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Nonstandard analysis based calculus

Kathleen Renae Gibson

NONSTANDARD ANALYSIS BASED CALCULUS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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ABSTRACT

This project seeks to present the basic approach to teaching the Calculus using concepts from Nonstandard Analysis, as developed in the text, "Elementary Calculus, An Infinitesimal Approach", by Jerome Keisler. In this first part of the project the elementary development of an extended number system called the Hyperreals is discussed. The Hyperreals, which contain infinitesimal and infinite numbers, are developed and it is shown how these numbers are used to replace limits in Calculus computations. This replacement of the limit concept is one of the foundations of Keisler's approach. The introduction to the Hyperreals in this section is limited and emphasis is placed on how they are used in the instruction of the Calculus. For the teaching of entry-level Calculus, no real understanding of the theory supporting the Hyperreals is needed.

The second half of this project develops the basics of Nonstandard Analysis, including the theory of ultrafilters, and the formal construction of the Hyperreals. Major theorems, definitions and axioms are presented. Proofs, generally using direct ultrafilter manipulations, are given in detail. The Transfer Principle is discussed briefly and examples are presented.

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CHAPTER 1: INTRODUCTION

When I discovered the Calculus text written by Jerome Keisler, I thought it would be interesting to learn more about using new methods in some basic Calculus problems. As a student and tutor I had always found that some students had a very difficult time comprehending and using the theory of limits. In Keisler's text an extended number system called the "Hyperreals" is used. The Hyperreals are the Reals with the added quantities of infinitesimal and infinite numbers. By using the Hyperreals, limit computations become simple calculations within this extended set of numbers, thereby reducing the complexity of learning the Calculus for many students.

For my purposes, which were to get an underlying understanding of the background behind the Calculus text, there was no need to delve too deeply into the major theory of Nonstandard Analysis. Therefore I studied only the first chapter, "Infinitesimals and the Calculus", in the text "An Introduction to Nonstandard Real Analysis" by A. E. Hurd and P. A. Loeb. This gave a basic background to the development of the Hyperreals, their construction as an extension of the Real numbers, and the formal language and simple methods used to do some standard proofs.

The Nonstandard Analysis approach is more of an intuitive computational approach as compared to the theory of limits, or the ε,δ (epsilon, delta) methods, as taught in colleges today. Unfortunately sometimes the more rigorous ε,δ theory is only touched on in a most cursory manner, if at all. Most all teaching of the Calculus at the beginning level is taught on an intuitive basis, however for some students it isn't intuitive at all. So if instructors are using an approach where they are not developing the theory, then they might as well teach using concepts which are a little easier to understand. The good thing is that this nonstandard approach also makes the limit concept a little more understandable since in the nonstandard approach "limit" computations work virtually the same as computations in the conventional or traditional methods. Note that the theory of limits is still taught in the Keisler text, though not until after derivatives are covered.

Another reason I was interested in the Nonstandard Analysis approach is that I feel that it is closer to the idea of "infinitely small numbers" [1] that Leibniz had in mind. When Leibniz was developing the Calculus (concurrently with Newton), he used a conceptual convenience which he referred to as "infinitely small numbers" [1] to allow him to manipulate what we refer to as limits without the complexity required by ε,δ (epsilon, delta) method, which was not developed until the 1800's. Neither he nor anyone else was able to provide a solid theoretical basis for these "infinitesimals" because at the time Leibniz was discovering the Calculus there still was some question as to the appropriate level of rigor which should be required of mathematics. The form for proofs and notation to be used were still in the process of being standardized. Although mathematicians were in agreement that this was necessary to comprehend each other's works, the work was not yet complete. Specifically, they had not at that time developed all the necessary mathematics for Leibniz to be able to explain precisely enough these infinitesimals and their appropriateness for the Calculus. In fact Leibniz' idea of infinitesimals caused a stir with one of the outstanding philosophers of the day, Bishop George Berkeley, who wrote:

> "And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" [2]

It was not until 1960 that Abraham Robinson discovered the means to provide a solid foundation for Nonstandard Analysis, and through this the concept and structure of infinitesimals were thoroughly established.

CHAPTER 2: THE HYPERREAL NUMBER SYSTEM

One of the student's greatest difficulties is conceptualizing what a "very small" number or "approaching a value" means and whether or not that number is "small" enough to disregard. So in the nonstandard approach we help them by defining a new set of numbers, an extension to the Reals, called Hyperreals. The Hyperreals is the set of the Real numbers to which have been added infinitesimals (infinitely small numbers) and infinites (infinitely large numbers). Infinitesimals will be denoted by $\Delta x, \Delta y, \varepsilon, \delta$, the infinites by H, K, and the set of the Hyperreals by $\mathcal{R}^*$.

STRUCTURE OF THE HYPERREALS

We begin building the Hyperreals, conceptually, by adding these infinitesimals to the Reals, somewhat like adding a decimal fraction to a whole number. This generates a collection of Hyperreal numbers infinitely close to each element in the Reals. This is reminiscent of the "fuzzy ball" idea that each Real number is surrounded by numbers that are really close in value. In this model however only one Real number is in each fuzzy ball. In the text Keisler uses an infinitesimal microscope to show this idea.

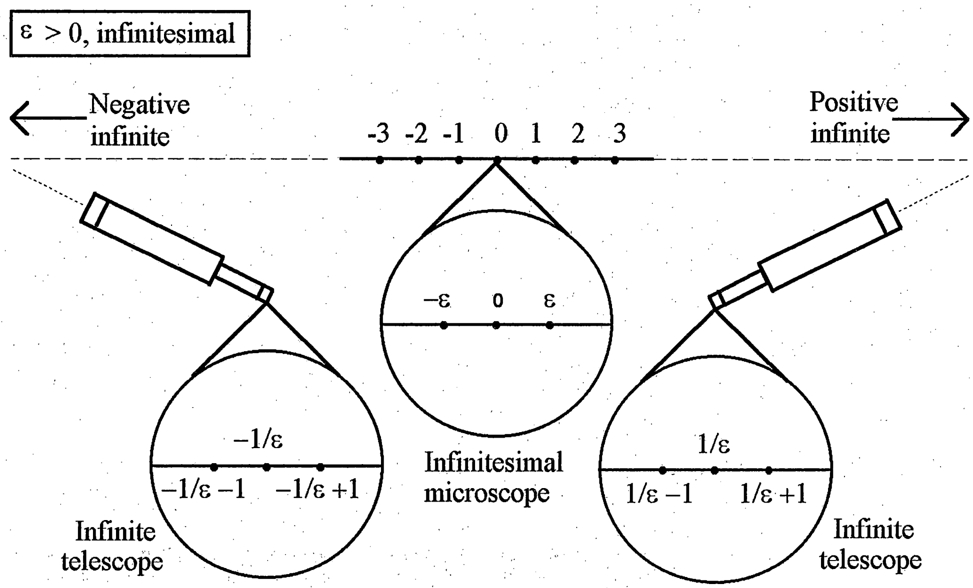

Figure 1: The Hyperreal Line [3]

We conceptually generate the infinites by taking reciprocals of the infinitesimals. The Hyperreals are said to be closed under addition, subtraction, multiplication and division, allowing the student to compute with this extended set of numbers exactly as they would with the Real numbers.

But this conceptual introduction will only take the student so far, so more formal definitions of what these Hyperreal numbers are and how they relate are provided. The most important definitions the student must learn relate directly or indirectly to infinitesimals.

Generally:

i) A number ε is said to be infinitely small or infinitesimal if for every positive a an element of the Reals, $-a < \varepsilon < a$. The only <u>Real</u> number that is infinitesimal is zero.

ii) If $a, b \in \mathcal{R}^*$ and $a - b$ is infinitesimal then $a \approx b$. This is read a is infinitely close to b.

The Hyperreal numbers infinitely close to 0 are infinitesimal, denoted by $a \approx 0$. These are the only infinitesimals.

iii) If ε is infinitesimal and is positive then $-\varepsilon$ is negative infinitesimal, $\frac{1}{\varepsilon}$ is infinite and positive, and $-\frac{1}{\varepsilon}$ is infinite and negative. A Hyperreal number that is not infinite is called a finite number.

Once we have identified what infinitesimals and infinites are, we need to know that we can compute with these new numbers in some reasonable way if the Hyperreals are going to be useful to us in the study of the Calculus. Three basic principles which establish the relationship between the Reals and the Hyperreals form the basis of the Calculus as developed in Keisler's textbook.

CHAPTER 3: THREE BASIC PRINCIPLES

The three basic principles on which Keisler relies for his development of the Calculus are the Extension Principle, the Transfer Principle and the Standard Parts Principle. In Hurd and Loeb, we should note, these results are actually presented as theorems. Presentation of these principles essentially as axioms allows Keisler to use the power they provide without the burden of detailed technical development or justification. Requiring such technical development would defeat the purpose of using the nonstandard approach to teach the Calculus and do little to enhance student understanding.

THE EXTENSION PRINCIPLE

The Extension Principle tells us that the Real numbers are a natural and proper subset of the Hyperreals, and it extends all Real functions to the Hyperreal numbers.

Principle I: Extension Principle

a) the Real numbers form a subset of the Hyperreal numbers, and the order relation $x < y$ for the Real numbers is a subset of the order relation for $\mathcal{R}^*$. In other words the Real line is a part of the Hyperreal line.

b) there exists an ε in $\mathcal{R}^*$ such that $\varepsilon > 0$ but $\varepsilon < a$ for all $a \in R^+$

c) for every Real function f of one or more variables we are given a corresponding Hyperreal function f^* of the same number of variables. f^* is called the natural extension of f.

From this we formulate a more precise description of infinitesimals.

Definition: A Hyperreal number b is said to be:

Positive Infinitesimal if b is positive but less than every positive real number.

Negative Infinitesimal if b is negative but greater than every negative Real number.

Infinitesimal if b is either positive infinitesimal, negative infinitesimal, or 0.

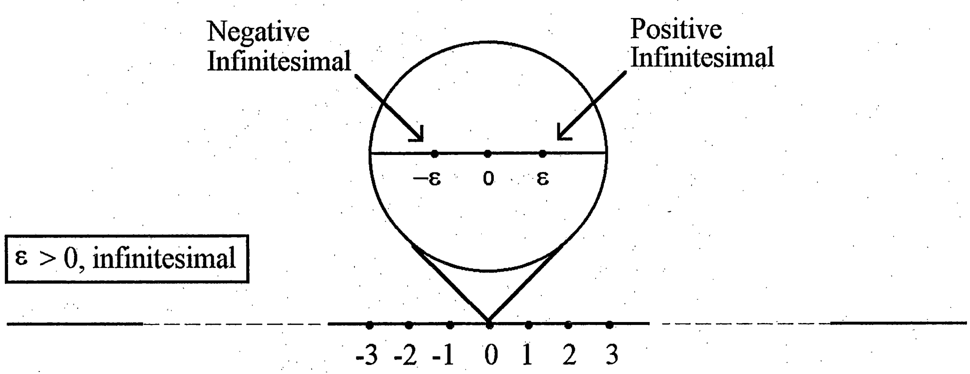

Figure 2: The Infinitesimals [3]

Then we do the same for the infinite numbers.

Definition: A Hyperreal number b is said to be:

<u>Finite</u> if b is between two Real numbers.

<u>Positive Infinite</u> if b is greater than every Real number.

<u>Negative Infinite</u> if b is less than every Real number.

<u>Infinite</u> if b is either positive infinite or negative infinite.

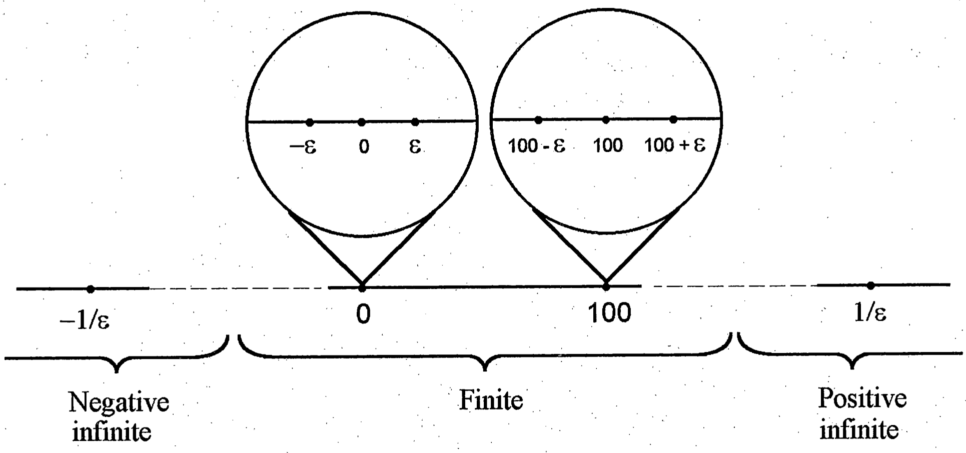

Figure 3: The Finites and Infinites [3]

The definitions for finite, infinite and infinitesimal numbers given above are stated in terms of comparisons to Real numbers. Usually when we are working with Hyperreal numbers it is more convenient to compare to other Hyperreals. The following theorem allows us to do this.

Theorem:

i) Every Hyperreal number which is between two infinitesimals is infinitesimal.

ii) Every Hyperreal number which is between two finite Hyperreal numbers is finite.

iii) Every Hyperreal number which is greater than some positive infinite number is positive infinite.

iv) Every Hyperreal number which is less than some negative infinite number is negative infinite.

These results are easy to prove. For instance, consider ii).

Proof of ii):

Let $a,b,c \in \mathcal{R}^*$ with $a<b<c$ and a,c finite.

To show b finite, i.e. $\exists$ r,s Real such that $r<b<s$.

Since a,c finite, $\exists$ r_1,s_1 Real such that $r_1<a<s_1$ and $\exists$ r_2,s_2 Real such that $r_2<c<s_2$.

Set $r=\min(r_1,r_2)$ and $s=\max(s_1,s_2)$, then $r \le r_1 < a < b < c < s_2 \le s$. Thus $r<b<s$ with r,s Real.

Therefore b is finite.

THE TRANSFER PRINCIPLE

The Transfer Principle states essentially that computations with, and properties of, the Hyperreal numbers are identical to those of the Real numbers.

Principle II: Transfer Principle

Every Real statement that holds for one or more particular Real function holds for the Hyperreal natural extensions of these functions.

The transfer principle basically states that everything we do with the Reals -- the rules, functions and operations -- we also can do in the Hyperreals, and they behave the way we expect them to.

Definition: Real Statement

A combination of equations or inequalities about Real expressions and statements specifying whether a Real expression is defined or undefined.

Specifically a Real statement involves real variables and particular Real functions.

Examples of Real Statements:

1) Associative laws for addition and multiplication
2) Distributive law
3) Properties of equality
4) The fact that division by zero is not defined
5) Additive inverses exist
6) Multiplicative inverses exist for non-zero elements
7) Properties of inequalities
8) Additive and multiplicative identities exist
9) Algebraic and trigonometric identities

Each of these statements transfers to an equivalent statement which holds for the Hyperreals.

Generally, any statement which you can say is true for all Real numbers is also true for all Hyperreal numbers, with an appropriate interpretation. For instance, the statement that $a^2 - 1 = (a+1)(a-1)$ holds true for all Real numbers a. Therefore it also holds true if a is any Hyperreal number. No change in the interpretation of the statement appeared to be necessary, but we did in fact have to remember that 1 is both a Real and a Hyperreal number. We treat 1 as a Real number when interpreting the Real statements and as a Hyperreal number when interpreting the transferred (Hyperreal) statements.

For the statement, "If $a \neq 0$ then $\frac{1}{a}$ exists as a Real number," we have to make a more explicit adjustment in our interpretation of the statement when we transfer it to the Hyperreals. The transferred statement should be, "If $a \neq 0$ then $\frac{1}{a}$ exists as a Hyperreal number." Though this seems obvious and trivial, it shows that being too casual in our handling of the Transfer Principle can lead to an error.

A good example is the Archimedean Principle: "For every Real a there exists an integer n such that $a < n$." If we are not careful, an incorrect transfer of this statement which we might generate could be: "For every Hyperreal a there exists an integer n such that $a < n$." Clearly this is not true because each positive infinite number is larger than any integer, since all integers are Real numbers. The correct transfer would be: "For every Hyperreal a there exists a Hyperinteger n such that $a < n$." The specific definition of Hyperinteger requires the precise construction of the Hyperreals from Hurd and Loeb and thus will be deferred until this is presented later in this document.

In Hurd and Loeb, the definition of the formal language which we must use to describe the Transfer Principle and the statements to which it applies is rather complicated to construct and understand, but it essentially says the same thing. Fortunately the language itself helps us avoid simple errors in using the Transfer Principle. By learning and using the formal language, transferring statements and interpreting the results becomes a straight-forward process.

The Transfer Principle basically states that everything we do with the Reals, be it rules, functions, or operations, behave the way we expect them to in the Hyperreals.

HYPERREAL ARITHMETIC

The arithmetic operation rules for the Hyperreals are the same as for the Reals with a few added details and special cases.

Rules for Infinitesimal, Finite, and Infinite Numbers

Assume that ε,δ are infinitesimals; b,c are Hyperreal numbers that are finite but not infinitesimal; and H,K are infinite Hyperreal numbers.

i) **Real numbers:**

The only infinitesimal Real number is 0

Every real number is finite.

ii) **Negatives:**

$-\varepsilon$ is infinitesimal

$-b$ is finite but not infinitesimal

$-H$ is infinite

iii) **Reciprocals:**

If $\varepsilon \neq 0$, $\frac{1}{\varepsilon}$ is infinite

$\frac{1}{b}$ is finite but not infinitesimal

$\frac{1}{H}$ is infinitesimal

iv) **Sums:**

$\varepsilon + \delta$ is infinitesimal

$b + \varepsilon$ is finite but not infinitesimal

$b + c$ is finite (possibly infinitesimal)

$H + \varepsilon$ and $H + b$ are infinite

v) **Products:**

$\varepsilon \cdot \delta$ and $b \cdot \varepsilon$ are infinitesimal

$b \cdot c$ is finite but not infinitesimal

$H \cdot b$ and $H \cdot K$ are infinite

vi) **Quotients:**

$\frac{\varepsilon}{b}, \frac{\varepsilon}{H}$ and $\frac{b}{H}$ are infinitesimal

$\frac{b}{c}$ is finite but not infinitesimal

$\frac{b}{\varepsilon}, \frac{H}{\varepsilon}$ and $\frac{H}{b}$ are infinite, provided that $\varepsilon \neq 0$

vii) **Roots:**

If $\varepsilon > 0$, $\sqrt[n]{\varepsilon}$ is infinitesimal

If $b > 0$, $\sqrt[n]{b}$ is finite but not infinitesimal

If $H > 0$, $\sqrt[n]{H}$ is infinite

There are four combinations which are not covered above because they may result in a variety of different answers depending on the relative "size" of the values being combined. Since we can not say for sure what the actual "size" of the answer will be without additional information about the numbers themselves, we refer to these combinations as "indeterminate forms". The four indeterminate forms are $\frac{\varepsilon}{\delta}$, $\frac{H}{K}$, $H\varepsilon$ and $H+K$. The following table illustrates how examples of each of these combinations can generate results which are infinitesimal, finite or infinite.

Figure 4: Indeterminate Forms of Various Values

Indeterminate Form	Infinitesimal	Finite (Equal to 1)	Infinite
$\frac{\varepsilon}{\delta}$	$\frac{\varepsilon^2}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon^2}$
$\frac{H}{K}$	$\frac{H}{H^2}$	$\frac{H}{H}$	$\frac{H^2}{H}$
$H\varepsilon$	$H \cdot \frac{1}{H^2}$	$H \cdot \frac{1}{H}$	$H^2 \cdot \frac{1}{H}$
$H+K$	$H+(-H)$	$(H+1)+(-H)$	$H+H$

So far we have discussed computations with the Hyperreals, but for these computations to be helpful to us in the Calculus we have to be able to get answers which are Real numbers. Generally the answer we need is the Real number closest to the Hyperreal number which is the result of our computations. Finding the appropriate Real number, and even knowing that such exists, involves a concept called the Standard Part of a number.

STANDARD PARTS

To define the Real number we are looking for, we need to define what it means to be "infinitely close" to a number.

Definition: Infinitely Close

Two Hyperreal numbers b and c are said to be infinitely close to each other, in symbols $b \approx c$, if their difference $b - c$ is infinitesimal.

Numbers which are infinitely close to each other satisfy certain properties, as described below.

Properties of Hyperreals:

1) If ε is infinitesimal then $b \approx b+\varepsilon$

(*b* is infinitely close to $b+\varepsilon$ because

$b-(b+\varepsilon)=-\varepsilon$ is infinitesimal)

2) b is infinitesimal if and only if $b \approx 0$

(this notation is used to indicate that b is infinitesimal)

3) If b and c are real and $b \approx c$ then $b=c$

($b-c$ is real <u>and</u> infinitesimal, hence 0)

Theorem:

Let a, b and c be Hyperreal numbers.

i) $a \approx a$

ii) If $a \approx b$, then $b \approx a$

iii) If $a \approx b$ and $b \approx c$ then $a \approx c$

Theorem:

Assume $b \approx c$

i) If $b \approx 0$ then so is c

ii) If b is finite then so is c

iii) If b is infinite then so is c

For convenience Real numbers are sometimes called "standard" numbers while Hyperreal numbers that are <u>not</u> Real are "nonstandard" numbers. This then leads us naturally to calling the Real number that is infinitely close to b the "standard part" of b.

Definition: Standard Part

Let b be a finite Hyperreal number. The <u>standard part</u> of b, denoted by $\mathrm{st}(b)$, is the Real number which is infinitely close to b. Infinite Hyperreal numbers do not have standard parts.

Example: If $b = a + \varepsilon$, $a \in R$, then $\mathrm{st}(b) = a$

Principle III: Standard Part Principle

Every finite Hyperreal number is infinitely close to <u>exactly</u> one Real number.

1) $\mathrm{st}(b)$ is Real
2) $b = \mathrm{st}(b) + \varepsilon$ for some ε
3) If b is Real then $b = \mathrm{st}(b)$

Theorem:

Let a and b be finite Hyperreal numbers. Then:

i) $\text{st}(-a) = -\text{st}(a)$

ii) $\text{st}(a+b) = \text{st}(a) + \text{st}(b)$

iii) $\text{st}(a-b) = \text{st}(a) - \text{st}(b)$

iv) $\text{st}(a \cdot b) = \text{st}(a) \cdot \text{st}(b)$

v) If $\text{st}(b) \neq 0$, then $\text{st}(a/b) = \text{st}(a)/\text{st}(b)$

vi) $\text{st}(a^n) = (\text{st}(a))^n$

vii) If $a \geq 0$, then $\text{st}(\sqrt[n]{a}) = \sqrt[n]{\text{st}(a)}$

viii) If $a \leq b$, then $\text{st}(a) \leq \text{st}(b)$

Note: Even if $a < b$, $\text{st}(a)$ may equal $\text{st}(b)$

Examples of Proofs of the Properties:

Given: r,s are Real numbers, and ε,δ are infinitesimals

$a = r + \varepsilon$ and $b = s + \delta$

Property (i): $\text{st}(-a) = -\text{st}(a)$

Proof:

$$\text{st}(-a) =$$
$$\text{st}(-(r+\varepsilon)) =$$
$$\text{st}(-r + (-\varepsilon)) =$$
$$-r =$$
$$-\text{st}(a) = -\text{st}(a)$$

Property (ii): $\text{st}(a+b) = \text{st}(a)+\text{st}(b)$

Proof: $\text{st}(a+b) =$

$$\text{st}((r+\varepsilon)+(s+\delta)) =$$

$$\text{st}(r+s+\varepsilon+\delta) =$$

$$\text{st}((r+s)+(\varepsilon+\delta)) =$$

$$\in R \qquad \approx 0$$

$$r+s =$$

$$\text{st}(a)+\text{st}(b) = \text{st}(a)+\text{st}(b)$$

Examples of Computations Using Standard Parts:

Example 1:

Compute the standard part of $2+\varepsilon+3\varepsilon^2$

$\text{st}(2+\varepsilon+3\varepsilon^2)$

$=\text{st}(2)+\text{st}(\varepsilon)+\text{st}(3\varepsilon^2)$

$=2+0+\text{st}(3)\cdot\text{st}(\varepsilon^2)$

$=2+0+3\cdot 0$

$=2$

Example 2:

$\text{st}\left((4+\varepsilon)^2+\sqrt{9\delta}\right)$

$=\text{st}\left((4+\varepsilon)^2\right)+\text{st}\left(\sqrt{9\delta}\right)$

$=\left(\text{st}(4+\varepsilon)\right)^2+\text{st}\left(\sqrt{9}\right)\cdot\text{st}\left(\sqrt{\delta}\right)$

$=4^2+3\cdot 0$

$=16$

CHAPTER 4: NONSTANDARD CALCULUS

STANDARD PARTS AND LIMITS

Many students have difficulty with limits. The nonstandard approach allows us to do the Calculus as a computational exercise with a new set of numbers instead of having to understand what limits are all about.

Yet the algebra involved is virtually identical and the rules for taking standard parts are the same as those for taking limits.

If you consider the following example, you should recognize the similarity between standard parts and limits. In fact, taking standard parts replaces the operation of taking limits in the Calculus as we shall see.

Example Comparing Rules for Standard Parts and Limits:

Standard Parts	Limits
$\mathrm{st}(a+b)=\mathrm{st}(a)+\mathrm{st}(b)$	$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

After getting a basic understanding of how standard parts computations go, when the students learn about limits, and even the ε,δ method, they will already have an understanding of the computations involved and should even have an intuitive idea of how limits behave.

DIFFERENTIATION, OR "LOOK MA, NO LIMITS!"

One of the most interesting aspects of the Keisler text is that Differentiation and Derivatives are introduced by using standard parts instead of limits. This, I feel, is what gives students the edge to a better understanding. Of course "limits" are introduced in the next chapter of Keisler's text, but only after a student has had the opportunity to do some work with derivatives by using standard parts.

SLOPE OF A LINE

The motivating diagrams and definitions for slope of a curve and derivative of a function used by Keisler are virtually identical to the traditional development, with the exception that limits are replaced by standard parts.

S is said to be the <u>slope</u> of f at a if

$$S = \mathrm{st}\left(\frac{f(a+\Delta x) - f(a)}{\Delta x}\right) \text{ for every infinitesimal } \Delta x \neq 0$$

Infinitesimal microscope makes idea of slope of tangent line intuitive.

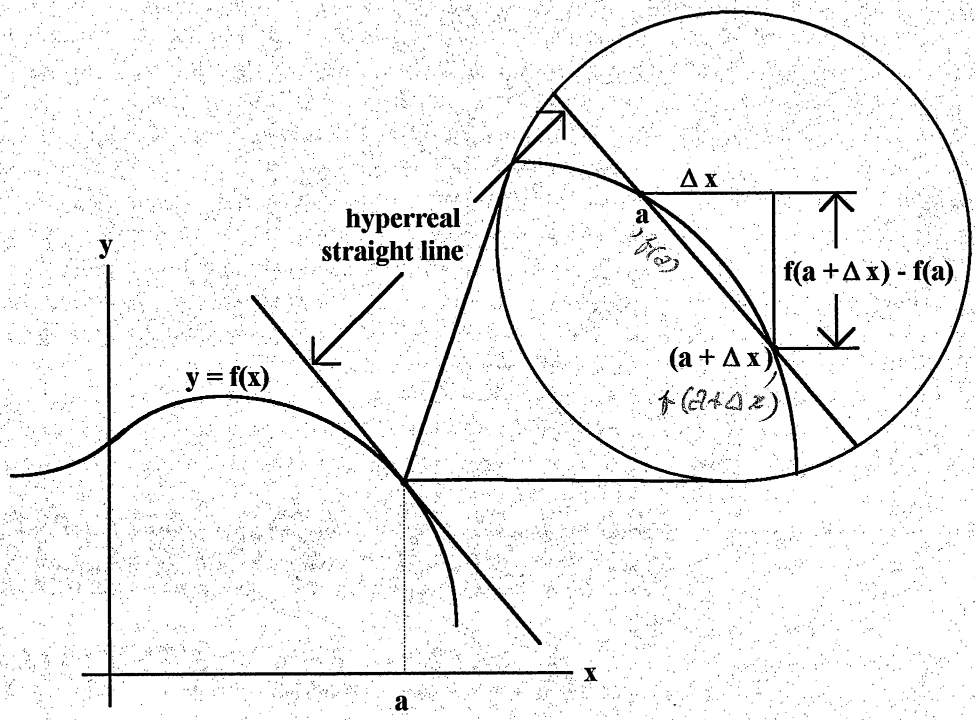

Figure 5: Slope of a Line [3]

As in the traditional method, there are several ways the slope of f at a can fail to exist. The details often look quite a bit different, though.

The following are all the possibilities that can occur when computing slopes.

1) The slope of f at a exists if the ratio $\left(\frac{f(a+\Delta x)-f(a)}{\Delta x}\right)$ is finite and has the same standard part for all infinitesimal $\Delta x \neq 0$

It has value $S = \mathrm{st}\left(\frac{f(a+\Delta x)-f(a)}{\Delta x}\right)$

2) The slope of f at a can fail to exist in any of four ways:

a) $f(a)$ is undefined.

b) $f(a+\Delta x)$ is undefined for some $\Delta x \neq 0$

c) The term $\dfrac{f(a+\Delta x)-f(a)}{\Delta x}$ is infinite for some $\Delta x \neq 0$

d) The term $\dfrac{f(a+\Delta x)-f(a)}{\Delta x}$ has different standard parts for different $\Delta x \neq 0$

The following are examples for the cases listed above of how computations of slopes can fail.

Example a): Slope of $f(1)$ doesn't exist for $f(x)=\frac{1}{x-1}$

Example b): $f(a+\Delta x)$ is undefined for some infinitesimal $\Delta x \neq 0$. The construction necessary for this example relies on a much more detailed knowledge of how functions transfer into the nonstandard domain. Because of this, it is not covered in Keisler and is not likely to be presented in a Calculus course. This will be covered in detail during the more formal discussion of nonstandard analysis as presented in Hurd and Loeb.

Example c): The term $\dfrac{f(a+\Delta x)-f(x)}{\Delta x}$ is infinite for some infinitesimal $\Delta x \neq 0$

$$f(x)=\begin{cases} 5 \ \text{if} \ x \geq 7 \\ -3 \ \text{if} \ x<7 \end{cases}$$

If $\Delta x>0$ then $\dfrac{f(7+\Delta x)-f(7)}{\Delta x}=\dfrac{5-5}{\Delta x}=\dfrac{0}{\Delta x}=0$

If $\Delta x<0$ then $\dfrac{f(7+\Delta x)-f(7)}{\Delta x}=\dfrac{-3-5}{\Delta x}=\dfrac{-8}{\Delta x}=$ infinite

Example d): The term $\dfrac{f(a+\Delta x)-f(x)}{\Delta x}$ has different standard parts for different infinitesimals $\Delta x \neq 0$

$$f(x)=|x|$$

Formula for slope at $a=0$,

$$\frac{f(0+\Delta x)-f(0)}{\Delta x}=\frac{f(\Delta x)-(0)}{\Delta x}=\frac{|\Delta x|}{\Delta x}$$

Case 1: $\Delta x>0$, $\dfrac{|\Delta x|}{\Delta x}=\dfrac{\Delta x}{\Delta x}=1$

Case 2: $\Delta x<0$, $\dfrac{|\Delta x|}{\Delta x}=\dfrac{-\Delta x}{\Delta x}=-1$

DERIVATIVES

Once the slope of a line is defined, the definition of the derivative is exactly the same as in the traditional method.

Definition:

Let f be a real function of one variable. The derivative of f is the new function f' whose value at x is the slope of f at x

$f'(x) = \mathrm{st}\left(\dfrac{f(x+\Delta x)-f(x)}{\Delta x}\right)$ wherever the slope exists

$f'(x)$ is undefined at x if the slope of f does not exist at x

For a given point a, the slope of f at a and the derivative of f at a are the same.

If we write $\Delta y = f(x+\Delta x) - f(x)$, we can identify Δy as a real function of two variables x and Δx.

The transfer principle implies that this equation also determines Δy as a Hyperreal function of the same two variables.

Δy is called the <u>increment</u> of y.

Sometimes we write $y' = f'(x)$ so that

$$f'(x) = \mathrm{st}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$$

takes the form $y' = \mathrm{st}\left(\dfrac{\Delta y}{\Delta x}\right)$

Whichever form we choose to use, with the nonstandard approach, differentiation becomes a simple computation.

Example: Compare finding the derivative of $f(x)=\frac{1}{x}$ using the limit concept to using standard parts.

Using Limits

$$\frac{d}{dx}f(x)= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\frac{1}{x} = \lim_{h\to 0}\frac{1/(x+h)-1/x}{h}$$

$$= \lim_{h\to 0}\frac{x-(x+h)}{xh(x+h)}$$

$$= \lim_{h\to 0}\frac{-h}{xh(x+h)}$$

$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$= \frac{-1}{x(x+0)}$$

$$f'(x) = -\frac{1}{x^2}$$

Standard Parts

$$f'(x)=\mathrm{st}\left(\frac{\Delta y}{\Delta x}\right)=\mathrm{st}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{1/(x+\Delta x)-1/x}{\Delta x}$$

$$= \frac{x-(x+\Delta x)}{x(x+\Delta x)\Delta x}$$

$$= \frac{-\Delta x}{x(x+\Delta x)\Delta x}$$

$$= \frac{-1}{x(x+\Delta x)}$$

$$\mathrm{st}\left(\frac{\Delta y}{\Delta x}\right) = \mathrm{st}\left(-\frac{1}{x(x+\Delta x)}\right)$$

$$= -\frac{1}{\mathrm{st}(x)\mathrm{st}(x+\Delta x)}$$

$$f'(x) = -\frac{1}{x\cdot x} = -\frac{1}{x^2}$$

Note that the algebra involved in the two different approaches is virtually identical. The only real difference is a conceptual one; the real question is will the student handle limits or standard parts more easily? Clearly the answer to this is somewhat dependent on the specific student, but it is known that limits cause some students difficulty. Perhaps other students would grasp limits, but have difficulty with standard parts. This would seem unlikely, but it may still happen. Until more classes are taught using the nonstandard approach, though, we will not be able to answer this question.

DERIVATIVES OF SIN AND COS

The computation of derivatives of the transcendental functions is another challenge which is made easier by use of the nonstandard method.

Many students have a very difficult time accepting that the derivative of the $\sin(x)$ is the $\cos(x)$, even after seeing the proof, because of two things: i) it is non-intuitive, and ii) it relies on the use of the Pinching Theorem, which involves an interesting property of limits which makes little sense unless you understand limits, which many of the students don't.

The nonstandard approach addressed both of these problems. It provides a nice, intuition-supporting image for why the derivative of the $\sin(x)$ is the $\cos(x)$ and, when you finally get around to the proof, it eliminates the need for the Pinching Theorem.

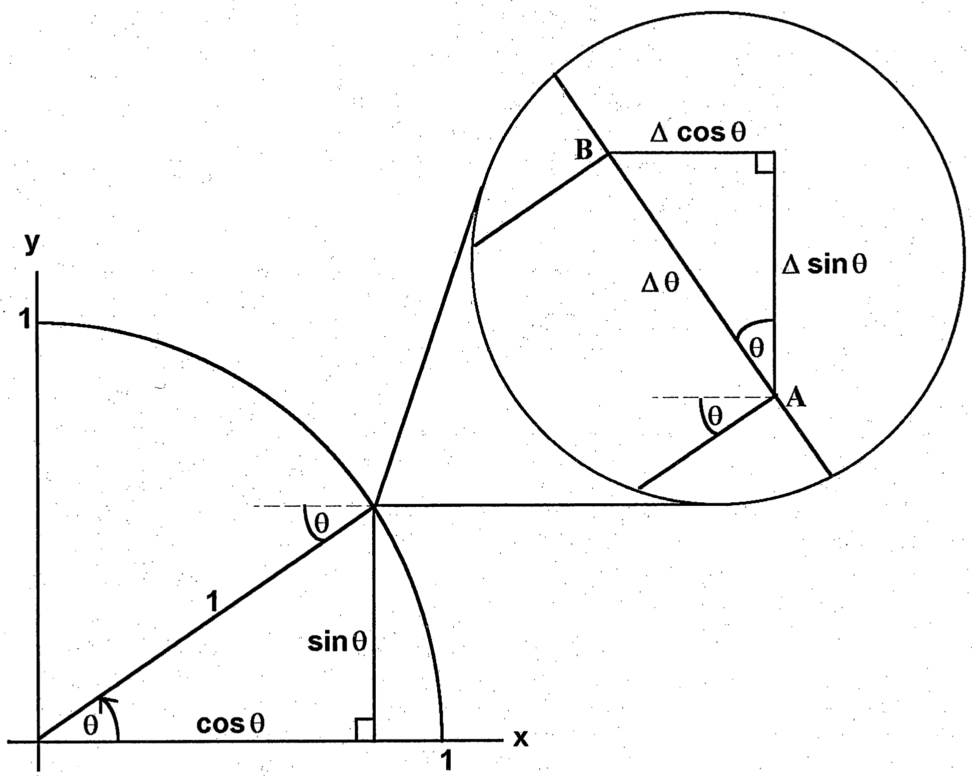

Figure 6: Derivatives of Sin(x) Under Microscope [3]

By looking at the above drawing, through the infinitesimal microscope one can see that by increasing θ by an infinitesimal amount, one moves an infinitesimal distance from point **A** to point **B**. In doing this, $\Delta\sin\theta$ is a small positive value and $\Delta\cos\theta$ is a small negative value. Thus we get $\cos\theta=\frac{\Delta\sin\theta}{\Delta\theta}=\frac{d}{d\theta}\sin\theta$ and $\sin\theta=\frac{-\Delta\cos\theta}{\Delta\theta}=-\frac{d}{d\theta}\cos\theta$, which are exactly the results we want.

If we desire, though, we can still do the complete proof. Like the traditional method, the key element of the proof that $\frac{d}{dx}\sin(x)=\cos(x)$ is the result that:

$$\mathrm{st}\left(\frac{\sin\theta}{\theta}\right)=1 \quad \left[\text{analogous to } \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

The same geometrical area comparison is used to prove $st\left(\frac{\sin\theta}{\theta}\right) = 1$, for any infinitesimal θ, as is used in the traditional approach. The difference is in the nonstandard method a simple standard parts computation is used, while the traditional method relies on the use of the Pinching Theorem. The inequality involves the areas of a nested sequence of objects, a triangle, a pie slice, and another triangle, determined by the same angle θ.

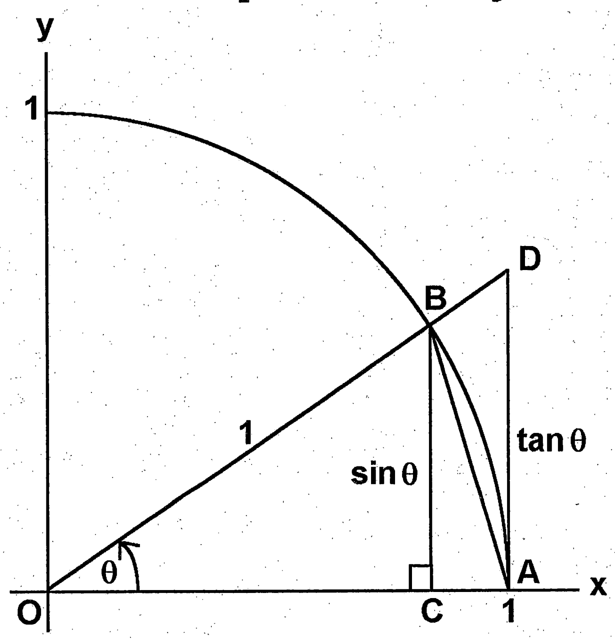

Figure 7: Diagram for the Geometric Inequality [3]

Proof:

Area(triangle OAB)<Area(pie slice OAB)<Area(triangle OAD)

$$\frac{1}{2}\cdot OA\cdot BC < \frac{1}{2}\cdot \theta\cdot r^2 < \frac{1}{2}\cdot OA\cdot AD$$

$$\rightarrow \frac{1}{2}\cdot BC < \frac{1}{2}\cdot \theta < \frac{1}{2}\cdot AD \quad (OA = r = 1)$$

$$\rightarrow BC < \theta < AD$$

$$\rightarrow \sin\theta < \theta < \tan\theta$$

$$\rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

taking standard parts with θ infinitesimal

(linear ordering is preserved)

$$\rightarrow st(1) > st\left(\frac{\sin\theta}{\theta}\right) > st(\cos\theta)$$

$\cos\theta$ is continuous by definition, so

$$st(\cos\theta) = \cos(st(\theta)) = \cos 0 = 1$$

$$\rightarrow 1 > st\left(\frac{\sin\theta}{\theta}\right) > 1$$

$$\rightarrow st\left(\frac{\sin\theta}{\theta}\right) = 1 \text{ for any infinitesimal } \theta$$

As we have come to expect, the algebra for the proof is identical to the traditional method. The primary difference is that we did not need the Pinching Theorem (see below) to complete the proof. In the nonstandard method, the Pinching Theorem is an automatic result of the linear ordering properties of the Hyperreal number system.

Theorem: Pinching Theorem

Given $f(x) < g(x) < h(x)$ on an interval around c

If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$

This Theorem is often stated and used in Calculus texts when computing the derivatives of the trigonometric functions, but rarely, if ever, is it proven in these texts. Instead it is treated as if it were intuitively obvious, which it probably is to people well versed in the theory of limits, but to many students it is not. These students often will find it easier and more understandable to do a simple computation using the rules of the Hyperreals than to even learn where to begin in applying the Pinching Theorem.

DIFFERENTIALS AND INCREMENT THEOREM

One of the nice, and reassuring, things about the nonstandard approach to the Calculus is that all of the properties and applications of the derivative are unchanged from the traditional approach. This makes it relatively easy for instructors experienced in teaching using the traditional approach to understand, accept and teach the Calculus using the nonstandard method. In some cases, the computational nature of the nonstandard approach makes it easier to understand not only the theory, but also the applications of the derivative.

One area of application of derivatives which has been difficult for some students to understand using the traditional method of teaching the Calculus is differentials. As with limits, the nonstandard method makes differentials a more intuitive, computational effort thus making them easier to understand and manipulate.

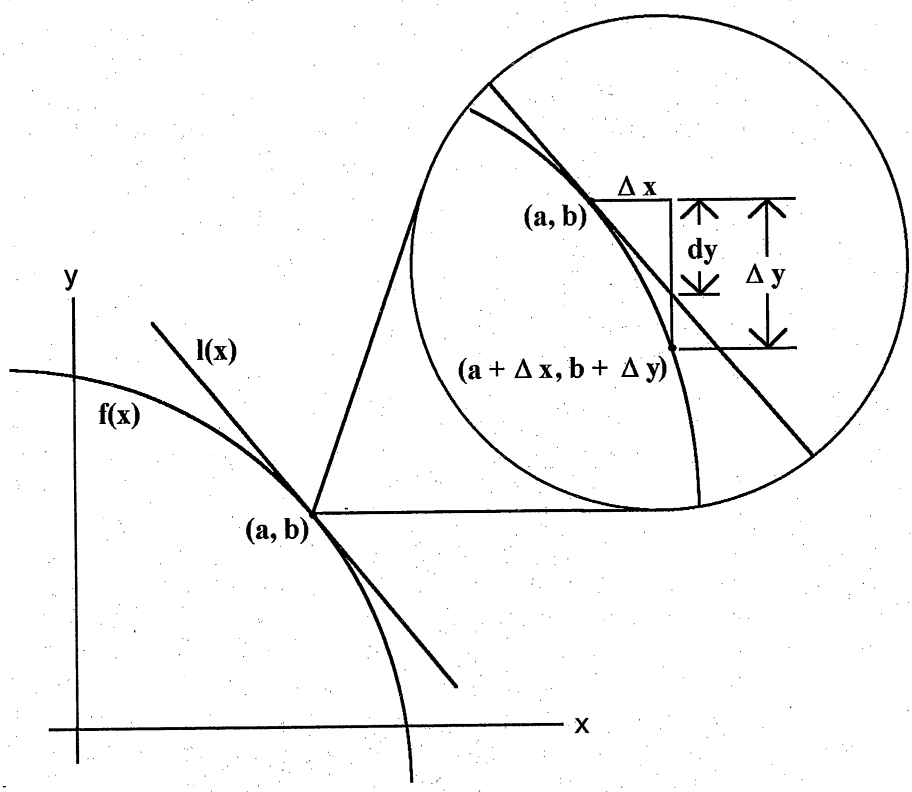

Figure 8: Differentials

Theorem: Increment Theorem

Let $y = f(x)$. Suppose $f'(x)$ exists at a certain point x, and Δx is infinitesimal. Then Δy is infinitesimal, and $\Delta y = f'(x)\Delta x + \varepsilon\Delta x$ for some infinitesimal ε, which depends on x and Δx.

Proof:

Case 1: $\Delta x = 0$. In this case, $\Delta y = f'(x)\Delta x = 0$,

and we put $\varepsilon = 0$

Case 2: $\Delta x \neq 0$. Then $\dfrac{\Delta y}{\Delta x} \approx f'(x)$, so for some

infinitesimal ε, $\dfrac{\Delta y}{\Delta x} = f'(x) + \varepsilon$

Multiplying both sides by Δx, we get,

$$\Delta y = f'(x)\Delta x + \varepsilon\Delta x$$

That is, the increment theorem provides us with a method for finding a good approximation of the change in the value of the function for small changes in the value of the variable. Intuitively, we can see that if Δx, an infinitesimal, is conceptually "very small", then $\varepsilon\Delta x$, the product of infinitesimals, is "extremely small" or "very small compared to Δx". Even though the "very small" change in x generates only a "very small" change in y, since we only want an approximation, it seems justifiable that we ignore the "extremely small" product-of-infinitesimals term.

In doing so, though, we must remember that our computations will not actually use infinitesimals and thus can only give us a good approximation if the change in x is kept as small as possible.

Example of Differentials

Find an approximation for $f(4.01)$, where $f(x) = \sqrt{x}$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $x = 4$, $f(x) = \sqrt{4} = 2$, and $\Delta x = 0.01$, so

$$\Delta y \cong f'(x)\Delta x = \frac{1}{2\sqrt{4}}(0.01) = 0.0025$$

and then $y \cong y_o + \Delta y = 2 + 0.0025 = 2.0025$

(Note: $y \cong$ 2.00249843945 by the calculator)

Notice that except for the nonstandard analysis style used in the statement of the Increment Theorem, as usual differentials are manipulated the same way as in the traditional method. The advantage of the nonstandard method to the student is that it provides a solid, intuitive basis for this computation.

CHAPTER 5: FOUNDATIONS OF NONSTANDARD ANALYSIS

The Hyperreal number system can help students learn and understand the Calculus without relying on a deep, technical understanding of why they work (or even real proof that such things as infinitesimals exist).

Unfortunately new methods are usually not adopted by the mathematical community just because "they seem to work". This is why a more rigorous development of the Calculus than Leibniz's was pursued. Abraham Robinson's development of infinitesimals in the 1960's finally provided the rigorous foundation for infinitesimals and the Hyperreals that allowed their use in the teaching of the Calculus, and mathematics generally, to be seriously considered.

FILTERS AND ULTRAFILTERS

In order to understand the construction of the Hyperreal number system, it is necessary to establish some basic definitions and methods related to filters and ultrafilters.

To begin with we need to understand what an ultrafilter is, and more importantly what a free ultrafilter might be. For it is with free ultrafilters that we build the extension to the Reals called the Hyperreals which is the basis for Nonstandard Analysis.

First we must start with the definition of a filter.

Definition: Filter

Let I be a nonempty set. A <u>filter</u> on I is a nonempty collection $\mathcal{U}$ of subsets of I having the following properties.

i) $\varnothing \notin \mathcal{U}$

ii) $A \in \mathcal{U}$ and $B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

iii) $A \in \mathcal{U}$ and $A \subseteq B \Rightarrow B \in \mathcal{U}$

A filter $\mathcal{U}$ is an <u>ultrafilter</u> if

iv) for any subset A of I either $A \in \mathcal{U}$ or its compliment $A^c = I - A \in \mathcal{U}$ (but not both)

By conditions i) and ii), an ultrafilter cannot contain both A and A^c, since if both were elements of $\mathcal{U}$, then by ii), $A \cap A^c \in \mathcal{U}$, but $A \cap A^c = \varnothing$, which contradicts i).

A filter therefore is closed under the operations of intersection and "supersetting", and does not contain the empty set. An ultrafilter has the additional property that either a set or its complement is always an element of the ultrafilter.

An alternate and equivalent definition for an ultrafilter is a maximal filter.

Definition: Maximal Filter

a filter F on I is <u>maximal</u> if whenever G is a filter on I and $F \subset G \rightarrow F = G$ [3]

Proposition: A filter F on I is maximal if and only if for every subset A of I either $A \in F$ or $A^c = I - A \in F$

Proof:

$\Rightarrow$ Suppose either $A \in F$ or $A^c \in F \quad \forall A \subseteq I$

Let G be some filter containing F, $G \supseteq F$, and suppose $B \in G$ and $B \notin F$ then $B^c \in F$ but since $G \supseteq F$ then $B^c \in G$. $B \in G$ and $B^c \in G$ implies $B \cap B^c \in G$. But $B \cap B^c = \varnothing$, by definition, which implies $\varnothing \in G$, a contradiction to G being a filter. Therefore there is no filter G properly containing F. Thus F is maximal.

$\Leftarrow$ Suppose F is a maximal filter on I and $A \notin F$

To show $A^c \in F$:

Suppose $\forall B \in F \quad A \cap B \neq \varnothing$

Let $G = \{X \subseteq I : A \cap H \subseteq X$ for some $H \in F\}$

$H \in G \quad \forall H \in F$ since $A \cap H \subseteq H$, therefore $F \subseteq G$

$A \in G$ since $A \cap H \subseteq A$

Therefore F is a proper subset of G

Check properties of filters:

i) $\varnothing \notin G$ by construction, $A \cap H \neq \varnothing \quad \forall H \in F$

ii) Pick $G_1, G_2 \in G$. Then $\exists H_1, H_2 \in F \ni A \cap H_1 \subseteq G_1$ and $A \cap H_2 \subseteq G_2$

Then $A \cap (H_1 \cap H_2) = (A \cap H_1) \cap (A \cap H_2) \subseteq G_1 \cap G_2$

Since $H_1 \cap H_2 \in F$, therefore $G_1 \cap G_2 \in G$

iii) Pick $G_1 \in G$, $G_1 \subseteq G_2$. Then $\exists H_1 \in F \ni A \cap H_1 \subseteq G_1$

Since $G_1 \subseteq G_2$, $A \cap H_1 \subseteq G_2$ and hence $G_2 \in G$

Therefore G is a filter. This contradicts our assumption that F is a maximal filter, therefore the assumption of this case cannot occur.

Since the previous case failed, $\exists B \in F \ni A \cap B = \varnothing$. This implies $B \subseteq A^C$, which implies $A^C \in F$, which is exactly what we wanted to show.

Definition: Fixed Ultrafilter

An ultrafilter $\mathcal{U}$ is <u>principal</u> or <u>fixed</u> if

$$\exists x \in I \ni \forall B \in \mathcal{U},\ x \in B$$

For each $x \in I$, then, there is a fixed ultrafilter

$$\mathcal{U}_x = \{B \subseteq I : x \in B\}$$

We will call x the <u>generating element</u> of the fixed ultrafilter $\mathcal{U}$.

Some collections of sets may satisfy some but not all of the conditions to be a filter (or ultrafilter), so one must be careful to verify the conditions accurately.

Given the set $S = \{1,2,3\}$, its power set is

$$\mathrm{P}(S) = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$

Clearly $\mathrm{P}(S)$, the power set of S, satisfies the intersection (ii) and superset (iii) conditions for a filter. It even satisfies the maximal condition for an ultrafilter (though not the mutual exclusiveness result). But it does contain the empty set. The empty set is a subset of every set, so it is an element of the power set, therefore the power set of a set is not a filter (and thus is certainly not an ultrafilter).

Let $I = \{a,b,c,d\}$ and consider the sets of subsets of I:

$$A=\{\{a,c\},\{a,b,c\},\{a,c,d\},\{a,b,c,d\}\}$$

$$B=\{\{c\},\{a,c\},\{b,c\},\{d,c\},\{a,b,c\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$$

Claim that A and B are filters on I.

Check the properties for A:

i) A does not contain the empty set

ii) Intersection of two elements of A is still an element of A, e.g. $\{a,b,c\} \cap \{a,c,d\} = \{a,c\} \in A$

iii) If a set is an element of A and that set is contained in a larger set then the larger set is also contained in A. That is,

$I = \{a,b,c,d,\}$ and let

$\alpha = \{b,c\}$ and $\beta = \{a,b,c\}$

Obviously $\beta \supset \alpha$ and $I \supset \beta \supset \alpha$

and since $\alpha \in A$

$\therefore \beta \in A$

By exhaustively checking all possible combinations, it can be verified that all the conditions hold, A is a filter on I. The set B is also a filter by the same properties, in fact it is a maximal filter or ultrafilter on I by:

iv) B is a fixed ultrafilter since every set contains the element c.

Fixed ultrafilters are of limited interest to us since they generate nothing new in our construction of the Hyperreals. Our construction requires the use of free ultrafilters. Free ultrafilters contain the Frechet Filter.

FRECHET FILTER

The Frechet Filter ($\mathcal{F}_1$) is a special filter defined on infinite sets. It is also called the cofinite filter and it is defined by:

$$\mathcal{F}_1 = \{A \subseteq I : I - A \text{ is finite}\}$$

The Frechet filter is a large collection of infinite sets. The base set must be infinite or the Frechet "filter" is not a filter at all, since if I is finite then $\mathcal{F}_1$ contains the null set (since it is cofinite) and thus is not a filter.

Proposition:

$\mathcal{F}_1 = \{A \subseteq I : I - A \text{ is finite}\}$ is a filter if I infinite

Proof:

By definition of $\mathcal{F}_1$, I is an infinite set.

i) Suppose $\varnothing \in \mathcal{F}_1$, then $I - \varnothing = I$ would be finite which would contradict the condition that I is infinite.

ii) Let $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_1$

To show $A \cap B \in \mathcal{F}_1$

(i.e., show that the intersection of the sets A and B in $\mathcal{F}_1$ is itself an element of $\mathcal{F}_1$)

By DeMorgan's Laws $A^c \cup B^c = (A \cap B)^c$

So $(I - A) \cup (I - B) = I - (A \cap B)$

finite finite finite

$\therefore (A \cap B)^c$ is finite and $A \cap B \in \mathcal{F}_1$

iii) If $A \in \mathcal{F}_1$ and $I \supseteq B \supseteq A$

To show $B \in \mathcal{F}_1$

$B \supseteq A \rightarrow B^c \subseteq A^c$, but A^c is finite

$\therefore B^c$ is finite which satisfies definition of an element of the Frechet filter

$\therefore B \in \mathcal{F}_1$

Examples of Elements of $\mathcal{F}_1$ on N

Let C={all primes less than 100}, C is finite and D=N−C $\in \mathcal{F}_1$. D is one element of the Frechet filter.

Other examples of elements of the Frechet filter

$N - \{k\}$, for each $k \in N$

$N - \{1,2,3\}$

N −{all numbers between 4 million and 5 billion which contain no 3's in their decimal form}

N −{all non-prime numbers with less than 487 digits}

N −{all specific natural numbers humans have ever spoken or written}

An ultrafilter $\mathcal{U}$ on I which contains the Frechet filter is called <u>free</u>. A free ultrafilter $\mathcal{U}$ cannot contain any finite set A, since then A^c would be cofinite and hence in $\mathcal{U}$, and then $A \cap A^c = \varnothing$ would be in $\mathcal{U}$, contradicting the first property of filters. Note that the Frechet filter is not an ultrafilter itself since we can always construct a set $A \subseteq I$ such that both A and A^c are infinite. Thus neither A nor A^c is an element of $\mathcal{F}_1$. For example, any ultrafilter on N would have to contain either the set of even numbers or the set of odd numbers, but not both since they are complements in N, but neither set is in $\mathcal{F}_1$.

Theorem:

If $\mathcal{U}$ is an ultrafilter on I, I infinite, then $\mathcal{U}$ is free if and only if $\mathcal{U}$ not fixed at any $x \in I$

Proof:

$\Rightarrow$ $\mathcal{U}$ free $\rightarrow$ not fixed

Suppose $\mathcal{U}$ free and fixed at $x \in I$,

i.e., $\mathcal{U} = \mathcal{U}_x = \{A \subseteq I : x \in A\}$

In specific $\{x\} \in \mathcal{U}_x = \mathcal{U}$

But if $\mathcal{U}$ is free $\rightarrow$ it contains the Frechet filter

$\rightarrow$ $\mathcal{U}$ contains $I - \{x\}$

$\rightarrow$ $\mathcal{U}$ contains $\{x\} \cap I - \{x\} = \varnothing$, a contradiction to the first property of filters.

$\Leftarrow$ $\mathcal{U}$ not fixed $\rightarrow$ free

To show if $\mathcal{U}$ not fixed $\rightarrow$ contains $\mathcal{F}_1$

Pick arbitrary $A \in \mathcal{F}_1$

A^c is finite, i.e., $A^c = \{a_1, a_2, \ldots, a_n\}$

Not fixed $\rightarrow$ $\forall x \in I \ \exists A_x \in \mathcal{U} \ni x \notin A_x$

Consider $\bigcap_{i=1}^{n} A_{a_i} = \tilde{A}, \ \tilde{A} \in \mathcal{U}, \ \tilde{A}^c \supseteq A^c$

$\tilde{A} \subseteq A \rightarrow A \in \mathcal{U}$

$\rightarrow \mathcal{U}$ contains $\mathcal{F}_1$

$\therefore \mathcal{U}$ is free

We have yet to prove that free ultrafilters exist. These, however, are the important ultrafilters for our construction.

THE ULTRAFILTER AXIOM

In order to obtain an understanding of ultrafilters one needs to learn some of the mathematics supporting and proving the Ultrafilter Axiom. Zorn's Lemma, which is a variation of the Axiom of Choice, will be used in the proof of the Ultrafilter Axiom. Zorn's Lemma involves the idea of a partially ordered set and related concepts as described below.

Definition:

A <u>partially ordered</u> set is a pair $(X,\leq)$, where X is a nonempty set and $\leq$ is a binary relation on X which is

i) reflexive, i.e., $x \leq x$ for all $x \in X$

ii) antisymmetric, i.e., if $x \leq y$ and $y \leq x$ then $x = y$

iii) transitive, i.e., if $x \leq y$ and $y \leq z$ then $x \leq z$

A subset C of X is a <u>chain</u> if for all $x,y \in C$ either $x \leq y$ or $y \leq x$. The element x is an <u>upper bound</u> for a subset $B \subseteq X$ if $b \leq x$ for all $b \in B$. An element $m \in X$ is <u>maximal</u> if, for any $x \in X$, $m \leq x$ implies $x = m$.

Zorn's Lemma

Let $(X,\leq)$ be a partially ordered set. If each chain in X has an upper bound then X has at least one maximal element.

Zorn' s lemma is equivalent to the Axiom of Choice and both can be used to prove the Ultrafilter Axiom but the preferred proof is the one that uses Zorn's Lemma.

Ultrafilter Axiom

If F is a filter on I then there is an ultrafilter $\mathcal{U}$ on I containing F.

Proof:

Let $\hat{F}$ be the set of all filters which contain F. $\hat{F}$ is nonempty since $F \in \hat{F}$. We partially order $\hat{F}$ by inclusion; i.e., if $A, B \in \hat{F}$ then we say that $A \leq B$ if $X \in A$ implies $X \in B$. It is easy to check that $\leq$ is a partial ordering on $\hat{F}$.

Now let $\widetilde{C}$ be a chain in $\hat{F}$. To show that $\widetilde{C}$ has an upper bound consider $\widetilde{F} = \bigcup C (C \in \widetilde{C})$. Then $C \leq \widetilde{F}$ for all $C \leq \widetilde{C}$. Also $\widetilde{F}$ is a filter.

Check Properties of Filters

i) $\varnothing \notin C$ for any C so $\varnothing \notin \bigcup C$.

ii) If $\mathbf{X}, \mathbf{Y} \in \widetilde{F}$ then $X \in C_1$ and $Y \in C_2$ for some C_1 and C_2 in $\widetilde{C}$. Since $\widetilde{C}$ is a chain, we may assume without loss of generality that $C_1 \leq C_2$, and so $X, Y \in C_2$ and $\mathbf{X} \cap \mathbf{Y} \in C_2 \subseteq \widetilde{F} \rightarrow \mathbf{Y} \in \widetilde{F}$.

iii) If $\mathbf{X} \in \widetilde{F}$ and $X \subseteq Y$ then $\exists C_1 \ni X \in C_1 \rightarrow Y \in C_1 \rightarrow \mathbf{Y} \in \widetilde{F}$.

By Zorn's Lemma $\hat{F}$ has a maximal element, call it $F_{\max}$ which both contains F, by construction of $\hat{F}$, and is an ultrafilter because $F_{\max}$ is a maximal filter.

Once we have the Ultrafilter Axiom, we can construct free ultrafilters on any infinite set.

Theorem:

Free ultrafilters exist on any infinite set I.

Proof:

Since I is infinite, construct $\mathcal{F}_1$, the Frechet filter on I. By the Ultrafilter Axiom, there exists a maximal filter $F_{\max}$ containing $\mathcal{F}_1$. $F_{\max}$ is the desired free ultrafilter.

For our construction of the Hyperreals we use the naturals numbers as the base set for our ultrafilter.

CHAPTER 6: CONSTRUCTING THE HYPERREAL NUMBERS

The construction of the Hyperreals, denoted $\mathcal{R}$, is similar to the construction of the Reals from the rational numbers by means of equivalence classes of Cauchy sequences. To begin the construction, let N denote the natural numbers, R denote the Reals, and $\hat{\mathrm{R}}$ denote the set of all sequences of Real numbers indexed by N.

Each element in $\hat{\mathrm{R}}$ is of the form $r = \langle r_1, r_2, r_3, \ldots \rangle$, where $r_i \in R \;\; \forall i$. For convenience we will denote $\langle r_1, r_2, r_3, \ldots \rangle$ by $\langle r_i \rangle$. Operations of addition, $\oplus$, and multiplication, $\otimes$, can be defined on $\hat{\mathrm{R}}$ as follows:

Definition:

If $r = \langle r_i \rangle$ and $s = \langle s_i \rangle$ are elements of $\hat{\mathrm{R}}$, we define $r \oplus s = \langle r_i + s_i \rangle$ and $r \otimes s = \langle r_i \cdot s_i \rangle$.

It is clear that $(\hat{\mathrm{R}}, \oplus, \otimes)$ is a commutative ring with an identity $\langle 1,1,\ldots \rangle$ and a zero $\langle 0,0,\ldots \rangle$ (where 1 and 0 are the unit and zero in R.

For instance, to verify that the operations are distributive:

Let $r = \langle r_i \rangle$, $s = \langle s_i \rangle$, $t = \langle t_i \rangle$

$$r \otimes (s \oplus t) = r \otimes \left(\langle s_i + t_i \rangle \right) = \langle r_i \cdot (s_i + t_i) \rangle =$$
$$\langle (r_i \cdot s_i) + (r_i \cdot t_i) \rangle = \langle r_i \cdot s_i \rangle \oplus \langle r_i \cdot t_i \rangle = (r \otimes s) \oplus (r \otimes t)$$

However, the ring is not a field. For example,

$\langle 1,0,1,0,1,\ldots\rangle \otimes \langle 0,1,0,1,0,\ldots\rangle = \langle 0,0,0,\ldots\rangle$, so the product of nonzero elements can be zero. This problem can be eliminated by introducing an equivalence relation on $\hat{\mathrm{R}}$ and defining operations and relations +, •, and < on the resulting set, $\mathbf{R}$, of equivalence classes which make $(\mathbf{R},+,\cdot,<)$ into a linearly ordered field.

THE HYPERREAL NUMBERS

The elements of the nonstandard or the Hyperreal numbers are equivalence classes of the elements of $\hat{\mathrm{R}}$ relative to the equivalence relation defined below.

Definition:

If $r=\langle r_i\rangle$ and $s=\langle s_i\rangle$ are in $\hat{\mathrm{R}}$, then $r \equiv s$ if and only if $\{i \in N : r_i = s_i\} \in \mathcal{U}$, where $\mathcal{U}$ is some free ultrafilter on N. We then say that $\langle r_i\rangle = \langle s_i\rangle$ almost everywhere (a.e.).

Since $\mathcal{U}$ is a free ultrafilter, not just a Frechet filter, note that $\langle r_i\rangle = \langle s_i\rangle$ a.e. does not mean simply $r_i \neq s_i$ for only a finite number of i's. This is a common error. Certainly two sequences that differ at only a finite number of places are always equivalent under ≡. In fact, for convenience, when comparing sequences one can always disregard any finite number of terms at the start of the sequence as long as you ignore the same number of terms for both (all) sequences being compared.

Sequences which are the same at only a finite number of places are <u>never equivalent</u> under ≡. Sequences which are the same at an infinite number of indices and also different at an infinite number of indices <u>may or may not be equivalent</u> under ≡, depending on the ultrafilter. [2]

For example $\langle 1,2,1,2,1,2,\ldots\rangle$ is equivalent to either $\langle 1,1,1,1,\ldots\rangle$ or $\langle 2,2,2,2,\ldots\rangle$, depending on whether $\mathcal{U}$ contains the set of odd numbers or the set of even numbers, respectively.

Lemma:

The relation ≡ is an equivalence relation on $\hat{\mathrm{R}}$

Proof:

We need to show that the relation ≡ is reflexive ($r \equiv r$), symmetric (if $r \equiv s$ then $s \equiv r$), and transitive (if $r \equiv s$ and $s \equiv t$ then $r \equiv t$).

i) To show $r \equiv r$ is trivial since

$$\{i \in N : r_i = r_i\} = N \in \mathcal{U}$$

ii) To show if $r \equiv s$ then $s \equiv r$

$$\{i \in N : r_i = s_i\} = \{i \in N : s_i = r_i\}$$

So if one is an element of $\mathcal{U}$, both are.

iii) To show if $r \equiv s$ and $s \equiv t$ then $r \equiv t$

If $U_1 = \{i \in N : r_i = s_i\}$,

$U_2 = \{i \in N : s_i = t_i\}$, and

$U_3 = \{i \in N : r_i = t_i\}$

$r \equiv s \rightarrow U_1 \in \mathcal{U}$ and $s \equiv t \rightarrow U_2 \in \mathcal{U}$

Then $U_1 \cap U_2 \in \mathcal{U}$, and since

$U_1 \cap U_2 = \{i \in N : r_i = s_i = t_i\} \subseteq U_3$

therefore $U_3 \in \mathcal{U}$ and $r \equiv t$

Definition:

Let $\mathbf{R}$ denote the set of all the equivalence class of $\hat{\mathrm{R}}$ induced by $\equiv$. The equivalence class containing a particular sequence $s = \langle s_i \rangle$ is denoted by $[s]$ or $\mathbf{s}$. Thus if $r \equiv s$ in $\hat{\mathrm{R}}$ then $\mathbf{r} = [r] = [s] = \mathbf{s}$.

Please note that two sequences can have the same limit as i approaches infinity and not be equivalent.

For example, $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$ is not equivalent to $\langle 0, 0, 0, \ldots \rangle$ since $\left\{ i \in N : \frac{1}{i} = 0 \right\} = \varnothing \notin \mathcal{U}$, though $\lim\limits_{i \to \infty} \frac{1}{i} = 0 = \lim\limits_{i \to \infty} 0$.

The set $\hat{\mathrm{R}}$ is divided into disjoint subsets called equivalence classes by $\equiv$. Each equivalence class consists of all sequences equivalent to any given sequence in the class.

Thus r and s are in the same equivalence class iff $r \equiv s$. [HL 5]

One of the difficulties with free ultrafilters is that you cannot completely determine which infinite sets with infinite complements are in the ultrafilter. For instance, how does one know, for a given ultrafilter $\mathcal{U}$ on N, whether the set of odd numbers or the set of even numbers is in $\mathcal{U}$? One of these sets must be in $\mathcal{U}$, since these sets are complements, yet ultrafilters on N exist for either case.

Fortunately this really does not matter. The ability to identify the exact equivalence class that one of these "borderline" sequences belongs to is generally irrelevant. We just need to know that every sequence in $\hat{\mathrm{R}}$ belongs to some equivalent class so that ≡ actually satisfies the definition for an equivalence relation on N. This is why we can not use just the Frechet filter to construct the Hyperreals. We need a structure which satisfies the mutual exclusivity property which characterizes an ultrafilter.

OPERATIONS AND FIELD STRUCTURE

We define the arithmetic operations and inequalities on the Hyperreal numbers as follows:

Definition:

Let $\mathbf{r}=[\langle r_i\rangle]$ and $\mathbf{s}=[\langle s_i\rangle]$. Then:

i) $\mathbf{r}+\mathbf{s}=[\langle r_i+s_i\rangle]$, i.e., $[r]+[s]=[r\oplus s]$

ii) $\mathbf{r}\cdot\mathbf{s}=[\langle r_i\cdot s_i\rangle]$, i.e., $[r]\cdot[s]=[r\otimes s]$

iii) $\mathbf{r} < \mathbf{s}$ ($\mathbf{s} > \mathbf{r}$) if and only if $\{i \in N : r_i < s_i\} \in \mathcal{U}$, and $\mathbf{r} \leq \mathbf{s}$ ($\mathbf{s} \geq \mathbf{r}$) if and only if $\mathbf{r} < \mathbf{s}$ or $\mathbf{r} = \mathbf{s}$. The structure $(\mathbf{R}, +, \cdot, <)$ is denoted by $\mathcal{R}$.

With the operations defined this way, we can proceed to prove that the Hyperreal number system is a linearly ordered field.

Theorem:

The structure $\mathcal{R}$ is a linearly ordered field.

Proof:

It is easy to prove that $\mathcal{R}$ is a commutative ring with unit. We will prove some of the properties as examples.

To show that multiplication is associative

$$
\begin{aligned}
(\mathbf{r} \cdot \mathbf{s}) \cdot \mathbf{t} &= \mathbf{r} \cdot (\mathbf{s} \cdot \mathbf{t}) \\
&= \left[\langle r_i \cdot s_i \rangle\right] \cdot \mathbf{t} \\
&= \left[\langle (r_i \cdot s_i) \cdot t_i \rangle\right] \\
&= \left[\langle r_i \cdot (s_i \cdot t_i) \rangle\right] && \text{associative property of Reals} \\
&= \mathbf{r} \cdot \left[\langle s_i \cdot t_i \rangle\right] \\
&= \mathbf{r} \cdot (\mathbf{s} \cdot \mathbf{t}) && \text{QED}
\end{aligned}
$$

To show that addition is commutative

$$
\begin{aligned}
\mathbf{r} + \mathbf{s} &= \mathbf{s} + \mathbf{r} \\
&= \left[\langle r_i + s_i \rangle\right] \\
&= \left[\langle s_i + r_i \rangle\right] && \text{commutative property of Reals} \\
&= \mathbf{s} + \mathbf{r} && \text{QED}
\end{aligned}
$$

To show that multiplicative identity exists

Define: $1 = [\langle 1,1,1,\ldots\rangle]$

$$\begin{aligned}\mathbf{r}\cdot 1 &= \mathbf{r}\\ &= [\langle r_i \cdot 1\rangle]\\ &= [\langle r_i\rangle] \quad \text{multiplicative identity of Reals}\\ &= \mathbf{r} \quad \text{QED}\end{aligned}$$

Clearly the proofs are similar for commutative property of multiplication, associative property of addition, distributive property and the additive inverse.

To complete the proof that $\mathcal{R}$ is a linearly ordered field we need to prove additionally that every nonzero element in R has a multiplicative inverse and that the field is ordered.

To show that if $\mathbf{r} \neq \mathbf{0}$ then there is an element $\mathbf{r}^{-1}$ in R such that $\mathbf{r}\cdot\mathbf{r}^{-1} = \mathbf{1}$.

Suppose that $\mathbf{r} = [\langle r_i\rangle] \neq [\langle 0,0,\ldots\rangle]$. Then $\{i \in N : r_i = 0\} \notin \mathcal{U}$ and so $\{i \in N : r_i \neq 0\} \in \mathcal{U}$ by the fourth property of filters. Define $\mathbf{r}^{-1} = [\langle \bar{r}_i\rangle]$, where $\bar{r}_i = r_i^{-1}$ if $r_i \neq 0$, and $\bar{r}_i = 0$ if $r_i = 0$.

Then $\mathbf{r}\cdot\mathbf{r}^{-1} = [\langle r_i \cdot \bar{r}_i\rangle]$, but $r_i \cdot \bar{r}_i = 1$ if $r_i \neq 0$, and $r_i \cdot \bar{r}_i = 0$ if $r_i = 0$, so $\{i \in N : r_i \cdot \bar{r}_i = 1\} = \{i \in N : r_i \neq 0\} \in \mathcal{U}$.

Therefore $\mathbf{r}\cdot\mathbf{r}^{-1} = 1$

Finally we must show that $\mathcal{R}$ is a linearly ordered field with the ordering given by $<$. We say that an element $\mathbf{r}$ of $\mathbf{R}$ is positive if $\mathbf{r} > 0$.

We must show that:

i) the sum of two positive elements is positive

ii) the product of two positive elements is positive

iii) Law of Trichotomy

By positive element we mean $\mathbf{r}>0$, where $0=[\langle 0,0,\ldots\rangle]$, that is $\mathbf{r}>0$ if and only if $\{i\in N: r_i>0\}\in\mathcal{U}$

The first two properties are easy to prove and follow directly from some of the previous sections.

To show, for example, that the sum of two positive elements is positive, pick $\mathbf{r},\mathbf{s}\in\mathbf{R}$ with $\mathbf{r}>0$ and $\mathbf{s}>0$. That is $U_1=\{i\in N: r_i>0\}\in\mathcal{U}$ and $U_2=\{i\in N: s_i>0\}\in\mathcal{U}$.

To show $\mathbf{r}+\mathbf{s}>0$, i.e., $\{i\in N: r_i+s_i>0\}\in\mathcal{U}$.

Clearly on $U_1\cap U_2$, r_i and s_i are both positive, so by the properties of the Reals, $r_i+s_i>0$ on $U_1\cap U_2$. But $U_3=\{i\in N: r_i+s_i>0\}\supseteq\{i\in N: r_i>0, s_i>0\}=U_1\cap U_2$.

So, since $U_1\cap U_2\in\mathcal{U}$, $U_3\in\mathcal{U}$.

Therefore $\mathbf{r}+\mathbf{s}>0$.

On the other hand, to prove the Law of Trichotomy we use the following theorem.

Theorem: Selection Theorem

Let $\mathcal{U}$ be an ultrafilter on I, $A_1,A_2,\ldots,A_n$ be a finite number of subsets of I, with $A_i\cap A_j=\varnothing$ for $i\neq j$ and $\bigcup A_i(1\le i\le n)=I$.

Then one and only one of the sets A_i is in $\mathcal{U}$.

i.e. $A_i \subseteq I \ \forall i = 1,2,\ldots,n$, $\bigcup_{i=1}^{n} A_i = I$, and

$A_i \cap A_j = \varnothing \ \forall i \neq j$

$\rightarrow \exists! i \ni A_i \in \mathcal{U}$

Proof:

Suppose there is no $A_i \in \mathcal{U}$.

Since $\mathcal{U}$ an ultrafilter then $A_c \in \mathcal{U} \ \forall i$.

This would imply that $\bigcap_{i=1}^{n} \left(A_i^c\right) \in \mathcal{U}$.

Then $\bigcap_{i=1}^{n} \left(A_i^c\right) = \left(\bigcup_{i=1}^{n} A_i\right)^c = I^c = \varnothing \in \mathcal{U}$.

But this would contradict $\mathcal{U}$ being an filter.

Therefore $\exists i \ni A_i \in \mathcal{U}$.

Now suppose $A_i, A_j \in \mathcal{U}$, $i \neq j$. Then $A_i \cap A_j = \varnothing \in \mathcal{U}$, again a contradiction to $\mathcal{U}$ being a filter.

Therefore $\exists! i \ni A_i \in \mathcal{U}$.

Proof of the Law of Trichotomy:

(i.e. if $\mathbf{r} \in \mathbf{R}$ then $\mathbf{r} < 0, \mathbf{r} = 0$, or $\mathbf{r} > 0$)

Fix $\mathcal{U}$ on N

Define $U_1 = \{i \in N : r_i < 0\}$

$U_2 = \{i \in N : r_i = 0\}$

$U_3 = \{i \in N : r_i > 0\}$

Clearly $U_i \subseteq N \ \forall i$

By Law of Trichotomy for the Reals, every r_i satisfies one, and only one, of these conditions. Thus $\bigcup_{i=1}^{3} U_i = N$ and $U_i \cap U_j = \varnothing$ for $i \neq j$.

By the previous theorem, then $\exists! i \ni A_i \in \mathcal{U}$.

Therefore either $\mathbf{r} < 0, \mathbf{r} = 0$, or $\mathbf{r} > 0$ depending on whether U_1, U_2, or U_3, respectively, is an element of $\mathcal{U}$.

So far all of this development could have been done whether the ultrafilter selected was free or fixed. We would like to understand why the use of a free ultrafilter is required. As we will see, using a fixed ultrafilter will generate nothing new beyond the standard Real numbers. To see why, it is necessary to better understand how the Real numbers themselves appear in the Hyperreals.

EMBEDDING MAP

The Reals can be embedded in the Hyperreals using what is called the embedding map. As we did earlier for the multiplicative identity $1 = [\langle 1 \rangle] = [\langle 1,1,1,\ldots \rangle]$ and zero $0 = [\langle 0 \rangle] = [\langle 0,0,0,\ldots \rangle]$, in general we will associate a Real number $\mathbf{r}$ with the equivalence class $[\langle r,r,r,\ldots \rangle]$ in the Hyperreals.

Definition:

If $r \in R$, we define $*(r) = *r$, where $*r = [\langle r,r,r,\ldots \rangle] \in \mathbf{R}$

Thus if $\mathbf{r} = *r = [\langle r,r,r,\ldots \rangle]$ for some $r \in R$ and $\mathbf{s} = [\langle s_i \rangle]$ then $\mathbf{r}=\mathbf{s}$ if and only if $\{i \in N : s_i = r\} \in \mathcal{U}$. In many of our examples, though, the terms of the sequences we use are all different, thus at most one term of such a sequence could match a given

Real number, forcing the sequence to be different than any embedded Real number.

For example, $\mathbf{s}=\left[\left\langle 1,\frac{1}{2},\frac{1}{3},\ldots\right\rangle\right]$ is not equal to any embedded Real number $\mathbf{r}={}^*r=\left[\langle r,r,r,\ldots\rangle\right]$ with $r\in R$ since $\left\{i\in N:\frac{1}{i}=r\right\}$, has at most one member and thus cannot possibly be an element of any free ultrafilter.

Theorem:

The mapping * is an order-preserving isomorphism of R into $\mathbf{R}$.

Proof:

The mapping * is 1-1, for if ${}^*r={}^*s$ then $\left[\langle r,r,\ldots\rangle\right]=\left[\langle s,s,\ldots\rangle\right]$ and so $\mathbf{r}=\mathbf{s}$. It is a trivial matter to show that * preserves the field and order properties.

For example, the mapping preserves addition,

$$\left[\langle r,r,\ldots\rangle\right]+\left[\langle s,s,\ldots\rangle\right]=\left[\langle r+s,r+s,\ldots\rangle\right] \text{ so } {}^*(r+s)={}^*r+{}^*s.$$

And also preserves ordering: If $r<s$ then ${}^*r<{}^*s$

Consider $\left\{i\in N:r_i<s_i\right\}=\left\{i\in N:r<s\right\}=N\in\mathcal{U}$

Hence ${}^*r<{}^*s$. QED

For convenience we will occasionally use r for both the Real number and its embedded Hyperreal value.

Definition: Standard Number

The image of the set of Real numbers under the embedding map, is called the set of <u>standard</u> numbers. Any Hyperreal number which is not a standard number, if any, is called a <u>non-standard</u> number.

The term "nonstandard Real numbers" is already being used for the entire set of Hyperreal numbers. Now we are defining the term "nonstandard number" to mean a Hyperreal number which is not "standard". Since these terms are so similar, we must be careful to distinguish between the two forms when used.

This brings us back to the notion of an ultrafilter. We can only build the extension of the Reals called the Hyperreals by using a free ultrafilter.

Fixed ultrafilters, "generated" by a single element, are inadequate for the construction of Hyperreals because comparing arbitrary sequences reduces to comparing them on the single element of each sequence which corresponds to the generating element of the ultrafilter.

If the ultrafilter on N is $\mathcal{U} = \mathcal{U}_k = \{A \subseteq N : k \in A\}$, $\mathbf{r} = [\langle r_i \rangle]$ and $\mathbf{s} = [\langle s_i \rangle]$ then $\mathbf{r} = \mathbf{s}$ if and only if $r_k = s_k$, because $A = \{i \in N : r_i < s_i\} \in \mathcal{U}_k$ if and only if $k \in A$. $\mathbf{r} = {}^*r_k = [\langle r_k, r_k, r_k, \ldots \rangle]$ for each $\mathbf{r} \in \mathbf{R}$ because $k \in \{i \in N : r_i = r_k\}$.

Thus, using the same sequences as in an earlier example, if the generating element, k, for the fixed ultrafilter is odd, then $[\langle 1,2,1,2,1,2,\ldots\rangle]$ is equal to $[\langle 1,1,1,1,\ldots\rangle]$, otherwise it is equal to $[\langle 2,2,2,2,\ldots\rangle]$.

For another example, consider the fixed ultrafilter $\mathcal{U}_3 = \{A \subseteq N : 3 \in A\}$ (i.e. $k = 3$). The sequences $\mathbf{r} = [\langle 5,6,7,8,\ldots\rangle]$ and $\mathbf{s} = [\langle 13,10,7,4,\ldots\rangle]$ are equal because $\{i \in N : r_i = s_i\} = \{3\}$, $(r_3 = s_3 = 7)$ and $\{3\} \in \mathcal{U}_3$. $\mathbf{r}$ is also equal to $\mathbf{t} = [\langle 7,10,7,10,\ldots\rangle]$ because $\{i \in N : r_i = t_i\} = \{3,6\} \in \mathcal{U}_3$ and $\mathbf{s} = \mathbf{t}$ because $\{i \in N : s_i = t_i\} = \{2,3\} \in \mathcal{U}_3$. In fact $\mathbf{r} = \mathbf{s} = \mathbf{t} = {}^*7$. The important thing to notice is that every sequence in a given equivalence class relative to $\mathcal{U}_3$ must have the same value in the third element of the sequence. So, $\mathrm{v} = [\langle \pi, 0, 1.9, 73, \ldots\rangle] = {}^*(1.9) = \left[\left\langle -5, \sqrt{2}, 1.9, 4e^2, \ldots\right\rangle\right] = \mathbf{w}$, no matter what the other terms are, because $\left\{i \in N : v_i = w_i\right\}$ at least contains the index 3, and thus is an element of $\mathcal{U}_3$.

Therefore, for fixed ultrafilters, $\equiv$ is simply a mapping from $\hat{\mathrm{R}}$ to R, the Real numbers, and every element of the new "Hyperreals" we are trying to construct would in fact be equal to some embedded Real number, i.e. we have gained nothing. Hence the requirement that we use a free ultrafilter for this construction.

But do we actually gain something by using a free ultrafilter? The answer, of course, must be yes or this has all been a waste of time. But we already have shown that $\mathbf{s} = \left[\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\rangle\right]$ is not equal to any embedded Real number and thus must be in some new equivalence class. Therefore the Hyperreal numbers are a proper superset of the (embedded) Real numbers.

In fact, **s** is our first non-trivial example of what we call infinitesimal numbers.

The definitions of infinite, finite, and infinitesimal numbers in Hurd and Loeb are equivalent to those given in Keisler, but are now described in terms of their <u>absolute values</u> in comparison to Real numbers.

The definition of absolute value for the Hyperreals is identical to that for the Reals.

Definition: Absolute Value

If $\mathbf{r} \in \mathbf{R}$, then the absolute value of $\mathbf{r}$, denoted by $|\mathbf{r}|$, is defined as follows:

$$|\mathbf{r}| = \begin{cases} \mathbf{r} \text{ if } \mathbf{r} > 0 \\ 0 \text{ if } \mathbf{r} = 0 \\ -\mathbf{r} \text{ if } \mathbf{r} < 0 \end{cases}$$

By the Law of Trichotomy, the absolute value of every Hyperreal number is defined. Often one will be manipulating a specific Hyperreal number, usually represented in the equivalence class as a sequence which is the result of a specific computation, or a sequence which is simply convenient. In such a case the following result is helpful.

Theorem:

If $\mathbf{r}=[\langle r_i \rangle]$ then $|\mathbf{r}|=[\langle |r_i| \rangle]$

Proof:

Pick $\mathbf{r} \in \mathbf{R}$ and define U_1, U_2 and U_3 as in the proof of the Law of Trichotomy. Exactly one of these sets is in $\mathcal{U}$. Which one determines whether $\mathbf{r}<0$, $\mathbf{r}=0$, or $\mathbf{r}>0$, respectively.

Let $\bar{\mathbf{r}}=[\langle |r_i| \rangle]$.

To show $|\mathbf{r}|=\bar{\mathbf{r}}$:

Case 1: $\mathbf{r}<0$ i.e., $U_1 \in \mathcal{U}$

$|\mathbf{r}|=-\mathbf{r}=\bar{\mathbf{r}}$ if and only if $\overline{U_1}=\{i \in N: -r_i=|r_i|\} \in \mathcal{U}$.

But $\overline{U_1}=U_1 \cup U_2$, $U_1 \in \mathcal{U}$, $U_1 \subseteq \overline{U_1} \rightarrow \overline{U_1} \in \mathcal{U}$.

Therefore $|\mathbf{r}|=\bar{\mathbf{r}}$.

Case 2: $\mathbf{r}=0$, i.e., $U_2 \in \mathcal{U}$

$|\mathbf{r}|=0=\bar{\mathbf{r}}$ if and only if $\overline{U_2}=\{i \in N: 0=|r_i|\} \in \mathcal{U}$.

But $\overline{U_2}=U_2 \in \mathcal{U}$.

Therefore $|\mathbf{r}|=\bar{\mathbf{r}}$.

Case 3: $\mathbf{r}>0$, i.e., $U_3 \in \mathcal{U}$

$|\mathbf{r}|=\mathbf{r}=\bar{\mathbf{r}}$ if and only if $\overline{U_3}=\{i \in N: r_i=|r_i|\} \in \mathcal{U}$.

But $\overline{U_3}=U_2 \cup U_3$, $U_3 \in \mathcal{U}$, $U_3 \subseteq \overline{U_3} \rightarrow \overline{U_3} \in \mathcal{U}$.

Therefore $|\mathbf{r}|=\bar{\mathbf{r}}$.

Since all possible cases verify, the theorem is true.

We now use absolute values to define the concepts of infinite, finite, and infinitesimal numbers.

Definition:

i) A number $\mathbf{s} \in \mathbf{R}$ is <u>infinite</u> if $n < |\mathbf{s}|$ for all standard natural numbers n.

ii) A number $\mathbf{s} \in \mathbf{R}$ is <u>finite</u> if $|\mathbf{s}| < n$ for all standard natural numbers n.

iii) A number $\mathbf{s} \in \mathbf{R}$ is <u>infinitesimal</u> if $|\mathbf{s}| < \frac{1}{n}$ for all standard natural numbers n.

To verify that $\mathbf{s} = \left[\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle\right]$ is infinitesimal, we need to show that $|\mathbf{s}| < \frac{1}{n}$, for all standard natural numbers n. First, notice that $|\mathbf{s}| = \mathbf{s}$ since all terms of $\mathbf{s}$ are positive. Now pick $n \in N$. Clearly all but the first n terms of the sequence $\mathbf{s} = \left[\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle\right]$ are less than $\frac{1}{n}$, i.e., $\left\{i \in N : \frac{1}{i} < \frac{1}{n}\right\}$ is cofinite and thus in $\mathcal{U}$. Therefore $\mathbf{s} = |\mathbf{s}| < \frac{1}{n}$, $\forall n \in N$, and $\mathbf{s}$ satisfies the definition of an infinitesimal.

Similarly we can create infinite and finite Hyperreal numbers. $\mathbf{s}^{-1} = \langle 1, 2, 3, 4, \ldots \rangle$ is an infinite Hyperreal number since $\{i \in N : i > n\}$ is cofinite for any $n \in N$ and thus in $\mathcal{U}$.

Finally pick $\mathrm{r} \in R$, $\mathrm{r} \neq 0$, so then ${}^*\mathrm{r} + \mathbf{s}$, is a finite $\left({}^*(\mathrm{r} - 1) < {}^*\mathrm{r} + \mathbf{s} < {}^*(r + 1)\right)$, non-Real, non-infinitesimal Hyperreal number.

NATURAL EXTENSIONS OF SETS

If $\mathrm{A} \subseteq R$, then we can construct a "Hyper-extension" to A in the same way we constructed the Hyperreals from the Reals.

Define $\hat{\mathrm{A}} = \{\langle a_1, a_2, a_3, \ldots \rangle : a_i \in \mathrm{A}\ \forall i \in N\}$. Let $\mathcal{U}$ be a free ultrafilter on N and construct the set ${}^*\mathrm{A}$ of equivalence classes of $\hat{\mathrm{A}}$ relative to the equivalence relation $\equiv$ ($\equiv$ is defined on $\hat{\mathrm{A}}$ because $\hat{\mathrm{A}} \subseteq \hat{R}$).

The properties of ${}^*\mathrm{A}$ are dependent on the properties of A, but clearly, $\mathrm{A} \subseteq {}^*\mathrm{A}$ relative to the same embedding map we used before. Further ${}^*\mathrm{A}$ is a subset of $\mathbf{R}$ and can be embedded using the obvious "identity" mapping $\mathrm{f}: {}^*\mathrm{A} \to \mathbf{R}$, $\mathrm{f}([\langle a_i \rangle]) = [\langle a_i \rangle]$, where we read $[\langle a_i \rangle]$ on the left side of the equation as equivalence classes of elements of $\hat{\mathrm{A}}$ and on the right side as equivalence classes of elements of $\hat{R}$. That is to say, the equivalence classes "look" the same, they are representable by the same elementary sequences involving only elements of A, but the sequences in the equivalence classes of the form $[\langle a_i \rangle]$ in $\mathbf{R}$ can contain values from $R - \mathrm{A}$ (just not too many of them).

$\mathbf{r} = [\langle r_i \rangle] \in \mathbf{R}$ is equal to some element $[\langle a_i \rangle] \in {}^*\mathrm{A}$ if and only if $\{i \in N : r_i \in \mathrm{A}\} \in \mathcal{U}$, i.e., if $\langle r_i \rangle$ is in A a.e.

Thus if any finite number of elements of a sequence $\langle r_i \rangle$ in an equivalence class ${}^*\mathrm{A}$ can be from outside the set A, and even some infinite collections of elements could be if the complement of the set of indices of the elements not in A is in $\mathcal{U}$, (i.e., $\{i \in N: r_i \notin A\}^c \in \mathcal{U}$ or $\{i \in N: r_i \notin A\} \notin \mathcal{U}$).

For example, if A is the set $\{\ldots,-14,-7,0,7,14,\ldots\}$ then $[\langle -3,49,-3,49,\ldots\rangle] \in {}^*A$ if and only if the set of even natural numbers is in $\mathcal{U}$.

As long as A is infinite, ${}^*A$ is a proper superset of A since we can construct a sequence $\langle a_1,a_2,a_3,\ldots\rangle$ such that $a_i \neq a_j \ \forall i \neq j$. Then $\langle a_i\rangle$ is not in the equivalence class $*(b)=[\langle b,b,b,\ldots\rangle]$ for any $b \in A$ because $\{i \in N: a_i = b\}$ can have at most one element and thus this set is not in $\mathcal{U}$.

If A is finite ${}^*A$ gives us nothing new. To show this, let $A=\{a_1,\ldots,a_n\}$, $a_i \neq a_j \ \forall i \neq j$, and consider the sequence $\langle b_1,b_2,b_3,\ldots\rangle = \langle b_k\rangle$ with $b_k \in A \ \forall k \in N$. Define $U_i = \{k \in N: b_k = a_i\}$ for each $i=1,\ldots,n$. Then $\bigcup_{i=1}^{n} U_i = N$ since every value in the sequence must be one of the a_i, and $U_i \cap U_j = \varnothing \ \forall i \neq j$ by construction since $a_i \neq a_j \ \forall i \neq j$.

By the Selection Theorem then exactly one of the $U_i \in \mathcal{U}$. That is <u>every</u> sequence $\langle b_k\rangle$ is in the equivalence class $*(a_i)$ for some $a_i \in A$. Thus no new elements are created.

There are two specific subsets of R whose Hyper-extensions are of interest to us. These sets are the Natural numbers, N, and the Rational numbers, Q. As we have shown before with the Reals, the Hyper-extension of a set has many of the properties of the original set.

Thus, ${}^*N$, the set of Hypernatural numbers, is closed under addition and multiplication, but there are no additive or multiplicative inverses. Unlike $\mathbf{R}$, ${}^*N$ contains no infinitesimals, since N contains no values near 0. All the finite values in ${}^*N$ are actually (images of) natural numbers. Yet ${}^*N$ does contain infinite values, though of course only positive infinites. Clearly, for instance, $[\langle 1,2,3,4,\ldots\rangle]$, is an infinite in ${}^*N$.

The set of Hyperrational numbers, ${}^*Q$, inherits the field structure of Q. In addition, ${}^*Q$ contains infinitesimals, infinites and non-trivial finite values. An example of a Hyperrational number is $\mathbf{r}=\left[\left\langle 1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\rangle\right]$, which is also one of our examples of an infinitesimal. Also note that all Hypernaturals are also Hyperrationals since $N \subseteq Q$.

Proofs for each of the properties of the Hyperrationals and Hypernaturals can be constructed by explicit handling of sequences and ultrafilters, or they can be verified by application of the Transfer Principle as we shall demonstrate in the next chapter. Since we have already provided several examples of direct verification of smaller properties for the Reals, we will not repeat this exercise here.

The complement of the Hyperrationals, $\left({}^*Q\right)^c$ is a set we would like to call the Hyperirrationals. But the Hyperirrationals should be ${}^*\left(Q^c\right)$. Fortunately $\left({}^*Q\right)^c={}^*\left(Q^c\right)$ as we can see from the following result.

Proposition:

If $A \subseteq R$, $\left({}^*A\right)^c = {}^*\left(A^c\right)$

Proof:

Let $\mathbf{r} = [\langle r_i \rangle] \in \left({}^*A\right)^c$.

Then $\mathbf{r} \notin {}^*A$, but this is true

$\leftrightarrow \{i \in N: r_k \in A\} \notin \mathcal{U}$

$\leftrightarrow \{i \in N: r_k \in A\}^c \in \mathcal{U}$

$\leftrightarrow \{i \in N: r_k \notin A\} \in \mathcal{U}$

$\mathbf{r} \in {}^*\left(A^c\right)$ QED

The Hyperirrationals also include finites, infinites, and infinitesimals. For instance, $\left[\left\langle \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \ldots \right\rangle\right]$ is an infinitesimal Hyperrational.

NATURAL EXTENSIONS OF FUNCTIONS

The natural extension ${}^*f$, of a Real function f, evaluated at the Hyperreal value $\mathbf{r} = [\langle r_i \rangle]$, is determined by finding the equivalence class of the sequence obtained by evaluating f at each of the r_i. That is to say: ${}^*f(\mathbf{r}) = [\langle f(r_1), f(r_2), f(r_3), \ldots \rangle]$, ${}^*f$ is defined at $\mathbf{r}$ if $\{i \in N: f(r_i)$ is defined $\} \in \mathcal{U}$, i.e., if $\langle f(r_i) \rangle$ is defined a.e. If ${}^*f(\mathbf{r})$ satisfies this rule for being defined, we ignore the elements of the sequence where $f(r_i)$ is undefined in determining the appropriate equivalence class.

Example 1:

$$f(x)=\begin{cases} x & \text{if } x<3 \\ 3 & \text{if } x\geq 3 \end{cases}$$

If $\mathbf{r}=[\langle 1,2,3,4,5,\ldots\rangle]$ then

$$^*f(\mathbf{r})=[\langle f(1),f(2),f(3),f(4),f(5),\ldots\rangle]$$

$=[\langle 1,2,3,3,3,\ldots\rangle]={}^*3=3$, since the sequence is identically 3 at all but the first two positions.

If $\mathbf{s}=[\langle -1,-2,-3,-4,\ldots\rangle]$ then

$$^*f(\mathbf{s})=[\langle f(-1),f(-2),f(-3),f(-4),\ldots\rangle]$$

$$=[\langle -1,-2,-3,-4,\ldots\rangle]=\mathbf{s}$$

Example 2:

$$f(x)=\sin(\pi\cdot x)$$

If $\mathbf{r}=[\langle 1,2,3,4,\ldots\rangle]$ then

$$^*f(\mathbf{r})=[\langle \sin(\pi),\sin(2\pi),\sin(3\pi),\ldots\rangle]$$

$$=[\langle 0,0,0,0,\ldots\rangle]={}^*0=0$$

If $\mathbf{t}=\frac{1}{2}\cdot\mathbf{r}=[\langle \frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\ldots\rangle]$ then

$$^*f(\mathbf{t})=\left[\left\langle \sin\left(\frac{\pi}{2}\pi\right),\sin(\pi),\sin\left(\frac{3\pi}{2}\right),\sin(2\pi),\ldots\right\rangle\right]$$

$$=[\langle 1,0,-1,0,1,0,-1,\ldots\rangle]$$

$^*f(\mathbf{t})=0,1$ or -1, depending on whether $\{2k:k\in N\}$, $\{4k-3:k\in N\}$ or $\{4k-1:k\in N\}$ is in $\mathcal{U}$.

If $\mathbf{s}=\mathbf{r}^{-1}=[\langle 1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\rangle]$ then

$$^*f(\mathbf{s})=\left[\left\langle \sin(\pi),\sin\left(\frac{\pi}{2}\right),\sin\left(\frac{\pi}{3}\right),\sin\left(\frac{\pi}{4}\right),\ldots\right\rangle\right]$$

$=\left[\left\langle 0,1,\frac{\sqrt{3}}{2},\frac{\sqrt{2}}{3},\ldots\right\rangle\right]\approx 0$, since $\lim_{\theta\to 0}\sin\theta=0$.

This relationship between sequences with limit 0 and infinitesimals can be shown with the following proposition.

Proposition:

If $\mathbf{s} = [\langle s_i \rangle]$ and $\lim_{i\to\infty} s_i = 0$, using the traditional definition of limits, then $\mathbf{s}$ is infinitesimal.

Proof:

$\lim_{i\to\infty} s_i = 0 \;\to\; \forall\, \varepsilon > 0,\ \exists\, n \in N \ni |s_i - 0| = |s_i| < \varepsilon \;\; \forall\, i > n.$

Pick an arbitrary $k \in N$ and let $\varepsilon = \frac{1}{k}$, then

$\exists n \in N \;\; \ni |s_i| < \frac{1}{k} \;\; \forall\, i > n.$

$\to \{ i \in N : |s_i| < \frac{1}{k} \} \in \mathcal{U} \;\; \forall\, k \in N,$

since each such set is cofinite.

$\therefore \mathbf{s}$ is infinitesimal.

The converse though is not necessarily true because of the nature of ultrafilters. As we discussed earlier, if $\mathbf{s} = [\langle 0,1,0,1,0,1,\ldots \rangle]$ and $\mathbf{t} = [\langle 1,0,1,0,1,0,\ldots \rangle]$ either $\mathbf{s} = 0$, i.e., is infinitesimal or $\mathbf{t} = 0$, i.e., is infinitesimal (but <u>not</u> both) depending on whether $\mathcal{U}$ contains the set of odd numbers or the set of even numbers, respectively. But both $\lim_{i\to\infty} s_i$ and $\lim_{i\to\infty} t_i$ are undefined because both $\{s_i\}$ and $\{t_i\}$ have two accumulation points $(0$ and $1)$.

Thus, just because a sequence is infinitesimal, it does not necessarily have a limit in the traditional sense (at best we might be able to say a sequence which is infinitesimal has a limit a.e.).

Perhaps a more interesting case is a classic example of a (very) discontinuous function.

Example 3:

$$f(x)=\begin{cases}1 & \text{if } x\in Q\\ 0 & \text{if } x\notin Q\end{cases}$$

If $\mathbf{r}=[\langle r_i\rangle]\in\mathbf{R}$ then

$$^*f(\mathbf{r})=[\langle f(r_1),f(r_2),f(r_3),\ldots\rangle]$$

This sequence will be 1 whenever r_i is Rational and 0 when r_i is Irrational. Thus $^*f(\mathbf{r})={}^*1=1$ if $\{i\in N: r_i\in Q\}\in\mathcal{U}$, that is, if $\mathbf{r}$ is Hyperrational. Otherwise $^*f(\mathbf{r})={}^*0=0$, since then $\{i\in N: r_i\in Q\}^c=\{i\in N: r_i\notin Q\}\in\mathcal{U}$, and $\mathbf{r}$ is then Hyperirrational.

CHAPTER 7: THE TRANSFER PRINCIPLE

The Transfer Principle provides us with a direct, straight-forward mechanism for identifying what properties of the Real numbers also hold true in the Hyperreals. Use of the Transfer Principle eliminates the need to resort to examining specific sets of indices where explicitly defined sequences satisfy the desired properties.

In many cases applying the Transfer Principle is quick and easy. The down side is that a relatively restrictive formal language must be used.

THE FORMAL LANGUAGE

For purposes of obtaining an understanding of the Transfer Principle it is necessary to introduce some elements of symbolic logic for relational systems. These are referred to as simple languages called $L_{\mathfrak{S}}$ and they are used to work with the properties of relations and functions that are extended from the Reals. For our purposes it will not be necessary to define the formal language completely. Except for a few peculiar details, much of the language is actually relatively easy to read, though as is usually the case with new languages, it is harder to learn to write.

One of the first things to note is that the names of most objects, such as sets, relations or functions, are underlined. This is used to indicate that we are in fact referring to the object in our language, not actually manipulating the object itself.

When we apply the Transfer Principle, the names of each of these objects will be further appended by adding a star, to indicate that we are referring to the nonstandard extension of the object in question.

The notation $\underline{R}\langle x\rangle$ states that x is an element of the Reals. The lower bar indicates the "name" of the Reals in this simple sentence. This is an example of what could be called "metamathematics", meaning that it describes the mathematics, it does not actually manipulate mathematical objects.

Many of the usual logic symbols apply: $\wedge$ (and), $\rightarrow$ (implies), and $\forall$ (for all). None of these symbols are underlined or starred since they are part of the language, not the mathematics being described. The standard arithmetic or linear ordering symbols also are not underlined or "starred" since they have exactly analogous meanings in both the Reals and the Hyperreals. Further, if necessary, it is easy to determine which operation we would be referring to in any specific situation.

However there is no use of the symbol $\exists$ (there exists). The symbols $\vee$ (or) and $\neg$ (not) are also not available for use in the simple language. One can get around these restrictions by creative use of the language, e.g., "not" can be replaced by set complements and "or" can be replaced by multiple statements. "There exists" is replaced by the use of Skolem functions. Imposing these limitations helps to make the proof of the "simple" form of the Transfer Principle possible to understand for people new to this area of study.

This does not mean that the proof is easy to explain. The proof involves a detailed dissection of interpreting true statements about the Real numbers, expressed in the formal language, to determine whether they will be true "almost everywhere" in the Hyperreals. Since the presentation of the complete proof would add little to the reader's understanding of how nonstandard Analysis can be used to teach entry-level Calculus, the proof is not be presented. Instead we will focus on examples of how Real statements are constructed in the formal language and how they are transferred to and interpreted in the Hyperreals.

SIMPLE SENTENCES

The Transfer Principle relies on the use of simple statements, or sentences as they are called in Hurd and Loeb, expressed in our formal language.

Our first example of a simple sentence is:

$$(\forall x)(\forall y)(\forall z)\left[\underline{R}\langle x\rangle \wedge \underline{R}\langle y\rangle \wedge \underline{R}\langle z\rangle \rightarrow x\cdot(y+z) = x\cdot y + x\cdot z\right]$$

Technically this is to be read "For all x, for all y, and for all z, if x is an element of the set called the Real numbers, and y is an element of the set called the Real numbers, and z is an element of the set called the Real numbers, then $x\cdot(y+z) = x\cdot y + x\cdot z$".

More reasonably this would be read "For all Real numbers x, y, and z, $x\cdot(y+z)=x\cdot y+x\cdot z$"

Another example of a Real Statement might be the Closure Rule for addition of Rational numbers, i.e., if $p,g\in Q$, then $p+g\in Q$.

This can be formally written as:

$$(\forall x)(\forall y)\left[\underline{Q}\langle x\rangle\wedge\underline{Q}\langle y\rangle\rightarrow\underline{Q}\langle x+y\rangle\right]$$

SKOLEM FUNCTIONS

Hurd and Loeb refer to Skolem functions as a "technical artifice" which is used to replace the phrase "there exists", usually denoted $\exists$. One of the examples they use is, "For each nonzero x in R there exists a y in R such that $x\cdot y=1$". In standard mathematical notation this could be written:

$$\forall x\in R, x\neq 0\ \exists y\in R\ni x\cdot y=1$$

This statement can be interpreted to assert that there is a function, call it ψ, of one variable whose domain is the set of nonzero Reals, which satisfies $x\cdot\psi(x)=1$, so that $\psi(x)=x^{-1}$.

We can combine the statements $x\in R$ and $x\neq 0$ into a single statement by defining $R_0=R-\{0\}=\{0\}^c$ and then stating $x\in R_0$. Also note that the definition of $R_0=R-\{0\}=\{0\}^c$ is an example of the methods which allows us to avoid using an "or" statement, i.e., $x \neq 0$ is the same as "$x < 0$ or $x > 0$."

Using such notation we translate our Real statement into the formal language of nonstandard analysis as follows:

$$(\forall x)\left[\underline{R}_0\langle x\rangle \to x \cdot \underline{\psi}(x) = 1\right]$$

In this case the function ψ is an example of a Skolem function. Another, more complicated example of the use of Skolem functions involves the Archimedean Principle.

One variation of the Archimedean Principle states that for each Real x there is a positive integer (i.e., a natural number) such that:

$$\forall x \in R \quad \exists m \in N \ni x < m$$

This transfers into our formal language as:

$$(\forall x)\left[\underline{R}\langle x\rangle \to \underline{N}\left\langle \underline{\psi}(x)\right\rangle \wedge x < \underline{\psi}(x)\right]$$

Note that $\psi(x)$ operates here as a selector function since may different values of $\psi(x)$ could be chosen for each x. For example, if $\psi(x)$ satisfies the given conditions, then so does $\psi_2(x) = \psi(x) + n$, for any fixed $n \in N$, and $\psi_3(x) = n \cdot \psi(x)$, for any fixed $n \in N$. Any one of these functions, and many more, would have adequately served the purpose here.

THE TRANSFER PRINCIPLE

The Transfer Principle is a basic assertion that any statement in the Reals, whether a function or a relation can be transferred to the Hyperreals.

Basically what happens is that a Real statement is translated into a simple sentence and then the Transfer Principle is applied. Notationally this essentially amounts to starring the names of the sets, functions and relations used. Conceptually, though, we are actually replacing the original Real sets, functions, and relations with their nonstandard extensions, the detailed construction of which we have already presented. The fact that the mechanical process of transferring Real statements to obtain their Hyperreal equivalents is so easy is what makes the Transfer Principle powerful.

The fact is that you must correctly express the desired Real statement(s) in an awkward, formal language, and then interpret the transferred statement(s) correctly, is what makes the process sometimes difficult and/or tricky.

The formal statement of the Transfer Principle, as presented in Hurd and Loeb, is as follows:

Theorem: Transfer Principle

If ϕ is a simple sentence in $L_{\mathcal{R}}$ (the language of the Reals) which is true in $\mathcal{R}$, Then ϕ^* is true in $\mathcal{R}^*$. [2]

The use of $\mathcal{R}$ and $\mathcal{R}^*$ here is a continuation of notation from the first section in which the Transfer Principle was initially introduced. $\mathcal{R}$ and $\mathcal{R}^*$ denote the Reals and the Hyperreals respectively. Proof is beyond scope of this paper and is not needed for understanding the basic principles described here.

Example 1: Distributive Property

$$\forall x, y, z \in R \rightarrow x \cdot (y+z) = x \cdot y + x \cdot z$$

Written in the formal language is:

$$(\forall x)(\forall y)(\forall z)\left[\underline{R}\langle x\rangle \wedge \underline{R}\langle y\rangle \wedge \underline{R}\langle z\rangle \rightarrow x \cdot (y+z) = x \cdot y + x \cdot z\right]$$

The transferred statement is:

$$(\forall x)(\forall y)(\forall z)\left[{}^*\underline{R}\langle x\rangle \wedge {}^*\underline{R}\langle y\rangle \wedge {}^*\underline{R}\langle z\rangle \rightarrow x \cdot (y+z) = x \cdot y + x \cdot z\right]$$

Example 2: Closure Rule for Addition of Rationals

If $p, g \in Q$, then $p + g \in Q$.

This can be formally written as:

$$(\forall x)(\forall y)[\underline{Q}\langle x\rangle \wedge \underline{Q}\langle y\rangle \rightarrow \underline{Q}\langle x+y\rangle]$$

The transferred statement is:

$$(\forall x)(\forall y)\left[{}^*\underline{Q}\langle x\rangle \wedge {}^*\underline{Q}\langle y\rangle \rightarrow {}^*\underline{Q}\langle x+y\rangle\right]$$

This says, in brief, that if x and y are Hyperrational numbers, then their sum is also a Hyperrational number.

Example 3: Existence of Multiplicative Inverses

$$\forall x \in R, x \neq 0 \;\; \exists y \in R \ni x \cdot y = 1$$

Written in the formal language is:

$$(\forall x)\left[\underline{R}_0\langle x\rangle \rightarrow x \cdot \underline{\psi}(x) = 1\right]$$

The transferred statement is:

$$(\forall x)\left[{}^*\underline{R}_0\langle x\rangle \rightarrow x \cdot {}^*\underline{\psi}(x) = 1\right]$$

${}^*R_0$ here is the set of non-zero Hyperreal numbers since the only Hyperreal number that cannot be represented using elements of R_0 is 0 itself. The transferred statement, then, is formally read, "For every x, if x is an element of the non-zero, Hyperreal numbers then x times the value of the function ${}^*\psi$, the Hyperreal extension of the Real function ψ, evaluated at x equals 1."

Example 4: Archimedean Principle

$$\forall x \in R \;\; \exists m \in N \ni x < m$$

Written in the formal language is:

$$(\forall x)\left[\underline{R}\langle x\rangle \rightarrow \underline{N}\left\langle \underline{\psi}(x)\right\rangle \wedge x < \underline{\psi}(x)\right]$$

The transferred statement is:

$$(\forall x)\left[{}^*\underline{R}\langle x\rangle \rightarrow {}^*\underline{N}\left\langle {}^*\underline{\psi}(x)\right\rangle \wedge x < {}^*\underline{\psi}(x)\right]$$

This transferred statement is read "For every x, if x is an element of the Hyperreals, then the values of the Hyperreal function ${}^*\psi(x)$ is a Hypernatural number and x is less than the value of ${}^*\psi(x)$.

STANDARD PART

In order to introduce the next section we need to introduce two important equivalence relations on $\mathbf{R}$ and the associated notions of monad and galaxy. Monads are of special importance for two reasons: for the nonstandard treatment of convergence and continuity, and also because Leibniz discussed monads as being the indivisibles essential to the understanding of his development of the Calculus.

Definition:

Let x and y be numbers in $\mathcal{R}^*$.

i) x and y are near or infinitesimally close if $x-y$ is infinitesimal. We write $x \approx y$. The monad of x is the set $m(x)=\{y \in \mathcal{R}^*: x \approx y\}$.

ii) x and y are finitely close if $x-y$ is finite. We write $x \cong y$. The galaxy of x is the set $G(x)=\{y \in \mathcal{R}^*: x \cong y\}$.

The monadic and galactic structure of $\mathcal{R}^*$ is easily visualized. Clearly $m(0)$ is the set of infinitesimals and $\mathrm{G}(0)$ is the set of finite numbers. It follows that any two monads $m(x)$ and $m(y)$ are either equal (if x is infinitely close to y, i.e., $x \approx y$) or disjoint (if x is not infinitely close to y). Specifically the monad of any infinitesimal is still $m(0)$. The relation $\approx$ is an equivalence relation on $\mathcal{R}^*$. An analogous statement can be said for two galaxies $\mathrm{G}(x)$ and $\mathrm{G}(y)$ are either equal (if $x \cong y$ is finite) or disjoint. The monad of any finite number, specifically any

Real number, is the same as G(0). The relation $\cong$ is also an equivalence relation on $\mathcal{R}^*$.

Examples of Monads and Galaxies:

Infinitesimals (elements of $m(0)$)

$$\mathbf{s}_0 = [\langle 0,0,0,\ldots\rangle] = [\langle 0\rangle] = 0$$

$$\mathbf{s}_1 = \left[\left\langle 1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\rangle\right] = \left[\left\langle\frac{1}{i}\right\rangle\right]$$

$$\mathbf{s}_2 = [\langle -0.4,0.04,-0.004,0.0004\ldots\rangle] = \left[\left\langle (4)(-0.1)^i\right\rangle\right]$$

$$\mathbf{s}_3 = [\langle 0.14159\ldots,0.04159\ldots,0.00159\ldots,\ \ldots\rangle]$$

$$= [\langle \text{all but the first } i \text{ digits of } \pi\rangle]$$

Finites (elements of $G(0)$)

$$\mathbf{t}_1 = [\langle 6.92,6.92,6.92,\ldots\rangle] = [\langle 6.92\rangle] = 6.92$$

$$\mathbf{t}_2 = [\langle 3,3.1,3.14,3.141,\ldots\rangle] = [\langle \text{the first } i \text{ digits of } \pi\rangle]$$

$$\mathbf{t}_2 \approx \pi \rightarrow \mathbf{t}_2 \in m(\pi)$$

$$\mathbf{t}_3 = \left[\left\langle \left(1+\tfrac{1}{1}\right),\left(1+\tfrac{1}{2}\right)^2,\left(1+\tfrac{1}{3}\right)^3,\ldots\right\rangle\right] = \left[\left\langle \left(1+\tfrac{1}{i}\right)^i\right\rangle\right]$$

$$\mathbf{t}_3 \approx e \rightarrow \mathbf{t}_3 \in m(e)$$

Infinites (elements of $G(0)^c$)

$$\mathbf{u}_1 = [\langle 1,2,3,\ldots\rangle] = [\langle i\rangle]$$

$$\mathbf{u}_2 = [\langle 1,-20,300,-4000,50000,\ldots\rangle] = \left[\left\langle i(-10)^{i-1}\right\rangle\right]$$

$$\mathbf{u}_3 = \left[\left\langle e,e^2,e^3,e^4,\ldots\right\rangle\right] = \left[\left\langle e^i\right\rangle\right]$$

Other Examples of Monads and Galaxies

Elements of $m(-8)$

$\mathbf{v}_1 = [\langle -8 \rangle] = -8$

$\mathbf{v}_2 = [\langle -7, -7\tfrac{1}{2}, -7\tfrac{2}{3}, -7\tfrac{3}{4}, \ldots \rangle] = \mathbf{v}_1 + \mathbf{s}_1$

$\mathbf{v}_3 = [\langle -8.4, -7.96, -8.004, -7.9996, \ldots \rangle] = \mathbf{v}_1 + \mathbf{s}_2$

Elements of $m(\mathbf{u}_1)$

$\mathbf{w}_1 = [\langle 2, 2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, \ldots \rangle] = \mathbf{u}_1 + \mathbf{s}_1$

$\mathbf{w}_2 = [\langle 1.14159\ldots, 2.04159\ldots, 3.00159\ldots, \ \ldots \rangle] = \mathbf{u}_1 + \mathbf{s}_3$

Elements of $G(\mathbf{u}_1)$

$\mathbf{u}_1 = [\langle 1, 2, 3, \ldots \rangle]$

$\mathbf{p}_1 = [\langle 101, 102, 103, 104, \ldots \rangle] = \mathbf{u}_1 + 100$

$\mathbf{p}_2 = [\langle 4, 5.1, 6.14, 7.141, 8.1415, \ldots \rangle] = \mathbf{u}_1 + \mathbf{t}_2$,

$\mathbf{p}_2 \approx \mathbf{u}_1 + \pi \rightarrow \mathbf{p}_2 \in m(\mathbf{u}_1 + \pi)$

Also, $\mathbf{p}_r = \mathbf{u}_1 + r, \ r \in R$

$\mathbf{p}_s = \mathbf{u}_1 + \mathbf{s}, \ \mathbf{s} \in G(0)$

Similar to our previous development, if $\mathbf{s} = [\langle s_i \rangle]$ and $\lim\limits_{i \to \infty} s_i = L \in R$, using the traditional definition of limits, then $\mathbf{s} \approx L$, or equivalently, $\mathbf{s} \in m(L)$.

To see this, consider $\mathbf{s} - L = [\langle s_i - L \rangle]$. Then $\lim\limits_{i \to \infty} (s_i - L) = 0$, and thus by the earlier result, $[\langle s_i - L \rangle] \approx 0$. This means $\mathbf{s} - L \approx 0$, hence $\mathbf{s} \approx L$ and $\mathbf{s} \in m(L)$.

We are now prepared to present a formal definition of the Standard Part of a Hyperreal number.

Theorem:

If $\rho \in \mathcal{R}^*$ is finite, there is a unique standard Real number $r \in R$ with $\rho \approx r$; i.e., every finite number is near a unique standard number.

Proof:

Let $A = \{x \in R: \rho \leq x\}$ and $B = \{x \in R: x < \rho\}$

Since ρ is finite, there exists a standard number s such that $-s < \rho < s$. It follows that B is nonempty and has an upper bound. Let r be the least upper bound of B (the existence of r is assured by the completeness of R). For each $\varepsilon > 0$ in R, $(r+\varepsilon) \in A$ and $(r-\varepsilon) \in B$, so $r-\varepsilon < \rho < r+\varepsilon$, and hence $|r-\rho| \leq \varepsilon$. It follows that $r \approx \rho$. If $r_1 \approx \rho$ then $|r_1 - r| \leq |r_1 - \rho| + |\rho - r| < 2\varepsilon$ for each standard $\varepsilon > 0$, whence $r = r_1$. QED [2]

Definition:

If $\rho \in \mathcal{R}^*$ is finite, the unique standard number $r \in R$ such that $\rho \approx r$ is called the <u>standard part</u> of ρ and is denoted by $\mathrm{st}(\rho)$. This defines a map $\mathrm{st}: \mathrm{G}(0) \to \mathrm{R}$ called the <u>standard part map</u>.

Clearly st maps $\mathrm{G}(0)$ onto R since $\mathrm{st}(r) = r$ when $r \in R$. That the map also preserves algebraic structure is shown by the following theorem. [3]

Theorem:

The map st is an order-preserving homomorphism of $G(0)$ onto R.

i) $\operatorname{st}(x \pm y) = \operatorname{st}(x) \pm \operatorname{st}(y)$

ii) $\operatorname{st}(x \cdot y) = \operatorname{st}(x) \cdot \operatorname{st}(y)$

iii) $\operatorname{st}\left(\frac{x}{y}\right) = \frac{\operatorname{st}(x)}{\operatorname{st}(y)}$ if $\operatorname{st}(y) \neq 0$

iv) $\operatorname{st}(x) \leq \operatorname{st}(y)$ if $x \leq y$

Proof:

Let $x = r + \varepsilon$ and $y = s + \delta$

$$\begin{aligned}
\operatorname{st}(x \cdot y) &= \operatorname{st}(x) \cdot \operatorname{st}(y) \\
\operatorname{st}((r+\varepsilon) \cdot (s+\delta)) &= \\
\operatorname{st}(r \cdot s + r \cdot \delta + s \cdot \delta + \varepsilon \cdot \delta) &= \\
\operatorname{st}(r \cdot s) + \operatorname{st}(r \cdot \delta) + \operatorname{st}(s \cdot \varepsilon) + \operatorname{st}(\varepsilon \cdot \delta) &= \\
r \cdot s + 0 + 0 + 0 &= \\
r \cdot s &= \operatorname{st}(x) \cdot \operatorname{st}(y)
\end{aligned}$$

QED

Note, the $\operatorname{st}(r \cdot s)$ is $r \cdot s$, because the product of real numbers is a real number, however, the product of real numbers and infinitesimals is infinitesimal and so is the product of infinitesimals, the Standard Part to those products is 0. Therefore $\operatorname{st}(r \cdot \delta)$, $\operatorname{st}(s \cdot \varepsilon)$ and $\operatorname{st}(\delta \cdot \varepsilon)$ are 0.

The development in Hurd and Loeb is as described in the first section of this paper. For the purposes of this discourse we need to restate the fact that Real numbers are sometimes called "standard" numbers while Hyperreal numbers that are <u>not</u> Real are "nonstandard" numbers, hence

Nonstandard Analysis. The Standard Part of a number is the Real number which is infinitely close to some Hyperreal number b. Infinite Hyperreal numbers do not have standard parts. This then leads us to calling the Real number that is infinitely close to b the "standard part" of b.

CHAPTER 8: CONCLUSIONS

As we have seen, the primary advantages of the nonstandard method of teaching the Calculus is that it provides the student with a more intuitive, conceptual approach to the subject. The results are the same and the algebra is virtually identical to the traditional limit based method, so instructors can easily make the transition in instruction to using the nonstandard method. Though covered later in the course and less heavily depended upon, limits are still taught, so no critical background is missing for students who choose to go on in their mathematics education. On the contrary, having been exposed to the nonstandard method may enhance the students' intuition about limits and provide them with another intellectual tool with which to solve other types of problems.

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