Coloring Curves That Cross a Fixed Curve*

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Abstract

We prove that for every integer $t \geqslant 1$, the class of intersection graphs of curves in the plane each of which crosses a fixed curve in at least one and at most t points is χ-bounded. This is essentially the strongest χ-boundedness result one can get for this kind of graph classes. As a corollary, we prove that for any fixed integers $k \geqslant 2$ and $t \geqslant 1$, every k-quasi-planar topological graph on n vertices with any two edges crossing at most t times has $O(n \log n)$ edges.

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1 Introduction

Overview

A *curve* is a homeomorphic image of the real interval $[0, 1]$ in the plane. The *intersection graph* of a family of curves has these curves as vertices and the intersecting pairs of curves as edges. Combinatorial and algorithmic aspects of intersection graphs of curves, known as *string graphs*, have been attracting researchers for decades. A significant part of this research has been devoted to understanding classes of string graphs that are χ-*bounded*, which means that every graph G in the class satisfies $\chi(G) \leqslant f(\omega(G))$ for some function $f\colon \mathbb{N} \to \mathbb{N}$, where $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number (the maximum size of a clique) of G, respectively. Recently, Pawlik et al. [24, 25] proved that the class of all string graphs is not χ-bounded. However, all known constructions of string graphs with small clique number and large chromatic number require a lot of freedom in placing curves around in the plane.

What restrictions on placement of curves lead to χ-bounded classes of intersection graphs? McGuinness [19, 20] proposed studying families of curves that cross a fixed curve *exactly once*. This initiated a series of results culminating in the proof that the class of intersection graphs of such families is indeed χ-bounded [26]. By contrast, the class of intersection graphs of curves each crossing a fixed curve *at least once* is equal to the class of all string graphs and therefore is not χ-bounded. We prove an essentially farthest possible generalization of the former result, allowing curves to cross the fixed curve *at least once and at most t times*, for any bound t.

▶ **Theorem 1.** *For every integer $t \geqslant 1$, the class of intersection graphs of curves each crossing a fixed curve in at least one and at most t points is χ-bounded.*

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Additional motivation for Theorem 1 comes from its application to bounding the number of edges in so-called k-quasi-planar graphs, which we discuss at the end of this introduction.

Context

Chromatic number of intersection graphs of geometric objects has been investigated since the 1960s. In a seminal paper, Asplund and Grünbaum [3] proved that intersection graphs of axis-parallel rectangles in the plane satisfy $\chi = O(\omega^2)$ and conjectured that for every integer $d \geqslant 1$, there is a function $f_d \colon \mathbb{N} \to \mathbb{N}$ such that intersection graphs of axis-parallel boxes in $\mathbb{R}^d$ satisfy $\chi \leqslant f_d(\omega)$. However, a few years later, a surprising construction due to Burling [5] showed that there are triangle-free intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ with arbitrarily large chromatic number. Since then, the upper bound of $O(\omega^2)$ and the trivial lower bound of $\Omega(\omega)$ on the maximum possible chromatic number of a rectangle intersection graph have been improved only in terms of multiplicative constants [11, 13].

Another classical example of a χ-bounded class of geometric intersection graphs is provided by circle graphs—intersection graphs of chords of a fixed circle. Gyárfás [10] proved that circle graphs satisfy $\chi = O(\omega^2 4^\omega)$. The best known upper and lower bounds on the maximum possible chromatic number of a circle graph are $O(2^\omega)$ [14] and $\Omega(\omega \log \omega)$ [12].

McGuinness [19, 20] proposed investigating the problem when much more general geometric shapes are allowed but the way how they are arranged in the plane is restricted. In [19], he proved that the class of intersection graphs of L-shapes crossing a fixed horizontal line is χ-bounded. Families of L-shapes in the plane are *simple*, which means that any two members of the family intersect in at most one point. McGuinness [20] also showed that triangle-free intersection graphs of simple families of curves each crossing a fixed line in exactly one point have bounded chromatic number. Further progress in this direction was made by Suk [27], who proved that simple families of x-monotone curves crossing a fixed vertical line give rise to a χ-bounded class of intersection graphs, and by Lasoń et al. [17], who reached the same conclusion without assuming that the curves are x-monotone. Finally, in [26], we proved that the class of intersection graphs of curves each crossing a fixed line in exactly one point is χ-bounded. These results remain valid if the fixed straight line is replaced by a fixed curve [28].

The class of string graphs is not χ-bounded. Pawlik et al. [24, 25] presented a construction of triangle-free intersection graphs of segments (or geometric shapes of various other kinds) with chromatic number growing as fast as $\Theta(\log \log n)$ with the number of vertices n. It was further generalized to a construction of string graphs with clique number ω and chromatic number $\Theta_\omega((\log \log n)^{\omega-1})$ [16]. The best known upper bound on the chromatic number of string graphs in terms of the number of vertices is $(\log n)^{O(\log \omega)}$, proved by Fox and Pach [8] using a separator theorem for string graphs due to Matoušek [18]. For intersection graphs of segments or, more generally, x-monotone curves, an upper bound of the form $\chi = O_\omega(\log n)$ follows from the above-mentioned result in [27] or [26] via recursive halving. Upper bounds of the form $\chi = O_\omega((\log \log n)^{f(\omega)})$ (for some function $f \colon \mathbb{N} \to \mathbb{N}$) are known for very special classes of string graphs: rectangle overlap graphs [15, 16] and subtree overlap graphs [16]. The former still allow the triangle-free construction with $\chi = \Theta(\log \log n)$ and the latter the construction with $\chi = \Theta_\omega((\log \log n)^{\omega-1})$.

Quasi-planarity

A *topological graph* is a graph with a fixed curvilinear drawing in the plane. For $k \geqslant 2$, a *k-quasi-planar graph* is a topological graph with no k pairwise crossing edges. In particular, a 2-quasi-planar graph is just a planar graph. It is conjectured that k-quasi-planar graphs with

n vertices have $O_k(n)$ edges [4, 23]. For $k = 2$, this asserts a well-known property of planar graphs. The conjecture is also verified for $k = 3$ [2, 22] and $k = 4$ [1], but it remains open for $k \geqslant 5$. Best known upper bounds on the number of edges in a k-quasi-planar graph are $n(\log n)^{O(\log k)}$ in general [7, 8], $O_k(n \log n)$ for the case of x-monotone edges [29], $O_k(n \log n)$ for the case that any two edges intersect at most once [28], and $2^{\alpha(n)^\nu} n \log n$ for the case that any two edges intersect in at most t points, where α is the inverse Ackermann function and ν depends on k and t [28]. We apply Theorem 1 to improve the last bound to $O_{k,t}(n \log n)$.

▶ **Theorem 2.** *Every k-quasi-planar topological graph G on n vertices such that any two edges of G intersect in at most t points has at most $\mu_{k,t} n \log n$ edges, where $\mu_{k,t}$ depends only on k and t.*

The proof follows the same line as the proof in [28] for the case $t = 1$ (see Section 3).

2 Proof of Theorem 1

Setup

Let $\mathbb{N}$ denote the set of positive integers. Graph-theoretic terms applied to a family of curves $\mathcal{F}$ have the same meaning as applied to the intersection graph of $\mathcal{F}$. In particular, the *chromatic number* of $\mathcal{F}$, denoted by $\chi(\mathcal{F})$, is the minimum number of colors in a *proper coloring* of $\mathcal{F}$ (a coloring that distinguishes pairs of intersecting curves), and the *clique number* of $\mathcal{F}$, denoted by $\omega(\mathcal{F})$, is the maximum size of a *clique* in $\mathcal{F}$ (a set of pairwise intersecting curves in $\mathcal{F}$).

▶ **Theorem 1** (rephrased). *For every $t \in \mathbb{N}$, there is a non-decreasing function $f_t \colon \mathbb{N} \to \mathbb{N}$ with the following property: for any fixed curve c_0, every family $\mathcal{F}$ of curves each intersecting c_0 in at least one and at most t points satisfies $\chi(\mathcal{F}) \leqslant f_t(\omega(\mathcal{F}))$.*

A point p is a *proper crossing* of curves c_1 and c_2 if c_1 passes from one side to the other side of c_2 in a sufficiently small neighborhood of p. From now on, without significant loss of generality, we make the following implicit assumption: any two distinct curves that we consider intersect in finitely many points, and each of their intersection points is a proper crossing. There is one exception to the latter condition: a curve c may have an endpoint on another curve if this is required by the definition of c (like for 1-curves defined below).

Initial reduction

We start by reducing Theorem 1 to a somewhat simpler and more convenient setting. We fix a horizontal line in the plane and call it the *baseline*. The upper half-plane bounded by the baseline is denoted by H^+. A 1-*curve* is a curve in H^+ that has one endpoint on the baseline and does not intersect the baseline in any other point. Intersection graphs of 1-curves are known as *outerstring graphs* and form a χ-bounded class of graphs—this result, due to the authors, is the starting point of the proof of Theorem 1.

▶ **Theorem 3** ([26]). *There is a non-decreasing function $f_0 \colon \mathbb{N} \to \mathbb{N}$ such that every family $\mathcal{F}$ of 1-curves satisfies $\chi(\mathcal{F}) \leqslant f_0(\omega(\mathcal{F}))$.*

An *even-curve* is a curve that has both endpoints above the baseline and intersects the baseline in at least two points (this is an even number, by the proper crossing assumption). For $t \in \mathbb{N}$, a $2t$-*curve* is an even-curve that intersects the baseline in exactly $2t$ points. The *basepoint* of a 1-curve s is the endpoint of s on the baseline. A *basepoint* of an even-curve c

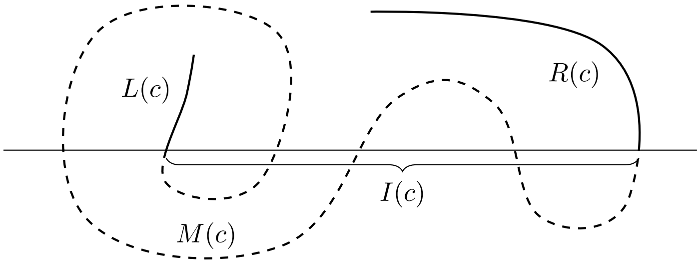

Figure 1 $L(c)$, $R(c)$, $M(c)$ (all the dashed part), and $I(c)$ for a 6-curve c.

is an intersection point of c with the baseline. Every even-curve c determines two 1-curves—the two parts of c from an endpoint to the closest basepoint. They are called the *1-curves of c* and denoted by $L(c)$ and $R(c)$ so that the basepoint of $L(c)$ lies to the left of the basepoint of $R(c)$ on the baseline (see Figure 1). A family $\mathcal{F}$ of even-curves is an *LR-family* if every intersection between two curves $c_1, c_2 \in \mathcal{F}$ is an intersection between $L(c_1)$ and $R(c_2)$ or between $L(c_2)$ and $R(c_1)$. The main effort in this paper goes to proving the following statement on LR-families of even-curves.

▶ **Theorem 4.** *There is a non-decreasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that every LR-family $\mathcal{F}$ of even-curves satisfies $\chi(\mathcal{F}) \leqslant f(\omega(\mathcal{F}))$.*

Theorem 4 makes no assumption on the maximum number of intersection points of an even-curve with the baseline. We derive Theorem 1 from Theorem 4 in two steps, first proving the following lemma, and then showing that Theorem 1 is essentially a special case of it.

▶ **Lemma 5.** *For every $t \in \mathbb{N}$, there is a non-decreasing function $f_t\colon \mathbb{N} \to \mathbb{N}$ such that every family $\mathcal{F}$ of $2t$-curves no two of which intersect on or below the baseline satisfies $\chi(\mathcal{F}) \leqslant f_t(\omega(\mathcal{F}))$.*

Proof of Lemma 5 from Theorem 4. The proof goes by induction on t. Let f_0 and f be the functions claimed by Theorem 3 and Theorem 4, respectively, and let $f_t(k) = f_{t-1}^2(k)f(k)$ for $t \geqslant 1$ and $k \in \mathbb{N}$. We establish the base case for $t = 1$ and the induction step for $t \geqslant 2$ simultaneously. Namely, fix an integer $t \geqslant 1$, and let $\mathcal{F}$ be as in the statement of the lemma. For every $2t$-curve $c \in \mathcal{F}$, enumerate the endpoints and basepoints of c as $p_0(c), \dots, p_{2t+1}(c)$ in their order along c so that $p_0(c)$ and $p_1(c)$ are the endpoints of $L(c)$ while $p_{2t}(c)$ and $p_{2t+1}(c)$ are the endpoints of $R(c)$. Build two families of curves $\mathcal{F}_1$ and $\mathcal{F}_2$ putting the part of c from $p_0(c)$ to $p_{2t-1}(c)$ to $\mathcal{F}_1$ and the part of c from $p_2(c)$ to $p_{2t+1}(c)$ to $\mathcal{F}_2$ for every $c \in \mathcal{F}$. If $t = 1$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of 1-curves. If $t \geqslant 2$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent to families of $2(t-1)$-curves, because the curve in $\mathcal{F}_1$ or $\mathcal{F}_2$ obtained from a $2t$-curve $c \in \mathcal{F}$ can be shortened a little at $p_{2t-1}(c)$ or $p_2(c)$, respectively, losing that basepoint but no intersection points with other curves. Therefore, by Theorem 3 or the induction hypothesis, we have $\chi(\mathcal{F}_k) \leqslant f_{t-1}(\omega(\mathcal{F}_k)) \leqslant f_{t-1}(\omega(\mathcal{F}))$ for $k \in \{1, 2\}$. For $c \in \mathcal{F}$ and $k \in \{1, 2\}$, let $\phi_k(c)$ be the color of the curve obtained from c in an optimal proper coloring of $\mathcal{F}_k$. Every subfamily of $\mathcal{F}$ on which ϕ_1 and ϕ_2 are constant is an LR-family and therefore, by Theorem 4 and monotonicity of f, has chromatic number at most $f(\omega(\mathcal{F}))$. We conclude that $\chi(\mathcal{F}) \leqslant \chi(\mathcal{F}_1)\chi(\mathcal{F}_2)f(\omega(\mathcal{F})) \leqslant f_{t-1}^2(\omega(\mathcal{F}))f(\omega(\mathcal{F})) = f_t(\omega(\mathcal{F}))$. ◀

A *closed curve* is a homeomorphic image of a unit circle in the plane. For a closed curve γ, the Jordan curve theorem asserts that the set $\mathbb{R}^2 \smallsetminus \gamma$ consists of two connected components: one bounded, denoted by $\operatorname{int}\gamma$, and one unbounded, denoted by $\operatorname{ext}\gamma$.

Proof of Theorem 1 from Theorem 4. We elect to present this proof in an intuitive rather than rigorous way. Let $\mathcal{F}$ be a family of curves each intersecting c_0 in at least one and at most t points. Let γ_0 be a closed curve surrounding c_0 very closely so that γ_0 intersects every curve in $\mathcal{F}$ in exactly $2t$ points (winding if necessary to increase the number of intersections) and all endpoints of curves in $\mathcal{F}$ and intersection points of pairs of curves in $\mathcal{F}$ lie in $\operatorname{ext}\gamma_0$. We "invert" $\operatorname{int}\gamma_0$ with $\operatorname{ext}\gamma_0$ to obtain an equivalent family of curves $\mathcal{F}'$ and a closed curve γ_0' with the same properties except that all endpoints of curves in $\mathcal{F}'$ and intersection points of pairs of curves in $\mathcal{F}'$ lie in $\operatorname{int}\gamma_0'$. It follows that some part of γ_0' lies in the unbounded component of $\mathbb{R}^2 \smallsetminus \bigcup\mathcal{F}'$. We "cut" γ_0' there and "unfold" it into the baseline, transforming $\mathcal{F}'$ into an equivalent family $\mathcal{F}''$ of $2t$-curves all endpoints of which and intersection points of pairs of which lie above the baseline. The "equivalence" of $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ means in particular that the intersection graphs of $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ are isomorphic, so the theorem follows from Lemma 5 (and thus Theorem 4). ◀

A statement analogous to Theorem 4 fails for families of objects each consisting of two 1-curves only, without the "middle part" connecting them. Specifically, we define a *double-curve* as a set $X \subset H^+$ that is a union of two disjoint 1-curves, denoted by $L(X)$ and $R(X)$ so that the basepoint of $L(X)$ lies to the left of the basepoint of $R(X)$, and we call a family $\mathcal{X}$ of double-curves an *LR-family* if every intersection between two double-curves $X_1, X_2 \in \mathcal{X}$ is an intersection between $L(X_1)$ and $R(X_2)$ or between $L(X_2)$ and $R(X_1)$.

▶ **Theorem 6.** *For every $\zeta \in \mathbb{N}$, there is a triangle-free LR-family of double-curves $\mathcal{X}$ such that $\chi(\mathcal{X}) \geqslant \zeta$.*

The proof of Theorem 6 is an easy adaptation of the construction from [24, 25]. We omit the details. The rest of this section is devoted to the proof of Theorem 4.

Overview of the proof of Theorem 4

Recall the assertion of Theorem 4: the LR-families of even-curves are χ-bounded. The proof is quite long and technical, so we find it useful to provide a high-level overview of its structure. The proof will be presented via a series of reductions. First, we will reduce Theorem 4 to the following statement (Lemma 7): the LR-families of 2-curves are χ-bounded. This statement will be proved by induction on the clique number. Specifically, we will prove the following as the induction step: if every LR-family of 2-curves $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k-1$ satisfies $\chi(\mathcal{F}) \leqslant \xi$, then every LR-family of 2-curves $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant \zeta$, where ζ is a constant depending only on k and ξ. The only purpose of the induction hypothesis is to infer that if $\omega(\mathcal{F}) \leqslant k$ and $c \in \mathcal{F}$, then the family of 2-curves in $\mathcal{F} \smallsetminus \{c\}$ that intersect c has chromatic number at most ξ. For notational convenience, LR-families of 2-curves with the latter property will be called *ξ-families*. We will thus reduce the problem to the following statement (Lemma 9): the ξ-families are χ-bounded, where the χ-bounding function depends on ξ.

We will deal with ξ-families via a series of technical lemmas of the following general form: every ξ-family with chromatic number large enough contains a specific configuration of curves. Two kinds of such configurations are particularly important: (a) a large clique, and (b) a 2-curve c and a subfamily $\mathcal{F}'$ with large chromatic number such that the basepoints of the 2-curves in $\mathcal{F}'$ lie between the basepoints of c. In the core of the argument are the proofs that

- every ξ-family with chromatic number large enough contains (a) or (b) (Lemma 16),
- assuming the above, every ξ-family with chromatic number large enough contains (a).

Combined, they complete the argument. Since the two proofs are almost identical, we introduce one more reduction—to *(ξ, h)-families* (Lemma 15). A (ξ, h)-family is just a ξ-family that satisfies an additional technical condition sufficient to carry both proofs at once.

More notation and terminology

Let $\prec$ denote the left-to-right order of points on the baseline ($p_1 \prec p_2$ means that p_1 is to the left of p_2). For convenience, we also use the notation $\prec$ for curves intersecting the baseline ($c_1 \prec c_2$ means that every basepoint of c_1 is to the left of every basepoint of c_2) and for families of such curves ($\mathcal{C}_1 \prec \mathcal{C}_2$ means that $c_1 \prec c_2$ for any $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$). For a family $\mathcal{C}$ of curves intersecting the baseline (even-curves or 1-curves) and two 1-curves x and y, let $\mathcal{C}(x, y) = \{c \in \mathcal{C} \colon x \prec c \prec y\}$ or $\mathcal{C}(x, y) = \{c \in \mathcal{C} \colon y \prec c \prec x\}$ depending on whether $x \prec y$ or $y \prec x$. For a family $\mathcal{C}$ of curves intersecting the baseline and a segment I on the baseline, let $\mathcal{C}(I)$ denote the family of curves in $\mathcal{C}$ with all basepoints on I.

For an even-curve c, let $M(c)$ denote the subcurve of c connecting the basepoints of $L(c)$ and $R(c)$, and let $I(c)$ denote the segment on the baseline connecting the basepoints of $L(c)$ and $R(c)$ (see Figure 1). For a family $\mathcal{F}$ of even-curves, let $L(\mathcal{F}) = \{L(c) \colon c \in \mathcal{F}\}$, $R(\mathcal{F}) = \{R(c) \colon c \in \mathcal{F}\}$, and $I(\mathcal{F})$ denote the minimal segment on the baseline that contains $I(c)$ for every $c \in \mathcal{F}$.

A *cap-curve* is a curve in H^+ that has both endpoints on the baseline and does not intersect the baseline in any other point. For a cap-curve γ, it follows from the Jordan curve theorem that the set $H^+ \smallsetminus \gamma$ consists of two connected components: one bounded, denoted by $\operatorname{int} \gamma$, and one unbounded, denoted by $\operatorname{ext} \gamma$. Any two cap-curves one with endpoints p_1, q_1 and the other with endpoints p_2, q_2 such that $p_1 \prec p_2 \prec q_1 \prec q_2$ intersect in an odd number of points.

Reduction to LR-families of 2-curves

We will reduce Theorem 4 to the following statement on LR-families of 2-curves, which is essentially a special case of Theorem 4.

▶ **Lemma 7.** *There is a non-decreasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that every LR-family $\mathcal{F}$ of 2-curves satisfies $\chi(\mathcal{F}) \leqslant f(\omega(\mathcal{F}))$.*

A *component* of a family of 1-curves $\mathcal{S}$ is a connected component of $\bigcup \mathcal{S}$ (the union of all curves in $\mathcal{S}$). The following easy but powerful observation reuses an idea from [17, 20, 27].

▶ **Lemma 8.** *For every LR-family of even-curves $\mathcal{F}$, if $\mathcal{F}^\star$ is the family of curves $c \in \mathcal{F}$ such that $L(c)$ and $R(c)$ lie in distinct components of $L(\mathcal{F}) \cup R(\mathcal{F})$, then $\chi(\mathcal{F}^\star) \leqslant 4$.*

Proof. Let G be an auxiliary graph where the vertices are the components of $L(\mathcal{F}) \cup R(\mathcal{F})$ and the edges are the pairs V_1V_2 of components such that there is a curve $c \in \mathcal{F}^\star$ with $L(c) \subseteq V_1$ and $R(c) \subseteq V_2$ or $L(c) \subseteq V_2$ and $R(c) \subseteq V_1$. Since $\mathcal{F}$ is an LR-family, the curves in $\mathcal{F}^\star$ cannot intersect "outside" the components of $L(\mathcal{F}) \cup R(\mathcal{F})$. It follows that G is planar and thus 4-colorable. Fix a proper 4-coloring of G, and assign the color of a component V to every curve $c \in \mathcal{F}^\star$ with $L(c) \subseteq V$. For any $c_1, c_2 \in \mathcal{F}^\star$, if $L(c_1)$ and $R(c_2)$ intersect, then $L(c_1)$ and $R(c_2)$ lie in the same component V_1 while $L(c_2)$ lies in a component V_2 such that V_1V_2 is an edge of G, so c_1 and c_2 are assigned distinct colors. The coloring of $\mathcal{F}^\star$ is therefore proper. ◀

Proof of Theorem 4 from Lemma 7. We show that $\chi(\mathcal{F}) \leqslant f(\omega(\mathcal{F})) + 4$, where f is the function claimed by Lemma 7. We have $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1 = \{c \in \mathcal{F} \colon L(c)$ and $R(c)$ lie in the same component of $L(\mathcal{F}) \cup R(\mathcal{F})\}$ and $\mathcal{F}_2 = \{c \in \mathcal{F} \colon L(c)$ and $R(c)$ lie in distinct components of $L(\mathcal{F}) \cup R(\mathcal{F})\}$. Lemma 8 yields $\chi(\mathcal{F}_2) \leqslant 4$. It remains to show that $\chi(\mathcal{F}_1) \leqslant f(\omega(\mathcal{F}))$.

Let $c_1, c_2 \in \mathcal{F}_1$. We claim that the intervals $I(c_1)$ and $I(c_2)$ are nested or disjoint. Suppose they are not. For $\varepsilon > 0$ and a component V of $L(\mathcal{F}) \cup R(\mathcal{F})$, let V^ε denote the ε-neighborhood of V in H^+. We assume that ε is small enough so that the sets V^ε for all components V of $L(\mathcal{F}) \cup R(\mathcal{F})$ and the curves $M(c)$ for all $c \in \mathcal{F}_1$ are pairwise disjoint (except at common basepoints). For $k \in \{1,2\}$, since $L(c_k)$ and $R(c_k)$ belong to the same component V_k of $L(\mathcal{F}) \cup R(\mathcal{F})$, there is a cap-curve $\gamma_k \subset V_k^\varepsilon$ that connects the basepoints of $L(c_k)$ and $R(c_k)$. We can assume without loss of generality that γ_1 and γ_2 intersect in a finite number of points and each of their intersection points is a proper crossing (this is why we take $\gamma_k \subset V_k^\varepsilon$ instead of $\gamma_k \subseteq V_k$). Since $I(c_1)$ and $I(c_2)$ are neither nested nor disjoint, the basepoints of $L(c_2)$ and $R(c_2)$ lie one in $\operatorname{int}\gamma_1$ and the other in $\operatorname{ext}\gamma_1$, so γ_1 and γ_2 intersect in an odd number of points. For $k \in \{1,2\}$, let $\tilde{\gamma}_k$ be the closed curve obtained as the union of γ_k and $M(c_k)$. It follows that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect in an odd number of points and each of their intersection points is a proper crossing, which is a contradiction.

Transform $\mathcal{F}_1$ into a family of 2-curves $\mathcal{F}_1'$ replacing the part $M(c)$ of every curve $c \in \mathcal{F}_1$ by the lower semicircle connecting the endpoints of $M(c)$. These semicircles are pairwise disjoint (because $I(c_1)$ and $I(c_2)$ are nested or disjoint for any $c_1, c_2 \in \mathcal{F}_1$), so $\mathcal{F}_1'$ is an LR-family with intersection graph isomorphic to that of $\mathcal{F}_1$. Lemma 7 yields $\chi(\mathcal{F}_1) = \chi(\mathcal{F}_1') \leqslant f(\omega(\mathcal{F}_1')) \leqslant f(\omega(\mathcal{F}))$. ◀

Reduction to ξ-families

For $\xi \in \mathbb{N}$, a *ξ-family* is an LR-family of 2-curves $\mathcal{F}$ with the following property: for every 2-curve $c \in \mathcal{F}$, the family of 2-curves in $\mathcal{F} \smallsetminus \{c\}$ that intersect c has chromatic number at most ξ. We reduce Lemma 7 to the following statement on ξ-families.

▶ **Lemma 9.** *For any $\xi, k \in \mathbb{N}$, there is a constant $\zeta \in \mathbb{N}$ such that every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant \zeta$.*

Proof of Lemma 7 from Lemma 9. Let $f(1) = 1$. For $k \geqslant 2$, let $f(k)$ be the constant claimed by Lemma 9 such that every $f(k-1)$-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant f(k)$. Let $k = \omega(\mathcal{F})$, and proceed by induction on k to prove $\chi(\mathcal{F}) \leqslant f(k)$. Clearly, if $k = 1$, then $\chi(\mathcal{F}) = 1$. For the induction step, assume $k \geqslant 2$. For every $c \in \mathcal{F}$, the family of 2-curves in $\mathcal{F} \smallsetminus \{c\}$ that intersect c has clique number at most $k-1$ and therefore, by the induction hypothesis, has chromatic number at most $f(k-1)$. That is, $\mathcal{F}$ is an $f(k-1)$-family, and the definition of f yields $\chi(\mathcal{F}) \leqslant f(k)$. ◀

Dealing with ξ-families

First, we establish the following special case of Lemma 9.

▶ **Lemma 10.** *For every $\xi \in \mathbb{N}$, every ξ-family $\mathcal{F}$ with $\bigcap_{c\in\mathcal{F}} I(c) \neq \emptyset$ satisfies $\chi(\mathcal{F}) \leqslant 4\xi + 4$.*

The proof of Lemma 10 is essentially the same as the proof of Lemma 19 in [28]. We need the following elementary lemma, which was also used in various forms in [17, 19, 20, 26, 27].

▶ **Lemma 11** (McGuinness [19, Lemma 2.1])**.** *Let G be a graph, $\prec$ be a total order on the vertices of G, and $\alpha, \beta \in \mathbb{N}$. If $\chi(G) > (2\beta + 2)\alpha$, then G has an induced subgraph H such that $\chi(H) > \alpha$ and $\chi(G(u, v)) > \beta$ for every edge uv of H. In particular, if $\chi(G) > 2\beta + 2$, then G has an edge uv with $\chi(G(u, v)) > \beta$. Here, $G(u, v)$ denotes the subgraph of G induced on the vertices strictly between u and v in the order $\prec$.*

Proof of Lemma 10. Suppose $\chi(\mathcal{F}) > 4\xi + 4$. Since $\bigcap_{c\in\mathcal{F}} I(c) \neq \emptyset$, the 2-curves in $\mathcal{F}$ can be enumerated as $c_1, \ldots, c_{|\mathcal{F}|}$ so that $L(c_1) \prec \cdots \prec L(c_{|\mathcal{F}|}) \prec R(c_{|\mathcal{F}|}) \prec \cdots \prec R(c_1)$. Apply Lemma 11 to the intersection graph of $\mathcal{F}$ and the order $c_1, \ldots, c_{|\mathcal{F}|}$ to obtain two indices $i, j \in \{1, \ldots, |\mathcal{F}|\}$ such that the 2-curves c_i and c_j intersect and $\chi\bigl(\{c_{i+1}, \ldots, c_{j-1}\}\bigr) > 2\xi + 1$.

Assume $L(c_i)$ and $R(c_j)$ intersect; the argument for the other case is analogous. There is a cap-curve $\gamma \subseteq L(c_i) \cup R(c_j)$ connecting the basepoints of $L(c_i)$ and $R(c_j)$. Every curve intersecting γ intersects c_i or c_j. Since $\mathcal{F}$ is a ξ-family, the 2-curves in $\{c_{i+1}, \ldots, c_{j-1}\}$ that intersect c_i have chromatic number at most ξ, and so do those that intersect c_j. Every 2-curve $c_k \in \{c_{i+1}, \ldots, c_{j-1}\}$ not intersecting γ satisfies $L(c_k) \subset \operatorname{int} \gamma$ and $R(c_k) \subset \operatorname{ext} \gamma$, so these 2-curves are pairwise disjoint. We conclude that $\chi(\{c_{i+1}, \ldots, c_{j-1}\}) \leqslant 2\xi + 1$, which is a contradiction. ◀

Lemma 11 easily implies that every family of 2-curves $\mathcal{F}$ with $\chi(\mathcal{F}) > (2\beta + 2)^2\alpha$ contains a subfamily $\mathcal{H}$ with $\chi(\mathcal{H}) > \alpha$ such that $\chi(\mathcal{F}(L(c_1), L(c_2))) > \beta$ and $\chi(\mathcal{F}(R(c_1), R(c_2))) > \beta$ for any two intersecting 2-curves $c_1, c_2 \in \mathcal{H}$. This is considerably strengthened by the following lemma. Its proof extends the idea used in [19] for the proof of Lemma 11.

▶ **Lemma 12.** *For every $\xi \in \mathbb{N}$, there is a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property: for any $\alpha, \beta \in \mathbb{N}$ and every ξ-family $\mathcal{F}$ with $\chi(\mathcal{F}) > f(\alpha, \beta)$, there is a subfamily $\mathcal{H} \subseteq \mathcal{F}$ such that $\chi(\mathcal{H}) > \alpha$ and $\chi(\mathcal{F}(x, y)) > \beta$ for any two intersecting 1-curves $x \in R(\mathcal{H})$ and $y \in L(\mathcal{H})$.*

Proof. Let $f(\alpha, \beta) = (2\beta + 12\xi + 20)\alpha$. Let $\mathcal{F}$ be a ξ-family with $\chi(\mathcal{F}) > f(\alpha, \beta)$. Construct a sequence of points $p_0 \prec \cdots \prec p_{m+1}$ on the baseline with the following properties:

- the points $p_0, \ldots, p_{m+1}$ are distinct from all basepoints of 2-curves in $\mathcal{F}$,
- p_0 lies to the left of and p_{m+1} lies to the right of all basepoints of 2-curves in $\mathcal{F}$,
- $\chi(\mathcal{F}(p_i p_{i+1})) = \beta + 1$ for $0 \leqslant i \leqslant m - 1$, and $\chi(\mathcal{F}(p_m p_{m+1})) \leqslant \beta + 1$.

This is done greedily by first choosing p_1 so that $\chi(\mathcal{F}(p_0 p_1)) = \beta + 1$, then choosing p_2 so that $\chi(\mathcal{F}(p_1 p_2)) = \beta + 1$, and so on. For $0 \leqslant i \leqslant j \leqslant m$, let $\mathcal{F}_{i,j} = \{c \in \mathcal{F} \colon p_i \prec L(c) \prec p_{i+1}$ and $p_j \prec R(c) \prec p_{j+1}\}$. In particular, $\mathcal{F}_{i,i} = \mathcal{F}(p_i p_{i+1})$ for $0 \leqslant i \leqslant m$. Since $\mathcal{F} = \bigcup_{0 \leqslant i \leqslant j \leqslant m} \mathcal{F}_{i,j}$, at least one of the following holds:

$$\chi\Big(\bigcup_{i=0}^{m} \mathcal{F}_{i,i}\Big) > (2\beta + 2)\alpha, \quad \chi\Big(\bigcup_{i=0}^{m-1} \mathcal{F}_{i,i+1}\Big) > (12\xi + 12)\alpha, \quad \chi\Big(\bigcup_{i=0}^{m-2} \bigcup_{j=i+2}^{m} \mathcal{F}_{i,j}\Big) > 6\alpha.$$

In each case, we will find a subfamily $\mathcal{H} \subseteq \mathcal{F}$ such that any two intersecting 1-curves $x \in R(\mathcal{H})$ and $y \in L(\mathcal{H})$ satisfy $x \in R(\mathcal{F}_{i,j})$ and $y \in L(\mathcal{F}_{r,s})$, where $0 \leqslant i \leqslant j \leqslant m$, $0 \leqslant r \leqslant s \leqslant m$, and $|j - r| \geqslant 2$. Then, $\chi(\mathcal{F}(x, y)) \geqslant \chi(\mathcal{F}(p_{\max(j,r)-1} p_{\max(j,r)})) = \beta + 1$, as required.

Suppose $\chi\big(\bigcup_{i=0}^{m} \mathcal{F}_{i,i}\big) > (2\beta + 2)\alpha$. We have $\chi(\mathcal{F}_{i,i}) \leqslant \beta + 1$ for $0 \leqslant i \leqslant m$. Color the 2-curves in every $\mathcal{F}_{i,i}$ properly using the same set of $\beta + 1$ colors on $\mathcal{F}_{i,i}$ and $\mathcal{F}_{r,r}$ whenever $i \equiv r \pmod 2$, thus using $2\beta + 2$ colors in total. It follows that $\chi(\mathcal{H}) > \alpha$ for some family $\mathcal{H} \subseteq \bigcup_{i=0}^{m} \mathcal{F}_{i,i}$ of 2-curves of the same color. To conclude, for any two intersecting 1-curves $x \in R(\mathcal{H})$ and $y \in L(\mathcal{H})$, we have $x \in R(\mathcal{F}_{i,i})$ and $y \in L(\mathcal{F}_{r,r})$ for some distinct indices $i, r \in \{0, \ldots, m\}$ with $i \equiv r \pmod 2$ and thus $|i - r| \geqslant 2$.

Now, suppose $\chi\big(\bigcup_{i=0}^{m-1} \mathcal{F}_{i,i+1}\big) > (12\xi + 12)\alpha$. By Lemma 10, we have $\chi(\mathcal{F}_{i,i+1}) \leqslant 4\xi + 4$ for $0 \leqslant i \leqslant m - 1$. Color the 2-curves in every $\mathcal{F}_{i,i+1}$ properly using the same set of $4\xi + 4$ colors on $\mathcal{F}_{i,i+1}$ and $\mathcal{F}_{r,r+1}$ whenever $i \equiv r \pmod 3$, thus using $12\xi + 12$ colors in total. It follows that $\chi(\mathcal{H}) > \alpha$ for some family $\mathcal{H} \subseteq \bigcup_{i=0}^{m-1} \mathcal{F}_{i,i+1}$ of 2-curves of the same color. To conclude, for any two intersecting 1-curves $x \in R(\mathcal{H})$ and $y \in L(\mathcal{H})$, we have $x \in R(\mathcal{F}_{i,i+1})$ and $y \in L(\mathcal{F}_{r,r+1})$ for some distinct indices $i, r \in \{0, \ldots, m - 1\}$ with $i \equiv r \pmod 3$ and thus $|i + 1 - r| \geqslant 2$.

Finally, suppose $\chi\big(\bigcup_{i=0}^{m-2} \bigcup_{j=i+2}^{m} \mathcal{F}_{i,j}\big) > 6\alpha$. It follows that $\chi\big(\bigcup_{i \in I} \bigcup_{j=i+2}^{m} \mathcal{F}_{i,j}\big) > 3\alpha$, where $I = \{i \in \{0, \ldots, m - 2\} \colon i \equiv 0 \pmod 2\}$ or $I = \{i \in \{0, \ldots, m - 2\} \colon i \equiv 1 \pmod 2\}$. Consider an auxiliary graph G with vertex set I and edge set $\{ij \colon i, j \in I,\ i < j$, and $\mathcal{F}_{i,j-1} \cup \mathcal{F}_{i,j} \neq \emptyset\}$. Since no two 2-curves in $\mathcal{F}$ cross below the baseline, G has no two edges $i_1 j_1$ and $i_2 j_2$ such that $i_1 < i_2 < j_1 < j_2$. In particular, G is an outerplanar graph, and

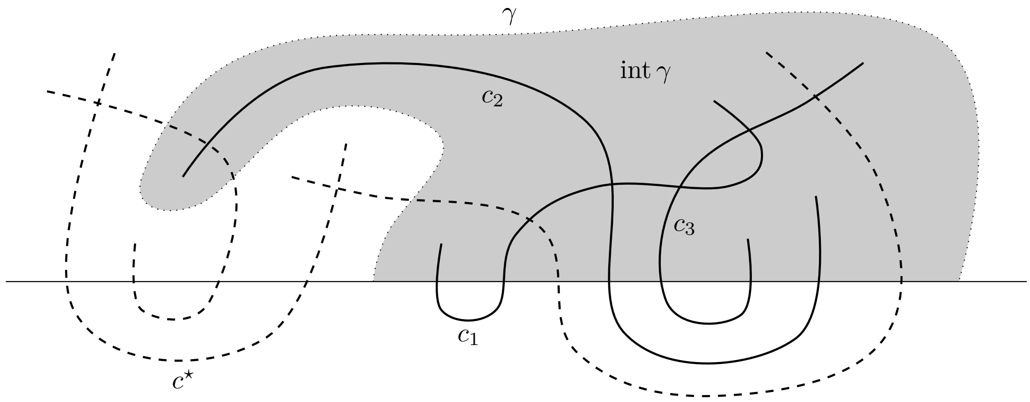

Figure 2 Illustration for Lemma 14: $\mathcal{G} = \{c_1, c_2, c_3\}$.

thus $\chi(G) \leqslant 3$. Fix a proper 3-coloring of G, and use the color of i on every 2-curve in $\bigcup_{j=i+2}^{m} \mathcal{F}_{i,j}$ for every $i \in I$. It follows that $\chi(\mathcal{H}) > \alpha$ for some family $\mathcal{H} \subseteq \bigcup_{i \in I} \bigcup_{j=i+2}^{m} \mathcal{F}_{i,j}$ of 2-curves of the same color. To conclude, for any two intersecting 1-curves $x \in R(\mathcal{H})$ and $y \in L(\mathcal{H})$, we have $x \in R(\mathcal{F}_{i,j})$ and $y \in L(\mathcal{F}_{r,s})$ for some indices $i, r \in I$, $j \in \{i+2, \ldots, m\}$, and $s \in \{r+2, \ldots, m\}$ such that $j \notin \{r-1, r\}$ (otherwise ir would be an edge of G), $j \neq r+1$ (otherwise two 2-curves, one from $\mathcal{F}_{i,r+1}$ and one from $\mathcal{F}_{r,s}$, would cross below the baseline), and thus $|j - r| \geqslant 2$. ◀

It is proved in [26] that for every family of 1-curves $\mathcal{S}$, there are a cap-curve γ and a subfamily $\mathcal{U} \subseteq \mathcal{S}$ with $\chi(\mathcal{U}) \geqslant \frac{1}{2}\chi(\mathcal{S})$ such that every 1-curve in $\mathcal{U}$ is contained in $\operatorname{int}\gamma$ and intersects some 1-curve in $\mathcal{S}$ that intersects $\operatorname{ext}\gamma$. The proof follows an idea from [10], used subsequently also in [17, 19, 20, 21, 27], where $\mathcal{U}$ is chosen as one of the sets of 1-curves at a fixed distance from an appropriately chosen 1-curve in the intersection graph of $\mathcal{S}$, and γ is a cap-curve surrounding $\mathcal{U}$ very closely. However, this method fails to imply an analogous statement for 2-curves. We will need a more powerful tool—part of the recent series of works on induced subgraphs that must be present in graphs with sufficiently large chromatic number.

▶ **Theorem 13** (Chudnovsky, Scott, Seymour [6, Theorem 1.8]). *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ with the following property: for every $\alpha \in \mathbb{N}$, every string graph G with $\chi(G) > f(\alpha)$ contains a vertex v such that $\chi(G_v^2) > \alpha$, where G_v^2 denotes the subgraph of G induced on the vertices within distance at most 2 from v.*

The special case of Theorem 13 for triangle-free intersection graphs of curves any two of which intersect in at most one point was proved earlier by McGuinness [21, Theorem 5.3].

▶ **Lemma 14** (see Figure 2). *For every $\xi \in \mathbb{N}$, there is a function $f\colon \mathbb{N} \to \mathbb{N}$ with the following property: for every $\alpha \in \mathbb{N}$ and every ξ-family $\mathcal{F}$ with $\chi(\mathcal{F}) > f(\alpha)$, there are a cap-curve γ and a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > \alpha$ such that every 2-curve $c \in \mathcal{G}$ satisfies $L(c), R(c) \subset \operatorname{int}\gamma$ and intersects some 2-curve in $\mathcal{F}$ that intersects $\operatorname{ext}\gamma$.*

Proof. Let $f(\alpha) = f_1(3\alpha+5\xi+5)$, where f_1 is the function claimed by Theorem 13. Let $\mathcal{F}$ be a ξ-family with $\chi(\mathcal{F}) > f(\alpha)$. It follows that there is a 2-curve $c^\star \in \mathcal{F}$ such that the family of curves within distance at most 2 from $c^\star$ in the intersection graph of $\mathcal{F}$ has chromatic number greater than $3\alpha + 5\xi + 5$. For $k \in \{1, 2\}$, let $\mathcal{F}_k$ be the 2-curves in $\mathcal{F}$ at distance exactly k from $c^\star$ in the intersection graph of $\mathcal{F}$. Since $\chi(\{c^\star\} \cup \mathcal{F}_1 \cup \mathcal{F}_2) > 3\alpha + 5\xi + 5$ and $\chi(\mathcal{F}_1) \leqslant \xi$ (because $\mathcal{F}$ is a ξ-family), we have $\chi(\mathcal{F}_2) > 3\alpha+4\xi+4$. We have $\mathcal{F}_2 = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, where

$\mathcal{G}_1 = \{c \in \mathcal{F}_2 \colon L(c) \prec R(c) \prec L(c^\star) \prec R(c^\star)\}$, $\mathcal{G}_2 = \{c \in \mathcal{F}_2 \colon L(c^\star) \prec L(c) \prec R(c) \prec R(c^\star)\}$, $\mathcal{G}_3 = \{c \in \mathcal{F}_2 \colon L(c^\star) \prec R(c^\star) \prec L(c) \prec R(c)\}$, $\mathcal{G}_4 = \{c \in \mathcal{F}_2 \colon L(c) \prec L(c^\star) \prec R(c^\star) \prec R(c)\}$. Since $\chi(\mathcal{F}_2) > 3\alpha + 4\xi + 4$ and $\chi(\mathcal{G}_4) \leqslant 4\xi + 4$ (by Lemma 10), we have $\chi(\mathcal{G}_k) > \alpha$ for some $k \in \{1, 2, 3\}$. Since neither basepoint of $c^\star$ lies on $I(\mathcal{G}_k)$, there is a cap-curve γ with $L(c^\star), R(c^\star) \subset \operatorname{ext}\gamma$ and $L(c), R(c) \subset \operatorname{int}\gamma$ for all $c \in \mathcal{G}_k$. The lemma follows with $\mathcal{G} = \mathcal{G}_k$. ◀

Reduction to (ξ, h)-families

For $\xi \in \mathbb{N}$ and a function $h \colon \mathbb{N} \to \mathbb{N}$, a *$(\xi, h)$-family* is a ξ-family $\mathcal{F}$ with the following additional property: for every $\alpha \in \mathbb{N}$ and every subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > h(\alpha)$, there is a subfamily $\mathcal{H} \subseteq \mathcal{G}$ with $\chi(\mathcal{H}) > \alpha$ such that every 2-curve in $\mathcal{F}$ with a basepoint on $I(\mathcal{H})$ has both basepoints on $I(\mathcal{G})$. We will prove the following lemma.

▶ **Lemma 15.** *For any $\xi, k \in \mathbb{N}$ and any function $h \colon \mathbb{N} \to \mathbb{N}$, there is a constant $\zeta \in \mathbb{N}$ such that every (ξ, h)-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant \zeta$.*

The notion of a (ξ, h)-family and Lemma 15 provide a convenient abstraction of what is needed to prove the next lemma and then to prove Lemma 9 with the use of the next lemma.

▶ **Lemma 16.** *For any $\xi, k \in \mathbb{N}$, there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every $\alpha \in \mathbb{N}$, every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > f(\alpha)$ contains a 2-curve c with $\chi(\mathcal{F}(I(c))) > \alpha$.*

Proof of Lemma 16 from Lemma 15. Let $h_\alpha \colon \mathbb{N} \ni \beta \mapsto \beta + 2\alpha + 2 \in \mathbb{N}$, and let $f(\alpha)$ be the constant claimed by Lemma 15 such that every (ξ, h_α)-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant f(\alpha)$. Let $\mathcal{F}$ be a ξ-family with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}(I(c))) \leqslant \alpha$ for every $c \in \mathcal{F}$. It is enough to show that $\mathcal{F}$ is a (ξ, h_α)-family. To this end, consider a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > h_\alpha(\beta)$ for some $\beta \in \mathbb{N}$. Take $\mathcal{G}_L, \mathcal{G}_R \subseteq \mathcal{G}$ so that $L(\mathcal{G}_L) \prec L(\mathcal{G} \smallsetminus \mathcal{G}_L)$, $\chi(\mathcal{G}_L) = \alpha + 1$, $R(\mathcal{G} \smallsetminus \mathcal{G}_R) \prec R(\mathcal{G}_R)$, and $\chi(\mathcal{G}_R) = \alpha + 1$. Let $\mathcal{H} = \mathcal{G} \smallsetminus (\mathcal{G}_L \cup \mathcal{G}_R)$. It follows that $\chi(\mathcal{H}) \geqslant \chi(\mathcal{G}) - 2\alpha - 2 > \beta$. If there is a 2-curve $c \in \mathcal{F}$ with one basepoint on $I(\mathcal{H})$ and the other basepoint not on $I(\mathcal{G})$, then $\mathcal{G}_L \subseteq \mathcal{F}(I(c))$ or $\mathcal{G}_R \subseteq \mathcal{F}(I(c))$, so $\chi(\mathcal{F}(I(c))) \geqslant \alpha + 1$, which is a contradiction. Therefore, every 2-curve in $\mathcal{F}$ with a basepoint on $I(\mathcal{H})$ has both basepoints on $I(\mathcal{G})$. This shows that $\mathcal{F}$ is a (ξ, h_α)-family. ◀

Proof of Lemma 9 from Lemma 15. Let h be the function claimed by Lemma 16 for ξ and k. Let $\mathcal{F}$ be a ξ-family with $\omega(\mathcal{F}) \leqslant k$. In view of Lemma 15, it is enough to show that $\mathcal{F}$ is a (ξ, h)-family. To this end, consider a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > h(\alpha)$ for some $\alpha \in \mathbb{N}$. Lemma 16 yields a 2-curve $c \in \mathcal{G}$ such that $\chi(\mathcal{G}(I(c))) > \alpha$. Every 2-curve in $\mathcal{F}$ with a basepoint on $I(c)$ has both basepoints on $I(c)$, otherwise it would cross c below the baseline. Therefore, the condition of a (ξ, h)-family is satisfied with $\mathcal{H} = \mathcal{G}(I(c))$. ◀

Dealing with (ξ, h)-families

The rest of the proof is inspired from the ideas in [26]. A family of 1-curves $\mathcal{S}$ *supports* a family of 2-curves $\mathcal{F}$ if every 2-curve in $\mathcal{F}$ intersects some 1-curve in $\mathcal{S}$. A *skeleton* is a pair $(\gamma, \mathcal{U})$ such that γ is a cap-curve and $\mathcal{U}$ is a family of pairwise disjoint 1-curves each of which has one endpoint (other than the basepoint) on γ and all the remaining part in $\operatorname{int}\gamma$ (see Figure 3). For a family of 1-curves $\mathcal{S}$, a skeleton $(\gamma, \mathcal{U})$ is an *$\mathcal{S}$-skeleton* if every 1-curve in $\mathcal{U}$ is a subcurve of some 1-curve in $\mathcal{S}$. A skeleton $(\gamma, \mathcal{U})$ *supports* a family of 2-curves $\mathcal{F}$ if every 2-curve $c \in \mathcal{F}$ satisfies $L(c), R(c) \subset \operatorname{int}\gamma$ and intersects some 1-curve in $\mathcal{U}$.

▶ **Lemma 17.** *For every function $h \colon \mathbb{N} \to \mathbb{N}$, there is a function $f \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $\alpha, \beta \in \mathbb{N}$, every (ξ, h)-family $\mathcal{F}$ with $\chi(\mathcal{F}) > f(\alpha, \beta)$ contains one of the following configurations:*

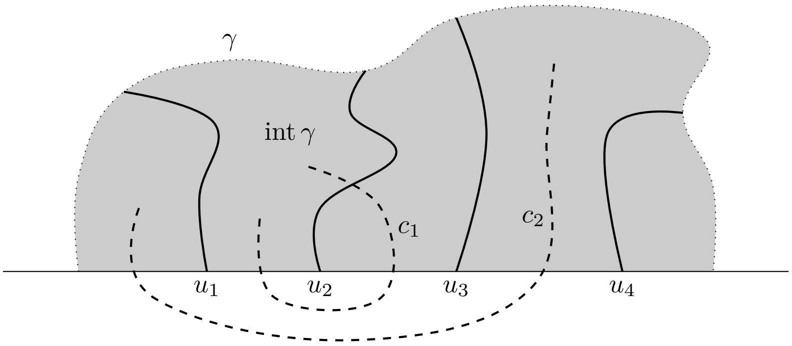

Figure 3 A skeleton $\big(\gamma, \{u_1, u_2, u_3, u_4\}\big)$, which supports c_1 but not c_2.

- *a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > \alpha$ supported by an $L(\mathcal{F})$-skeleton or an $R(\mathcal{F})$-skeleton,*
- *a subfamily $\mathcal{H} \subseteq \mathcal{F}$ with $\chi(\mathcal{H}) > \beta$ supported by a family of 1-curves $\mathcal{S}$ with $\mathcal{S} \subseteq L(\mathcal{F})$ or $\mathcal{S} \subseteq R(\mathcal{F})$ such that $s \prec \mathcal{H}$ or $\mathcal{H} \prec s$ for every 1-curve $s \in \mathcal{S}$.*

Proof. Let $f(\alpha, \beta) = f_1(2\alpha + h(2\beta) + 4)$, where f_1 is the function claimed by Lemma 14. Apply Lemma 14 to obtain a cap-curve γ and a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > 2\alpha + h(2\beta) + 4$ such that every 2-curve $c \in \mathcal{G}$ satisfies $L(c), R(c) \subset \operatorname{int}\gamma$ and intersects some 2-curve in $\mathcal{F}_{\mathrm{ext}}$. Here and further on, $\mathcal{F}_{\mathrm{ext}}$ denotes the family of 2-curves in $\mathcal{F}$ that intersect $\operatorname{ext}\gamma$. Let $\mathcal{U}_L$ be the 1-curves that are subcurves of 1-curves in $L(\mathcal{F})$, have one endpoint (other than the basepoint) on γ, and have all the remaining part in $\operatorname{int}\gamma$. Let $\mathcal{U}_R$ be the 1-curves that are subcurves of 1-curves in $R(\mathcal{F})$, have one endpoint (other than the basepoint) on γ, and have all the remaining part in $\operatorname{int}\gamma$. Thus $(\gamma, \mathcal{U}_L)$ is an $L(\mathcal{F})$-skeleton, and $(\gamma, \mathcal{U}_R)$ is an $R(\mathcal{F})$-skeleton. Let $\mathcal{G}_L$ be the 2-curves in $\mathcal{G}$ that intersect some 1-curve in $\mathcal{U}_L$, and let $\mathcal{G}_R$ be those that intersect some 1-curve in $\mathcal{U}_R$. If $\chi(\mathcal{G}_L) > \alpha$ or $\chi(\mathcal{G}_R) > \alpha$, then the first conclusion of the lemma holds. Thus assume $\chi(\mathcal{G}_L) \leqslant \alpha$ and $\chi(\mathcal{G}_R) \leqslant \alpha$. Let $\mathcal{G}' = \mathcal{G} \smallsetminus (\mathcal{G}_L \cup \mathcal{G}_R)$. It follows that $\chi(\mathcal{G}') \geqslant \chi(\mathcal{G}) - 2\alpha > h(2\beta) + 4$.

By Lemma 8, the 2-curves $c \in \mathcal{G}'$ such that $L(c)$ and $R(c)$ lie in distinct components of $L(\mathcal{G}') \cup R(\mathcal{G}')$ have chromatic number at most 4. Therefore, there is a component V of $L(\mathcal{G}') \cup R(\mathcal{G}')$ such that $\chi(\mathcal{G}'_V) \geqslant \chi(\mathcal{G}') - 4 > h(2\beta)$, where $\mathcal{G}'_V = \{c \in \mathcal{G}' \colon L(c), R(c) \subseteq V\}$. There is a cap-curve $\nu \subseteq V$ connecting the two endpoints of the segment $I(\mathcal{G}'_V)$. Suppose there is a 2-curve $c \in \mathcal{F}_{\mathrm{ext}}$ with both basepoints on $I(\mathcal{G}'_V)$. If $L(c)$ intersects $\operatorname{ext}\gamma$, then the part of $L(c)$ from the basepoint to the first intersection point with γ, which is a 1-curve in $\mathcal{U}_L$, must intersect ν (as $\nu \subseteq V \subset \operatorname{int}\gamma$) and thus a curve in $\mathcal{G}'$ (as V is a component of $\mathcal{G}'$). Thus $\mathcal{G}' \cap \mathcal{G}_L \neq \emptyset$, which is a contradiction. An analogous contradiction is reached if $R(c)$ intersects $\operatorname{ext}\gamma$. This shows that no curve in $\mathcal{F}_{\mathrm{ext}}$ has both basepoints on $I(\mathcal{G}'_V)$.

Since $\mathcal{F}$ is a (ξ, h)-family and $\chi(\mathcal{G}'_V) > h(2\beta)$, there is a subfamily $\mathcal{H}' \subseteq \mathcal{G}'_V$ such that $\chi(\mathcal{H}') > 2\beta$ and every 2-curve in $\mathcal{F}$ with a basepoint on $I(\mathcal{H}')$ has the other basepoint on $I(\mathcal{G}'_V)$. This and the above imply that no curve in $\mathcal{F}_{\mathrm{ext}}$ has a basepoint on $I(\mathcal{H}')$. Since every curve in $\mathcal{H}'$ intersects some curve in $\mathcal{F}_{\mathrm{ext}}$, we have $\mathcal{H}' = \mathcal{H}_L \cup \mathcal{H}_R$, where $\mathcal{H}_L$ are the 2-curves in $\mathcal{H}'$ that intersect some 1-curve in $L(\mathcal{F}_{\mathrm{ext}})$ and $\mathcal{H}_R$ are those that intersect some 1-curve in $R(\mathcal{F}_{\mathrm{ext}})$. Since $\chi(\mathcal{H}') > 2\beta$, we conclude that $\chi(\mathcal{H}_L) > \beta$ or $\chi(\mathcal{H}_R) > \beta$ and thus the second conclusion of the lemma holds with $(\mathcal{H}, \mathcal{S}) = (\mathcal{H}_L, L(\mathcal{F}_{\mathrm{ext}}))$ or $(\mathcal{H}, \mathcal{S}) = (\mathcal{H}_R, R(\mathcal{F}_{\mathrm{ext}}))$. ◀

▶ **Lemma 18.** *For every function $h \colon \mathbb{N} \to \mathbb{N}$, there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every $\alpha \in \mathbb{N}$, every (ξ, h)-family $\mathcal{F}$ with $\chi(\mathcal{F}) > f(\alpha)$ contains a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > \alpha$ supported by an $L(\mathcal{F})$-skeleton or an $R(\mathcal{F})$-skeleton.*

Proof. Let $f(\alpha) = f_1(\alpha, f_1(\alpha, f_1(\alpha, 4\xi)))$, where f_1 is the function claimed by Lemma 17. Suppose to the contrary that no such subfamily $\mathcal{G}$ exists. Let $\mathcal{F}_0 = \mathcal{F}$. Apply Lemma 17 three times to obtain families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ with the following properties:

- $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3$,
- for $i \in \{1, 2, 3\}$, we have $\mathcal{S}_i \subseteq L(\mathcal{F}_{i-1})$ or $\mathcal{S}_i \subseteq R(\mathcal{F}_{i-1})$, $\mathcal{F}_i$ is supported by $\mathcal{S}_i$, and $s \prec \mathcal{F}_i$ or $\mathcal{F}_i \prec s$ for every 1-curve $s \in \mathcal{S}_i$.
- $\chi(\mathcal{F}_1) > f_1(\alpha, f_1(\alpha, 4\xi))$, $\chi(\mathcal{F}_2) > f_1(\alpha, 4\xi)$ and $\chi(\mathcal{F}_3) > 4\xi$.

There are indices $i, j \in \{1, 2, 3\}$ with $i < j$ such that $\mathcal{S}_i$ and $\mathcal{S}_j$ are of the same "type": either $\mathcal{S}_i \subseteq L(\mathcal{F}_{i-1})$ and $\mathcal{S}_j \subseteq L(\mathcal{F}_{j-1})$ or $\mathcal{S}_i \subseteq R(\mathcal{F}_{i-1})$ and $\mathcal{S}_j \subseteq R(\mathcal{F}_{j-1})$. Assume for the rest of the proof that $\mathcal{S}_i \subseteq R(\mathcal{F}_{i-1})$ and $\mathcal{S}_j \subseteq R(\mathcal{F}_{j-1})$; the argument for the other case is analogous.

Let $\mathcal{S}_\prec = \{s \in \mathcal{S}_j : s \prec \mathcal{F}_j\}$, $\mathcal{S}_\succ = \{s \in \mathcal{S}_j : \mathcal{F}_j \prec s\}$, $\mathcal{F}_\prec$ be the 2-curves in $\mathcal{F}_j$ that intersect some 1-curve in $\mathcal{S}_\prec$, and $\mathcal{F}_\succ$ be those that intersect some 1-curve in $\mathcal{S}_\succ$. Thus $\mathcal{F}_\prec \cup \mathcal{F}_\succ = \mathcal{F}_j$. This and $\chi(\mathcal{F}_j) \geqslant \chi(\mathcal{F}_3) > 4\xi$ yield $\chi(\mathcal{F}_\prec) > 2\xi$ or $\chi(\mathcal{F}_\succ) > 2\xi$. Assume for the rest of the proof that $\chi(\mathcal{F}_\prec) > 2\xi$; the argument for the other case is analogous.

Let $\mathcal{S}_\prec^{\min}$ be an inclusion-minimal subfamily of $\mathcal{S}_\prec$ with the property that $\mathcal{S}_\prec^{\min}$ still supports $\mathcal{F}_\prec$. Let $s^\star$ be the 1-curve in $\mathcal{S}_\prec^{\min}$ with rightmost basepoint, and let $\mathcal{F}_\prec^\star = \{c \in \mathcal{F}_\prec : L(c) \text{ intersects } s^\star\}$. Since $\mathcal{F}$ is a ξ-family, we have $\chi(\mathcal{F}_\prec^\star) \leqslant \xi$. By the choice of $\mathcal{S}_\prec^{\min}$, there exists a 2-curve $c^\star \in \mathcal{F}_\prec^\star$ disjoint from every 1-curve in $\mathcal{S}_\prec^{\min}$ other than $s^\star$. Since $\mathcal{F}_\prec$ is supported by $\mathcal{S}_i$, there is a 1-curve $s_i \in \mathcal{S}_i$ that intersects $L(c^\star)$. We show that every 2-curve in $\mathcal{F}_\prec \smallsetminus \mathcal{F}_\prec^\star$ intersects s_i.

Let $c \in \mathcal{F}_\prec \smallsetminus \mathcal{F}_\prec^\star$, and let s be a 1-curve in $\mathcal{S}_\prec^{\min}$ that intersects $L(c)$. Thus $s \neq s^\star$, by the definition of $\mathcal{F}_\prec^\star$. There is a cap-curve $\gamma \subseteq L(c) \cup s$. Since $s \prec s^\star \prec L(c)$ and $s^\star$ intersects neither s nor $L(c)$, we have $s^\star \subset \operatorname{int}\gamma$. Since $L(c^\star)$ intersects $s^\star$ but neither s nor $L(c)$, we also have $L(c^\star) \subset \operatorname{int}\gamma$. Since $s_i \prec \mathcal{F}_i$ or $\mathcal{F}_i \prec s_i$, the basepoint of s_i lies in $\operatorname{ext}\gamma$. Therefore, since s_i intersects $L(c^\star)$, the 1-curve s_i must enter $\operatorname{int}\gamma$ through a point on $L(c)$. This shows that every 2-curve in $\mathcal{F}_\prec \smallsetminus \mathcal{F}_\prec^\star$ intersects s_i. This and the assumption that $\mathcal{F}$ is a ξ-family yield $\chi(\mathcal{F}_\prec \smallsetminus \mathcal{F}_\prec^\star) \leqslant \xi$. We conclude that $\chi(\mathcal{F}_\prec) \leqslant \chi(\mathcal{F}_\prec^\star) + \chi(\mathcal{F}_\prec \smallsetminus \mathcal{F}_\prec^\star) \leqslant 2\xi$, which is a contradiction. ◀

A *chain* of length n is a sequence $\big((a_1, b_1), \ldots, (a_n, b_n)\big)$ of pairs of 2-curves such that

- for $1 \leqslant i \leqslant n$, the 1-curves $R(a_i)$ and $L(b_i)$ intersect,
- for $2 \leqslant i \leqslant n$, the basepoints of $R(a_i)$ and $L(b_i)$ lie between the basepoints of $R(a_{i-1})$ and $L(b_{i-1})$, and $L(a_i)$ intersects $R(a_1), \ldots, R(a_{i-1})$ or $R(b_i)$ intersects $L(b_1), \ldots, L(b_{i-1})$.

▶ **Lemma 19.** *For every $\xi \in \mathbb{N}$ and every function $h\colon \mathbb{N} \to \mathbb{N}$, there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, every (ξ, h)-family $\mathcal{F}$ with $\chi(\mathcal{F}) > f(n)$ contains a chain of length n.*

Proof (see Figure 4). We define the function f by induction. Let $f(1) = 1$; if $\chi(\mathcal{F}) > 1$, then $\mathcal{F}$ contains two intersecting 2-curves, which form a chain of length 1. For the induction step, fix $n \geqslant 1$, and assume that every (ξ, h)-family $\mathcal{H}$ with $\chi(\mathcal{H}) > f(n)$ contains a chain of length n. Let $\beta = f_1\big(f(n), h(2\xi) + 4\xi + 2\big)$ and $f(n+1) = f_2(f_2(f_2(\beta)))$, where f_1 is the function claimed by Lemma 12 and f_2 is the function claimed by Lemma 18. Let $\mathcal{F}$ be a (ξ, h)-family with $\chi(\mathcal{F}) > f(n+1)$. We claim that $\mathcal{F}$ contains a chain of length $n+1$.

Let $\mathcal{F}_0 = \mathcal{F}$. Apply Lemma 18 three times to find families of 2-curves $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and skeletons $(\gamma_1, \mathcal{U}_1)$, $(\gamma_2, \mathcal{U}_2)$, $(\gamma_3, \mathcal{U}_3)$ with the following properties:

- $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3$,
- for $i \in \{1, 2, 3\}$, $(\gamma_i, \mathcal{U}_i)$ is an $L(\mathcal{F}_{i-1})$-skeleton or an $R(\mathcal{F}_{i-1})$-skeleton supporting $\mathcal{F}_i$,
- $\chi(\mathcal{F}_1) > f_2(f_2(\beta))$, $\chi(\mathcal{F}_2) > f_2(\beta)$, and $\chi(\mathcal{F}_3) > \beta$.

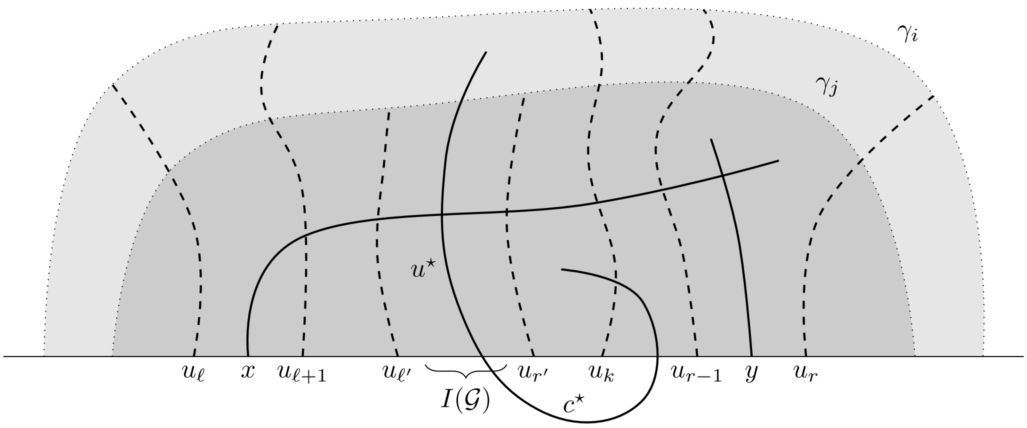

Figure 4 Illustration for the proof of Lemma 19.

There are two indices $i, j \in \{1, 2, 3\}$ with $i < j$ such that the skeletons $(\gamma_i, \mathcal{U}_i)$ and $(\gamma_j, \mathcal{U}_j)$ are of the same "type": either an $L(\mathcal{F}_{i-1})$-skeleton and an $L(\mathcal{F}_{j-1})$-skeleton or an $R(\mathcal{F}_{i-1})$-skeleton and an $R(\mathcal{F}_{j-1})$-skeleton. Assume for the rest of the proof that $(\gamma_i, \mathcal{U}_i)$ is an $L(\mathcal{F}_{i-1})$-skeleton and $(\gamma_j, \mathcal{U}_j)$ is an $L(\mathcal{F}_{j-1})$-skeleton; the argument for the other case is analogous.

By Lemma 12, since $\chi(\mathcal{F}_j) \geqslant \chi(\mathcal{F}_3) > \beta$, there is a subfamily $\mathcal{H} \subseteq \mathcal{F}_j$ such that $\chi(\mathcal{H}) > f(n)$ and $\chi(\mathcal{F}_j(x, y)) > h(2\xi) + 4\xi + 2$ for any two intersecting 1-curves $x, y \in L(\mathcal{H}) \cup R(\mathcal{H})$. Since $\chi(\mathcal{H}) > f(n)$, the family $\mathcal{H}$ contains a chain $\big((a_1, b_1), \ldots, (a_n, b_n)\big)$ of length n. Let x and y be the 1-curves $R(a_n)$ and $L(b_n)$ assigned so that $x \prec y$. By the definition of a chain, x and y intersect, and therefore $\chi(\mathcal{F}_j(x, y)) > h(2\xi) + 4\xi + 2$.

Enumerate the 1-curves in $\mathcal{U}_i$ as $u_1, \ldots, u_m$ so that $u_1 \prec \cdots \prec u_m$, where $m = |\mathcal{U}_i|$. Assume $u_1 \prec x \prec y \prec u_m$ for simplicity (adjusting the proof to the general case is straightforward). There are indices ℓ and r with $1 \leqslant \ell < r \leqslant m$, $u_\ell \prec x \prec u_{\ell+1}$, and $u_{r-1} \prec y \prec u_r$. Let $\mathcal{F}_j^L = \{c \in \mathcal{F}_j : x \prec L(c) \prec u_{\ell+1}\}$ and $\mathcal{F}_j^R = \{c \in \mathcal{F}_j : u_{r-1} \prec R(c) \prec y\}$. It follows that $\mathcal{F}_j(x, y) \subseteq \mathcal{F}_j^L \cup \mathcal{F}_j(u_{\ell+1}, u_{r-1}) \cup \mathcal{F}_j^R$.

Since $\mathcal{F}$ is a ξ-family, the 2-curves in $\mathcal{F}_j^L$ that intersect u_ℓ have chromatic number at most ξ, and so do the 2-curves in $\mathcal{F}_j^L$ that intersect $u_{\ell+1}$. The remaining 2-curves $c \in \mathcal{F}_j^L$ (intersecting neither u_ℓ nor $u_{\ell+1}$) are pairwise disjoint, because their 1-curves $L(c)$ are contained in and $R(c)$ are disjoint from the part of $\operatorname{int} \gamma_i$ between u_ℓ and $u_{\ell+1}$. Thus $\chi(\mathcal{F}_j^L) \leqslant 2\xi + 1$. Similarly, $\chi(\mathcal{F}_j^R) \leqslant 2\xi + 1$. This yields $\ell + 1 \leqslant r - 1$ and $\chi(\mathcal{F}_j(u_{\ell+1}, u_{r-1})) \geqslant \chi(\mathcal{F}_j(x, y)) - 4\xi - 2 > h(2\xi)$.

Since $\mathcal{F}$ is a (ξ, h)-family, there is a subfamily $\mathcal{G} \subseteq \mathcal{F}_j(u_{\ell+1}, u_{r-1})$ with $\chi(\mathcal{G}) > 2\xi$ such that every 2-curve $c \in \mathcal{F}$ with a basepoint on $I(\mathcal{G})$ satisfies $u_{\ell+1} \prec c \prec u_{r-1}$.

Let $u_{\ell'}$ be the 1-curve in $\mathcal{U}_j$ with rightmost basepoint to the left of $I(\mathcal{G})$, and let $u_{r'}$ be the 1-curve in $\mathcal{U}_j$ with leftmost basepoint to the right of $I(\mathcal{G})$. Every 2-curve in $\mathcal{G}$ must intersect $u_{\ell'}$, some 1-curve in $\mathcal{U}_j(I(\mathcal{G}))$, or $u_{r'}$. Since $\mathcal{F}$ is a ξ-family, the 2-curves in $\mathcal{G}$ that intersect $u_{\ell'}$ have chromatic number at most ξ, and so do the 2-curves in $\mathcal{G}$ that intersect $u_{r'}$. Therefore, since $\chi(\mathcal{G}) > 2\xi$, some 2-curve in $\mathcal{G}$ must intersect a 1-curve in $\mathcal{U}_j(I(\mathcal{G}))$. In particular, the family $\mathcal{U}_j(I(\mathcal{G}))$ is non-empty.

Let $u^\star \in \mathcal{U}_j(I(\mathcal{G}))$. The 1-curve $u^\star$ is a subcurve of $L(c^\star)$ for some 2-curve $c^\star \in \mathcal{F}_{j-1}$. Since the basepoint of $L(c^\star)$ lies on $I(\mathcal{G})$, the property of $\mathcal{G}$ implies $u_{\ell+1} \prec c^\star \prec u_{r-1}$. Since $c^\star \in \mathcal{F}_{j-1} \subseteq \mathcal{F}_i$ and $\mathcal{F}_i$ is supported by $(\gamma_i, \mathcal{U}_i)$, the 1-curve $R(c^\star)$ intersects at least one of the 1-curves $u_{\ell+1}, \ldots, u_{r-1}$, say u_k. Let $a_{n+1} = c^\star$ and b_{n+1} be the 2-curve in $\mathcal{F}_{i-1}$ such that u_k is a subcurve of $L(b_{n+1})$. For $1 \leqslant t \leqslant n$, the 1-curves $R(a_t)$ and $L(b_t)$ intersect and they are both contained in $\operatorname{int}\gamma_j$ (because $a_t, b_t \in \mathcal{H}$), the basepoint of $L(a_{n+1})$ is between the basepoints of $R(a_t)$ and $L(b_t)$, and $L(a_{n+1})$ intersects γ_j (as it contains $u^\star$). Therefore, $L(a_{n+1})$ intersects all $R(a_1), \dots, R(a_n)$. We conclude that $\bigl((a_1, b_1), \dots, (a_{n+1}, b_{n+1})\bigr)$ is a chain of length $n+1$. ◀

Proof of Lemma 15. Let $\zeta = f(2k+1)$, where f is the function claimed by Lemma 19 for ξ and h. Suppose $\chi(\mathcal{F}) > \zeta$. It follows that $\mathcal{F}$ contains a chain of length $2k+1$. This chain contains a subchain $\bigl((a_1, b_1), \dots, (a_{k+1}, b_{k+1})\bigr)$ of pairs of the same "type": $L(a_i)$ intersects $R(a_1), \dots, R(a_{i-1})$ for $2 \leqslant i \leqslant k+1$ and thus $\{a_1, \dots, a_{k+1}\}$ is a clique, or $R(b_i)$ intersects $L(b_1), \dots, L(b_{i-1})$ for $2 \leqslant i \leqslant k+1$ and thus $\{b_1, \dots, b_{k+1}\}$ is a clique. Thus $\omega(\mathcal{F}) > k$. ◀

3 Proof of Theorem 2

▶ **Lemma 20** (Fox, Pach, Suk [9, Lemma 3.2]). *For every $t \in \mathbb{N}$, there is a constant $\nu_t > 0$ such that every family of curves $\mathcal{F}$ any two of which intersect in at most t points has subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_d \subseteq \mathcal{F}$ with the following properties:*

- *for $1 \leqslant i \leqslant d$, there is a curve $c_i \in \mathcal{F}_i$ intersecting all curves in $\mathcal{F}_i \smallsetminus \{c_i\}$,*
- *for $1 \leqslant i < j \leqslant d$, every curve in $\mathcal{F}_i$ is disjoint from every curve in $\mathcal{F}_j$,*
- *$|\mathcal{F}_1 \cup \dots \cup \mathcal{F}_d| \geqslant \nu_t |\mathcal{F}| / \log |\mathcal{F}|$.*

Proof of Theorem 2. Let $\mathcal{F}$ be a family of curves obtained from the edges of G by shortening them slightly so that they do not intersect at the endpoints but all other intersection points are preserved. If follows that $\omega(\mathcal{F}) \leqslant k-1$ (as G is k-quasi-planar) and any two curves in $\mathcal{F}$ intersect in at most t points. Let ν_t, $\mathcal{F}_1, \dots, \mathcal{F}_d$, and $c_1, \dots, c_d$ be as claimed by Lemma 20. For $1 \leqslant i \leqslant d$, since $\omega(\mathcal{F}_i \smallsetminus \{c_i\}) \leqslant \omega(\mathcal{F}) - 1 \leqslant k-2$, Theorem 1 yields $\chi(\mathcal{F}_i \smallsetminus \{c_i\}) \leqslant f_t(k-2)$. Thus $\chi(\mathcal{F}_1 \cup \dots \cup \mathcal{F}_d) \leqslant f_t(k-2)+1$. For every color class $\mathcal{C}$ in a proper coloring of $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_d$ with $f_t(k-2)+1$ colors, the vertices of G and the curves in $\mathcal{C}$ form a planar topological graph, and thus $|\mathcal{C}| < 3n$. Thus $|\mathcal{F}_1 \cup \dots \cup \mathcal{F}_d| < 3(f_t(k-2)+1)n$. This, the third property in Lemma 20, and the fact that $|\mathcal{F}| < n^2$ yield $|\mathcal{F}| < 3\nu_t^{-1}(f_t(k-2)+1)n \log |\mathcal{F}| < 6\nu_t^{-1}(f_t(k-2)+1)n \log n$. ◀

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