1994

Arguably the scheme that conquered the infinite

Timothy Michael Curran

ARGUABLY THE SCHEME THAT

CONQUERED THE INFINITE

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Timothy Michael Curran

June 1994

ARGUABLY THE SCHEME THAT

CONQUERED THE INFINITE

A Project

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Timothy Michael Curran

June 1994

Approved by:

[redacted] 29 June 1994

Dr. Robert Stein, Project Committee Chair, Mathematics   Date

[redacted]

Dr. Gary Griffing, Project Committee Member

ABSTRACT

This project involves an in-depth look at some of the historical developments that preceded the formal invention of calculus. It focuses chronologically on the events that led to Newton's discovery of the binomial series. The primary purpose of this project is to provide high school calculus students with both written historical background and discovery-based activities that involve them in the developmental stages of the calculus. The discovery-based activities contain exercises on Pascal's Triangle, Alhazen's method for acquiring formulas for $1^k + 2^k + 3^k + \ldots + n^k$, John Wallis' characteristic ratio of index k, Wallis' famous representation of π as an infinite product, and Newton's discovery of the binomial series.

TABLE OF CONTENTS

Chapter 1

Introduction

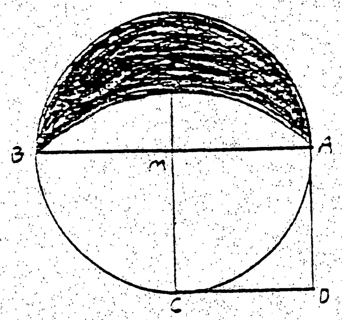
Mathematics, not unlike any other subject of study, has in its historical background certain developments that stand out among all others as critical turning points. This paper involves one such development, namely, Wallis' interpolation scheme which leads to his famous representation of π as an infinite product. In fact, Wallis himself coined the term "interpolation" (originating from words meaning "to polish in between") in his most celebrated work *Arithmetica Infinitorum*, published in 1656. The new method of experimentation continuously employed by Wallis in its pages involves interpolation and induction, leading to multitudes of generalizations. Wallis' use of induction was the first of its kind and it led to an abundant harvest of fresh revelations in mathematics. Arguably, the primary reason for such an explosion of newly discovered outcomes in mathematics following the work in *Arithmetica Infinitorum* by Wallis lies in the fact that, from ancient times, until the middle of the seventeenth century, mathematicians avoided working with the infinite due to the horrific methodology required. Wallis prepared the way to modern-day approaches to infinite processes. He admitted both the concepts of infinite series and limit theory into mathematical reasoning. The following paragraphs act as a brief chronology of significant events in mathematics which ultimately directed Wallis to his profound investigations.

The earliest written inquiries into the infinite process date back to about 500 BC.

with the paradoxes of Zeno. If Zeno were to be reincarnated into today's world, it is likely that he would believe that he could jump from an airplane without a parachute and, even more frightening, without the fear of hitting the ground. After all, there are an infinite number of "half-way points" he would have to bypass and, naturally, that would take an infinite amount of time.

At about the same time Aristotle wrote of Zeno's paradoxes, the Pythagoreans discovered the incommensurable (segments that lacked a common measure, so that the ratio of their lengths was what we now call irrational). This escalated interest in the idea of the infinite process and led to the next stage in the cultivation of infinite theories.

In about 450 BC Hippocrates squared the area under a curve. He ascertained that the area of the shaded region below was equivalent to the area of square AMCD.



This problem was the first of its kind. It showed that the area under a curve could be squared. Consequently, it sparked interest in one of the most tantalizing problems encountered in mathematics, that of "squaring the circle."

Shortly thereafter, the Sophist Antiphon proposed that there existed a regular polygon, with a sufficient number of sides, whose area matched that of a circle having a radius equal to the length of the polygon's apothem. Already and quite understandably, we see the confusion perpetrated by processes involving the infinite.

The Pythagoreans' discovery of the existence of incommensurable geometric magnitudes (lengths, areas, volumes) ultimately led to the next stage in the development of the theory of the infinite process, the writing of the 13 books of the *Elements* by Euclid in about 300 BC. Incommensurable geometric magnitudes forced a thorough reexamination of the foundations of mathematics and this task was undertaken by Euclid. He created a "continuity axiom" that allowed the Greeks to deal with geometric magnitudes that could not be "measured by numbers." An important application of the continuity axiom involved the Greeks "method of exhaustion."

The method of exhaustion was utilized by the Greeks as they attempted to calculate the area of a curvilinear figure. This application involved filling up, or exhausting, the curvilinear figure by means of a sequence of polygons. It was devised, apparently by Eudoxus, to provide a geometric approach to acquiring certain limits. This was an early form of the modern-day means used which is informally called "taking the limit." The method of exhaustion was logically clear, but was very cumbersome, for it, in practice, hinged on the idea that the difference between the area of the curvilinear figure and the last polygon in the sequence could be made as small as desired by making the sequence of polygons sufficiently large. In many instances, this led to difficult geometry problems.

The classical era of Greek mathematics probably reached its climax in the third century BC. with the squaring of the parabola by Archimedes. In the Preface to his Treatise on the Quadrature of the Parabola, Archimedes writes: "Many mathematicians have endeavored to square the circle, the ellipse, or the segment of a circle, of an ellipse, or of a hyperbola. No one, however, seems to have thought of attempting the quadrature of the segment of a parabola, which is precisely the one that can be carried out." This was a significant event because since Hippocrates' squaring of a particular crescent in 450 BC., many of the greatest minds over the course of the next 190 years tried in vain to square other curvilinear figures. Essentially, Archimedes' work constituted the very beginning of the calculus, and indeed one of his proofs (he gave two) anticipated methods developed in the theory of integration, nearly 2000 years later. Archimedes' quadrature of the parabola was equivalent, in modern terms, to evaluation of $\int_a^b (px^2 + q)dx$ .Generalization of this result to other functions involved the function concept and tools of algebra and analytic geometry.

Nearly 1900 years later a discovery related to Archimedes quadrature of the parabola was made by Cavalieri. In about 1630, he calculated the area under the curves $y = x^k$ for $k = 3, 4, 5, \ldots, 9$. Cavalieri's work was based on formulas developed by the Arab mathematician Alhazen in about 1000 AD. for the sums of the first n cubes and fourth powers. Alhazen's brilliant acquisition of a means for obtaining these formulas helped bridge the gap between early Greek mathematics and the explosion of ideas in the seventeenth century, including this one by Cavalieri, that simplified further work involving infinite processes.

Another bridge was the translation, in the thirteenth century, of Aristotle's *Physics.* Aristotle's work by no means represented the best scientific thinking of his day. However, its translation had a profound influence on European thinking. He explored the nature of the infinite along with the existence of indivisibles or infinitesimals. He summoned scholars "to discuss the infinite and to inquire whether there is such a thing or not, and, if there is, what it is"{ Book III, Ch. 4 }. Because of this summoning, thirteenth century philosophers and mathematicians became fascinated with the mysteries of the infinite. A sample of the problems solved at this time follows:

1. If a point moves throughout the first half of a certain time interval with a constant velocity, throughout the next quarter of the interval at double the initial velocity, throughout the following eighth at triple the initial velocity, and so on ad infinitum; then the average velocity during the whole time interval will be double the initial velocity.

2. If an aliquot part (one *k*th) should be taken from some quantity (a), and from the first remainder such a part is taken, and from the second remainder such a part is taken, and so on into infinity, such a quantity would be consumed exactly - no more, no less - by such a mode of subtraction.

The first problem is equivalent to the summation $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\ldots+\frac{n}{2^n}+\ldots=2$, where both the initial velocity and time interval are taken as unity. The second has the *n*th remainder $a(1-1/k)^n$ approaching zero as n goes to infinity in the expression

$$\frac{a}{k}\left[1+\left(1-\frac{1}{k}\right)+\left(1-\frac{1}{k}\right)^2+\ .\ .\ .\ +\left(1-\frac{1}{k}\right)^{n-1}\right]+a\left(1-\frac{1}{k}\right)^n=a$$

Also, at this time, the first notions of instantaneous velocity were recorded, but fell short without thorough understanding and tools for working with limits. These medieval speculations on infinity and the continuum helped weaken the grip the Greeks' "horror of the infinite" had on the mathematical world.

By the beginning of the seventeenth century, the subjects of Indivisibles and Infinitesimals had gained popularity. Kepler initiated extensive work in the subject of Infinitesimals (basically the notion that an area consists of an infinite number of say, rectangles, each of whose area is infinitely small) with the publishing of *Stereometria* in 1615. Greatly influenced by this work, Cavalieri published *Geometria Indivisibilibus* in 1635. His method of Indivisibles hinged on the idea that an area (surface) consisted of an infinite number of lines and that a volume (solid) consisted of an infinite number of surfaces. These treatises constituted major advances toward breaking away from the Greek's oppressive Method of Exhaustion.

Roberval, Torricelli, Fermat, and Pascal pursued further the concepts of area outlined above in the first half of the seventeenth century, thus helping to set the stage for the work of John Wallis' *Arithmetica Infinitorum,* published in 1656. Abound with fresh points of view, new methodology, and stimulating discoveries, Wallis' *Arithmetica Infinitorum* put him at the forefront of mathematics. Quite arguably, it is this treatise by Wallis which ultimately led to the "conquering of the infinite" by the mathematical world. This was largely due to the fact that Wallis' impact on Newton was immense. He stood as Newton's mighty predecessor and many discoveries of the subsequent three centuries can somehow be traced back to Newton and hence, to Wallis.

John Wallis was most aroused by the work of Cavalieri, which brought him to the belief that the quadrature of the circle could be affected. In 1652, Wallis embarked on a venture that eventually led him to his celebrated representation of π as an infinite product.

Wallis began by investigating ratios of the form $\frac{0^k + 1^k + 2^k + \ldots + n^k}{n^k + n^k + n^k + \ldots + n^k}$. He was particularly interested in these ratios for large values of n. He called this the characteristic ratio of index k. The table patterns and analogies he was able to formulate in his work here encouraged him to undertake investigations directed toward the age-old problem of squaring the circle. The interpolation scheme utilized by Wallis in his attempt to square the circle was the first of its kind. The intuition, use of analogy, method of induction, and means of generalization employed by Wallis throughout his work became the vehicle for further discoveries for generations to come.

Wallis' interpolation scheme had a particularly important impact on Newton. Newton's discovery of the Binomial Theorem was a direct consequence of Wallis' influence. The methodology originated by Wallis also had a profound effect on Leibnitz and the limit concepts he developed.

These men, Newton and Leibnitz, played especially important roles in the discovery of calculus, the synthesis of a new and powerful way to arithmetically (as opposed to the geometrical methods of the Greeks) analyze infinite processes. However, it is difficult to deny the debt these men owed to John Wallis and his *Arithmetica Infinitorum*. This was the key that unlocked the chains that bound the mathematical universe to the Greek's "horror of the infinite." The conquering of the infinite by gaining the tools for working with, and the thorough understanding necessary for, limit theory and its applications, took place during the late seventeenth, eighteenth, and nineteenth centuries.

John Wallis was born at Ashford, in East Kent, on November 23, 1616. His father died when he was six years old, and it was at this time he began school at Ashford. His enthusiasm for learning persisted from this point onward until his death in 1703. He once wrote "It was always my affection, even from a child in all pieces of Learning and Knowledge, not merely to learn by rote, which is soon forgotten, but to know the grounds or reasons of what I learn; to inform my Judgement as well as furnish my Memory, and thereby make a better Impression on both" [Scott, pg. 3].

Mathematics made its first impression on him in 1630 when his younger brother had been learning to write, to cipher, and to cast account. Wallis asked what this meant and was told that it dealt with "The Practical Parts of Common Arithmetick in Numeration, Addition, Substraction, Multiplication, Division, the Rule of Three (Direct and Inverse), the Rule of Fellowship (with and without Time), the Rule of False-Position, Rules of Practise, and Reduction of Coins, and some other little things" [Scott, pg. 4].

In 1632, John Wallis decided to attend Emmanuel College, Cambridge. This was the real birthplace of his mathematical expertise, just as it was Newton's thirty years later. He graduated with a Bachelor of Arts degree in 1637 and was admitted to the Master's degree program four years later. In spite of the fact that his formal schooling involved primarily mathematics, Divinity remained his principle interest. In 1941, he was appointed chaplain to Lady Vere, the widow of Lord Horatio Vere.

During his tenure with this family, Wallis exhibited a skill in the art of deciphering cryptic messages. The country was involved in the Civil War and the speed in which Wallis could decode messages written in cipher caught the attention of the Parliamentary party. The Parliamentary party called on Wallis to decipher letters for many years. As a master of this very dangerous art, Wallis made many enemies and on occasion he was accused of exercising this skill carelessly, without regard to the potential consequences. Nevertheless, Wallis was able to rise above the very turbulent surroundings of this time and endure the negative aspects of deciphering coded messages for government officials.

In the 1650's, Wallis played an important role in the climb to prosperity of the Royal Society. He was the most faithful of its members to adhering to the Society's original plan of developing the technique of experimentation. Wallis applied himself to virtually every branch of learning and continually submitted his observations and experiments to the Royal Society. These included astronomical observations, experiments in the theory of the Flux and the Reflux of the Sea, observations on Gravity and on the height of the barometer at different seasons, and experiments in blood transfusions, to name a few. It was the devotion of his energies in its infancy that enabled Wallis to stimulate interest and enthusiasm in the newly formed learning establishment called the Royal Society.

Though his interests and work ranged over almost every facet of human activity, Wallis added fame to both his own name and country with his endeavors in the field of mathematics. With his appointment to the post of Savilian Professor of Geometry in 1649, mathematics became the subject of serious study. About this time, Wallis' interest in the subject of Indivisibles, spurred on by the works of Torricelli, in which Cavalieri's methods were constantly used, prompted the thought that in it was a way by which the circle could be squared. The result of the energy directed toward solving this centuries old problem arrived in 1656 with the publishing of John Wallis' most famous treatise, *Arithmetica Infinitorum.* This treatise relates primarily to the quadrature of curves by Cavalieri's *method of indivisibles.* However, it goes far beyond Cavalieri's geometrical exposition with the use by Wallis of Analytical Geometry. As noted earlier, it played an important role in the development of the calculus, especially integral calculus.

The full title of John Wallis' famous treatise is:

Arithmetica Infinitorum siv Nova Methodus
Inquirendi in Curvilineorum Quadraturam, aliaque
difficiliora Matheseos Problemata

It's translation is as follows:

THE ARITHMETIC of INFINITIES, or a NEW METHOD
of studying the QUADRATURE of CURVES, and other more
DIFFICULT MATHEMATICAL PROBLEMS

Appendix A

The Activities

The following activities are of a heuristic nature and are intended for use by first year calculus students. The rationale for the involvement by calculus students in these activities is that no student, who gives considerable attention to calculus, should fail to make acquaintance with the historical phases and logical transitions which occurred in the developing stages of the calculus. All too often, subjects, particularly in mathematics, are taught to students as finished products. However, the high school AP Calculus course affords a great opportunity for students to engage in activities which are of an investigative nature, thereby leading to discovery. This opportunity occurs in the month of school following the AP Calculus exam. These activities will serve to put the "finishing touches" on the AP course by letting today's calculus students in on the methods, history, and above all, the excitement of the work by Wallis and others which preceded the formal invention of the calculus.

IMPORTANT

As a student works through these exercises, they should think of themselves as a cryptographer attempting to break a code. After all, as stated previously, that is what John Wallis primarily did for a living. He was very good at it and thus made many enemies. This might be the primary reason for Wallis' not receiving the notoriety he deserved for writing *Arithmetica Infinitorum.* In simulating a cryptographer, one should look for patterns and use intuition to predict future outcomes. Then, verify or test the predictions and try to generalize the results (i.e. "break the code").

THE ACTIVITIES

SET I: Acts as an introduction to the style and nature of succeeding activities.

SET II: Delves into the fascinating method by which Alhazen determined formulas for the sums of the first n integers, the first n squares, the first n cubes, etc.

SET III: Involves the student in Wallis' investigations into the value of ratios of the form $\frac{0^k + 1^k + 2^k + \ldots + n^k}{n^k + n^k + n^k + \ldots + n^k}$.

SET IV: Embarks the student on a venture similar to that undertaken by Wallis in his renowned discovery of π as an infinite product.

SET V: Depicts how Newton extended Wallis' concept of interpolation to include areas under curves having negative powers associated with them, to find a new representation of π, and to create the binomial series.

SET I

A

CRYPTOGRAPHIC

WARM-UP

ACTIVITY 1: PASCAL'S TRIANGLE

DIRECTIONS: Use the space provided to expand and simplify the following binomial expressions.

1) $(a + b)^1 =$ ____________

2) $(a + b)^2 =$ ____________________

3) $(a + b)^3 =$ ________________________________

Now, transfer to the blanks below the coefficients of the terms you got in your answers.

0) $(a + b)^0 =$ 1

1) $(a + b)^1 =$ _a + _b

2) $(a + b)^2 =$ $\_a^2 + \_ab + \_b^2$

3) $(a + b)^3 =$ $\_a^3 + \_a^2b + \_ab^2 + \_b^3$

At this point, a cryptographer would probably make a prediction as to what the answer is for problem 4. Verifying the prediction, the next step might be to predict and then verify the result to be obtained in problem 5. Do this below.

Predict the answer to problem 4 :

4) $(a + b)^4 = \_a^4 + \_a^3b + \_a^2b^2 + \_ab^3 + \_b^4$

Verify your prediction by carrying out the required expansion and simplification below.

____

____

Predict the answer to problem 5 :

5) $(a + b)^5 =$ ________________________________

Verify your prediction by carrying out the required expansion and simplification below.

____

____

The subsequent step which might be taken by a cryptographer would involve predicting answers to the expansion and simplification of what would be problems 6, 7, 8, and 9. Then he/she may attempt to verify problem 9 by actually carrying out the expansion. If this continued to fit the established pattern, the next step would most likely be a prediction of the answer to say, problem 15. If the cryptographer attained verification of this prediction, he/she may very well move on to an attempt to generalize the entire situation. In other words, further effort would be directed towards deriving a formula for an arbitrary problem (this is usually denoted by the letter *n*). The use of analogy, experience, intuition, and sometimes just plain common sense is important in the initial stages of pattern recognition. Once a pattern is recognized (or thought to be recognized), the next phase usually involves predicting future outcomes and then obtaining verification of the predictions. Finally, the last stage is to try to generalize the pattern developed so that an arbitrary case fitting the particular situation can be solved without too much effort.

ACTIVITY 2 : NUMBER PATTERNS

DIRECTIONS: Fill in the blanks below by following the established number patterns.

1 8 36 120 330 792 1716 3432

1 7 28 ___ 210 ___ 924 ___

1 ___ 21 ___ ___ ___ ___ 792

1 ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

In the space below, list or describe any patterns that you recognize in the chart completed above.

Now, complete the problems given on the next page. But first, recall that in mathematics

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot r},$$ where $n \geq r$ and both are whole numbers.

$\left[\text{Note that } \binom{n}{r} \text{ is read "}n\text{ choose }r\text{"}\right]$. By definition, $\binom{n}{0} = 1 \;\; \forall n \in \{0,1,2,3,\ldots\}$.

(continued next page)

EXAMPLE: $\binom{5}{3} = \frac{5(5-1)(5-2)}{1\cdot2\cdot3} = \frac{5\cdot4\cdot3}{1\cdot2\cdot3} = 10$. Notice that in the numerator we stopped at 5 - 2. This is because $n-r+1=5-3+1=5-2$. EXAMPLE:

$\binom{8}{5} = \frac{8\cdot7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5} = 56$. Again, we stopped at 4 in the numerator because $n - r + 1 = 4$.

⓿ $\binom{0}{0} = \_\_$ ❶ $\binom{1}{0} = \_\_$ and $\binom{1}{1} = \_\_$ ❷ $\binom{2}{0} = \_\_$, $\binom{2}{1} = \_\_$, and $\binom{2}{2} = \_\_$

❸ $\binom{3}{0} = \_\_$, $\binom{3}{1} = \_\_$, $\binom{3}{2} = \_\_$, and $\binom{3}{3} = \_\_$

❹ $\binom{4}{0} = \_\_$, $\binom{4}{\_} = \_\_$, $\binom{4}{2} = \_\_$, $\binom{-}{\_} = \_\_$, and $\binom{4}{4} = \_\_$

❺ $\binom{5}{0} = \_\_$, $\binom{5}{1} = \_\_$, $\binom{5}{2} = \_\_$, $\binom{5}{3} = \_\_$, $\binom{5}{4} = \_\_$, and $\binom{5}{5} = \_\_$

On your own paper, predict what would be problem ❻ and also predict the answers to the individual parts of this problem. Then, do the same thing for problems ❼, ❽, and ❾. Finally, verify that some of the individual parts of problem ❾ hold true to your prediction.

In other words, verify that the values of say, $\binom{9}{2}$, $\binom{9}{5}$, and $\binom{9}{7}$ match your prediction.

When finished with the above work, answer the questions below in the space provided.

What patterns did you discover in the above exercises that made your work less tedious and enabled you to eventually make predictions?

In what ways do these problems relate to the first part of this activity? to activity #1?

ACTIVITY 3: INTEGRATION WITHIN THE UNIT SQUARE

DIRECTIONS: Complete the following chart by evaluating each definite integral.

k	$\int_0^1 x^k \, dx$	$\int_0^1 x^{1/k} \, dx$
0		
1/2		
1		
3/2		
2		
5/2		
3		
7/2		
4		

What do you notice about the sum of each pair of answers in any particular row?

____

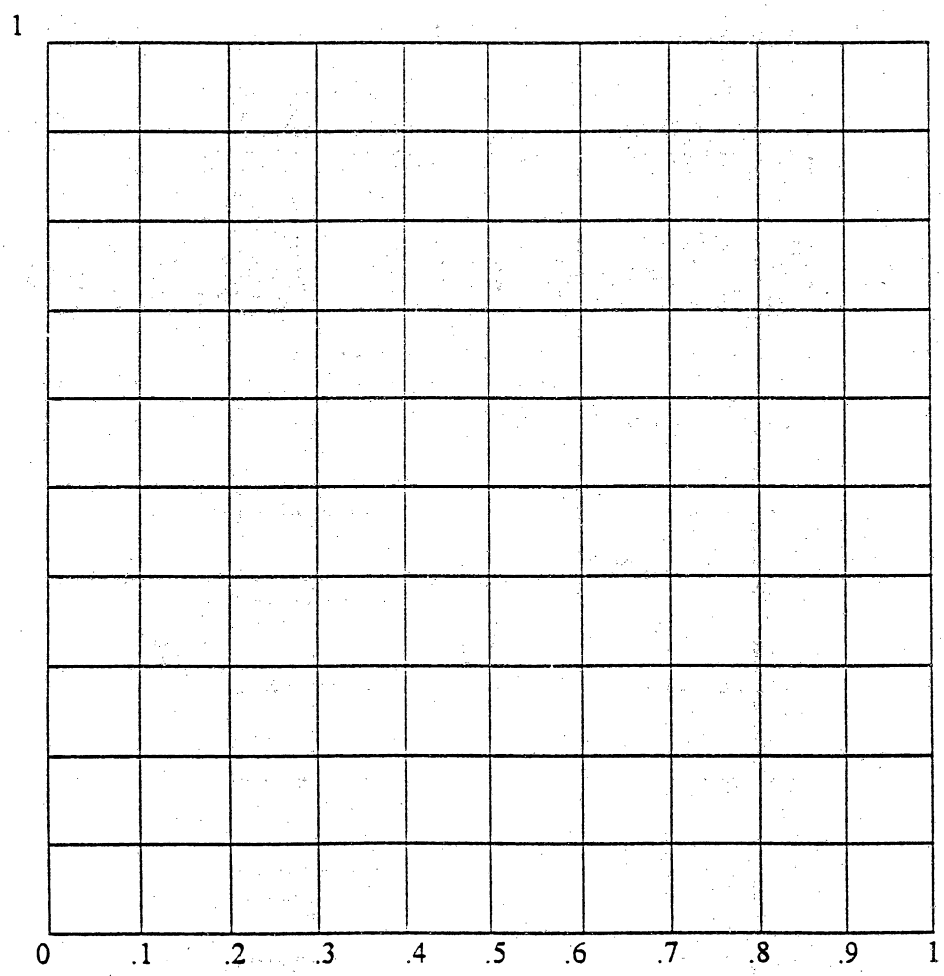
Make a mental note of the results above, especially those achieved when $k = 0$, 1, 2, 3, and 4, as they are bound to reappear in the near future. Then, continue the chart below.

4/5		
7/3		
8/9		
n ($n > 0$) (that is, generalize)	$\int_0^1 x^n dx =$	$\int_0^1 x^{1/n} dx =$

Does what you noticed above about the sum of each pair of answers in any particular row still hold true, even in the general case where $k = n$? _______

ACTIVITY 4: GRAPHING IN THE UNIT SQUARE

DIRECTIONS: Illustrate geometrically the relationship between the regions in the unit square having areas equal to $\int_0^1 x^k dx$ and $\int_0^1 x^{1/k} dx$ for the $k = 2$ row of the previous activity. Do this by graphing $y = x^2$ above and then shading the area below the curve. Then do the same thing for $y = x^{1/2}$ on the back of this paper. You should notice two important things as you complete this activity. One concerns symmetry and the other deals with the combined area of the two regions graphed. Recall that two graphs are symmetrical with respect to the line $y = x$ if, when holding a picture of one of the graphs up to the light with one hand and viewing it normally, turning the paper over and rotating it ninety degrees clockwise gives you the proper view of the other graph. Try this after completing both graphs. What do you notice about the combined areas of the two graphs?



ACTIVITY 5: GRAPHING CONTINUED

DIRECTIONS: Graph $y = x^{4/5}$ below and $y = x^{5/4}$ on back. Then shade the region below each curve. Finally, answer the questions below concerning this activity.

A) What is the area of the shaded region below? $\int_0^1 x^{4/5}dx =$ ______

B) What is the area of the shaded region on back? ________________

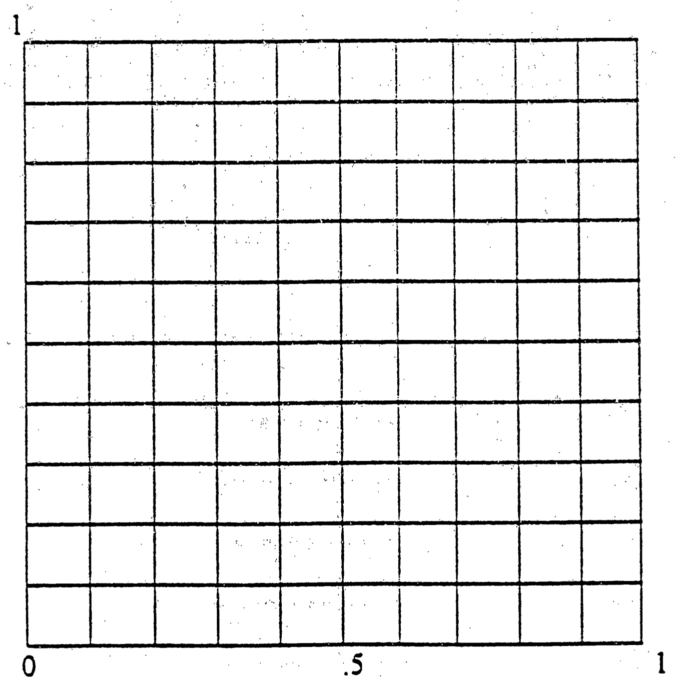
C) What is the sum of these two regions? ____

D) What is the area of the unit square? ____

E) Why is your answer to question D the same as your answer to question C?

____

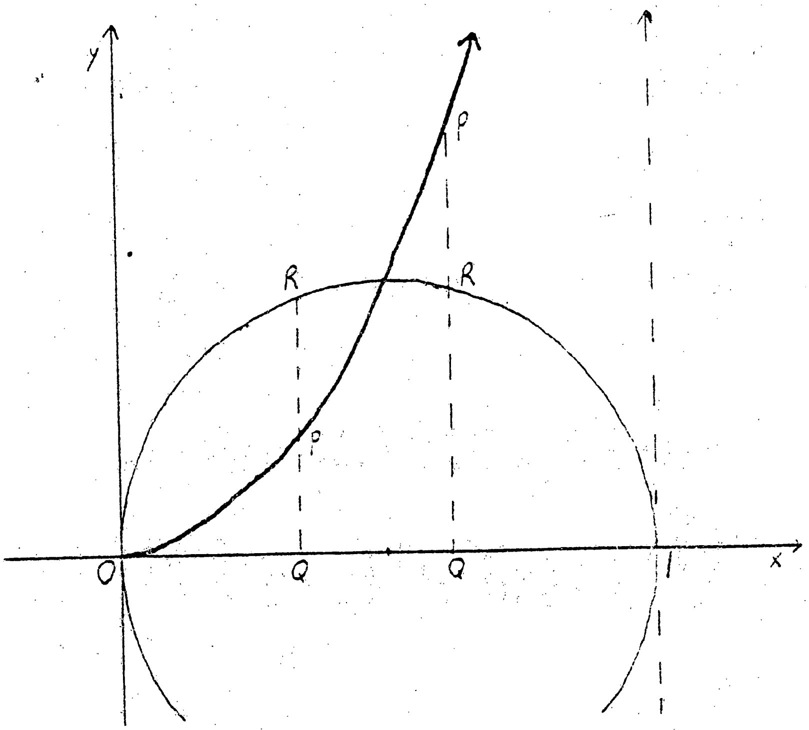
____



ACTIVITY 6 : WALLIS' QUADRATURE OF THE CISSOID

EXERCISE 1:

The "cissoid," shown below, is the set of points P such that $\frac{PQ}{OQ} = \frac{OQ}{RQ}$. Show that its equation is $y = x^{3/2}(1-x)^{-1/2}$, $x \in (0,1)$.



EXERCISE 2:

Note first that the area under the cissoid is $\int_0^1 x^{3/2}(1-x)^{-1/2}\,dx$. Can you evaluate this definite integral? If so, feel free to skip the remaining exercises. If not, let

$a_m = \int_0^1 x^{1/2}(1-x)^{m/2}\,dx = \int_0^1 x^{m/2}(1-x)^{1/2}\,dx$. Prove that the two integrals in this expression are in fact equal. Hint: Let $u = 1-x$ and get one integrand in terms of u, including the limits of integration. Then do a reverse substitution (i.e. let $x = u$) to obtain the second integral.

(continued next page)

EXERCISE 3:

Evaluate the definite integrals a_0, a_2, a_4, and a_6. Show that the pattern appears to imply that $a_m = \left(\frac{m}{m+3}\right)a_{m-2}$ for even values of m. Now assume, just as Wallis did, that this recursion relation holds for odd values of m as well. What is the value of a_3?

EXERCISE 4:

Let $b_n = \int_0^1 x^{3/2}(1-x)^{n/2}\,dx$, and note two things: 1. That $b_1 = a_3$

2. That b_{-1} is the area under the cissoid.

Evaluate the definite integrals b_0, b_2, b_4, and b_6. Show that the pattern appears to imply that $b_n = \left(\frac{n}{n+5}\right)b_{n-2}$ for even values of n. Now assume, just as Wallis did, that this recursion relation holds for odd values of n as well.

What is the value of b_{-1}? Hint: Get b_{-1} in terms of b_1.

EXERCISE 5:

How many times larger is the area of the cissoid than the area of the generating semi-circle?

SET II

ALHAZEN'S METHOD FOR CALCULATING $1^k+2^k+3^k+......+n^k$ FOR ANY POSITIVE INTEGER k

ACTIVITY 7 : SUM OF THE FIRST n INTEGERS

DIRECTIONS: Fill in the blanks. BE AWARE! Only even n is explored in this activity.

Problem number (n)	Figurate number	Sum of first n integers	Sum as a product
2	3 =	1 + 2 =	(__)(__)
4	10 =	1 + 2 + 3 + 4 = (1 + 4) + (2 + 3) = 5 + 5 =	(2)(5)
6	21 =	1 + __ + 3 + 4 + __ + 6	
	=	(1 + 6) + (2 + __) + (__ + 4) = 7 + 7 + 7 =	(3)(7)
8	36 =	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8	
	=	(1 + __) + (2 + 7) + (__ + __) + (4 + __)	
	=	9 + __ + __ + 9 =	(4)(__)
10	55 =	1 + __ + 3 + 4 + __ + __ + 7 + __ + __ + 10	
	=	(__ + __)+(__ + 9)+(3 + __)+(__ + 7)+(__ + __)	
	=	___ + ___ + ___ + ___ + ___ =	(__)(__)
__	78 =	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 =	(__)(13)
14	___ =	1 + 2 + 3 + 4 + ... + 12 + 13 + 14 =	(7)(__)
16	136 =	1 + 2 + 3 + ... + __ + __ + __ =	(__)(17)
18	___ =	1 + 2 + 3 + ... + 18 =	(__)(__)
__	___ =	__ + __ + __ + ... + __ =	(__)(__)
:	:		
:	:	"GENERALIZE"	
:	:		
n	____ =	1 + 2 + 3 + ... + __ =	(__)(___)

↗ n even ←---→

VERIFY YOUR RESULT IN PROBLEM "n":

A) 1 + 2 + 3 + 4 + ... + 100 = (______)(______) = 5050

B) $\sum_{i=1}^{500} i$ = (________)(________) = 125,250

Does your result work? ______ If so, let's look at the sum of the first n integers where n is ______, as opposed to even.

DIRECTIONS: First, fill in as many blanks as possible **using your answers from the previous page**. Then, **interpolate** to fill in the remaining blanks. Recall that interpolate means "to polish in between."

1) $1 = (\_\_)(\_\_)$

2) $3 = 1 + 2 = (\_\_)(\_\_)$

3) $6 = 1 + 2 + 3 = (\_\_)(\_\_)$

4) $10 = 1 + 2 + 3 + 4 = (\_\_)(\_\_)$

5) $\_ = 1 + 2 + 3 + 4 + 5 = (\_\_)(\_\_)$

6) $21 = 1 + 2 + 3 + 4 + 5 + 6 = (\_\_)(\_\_)$

7) $\_ = 1 + 2 + 3 + \ldots + 7 = (\_\_)(\_\_)$

8) $36 = 1 + 2 + 3 + \ldots + 8 = (\_\_)(\_\_)$

9) $\_ = 1 + 2 + 3 + \ldots + 9 = (\_\_)(\_\_)$

10) $55 = 1 + 2 + 3 + \ldots + 10 = (\_\_)(\_\_)$

11) $\_ = 1 + 2 + 3 + \ldots + 11 = (5.5)(\_\_)$

•
•
•

102) $\_\_\_\_\_\_ = 1 + 2 + 3 + \ldots + 102 = (\_\_\_\_)(\_\_\_\_)$

195) $\_\_\_\_\_\_ = 1 + 2 + 3 + \ldots + 195 = (\_\_\_\_)(\_\_\_\_)$

"GENERALIZE"

n) $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

n can be odd or even ✎

VERIFY YOUR RESULTS FROM THE PREVIOUS PAGE:

A) $\sum_{i=1}^{50} i = (\_\_\_\_\_\_)(\_\_\_\_\_\_) = 1275$

B) $\sum_{i=1}^{51} i = (\_\_\_\_\_\_)(\_\_\_\_\_\_) = \sum_{i=1}^{50} i + \_\_\_ = \_\_\_\_\_\_\_$

C) $\sum_{i=1}^{49} i = (\_\_\_\_\_\_)(\_\_\_\_\_\_) = \sum_{i=1}^{50} i - \_\_\_ = \_\_\_\_\_\_\_$

A different approach to the same summation follows.

1) $2(1) = 1 + 1 = (1)(2)$

2) $2(1 + 2) = (1 + 2) + (1 + 2) = (2)(\_\_)$

3) $2(1 + 2 + 3) = (1 + 2 + 3) + (1 + 2 + 3)$
$= (1 + 3) + (2 + 2) + (3 + 1) = 4 + 4 + 4 = (\_\_)(\_\_)$

4) $2(1 + \_\_ + \_\_ + 4) = (1 + 2 + \_\_ + \_\_) + (1 + \_\_ + 3 + \_\_)$
$= (1 + \_\_) + (2 + 3) + (3 + 2) + (4 + \_\_) = (\_\_)(\_\_)$

5) $2(1 + 2 + \_\_ + \_\_ + \_\_) = 1 + 2 + 3 + \_\_ + \_\_ + \_\_ + 2 + \_\_ + 4 + 5$
$= (\_\_ + \_\_) + (2 + \_\_) + (3 + 3) + (4 + \_\_) + (5 + 1)$

* pairs must add up to 6 ↗ ↓ ↓ ↓ ↓ ↓

$= \_\_ + \_\_ + \_\_ + \_\_ + \_\_$

$= (\_\_)(\_\_)$

$\_\_) \; \_\_(1 + \_\_ + \_\_ + \_\_ + \_\_ + \_\_)$

$= (1 + 2 + 3 + 4 + 5 + 6) + (1 + 2 + 3 + 4 + 5 + 6)$

$= (\_\_ + 6) + (2 + 5) + (\_\_ + \_\_) + (4 + 3) + (\_\_ + 2) + (6 + \_\_)$

↓ ↓ ↓ ↓ ↓ ↓

$= \_\_ + \_\_ + \_\_ + \_\_ + \_\_ + \_\_$

$= (\_\_)(\_\_)$

➪ **On your own:** Attempt setting up one or two more of these, then generalize.

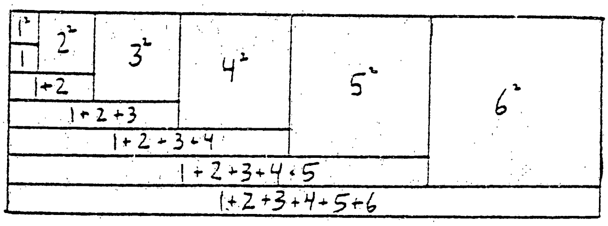
LECTURE 7A : DISCOVERING ALHAZEN'S METHOD FOR OBTAINING FORMULAS TO $1^k + 2^k + 3^k + ... + n^k$

This lecture describes an activity that, if carried out to the extreme, may replace or at the very least enhance the remainder of activities in this set. Thus, two approaches may be taken:

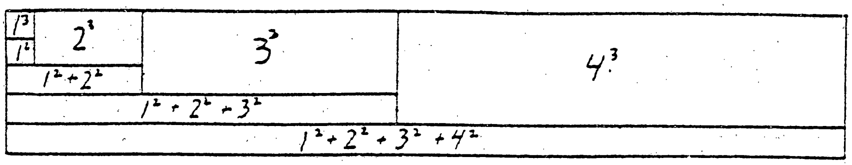
1) Students answer the first group of questions given below and then move on to activity number eight. Because of the "hands-on" experience here students will be able to move through the subsequent activities more rapidly. They will also have a greater "feel" and understanding of what should be learned throughout the activities.
2) Students in effect go to the extreme with this activity. They answer the first set of questions and continue on to the second set with the overall objective of finding a means to acquiring formulas for $1^k + 2^k + 3^k + ... + n^k$ in terms of n given any positive integer k.

Materials needed: An envelope containing the pieces to both the "$k = 2$, $n = 6$" rectangle and the "$k = 3$, $n = 4$" rectangle as shown below.

The "$k = 2$, $n = 6$" rectangle.



The "$k = 3$, $n = 4$" rectangle.



Instructions:

1) Take out the pieces in the envelope containing the "$k = 2$, $n = 6$" rectangle and assemble all of them to form a rectangle. There are several ways in which this can be done. The best way to assemble the pieces is the one that is most "organized." If students do not assemble the pieces in the most organized fashion right away, they will once they start answering some of the questions. The most "organized" way is illustrated above.

2) Do the same thing as described above with the "$k = 3$, $n = 4$" rectangle.

(continued next page)

QUESTIONS - GROUP I

1) What are the length, width, and area of the first rectangle assembled (the "$k = 2$, $n = 6$" rectangle) in terms of expressions containing n and/or k or expressions written on the smaller individual pieces (rectangles)?

2) What are the length, width, and area of the second rectangle assembled (the "$k = 3$, $n = 6$" rectangle) in terms of expressions containing n and/or k or expressions written on the smaller individual pieces (rectangles)?

3) What do the "$k = 2$, $n = 5$", "$k = 2$, $n = 4$", "$k = 2$, $n = 3$", "$k = 2$, $n = 2$", and "$k = 2$, $n = 1$" rectangles look like? Sketch the most organized version of these rectangles on your own paper.

4) What does the "$k = 2$, $n = 7$" rectangle look like? Sketch this on your paper.

5) What do the "$k = 3$, $n = 3$", "$k = 3$, $n = 2$", "$k = 3$, $n = 1$", "$k = 3$, $n = 5$", and "$k = 3$, $n = 6$" rectangles look like? Sketch the most organized version of these rectangles on your own paper.

6) What are the length, width, and area of each of the rectangles sketched in numbers three through five above? Be organized in your approach to answering this question. Look for patterns to develop. You will recognize patterns sooner if you write your answers in terms of expressions containing n and/or k or expressions written on the smaller individual pieces (rectangles)?

GROUP II:

1) What do the "$k = 1$, $n = 1$", "$k = 1$, $n = 2$", "$k = 1$, $n = 3$", "$k = 1$, $n = 4$", and "$k = 1$, $n = 5$" rectangles look like? Sketch the most organized version of these rectangles on your own paper.

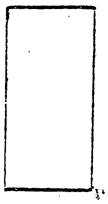
2) What do the "$k = 4$, $n = 1$", "$k = 4$, $n = 2$", "$k = 4$, $n = 3$", "$k = 4$, $n = 4$", and "$k = 4$, $n = 5$" rectangles look like? Sketch the most organized version of these rectangles on your own paper.

3) Use the "$k = 1$" set of rectangles to determine a formula for $\sum_{i=1}^{n} i$. To do this, you will have to construct the "generalized" version of a "$k = 1$" rectangle, determine its length (l) and width (w) strictly in terms of n, and note that $\sum_{i=1}^{n} i$ appears twice inside the "generalized" version of the "$k = 1$" rectangle. Therefore, $2\sum_{i=1}^{n} i = (l)(w)$.

4) Use the formula developed in problem three along with the "$k = 2$" set of rectangles (including its "generalized" version) to devise a means for obtaining a formula for $\sum_{i=1}^{n} i^2$.

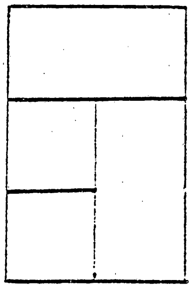
5) Continue the recursive relationships established above to find formulas in terms of n for $\sum_{i=1}^{n} i^3$, $\sum_{i=1}^{n} i^4$, and $\sum_{i=1}^{n} i^5$.

ACTIVITY 8 : ALHAZEN'S GEOMETRICAL APPROACH TO THE SUM OF THE FIRST n INTEGERS

1) 

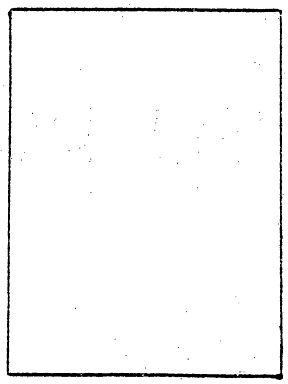
b = ___ h = ___ A = ___ = (b)(h) = (1)(2) = 2(1) = 1 + 1

Draw one line segment inside this figure to show that its area is 1 + 1.

2) 

b = ___ h = ___ A = ___ = (b)(h) = (__)(__) = 2(1 + 2)
= (1 + 2) + (1 + __)

Insert the area of each inner rectangle to show that the area of the original rectangle is (1 + 2) + (1 + 2).

3) 

A = ___ = (b)(h) = (__)(__) = 2(1 + 2 + 3)

Draw the line segments required to show the area of this rectangle is (1 + __ + __) + (__ + __ + __).

HINT: The figure in #2 above should be contained in this rectangle starting in the lower left corner.

4) Do this next case on your own paper by continuing the patterns established above.

5) Same instructions as #4.

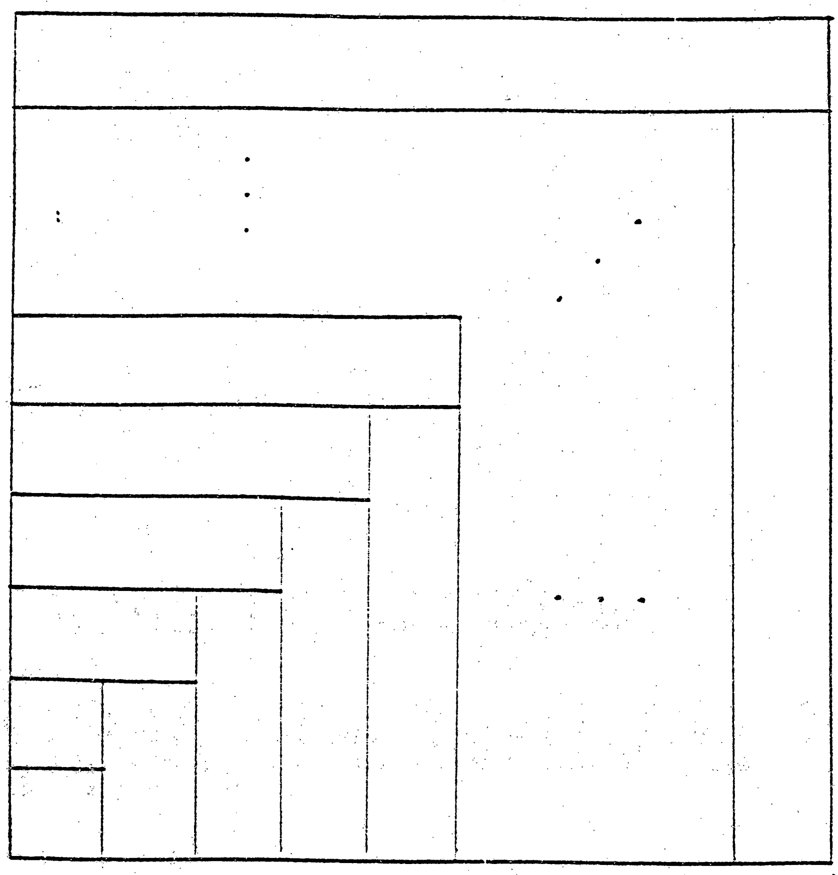
6) You may skip this problem and try, say, problem #9 and then "generalize" if you feel you are ready. If you find difficulties in your attempt to generalize, a good idea would be to return to this problem, work through it, and refresh your memory of the patterns involved here. The generalization procedure for Alhazen's geometrical approach to the sum of the first n integers is contained in the next activity. However, you should try this on your own before advancing. Good luck.

ACTIVITY 9 : GENERALIZING ALHAZEN'S SUM OF THE FIRST *n* INTEGERS

b = ___

n)

h = _______



AREA = (b)(h) = (___)(_______)

= (1 + 2 + 3 + ... + ___) + (___ + ___ + ___ + ... + ___)

$$= \sum_{i=\_}^{\_} \_ + \sum_{i=\_}^{\_} \_ = 2\sum_{i=\_}^{\_} \_$$

Does the geometrical method here verify further the algebraic result achieved earlier? _____

Why? ____

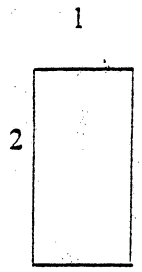
____

____

ACTIVITY 10 : ALHAZEN'S GEOMETRICAL APPROACH TO THE SUM OF THE SQUARES OF THE FIRST n INTEGERS: A WARM-UP

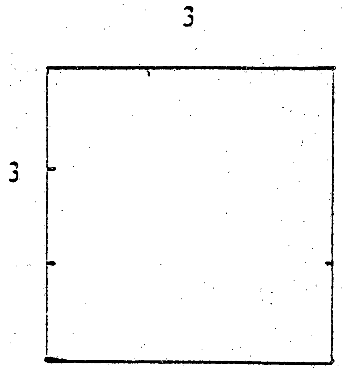
DIRECTIONS: Divide each rectangle below in such a way so that the original contains smaller rectangles having areas a_1, a_2, a_3, etc.

1)



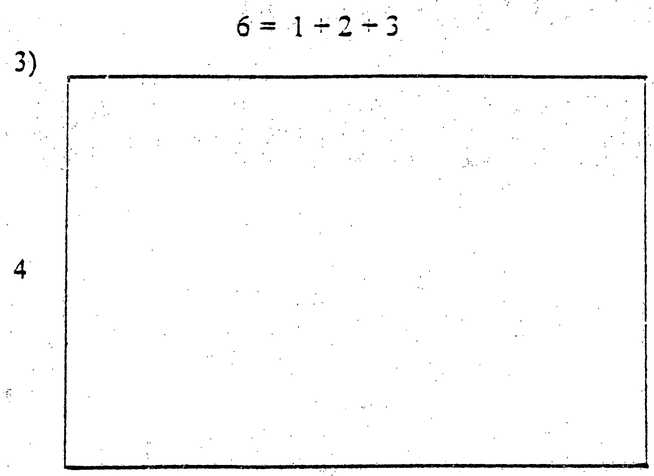
$a_1 = 1,\ a_2 = 1^2$

2)



$a_1 = 1,\ a_2 = 1^2\ ,\ a_3 = 1+2,\ a_4 = 2^2$

3)



HINT: The picture you formed in number 2 above should be inserted into this rectangle.

$a_1 = 1,\ a_2 = 1^2,\ a_3 = 1+2,\ a_4 = 2^2,$

$a_5 = 1+2+3,\ a_6 = 3^2$

4) Sketch a rectangle with an area of 5 units by ____ = 1 + __ + 3 + __ and do as you did on the previous page with:

$a_1 = 1,\qquad a_2 = \_$

$a_3 = 1+2,\qquad a_4 = \_$

$a_5 = 1+2+\_,\qquad a_6 = \_$

$a_{\_} = 1+\_+\_+\_,\qquad a_{\_} = \_$

5) On your own paper, sketch a rectangle with an area of ___ units by ____ = ___ + ___ + ___ + ___ + ___ and complete this next case by continuing the patterns established above. Here we have

$a_1 = 1,\ a_2 = 1^2,\ a_3 = 1+2,\ a_4 = 2^2,$

$a_5 = 1+2+\_,\quad a_6 = 3^2,\quad a_7 = 1+2+3+\_,\quad a_8 = 4^2,$

$a_9 = \_+\_+\_+\_+\_,\qquad a_{10} = \_\_\_$

6) Complete this case in the same manner as you did those above entirely on your own paper.

ACTIVITY 11 : SUMMARY OF ACTIVITY 10

☒ NOTE: The problem numbers below correspond to the same problem numbers of activity #10.

1) $A = a_1 + a_2 = 1 + 1^2$

2) $A = a_1 + a_3 + a_2 + a_4 = \underline{1} + \underline{1 + 2} + 1^2 + 2^2$

3) $A = a_1 + a_3 + a_5 + a_2 + a_4 + a_6$ = __ + _____ + _________ + 1^2 + ___ + ___

4) $A = a_1 + a_3 +$ ___ $+ a_7 + a_2 +$ ___ + ___ + ___

= $\underline{1}$ + _____ + _________ + _____________ + 1^2 + ___ + ___ + ___

$$= \sum_{i=1}^{4}\left\{\sum_{k=1}^{i} k\right\} + \sum_{i=1}^{4} i^2$$

5) Try doing this case in the same fashion as those above.

DIRECTIONS: Prior to continuing, go back to activity #10 and shade in all regions whose area is represented by a natural number raised to the second power (i.e. $1^2, 2^2, 3^2, \ldots$). Again, the problem numbers below correspond to those in activity #10.

1) A = (b)(h) = (1)(2)

2) A = (b)(h) = (3)(3) = (1+2)(3)

3) A = (b)(h) = (__)(__) = (1+ 2 + __)(__)

4) A = (10)(5) = (__ + __ + __ + __)(__) = $\left(\sum_{i=1}^{4} i\right)(4+1)$

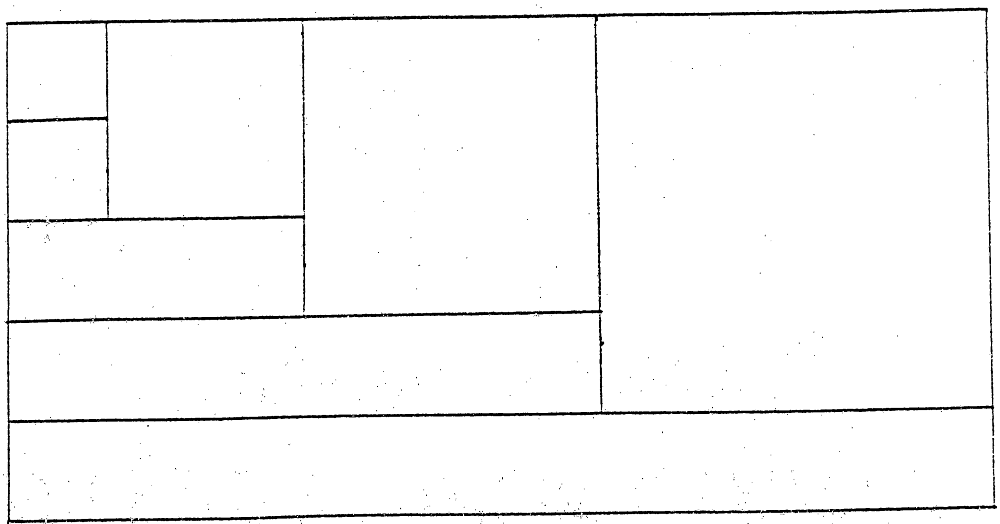
5) A = (__)(__) = (__ + __ + __ + __ + __)(__) = $\left(\sum_{i=1}^{5} i\right)$(__+1)

6) Complete this case in the same fashion as you did those above. Do this below.

ACTIVITY 12 : THE GENERALIZATION OF ALHAZEN'S GEOMETRICAL APPROACH TO THE _____ OF THE __________ OF THE FIRST ___ INTEGERS.

DIRECTIONS: Fill in the blanks below as we carry-out problem 4 in activities 10 and 11 to its fullest extent.

4)



Insert the areas 1, 1+2, 1+2+3, 1+2+3+4, 1^2, 2^2, 3^2, and 4^2 into their proper places above. Next, shade in the 1^2, 2^2, 3^2, and 4^2 regions. Note that the area of the figure above is equal to its base times its height which in turn equals the sum of the shaded regions plus the sum of the non-shaded regions. In other words (or symbols),

$$(b)(h) = \Sigma \text{ shaded regions} + \Sigma \text{ non-shaded regions}$$

THUS,

$$(1+2+3+4)(4+1) = \left(1^2+2^2+3^2+4^2\right)+\left[(1)+(1+2)+(1+2+3)+(1+2+3+4)\right]$$

$$\Rightarrow \left(\sum_{i=1}^{4} i\right)(4+1) = \sum_{i=1}^{4} i^2 + \left(\sum_{i=1}^{4}\left\{\sum_{k=1}^{i} k\right\}\right)$$

☒Note that $\sum_{k=1}^{i} k =$ ______________

$$\Rightarrow \left(\sum_{i=1}^{4} i\right)(5) = \sum_{i=1}^{4} i^2 + \left(\sum_{i=1}^{4}\left\{\frac{1}{2}i(i+1)\right\}\right)$$

(continued on the next page)

$$\Rightarrow\ 5\sum_{i=1}^{4} i = \sum_{i=1}^{4} i^2 + \frac{1}{2}\sum_{i=1}^{4} i^2 + \frac{1}{2}\sum_{i=1}^{4} i$$

$$\Rightarrow\ \frac{9}{2}\sum_{i=1}^{4} i = \frac{3}{2}\sum_{i=1}^{4} i^2 \qquad \Rightarrow\ \sum_{i=1}^{4} i^2 = 3\sum_{i=1}^{4} i = 3\left[\frac{1}{2}(4)(4+1)\right] = \_\_\_$$

Can you think of a quick "check" to the result above? If so, show it below.

Fill in the blanks below as we do the same thing for problem #5 of activity # 11. Since (b)(h) = Σ shaded regions + Σ non-shaded regions. We have, $(1+2+3+4+5)(5+1)$
$=(1^2+2^2+3^2+4^2+5^2)+[(1)+(1+2)+(1+2+3)+(1+2+3+4)+(\_+\_+\_+\_+\_)]$

$$\Rightarrow \left(\sum_{i=\_}^{\_} i\right)(\_+1) = \sum_{i=1}^{\_} i^2 + \left(\sum_{i=1}^{\_}\left\{\sum_{k=1}^{\_} k\right\}\right)$$

$$\Rightarrow \left(\sum_{i=1}^{\_} i\right)(\_) = \sum_{i=\_}^{\_} i^2 + \left(\sum_{i=1}^{\_}\left\{\frac{1}{2}i(i+1)\right\}\right)$$

$$\Rightarrow (\_)\sum_{i=1}^{\_} i = \sum_{i=1}^{\_} i^2 + \left(\frac{\_}{\_}\right)\sum_{i=1}^{\_} i^2 + \left(\frac{\_}{\_}\right)\sum_{i=1}^{\_} i$$

$$\Rightarrow \left(\frac{\_}{2}\right)\sum_{i=1}^{5} i = \frac{3}{2}\sum_{i=1}^{5} i^2$$

$$\Rightarrow \sum_{i=1}^{5} i^2 = \left(\frac{\_}{\_}\right)\sum_{i=1}^{5} i = \left(\frac{\_}{\_}\right)\left[\frac{1}{2}(\_)(\_+\_)\right] = \_\_\_$$

As soon as you understand every step above, do case 6 and case *n* (i.e. generalize) in the same manner as cases 4 and 5 above on your own paper.

⊠ When doing problem *n*, you will want to prove that $\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$

ACTIVITY 13 : ALHAZEN'S GEOMETRICAL APPROACH TO THE SUM OF THE CUBES OF THE FIRST *n* INTEGERS

Note that in obtaining a formula for the sum of the squares of the first n integers (end of activity #12), the formula for the sum of the first n integers was used. In mathematics, this is called a recursive relationship. When working through this activity, look (👀) for all the familar patterns above to occur, including, eventually, the two formulas you have discovered above.

DIRECTIONS: Divide each rectangle below in such a way so that the original contains smaller rectangles having areas a_1, a_2, a_3, etc.

1)

$a_1 = 1^2$, $a_2 = 1^3$

2)

$a_1 = 1^2$, $a_2 = 1^3$, $a_3 = 1^2 + 2^2$, $a_4 = 2^3$

3)

$a_1 = 1^2$, $a_2 = 1^3$, $a_3 = 1^2 + 2^2$, $a_4 = 2^3$, $a_5 = 1^2 + 2^2 + 3^2$, $a_6 = 3^3$

4) Complete this case four here on your own in the same way as those above.

5) Complete this case and then skip to, say, problem 9, and complete its picture. Then try to create a formula for the sum of the cubes of the first n integers by "generalizing." If you run into problems, go to the next activity. It contains the procedures required for generalizing.

ACTIVITY 14 : THE GENERALIZATION PROCEDURE FOR ACTIVITY 13

☒ NOTE: The problem numbers below correspond to the same problem numbers in acivity #13. Also, prior to continuing, go back to activity #13 and shade in all of the regions whose area is represented by a natural number raised to the third power. Keep in mind too that

(b)(h) = Σ shaded regions + Σ non-shaded regions

1) $b = 1^2$ $h = 1+1$ A = Σ shaded regions + Σ non-shaded regions

$\therefore \left(1^2\right)(1+1) = 1^3 + 1^2$

2) $b = 1^2 + \_^2$ $h = \_ + 1$

$\therefore \left(1^2 + 2\text{-}\right)(\_+1) = 1^3 + \_^3 + \left(1^2\right) + \left(1^2 + 2\text{-}\right)$

✓Check your result above before continuing.

3) $b = \_^2 + 2^2 + \_\text{-}$ $h = 3 + \_$

$\therefore \left(1^2 + 2^2 + 3^2\right)(\_+1) = \underline{1^3 + 2^3 + \_^3} + \left[(\_) + (\_+\_) + \left(1^2 + \_ + \_\right)\right]$

$\therefore \left(\sum_{i=1}^{3} i^2\right)(3+1) = \sum_{i=1}^{3} i^3 + \left\{\sum_{i=1}^{3}\left(\sum_{k=1}^{i} k^2\right)\right\}$ ☞Look familiar?

4 $)b = 1^2 + 2^2 + 3^2 + 4^2, h = 4 + 1$

$\therefore \left(\sum_{i=\_}^{-} i^{-}\right)(\_+\_) = \sum_{i=\_}^{-} \_^{-} + \left\{\sum_{i=\_}^{-}\left(\sum_{k=\_}^{-} \_^{-}\right)\right\}$

↖Recall the formula for this:

(continued on next page)

$$\Rightarrow 5\sum_{i=1}^{4} i^2 = \sum_{i=1}^{4} i^3 + \sum_{i=1}^{4}\left\{\frac{1}{3}i^3 + \frac{1}{2}i^2 + \frac{1}{6}i\right\}$$

$$\Rightarrow (\_)\sum_{i=1}^{4} i^2 = \sum_{i=1}^{4} i^3 + \left(\frac{=}{\_}\right)\sum_{i=1}^{4} i^3 + \left(\frac{=}{\_}\right)\sum_{i=1}^{4} i^2 + \left(\frac{=}{\_}\right)\sum_{i=1}^{4} i$$

$$\Rightarrow \quad \frac{4}{3}\sum_{i=1}^{4} i^3 = \left(\frac{=}{2}\right)\sum_{i=1}^{4} i^2 - \left(\frac{=}{\_}\right)\sum_{i=1}^{4} i$$

Complete this problem in the space provided below (i.e. Find $\sum_{i=1}^{4} i^3$).

____

____

✓ Does your solution obtained in the manner above equate to $\sum_{i=1}^{4} i^3$?

5) Find $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ using the same approach as number 4 above.

n) The general case! Prove $1^3 + 2^3 - 3^3 + \ldots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

ACTIVITY 15 : SUMMARY OF ALHAZEN

SUMMARY OF RESULTS TO DATE :

Let S_k be the sum of the k^{th} powers of the first n integers. Then

❶ $S_1 = 1 + 2 + 3 + \ldots + n =$ ______________________________

❷ $S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 =$ ______________________________

❸ $S_3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 =$ ______________________________

❹ $S_4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

A) Prove number 4 above using Alhazen's geometric method.

B) S_5, S_6, and S_7 can be written in terms of the formulas above as shown below. Use these equivalent expressions to determine formulas (in terms of n only) for S_5, S_6, and S_7. You will need these formulas in subsequent activities.

❺ $S_5 = \frac{4}{3}S_1^3 - \frac{1}{3}S_3 =$ ______________________________

____

____

❻ $S_6 = \frac{1}{7}S_2\left(12S_1^2 - 6S_1 + 1\right) =$ ______________________________

____

____

❼ $S_7 = 2S_1^4 - S_5 =$ ______________________________

____

ACTIVITY 16 : SUMS OF SUMS AND FIGURATE NUMBERS

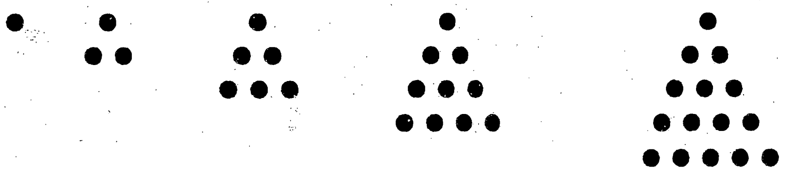
A) List the first ten natural numbers: ___, ___, ___, ___, ___, ___, ___, ___, ___, ___

B) The triangular numbers: These are obtained by summing the first n integers. Thus, the first ten are 1, 1+2, 1+2+3, 1+2+3+4, ... , 1+2+3+...+10. Write the first ten of these below:

____, ____, ____, ____, ____, ____, ____, ____, ____, ____

↳ The formula for obtaining the n^{th} triangular number is: ____________________

The set of triangular numbers can be illustrated in the manner shown below.



ETC.

C) The tetrahedral numbers: These are derived by summing the sums of the first n integers. In other words, by summing the first n triangular numbers. The first ten are __, 1+3, 1+3+__, __+__+6+10, ..., 1+3+6+...+___. Write the first ten of these below.

____, ____, ____, ____, ____, ____, ____, ____, ____, ____,

↳ The formula for deriving the n^{th} tetrahedral number is:

$$\sum_{i=1}^{n}\left\{\sum_{k=1}^{i} k\right\}$$

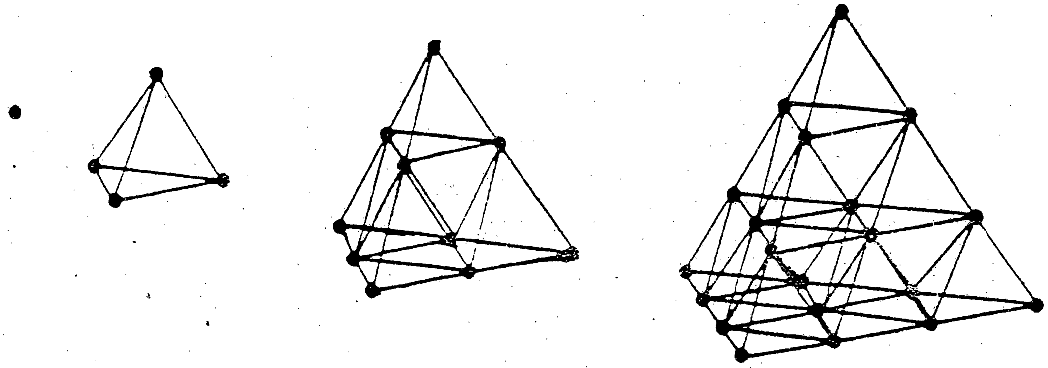
$$= \sum_{i=1}^{n}\left\{\frac{1}{2}i(i+1)\right\} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i$$

$$= \frac{1}{2}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) + \frac{1}{2}\left(\frac{1}{2}n^2 + \frac{1}{2}n\right)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(continued next page)

The set of tetrahedral numbers (sometimes called the pyramidal numbers) can be depicted in the manner shown below.



D) The next set of <u>figurate</u> numbers along with their formula can be acquired by summing the sums of the sums of the first n integers. That is, by summing the first n tetrahedral numbers. The first ten are 1, 1 + 4, 1 + 4 + 10, etc. List the first ten below.

_____, _____, _____, _____, _____, _____, _____, _____, _____, _____

EXERCISE: Produce a formula for these starting with $\sum_{i=1}^{n}\left\{\sum_{j=1}^{i}\left(\sum_{k=1}^{j} k\right)\right\}$

ACTIVITY 17 : CONCLUSION TO ALHAZEN

DIRECTIONS: The following is a continuation of activity 16. Make a mental note of the results both here and in activity 16 since they will become very useful in future activities.

E) List the first ten numbers in the next set of figurate numbers below. You do not need to develop the formula.

_____, _____, _____, _____, _____, _____, _____, _____, _____, _____

F) Construct Pascal's Triangle below.

___

___ ___

___ ___ ___

___ ___ ___ ___

___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

G) Describe the relationship between Pascal's triangle and the previous activity.

____

____

____

____

SET III

WALLIS' CHARACTERISTIC RATIO OF INDEX k

ACTIVITY 18 : A TI-81 PROGRAM NECESSARY FOR FUTURE INVESTIGATIONS

THE EVALUATE PROGRAM

```
:Lbl 1
:Disp "X"
:Input X
:Disp "Y="
:Disp Y1
:Goto 1
```

This program will allow you to evaluate any function in the variable "x" for any real number in the domain of the function. The key step involves entering the expression in "x" as a function of Y_1 in the Y= part of the keyboard.

EXAMPLE

Suppose you want to find the numerical value of $\frac{0^2+1^2+2^2+\ldots+50^2}{50^2+50^2+50^2+\ldots+50^2}$

Using the formulas developed earlier, we know we can evaluate the following function at x = 50 to obtain the desired solution.

$$Y_1 = \frac{\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{6}x}{x^2(x+1)}$$

Simply enter this into its proper place in the TI-81, run this program and, when the "x = ?" prompt appears on the screen, enter 50. You should immediately see

Y = .3366666667

This is the value of the ratio above. More importantly, you can now find the value of similarily designed ratios quite rapidly. Now, try these on your own.

X	$Y_1=((1/3)X^3+(1/2)X^2+(1/6)X)/(X^2(X+1))$
2	
2000	
200,000	
.5	
-1	
-5	
-5000	
-50,000	
0	

ACTIVITY 19 : $k = 2$

Consider the ratio of the form $\dfrac{0^k + 1^k + 2^k + \ldots + n^k}{n^k + n^k + n^k + \ldots + n^k}$ when $k = 2$.

① $\dfrac{0^2 + 1^2}{1^2 + 1^2} = \dfrac{(\_)}{(\_)}$ when $n = 1$

② $\dfrac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \dfrac{(\_)}{(\_\_)}$ when $n = 2$

③ $\dfrac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2} = \dfrac{(\_\_)}{(\_\_)} = \dfrac{(\_)}{(\_\_)}$ when $n = 3$

④ $\dfrac{0^2 + 1^2 + 2^2 + 3^2 + 4^2}{4^2 + 4^2 + 4^2 + 4^2 + 4^2} = \dfrac{(\_\_)}{(\_\_)} = \dfrac{(\_)}{(\_)}$ when $n =$ __

⑤ $\dfrac{0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5^2 + 5^2 + 5^2 + 5^2 + 5^2 + 5^2} = \dfrac{(\_\_)}{(\_\_\_)} = \dfrac{(\_\_)}{(\_\_)}$ when $n = 5$

At this point ask yourself whether the answers above are converging to a non-zero limit. List the decimal equivalents for your solutions to the problems above.

①__________ ②__________ ③__________ ④__________ ⑤__________

Use Alhazen's formula and a graphics calculator to find both the fractional and decimal forms of the ratios given below. Hint: Use the "up" arrow on the calculator to bring back previous expressions along with the "insert" button to insert 0's when needed. The "evaluate" program may also be very handy.

⑥ $\dfrac{0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2} = \dfrac{(\_\_)}{(\_\_\_)} = .\_\_\_$

⑦ $\dfrac{0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2}{7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2} = \dfrac{(\_\_\_)}{(\_\_\_)} = .\_\_\_\_\_$

⑤⓪ $\dfrac{0^2 + 1^2 + 2^2 + \ldots + 50^2}{50^2 + 50^2 + 50^2 + \ldots + 50^2} = \dfrac{(\_\_\_\_\_)}{(\_\_\_\_\_\_)} = .\_\_\_\_\_$

⑤⓪⓪ $\dfrac{0^2 + 1^2 + 2^2 + \ldots + 500^2}{500^2 + 500^2 + 500^2 + \ldots + 500^2} = \dfrac{(\_\_\_\_\_\_\_\_)}{(\_\_\_\_\_\_\_\_\_)} = .\_\_\_\_\_\_$

ACTIVITY 19 -CONTINUED-

②⓪⓪⓪ $\dfrac{0^2+1^2+2^2+\ldots+2000^2}{2000^2+2000^2+2000^2+\ldots+2000^2} = $.____

⑤⓪⓪⓪ $\dfrac{0^2+1^2+2^2+\ldots+5000^2}{5000^2+5000^2+5000^2+\ldots+5000^2} = $.____

At this time, do you believe that these ratios are approaching a limit as n increases? ______
If so, what do you think the limit is? ______ Now, it's time to generalize.

n) $\dfrac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+n^2+\ldots+n^2} = \dfrac{\left(\frac{=}{\_}\right)n^3+\left(\frac{=}{\_}\right)n^2+\left(\frac{=}{\_}\right)n}{(n^2)(\_+\_)}$

↳HINT: Simplify the numerator and then use long division.

= ________________________ = __________

AN ALTERNATIVE WAY TO OBTAIN THE SAME LIMIT

DIRECTIONS: Transfer the fractional form of your answers to the $k = 2$ case for $n = 1$, $n = 2$, $n = 3$, ... , $n = 7$ into the spaces provided below. Then fill in the blanks.
☒NOTE: The problem numbers below correspond to those in the previous activities.

❶ $\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{(\_)}{(\_)}$ ❷ $\dfrac{5}{12} = \dfrac{1}{3} + \dfrac{(\_)}{(\_\_)}$ ❸ $\dfrac{(\_\_)}{(\_\_)} = \dfrac{7}{18} = \dfrac{1}{3} + \dfrac{(\_)}{(\_\_)}$

❹ $\dfrac{(\_\_)}{(\_\_)} = \dfrac{(\_)}{8} = \dfrac{1}{3} + \dfrac{(\_)}{(\_\_)}$ ❺ $\dfrac{(\_\_)}{(\_\_\_)} = \dfrac{(\_\_)}{(\_\_)} = \dfrac{1}{3} + \dfrac{(\_)}{(\_\_)}$

❻ $\dfrac{\_\_}{\_\_\_} = \dfrac{1}{3} + \dfrac{\_}{\_\_}$ ❼ $\dfrac{\_\_}{\_\_\_} = \dfrac{1}{3} + \dfrac{\_}{\_\_}$

CONCLUSION: AS $n \rightarrow \infty$, ______________________________

____

____

ACTIVITY 20 : $k = 3$, DECIMAL APPROACH

INVESTIGATING WALLIS' CHARACTERISTIC RATIO OF INDEX $k = 3$

n	$\frac{0^3+1^3+2^3+\ldots+n^3}{n^3+n^3+n^3+\ldots+n^3}$	RATIO FROM COLUMN AT LEFT AS A DECIMAL
1	$\frac{0^3+1^3}{1^3+1^3}$	
2	$\frac{0^3+1^3+2^3}{2^3+2^3+2^3}$	
3	$\frac{0^3+1^3+2^3+3^3}{3^3+3^3+3^3+3^3}$	
4	$\frac{0^3+1^3+2^3+3^3+4^3}{4^3+4^3+4^3+4^3+4^3}$	
5	$\frac{0^3+1^3+2^3+3^3+4^3+5^3}{5^3+5^3+5^3+5^3+5^3+5^3}$	
50	$\frac{0^3+1^3+2^3+\ldots+50^3}{50^3+50^3+50^3+\ldots+50^3}$	
500	skip	
5000	skip	
50,000	skip	
500,000	skip	
5,000,000	skip	
50,000,000	skip	
.5 billion	skip	
1 billion	skip	
n	$\lim_{n\to\infty}\left(\frac{0^3+1^3+2^3+\ldots+n^3}{n^3+n^3+n^3+\ldots+n^3}\right)$	Use the space below to prove that this limit is 1/4.

ACTIVITY 21 : FRACTIONAL APPROACH TO $k=3$

Without the formal theory we now have at our disposal to understand limit processes, John Wallis could not "take the limit as $n \to \infty$." So he argued that the characteristic ratio of index $k=3$ was 1/4 in much the manner illustrated below.

n	$\dfrac{0^3+1^3+2^3+\ldots+n^3}{n^3+n^3+n^3+\ldots+n^3}$	RATIO'S VALUE AS A REDUCED FRACTION	YOUR FRACTION IS 1/4 + ?
1	$\dfrac{0^3+1^3}{1^3+1^3}$		
2	$\dfrac{0^3+1^3+2^3}{2^3+2^3+2^3}$		
3	$\dfrac{0^3+1^3+2^3+3^3}{3^3+3^3+3^3+3^3}$		
4	$\dfrac{0^3+1^3+2^3+3^3+4^3}{4^3+4^3+4^3+4^3+4^3}$		
5	$\dfrac{0^3+1^3+2^3+3^3+4^3+5^3}{5^3+5^3+5^3+5^3+5^3+5^3}$		
50	$\dfrac{0^3+1^3+2^3+\ldots+50^3}{50^3+50^3+50^3+\ldots+50^3}$		

GENERALIZE: Use the space below to prove that $\lim\limits_{n\to\infty}\left(\dfrac{0^3+1^3+2^3+\ldots+n^3}{n^3+n^3+n^3+\ldots+n^3}\right) = \dfrac{1}{4}$.

However, do this in such a way that the results obtained in the chart above are, at the same time, verified. HINT: After substituting formulas for the numerator and denominator in the limit given above, there are two routes you can take: a) use long division or b) absorb 1/4 into the limit, get a common denominator, and then carry out the subtraction.

____

____

____

ACTIVITY 22 : $k = 4$

Complete the following chart.

k	2	3	4	5	6	7	8	9
C.Ratio of index k								

make predictions here

Verify your prediction numerically for $k = 4$ below.

n	$\frac{0^4+1^4+2^4+\ldots+n^4}{n^4+n^4+n^4+\ldots+n^4}$	SOLUTION AS A DECIMAL	SOLUTION AS A FRACTION	YOUR FRACTION IS
1	$\frac{0^4+1^4}{1^4+1^4}$			$\frac{1}{5}+\frac{}{}$
2	$\frac{0^4+1^4+2^4}{2^4+2^4+2^4}$			$\frac{1}{5}+\frac{}{}$
3	$\frac{0^4+1^4+2^4+3^4}{3^4+3^4+3^4+3^4}$			$\frac{1}{5}+\frac{}{}$
4	$\frac{0^4+1^4+2^4+3^4+4^4}{4^4+4^4+4^4+4^4+4^4}$			$\frac{1}{5}+\frac{}{}$
5	$\frac{0^4+1^4+2^4+3^4+4^4+5^4}{5^4+5^4+5^4+5^4+5^4+5^4}$			$\frac{1}{5}+\frac{}{}$
50	$\frac{0^4+1^4+2^4+\ldots+50^4}{50^4+50^4+50^4+\ldots+50^4}$		skip	$\frac{1}{5}+\frac{}{}$
5000	$\frac{0^4+1^4+2^4+\ldots+5000^4}{5000^4+5000^4+\ldots+5000^4}$		skip	$\frac{1}{5}+\frac{}{}$
n	$\frac{0^4+1^4+2^4+\ldots+n^4}{n^4+n^4+n^4+\ldots+n^4}$	skip		$\frac{1}{5}+\frac{}{}$

At this time, it would be wise to take a step back to see if the $k = 0$ and $k = 1$ cases fit the pattern that has been established in the $k =$ 2, 3, and 4 cases. But first, make a prediction as to what each characteristic ratio will be.

The C.R. of index $k = 0$ will be _____ and the C.R. of index $k = 1$ will be _____.

ACTIVITY 23 : $k = 0$ and $k = 1$

① $\dfrac{0^0 + 1^0}{1^0 + 1^0} = \dfrac{(\_)}{(\_)} = \_$ when $n = 1$

② $\dfrac{0^0 + 1^0 + 2^0}{2^0 + 2^0 + 2^0} = \dfrac{(\_)}{(\_)} = \_$ when $n = 2$

③ $\dfrac{0^0 + 1^0 + 2^0 + 3^0}{3^0 + 3^0 + 3^0 + 3^0} = \dfrac{(\_)}{(\_)} = \_$ when $n = 3$

⑦⓪ $\dfrac{0^0 + 1^0 + 2^0 + \ldots + 70^0}{70^0 + 70^0 + 70^0 + \ldots + 70^0} = \dfrac{(\_\_)}{(\_\_)} = \_$ when $n = \_\_\_\_$

$n)$ $\dfrac{0^0 + 1^0 + 3^0 + \ldots + n^0}{n^0 + n^0 + n^0 + \ldots + n^0} = \dfrac{(\_ + \_)}{(\_ + \_)} = \_$

CONCLUSION: The characteristic ratio of index $k = 0$ is ____

① $\dfrac{0^1 + 1^1}{1^1 + 1^1} = \dfrac{(\_)}{(\_)}$ when $n = 1$

② $\dfrac{0^1 + 1^1 + 2^1}{2^1 + 2^1 + 2^1} = \dfrac{(\_)}{(\_)} = \dfrac{(\_)}{(\_)}$ when $n = 2$

③ $\dfrac{0^1 + 1^1 + 2^1 + 3^1}{3^1 + 3^1 + 3^1 + 3^1} = \dfrac{(\_)}{(\_\_)} = \dfrac{(\_)}{(\_)}$ when $n = \_\_\_$

④ $\dfrac{0^1 + 1^1 + 2^1 + 3^1 + 4^1}{4^1 + 4^1 + 4^1 + 4^1 + 4^1} = \dfrac{(\_\_)}{(\_\_)} = \dfrac{(\_)}{(\_)}$ when $n = \_\_\_$

⑦⓪ $\dfrac{0^1 + 1^1 + 2^1 + \ldots + 70^1}{70^1 + 70^1 + 70^1 + \ldots + 70^1} = \dfrac{\frac{1}{2}[\_\_(\_\_+1)]}{(\_\_)(70)} = \dfrac{(\_\_\_\_)}{(\_\_\_\_)} = \dfrac{(\_)}{(\_)}$

$n)$ $\dfrac{0^1 + 1^1 + 3^1 + \ldots + n^1}{n^1 + n^1 + n^1 + \ldots + n^1} = \dfrac{\frac{1}{2}(\_)(\_ + \_)}{(\_)(\_ + \_)} = \dfrac{(\_)}{(\_)}$

ACTIVITY 24 : MORE ON WALLIS' C.R.

SUMMARY TO DATE

⓿ Wallis' characteristic ratio of index $k = 0$ is _______

❶ Wallis' characteristic ratio of index $k = 1$ is _______

❷ Wallis' characteristic ratio of index $k = 2$ is _______

❸ Wallis' characteristic ratio of index $k = 3$ is _______

❹ Wallis' characteristic ratio of index $k = 4$ is _______

DIRECTIONS: Define Wallis' characteristic ratio of index k in your own words.

____

____

____

EXERCISES: Verify that your definition above holds true for $k = 5$, $k = 6$, and $k = 7$ by just jumping to the general case and considering the limit as n approaches infinity. You will want to return to activity 16 for the necessary formulas. Show all your work below.

$k = 5$: ____

____

____

$k = 6$: ____

____

____

$k = 7$: ____

____

____

____

LECTURE 24A : GROUP WORK ON WALLIS' CHARACTERISTIC RATIO

Group work to be initiated by instructor at this point in time in the investigations

Have students list questions, concerns, discoveries, and possible directions of future investigations. Have them answer how one might expand on what has been learned so far. Have them list "what ifs."

Promising questions that might arise as a result of the above exercise.

-formula for general case: $\frac{1}{k+1}$?

-what happens when k is negative?

-what happens when k is a non-integer?

-can one change the interval by which each base in the numerator increases?

e.g. $$\frac{0^2+3^2+6^2+\ldots+(3n)^2}{(3n)^2+(3n)^2+(3n)^2+\ldots+(3n)^2}$$

or

e.g. $$\frac{0^2+(.1)^2+(.2)^2+\ldots+(n/10)^2}{(n/10)^2+(n/10)^2+(n/10)^2+\ldots+(n/10)^2}$$

Instructor's Response

-k negative or irrational will be taken care of by Newton later in these investigations

-changing the interval does not affect the value of the ratio (simply factor out the change)

-k fractional "let's investigate-next activity"

-formula being $\frac{1}{k+1}$? This can be verified through the upcoming investigations.

ACTIVITY 25 : A TI-81 PROGRAM FOR EXAMINING WALLIS' C.R. FOR FRACTIONAL k

Program for securing decimal approximations to ratios of the form

$$\frac{0^k+1^k+2^k+\ldots+n^k}{n^k+n^k+n^k+\ldots+n^k},\quad k>0,\quad n\in\{1,2,3,\ldots\}$$

```
:Lbl 2
:0→X
:Disp "N"
:Input N
:0→S
:Lbl 1
:S+Y1→S
:IS> (X,N)
:Goto 1
:Disp "INDEX k="
:Input k
:S/(N^K(N+1))→R
:Disp "APPX SUM ="
:Disp R
:Goto 2
```

-A SAMPLE SCREEN
($n = 20,\ k = 2/3$)

```
N
? 20
INDEX K =
? 2/3
APPX SUM =
         .594367346
N
?
```

☒NOTES ABOUT THIS PROGRAM

① Before running this program, you must enter the general term of the numerator in

$\frac{0^k+1^k+2^k+\ldots+n^k}{n^k+n^k+n^k+\ldots+n^k}$ into Y_1 (graphing portion of the calculator) as a function of x.

(i.e. let $Y_1 = X^{2/3}$ for our example above)

② The loop created with "Lbl 2" and "Goto 2" allows you to continue running this program for different values of n. To escape, you must "quit."

③ A nice feature includes what is displayed on the screen after one run, viz. everything needed. A sample screen after a run with $n = 20$ and $k = 2/3$ is illustrated above.

④ WARNING! A run with $n = 10{,}000$ and $k = 2/3$ takes about 10-13 minutes depending on the freshness of the batteries.

LECTURE 25A : INVESTIGATING RATIONAL k

INSTUCTOR'S NOTES:

-Hopefully students have realized that Alhazen's formulas will not help in the generalization to cases where k is not a whole number. Thus, the need for the TI-81 program in activity 25.

-The chart in activity 26 is meant to be completed by groups of students assigned just part of the chart. Each group gets a different part and then reports solutions to each of the other groups as they are attained.

-The chart in activity 26 should be completed in 1-2 class periods.

-The following is a "generic" investigation worksheet, meant to be completed with each new k.

GENERIC INVESTIGATION WORKSHEET

Group names:____

$k =$ ____ Prediction of the characteristic ratio of index k : ______

n	$\frac{0^k+1^k+2^k+\ldots+n^k}{n^k(n+1)}$	DECIMAL APPROXIMATION
1		
2		
3		
4		
5		
10		
100		
1000		
10,000(optional)		

☜Be aware that this may take awhile.

ACTIVITY 26 : INVESTIGATING RATIONAL k

k	Initial Prediction	Best appx. obtained (n)	Actual C.R. of index k	Initials of students
0		$1(n \to \infty)$	1	class
1/5				
1/4				
1/3				
2/5				
1/2		.599898605(n=1000)		
3/5				
2/3				
3/4				
4/5				
1		$1/2\ (n \to \infty)$	1/2	class
6/5				
5/4				
4/3				
7/5				
3/2		.4010022229(n=100)		
8/5				
5/3				
7/4				
9/5				
2		$1/3(n \to \infty)$	1/3	class
11/5				
9/4				
7/3				
12/5				
5/2				
13/5				
8/3				
11/4				
14/5				
3		$1/4(n \to \infty)$		class
7/2				
4		$1/5(n \to \infty)$		class
9/2				
5		$1/6(n \to \infty)$		class
11/2				

LECTURE 26A : OBJECTIVES (CHECK-UP)

MAIN OBJECTIVE OF INVESTIGATIONS INTO JOHN WALLIS' C.R.

To have students realize that the formula for John Wallis' characteristic ratio of index k ($k > 0$ and rational) is $1/(k+1)$ and that this relates directly back to

$$\int_0^1 x^k dx, \text{ and more generally to } \int_0^b x^k dx = \frac{b^{k+1}}{k+1}$$

SECONDARY OBJECTIVES

1) Students discover that the above is true $\forall\ k > 0$ (without proof).

2) Students investigate $k = 0$ or $k < 0$ using their program and realize this creates an "error" in the calculations. This can be remedied by storing .1 into X (line 2 of program) instead of 0. Students who have discovered that Wallis' characteristic ratio of index k relates directly to the area under the curve x^k from $x = 0$ to $x = 1$ may realize this on their own. That would be fantastic!

3) Students recognize that when $0 < k < 1$, approximations come from below, and vice-versa for $k > 1$.

ASSIGNMENT

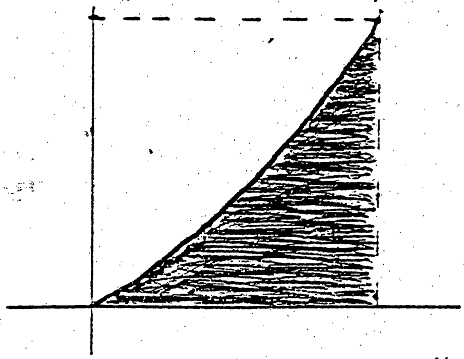
Assignment to find out if these objectives have been met (and anything else that may have been discovered):

> Write a 1-3 page essay outlining anything discovered or even suspected in relation to these investigations.

LECTURE 26B : HOW WALLIS TIED HIS CHARACTERISTIC RATIO TO AREA UNDER THE CURVE X^k

DEFINITION: Wallis' characteristic ratio of index k is the ratio of the area under x^k (or for that matter, under cx^k $\forall c \neq 0$) to the area of the rectangle containing the curve.

e.g.



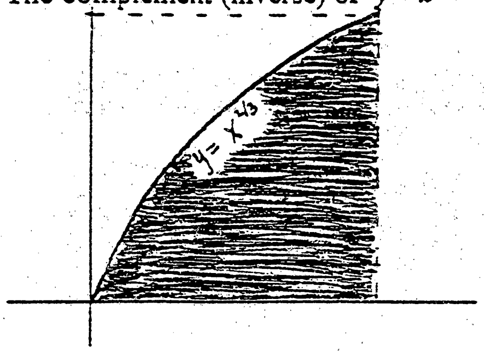
$y = x^{3/2}$

Area under curve = 2/5

Area of rectangle = 1

Ratio = 2/5

e.g. The complement (inverse) of $y = x^{3/2}$

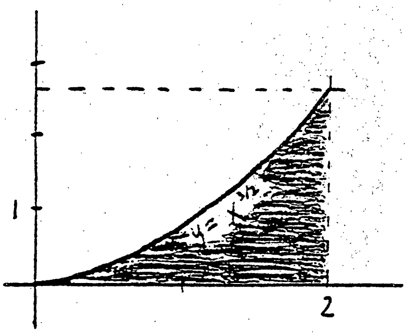


Area under curve = 3/5

Area of rectangle = 1

Ratio = 3/5

e.g.

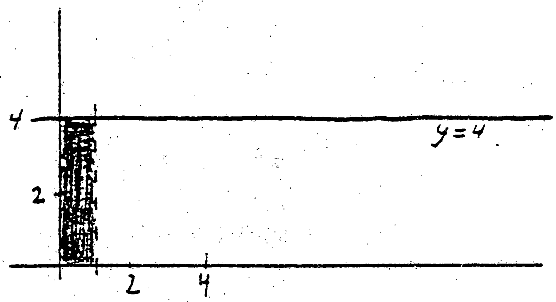


Area under curve = $(2/5)(2^{5/2})$

Area of rectangle = $2^{5/2}$

Ratio = 2/5

e.g.



Area under curve = 4

Area of rectangle = 4

Ratio = 1

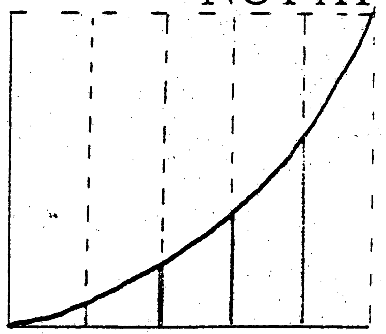
LECTURE 26C : HOW WALLIS VIEWED THE CONCEPT OF AREA (INTEGRAL NOTATION WAS NOT AT HIS DISPOSAL)

e.g.

$n = 5$

$y = x^2$

$\therefore k = 2$



Wallis' notion of area was taken from Cavalieri; That area is the sum of an infinite number of parallel line segments. The length of the six line segments erected above (the first segment is the one at 0, length is 0^2, the second is at .2, length is $.2^2$, etc.), when summed, yield the numerator of Wallis' characteristic ratio.

i.e. $0^2 + .2^2 + .4^2 + .6^2 + .8^2 + 1^2 = .2^2(0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2)$

The sum of the lengths of the six segments corresponding to the above segments which make up the rectangle the curve lies within is

$$1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = .2^2(5^2 + 5^2 + 5^2 + 5^2 + 5^2 + 5^2)$$

Thus, the ratio of the area under the curve (approximated since $n = 5$) to the area of the proper rectangle is $\dfrac{.2^2(0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2)}{.2^2(5^2 + 5^2 + 5^2 + 5^2 + 5^2 + 5^2)} \approx .3\overline{6}$

The students will recognize how this concept can be directly related to the modern-day concepts of finding area under a curve (i.e. inscribed and circumsribed rectangles).

EXERCISES (based on material covered in the previous two lectures):

A) Choose two different characteristic ratios and their respective indices for each of these, show a result like that which was shown in Lecture 26B.

B) Explain why negative indices will not work in the situation presented in lecture 26B.

C) Show how the ratio of the area of any right triangle with base b and height h to the area of the rectangle with base b and height h is 1/2 using Wallis' concept of area.

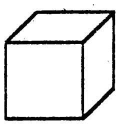
D) Choose a curve and its complement and do with each what was done in the example above. For each curve, let $n = 8$.

ACTIVITY 27 : CONCLUSION TO WALLIS' CHARACTERISTIC RATIO

HOW WALLIS' CHARACTERISTIC RATIO OF INDEX $k = 2$ CAN BE USED TO SHOW THAT THE VOLUME OF A PYRAMID IS 1/3 OF THE BOX THAT CONTAINS IT.

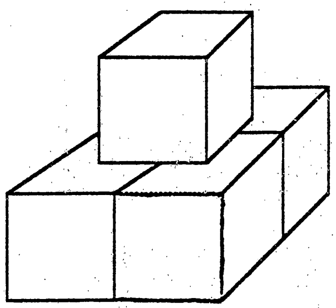
DIRECTIONS: Build the following using sugar cubes. Then continue the pattern on your own until you run out of sugar cubes.

❶ VOLUME OF PYRAMID : VOLUME OF BOX



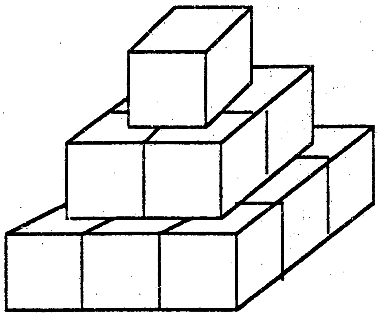
$1 : 1$

❷



$1^2 + 2^2 : 2^2 + 2^2$

❸



$1^2 + 2^2 + 3^2 : 3^2 + 3^2 + 3^2$

❹ You can picture the drawing in your head far better than I can draw it.

CONCLUSIONS:

____

____

____

SET IV

WALLIS' REPRESENTATION OF π AS AN INFINITE PRODUCT

ACTIVITY 28 : Curves of the form $\left(1-x^{1/q}\right)^p$

Background notes and summary:

After extensive work with the formulas derived by Alhazen and the ratio

$\frac{0^k+1^k+2^k+\ldots+n^k}{n^k+n^k+n^k+\ldots+n^k}$ and, after determining that the value of this ratio as n gets large tends

to $\frac{1}{k+1}$, John Wallis had the confidence that he could solve the age-old problem of "squaring the circle." That is, he felt that he could use his notion of characteristic ratio along with interpolation to derive a constructible value for $\sqrt{\pi}$, one which would allow for the construction with ruler and compass of a square having the same area of a circle with, say, radius equal to one. Note that if $\sqrt{\pi}$ were constructable, then one could duplicate the area of the unit circle by constructing a square whose sides each measured $\sqrt{\pi}$.

Wallis knew that to solve this problem, he would have to find the area of the unit circle. It should be noted here that the method for approximating a given circle's area employed by mathematicians at this time was that which was developed by Archimedes in the third century BC. Enhanced by Eudoxus, it became known as the "method of exhaustion." For nearly two thousand years, mathematicians failed at their attempts to "square the circle" using the Greeks' method of exhaustion. Wallis felt he had a fresh approach to this problem and began his venture by considering the equation of the unit circle, namely,
$x^2+y^2=1$. He decided to work with a family of curves represented by the equation $y=\left(1-x^{1/q}\right)^p$. Observe that the upper half of the unit circle ($p = q = 1/2$) is one member of this family.

EXERCISE 1: Sketch various members of the family of curves discussed above for whole number values of p and q. Limit the portions of the curves sketched to the unit square and be "organized," as a code breaker would be, in your approach to the sketching of these curves. Write your observations concerning this activity in the space below.

ACTIVITY 29: EXPANDING $\left(1-x^{\frac{1}{q}}\right)^p$

The most important observation which should have been made in the previous activity is the fact that $y=\left(1-x^{\frac{1}{q}}\right)^p$ is symmetric to $y=\left(1-x^{\frac{1}{p}}\right)^q$ about the line $y=x$. Did you happen to make this observation? _______ Of course, symmetry with respect to the line $y=x$ occurs for graphs of inverse functions, so a good exercise is to check that $y=\left(1-x^{\frac{1}{q}}\right)^p$ and $y=\left(1-x^{\frac{1}{p}}\right)^q$ are inverse functions. Just solve one of these for x in terms of y, interchange the variables x and y, and you have the other. Wallis also realized the symmetry that occurs when positive whole number values of p and q are interchanged. This led him to the development of the following chart (the directions for filling out this chart will be given on the next page):

p / q	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

In the previous set of activities, you learned that Wallis' characteristic ratio of index k is $\frac{1}{k+1}$, and this repesents the value of the ratio $\frac{0^k+1^k+2^k+...+n^k}{n^k+n^k+n^k+...+n^k}$ as n approaches infinity. You also learned, as he did, that this characteristic ratio is directly related to the ratio of the area under the curve $y=x^k$ to the area of the rectangle containing the curve.

(continued next page)

With these ideas in mind, Wallis filled in the grid above with values which represented the ratios of the areas under the curves $y=\left(1-x^{1/q}\right)^{p}$ to the areas of the rectangles containing them. He did this using algebraic expansion on curves where p is a natural number. The manner in which Wallis determined the values to be entered in the chart is illustrated below.

Consider the curve $y=\left(1-x^{1/q}\right)^{p}$ where $q=3$ and $p=2$. Then, via expansion, $y=\left(1-x^{1/3}\right)^{2}=1-2x^{1/3}+x^{2/3}=1x^{0}-2x^{1/3}+x^{2/3}$. Wallis claimed that the ratio of the area under this curve to the area of the rectangle that contains it could be found by summing the characteristic ratios of each term in the expansion. Thus, the desired ratio is

1 times the characteristic ratio of x^{0}
minus 2 times the characteristic ratio of $x^{1/3}$
plus 1 times the characteristic ratio of $x^{2/3}$.

That is, $1\left(\frac{1}{0+1}\right)-2\left(\frac{1}{1/3+1}\right)+1\left(\frac{1}{2/3+1}\right)=\frac{1}{10}$

DIRECTIONS: Complete the chart on the previous page row by row, beginning with the $q=1$ row, using the method of expansion employed by Wallis as illustrated above. ✍ Keep an eye out for patterns. If you recognize one, say, by the time you get to the $p=4$ column, feel free to skip the expansions required for the $p=5$, $p=6$, and $p=7$ columns, predict a value for the $p=8$ column, and then verify your prediction through expansion.

✉ Please note that you are not expected to carry out all 64 expansions in the chart. The sooner you recognize a pattern, the less work you'll have to do. In fact, you will recognize the desired patterns more quickly if you <u>invert the values</u> obtained by expansion. Therefore, for the example above, the entry which should be placed in the $q=3$, $p=2$ box is 10 rather than 1/10.

Finally, make a list below of <u>any</u> patterns you recognize as you carry out the expansions and enter the values representing the characteristic ratios of the curves of the form $y=\left(1-x^{1/q}\right)^{p}$ into the chart.

____

____

____

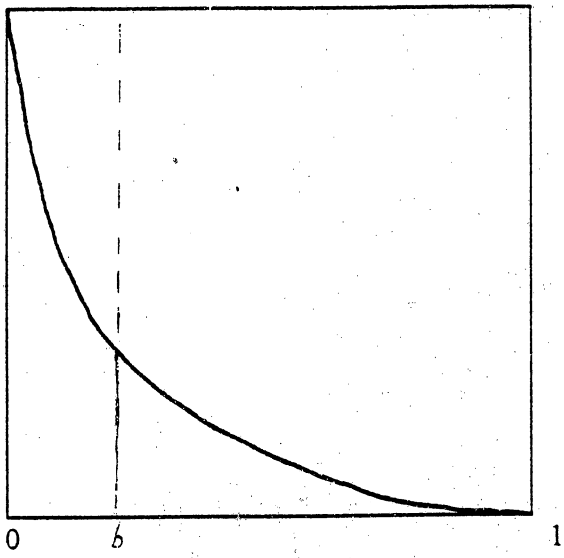
____

____

____

LECTURE 29A: WHY WALLIS' METHOD OF EXPANSION WORKS

EXAMPLE: $y = \left(1 - x^{1/3}\right)^2 = 1 - 2x^{1/3} + x^{2/3} = 1x^0 - 2x^{1/3} + x^{2/3}$



Setting up Wallis' characteristic ratio for this curve yields the following ratio:

$$\frac{\text{lengths of } n \text{ parallel line segments drawn to curve}}{\text{lengths of } n \text{ parallel line segments making up rectangle}}$$

$$= \frac{\left[1-\left(\frac{0}{n}\right)^{1/3}\right]^2 + \left[1-\left(\frac{1}{n}\right)^{1/3}\right]^2 + \left[1-\left(\frac{2}{n}\right)^{1/3}\right]^2 + \ldots + \left[1-\left(\frac{n}{n}\right)^{1/3}\right]^2}{1^2+1^2+1^2+\ldots+1^2}$$

$$= \frac{1^2 - 2\left(\frac{0}{n}\right)^{1/3} + \left(\frac{0}{n}\right)^{2/3} + 1^2 - 2\left(\frac{1}{n}\right)^{1/3} + \left(\frac{1}{n}\right)^{2/3} + 1^2 - 2\left(\frac{2}{n}\right)^{1/3} + \left(\frac{2}{n}\right)^{2/3} + \ldots + 1^2 - 2\left(\frac{n}{n}\right)^{1/3} + \left(\frac{n}{n}\right)^{2/3}}{1^2+1^2+1^2+\ldots+1^2}$$

$$= \frac{1^2+1^2+1^2+\ldots+1^2}{1^2+1^2+1^2+\ldots+1^2} - \frac{2\left(0^{1/3}+1^{1/3}+2^{1/3}+\ldots+n^{1/3}\right)}{n^{1/3}+n^{1/3}+n^{1/3}+\ldots+n^{1/3}} + \frac{0^{2/3}+1^{2/3}+2^{2/3}+\ldots+n^{2/3}}{n^{2/3}+n^{2/3}+n^{2/3}+\ldots+n^{2/3}}$$

$$= 1 - 2\left(\frac{1}{1/3+1}\right) + \frac{1}{2/3+1}$$

$$= \frac{1}{10}$$

ACTIVITY 30 : ABANDONING AREA NOTION IN FAVOR OF INTERPOLATION

Due to the fact that Wallis was incapable of expanding the expression $\left(1-x^{1/q}\right)^p$ for fractional values of p, he was forced to abandon this method for finding characteristic ratios of these curves in favor of interpolation.

DIRECTIONS: Use the patterns developed in the foregoing activity along with interpolation to complete as much of the chart (located on the next page) as possible.

☒You may want to return to activity #16 to obtain the formulas necessary for correct interpolation. Keep in mind the symmetrical features of the chart and be aware of the slight twist that is required prior to using the formulas from activity #16. An example follows:

The known entries of the $q=2$ row are 3, 6, 10, 15, 21, 28, 36, 45, and 55. It was established earlier in these activities that these are the "triangular numbers." Thus, it would make sense to employ the formula for the nth triangular number, which is

$\frac{n(n+1)}{2}$. However, 3 is the 2nd triangular number, and it is situated in the $p=1$ column. Also, 6 is the 3rd triangular number, situated in the $p=2$ column. Hence, to get the value required in the $p=3$ column of the $q=2$ row, one would need to use

$4=p+1$ in the formula $\frac{n(n+1)}{2}$ in order to obtain the 4th triangular number.Further, to get the value required in the $p=4$ column of the $q=2$ row, one would need to use

$5=p+1$ in the formula $\frac{n(n+1)}{2}$ in order to obtain the 5th triangular number. Once more, to get the value required in the $p=5$ column of the $q=2$ row, one would need

to use $6=p+1$ in the formula $\frac{n(n+1)}{2}$ in order to obtain the 6th triangular number.This is the "slight twist" discussed above. More importantly, to find the value of, say, the $q=2, p=5/2$ entry, one would need to use $5/2+1=7/2$ as n in the formula given above. This yields 63/8 as a solution, which is the proper value for the $q=2$, $p=5/2$ box in the chart.

IMPORTANT: The exact fractional solutions to entries in the chart are necessary for future interpolations. You do not need to reduce the fractions. In fact, reducing will only make it harder to recognize patterns.

✓ If you would like to check your solutions as you complete the chart on the next page, you may use the "integral" program designed for the TI-81 calculator. This program is also on the adjacent page.

Program for checking the entries to the chart below:

Prgm: AREA

This program uses "midpoint approximations" to approximate the definite integral $\int_0^1 f(x)dx$ where $f(x) = \left(1 - x^{1/q}\right)^p$ for any particular p and q.

```
: 0 → A
: 1 → B
: Lbl 1
: Disp "N"
: Input N
: .5(B-A)/N → H
: 0 → M
: 0 → K
: Lbl 2
: A+(2K+1)H → X
: M+2HY1 → M
: IS>(K,N-1)
: Goto 2
: Disp "MIDPT APPX TO    AREA IS"
: Disp M
: Goto 1
```

☒ NOTE: The curve whose area is being approximated must be entered into Y_1 of the graphing portion of the calculator.

☒ Also note that entries in the chart below represent reciprocals of the actual area under the curve over the interval from 0 to 1 on the x-axis. Be aware of this fact when you do decide to check an entry in the chart with this "area" program.

CHART IS LOCATED ON THE NEXT PAGE

In the space below, write down any patterns, methods, or formulas used to determine missing entries.

p / *q*	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5
0											
1/2											
1			2		3		4		5		6
3/2											
2			3		6		10		15		21
5/2											
3			4		10		20		35		56
7/2											
4			5		15		35		70		126
9/2											
5			6		21		56		126		252

ACTIVITY 31 : SUMMARY OF RESULTS

p / q	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5
0	**1**	1	**1**	1	**1**	1	**1**	1	**1**	1	**1**
1/2	1		3/2		15/8		105/48		945/384		10395/3840
1	**1**	3/2	**2**	5/2	**3**	7/2	**4**	9/2	**5**	11/2	**6**
3/2	1		5/2		35/8		315/48		3465/384		45045/3840
2	**1**	15/8	**3**	35/8	**6**	63/8	**10**	99/8	**15**	143/8	**21**
5/2	1		7/2		63/8		693/48		9009/384		135135/3840
3	**1**	105/48	**4**	315/48	**10**	693/48	**20**	1287/48	**35**	2145/48	**56**
7/2	1		9/2		99/8		1287/48		19305/384		328185/3840
4	**1**	945/384	**5**	3465/384	**15**	9009/384	**35**	19305/384	**70**	36465/384	**126**
9/2	1		11/2		143/8		2145/48		36465/384		692835/3840
5	**1**	10395/3840	**6**	45045/3840	**21**	135135/3840	**56**	328185/3840	**126**	692835/3840	**252**

SOMETHING TO PONDER: Try your best at answering the following questions. What should the plan of attack be from here on? What do the **bold-faced** numbers represent? Why is it possible for 1's to exist across the $q = 0$ row and down the $p = 0$ column? Will arithmetic average work to get the entries for any remaining boxes? Which box represents the reason all this work in the chart was undertaken in the first place?

ACTIVITY 32: ATTACKING THE $q = 1/2$ ROW

How did you do in answering the questions at the end of the previous activity?

A) What should the plan of attack be from here on?
ANSWER: Try to get the solutions to the first two empty boxes in the $q = 1/2$ row. If this can be done, then because of the symmetry and the fact that each entry in the table is the sum of the entries of the boxes two up from and two to the left of the one in question, the entire chart can be affected.

B) What do the **bold-faced** numbers represent?
ANSWER: The entries of Pascal's Triangle. Turn the grid 45° clockwise and you will notice Pascal's Triangle.

C) Why is it possible for 1's to exist in both the first row and column of the chart?
ANSWER: When $p = 0$, the curve is constant line $y = 1$ since anything to the 0th power is 1. Thus, the ratio of the area of the rectangle to the area under this curve is 1. When $q = 0$, one must consider the limit of the curve

$y = \left(1 - x^{1/q}\right)^p$ as $q \to 0$. Try seeing what this curve approaches as q approaches 0 for some fixed $p>0$ using your TI-81 calculator. Describe your observations below.

____

____

____

D) Will arithmetic average work to get the entries for any remaining boxes?
ANSWER: No. This breaks down immediately in any row or column with missing entries.

E) Which box represents the reason all this work in the chart was undertaken in the first place?
ANSWER: The $p = q = 1/2$ box. It's the box that depicts the ratio of the area of the unit square to the area of one-fourth of the unit circle. What is this ratio?

____

☒ Wallis did not know the answer to question C. He simply applied his interpolation scheme and concluded that both the first row and column must contain 1's because of patterns developed through expansion. However, he did know the answer to question E (along with the others). Because the area of a circle was known to be πR^2 at this time, Wallis knew that the ratio of the area of the unit square to the area of one-fourth the unit circle was $4/\pi$. But remember that his goal was to find an exact value for this ratio, so that the problem of "squaring the circle" could be resolved once and for all. Thus, he continued interpolating in the manner illustrated on the next page.

Wallis used the symbol □ to represent the value of the $p = q = 1/2$ entry in the chart. He then realized a pattern which existed in the $q = 1/2$ row and this in turn led to the filling in of the remaining boxes, each in terms of □. Take a close look now at the $q = 1/2$ row below and see if you can spot the same pattern Wallis did.

$\frac{1}{1}$	□	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{985}{384}$

DON'T LOOK BELOW UNTIL YOU "GIVE UP" ON TRYING TO ESTABLISH A PATTERN ABOVE! Hints follow, but the goal has to be to use as few of the hints below as possible.

HINT #1

What do you multiply by to get from one known entry to another? Try your best on finding all the unknown entries above given this hint before moving on to the next hint.

HINT #2

Row $q = 1/2$ can be changed to

$\frac{1}{1}$	□	$\frac{1\cdot 3}{1\cdot 2}$		$\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 4}$		$\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}$		$\frac{3\cdot\_\cdot\_\cdot\_}{\_\cdot\_\cdot\_\cdot\_}$

HINT #3

$\frac{1}{1}$	□	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{985}{384}$

To get from the first box to the third box, one must multiply by 3/2.

To get from the third box to the fifth box, one must multiply by __/4.

To get from the fifth box to the seventh box, one must multiply by 7/__.

To get from the seventh box to the ninth box, one must multiply by __/__.

(continued next page)

HINT #4

$\frac{1}{1}$	□	$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{985}{384}$

To get from the first box to the third box, one must multiply by 3/2.

To get from the second box to the fourth box, one must multiply by __/__.

To get from the third box to the fifth box, one must multiply by 5/4.

To get from the fourth box to the sixth box, one must multiply by __/__.

To get from the fifth box to the seventh box, one must multiply by 7/6.

To get from the _________ box to the_________ box, one must multiply by __/__.

To get from the seventh box to the ninth box, one must multiply by __/__.

DIRECTIONS: Complete the chart below using the patterns found above and symmetry.

p / q	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5
0	1	1	1	1	1	1	1	1	1	1	1
1/2	1	□	3/2	$\frac{4}{3}$□	15/8		105/48		945/384		$\frac{10395}{3840}$
1	1	3/2	2	5/2	3	7/2	4	9/2	5	11/2	6
3/2	1		5/2		35/8		315/48		$\frac{3465}{384}$		$\frac{45045}{3840}$
2	1	15/8	3	35/8	6	63/8	10	99/8	15	143/8	21

ACTIVITY 33 : THE DAWN BEFORE THE SQUEEZE

DIRECTIONS: Complete the table below. Recall that each entry is the sum of the entries two boxes up from and two boxes to the left of that particular entry.

q \ p	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5
0	1	1	1	1	1	1	1	1	1	1	1
1/2	1	□	3/2	4/3 □	15/8	8/5 □	105/48	64/35 □	945/384	640/315 □	10395/3840
1	1	3/2	2	5/2	3	7/2	4	9/2	5	11/2	6
3/2	1	4/3 □	5/2	8/3 □	35/8	64/15 □	315/48		3465/384		45045/3840
2	1	15/8	3	35/8	6	63/8	10	99/8	15	143/8	21
5/2	1	8/5 □	7/2	64/15 □	63/8		693/48		9009/384		135135/3840
3	1	105/48	4	315/48	10	693/48	20	1287/48	35	2145/48	56
7/2	1	64/35 □	9/2		99/8		1287/48		19305/384		328185/3840
4	1	945/384	5	3465/384	15	9009/384	35	19305/384	70	36465/384	126
9/2	1	640/315 □	11/2		143/8		2145/48		36465/384		692835/3840
5	1	10395/3840	6	45045/3840	21	135135/3840	56	328185/3840	126	692835/3840	252

ACTIVITY 34 : THE SQUEEZE

It's time to take a deep breath and realize that you are about to conclude activities with John Wallis' scheme that, quite arguably, conquered the infinite. The only task left undone is that which determines the value of □. Now, as stated earlier, Wallis knew that the value of □ was $4/\pi$. However, he was one of the best "code breakers" or "cryptographers" of the 1600's. The chart unraveled here was a code to him, a code which was on the verge of being broken. Thus, Wallis was not about to dispose of all the work done to get this far in the table and settle for □ being $4/\pi$. He wanted to find out the value of □ using the skills of pattern recognition, interpolation, induction, and generalization that had worked to get him this far. Wallis was determined to find □ in his own unique way and the brilliant manner in which he did this is the goal of this activity.

Since this interpolation scheme arguably conquered the stranglehold the Greeks' "horror of the infinite" had on the mathematical world, let's pay one final tribute to early Greek mathematics by using Greek letters to name the entries of the $q = 1/2$ row. A carefully constructed list of these is required to understand the steps necessary for finding □. Thus,

$\alpha = 1$

$\beta = \square$

$\chi = \frac{3}{2}$

$\delta = \frac{4}{3}\square$

$\varepsilon = \frac{15}{8} = \frac{3\cdot 5}{2\cdot 4}$

$\phi = \frac{24}{15}\square = \frac{4\cdot 6}{3\cdot 5}$

$\varphi = \frac{105}{48} = \frac{3\cdot\_\cdot 7}{2\cdot 4\cdot\_}$

$\gamma = \frac{192}{105}\square = \frac{4\cdot\_\cdot\_}{\_\cdot\_\cdot 7}$

$\eta = \frac{945}{384} = \frac{\_\cdot\_\cdot 7\cdot\_}{2\cdot\_\cdot\_\cdot\_}$

$\iota = \frac{1920}{945}\square = \frac{\_\cdot\_\cdot\_\cdot\_}{\_\cdot\_\cdot\_\cdot\_}$

DIRECTIONS:
Continue the list constructed in column one by finding the next four entries in the space provided below.

κ = ______________________________

____

λ = ______________________________

____

μ = ______________________________

____

ν = ______________________________

____

The critical observation that must be made at this time lies in the fact that the sequence $\alpha, \beta, \chi, \delta, \varepsilon, \phi, \varphi, \gamma, \eta, \iota, \kappa, \lambda, \mu, \nu, o, \pi, \varpi, \theta, \vartheta, \rho, \sigma, \varsigma, \tau, \upsilon, \omega, \xi, \psi, \zeta, \ldots$ is monotonically increasing. This means that $\alpha < \beta,\ \beta < \chi,\ \chi < \delta,\ \delta < \varepsilon, \ldots$ Therefore, an entry in terms of $\square$ can be "squeezed" between the entries on each side of it. The "squeezing" of $\square$ follows.

SQUEEZE #1

$$1 < \square < \frac{3}{2}$$

SQUEEZE #2

$$\frac{3}{2} < \frac{4}{3}\square < \frac{3\cdot5}{2\cdot4} \quad\Rightarrow\quad \frac{3\cdot3}{2\cdot4} < \square < \frac{3\cdot3\cdot5}{2\cdot4\cdot4}$$

↗via multiplying through by 4/3↖

SQUEEZE #3

$$\frac{3\cdot5}{2\cdot4} < \frac{4\cdot6}{3\cdot5}\square < \frac{3\cdot5\cdot7}{2\cdot4\cdot6} \quad\Rightarrow\quad \frac{\_\cdot\_\cdot\_\cdot\_}{\_\cdot\_\cdot\_\cdot\_} < \square < \frac{\_\cdot\_\cdot\_\cdot\_\cdot\_\cdot\_}{\_\cdot\_\cdot\_\cdot\_\cdot\_\cdot\_}$$

SQUEEZE #4

$$\frac{3\cdot5\cdot7}{2\cdot4\cdot6} < \frac{4\cdot6\cdot8}{3\cdot5\cdot7}\square < \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8} \quad\Rightarrow\quad \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7}{2\cdot4\cdot4\cdot6\cdot6\cdot8} < \square < \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9}{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8}$$

↳ON YOUR OWN: Complete the next five "squeezes" on $\square$. Then, try to put together an argument as to what the value of $\square$ must be. You will have to picture carrying out the squeezing process to infinity. Look for a way to argue that $\square$ must be between two fractions that are equal, since $a < b < c$ and $a = c$ implies $b = a = c$. The key to this involves the fact that something tends to 1 as you go farther out in the "squeezing" process. A way to see just what does tend to 1 is by focusing on the factor by which the two numbers surrounding $\square$ differ. Finally, the value of $\square$ will be an infinite product. More specifically, it will be a fraction where both the numerator and denominator are infinite products.

LECTURE 34A: π AS AN INFINITE PRODUCT

If we look closely at the fourth squeeze,

$$\frac{3\cdot5\cdot7}{2\cdot4\cdot6} < \frac{4\cdot6\cdot8}{3\cdot5\cdot7}\square < \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8} \quad\Rightarrow\quad \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7}{2\cdot4\cdot4\cdot6\cdot6\cdot8} < \square < \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9}{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8},$$

it is easy to see that the two fractions which surround $\square$ differ by a factor of 9/8. After carrying out the subsequent "squeeze," it will be easy to see that the fractions surrounding $\square$ differ by a factor of 11/10. Next, they will differ by a factor of 13/12. The factor by which the two fractions on either side of $\square$ differ approaches 1 as the "squeezing" process is taken to infinity. Therefore,

$$\square = \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11\cdot\ \ldots}{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12\cdot\ \ldots}$$

$$\Rightarrow \pi = 2\left(\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot\ \ldots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot\ \ldots}\right)$$ since it was already known that $\square = 4/\pi$.

This representation of π as an infinite product was the first of its kind. It bears John Wallis' name and is the benchmark of his notoriety. Wallis' explorations into the realms of the infinite with uncanny analytic tactics led to increased attention being given this methodology. In turn, attention shifted away from the much more burdensome, geometrical means for understanding infinite processes. Although the results obtained here by Wallis were not formally proved until the nineteenth century, they had a profound effect on Sir Isaac Newton.

Newton modeled Wallis' interpolation procedure in his discovery of the binomial series. He too was not able to prove his results rigorously. However, the mere discovery of the binomial series played a profound role in substantiating the use of infinite series as a tool for working with infinite processes and limit theory. The set of activities that follow will take you through Newton's interpolation scheme.

One final note about Wallis' investigations hinges on the frequently unexpected nature of mathematical invention. Wallis began his mystical and very original interpolation scheme with the confidence that he could either prove or disprove the problem of "squaring the circle." Though he could not accomplish this goal, the discoveries afforded him in his quest are certainly no less valuable.

SET V

NEWTON'S DISCOVERY OF THE BINOMIAL SERIES

ACTIVITY 35 : NEWTON AFTER WALLIS

In 1661, at nineteen years of age, Isaac Newton read John Wallis' *Arithmetica Infinitorum* and subsequently began a venture that, by 1665, led to his first mathematical discovery of lasting significance. This was the formulation of the binomial series illustrated below.

$$(1+x)^\alpha = 1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\ldots = 1+\sum_{n=1}^{\infty}\binom{\alpha}{n}x^n \quad \text{where} \quad \binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdot\ \ldots\ \cdot(\alpha-n+1)}{n!}$$

Influenced by Wallis' interpolation technique and, at the same time frustrated by geometrical means by which to calculate the area under a hyperbola, Newton extended Wallis' table to the negative side with confidence that he too could utilize Wallis' methodology and effectively devise a way to obtain area under a hyperbola. He constructed a table similar in design to Wallis', but which instead depicted the area brought on by each term following expansion. For brevity, let's use modern notation for areas under curves represented by expressions of the form $y=(1+x)^p$.

0) $$\int_0^x (1+t)^0 dt = \int_0^x 1dt = \left.\frac{t}{1}\right|_0^x = 1\left(\frac{x}{1}\right)$$

1) $$\int_0^x (1+t)^1 dt = \left.\frac{t}{1}+\frac{t^2}{2}\right|_0^x = \frac{x}{1}+\frac{x^2}{2}$$

2) $$\int_0^x (1+t)^2 dt = \int(1+2t+t^2)dt = \left.\frac{t}{1}+2\left(\frac{t^2}{2}\right)+\frac{t^3}{3}\right|_0^x = \frac{x}{1}+2\left(\frac{x^2}{2}\right)+\frac{x^3}{3}$$

3) $$\int_0^x (1+t)^3 dt = \left.\frac{t}{1}+3\left(\frac{t^2}{2}\right)+3\left(\frac{t^3}{3}\right)+\frac{t^4}{4}\right|_0^x = \frac{x}{1}+3\left(\frac{x^2}{2}\right)+3\left(\frac{x^3}{3}\right)+\frac{x^4}{4}\Bigg|$$

The area under the curve $(1+t)^3$ is of course dependent on the choice of x. But without regards to x, we can say that the area under this curve is 1 times the $\frac{x}{1}$ term, plus 3 times the $\frac{x^2}{2}$ term, plus 3 times the $\frac{x^3}{3}$ term, plus 1 times the $\frac{x^4}{4}$ term.

Continue this sequence of problems by completing the cases where $p = 4$, $p = 5$, and $p = 6$, and then finish the right side of Newton's chart shown in the next activity. Remember that entries in the chart are the coefficients for the terms in the area expressions derived after definite integration.

ACTIVITY 36 : NEWTON'S TABLE FOR $(1+x)^p$

DIRECTIONS: Complete the right side of this chart by determining the coefficients of the terms $\frac{x}{1}, \frac{x^2}{2}, \frac{x^3}{3}, \ldots$ brought on by definite integration of $\int_0^x (1+t)^p\,dt$ with $p = 4, 5,$ and 6.

Next, note that the $p = -1$ column represents the area under a hyperbola $\frac{1}{1+x}$. This is the curve Newton was having trouble with. Just what is the value of $\int_0^x (1+t)^{-1}\,dt$?

Yes, you're correct, it is ln $(1+x)$. But this fact was unknown to Newton. So instead, he recognized that the columns of his table were the diagonals of Wallis' table and, consequently, that each entry is the sum of the entry to the left of it and one up from that one. Now use this binomial pattern of formulation to fill in the empty boxes on the left side of this chart. Before doing this however, you must assume, as Newton did, that the entire top row remains constant at 1.

p / term	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x}{1}$							1	1	1	1	1	1	1
$\frac{x^2}{2}$							0	1	2	3	4	5	6
$\frac{x^3}{3}$							0	0	1	3			
$\frac{x^4}{4}$							0	0	0	1			
$\frac{x^5}{5}$							0	0	0	0			
$\frac{x^6}{6}$							0	0	0	0			
$\frac{x^7}{7}$							0	0	0	0			

LECTURE 36A: NEWTON'S CHART

With the completion of the chart below, Newton was able to formulate an expression for the area (by way of placing coefficients on respective terms in the table) under the

hyperbola $\frac{1}{1+x}$, namely, for $x > 0$, area $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \ldots$
We now call this area function the natural logarithm of $(1 + x)$. Following much detailed work with this area function, Newton realized its logaritmic properties and thus it could be used to manufacture tables of common logarithms.

p / term	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\frac{x^2}{2}$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x^3}{3}$	21	15	10	6	3	1	0	0	1	3	6	10	15
$\frac{x^4}{4}$	-56	-35	-20	-10	-4	-1	0	0	0	1	4	10	20
$\frac{x^5}{5}$	126	70	35	15	5	1	0	0	0	0	1	5	15
$\frac{x^6}{6}$	-252	-126	-56	-21	-6	-1	0	0	0	0	1	6	21
$\frac{x^7}{7}$	462	210	84	28	7	1	0	0	0	0	1	7	28

Having addressed and then solved the problem of finding area under a hyperbola through the extensive work done here, which allows for the calculation of the area under any member in the family of curves denoted by $(1+x)^p$ for any integer p, Newton engaged himself in the same family of curves attacked by Wallis in his $q = 1/2$ row, that is, $\left([1-x^2]^p\right)$. He concerned himself with areas over the interval $[1-x^2, 1]$ and, from continually returning to Wallis' work on characteristic ratios together, with an ingenious way to determine solutions to missing entries by solving a system of linear equations composed from known entries, he was able to constuct the table shown on the next page.

LECTURE 36B: A NEW WAY TO CALCULATE π

p term	-1	$\frac{-1}{2}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{-x^3}{3}$	-1	$\frac{-1}{2}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$\frac{-1}{9}$	$\frac{-1}{8}$	$\frac{-1}{9}$	0	$\frac{2}{9}$	$\frac{3}{8}$	$\frac{5}{9}$	1
$\frac{-x^7}{7}$	-1	$\frac{-5}{16}$	0	$\frac{5}{81}$	$\frac{3}{48}$	$\frac{4}{81}$	0	$\frac{-4}{81}$	$\frac{-1}{16}$	$\frac{-5}{81}$	0
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$\frac{-10}{243}$	$\frac{-15}{384}$	$\frac{-7}{243}$	0	$\frac{5}{243}$	$\frac{3}{128}$	$\frac{5}{243}$	0
$\frac{-x^{11}}{11}$	-1	$\frac{-63}{256}$	0	$\frac{22}{729}$	$\frac{105}{3840}$	$\frac{14}{729}$	0	$\frac{-8}{729}$	$\frac{-3}{256}$	$\frac{-7}{729}$	0

With this table allowing for the computation of area under $(1-x^2)^p$ over $[1-x^2,1]$, letting $x = 1$ and using the $p = 1/2$ column yields a new way to calculate π, since

AREA of $(1-x^2)^{1/2}$ over $[0,1]$ = AREA of the portion of the unit circle in quadrant I

$$= 1+\left(\frac{1}{2}\right)\left(\frac{-1}{3}\right)-\left(\frac{1}{8}\right)\left(\frac{1}{5}\right)+\left(\frac{3}{48}\right)\left(\frac{-1}{7}\right)-\left(\frac{15}{384}\right)\left(\frac{1}{9}\right)+\left(\frac{105}{3840}\right)\left(\frac{-1}{11}\right)-\left(\frac{945}{46080}\right)\left(\frac{1}{13}\right)+\ \ldots$$

$$=1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}-\ \ldots\ = \frac{\pi}{4}$$

$$\Rightarrow\ \pi = 4\left(1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}-\ \ldots\right)$$

Checking that this series does indeed converge to π gave Newton the satisfaction that his interpolation scheme for finding area under a curve was correct. This triggered a search for a way to ease the burden of tabulating entries in the columns of his chart row by row. This search in turn leads Newton to the discovery of the formula for the binomial coefficient which is the focus of the subsequent activity.

ACTIVITY 37 : NEWTON'S $p = 1/2$ COLUMN

With the $p = 1/2$ column of his $(1-x^2)^p$ chart (which again contains entries that represent coefficients,of the terms of the row they are situated in, for formulating area expressions), Newton concentrates all his energies on finding a pattern. What he found is shown below. Carefully follow the patterns being developed and fill in the blanks as you go along.

ROW	COEFFICIENT
$\frac{x}{1}$	$\frac{1}{1}$
$\frac{-x^3}{3}$	$\frac{1}{2}$
$\frac{x^5}{5}$	$\frac{-1}{8} = \frac{1}{2}\cdot\frac{(-\_)}{(\_)} = \frac{1}{2}\cdot\frac{(1-1\cdot 2)}{2\cdot 2}$
$\frac{-x^7}{7}$	$\frac{3}{48} = \frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{(-\_)}{(\_)} = \frac{1}{1\cdot 2}\cdot\frac{(1-1\cdot 2)}{\_\cdot 2}\cdot\frac{(1-2\cdot\_)}{3\cdot 2}$
$\frac{x^9}{9}$	$\frac{-15}{384} = \frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdot\frac{(-\_)}{(\_)} = \frac{1}{1\cdot 2}\cdot\frac{(1-1\cdot 2)}{\_\cdot 2}\cdot\frac{(1-2\cdot\_)}{3\cdot 2}\cdot\frac{(1-\_\cdot 2)}{\_\cdot\_}$
$\frac{-x^{11}}{11}$	$\frac{105}{3840} = \frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_\_)} = \frac{1}{1\cdot 2}\cdot\frac{(1-1\cdot 2)}{\_\cdot 2}\cdot\frac{(1-2\cdot\_)}{3\cdot 2}\cdot\frac{(1-\_\cdot 2)}{\_\cdot\_}\cdot\frac{(\_-4\cdot\_)}{5\cdot 2}$
$\frac{x^{13}}{13}$	$\frac{-945}{46080} = \frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdot\frac{-5}{8}\cdot\frac{-7}{10}\cdot\frac{(-\_)}{\_\_} =$ Carry this out on your own.
$\frac{-x^{15}}{15}$	$\frac{10395}{645120} = \frac{1}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{-7}{10}\cdot\frac{(-\_)}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} =$ Do this on your own.

Note again that the patterns devised above are for the $p = 1/2$ column of Newton's $(1-x^2)^p$. Then, move on to the next activity which involves the $p = 1/3$ column of his chart. Keep an eye out for how the change from $p = 1/2$ to $p = 1/3$ affects the patterns.

ACTIVITY 38 : NEWTON'S $p = 1/3$ COLUMN

DIRECTIONS: Carefully follow the patterns being established in the chart below and fill in the blanks as you go along. CAUTION: Some fractions in this chart are reduced.

ROW	COEFFICIENT
$\frac{x}{1}$	$\frac{1}{1}$
$\frac{-x^3}{3}$	$\frac{1}{3}$
$\frac{x^5}{5}$	$\frac{-1}{9} = \frac{1}{3}\cdot\frac{-2}{6} = \frac{1}{1\cdot 3}\cdot\frac{(1-1\cdot 3)}{2\cdot 3}$
$\frac{-x^7}{7}$	$\frac{5}{81} = \frac{1}{3}\cdot\frac{-2}{6}\cdot\frac{(-\_)}{(\_)} = \frac{1}{1\cdot 3}\cdot\frac{(1-1\cdot 3)}{\_\cdot 3}\cdot\frac{(1-2\cdot\_)}{3\cdot 3}$
$\frac{x^9}{9}$	$\frac{-10}{243} = \frac{1}{3}\cdot\frac{-2}{6}\cdot\frac{-5}{9}\cdot\frac{(-\_)}{(\_\_)} = \frac{1}{1\cdot\_}\cdot\frac{(1-1\cdot 3)}{\_\cdot 3}\cdot\frac{(1-2\cdot\_)}{\_\cdot\_}\cdot\frac{(1-\_\cdot 3)}{4\cdot 3}$
$\frac{-x^{11}}{11}$	$\frac{22}{729} = \frac{1}{3}\cdot\frac{-2}{6}\cdot\frac{-5}{9}\cdot\frac{(-\_)}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} = \frac{1}{1\cdot 3}\cdot\frac{(1-1\cdot 3)}{\_\cdot 3}\cdot\frac{(1-2\cdot\_)}{3\cdot 3}\cdot\frac{(1-\_\cdot 3)}{\_\cdot\_}\cdot\frac{(\_-4\cdot\_)}{5\cdot\_}$
$\frac{x^{13}}{13}$	$\frac{-154}{6561} = \frac{1}{3}\cdot\frac{-2}{6}\cdot\frac{-5}{9}\cdot\frac{-8}{12}\cdot\frac{-11}{15}\cdot\frac{(-\_\_)}{\_\_} =$ ______ (Carry this out on your own)
$\frac{-x^{15}}{15}$	$\frac{374}{19683} = \frac{1}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_\_)}\cdot\frac{-11}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} =$ ____(On your own)

In the space below, write down what you believe was the effect of changing from the $p = 1/2$ column to the $p = 1/3$ column.

____

____

____

____

____

ACTIVITY 39 : NEWTON'S $p = 2/3$ COLUMN

DIRECTIONS: Carefully follow the patterns being established in the chart below and fill in the blanks as you go along. CAUTION: Some fractions in this chart are reduced.

ROW	COEFFICIENT
$\frac{x}{1}$	$\frac{1}{1}$
$\frac{-x^3}{3}$	$\frac{(\_)}{(\_)}$
$\frac{x^5}{5}$	$\frac{-1}{9} = \frac{2}{3}\cdot\frac{-1}{6} = \frac{2}{1\cdot 3}\cdot\frac{(2-1\cdot 3)}{2\cdot 3}$
$\frac{-x^7}{7}$	$\frac{4}{81} = \frac{2}{3}\cdot\frac{-1}{6}\cdot\frac{(-\_)}{(\_)} = \frac{1}{1\cdot 3}\cdot\frac{(2-1\cdot 3)}{\_\cdot 3}\cdot\frac{(2-2\cdot\_)}{3\cdot 3}$
$\frac{x^9}{9}$	$\frac{-7}{243} = \frac{2}{3}\cdot\frac{-1}{6}\cdot\frac{-4}{9}\cdot\frac{(-\_)}{(\_\_)} = \frac{2}{1\cdot\_}\cdot\frac{(\_-1\cdot 3)}{\_\cdot 3}\cdot\frac{(2-2\cdot\_)}{\_\cdot\_}\cdot\frac{(2-\_\cdot 3)}{4\cdot 3}$
$\frac{-x^{11}}{11}$	$\frac{14}{729} = \frac{2}{3}\cdot\frac{-1}{6}\cdot\frac{-4}{9}\cdot\frac{(-\_)}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} = \frac{2}{1\cdot 3}\cdot\frac{(2-1\cdot 3)}{\_\cdot 3}\cdot\frac{(2-2\cdot\_)}{3\cdot 3}\cdot\frac{(2-\_\cdot 3)}{\_\cdot\_}\cdot\frac{(\_-4\cdot\_)}{5\cdot\_}$
$\frac{x^{13}}{13}$	$\frac{-91}{6561} = \frac{(\_)}{3}\cdot\frac{-1}{6}\cdot\frac{-4}{9}\cdot\frac{-7}{12}\cdot\frac{-10}{15}\cdot\frac{(-\_\_)}{(\_\_)} =$ _?_(Carry this out on your own)
$\frac{-x^{15}}{15}$	$\frac{208}{19683} = \frac{(\_)}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_)}\cdot\frac{(-\_)}{(\_\_)}\cdot\frac{-10}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} =$ _?_(On your own)

In the space below, write down what you believe was the effect of changing from the $p = 1/2$ column and/or the $p = 1/3$ column to the $p = 2/3$ column.

____

____

____

____

____

ACTIVITY 40 : NEWTON'S $p = 3/5$ COLUMN

Yes, I realize that this column is not displayed in the chart, but let's attack it anyway. DIRECTIONS: Carefully follow the patterns being established in the chart below and fill in the blanks as you go along. CAUTION: Some fractions in this chart are reduced.

ROW	COEFFICIENT
$\frac{x}{1}$	$\frac{1}{1}$
$\frac{-x^3}{3}$	$\frac{(\_)}{(\_)}$
$\frac{x^5}{5}$	$\frac{-3}{25} = \frac{3}{5}\cdot\frac{-2}{10} = \frac{3}{1\cdot5}\cdot\frac{(3-1\cdot5)}{2\cdot5}$
$\frac{-x^7}{7}$	$\frac{-7}{125} = \frac{3}{5}\cdot\frac{-2}{10}\cdot\frac{(-\_)}{(\_\_)} = \frac{3}{1\cdot5}\cdot\frac{(3-1\cdot5)}{\_\cdot5}\cdot\frac{(2-2\cdot\_)}{3\cdot5}$
$\frac{x^9}{9}$	$\frac{-21}{5^4} = \frac{3}{5}\cdot\frac{-2}{10}\cdot\frac{-7}{15}\cdot\frac{(-\_\_)}{(\_\_)} = \frac{3}{1\cdot\_}\cdot\frac{(\_-1\cdot5)}{\_\cdot5}\cdot\frac{(3-2\cdot\_)}{\_\cdot\_}\cdot\frac{(3-\_\cdot5)}{4\cdot5}$
$\frac{-x^{11}}{11}$	$\frac{(\_\_\_)}{5^6} = \frac{3}{5}\cdot\frac{-2}{10}\cdot\frac{-7}{15}\cdot\frac{-12}{(\_\_)}\cdot\frac{(-\_\_)}{(\_\_)} = \frac{3}{5}\cdot\frac{(3-1\cdot\_)}{\_\cdot5}\cdot\frac{(\_-2\cdot\_)}{3\cdot5}\cdot\frac{(3-\_\cdot5)}{\_\cdot\_}\cdot\frac{(\_-4\cdot\_)}{5\cdot\_}$
$\frac{x^{13}}{13}$	$\frac{(\_\_\_\_)}{2\cdot5^7} = \frac{(\_)}{5}\cdot\frac{-2}{10}\cdot\frac{-7}{15}\cdot\frac{-12}{20}\cdot\frac{-17}{25}\cdot\frac{(-\_\_)}{(\_\_)} =$ _____(do this on your own)

Now, complete a similarly designed chart for Newton's $p = 6/7$ column on your own. Then, do the exact same thing for Newton's $p = x/y$ column.

ACTIVITY 41 : NEWTON'S FORMULA FOR THE BINOMIAL COEFFICIENT

NEWTON'S $p = x/y$ COLUMN

TERM	COEFFICIENT
1st	$\frac{1}{1}$
2nd	$\frac{x}{y}$
3rd	$\frac{x}{y}\cdot\frac{x-y}{2y}$
4th	$\frac{x}{y}\cdot\frac{x-y}{2y}\cdot\frac{x-2y}{3y}$
5th	$\frac{x}{y}\cdot\frac{x-y}{2y}\cdot\frac{x-2y}{3y}\cdot\frac{x-3y}{4y}$
6th	$\frac{x}{y}\cdot\frac{x-y}{2y}\cdot\frac{x-2y}{3y}\cdot\frac{x-3y}{4y}\cdot\frac{x-4y}{5y}$

DIRECTIONS: Let $n = \frac{x}{y}$ and use your algebraic skills to get each of the coefficients in the chart in terms of n. Hint: $\frac{x-3y}{4y} = \frac{x}{4y} - \frac{3y}{4y} = \frac{x/y}{4} - \frac{3}{4} = \frac{n-3}{4}$.

If this exercise is done correctly, you should wind up with the general binomial coefficients. That is, the coefficients of the terms created during binomial expansion of $(a+b)^n$, n being a rational number. If you wish to check your work, you should be able to find the Binomial Thereom in any algebra textbook.

This culminates our work with Newton's creation of the binomial series. However, of paramount significance is the lasting result of these investigations. Commencing with Wallis' daring numerical approaches to working with infinite processes, leading to Newton's extension to negative exponents of Wallis' interpolation scheme, and coming to a head with the discovery of the binomial series, the "horror of the infinite" that so impeded the Greeks was exiled forever. As Boyer said, Newton "had found that analysis by infinite series had the same inner consistency, and was subject to the same general laws, as the algebra of finite quantities" ([b], p. 432).One final comment is that this work for Newton, in large part influenced by John Wallis, as we have seen, became the catalyst with which he would later generate his version of the calculus.

BIBLIOGRAPHY

Dennis/Confrey, "The Creation of Binomial Series: A Study of the Methods and Epistemology of Wallis, Newton, and Euler." A paper presented at the Joint Mathematics Meetings. Vancouver, Canada 1993.

Boyer, C.B. A History of Mathematics, New York: Wiley, 1968.

Edwards, C.H., The Historical Development of Calculus, New York: Springer-Verlag, 1979

Grant, E. A Source Book in Medieval Science, Cambridge, MA, Harvard Univ. Press, 1974.

Nunn, T.P. " The Arithmetic of Infinites", Mathematical Gazette, Vol. 5, pgs. 345-356 and 377-386, 1909-11.

Polya, G. Mathematical Discovery, John Wiley & Sons, New York, 1962.

Scott, J.G. The Mathematical Work of John Wallis, New York: Chelsea Publishing Co., 1981.

Struik, F.J. A Source Book in Mathematics, 1200-1800, Cambridge, Mass: Harvard University Press, 1969.

Toeplitz, The Calculus: A Genetic Approach, University of Chicago Press, 1963.

Wallis, J. Arithmetica Infinitorum in Opera Mathematica, vol. 1, p. 355-478. New York: Georg Olms Verlag 1972.