Totally non-proper ordinals beyond $L(V_{\lambda+1})$

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December 12, 2014

Abstract

In recent work Woodin has defined new axioms stronger than I0 (the existence of an elementary embedding j from $L(V_{\lambda+1})$ to itself), that involve elementary embeddings between slightly larger models. There is a natural correspondence between I0 and Determinacy, but to extend this correspondence in this new framework we must insist that these elementary embeddings are proper. While at first this seemed to be a common property, in this paper will be provided a model in which all such elementary embeddings are not proper. This result fills a gap in a theorem by Woodin and justifies the definition of properness. *Keywords*: Large Cardinals, Elementary Embeddings, Sharp, Relative Ordinal-Definability *2010 Mathematics Subject Classifications*: 03E55, (03E45)

1 Introduction

In 1971, a result by K. Kunen ([3]) threatened to shatter the fragile top of the soaring skyscraper of large cardinal hypotheses. Following a remark in W. Reinhardt's thesis, Kunen proved that there are no non–trivial elementary embeddings $j : V \prec V$. This was unprecedented in the history of large cardinal hypotheses, the first (and, in fact, the last) inconsistency result. After this dramatic discovery, many tried to check how deep the cracks of inconsistency pervaded the large cardinals structure. A crucial point in every proof of this result (many can be found in [2]) is the use of AC, in particular the use of a well-order of $V_{\lambda+1}$, where λ is the supremum of the critical sequence. While these efforts of finding another inconsistency were fruitless,

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a map of new hypotheses that cover the remaining possibilities arised, and the confidence in these axioms now is quite strong.

Among all these axioms, probably the most interesting one is I0, that states the existence of an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$, where λ is the supremum of the critical sequence of j. This axiom, proposed by Woodin in 1984, has several interesting entailments, since it produces a detailed and coherent structural theory for an inner model of ZF, i.e., $L(V_{\lambda+1})$, that is strikingly similar to the structural theory of $L(\mathbb{R})$ under Determinacy. For example, under I0 the Coding Lemma holds, λ^+ is measurable, and there are also nice reflection properties. This creates a new kind of tie between large cardinal hypotheses and Determinacy which has not yet been thoroughly understood.

In his [9], Woodin explores the dangerous and fascinating territory between I0 and the inconsistency proved by Kunen. The axioms he considered are of the form "There exists an elementary embedding $j : L(N) \prec L(N)$, with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $\mathrm{crt}(j) < \lambda$": generally the larger the set N, the stronger the axiom. He creates a nicely absolute increasing sequence of such sets, in this paper called E_α-sequence, that in a certain sense can be considered standard in the analysis of hypotheses stronger than I0. The main motivation in this definition is the search for a hypothesis stronger than ever, corresponding to $\mathsf{AD}_\mathbb{R}$ just like I0 corresponded to $\mathsf{AD}^{L(\mathbb{R})}$. In [9] one can find a captivating discussion on the similarities of this new axiom with $\mathsf{AD}_\mathbb{R}$ and on its credibility.

The main problem with these new axioms is in maintaining the tie with Determinacy. Since this tie was the driving force behind the exploration of I0, it is very desirable to have similar results: Woodin proved that this is true (for specific N's) if the elementary embedding considered is *proper*. Properness is a particular instance of the Axiom of Replacement that involves the elementary embedding and subsets of $V_{\lambda+1}$, and not only gives Determinacy-like results, but also iterability. However, one can ask if this is not a vacuous definition, i.e., whether every elementary embedding is proper, and Theorem 2.21 seems to push in this direction, since it proves that almost all elementary embeddings are proper. The main theorem of this article, Theorem 2.23, deals with this doubt, not only providing a non-proper elementary embedding, but showing an example of N as above such that *every* $j : L(N) \prec L(N)$ is not proper, and such an N will be an element of the E_α-sequence. This Theorem both fills a gap in Theorem 2.21 and validates the definition of properness.

The strategy is the following. Theorem 2.24 gives a criterion of non-properness: if the fixed points of j are bounded in the Θ of $L(N)$, that is the supremum of the ordertypes of the prewellorderings of $V_{\lambda+1}$ in $L(N)$, then j can't be proper. The idea is to find a sequence cofinal in Θ whose ordertype is an ordinal which we know is a discontinuity point between the fixed points of j (for example λ). At first it seems that like we should require this sequence should be definable, but in fact it it suffices that j behaves on this sequence *like* a definable one.

In the first section are collected the basic notations, all the definitions and theorems from [9] that will be useful for the paper, and the main theorem, Theorem 2.23, is stated. The second section is dedicated to a general analysis of the relationship between sharps and elementary embeddings in the E_α–sequence: the aim is not only proving Theorem 2.23, but also giving an idea of the techniques that are useful in this new large cardinals framework. The third and fourth sections will each deal with half of the proof of Theorem 2.23: Theorem 4.2 provides a sufficient condition for the existence of many non–proper elementary embedding, and Theorem 5.2 states the consistence of that sufficient condition under the assumption of a very large cardinal hypothesis.

2 Preliminaries

To avoid confusion or misunderstandings, here are collected all notations and standard basic results.

The double arrow (e.g. $f : a \twoheadrightarrow b$) denotes a surjection.

If M and N are sets or classes, $j : M \prec N$ denotes that j is an elementary embedding from M to N, that is an injective function whose range in an elementary submodel of N. The case in which j is the identity, i.e., if M is an elementary submodel of N, is simply written as $M \prec N$.

If $j : M \prec N$ is not the identity, then it moves at least one ordinal. The *critical point*, $\mathrm{crt}(j)$, is the least ordinal moved by j.

Let j be an elementary embedding and $\kappa = \mathrm{crt}(j)$. Define $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. Then $\langle \kappa_n : n \in \omega \rangle$ is the *critical sequence* of j.

Kunen ([3]) proved that if $M = N = V_\eta$ for some ordinal η, and λ is the supremum of the critical sequence, then η cannot be bigger than $\lambda + 1$ (and of course cannot be smaller than λ).

If X is a set, then $L(X)$ denotes the smallest inner model that contains X, it is defined like L but starting with the transitive closure of $\{X\}$ as $L_0(X)$.

If X is a set, then OD_X denotes the class of the sets that are *ordinal-definable over* X, i.e., the sets that are definable using ordinals, X and elements of X as parameters. A set is in HOD_X iff it is in OD_X and all the elements of its transiive closure are in OD_X. For example, $L(X) \vDash V = \mathrm{HOD}_X$.

One advantage in considering models of HOD_X is the possibility of defining partial Skolem functions. Let $\varphi(v_0, v_1, \ldots, v_n)$ be a formula with $n+1$ free variables and let $a \in X$. Then:

$$h_{\varphi,a}(x_1,\ldots,x_n) = \begin{cases} y & \text{where } y \text{ is the least in } OD_a \text{ such that} \\ & \quad \phi(y, x_1, \ldots, x_n) \\ \emptyset & \text{if } \forall x \neg\phi(x, x_1, \ldots, x_n) \\ \text{not defined} & \text{else} \end{cases}$$

are partial Skolem functions. For every set or class y, $H^{L(X)}(y)$ denotes the closure of y under partial Skolem functions for $L(X)$, and $H^{L(X)}(Y) \prec L(X)$.

Let X a set. $X^\sharp$ can be defined in different ways, but in this paper it is considered as a complete theory in the language $\mathcal{L}^+_X = \{\in\} \cup \{X\} \cup X \cup \{i_n\}_{n\in\omega}$, where i_n are indiscernibles, similarly to the original definition by Solovay ([6]). Informally, $X^\sharp$ exists iff there is a class I of indiscernibles in $(L(X), \in, X, (x : x \in X))$ such that every cardinal bigger than $|X|$ is in I and $H^{L(X)}(I, X) = L(X)$. Then $X^\sharp$ is the set of formulas, in the language LST with constants for X and elements of X, satisfied by finite sequences of indiscernibles. With the usual methods, $X^\sharp$ can be coded as a subset of $V_\omega \times X$ using Gödel numbers. For future reference, here are two standard facts on sharps:

Lemma 2.1. *Let X be a set. If $X^\sharp$ exists, then*

- *every subset of X that is in $L(X)$ is definable from a finite set of elements of X, X and the first ω indiscernibles;*
- *for every $Y \subseteq X$, if $X^\sharp \cap \mathcal{L}^+_Y \prec X^\sharp$, then $X^\sharp \cap L^+_Y = Y^\sharp$.*

The territory of very large hypotheses is probably less standard, since its research is based mostly on unpublished results, so it is worth spending some word on that, starting with what was the largest hypothesis before the work of Woodin in [9], that is I0:

I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$.

The elementary embeddings are considered with critical point less than λ to follow the thread of rank-to-rank axioms: in this case, in fact, I0 implies I1, the existence of an elementary embedding from $V_{\lambda+1}$ to itself. By Kunen's Theorem in this case λ must be the supremum of the critical sequence of j. This means that λ is in particular limit of inaccessible cardinals, so $|V_\lambda| = \lambda$ and V_λ is closed by finite sequences. Therefore every λ-sequence of elements of $V_{\lambda+1}$ can be codified in $V_{\lambda+1}$, and this fact will be used throughout the paper without notice.

Also by Kunen's Theorem if I0 holds then $L(V_{\lambda+1}) \nvDash \mathsf{AC}$. This is just the first of many analogies between I0 and $\mathsf{AD}^{L(\mathbb{R})}$.

The first step in finding this analogies is considering the similarities between $L(V_{\lambda+1})$ and $L(\mathbb{R})$. The following are generalization of the classic Θ (the supremum of the ordertypes of prewellorderings in $L(\mathbb{R})$) and DC.

$$\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})} = \sup\{\gamma : \exists f : V_{\lambda+1} \twoheadrightarrow \gamma, f \in L(V_{\lambda+1})\};$$
$$\mathsf{DC}_\lambda : \quad \forall X\ \forall F : (X)^{<\lambda} \to \mathcal{P}(X) \setminus \emptyset\ \exists g : \lambda \to X\ \forall \gamma < \lambda\ g(\gamma) \in F(g \upharpoonright \gamma).$$

Note that $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ is also the supremum of the ordertypes of prewellorderings in $V_{\lambda+1}$, that are reflective, transitive and well-founded relations (informally well-orders without anti-simmetry), because every surjection like the ones in the definition corresponds to one prewellordering and vice versa.

Theorem 2.2. *In* $L(\mathbb{R})$*:*

- *there exists a definable surjection* $\Phi : \mathrm{Ord} \times \mathbb{R} \to L(\mathbb{R})$*;*
- Θ *is regular;*
- DC *holds.*

In $L(V_{\lambda+1})$*:*

- *there exists a definable surjection* $\Phi : \mathrm{Ord} \times V_{\lambda+1} \to L(V_{\lambda+1})$*;*
- $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ *is regular;*
- DC_λ *holds.*

The first part of Theorem 2.2 is a classic result, whose proof is for example in [2]. The proof of the second part is a direct generalization of the first proof.

If we assume I0, then the analogies are stronger:

Theorem 2.3. *Suppose* $L(\mathbb{R}) \vDash \mathsf{AD}$*. Then in* $L(\mathbb{R})$*:*

- ω_1 *is measurable;*
- *the Coding Lemma holds.*

Suppose $\exists j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ *with* $\mathrm{crt}(j) < \lambda$*. Then in* $L(V_{\lambda+1})$*:*

- *λ^+ is measurable;*
- *a generalization of the Coding Lemma holds.*

For a description of the Coding Lemma and a proof of the first part see [4]. For a detailed enunciation of the generalization and the proof of the second part see [1].

In his [9] Woodin carried out the analogy even further:

Theorem 2.4. *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is a limit of γ such that:*

- *γ is weakly inaccessible in $L(V_{\lambda+1})$;*
- *$\gamma = \Theta^{L_\gamma(V_{\lambda+1})}_{V_{\lambda+1}}$ and $j(\gamma) = \gamma$;*
- *for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;*
- *for cofinally $\kappa < \gamma$, κ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;*
- *$L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.*

Most of the proof of the $L(\mathbb{R})$ relative theorem can be found in [5].

The task of finding hypotheses stronger than I0 is not trivial. A natural form for a stronger Hypothesis must be something like "There exists an elementary embedding $j : M \prec M$", with $L(V_{\lambda+1}) \subset M$ and $\mathrm{crt}(j) < \lambda$, but by Kunen's Theorem $M = L(V_{\lambda+2})$ is already inconsistent. So the most immediate attempt should be adding one subset of $V_{\lambda+1}$, that is considering elementary embeddings $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subset V_{\lambda+1}$ and $\mathrm{crt}(j) < \lambda$. Naturally not all the subsets of $V^{\lambda+1}$ are eligible, for example if X is a well-ordering of $V_{\lambda+1}$ that would lead to a contradiction. To avoid repetitions, an $X \subset V_{\lambda+1}$ such that there exists an elementary embedding from $L(X, V_{\lambda+1})$ to itself with critical point less than λ will be called an *Icarus set*.

One of the defining feature of I0 is its analogy with AD, so it is natural to investigate whether this analogy will continue to hold with these new hypotheses, i.e., whether the previous theorems hold with $L(X, V_{\lambda+1})$ instead of $L(V_{\lambda+1})$. The proof of the analogue of Theorem 2.2 is immediate, and Woodin in [9] proved the equivalent of Theorem 2.3. However, the appropriate generalization of Theorem 2.4 resisted all attempts to prove without further hypotheses.

Definition 2.5. *Let $X \subset V_{\lambda+1}$. Then*

$$\Theta^{L(X,V_{\lambda+1})} = \sup\{\gamma : \exists f : V_{\lambda+1} \twoheadrightarrow \gamma, f \in L(X, V_{\lambda+1})\}.$$

Theorem 2.6 ([9]). *Let $X \subset V_{\lambda+1}$ be Icarus and j witnessing it. Then there exists an ultrafilter $U \subset L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ such that* $\mathrm{Ult}(L(X, V_{\lambda+1}), U)$ *us well-founded and $j_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$, the associated embedding, is an elementary embedding. Moreover, there is an elementary embedding $k_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with* $\mathrm{crt}(k_U) > \Theta^{L(X,V_{\lambda+1})}$ *such that $j = k_U \circ j_U$.*

Definition 2.7 ([9]). *Let $X \subset V_{\lambda+1}$ be Icarus and j witnessing it. Then*

- *j is* weakly proper *if $j = j_U$;*
- *j is* proper *if it is weakly proper and if $\langle X_i : i < \omega\rangle \in L(X, V_{\lambda+1})$ where $X_0 = X$ and for all $i < \omega$, $X_{i+1} = j(X_i)$.*

Woodin proved in [9] that if j is proper, then the corresponding Theorem 2.4 holds, and this is the main motivation for considering proper elementary embeddings.

The following theorem introduces a natural criterion to order the X's that are Icarus:

Theorem 2.8 ([9]). *Let $X \subset V_{\lambda+1}$ be Icarus. Let $Y \in L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ such that $\Theta^{L(Y,V_{\lambda+1})} < \Theta^{L(X,V_{\lambda+1})}$. Then $(Y, V_{\lambda+1})^\sharp$ exists and $(Y, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$.*

Note that also the converse is true, since if $(Y, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$ then in $L(X, V_{\lambda+1})$, $\Theta^{L(Y,V_{\lambda+1})}$ has cofinality ω (by Lemma 2.1) and $\Theta^{L(X,V_{\lambda+1})}$ is regular (considering $(X, V_{\lambda+1})^\sharp$ a subset of $V_{\lambda+1}$ and using Theorem 2.2).. So, when dealing with Icarus $X \subset V_{\lambda+1}$, the intuitive notion of "larger“ derived from the sharp (if $(Y, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$, then $L(X, V_{\lambda+1})$ is "larger“ than $L(Y, V_{\lambda+1})$) corresponds exactly to the largeness of the Θ's. Is it possible to find a standard representative for every possible Θ?

A different approach on this subject will give the right answer. Since the main point of I0 is its similarity with $\mathsf{AD}^{L(\mathbb{R})}$, one can try to find stronger axioms that follow this thread, i.e., axioms similar to hypotheses stronger than $\mathsf{AD}^{L(\mathbb{R})}$, like $\mathsf{AD}_\mathbb{R}$. For the latter case, the idea is to generalize the definition of the minimum model for $\mathsf{AD}_\mathbb{R}$ (that it is possible to find in [7]), constructing a sequence of $E_\alpha(V_{\lambda+1})$ sets such that $V_{\lambda+1} \subseteq E_\alpha(V_{\lambda+1}) \subset V_{\lambda+2}$.

Definition 2.9. *Suppose $V_{\lambda+1} \subset N \subset V_{\lambda+2}$.*

- *$\mathcal{E}(N)$ denotes the set of all the elementary embeddings $k : N \prec N$.*

- *Suppose that* $X \subseteq V_{\lambda+1}$. *Then* $N < X$ *if there exists a surjection* $\pi : V_{\lambda+1} \twoheadrightarrow N$ *such that* $\pi \in L(X, V_{\lambda+1})$.

The definition of the E_α-sequence it is by induction with four steps: 0, limit, successor of a limit and successor of a successor. First a longer sequence is defined:

Definition 2.10 ([9])**.** *Let* λ *be a limit ordinal such that* $\mathrm{cof}(\lambda) = \omega$. *The sequence*

$$\langle E_\alpha(V_{\lambda+1}) : \alpha < \Upsilon'_{V_{\lambda+1}} \rangle$$

is defined as:

- $E_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}$;
- *for* α *limit,* $E_\alpha(V_{\lambda+1}) = L(\bigcup_{\beta<\alpha} E_\beta(V_{\lambda+1})) \cap V_{\lambda+2}$;
- *for* α *limit,*
  - *if* $(\mathrm{cof}(\Theta^{L(E_\alpha(V_{\lambda+1}))}) < \lambda)^{L(E_\alpha(V_{\lambda+1}))}$ *then*
  $$E_{\alpha+1}(V_{\lambda+1}) = L((E_\alpha(V_{\lambda+1}))^\lambda) \cap V_{\lambda+2};$$
  - *if* $(\mathrm{cof}(\Theta^{L(E_\alpha(V_{\lambda+1}))}))^{L(E_\alpha(V_{\lambda+1}))} > \lambda$ *then*
  $$E_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(E_\alpha(V_{\lambda+1}))) \cap V_{\lambda+2};$$
- *for* $\alpha = \beta + 2$, *if there exists* $X \subseteq V_{\lambda+1}$ *such that* $E_{\beta+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ *and* $E_\beta(V_{\lambda+1}) < X$, *then*
  $$E_{\beta+2} = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$$
  otherwise we stop the sequence.

This definition is unsatisfactory, because this sequence is too long. It would be preferable, for example, to have elementary embeddings from $L(E_\alpha(V_{\lambda+1}))$ to itself, but this is not guaranteed by the above definition. So the sequence is shortened following this definition:

Definition 2.11 ([9])**.** *We call* $\Upsilon_{V_{\lambda+1}}$ *the maximum ordinal* $\leq \Upsilon'_{V_{\lambda+1}}$ *such that*

1. $\forall \alpha < \Upsilon_{V_{\lambda+1}}\ \exists X \subseteq V_{\lambda+1}$ *such that* $E_\alpha(V_{\lambda+1}) \subseteq L(X, V_{\lambda+1})$ *and* $\exists j \colon L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ *proper;*

2. $\forall\alpha$ *limit* $\alpha+1<\Upsilon_{V_{\lambda+1}}$ *iff*

$$(\mathrm{cof}(\Theta^{E_\alpha(V_{\lambda+1})}))^{L(E_\alpha(V_{\lambda+1}))}>\lambda\to \exists Z\in E_\alpha(V_{\lambda+1})\ L(E_\alpha(V_{\lambda+1}))=(\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(E_\alpha(V_{\lambda+1}))}.$$

The $E_\alpha(V_{\lambda+1})$*-sequence is:*

$$\langle E_\alpha(V_{\lambda+1}):\alpha<\Upsilon_{V_{\lambda+1}}\rangle. \tag{1}$$

From now on, we will write just E_α and Υ instead of $E_\alpha(V_{\lambda+1})$ and $\Upsilon_{V_{\lambda+1}}$. This is a slight abuse of notation, since in fact these objects depend on $V_{\lambda+1}$, but this will be considered always fixed. Note that the E_α-sequence is strictly increasing, and that at the limit point *not* necessarily $E_\eta=\bigcup_{\beta<\eta}E_\beta$. Note also that if α is a successor, then there exists $X\subset V_{\lambda+1}$ such that $L(E_\alpha)=L(X,V_{\lambda+1})$. If α is a limit, this can also happen:

Lemma 2.12 ([9]). *Let* $\alpha<\Upsilon$ *and suppose that* $\Theta^{E_\alpha}>\sup_{\beta<\alpha}\Theta^{E_\beta}$. *Then there exists* $X\subset V_{\lambda+1}$ *such that* $L(E_\alpha)=L(X,V_{\lambda+1})$.

Even when this is not true, by Definition 2.11(1) E_α can be codified as a subset of $V_{\lambda+1}$ in V anyway.

Lemma 2.13 ([9]). *Let* $\alpha<\Upsilon$. *Then there exists an elementary embedding* $j:L(E_\alpha)\prec L(E_\alpha)$ *with* $\mathrm{crt}(j)<\lambda$.

Theorem 2.14 ([9]). *Suppose* $X\subset V_{\lambda+1}$ *and there is a proper elementary embedding* $j:L(X,V_{\lambda+1})\prec L(X,V_{\lambda+1})$. *Define* E^X_α *and* Υ^X *as* $(E_\alpha)^{L(X,V_{\lambda+1})}$ *and* $(\Upsilon)^{L(X,V_{\lambda+1})}$ *but without the condition 2.11(1). Then, either* $\Upsilon^X=\Upsilon$ *and* $\sup_{\eta<\Upsilon}\Theta^{E_\eta}\le\Theta^{L(X,V_{\lambda+1})}$, *or there exists* $\eta<\Upsilon$ *such that* $\Upsilon^X=\eta+1$ *and* $\Theta^{E_\eta}=\Theta^{L(X,V_{\lambda+1})}$. *Moreover, if* $\alpha<\Upsilon^X$ *then* $E^X_\alpha=E_\alpha$.

This guarantees that the E_α-sequence gives one standard representative for at least an initial segment of the $\Theta^{L(X,V_{\lambda+1})}$ such that there exists an elementary embedding from $L(X,V_{\lambda+1})$ to itself. Let X be as such. There are two possibilities: if $\Theta^{L(X,V_{\lambda+1})}$ is bigger than any Θ^{E_α} with $\alpha<\Upsilon$ then it is not possible to have any useful information. Otherwise there exists $\eta<\Upsilon$ such that the E_α-sequence defined in $L(X,V_{\lambda+1})$ goes up until η, and there exists $j:L(E_\eta)\prec L(E_\eta)$ with $\mathrm{crt}(j)<\lambda$. Since $\Theta^{E_\eta}=\Theta^{L(X,V_{\lambda+1})}$, this is the standard representative desired.

It is immediate to see that there can be cases such that there is no $X\subset V_{\lambda+1}$ with $L(X,V_{\lambda+1})=L(E_\eta)$, so it is worth widening the horizons and considering also elementary embeddings from $L(N)\prec L(N)$, with $V_{\lambda+1}\subset N\subset V_{\lambda+2}$. Some results generalize:

Lemma 2.15 ([9]). *Suppose that N is transitive, $V_{\lambda+1} \subset N \subset V_{\lambda+2}$, $N = L(N) \cap V_{\lambda+2}$, $j : L(N) \prec L(N)$ is an elementary embedding with* $\mathrm{crt}(j) < \lambda$. *Then there exist an ultrafilter $U \subset N$ such that* $\mathrm{Ult}(L(N), U)$ *is well-founded and $j_U : L(N) \to \mathrm{Ult}(L(N), U)$, the associated embedding, is an elementary embedding. Moreover, there is an elementary embedding $k_U : \mathrm{Ult}(L(N), U) \prec L(N)$ such that $k_U \upharpoonright N$ is the identity and $j = k_U \circ j_U$.*

Anyway, not all results generalize. For example, it is not necessary for the analogue of Theorem 2.2 to hold. However, when there exists $X \subset V_{\lambda+1}$ such that $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1} \cup \{X\}}$, the equivalent of Theorems 2.2 and 2.3 hold. In the case of the E_α-sequence, this is true in trivial cases (for example when α is a successor) or when α is a limit such that $(\mathrm{cof}(\Theta^{E_\alpha}))^{L(E_\alpha)} > \lambda$, because of condition 2.11(2). For future reference, the first result is summarized in a Lemma:

Lemma 2.16 ([9]). *Let N such that $V_{\lambda+1} \subset N \subset V_{\lambda+2}$. Suppose that there exists $Z \subset V_{\lambda+1}$ such that $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1} \cup \{Z\}}$. Then*

- $\Theta^{L(N)} = \sup\{\alpha : \exists \pi : V_{\lambda+1} \twoheadrightarrow \alpha,\ \pi \in L(N)\}$ *is regular in $L(N)$;*
- $L(N) \vDash \mathsf{DC}_\lambda$.

It is also possible to define a generalized definition of properness:

Definition 2.17 ([9]). *Suppose N transitive, $V_{\lambda+1} \subset N \subset V_{\lambda+2}$, $N = L(N) \cap V_{\lambda+2}$ and let $j\colon L(N) \prec L(N)$ be an elementary embedding with critical point $< \lambda$.*

- *j is* weakly proper *if $j = j_U$;*
- *j is* proper *if for all $X \in N$, $\langle X_i : i < \omega \rangle \in L(N)$ where $X_0 = X$ and for all $i < \omega$, $X_{i+1} = j(X_i)$;*

The following are three lemmas that complete the miscellanea of results from [9] and help to have a better understanding on the structure of the E_α-sequence. The first one is a result of condensation, the second one deals with definability inside $L(E_\alpha)$, the third one is a result of absolutness.

Lemma 2.18 ([9]). *Let $\beta < \Upsilon$ and M be a transitive class of* ZF *such that $E_\beta \subseteq M$. If there exists a $\eta < \Upsilon$ such that $M \prec L(E_\eta)$, then there exists $\beta \leq \gamma \leq \eta$ such that $M = L(E_\gamma)$ or exists ζ such that $M = L_\zeta(E_\gamma)$.*

Lemma 2.19 ([9]). *Suppose $\alpha < \Upsilon$ is a limit ordinal and* $(\mathrm{cof}(\Theta^{E_\alpha}))^{L(E_\alpha)} >$ *λ. Then there exists $Z \in E_\alpha$ such that for each $Y \in E_\alpha$, Y is Σ_1-definable in $L(E_\alpha)$ with parameters from $\{Z\} \cup \{V_{\lambda+1}\} \cup V_{\lambda+1} \cup \Theta^{E_\alpha}$. Moreover, if $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, then for every $Y \in E_\alpha$, Y is Σ_1-definable in $L(E_\alpha)$ with parameters from $\{V_{\lambda+1}\} \cup V_{\lambda+1} \cup \Theta^{E_\alpha}$.*

Lemma 2.20 ([9]). *Suppose* $\alpha < \Upsilon$ *limit such that* $(\mathrm{cof}(\Theta^{E_\alpha}) > \lambda)^{E_\alpha}$. *Then* $(\langle E_\beta : \beta < \Upsilon \rangle)^{E_\alpha} = \langle E_\beta : \beta < \alpha \rangle$.

So, Theorem 2.4 gives a motivation to investigate further the concept of properness. However, looking only at the definition, it is unclear if this is a proper definition. What if every elementary embedding is proper? A theorem by Woodin seems to push in this direction:

Theorem 2.21 ([9]). *Suppose* $\alpha < \Upsilon$. *If*

- $\alpha = 0$, *or*
- α *is a successor ordinal, or*
- α *is a limit ordinal with cofinality* $> \omega$

then every weakly proper elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ *is proper.*

However, this theorem deals only with models from the E_α-sequence, and not even all of them, leaving a gap for the $L(E_\alpha)$ such that α is a limit and $\mathrm{cof}(\alpha) = \omega$. The main result of this article will exploit exactly this gap.

Definition 2.22. *Let* $\alpha < \Upsilon$ *limit such that* $\mathrm{cof}(\alpha) = \omega$. *Then* α *is a* totally non-proper ordinal *if every weakly proper elementary embedding* $j : L(E_\alpha) \prec L(E_\alpha)$ *is not proper.*

Theorem 2.23. *If there exists a* $\xi < \Upsilon$ *such that* $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$, *then there exists a totally non-proper ordinal.*

Note that the existence of a totally non-proper ordinal is not provable in ZFC and there are no known large cardinal hypothesis that implies it, since this is true also for the consistency of $\Upsilon > 0$, but supposing that Υ is big enough (i.e., if there exists $X \subset V_{\lambda+1}$ Icarus such that $\Theta^{L(X,V_{\lambda+1})}$ is big enough) then a totally non-proper ordinal exists. However, after Section 4 it will be clear that the existence of that totally non-proper elementary embedding is a consequence of a much weaker hypothesis.

The main tool for the proof of Theorem 2.23 is the following criterion for non-properness:

Lemma 2.24 ([9]). *Let* M *be* $L(X, V_{\lambda+1})$ *with* $X \subset V_{\lambda+1}$ *or* $L(N)$ *with* $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ *such that* $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. *If* $j : M \prec M$ *is proper, then the fixed points of* j *are cofinal in* Θ^M.

3 Slicing Sharps

It is easy to see that sharps play a big role in the definition of the E_α-sequence, and in fact it turns out they're important for a whole analysis of such sequence. In this section some particular characteristics of the sequence $\langle (E_\beta)^\sharp : \beta < \Upsilon \rangle$ are summarized, with an accent on the reflection properties, that will be the key for proving Theorem 2.23

The following notations are useful for this analysis. For every $\alpha < \Upsilon$, by definition $(E_\alpha)^\sharp$ is a set of formulas in the language

$$\mathcal{L}^+_\alpha := \{\in\} \cup \{c_a\}_{a \in E_\alpha} \cup \{d_i\}_{i \in \omega} \cup \{C\},$$

where in $L(E_\alpha)$ every c_a is interpreted as a, every d_i is interpreted as an indiscernible and C is interpreted as E_α. The language

$$\mathcal{L}^+_{\alpha,n} := \{\in\} \cup \{c_a\}_{a \in E_\alpha} \cup \{d_1, \dots, d_n\} \cup \{C\}.$$

is the restriction of $\mathcal{L}^+_\alpha$ to a language that uses at most n constants for indiscernibles.

The first step is to outline the relationship between sharps and elementary embeddings. Fix $\alpha < \Upsilon$. In a certain sense the sharp $(E_\alpha)^\sharp$ contains the truth of $L(E_\alpha)$ and any elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ preserves such truth, so there must be a connection. However, it is not possible to directly apply j to $(E_\alpha)^\sharp$, since $(E_\alpha)^\sharp \notin L(E_\alpha)$, but when E_α is easily disassembled in smaller well-ordered parts then the sharp can be sliced in pieces digestible by $L(E_\alpha)$:

Definition 3.1. *For $\gamma, \alpha < \Upsilon$ define the* (γ,n)-fragment *of $(E_\alpha)^\sharp$ as $(E_\alpha)^\sharp \cap \mathcal{L}^+_{\gamma,n}$, and denote it as $(E_\alpha)^\sharp_{\gamma,n}$*

Define the γ-fragment *of $(E_\alpha)^\sharp$ as $(E_\alpha)^\sharp \cap \mathcal{L}^+_\gamma$, and denote it as $(E_\alpha)^\sharp_\gamma$.*

The structure of the E_α-sequence gives information on the largeness of these fragments. The set $(E_\beta)^\sharp$ is a subset of $V_\omega \times E_\alpha$, so it can be coded directly as a subset of E_α. Moreover, this coding is definable. By Definition 2.10, for every $\beta < \alpha < \Upsilon$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow E_\beta$ with $\pi \in L(E_\alpha)$, so E_β (and its sharp) can be coded as a subset of $V_{\lambda+1}$ in $L(E_\alpha)$, i.e., as an element of E_α. This means that for every $\beta < \alpha < \Upsilon$ and every $n \in \omega$, $(E_\alpha)^\sharp \in E_{\alpha+1}$ and $(E_\alpha)^\sharp_{\beta,n} \in E_\alpha$.

Since all the fragments of $(E_\alpha)^\sharp$ are in E_α, it is quite natural to ask, for an elementary embedding $k : E_\alpha \prec E_\alpha$, if it preserves the fragments, and what happens in that case.

Definition 3.2. *Let $\alpha < \Upsilon$ be a limit ordinal. Given a Σ_1-elementary embedding $k \colon E_\alpha \prec_1 E_\alpha$ we call k* $\sharp$-friendly *if for every $\gamma < \alpha$*

$$k((E_\alpha)^\sharp_{\gamma,n}) = (E_\alpha)^\sharp_{k(\gamma),n}.$$

More generally, given $\beta \leq \alpha < \Upsilon$ limit ordinals, a Σ_1-elementary embedding $k \colon E_\beta \prec_1 E_\alpha$ is called $\sharp$-friendly *if for every $n \in \omega$ and $\gamma < \beta$*

$$k((E_\beta)^\sharp_{\gamma,n}) = (E_\alpha)^\sharp_{k(\gamma),n}.$$

The following theorem is the key of this section:

Theorem 3.3. *Let $\beta \leq \alpha < \Upsilon$ limit ordinals, $k : E_\beta \prec E_\alpha$. Then k is $\sharp$-friendly iff it is possible to extend it to $\hat{k} : L(E_\beta) \prec L(E_\alpha)$ such that $k \subset \hat{k}$.*

The proof of the theorem is split in two parts.

Lemma 3.4. *Let $\beta \leq \alpha$ be limit ordinals less than Υ. If $k \colon E_\beta \prec_1 E_\alpha$ is $\sharp$-friendly, then it is possible to extend k to an elementary embedding $\hat{k} \colon L(E_\beta) \prec L(E_\alpha)$.*

Proof. Temporarily call J the class of indiscernibles for $L(E_\beta)$, and K the class of indiscernibles for $L(E_\alpha)$. Let b be the only bijection from I to J that is order-preserving.

Since $(E_\beta)^\sharp$ exists, for every element Y of $L(E_\beta)$ there are $a_1, \dots, a_n \in E_\beta$, $i_1, \dots, i_m \in J$ and a formula $\varphi(x, a_1, \dots, a_n, E_\beta, i_1, \dots, i_m)$ that defines Y in $L(E_\beta)$.

Therefore

$$L(E_\beta) \vDash \exists! y \ \varphi(y, a_1, \dots, a_n, , E_\beta, i_1, \dots, i_m).$$

But then

$$\exists! y \varphi(y, c_{a_1}, \dots, c_{a_n}, C, d_1, \dots, d_m) \in (E_\beta)^\sharp.$$

Since by definition

$$E_\beta = L\big(\bigcup_{\gamma<\beta} E_\gamma\big) \cap V_{\lambda+2},$$

it is possible to suppose $a_1, \dots, a_n \in \bigcup_{\gamma<\beta} E_\gamma$, thus there exists $\gamma < \beta$ such that $a_1, \dots, a_n \in E_\gamma$. Then

$$\exists! y \ \varphi(y, c_{a_1}, \dots, c_{a_n}, C, d_1, \dots, d_m) \in (E_\beta)^\sharp_{\gamma,m}.$$

As k is $\sharp$-friendly and the coding of the sharp is definable[1],

$$\exists! y\ \varphi(y, c_{k(a_1)}, \dots, c_{k(a_n)}, C, d_1, \dots, d_m) \in (E_\alpha)^\sharp_{k(\gamma),m},$$

and then

$$L(E_\alpha) \vDash \exists! y\ \varphi(y, k(a_1), \dots, k(a_n), E_\alpha, b(i_1), \dots, b(i_m)).$$

Finally $\hat{k}(Y)$ is defined as the unique set such that

$$L(E_\alpha) \vDash \varphi(\hat{k}(Y), k(a_1), \dots, k(a_n), E_\alpha, b(i_1), \dots, b(i_m)).$$

Using the same method, replacing an element of $L(E_\beta)$ with the formula that defines it when needed, it is immediate to prove that $\hat{k}$ is well-defined, injective and an elementary embedding. □

Note that $\hat{k}$ is not unique, b can be any order-preserving injection from J to K. However, the $\hat{k}$ constructed in the proof has the benefit of being definable in a larger model, for example $L((E_\alpha)^\sharp)$.

Lemma 3.5. *Let $\beta \leq \alpha < \Upsilon$ limit ordinals and $j : L(E_\beta) \prec L(E_\alpha)$. Then $j \upharpoonright E_\beta : E_\beta \prec E_\alpha$ is $\sharp$-friendly.*

Proof. The case $\beta = \alpha$ is easier. Because of Lemma 2.15, without loss of generality it is possible to assume that j is weakly proper, so j is an elementary embedding associated to an ultrapower. In particular every strong limit cardinal in V with cofinality bigger than Θ^{E_α} is a fixed point of j. But it is also an indiscernible in $L(E_\alpha)$. So let $\eta_1, \dots, \eta_n$ be strong limit cardinals in V with cofinality bigger than Θ^{E_α}. Then

$$(E_\alpha)^\sharp_{\gamma,n} = \{\varphi(a_1, \dots, a_n, E_\alpha, \eta_1, \dots, \eta_n) : a_1, \dots, a_n \in E_\gamma, \\ L(E_\alpha) \vDash \varphi(a_1, \dots, a_n, E_\alpha, \eta_1, \dots, \eta_n)\},$$

so

$$j((E_\alpha)^\sharp_{\gamma,n}) = \{\varphi(a_1, \dots, a_n, E_\alpha, \eta_1, \dots, \eta_n) : a_1, \dots, a_n \in E_{j(\gamma)}, \\ L(E_\alpha) \vDash \varphi(a_1, \dots, a_n, E_\alpha . \eta_1, \dots, \eta_n)\} = (E_\alpha)^\sharp_{j(\gamma),n}.$$

If $\beta < \alpha$, then the previous proof is almost valid, the problem is that in this case it is not possible to express j as an ultrapower embedding. The idea is to construct something similar to the ultrapower construction.

[1]The Gödel numbering is important here. The proof works only if it is definable, or at least preserved by k. For example, the most natural coding $\ulcorner\varphi(y, c_{a_1}, \dots, c_{a_n}, C, d_1, \dots, d_m)\urcorner = \langle \ulcorner\varphi(y, x_1, \dots, x_n, C, d_1, \dots, d_n)\urcorner, a_1, \dots, a_n\rangle$ works.

Let

$$\mathcal{Z} = \{j(F)(a) : a \in E_\alpha, F \in L(E_\beta), F \text{ is a function, } \operatorname{dom}(F) = E_\beta\}.$$

Then

1. $j``L(E_\beta) \subseteq \mathcal{Z}$: for $x \in L(E_\beta)$ consider c_x the constant x on E_β. Then $j(c_x)(a) = j(x)$ for every $a \in E_\alpha$.

2. $E_\alpha \subseteq \mathcal{Z}$: consider id the identity on E_β, then $j(id)(b) = b$ for every $b \in E_\alpha$.

3. $\mathcal{Z} \prec L(E_\alpha)$. Let $a_1, \dots, a_n \in E_\alpha$, $F_1, \dots, F_n$ functions in $L(E_\beta)$ with dominion E_β such that

$$L(E_\alpha) \vDash \exists x\, \varphi(x, j(F_1)(a_1), \dots, j(F_n)(a_n)).$$

The objective is to find a function $H \in L(E_\beta)$ with dominion E_β and $b \in E_\alpha$ such that

$$L(E_\alpha) \vDash \varphi(j(H)(b), j(F_1)(a_1), \dots, j(F_n)(a_n)).$$

Fix a $c \in E_\alpha$ such that a witness for φ is definable in $L(E_\alpha)$ with an ordinal and c as parameters. Note that E_β and E_α are closed for finite sequences, so it is possible to define

$$G(\langle \vec{b}, d\rangle) = \min\{\delta : \exists x \text{ definable from } \delta, d, \\ L(E_\beta) \vDash \varphi(x, F_1(b_1), \dots, F_n(b_n))\}$$

if it exists, otherwise $G(\langle \vec{b}, d\rangle) = \emptyset$. Then $G \in L(E_\beta)$, and define $H(\langle \vec{b}, d\rangle)$ as the least x definable from $G(\langle \vec{b}, d\rangle)$ and d such that $L(E_\beta) \vDash \varphi(x, F_1(b_1), \dots, F_n(b_n))$. Therefore

$$L(E_\alpha) \vDash \varphi(j(H)(\langle \vec{a}, c\rangle), j(F_1)(a_1), \dots, j(F_n)(a_n)).$$

and $\mathcal{Z} \prec L(E_\alpha)$ is proved.

By Lemma 2.18, then, the collapse of $\mathcal{Z}$ is $L(E_\alpha)$. Let $k : L(E_\alpha) \prec L(E_\alpha)$ be the inverse of the collapse. Since E_α is not collapsed, then $k \upharpoonright E_\alpha$ is the identity. Note that by (1) $j''L(E_\alpha) \subseteq k``L(E_\alpha)$ so there exists $j_{\mathcal{Z}} : L(E_\beta) \prec L(E_\alpha)$ such that $j = k \circ j_{\mathcal{Z}}$. But then $j \upharpoonright E_\alpha = j_{\mathcal{Z}} \upharpoonright E_\alpha$, so without loss of generality $j = j_{\mathcal{Z}}$.

Let η be a cardinal closed under $j_{\mathcal{Z}}$ such that $\operatorname{cof}(\eta) > |V_{\lambda+1}|$. Note that by the definition of $j_{\mathcal{Z}}$, $j_{\mathcal{Z}}(\gamma) = \operatorname{ot}(j(\gamma) \cap \mathcal{Z})$, so this means that for every

$\delta < \eta$, $\mathrm{ot}(j(\delta) \cap \mathcal{Z}) < \eta$. For every $F : E_\beta \to L(E_\beta)$ if $\mathrm{ran}(F) \subseteq \eta$ then $\mathrm{ran}(F)$ is bounded in η, i.e., there exists $\delta < \eta$ such that $F(a) < \delta$ for every $a \in E_\beta$, but this means that $j(F)(a) < j(\delta) < j(\eta)$ for every $a \in E_\alpha$, because by Definition 2.10 there exists in $L(E_\alpha)$ a $\pi : V_{\lambda+1} \twoheadrightarrow E_\beta$. But for every element $j(F)(a)$ in $j(\eta)$ we can suppose that $\mathrm{ran}(F) \subseteq \eta$, so $\{j(\delta) : \delta < \eta\}$ is cofinal in $j(\eta)$ and

$$\eta \leq j_{\mathcal{Z}}(\eta) = \mathrm{ot}(j(\eta) \cap \mathcal{Z}) = \bigcup_{\delta<\eta} \mathrm{ot}(j(\delta) \cap \mathcal{Z}) \leq \eta.$$

This proof provides a class of indiscernibles fixed by j, and like before this suffices to prove that j is $\sharp$-friendly. □

4 Sharp Reflection and Totally Non-proper Ordinals

While in the previous section were used mostly (γ, n)-fragments, it is worth considering also γ-fragments: note that if $\gamma < \alpha < \Upsilon$, both $(E_\alpha)^\sharp_\gamma$ and (E_γ) are theories in the same language, $\mathcal{L}^+_\gamma$. Can they be equal?

Definition 4.1. *Let $\gamma < \alpha < \Upsilon$. Then $(E_\alpha)^\sharp$* reflects *on γ if $(E_\alpha)^\sharp_\gamma = (E_\gamma)^\sharp$. For every $\alpha < \Upsilon$ limit define*

$$I_\alpha = \{\gamma < \alpha : (E_\alpha)^\sharp_\gamma = (E_\gamma)^\sharp\}$$

the set of the γ's in which $(E_\alpha)^\sharp$ is reflected.

Now it is possible to state the second (and most important) half of Theorem 2.23:

Theorem 4.2. *Let $\alpha < \Upsilon$ limit ordinal such that Θ^{E_α} is regular in $L(E_\alpha)$ and $\mathrm{ot}(I_\alpha) = \lambda$. Then α is totally non-proper.*

The other part of Theorem 2.23 will be a sufficient condition for the existence of an $\alpha < \Upsilon$ limit ordinal such that Θ^{E_α} is regular in $L(E_\alpha)$ and $\mathrm{ot}(I_\alpha) = \lambda$ (see Theorem 5.2).

The main point of the proof will be that, even if I_α is not an element of $L(E_\alpha)$, for every elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ by Lemma 3.5 the image of I_α under j is in I_α. Proving that the initial segments of I_α are definable will suffice.

Theorem 3.3 will be useful to state Definition 4.1 without using the sharp.

Lemma 4.3. *Let $\gamma < \alpha < \Upsilon$ limit ordinals. The following are equivalent:*

1. $\gamma \in I_\alpha$;

2. $E_\gamma \prec E_\alpha$ *and the identity is a* $\sharp$*-friendly elementary embedding;*

3. *there exists an elementary embedding* $j : L(E_\gamma) \prec L(E_\alpha)$ *with* $j \restriction E_\gamma = \mathrm{id}$.

Proof. The equivalence between (2) and (3) is a direct consequence of Theorem 3.3, so it is sufficient to prove the equivalence between (1) and (2).

Suppose that $\gamma \in I_\alpha$, let $\varphi(x_1, \ldots, x_n)$ be a formula and pick $a_1, \ldots, a_n \in E_\gamma$. Then $E_\gamma \vDash \varphi(a_1, \ldots, a_n)$ iff $L(E_\gamma) \vDash (E_\gamma \vDash \varphi(a_1, \ldots, a_n))$ iff

$$\ulcorner C \vDash \varphi(c_{a_1}, \ldots, c_{a_n})\urcorner \in (E_\gamma)^\sharp$$

iff

$$\ulcorner C \vDash \varphi(c_{a_1}, \ldots, c_{a_n})\urcorner \in (E_\alpha)^\sharp_\gamma$$

iff $L(E_\alpha) \vDash (E_\alpha \vDash \varphi(a_1, \ldots, a_n))$ iff $E_\alpha \vDash \varphi(a_1, \ldots, a_n)$. Moreover for every $\beta < \gamma$, $n \in \omega$,

$$(E_\gamma)^\sharp_{\beta,n} = (E_\gamma)^\sharp \cap \mathcal{L}^+_{\beta,n} = (E_\alpha)^\sharp \cap \mathcal{L}^+_\gamma \cap \mathcal{L}^+_{\beta,n} = (E_\alpha)^\sharp_{\beta,n}.$$

If $E_\gamma \prec E_\alpha$ and the identity is a $\sharp$-friendly elementary embedding, then

$$(E_\gamma)^\sharp = \bigcup_{\beta<\gamma, n\in\omega} (E_\gamma)^\sharp_{\beta,n} = \bigcup_{\beta<\gamma, n\in\omega} (E_\alpha)^\sharp_{\beta,n} = (E_\alpha)^\sharp_\gamma.$$

□

Using this equivalence it is possible to describe a necessary condition for an ordinal to be in I_β:

Lemma 4.4. *If* $\gamma \in I_\alpha$*, then there cannot exist an* $X \subseteq V_{\lambda+1}$ *such that* $L(E_\gamma) = L(X, V_{\lambda+1})$ *or* $L(E_\alpha) = L(X, V_{\lambda+1})$*.*

Proof. Suppose that there exists an $X \subseteq V_{\lambda+1}$ such that $L(E_\gamma) = L(X, V_{\lambda+1})$. Since $\gamma \in I_\alpha$ there exists an elementary embedding from $L(X, V_{\lambda+1})$ to $L(E_\alpha)$, so there exists an $Y \subseteq V_{\lambda+1}$ such that $L(E_\alpha) = L(Y, V_{\lambda+1})$. But by Lemma 4.3 there exists an elementary embedding $j : L(X, V_{\lambda+1}) \prec L(Y, V_{\lambda+1})$ such that $j(X) = X$. Therefore $L(Y, V_{\lambda+1}) = L(X, V_{\lambda+1})$ and this is a contradiction because $\gamma < \alpha$. If $L(E_\alpha) = L(X, V_{\lambda+1})$, then again for elementarity there exists $Y \subseteq V_{\lambda+1}$ such that $L(E_\gamma) = L(Y, V_{\lambda+1})$, and this is a contradiction. □

So by Lemma 2.12 if $I_\alpha \neq \emptyset$ it must be that $\Theta^{E_\alpha} = \sup_{\gamma<\alpha} \Theta^{E_\gamma}$.

The following lemma will establish two useful properties of I_α:

Lemma 4.5. *For every $\alpha \in I$:*

1. *for every $\gamma < \alpha$, if $\gamma \in I_\alpha$ then $I_\alpha \cap \gamma = I_\gamma$;*
2. *I_α is closed.*

Proof. 1. Let $\eta < \gamma$. Then

$$(E_\alpha)^\sharp_\eta = (E_\alpha)^\sharp \cap \mathcal{L}^+_{E_\eta} = (E_\alpha)^\sharp \cap \mathcal{L}^+_{E_\gamma} \cap \mathcal{L}^+_{E_\eta} = \\ (E_\alpha)^\sharp_\gamma \cap \mathcal{L}^+_{E_\eta} = (E_\gamma)^\sharp \cap \mathcal{L}^+_{E_\eta} = (E_\gamma)^\sharp_\eta.$$

So $(E_\eta)^\sharp = (E_\alpha)^\sharp_\eta$ iff $(E_\eta)^\sharp = (E_\gamma)^\sharp_\eta$ and $\eta \in I_\alpha$ iff $\eta \in I_\gamma$.

2. Let γ be a limit point of I_α. By Lemma 4.3 for every $\eta \in I_\alpha \cap \gamma$ there exists $\pi_{\eta,\alpha} : L(E_\eta) \prec L(E_\alpha)$ elementary embedding such that $\pi_{\eta,\alpha} \upharpoonright E_\eta = \mathrm{id}$, and by the previous point for every $\eta_1 < \eta_2 \in I_\alpha \cap \gamma$ there exists $\pi_{\eta_1,\eta_2} : L(E_{\eta_1}) \prec L(E_{\eta_2})$ such that $\pi_{\eta_1,\eta_2} \upharpoonright E_{\eta_1} = \mathrm{id}$. It is easy to see that all the $\pi_{\eta,\alpha}$'s and π_{η_1,η_2} commute, so

$$(\{L(E_\eta) : \eta \in I_\alpha\}, \{\pi_{\eta_1,\eta_2} : \eta_1 < \eta_2 \in I_\alpha\})$$

is a directed system that commutes with $\{\pi_{\eta,\alpha} : \eta \in I_\alpha\}$.

Let M be the direct limit of this system, with corresponding elementary embeddings $\pi_\eta : L(E_\eta) \prec M$ and $\pi_\alpha : M \prec L(E_\alpha)$. By elementarity there exists $N \subset V_{\lambda+2}$ such that $M = L(N)$. Since for every $\eta \in I_\alpha \cap \gamma$, π_η is the identity on E_η, it is clear that $N = \bigcup_{\eta \in I_\alpha \cap \gamma} E_\eta$. But γ is a limit point in I_α, therefore $N = E_\gamma$ and π_α can witness that $\gamma \in I_\alpha$.

□

Proof of Theorem 4.2. Let $\alpha < \Upsilon$ be a limit ordinal such that Θ^{E_α} is regular in $L(E_\alpha)$ and $\mathrm{ot}(I_\alpha) = \lambda$, and fix $j : L(E_\alpha) \prec L(E_\alpha)$. By Lemma 2.24 it suffices to prove that the fixed points of j are bounded under Θ^{E_α}. By Lemma 4.4 and Lemma 2.12 $\Theta^{E_\alpha} = \sup_{\beta<\alpha} \Theta^{E_\beta}$, and since Θ^{E_α} is regular in $L(E_\alpha)$ and by Lemma 2.20 $\langle \Theta^{E_\beta} : \beta < \alpha \rangle \in L(E_\alpha)$, $\alpha = \Theta^{E_\alpha}$. Since the ordertype of I_α is a limit and I_α is closed, it must be that $\sup I_\alpha = \alpha = \Theta^{E_\alpha}$.

Note that I_γ is definable with parameters γ, $\langle E_\eta : \eta < \gamma \rangle$ and $(E_\gamma)^\sharp$, so if $\gamma < \alpha$, then I_γ is definable in $L(E_\alpha)$ and $j(I_\gamma) = I_{j(\gamma)}$. Moreover, since by Lemma 3.5 j is $\sharp$-friendly, if $\gamma \in I_\alpha$ then $j(\gamma) \in I_\alpha$. Let $\langle \eta_\zeta : \zeta < \lambda \rangle$ be the enumeration of I_α. Then for every $\zeta < \lambda$, $j(\eta_\zeta) = \eta_{j(\zeta)}$. Consider $\langle \kappa_n : n \in \omega \rangle$ the critical sequence of j and let $\beta > \eta_{\kappa_0}$. Then there exists n such that $\eta_{\kappa_n} \leq \beta < \eta_{\kappa_{n+1}}$, and therefore $j(\eta_{\kappa_n}) = \eta_{\kappa_{n+1}} \leq j(\beta)$, so β is not a fixed point, and every fixed point of j must be under η_{κ_0}. □

5 HOD-Part and Sharp Reflection

In the last section Theorem 4.2 provides a criterion under which α is totally non-proper. One can ask if it is possible to meet this criterion. Theorem 5.2 gives an answer to this doubt, stating that if the E_α-sequence is long enough, then there exists an ordinal α such that I_α is big enough.

The following special initial segment of Υ clarifies the concept of "big enough":

Definition 5.1.

$$I = \{\alpha < \Upsilon : \forall \beta \leq \alpha \ \ L(E_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}.$$

There are many advantages in considering ordinals in I, because of the similiarities with $L(V_{\lambda+1})$ outlined in Proposition 2.16, Lemma 2.19 and Lemma 2.24.

Theorem 5.2. *Suppose there exists $\xi \notin I$. Then there exists $\eta \in I$ such that $\Theta^{E_\eta} = \eta$ and $\mathrm{ot}(I_\eta) = \eta$.*

Note that η is much more bigger then λ, so this Theorem gives a condition much stronger then what really is needed to have a totally non-proper ordinal. The advantage is that this condition is easily expressible and the proof shows quite clearly why there should be an α such that $\mathrm{ot}(I_\alpha) = \lambda$.

A ξ such that $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$ is interesting because its $\mathrm{HOD}_{V_{\lambda+1}}$-part has important closure properties.

Definition 5.3. *Let $\xi < \Upsilon$. Define*

$$H_\xi = \{\gamma \leq \xi : E_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}\}.$$

By definition if $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$ then $H_\xi \neq \xi$.

Lemma 5.4. *For $\xi < \Upsilon$ let $\eta = \sup H_\xi < \xi$. Then:*

1. *H_ξ is a closed initial segment of ξ and η is a limit ordinal, i.e., $H_\xi = \eta \cup \{\eta\}$;*
2. *$\Theta^{E_\eta} = \Theta^{(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}}$;*
3. *Θ^{E_η} is regular in $L(E_\eta)$;*
4. *$\Theta^{E_\eta} = \sup_{\beta<\eta} \Theta^{E_\beta}$;*
5. *$\eta = \Theta^{E_\eta}$.*

Proof. Define $N = (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)} \cap V_{\lambda+2}$. Then

$$\Theta^{(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}} = \Theta^N.$$

Let j an elementary embedding from $L(E_\xi)$ to itself with $\mathrm{crt}(j) < \lambda$ whose existence is granted by Lemma 2.13.

1. If $\gamma_1 < \gamma_2 \in H_\xi$, then by definition $E_{\gamma_1} \subseteq E_{\gamma_2} \subseteq N$, so $\gamma_1 \in H_\xi$. Suppose that $\gamma + 1 \in H_\xi$. Then by Definition there exists $X \in N$ such that $L(E_{\gamma+1}) = L(X, V_{\lambda+1})$. Every element of $(X, V_{\lambda+1})^\sharp$ is a formula that uses as parameters elements of $V_{\lambda+1}$, X and ordinals, so $(X, V_{\lambda+1})^\sharp \subset N$. Moreover, $(X, V_{\lambda+1})^\sharp$ is definable in $L(E_\xi)$ using $V_{\lambda+1}$ and X as parameters, and as these are elements of N, it follows that $(X, V_{\lambda+1})^\sharp \in N$. Since $L(E_{\gamma+2}) = L((X, V_{\lambda+1})^\sharp)$ then $\gamma + 2 \in H_\xi$. If γ is a limit ordinal and for every $\beta < \gamma$, $\beta \in H_\xi$, then by definition $E_\eta = L(\bigcup_{\beta<\eta} E_\beta) \cap V_{\lambda+2}$. By hypothesis $\bigcup_{\beta<\eta} E_\beta \subset N$, so $\gamma \in H_\xi$.

2. Looking for a contradiction, suppose that $\Theta^{E_\eta} < \Theta^N$. Then there exists a surjection $\pi\colon V_{\lambda+1} \twoheadrightarrow \Theta^{E_\eta}$, with $\pi \in N$, therefore definable in $L(E_\xi)$ with ordinals and elements of $V_{\lambda+1}$ as parameters. Let φ be the formula that defines π, with parameters $x_1, \dots, x_n \in V_{\lambda+1}$ and $\beta_1, \dots, \beta_m \in \mathrm{Ord}$. So $\pi(x) = y$ iff $L(E_\xi) \vDash \varphi(y, x, x_1, \dots, x_n, \beta_1, \dots, \beta_m)$. Define:

$$\bar\pi(\langle x, y_1, \dots, y_n\rangle) = \begin{cases} y & \text{if there exists } y \in E_\eta \text{ such that} \\ & \quad L(E_\xi) \vDash \varphi(y, x, y_1, \dots, y_n, \beta_1, \dots, \beta_m) \\ & \quad \text{and is unique;} \\ \emptyset & \text{otherwise.} \end{cases}$$

   Clearly $\bar\pi$ is in N, since the only parameters in its definition are $\beta_1, \dots, \beta_m, \eta$. It is a surjection, because in a subset of $V_{\lambda+1}$ (specifically $\{z \in V_{\lambda+1} : \exists x\ z = \langle x, x_1, \dots, x_n\rangle\}$) is already a surjection, so $\bar\pi$ definable in $L(E_\xi)$ only with ordinal parameters. Between all the surjections, we pick the one that uses the minimum ordinal parameters. Then without loss of generality π is definable in $L(E_\xi)$ without parameters, therefore $j(\pi) = \pi$. Thus the restriction of j in $L(\pi, V_{\lambda+1})$ is a proper elementary embedding, and $\Theta^{E_\eta} < \Theta^{L(\pi, V_{\lambda+1})}$. So by Theorem 2.14, $\Upsilon^\pi > \eta + 1$, and then $E_{\eta+1} \subseteq L(\pi, V_{\lambda+1})$. But, since π is definable in $L(E_\xi)$, we have that $L(\pi, V_{\lambda+1}) \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, and therefore $E_{\eta+1} \subseteq N$, contradiction because η was the maximum one.

3. Since by Lemma 2.16 Θ^N is regular in $(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$ and as $E_\eta \subseteq N$, we have by (2) that Θ^{E_η} is regular in $L(E_\eta)$ (otherwise a cofinal

sequence in $L(E_\eta) \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$ would witness that Θ^N is not regular).

4. As Θ^{E_η} is regular in $L(E_\eta)$, by Definition 2.10 $E_{\eta+1} = L(\mathcal{E}(E_\eta)) \cap V_{\lambda+2}$. If Θ^{E_η} could be strictly bigger than $\sup_{\beta<\eta} \Theta^{E_\beta}$, then by Lemma 2.12 it would exist $Y \in E_\eta$ such that $L(E_\eta) = L(Y, V_{\lambda+1})$. Let $\langle i_n : n \in \omega\rangle$ be the sequence of the first ω indiscernibles in $L(Y, V_{\lambda+1})$ and define

$$X_n = \{x \in E_\eta : x \text{ is definable with parameters from } V_{\lambda+1} \cup \{Y, i_1, \dots, i_n\}\}.$$

Note that it is possible to codify X_n as an element of E_η. By Lemma 2.1 every element of E_η is defined with parameters from $\{Y\} \cup V_{\lambda+1}$ and the first ω indiscernibles in $L(Y, V_{\lambda+1})$, so $E_\eta \subseteq \bigcup_{n\in\omega} L_\omega(X_n, V_{\lambda+1})$. Therefore every $k \in \mathcal{E}(E_\eta)$ is uniquely specified by the sequence $\langle k(X_n) : n \in \omega\rangle$. Since $\langle X_n : n \in \omega\rangle \in L((Y, V_{\lambda+1})^\sharp)$, it is possible to codify E_η as a subset of $V_{\lambda+1}$ in $L((Y, V_{\lambda+1})^\sharp)$, so $(E_\eta)^\omega \in L((Y, V_{\lambda+1})^\sharp)$ and

$$\langle k(X_n) : n \in \omega\rangle \in L((Y, V_{\lambda+1})^\sharp).$$

But then $k \in L((Y, V_{\lambda+1})^\sharp)$, thus $L(\mathcal{E}(E_\eta)) \subseteq L((Y, V_{\lambda+1})^\sharp)$. Therefore

$$E_{\eta+1} = L(\mathcal{E}(E_\eta)) \cap V_{\lambda+2} \subseteq L((Y, V_{\lambda+1})^\sharp).$$

Since $Y \in (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, then $(Y, V_{\lambda+1})^\sharp \in (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, and therefore $E_{\eta+1} \subseteq N$. This is a contradiction, because η was the maximum one.

5. It follows directly from (3) and (4).

□

The first part of the proof of Theorem 5.2 uses the properties in Lemma 5.4:

Lemma 5.5. *Let $\xi < \Upsilon$. If $\eta \in H_\xi$ and $\eta = \Theta^{E_\eta} = \Theta^{(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}}$, for example when $\eta = \sup H_\xi < \xi$, then*

1. *$E_\eta = \bigcup_{\beta<\eta} E_\beta$;*

2. *if $\eta < \xi$, then $L((E_\eta)^\sharp) \cap V_{\lambda+2} \subseteq E_\eta$.*

Proof. Again, define $N = (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)} \cap V_{\lambda+2}$.

1. Let $Y \in E_\eta = L(\bigcup_{\beta<\Theta^{E_\eta}} E_\beta) \cap V_{\lambda+2}$. Call $\mathcal{X}$ the Skolem closure of $\{Y\} \cup V_{\lambda+1}$ in $L(\bigcup_{\beta<\Theta^{E_\eta}} E_\beta)$. Then the collapse of $\mathcal{X}$ is isomorphic to $L(\bigcup_{\beta<\Theta^{E_\eta}} E_\beta)$, i.e., $Coll(\mathcal{X}) \vDash V = L(\bigcup_{\beta<\Theta} E_\beta)$, and so by Lemma 2.18 is a set like $L_\gamma(\bigcup_{\beta<\bar{\Theta}} E_\beta)$. As $E_\eta \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, we have that $L(E_\eta) \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, therefore one can construct partial Skolem functions:

$$h_{\varphi,a}(x_1,\dots,x_n) = y \quad \text{where } y \text{ is the minimum in } (\mathrm{OD}_a)^{L(E_\xi)} \text{ such that } L(E_\eta) \vDash \varphi(y,x_1,\dots,x_n)$$

for $a \in V_{\lambda+1}$, φ formula and $x_1,\dots,x_n \in L(E_\eta)$. All these Skolem functions are in $(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, and for all $y \in \mathcal{X}$ there exist φ formula, $a \in V_{\lambda+1}$ and $x_1,\dots,x_n \in \{Y\}\cup V_{\lambda+1}$ such that $y = h_{\phi,a}(x_1,\dots,x_n)$. Therefore it is possible to construct $\rho\colon V_{\lambda+1} \twoheadrightarrow \mathcal{X}$, with $\rho \in (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$. But then $\gamma, \bar{\Theta} < \Theta^N$. Both Y and its elements are in $\mathcal{X}$, so Y is not moved by the collapsing map. Then

$$Y \in L_\gamma(\bigcup_{\beta<\bar{\Theta}} E_\beta) \cap V_{\lambda+2} \subseteq E_{\bar{\Theta}} \subseteq \bigcup_{\beta<\Theta^N} E_\beta.$$

2. Let $Y \in L((E_\eta)^\sharp) \cap V_{\lambda+2}$. Let $\mathcal{X}$ be the Skolem closure of $\{Y\} \cup V_{\lambda+1}$ in $L((E_\eta)^\sharp) = L((\bigcup_{\beta<\Theta^N} E_\beta)^\sharp)$. Then $Y \in \mathcal{X}$, $V_{\lambda+1} \in \mathcal{X}$ and as in the previous proof Y is in the collapse of $\mathcal{X}$, that is a set like $L_\gamma((\bigcup_{\beta<\bar{\Theta}} E_\beta)^\sharp)$, with $\gamma, \bar{\Theta} < \eta$ (note that $(E_\eta)^\sharp \in (\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$, so the construction of $\mathcal{X}$ can be carried out in $(\mathrm{HOD}_{V_{\lambda+1}})^{L(E_\xi)}$ with "few" partial Skolem functions). But $L((\bigcup_{\beta<\bar{\Theta}} E_\beta)^\sharp) \cap V_{\lambda+2} \subseteq E_{\bar{\Theta}}$, therefore $Y \in \bigcup_{\beta<\eta} E_\beta$.

□

Such η has the advantage that $L((E_\eta)^\sharp)$ will inherit the properties of $L(E_\eta)$ that depend on its $V_{\lambda+2}$ part.

Lemma 5.6. *Let η such that $L(E_\eta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, $\eta = \Theta^{E_\eta}$ and $L((E_\eta)^\sharp) \cap V_{\lambda+2} = E_\eta$. Then*

1. $\eta = \Theta^{(E_\eta)^\sharp}$;

2. $L((E_\eta)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$.

Proof. 1. Every ordinal under $\Theta^{(E_\eta)^\sharp}$ can be coded in $L((E_\eta)^\sharp)$ as an element of $V_{\lambda+2}$. But then it can be coded as an element of E_η too, so $\Theta^{(E_\eta)^\sharp} = \Theta^{E_\eta} = \eta$.

2. By Lemma 2.19 every element of E_η is definable in $L(E_\eta)$ with parameters from $\Theta^{E_\eta} \cup V_{\lambda+1}$. Since E_η is definable in $L((E_\eta)^\sharp)$ this is also true in $L((E_\eta)^\sharp)$. Every element of $(E_\eta)^\sharp$ is definable with parameters from E_η, and this concludes the proof.

□

The following lemma will complete the proof of Theorem 5.2:

Lemma 5.7. *Let $\eta \in I$ such that $L((E_\eta)^\sharp) \cap V_{\lambda+2} = E_\eta$. Then* $\mathrm{ot}(I_\eta) = \eta$.

Proof. Code $(E_\eta)^\sharp$ as a subset of E_η and let $\gamma \in \eta$. Then consider $H^{(E_\eta)^\sharp}((E_\eta)^\sharp_\gamma)$. Since $L((E_\eta)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$ like in the proof of Lemma 5.5 there are "few" partial Skolem functions, so there is a surjection from $V_{\lambda+1}$ to $H^{(E_\eta)^\sharp}((E_\eta)^\sharp_\gamma)$, and this means that there exists $\gamma_1 < \eta$ such that $H^{(E_\eta)^\sharp}((E_\eta)^\sharp_\gamma) \subset (E_\eta)^\sharp_{\gamma_1}$. Iterating this process ω times, there exists $\beta < \eta$ such that $(E_\eta)^\sharp_\beta \prec (E_\beta)^\sharp$, so by Lemma 2.1 $\beta \in I_\eta$.

This means that I_η is cofinal in η. Since I_η is definable in $L((E_\eta)^\sharp)$ and $\eta = \Theta^{(E_\eta)^\sharp}$ is regular in $L((E_\eta)^\sharp)$ by Lemma 2.16, $\mathrm{ot}(I_\eta) = \eta$. □

This proof suggests that the sharp reflection is a *closure property*, so as long as it is possible to find α's that are regular in $L((E^0_\alpha)^\sharp)$ (and that seems reasonable), it is also possible to reflect the sharp for any desired times less than α. The important point is that this should happen before leaving the safe haven of I, and Lemma 5.7 assures it.

Proof of Theorem 5.2. Suppose that there exists $\xi \notin I$. Then by Lemma 5.5 there exists an η such that $L((E_\eta)^\sharp) \cap V_{\lambda+2} = E_\eta$ and $\eta = \Theta^{E_\eta}$. Consider the smallest of such η. Then η must be in I, because if $\beta \leq \eta$ weren't in I then $\sup H_\beta$ would be strictly smaller than η but with the same properties, and this is a contradiction because η was the minimum one. So by Lemma 5.7 η satisfies the Theorem. □

Proof of Theorem 2.23. Suppose that there exists $\xi \notin I$. Then by Theorem 5.2 there exists $\eta \in I$ such that $\mathrm{ot}(I_\eta) = \eta$, with $\eta > \lambda$. Therefore if α is the λ-th element of I_η then $I_\alpha = I_\eta \cap \alpha$ and $\mathrm{ot}(I_\alpha) = \lambda$. By Theorem 4.2 α is totally non-proper. □

Note that in fact as a consequence of the existence of a $\xi \notin I$ there are many totally non-proper ordinals, and not just only one. Theorem 4.2 could work even if the ordertype of I_α would be, for example, $\lambda + \lambda$, or λ^2, or λ^λ (ordinal exponential). The proof of the Theorem can be generalized to state that if the ordertype of I_α is a limit, then the elementary embedding lifts its behaviour to Θ^{E_α}, so one needs only an ordinal under which the fixed points are bounded for *every* elementary embedding, like the examples above.

6 Acknowledgments

I would like to thank Prof. Hugh W. Woodin for his invaluable help during the last two years and for introducing me to this new, fascinating subject; Prof. Alessandro Andretta for his collaboration, encouragement and for giving me the right hint at the right time; Prof. Sy-David Friedman for his support; Dr. Clinton Conley for providing language help. I would like also to acknowledge the support of the Austrian Science Fund FWF under Project P 20835-N13.

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