Filtering in Stochastic Systems: Analysis for the case of continuous observations with memory of arbitrary multiplicity

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Abstract. We consider stochastic systems with continuous time over observations with memory in the presence of an anomalous noise. The paper is devoted to analysis of some properties of an optimal unbiased in mean-square sense filter. In the case of anomalous noises action in the observation channel with memory, we have proved insensitivity of the filter to inaccurate knowledge of the matrix of anomalous noise intensity and its equivalence to a truncated filter constructed only over non-anomalous components of an observation vector.

1. Introduction

The article considers the topical estimations of multidimensional dynamic systems, the behavior of which is described by stochastic differential equations. In this paper some properties of the filter, the synthesis of which was carried out in [1], are investigated. Problems of estimation of casual processes, and also problems of detection with anomalous noises, represent considerable interest both from the theoretical and practical points of view. The analysis of efficiency of the received algorithm of estimation is given for the model of measuring channels with memory of arbitrary multiplicity on the basis of the offered algorithm of the synthesis optimal in mean square sense of not displaced filter - interpolator.

Later: $P\{\cdot\}$ is a probability of an event; $M\{\cdot\}$ is a mathematical expectation; $tr[\cdot]$ is the trace of a matrix, "T" and "+" are transposition and pseudo-inversion of a matrix, if they stand as right superscripts; $\delta(\cdot)$ is Dirac delta-function; 0 is zero vector of the appropriate size; O is zero matrix of an appropriate size, I_k is identity $(k\times k)$-matrix, $A>0\,(\geq 0)$ is a positively (non-negatively) determined matrix.

2. Problem formulation

The system described by equations (a dot on top means derivative then everywhere on t)

$$\dot{x}(t) = F(t)x(t) + \omega(t), t \geq 0, \tag{1}$$

where $x(t)$ is an n-dimensional process, which is a Gaussian Markov process (n-dimensional state vector), $\omega(t)$ is an n-dimensional vector of perturbations, which is a white Gaussian process with $M\{\omega(t)\}=0$ and $M\{\omega(t)\omega^T(t)\}=Q(t)\delta(t-s)$, where $Q(t)$ is the intensity matrix.

The channel monitoring output is l-dimensional process $z(t)$ which describes the state of the system and has the following form:

$$z(t)=H_0(t)x(t)+\sum_{k=1}^{N}H_k(t)x(\tau_k)+\upsilon(t)+C\,f(t), \qquad (2)$$

where $0<\tau_N<\tau_{N-1}<\ldots<\tau_1<t$. In (2), $\upsilon(t)$ is a white l-dimensional Gaussian process with intensity matrix $R(t)$, which is a regular noise, $f(t)$ is an r-dimensional $(r\le l)$ white Gaussian process with unknown mean $f_0(t)$ and intensity matrix $\Theta(t)$, which is anomalous noise. Moreover, $M\{\upsilon(t)\}=0$,

$$M\{\upsilon(t)\upsilon^T(s)=R(t)\delta(t-s)\},\ M\{f(t)\}=f_0(t),$$

$$M\{[f(t)-f_0(t)][f(s)-f_0(s)]^T\}=\Theta(t)\delta(t-s).$$

Matrix C $(l\times r)$, determining the structure of the effect of anomalous noise components on the components of the observed process, is Boolean one of the following type: if $i_1,i_2,\ldots i_r$ are numbers of vector components $z(t)$, on which anomalous components produce an effect that $f(t)$, in a column with numbers j, the unit is placed on i_j position $(1\le j\le r;1\le i_j\le l)$. It is supposed that: 1) $x(0)=x_0$ has normal distribution with parameters μ_0 and $\tilde{A}_0$; 2) $x_0,\omega(t),\upsilon(t),f(t)$ are independent in the aggregate; 3) matrices $\tilde{A}(t),Q(t),R(t),\Theta(t)$ are positively determined; 4) $f_0(t)$ is unknown.

The following problem is formulated: the sensitivity of the filter to check to inaccurate knowledge of the matrix of intensity $\Theta(t)$ for $f(t)$, where filter synthesized in [1].

We will enter white processes $\tilde{\omega}(t)$, $\tilde{x}_{N+1}(\tilde{\tau}_N,t)$, $\tilde{\mu}^0_{N+1}(\tilde{\tau}_N,t)$ $(\tilde{\tau}_N=[\tau_1,\tau_2,\cdots\tau_N])$ $(\tilde{\tau}_N=[\tau_1,\tau_2,\cdots\tau_N])$ of sizes $(N+1)n$ of a look into consideration

$$\tilde{\omega}(t)=\begin{bmatrix}\omega(t)\\ \cdots\\ 0\end{bmatrix},\quad \tilde{x}_{N+1}(\tilde{\tau}_N,t)=\begin{bmatrix}x(t)\\ \cdots\\ x(\tau_k)\end{bmatrix},\quad \tilde{\mu}_{N+1}(\tilde{\tau}_N,t)=\begin{bmatrix}\mu(t)\\ \cdots\\ \mu(\tau_k,t)\end{bmatrix},\quad k=\overline{1;N}, \qquad (3)$$

where $\tilde{\mu}^0_{N+1}(\tilde{\tau}_N,t)=\tilde{x}_{N+1}(\tilde{\tau}_N,t)-\tilde{\mu}_{N+1}(\tilde{\tau}_N,t)$, also block matrixes

$$\tilde{F}(t)=\begin{bmatrix}F(t) & 0\\ 0 & 0\end{bmatrix},\quad K(t)=\begin{bmatrix}\tilde{H}_0^T(t)\tilde{R}^{-1}(t)\\ \tilde{H}_k^T(t)\tilde{R}^{-1}(t)\end{bmatrix}=\begin{bmatrix}K_0(t)\\ K_k(t)\end{bmatrix},\qquad \tilde{Q}(t)=\left[\begin{array}{c|c}Q(t) & 0\\ \hline 0 & 0\end{array}\right]$$

$$H(t)=\left[H_0(t)\,\middle|\,H_k(t)\right],\ k=\overline{1;N}. \qquad (4)$$

Sizes are $[(N+1)n]\times[(N+1)n)]$, $[(N+1)n]\times l, l\times[(N+1)n)]$, $[(N+1)n]\times[(N+1)n)]$, $l\times n(N+1)$ respectively.

$$\tilde{K}(t)=K(t)\tilde{S}(t),\quad \tilde{S}(t)=I-CS(t), \qquad (5)$$

where $S(t)$ is the matrix, the choice of which is determined by the condition of estimates unbiasedness in [1].

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3. Analysis of sensitivity

Filter sensitivity, determined by theorem in [1], to inaccurate knowledge of the matrix of anomalous noise intensity is investigated using the technique [2,3]. Let us suppose $\Theta^*(t)$ is correct, $\Theta(t)$ is the noise intensity matrix used in the filter, , and $\tilde{\mu}_r^0(\tilde{\tau}_N,t)$ is real estimation error $\tilde{\mu}_r(\tilde{\tau}_N,t)$. The equation for $\tilde{\mu}_r(\tilde{\tau}_N,t)$ follows from [1]:

$$\dot{\tilde{\mu}}_r(\tilde{\tau}_N,t)=\tilde{F}(t)\tilde{\mu}_r(\tilde{\tau}_N,t)+\tilde{K}(t)[z_r(t)-H(t)\tilde{\mu}_r(\tilde{\tau}_N,t)], \tag{6}$$

where $z_r(t)$ is real observations with the real matrix of intensity $\Theta^*(t)$ for $f(t)$,

$$\tilde{K}(t)=K(t)\tilde{S}(t)=\begin{bmatrix}\tilde{K}_0(t)\\ \tilde{K}_k(t)\end{bmatrix}=\begin{bmatrix}K_0(t)\tilde{S}(t)\\ K_k(t)\tilde{S}(t)\end{bmatrix}=\begin{bmatrix}\tilde{H}_0^T(t)\tilde{R}^{-1}(t)\tilde{S}(t)\\ \tilde{H}_k^T(t)\tilde{R}^{-1}(t)\tilde{S}(t)\end{bmatrix}, \tag{7}$$

$$\tilde{R}(t)=R(t)+C\Theta(t)C^T, \tag{8}$$

where $\tilde{H}_0(t)$ and $\tilde{H}_k(t)$ are determined by formulas

$$\tilde{H}_0(t)=H_0(t)\Gamma(t)+\sum_{j=1}^{N}H_j(t)\Gamma_{0j}^T(\tau_j,t),\qquad \tilde{H}_k(t)=H_k(t)\Gamma_{kk}(\tau_k,t)+\sum_{j\neq k}^{N}H_j(t)\Gamma_{kj}^T(\tau_j,\tau_k,t),$$

in [1]. Process $\tilde{x}_{N+1}(\tilde{\tau}_N,t)=\left[x(t)\ \vdots\ \ldots|\ \ x(\tau)\right]$, as it follows from [1], is determined by equation

$$\dot{\tilde{x}}_{N+1}(\tilde{\tau}_N,t)=\tilde{F}(t)\tilde{x}_{N+1}(\tilde{\tau}_N,t)+\tilde{\omega}(t). \tag{9}$$

As $z_r(t)=H(t)\tilde{x}_{N+1}(\tilde{\tau}_{N+1},t)+\tilde{\upsilon}(t)$, where $\tilde{\upsilon}(t)=\upsilon(t)+Cf(t)$, then from (6), (9) it follows that error $\tilde{\mu}_r^0(\tilde{\tau}_N,t)$ of real estimation $\tilde{\mu}_r(\tilde{\tau}_n,t)$ is determined by equation

$$\dot{\tilde{\mu}}_r^0(\tilde{\tau}_N,t)=\bar{F}(t)\tilde{\mu}_r^0(\tilde{\tau}_N,t)+\tilde{\omega}(t)-\tilde{K}(t)\tilde{\upsilon}(t), \tag{10}$$

where $\bar{F}(t)=\tilde{F}(t)-\tilde{K}(t)H(t)$. Let $\Phi(t,\sigma)$ be the transition matrix corresponding to matrix $\bar{F}(t)$. Then equation solution (10) can be written as

$$\tilde{\mu}_r^0(\tilde{\tau}_N,t)=\Phi(t,\tau_1+0)\tilde{\mu}_r^0(\tilde{\tau}_N,\tau_1+0)+\int_{\tau_1+0}^{t}\Phi(t,\sigma)\left[\tilde{\omega}(\sigma)-K(\sigma)\tilde{\upsilon}(\sigma)\right]d\sigma. \tag{11}$$

Hence, taking independence into account $x_{0,}\ \omega(t),\upsilon(t),\ f(t)$, it follows that the matrix of second moments of real estimation error $\tilde{\Gamma}_{N+1}^r(\tilde{\tau}_N,t)=M\left\{\tilde{\mu}_r^0(\tilde{\tau}_N,t)\left(\tilde{\mu}_r^0(\tilde{\tau}_N,t)\right)^T\right\}$ is expressed by formula

$$\tilde{\Gamma}_{N+1}^r(\tilde{\tau}_N,t)=\bar{\Phi}(t,\tau_1+0)M\left\{\tilde{\mu}_r^0(\tilde{\tau}_N,\tau_1+0)\left(\tilde{\mu}_r^0(\tilde{\tau}_N,\tau_1+0)\right)^T\right\}\bar{\Phi}^T(t,\tau_1+0)+$$
$$+\int_{\tau_1+0}^{t}\int\bar{\Phi}(t,\sigma)\left[M\left\{\tilde{\omega}(\sigma)\tilde{\omega}^T(\xi)\right\}+\tilde{K}(\sigma)M\left\{\tilde{\upsilon}(\sigma)\tilde{\upsilon}^T(\xi)\right\}\tilde{K}^T(\xi)\right]\bar{\Phi}^T(t,\xi)d\sigma d\xi.$$

(12)

According to [1]

$$M\left\{\tilde{\omega}(\sigma)\tilde{\omega}^T(\xi)\right\}=\tilde{Q}(\sigma)\delta(\sigma-\xi). \tag{13}$$

As $\upsilon(t)$ and $f(t)$ are independent in the problem formulation, and a correct value of intensity matrix $f(t)$ is equal to $\Theta^*(t)$, then according to [1]

$$M\left\{\tilde{\upsilon}(\sigma)\tilde{\upsilon}^T(\xi)\right\}=R^*(\sigma)\delta(\sigma-\xi)+Cf_0(\sigma)f_0^T(\xi)C^T, \tag{14}$$

where $R^*(\sigma)=R(\sigma)+C\Theta(\sigma)C^T$. As $\tilde{K}(t)=K(t)\tilde{S}(t)$, after substitution (13), (14) in (12), taking unbiasedness property $\tilde{S}(t)C=0$ and properties of δ-Dirac function into account, we get:

$$\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)=\bar{\Phi}(t,\tau_1+0)M\left\{\tilde{\mu}^0_r(\tilde{\tau}_N,t)\left(\tilde{\mu}^0_r(\tilde{\tau}_N,t)\right)^T\right\}\bar{\Phi}^T(t,\tau_1+0)+\int_{\tau_1+0}^{t}\bar{\Phi}(t,\sigma)Q^*(\sigma)\bar{\Phi}^T(t,\sigma)d\sigma=$$
$$=\bar{\Phi}(t,\tau_1+0)\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)\bar{\Phi}^T(t,\tau_1+0)+\int_{\tau_1+0}^{t}\bar{\Phi}(t,\sigma)Q^*(\sigma)\bar{\Phi}^T(t,\sigma)d\sigma, \quad (15)$$

where

$$Q^*(t)=\tilde{Q}(t)+\tilde{K}(\sigma)R^*(t)\tilde{K}^T(\sigma). \quad (16)$$

Differentiating (15) over t leads to the equation of the following form:

$$\dot{\tilde{\Gamma}}^r_{N+1}(\tilde{\tau}_N,t)=\bar{F}_0(t)\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)+\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)\bar{F}_0(t)+\tilde{K}(t)R^*(t)\tilde{K}^T(t)+\tilde{Q}(t). \quad (17)$$

Let us introduce and consider sensitivity function

$$\Psi_{ij}(t)=\left.\frac{\partial\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)}{\partial\Theta^*_{ij}}\right|_{\Theta^*_{ij}=\Theta_{ij}} \quad (18)$$

This is a matrix of second moment's matrix $\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)=M\left\{\tilde{\mu}^0_r(\tilde{\tau}_N,t)\left(\tilde{\mu}^0_r(\tilde{\tau}_N,t)\right)^T\right\}$ of real estimation error to (i,j)-element of the matrix of anomalous noise intensity. Then from (17) and (18) the following equation for $\Psi_{ij}(t)$ follows:

$$\dot{\Psi}_{ij}(t)=\tilde{K}(t)CI_{ij}C^T\tilde{K}^T(t),\quad \Psi_{ij}(\tau_1)=O, \quad (19)$$

where I_{ij} is a Boolean $(r\times r)$-matrix, which unit is situated at a (i,j)-place, and other elements are equal to zero. As $\tilde{K}(t)=K(t)\tilde{S}(t)$, then taking unbiasedness property into account $\tilde{S}(t)C=0$, we get $\psi_{ij}(t)=0$ for all $i=\overline{1;r}$, $j=\overline{1;r}$. Thus, we have got the following statement.

***T h e o r e m* 1**. An optimal unbiased filter, in the mean-square sense, synthesized in [1], is insensitive to inaccurate knowledge of matrix of anomalous noise intensity.

Let us investigate the Bayesian unbiased filter with memory for the property of sensitivity to inaccurate knowledge of the matrix of anomalous noise intensity. The filter is determined by equation in [1]:

$$\dot{\tilde{\mu}}_{N+1}(\tilde{\tau}_N,t)=\tilde{F}(t)\tilde{\mu}_{N+1}(\tilde{\tau}_N,t)+K(t)\tilde{S}(t)\tilde{\tilde{z}}(t),\quad \tilde{\tilde{z}}(t)=\tilde{z}(t)-Cf_0(t),$$
$$\tilde{z}(t)=z(t)-[H_0(t)\mu(t)+\sum_{j=1}^{N}H_j(t)\mu(\tau_j,t)]=z(t)-H(t)\tilde{\mu}_{N+1}(\tilde{\tau}_N,t) \quad (20)$$

Then the real estimation for it is $\tilde{\mu}_r(\tilde{\tau}_N,t)$ and error $\tilde{\mu}^0_r(\tilde{\tau}_N,t)$ will be determined by equations:

$$\dot{\tilde{\mu}}_r(\tilde{\tau}_N,t)=\tilde{F}(t)\tilde{\mu}_r(\tilde{\tau}_N,t)+K(t)\left[z_r(t)-H(t)\tilde{\mu}_r(\tilde{\tau}_N,t)-Cf_0(t)\right], \quad (21)$$

$$\dot{\tilde{\mu}}^0_r(\tilde{\tau}_N,t)=\tilde{\tilde{F}}(t)\tilde{\mu}^0_r(\tilde{\tau}_N,t)+\tilde{\omega}(t)-K(t)\left[\tilde{\upsilon}(t)+C\tilde{f}(t)\right], \quad (22)$$

where $\tilde{\tilde{F}}(t)=\tilde{F}(t)-K(t)H(t)$, $\tilde{f}(t)=f(t)-f_0(t)$, which are obtained similarly to (6), (10). Introducing transfer matrix $\tilde{\Phi}(t,\sigma)$, corresponding to matrix $\tilde{\tilde{F}}(t)$, into consideration, we get equation for $\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)=M\left\{\tilde{\mu}^0_r(\tilde{\tau}_N,t)\left(\tilde{\mu}^0_r(\tilde{\tau}_N,t)\right)^T\right\}$ Bayesian filter with memory:

$$\dot{\tilde{\Gamma}}^r_{N+1}(\tilde{\tau}_N,t)=\tilde{\tilde{F}}(t)\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)+\tilde{\Gamma}^r_{N+1}(\tilde{\tau}_N,t)\tilde{\tilde{F}}_0(t)+K(t)R^*(t)K^T(t)+\tilde{Q}(t). \quad (23)$$

The derivation of this equation is similar to derivation of (17). The equation for sensitivity function (18) according to (23) is written as

$$\dot{\Psi}_{ij}(t)=K(t)CI_{ij}C^TK^T(t),\quad \Psi_{ij}(\tau_1)=0. \quad (24)$$

Thus,

$$\Psi_{ij}(t) = \int_{\tau_1+0}^{t} \tilde{\Phi}(t,\sigma)K(\tau)CI_{ij}C^T K^T(\tau)\tilde{\Phi}^T(t,\sigma)d\tau, \tag{25}$$

i.e. Bayesian filter with memory is insensitive to inaccurate knowledge of the matrix of anomalous noise intensity.

4. Filter structure

Let us suppose that $i_1, i_2, ..., i_r$ are components numbers of the vector of observation $z(t)$, in which components $f_1(t), f_2(t), ..., f_r(t)$ are the vector of the anomalous noise $f(t)$. We will suppose that $\bar{z}(t)$ is a vector of size $(l-r)$, which is obtained from observation vector $z(t)$ excluding anomalous components $z_{i_1}(t), z_{i_2}(t), ..., z_{i_r}(t)$. Let us suppose that $\bar{H}_0(t), \bar{H}_k(t)$ are matrices of size $[(l-r)\times n]$, and $\bar{R}(t)$ is a matrix of size $[(l-r)\times(l-r)]$, which can be obtained from matrices $H_0(t), H_k(t)$ excluding rows and correspondingly excluding rows and columns with numbers $i_1, i_2, ..., i_r$. Thus, the observed process is of a $\bar{z}(t)$ size of a $(l-r)$ form:

$$\bar{z}(t) = \bar{H}_0(t)x(t) + \sum_{k=1}^{N} \bar{H}_k(t)x(\tau_k) + \bar{\upsilon}(t), \tag{26}$$

where $\bar{H}(t) = [\bar{H}_0(t) \mid \bar{H}_k(t)]$, $(k = \overline{1;N})$, $\bar{\upsilon}(t)$ is a $(l-r)$ -vector, which is obtained from the vector of regular noises excluding components with numbers $i_1, i_2, ..., i_r$, which will be free from anomalous noises. Process $\bar{z}(t)$ will be called as a truncated vector of observations, and Bayesian optimal filter, in the mean-square sense, constructed over $\bar{z}(t)$, will be called as a truncated filter.

S t a t e m e n t. The truncated filter is determined by equations:

$$\dot{\mu}(t) = F(t)\bar{\mu}(t) + \tilde{\bar{H}}_0^T(t)\bar{R}^{-1}(t)\tilde{\bar{z}}(t), \tag{27}$$

$$\dot{\mu}(\tau_k, t) = \tilde{\bar{H}}_k^T(t)\bar{R}^{-1}(t)\tilde{\bar{z}}(t),\ k = \overline{1;N} \tag{28}$$

$$\dot{\tilde{\bar{A}}}(t) = F(t)\bar{\Gamma}(t) + \bar{\Gamma}(t)F^T(t) + Q(t) - \tilde{\bar{H}}_0^T(t)\bar{R}^{-1}(t)\tilde{\bar{H}}_0(t), \tag{29}$$

$$\dot{\bar{\Gamma}}_{kk}(\tau_k, t) = -\tilde{\bar{H}}_k^T(t)\bar{R}^{-1}(t)\tilde{\bar{H}}_k(t),\ k = \overline{1;N} \tag{30}$$

$$\dot{\bar{\Gamma}}_{0k}(\tau_k, t) = F(t)\bar{\Gamma}_{0k}(\tau_k, t) - \tilde{\bar{H}}_0^T(t)\bar{R}^{-1}(t)\tilde{\bar{H}}_k(t),\ k = \overline{1;N} \tag{31}$$

$$\dot{\bar{\Gamma}}_{kl}(\tau_l, \tau_k, t) = -\tilde{\bar{H}}_k^T(t)\tilde{R}^{-1}(t)\tilde{\bar{H}}_l(t),\quad k = \overline{1;N-1},\quad l = \overline{2;N},\quad l > k, \tag{32}$$

where

$$\tilde{\bar{z}}(t) = \bar{z}(t) - [\tilde{\bar{H}}_0(t)\bar{\mu}(t) + \sum_{j=1}^{N} \tilde{\bar{H}}_j(t)\bar{\mu}(\tau_j, t)], \tag{33}$$

$$\tilde{\bar{H}}_0(t) = \bar{H}_0(t)\bar{\Gamma}(t) + \sum_{j=1}^{N} \bar{H}_j(t)\bar{\Gamma}_{0j}^T(\tau_j, t), \tag{34}$$

$$\tilde{\bar{H}}_k(t) = \bar{H}_k(t)\bar{\Gamma}_{kk}(\tau_k, t) + \sum_{j\neq k}^{N} \bar{H}_j(t)\bar{\Gamma}_{kj}^T(\tau_j, \tau_k, t), \tag{35}$$

This follows directly from the statement of [1].

T h e o r e m 2. The filter determined by theorem in [1], and the truncated filter are equivalent.

Proof. It is evident that the truncated filter and the filter, determined in [1] are written in the form of

$$\frac{d\tilde{\bar{\mu}}_{N+1}(\tilde{\tau}_N, t)}{dt} = \tilde{F}(t)\tilde{\bar{\mu}}_{N+1}(\tilde{\tau}_N, t) + \bar{K}(t)\tilde{\bar{z}}(t), \tag{36}$$

$$\frac{d\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)}{dt}=\tilde{F}(t)\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)+\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)\tilde{F}^T(t)-$$
$$-\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)\bar{H}^T(t)\bar{R}^{-1}(t)\bar{H}(t)\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)+\tilde{Q}(t), \quad (37)$$

$$\frac{d\dot{\tilde{\mu}}_{N+1}(\tilde{\tau}_N,t)}{dt}=\tilde{F}(t)\tilde{\mu}_{N+1}(\tilde{\tau}_N,t)+\tilde{K}(t)\tilde{z}(t), \quad (38)$$

$$\frac{d\dot{\tilde{\Gamma}}_{N+1}(\tilde{\tau}_N,t)}{dt}=\tilde{F}(t)\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)+\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)\tilde{F}^T(t)-$$
$$-\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)H^T(t)\tilde{R}^{-1}(t)\tilde{S}(t)H(t)\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)+\tilde{Q}(t), \quad (39)$$

where

$$\bar{K}(t)=\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)\bar{H}(t)\bar{R}^{-1}(t), \quad (40)$$

$$\tilde{K}(t)=K(t)\tilde{S}(t), \quad K(t)=\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)H^T(t)\tilde{R}^{-1}(t)\tilde{S}(t) . \quad (41)$$

From (36)-(37) it follows that the proof of the theorem can be reduced to proving relations

$$\bar{K}(t)\tilde{\bar{z}}(t)=\tilde{K}(t)\tilde{z}(t), \quad (42)$$

$$\bar{H}^T(t)\bar{R}^{-1}(t)\bar{H}(t)=H^T(t)\tilde{R}^{-1}(t)\tilde{S}(t)H(t). \quad (43)$$

Let us first prove (43). We will derive and consider Boolean $\left[(l-r)\times l\right]$-matrix E, matrix E, which is obtained from the identity matrix of size $(l\times l)$ excluding rows with numbers $i_1,i_2,...,i_r$. As $\bar{H}(t)=EH(t)$, $\bar{R}(t)=ER(t)E^T$, then the proof of (43) can be reduced to the proof of

$$E^T\left[ER(t)E^T\right]^{-1}E=\tilde{R}^{-1}(t)\tilde{S}(t). \quad (44)$$

As $\tilde{R}(t)=R(t)+C\Theta(t)C^T$ square, using the matrix identities from [4]

$$\left[A+BCB^T\right]^{-1}=A^{-1}-A^{-1}B\left[C^{-1}+B^TA^{-1}B\right]^{-1}B^TA^{-1},$$

we get

$$\tilde{R}^{-1}(t)=R^{-1}(t)-R^{-1}(t)C\left[\Theta^{-1}(t)+C^TR^{-1}(t)C\right]^{-1}C^TR^{-1}(t). \quad (45)$$

Multiplying both sides of (45) by C^T on the left and by C on the right, and then removing the right side according to the formula of form (45), we get that

$$C^T\tilde{R}^{-1}(t)C=\left[\Theta(t)+\left[C^TR^{-1}(t)C\right]^{-1}\right]^{-1}. \quad (46)$$

From (46), it follows that

$$\Theta(t)=\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}-\left[C^TR^{-1}(t)C\right]^{-1}. \quad (47)$$

Multiplication of both sides of (47) by C on the left and by C^T on the right, taking $\tilde{R}(t)=R(t)+C\Theta(t)C^T$ into account, leads to the formula of the form of

$$\tilde{R}(t)=R(t)+C\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}C^T-C\left[C^TR^{-1}(t)C\right]^{-1}C^T . \quad (48)$$

Let us rewrite (48) as

$$\tilde{R}(t)-C\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}C^T=R(t)-C\left[C^TR^{-1}(t)C\right]^{-1}C^T . \quad (49)$$

From (49), taking into account $\tilde{R}(t)\tilde{R}^{-1}(t)=I_l$, $R(t)R^{-1}(t)=I_l$, we can obtain

$$\tilde{R}(t)\left[\tilde{R}^{-1}(t)-\tilde{R}^{-1}(t)C\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}C^T\tilde{R}^{-1}(t)\right]\tilde{R}(t)=$$
$$=R(t)\left[R^{-1}(t)-R^{-1}(t)C\left[C^TR^{-1}(t)C\right]^{-1}C^TR^{-1}(t)\right]R(t). \quad (50)$$

Let us suppose that $\tilde{\Psi}(t)$ is the left side of (44). Using formula (12) from [1] for $\tilde{R}(t)$, standing as factors at a square bracket on the left and on the right in formula (50), we get that $\tilde{\Psi}(t)=R(t)\left[\tilde{R}^{-1}(t)-\tilde{R}^{-1}(t)C\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}C^T\tilde{R}^{-1}(t)\right]R(t)$, and thus, from (50), it follows that

$$\begin{gathered}\tilde{R}^{-1}(t)-\tilde{R}^{-1}(t)C\left[C^T\tilde{R}^{-1}(t)C\right]^{-1}C^T\tilde{R}^{-1}(t)= \\ =R^{-1}(t)-R^{-1}(t)C\left[C^TR^{-1}(t)C\right]^{-1}C^TR^{-1}(t).\end{gathered} \tag{51}$$

Using (51) in (44), taking into account (50), $\tilde{S}(t)=\left[I_l-CS(t)\right]$ from [1], we get that the proof of (43) can be reduced to the proof of identity

$$R(t)E^T\left[ER(t)E^T\right]^{-1}E+C\left[C^TR^{-1}(t)C\right]^{-1}C^TR^{-1}(t)=I_l\ . \tag{52}$$

Let us denote

$$R(t)E^T\left[ER(t)E^T\right]^{-1}E=A_1\ ,\ C\left[C^TR^{-1}(t)C\right]^{-1}C^TR^{-1}(t)=A_2\ . \tag{53}$$

According to the construction of matrices *C* and *E*, we get that *EC*=*O*. Using this property leads to the fact that for matrices A_1 and A_2

$$A_1A_2=O\ ,\quad A_2A_1=O\ . \tag{54}$$

For the ranks of arbitrary matrices A and B, the properties are valid [5]:

$$rk[AB]=rk\left[A^+AB\right]=rk\left[ABB^+\right]. \tag{55}$$

Taking into account that for invertible matrix $D^+=D^{-1}$, as a result of consistent application of (55) to A_1 and A_2, we get:

$$rk[A_1]=rk\left[E^T\left[ER(t)E^T\right]^{-1}EE^+\right], \tag{56}$$

$$rk[A_2]=rk\left[C^+C\left[C^TR^{-1}(t)C\right]^{-1}C^T\right]. \tag{57}$$

Since by construction the E -matrix is with linearly independent rows and the C -matrix is with linearly independent columns, then [5]

$$EE^+=I_{l-r}\ ,\quad C^+C=I_r\ . \tag{58}$$

Using (58) and (55) in (56) and (57), we get that $rk[A_1]=rk\left[E^T\right]=l-r,\ rk[A_2]=rk\left[C^T\right]=r$.

Hence,

$$rk[A_1]+rk[A_2]=l\ . \tag{59}$$

Directly from (53), we get that $A_1^2=A_1$, $A_2^2=A_2$, i.e. matrices A_1 and A_2 are projection matrices [6]. Since the projection matrices, satisfying conditions of (54) and (59), have property $A_1+A_2=I_l$, then, taking into account (53), it proves (52) as well as (43).

Let us prove (42). Since having proved (43), we have proved equality $\tilde{\bar{\Gamma}}_{N+1}(\tilde{\tau}_N,t)=\tilde{\Gamma}_{N+1}(\tilde{\tau}_N,t)$ then from (40)-(41) it follows that the proof of (42) is equivalent to the proof of the following relation:

$$\bar{H}^T(t)\bar{R}^{-1}(t)\tilde{\bar{z}}(t)=H^T(t)\tilde{R}^{-1}(t)\tilde{S}(t)\tilde{z}(t). \tag{60}$$

As $\tilde{\bar{z}}(t)=E\tilde{z}(t),\bar{H}(t)=EH(t),\bar{R}(t)=ER(t)E^T(t),$ then from (60), it follows that the proof of (42) can be reduced to the proof of (44), which completes the proof of the Theorem.

The special case of the filter with memory unit multiplicity is considered in works [4], [5].

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