_c, then J_c is connected. If the critical point is not in K_c then J_c is a Cantor set.

Hubbard and Douady used electrostatics to developed their theory of external rays. Since M is connected, they thought of M as a charged region and gave M an electric field to create equipotential level sets. These level sets are related to the different speeds at which the orbit of 0 escapes to infinity under iteration for each c-value outside of M.. The closer c is to the boundary of the M-set the longer it takes for the iteration of 0 to escape. Recall that the map z to z^2 is conjugate to a neighborhood $\mathbf{V}_R$, near infinity, where infinity is a super-attracting fixed point.

Previously, we defined a map $\mathbf{P_c}$: $\mathbf{U}_c$ to $\mathbf{V}_c$ such that $\mathbf{P_c}(f_c(z))=(\mathbf{P_c}(z))^2$. This mapping conjugates the dynamics of $f_c(z)$ near infinity to (z to z^2) outside of some large circle. If c is not in the M-set, then the iteration $f_c{}^n(z)$ goes to infinity as n goes to infinity.

It can be shown that $\mathbf{P_c}$ is analytic in both c and z. The map $\mathbf{P_c}$ is defined in a neighborhood of infinity in the Reiman sphere. The map, $\mathbf{P_c}$, extends to a larger domain via the conjugacy by the folowing method. Suppose 0 is not in the boundary of U_c. Let z be $bd(U_c)$. Then there exists z' in the $bd(U_c)$ such that z does not equal z' and $f_c(z)=f_c(z')$ are in U_c. Choose a neighborhood W of $f_c(z)$ so that the pre-image of W consists of two open sets W_1 and W_2 satisfying:

1. z is in W_1, and z' is in W_2.
2. The intersection of W_1 and W_2 is empty.

The intersection of W_1 and W_2 is guaranteed to be empty provided that $f_c(0)=c$, where c is not in W.

One can extend P_c to all of W_1 and W_2 by requiring:

1. $(P_c(z))^2=P_c(f_c(z))$
2. P_c is continuous.

If our critical orbit (the orbit of zero) escapes to infinity then this method allows us to extend P_c to an open neighborhood of infinity that contains the critical point (zero). Otherwise, the method above allows us to define P_c on an open neighborhood U of infinity such that zero is contained in bd(U).

Theorem 7.1: Q: $\overline{C}$ - M to $\overline{C}$ - [z: |z| = 1], where $Q(c)=P_c(c)$, for c in $\overline{C}$ - M is an analytic isomorphism onto $\overline{C}$ - [z: |z| = 1].

Corollary: The Mandelbrot set is connected.

Let D denote the open unit disc, and let Q_M: $\overline{C}$ - M to $\overline{C}$ - D be the conformal mapping which maps infinity to infinity and is tangent to the identity at infinity. The proof is not obvious. First assume connected if its complement is a simply connected open subset. The Riemann mapping theorem says that a simply connected open subset of the sphere, that dosen't represent the entire plane, is equivalent to homeomrphic and analytic to the unit disc, **D**. Hubbard and Douady showed that there must exist a homeomorphic and analytic map from the unit disc to the complement of M. Which showed that the complement of M was simply connected and consequently M is connected.

Their map **T**: **D** to $\bar{C}$ - **M**. $T(z)=1/z + b_0 + b_1 z + b_2 z^2 + \cdots$

The 1/z term maps the interior of the unit disk onto the exterior of **D** (where the origin is sent to infinity). The power series terms add a small distortion to make the image of the map **T** the exterior of the **M** (rather than the exterior of **D**). The image under **T** (as r tends to 1) of a circle($|z|=r$, where r is less than 1) is sent to a simple closed curve bounding **M**.

There are still many open questions and conjectures concerning **M**. For example, the map **T** is a homemorphism that sends the open disk to the complement of **M**, however the map does not extend to a homeomorphism on the boundaries. It is still unknown whether or not **T** will extend to a continuous map. There is a conjecture concerning whether or not **M** is locally connected. If **M** is locally connected then one could show that **T** would extend to a continuous map. In addition, one could show that all irrational external rays land on the boundary of **M**. Currently there is much work being done on this conjecture.

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