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On almost complex structures from classical linear connections

Abstract. Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms and let T be the tangent functor on $\mathcal{M}f_m$. Let $\mathcal{V}$ be the category of real vector spaces and linear maps and let $\mathcal{V}_m$ be the category of m-dimensional real vector spaces and linear isomorphisms. We characterize all regular covariant functors $F : \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{J}$ transforming classical linear connections ∇ on m-dimensional manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $F(T)M = \bigcup_{x\in M} F(T_xM)$.

Introduction. All manifolds considered in the paper are assumed to be Hausdorff, finite dimensional, second countable, without boundaries and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (of class C^∞).

The category of m-dimensional manifolds and local diffeomorphisms is denoted by $\mathcal{M}f_m$. The category of vector bundles and vector bundle homomorphisms is denoted by $\mathcal{VB}$. The category of m-dimensional real vector spaces and linear isomorphisms is denoted by $\mathcal{V}_m$. The category of finite dimensional real vector spaces and linear maps is denoted by $\mathcal{V}$.

The concepts of natural bundles and natural operators can be found in [3].

Let $F : \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor. The regularity of F means that F transforms smoothly parametrized families of morphisms into smoothly parametrized ones. Let $T : \mathcal{M}f_m \to \mathcal{VB}$ be the tangent functor

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sending any m-manifold M into the tangent bundle TM of M and any $\mathcal{M}f_m$-map $\varphi : M \to M_1$ into the tangent map $T\varphi : TM \to TM_1$. Applying F to fibers T_xM of TM, one can define a vector natural bundle $F(T)$ over m-manifolds of order 1 by

$$F(T)M = \bigcup_{x\in M} F(T_xM) \quad \text{and} \quad F(T)\varphi = \bigcup_{x\in M} F(T_x\varphi) : F(T)M \to F(T)M_1$$

for any m-manifold M and any $\mathcal{M}f_m$-map $\varphi : M \to M_1$ between m-manifolds. In particular, if F is the identity functor, then $F(T) = T$. If $FV = \otimes^r V \otimes \otimes^q V^*$ then $F(T)M = \otimes^r TM \otimes \otimes^q T^*M$.

It is well known (see e.g. [1]), that if ∇ is a classical linear connection on a manifold M, then the tangent bundle TM of M possesses the (canonical) almost complex structure $\tilde{J}^\nabla$ such that $\tilde{J}^\nabla(X^H) = X^V$ and $\tilde{J}^\nabla(X^V) = -X^H$ for any vector field $X \in \mathcal{X}(M)$ on M, where $X^H \in \mathcal{X}(TM)$ is the ∇-horizontal lift of X and $X^V \in \mathcal{X}(TM)$ is the vertical lift of X to TM.

In the present note we study the following problem.

Problem 1. Characterize all covariant regular functors $F : \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{J}$ transforming classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $F(T)M$.

1. Basic definitions. A classical linear connection on a manifold M is a $\mathbf{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X fY = XfY + f\nabla_X Y$ for any smooth map $f : M \to \mathbf{R}$ and any vector fields $X, Y \in \mathcal{X}(M)$ on M. Equivalently, a classical linear connection on M is a right invariant decomposition $TLM = H^\nabla \oplus VLM$ of the tangent bundle TLM of LM, where LM is the principal bundle (with the structural group $GL(m)$) of linear frames of M and VLM is the vertical bundle of LM, see [2].

A complex structure on a real vector space W is a linear endomorphism $J : W \to W$ such that $J^2 = -id_W$.

An almost complex structure on a manifold N is a tensor field $\tilde{J} : TN \to TN$ on N of type $(1, 1)$ (affinor) such that $\tilde{J} \circ \tilde{J} = -id_{TN}$.

The general concept of natural operators can be found in the fundamental monograph [3]. We need the following particular case of the one, only.

Let $F : \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor. A $\mathcal{M}f_m$-natural operator transforming classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}(\nabla) : TF(T)M \to TF(T)M$ on $F(T)M$ is a $\mathcal{M}f_m$-invariant family $\tilde{J} : Q \rightsquigarrow (ACS)F(T)$ of operators

$$\tilde{J} : Q(M) \to (ACS)(F(T)M)$$

for m-manifolds M, where $Q(M)$ is the set of classical linear connections on M and $(ACS)(N)$ is the set of almost complex structures on N. The invariance of $\tilde{J}$ means that if $\nabla \in Q(M)$ and $\nabla_1 \in Q(M_1)$ are φ-related by an embedding $\varphi : M \to M_1$ (i.e. if φ is (∇, ∇_1)-affine embedding), then $\tilde{J}(\nabla)$ and $\tilde{J}(\nabla_1)$ are $F(T)\varphi$-related (i.e. $TF(T)\varphi \circ \tilde{J}(\nabla) = \tilde{J}(\nabla_1) \circ TF(T)\varphi$).

Let F be as above. A $\mathcal{V}_m$-canonical complex structure on $V \oplus FV$ is a $\mathcal{V}_m$-invariant system J of complex structures

$$J : V \oplus FV \to V \oplus FV$$

on (vector spaces) $V \oplus FV$ for m-dimensional real vectors spaces V. The invariance means that $(\varphi \oplus F\varphi) \circ J = J \circ (\varphi \oplus F\varphi)$ for any linear isomorphism $\varphi : V \to V_1$ between m-dimensional vector spaces.

2. The main result. The main result of the present note is the following theorem.

Theorem 1. *Let $F : \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor. The following conditions are equivalent:*

(i) There exists a $\mathcal{M}f_m$-natural operator $\tilde{J} : Q \rightsquigarrow (ACS)F(T)$.

(ii) There exists a $\mathcal{V}_m$-canonical complex structure J on $V \oplus FV$.

Proof. (i) $\Rightarrow$ (ii). Let $\tilde{J} : Q \rightsquigarrow (ACS)F(T)$ be a $\mathcal{M}f_m$-natural operator in question. Let V be a $\mathcal{V}_m$-object and let ∇^V be the ($\mathcal{V}_m$-canonical) torsion free flat classical linear connection on V. Then the almost complex structure $\tilde{J}(\nabla^V) : TF(T)V \to TF(T)V$ on $F(T)V$ restricts to the complex structure

$$J := \tilde{J}(\nabla^V)_{0_{0_V}} : T_{0_{0_V}} F(T)V \to T_{0_{0_V}} F(T)V$$

on the tangent space $T_{0_{0_V}} F(T)V$ of $F(T)V$ at $0_{0_V} \in F(T)V$, where 0_V is the zero in V and 0_{0_V} is the zero in $F(T)_{0_V} V$. We see that $T_{0_{0_V}} F(T)V = V \oplus FV$ modulo the usual identifications. (For, $TV = V \oplus V$, then $F(T)V = V \oplus FV$, and then $T_{0_{0_V}} F(T)V = V \oplus FV$.) So,

$$J : V \oplus FV \to V \oplus FV$$

is the complex structure on $V \oplus FV$ for any $\mathcal{V}_m$-object V. Because of the canonical character of the construction of J, J is $\mathcal{V}_m$-canonical.

(ii) $\Rightarrow$ (i). Suppose $J : V \oplus FV \to V \oplus FV$ is a $\mathcal{V}_m$-canonical complex structure. Let $\nabla \in Q(M)$ be a classical linear connection on an m-manifold M. Let $v \in F(T)_x M$, $x \in M$. Since $F(T)$ is of order 1, $F(T)M = LM[F(T)_0 \mathbf{R}^m]$ (the associated space). Then the ∇-decomposition $TLM = H^\nabla \oplus VLM$ induces (in obvious way) the ∇-decomposition $TF(T)M = \tilde{H}^\nabla \oplus VF(T)M$. Then we have the identification

$$T_v F(T)M = \tilde{H}^\nabla_v \oplus V_v F(T)M \tilde{=} T_x M \oplus F(T)_x M = T_x M \oplus F(T_x M)$$

canonically depending on ∇, where the equality is the connection decomposition, the identification $\tilde{=}$ is the usual one (namely, $\tilde{H}^\nabla_v = T_x M$ modulo the tangent of the projection of $F(T)M$, and $V_v F(T)M = T_v(F(T)_x M) =$

$F(T)_xM$ modulo the standard identification) and the second equality is by the definition of $F(T)M$. We define $\tilde{J}(\nabla)_{|v} : T_vF(T)M \to T_vF(T)M$ by

$$\tilde{J}(\nabla)_{|v} := J : T_xM \oplus F(T_xM) \to T_xM \oplus F(T_xM)$$

modulo the above identification $T_vF(T)M \tilde{=} T_xM \oplus F(T_xM)$. Then $\tilde{J}(\nabla) : TF(T)M \to TF(T)M$ is an almost complex structure on $F(T)M$. By the canonical character of $\tilde{J}(\nabla)$, the (resulting) family $\tilde{J} : Q \rightsquigarrow (ACS)F(T)$ is a $\mathcal{M}f_m$-natural operator. □

3. Corollaries. We start with the following two lemmas.

Lemma 1. *Let $F : \mathcal{V}_m \to \mathcal{V}$ be a regular covariant functor. Suppose that there is no non-zero $\mathcal{V}_m$-canonical linear map $V \to FV$. Then there is no $\mathcal{V}_m$-canonical complex structure on $V \oplus FV$.*

Proof. Suppose $J : V \oplus FV \to V \oplus FV$ is a $\mathcal{V}_m$-canonical complex structure. Since there is no non-zero $\mathcal{V}_m$-canonical linear map $V \to FV$, then $J_{|V\times\{0\}} : V \to V$ (modulo the identification $V = V \times \{0\}$) is a canonical complex structure on V. On the other hand, any $\mathcal{V}_m$-canonical linear map $V \to V$ is a constant multiple of the identity map id_V. It is a contradiction. □

Lemma 2. *Let r, q be non-negative integers. If $r - q \neq 1$, there is no non-zero $\mathcal{V}_m$-canonical linear map $k : V \to \otimes^r V \otimes \otimes^q V^*$.*

Proof. Let $k : V \to \otimes^r V \otimes \otimes^q V^*$ be a $\mathcal{V}_m$-canonical linear map. Let $v \in V$. The invariance of k with respect to homotheties $t id_V$ (for $t > 0$) gives the homogeneity condition $tk(v) = t^{r-q}k(v)$. Then $k(v) = 0$ if $r - q \neq 1$. □

Then we have the following corollary of Theorem 1.

Corollary 1. *Let r, q be non-negative integers. If $r - q \neq 1$, there is no $\mathcal{M}f_m$-natural operator $\tilde{J} : Q \rightsquigarrow (ACS) \otimes^r T \otimes^q T^*$ sending classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $\otimes^r TM \otimes \otimes^q T^*M$.*

Proof. We have $F(T)M = \otimes^r TM \otimes \otimes^q T^*M$ for $FV = \otimes^r V \otimes \otimes^q V^*$. By Lemma 2, there is no $\mathcal{V}_m$-canonical non-zero linear map $V \to FV$ if $r - q \neq 1$. So, by Lemma 1, there is no $\mathcal{V}_m$-canonical complex structure on $V \oplus FV = V \oplus (\otimes^r V \otimes \otimes^q V^*)$ if $r - q \neq 1$. Then there is no $\mathcal{M}f_m$-natural operator $\tilde{J}$ in question if $r - q \neq 1$ because of Theorem 1. □

We have also the following two lemmas.

Lemma 3. *Let p be a positive integer. Let $F : \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor given by $FV = V \times \cdots \times V$ ($(p-1)$ times of V) and $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). There is a $\mathcal{V}_m$-canonical complex structure on $V \oplus FV$ if and only if p is even.*

Proof. Any $\mathcal{V}_m$-canonical map $J : V \times \cdots \times V$ (p times of V) $\to V \times \cdots \times V$ (p times of V) is of the form $J(w) = Cw^T$, $w = (w_1, \ldots, w_p)$ for some $p \times p$ matrix C with real coefficients. If $J \circ J = -id_V$, then $C^2 = -id$, and then p is even. On the other hand, if p is even we have the $\mathcal{V}_m$-canonical almost complex structure on $V \times \cdots \times V$ (p times of V). Namely, the $\frac{p}{2}$ copies of the canonical almost complex structure on $V \times V$, $(v, w) \to (-w, v)$. □

Lemma 4 (Lemma 5.1 in [4]). *Let A be a p-dimensional Weil algebra and let T^A be the corresponding Weil functor. For any classical linear connection ∇ on an m-manifold M, we have the base-preserving fibred diffeomorphism $I^A_\nabla : T^A M \to TM \otimes \mathbf{R}^{p-1}$ canonically depending on ∇.*

We see that $TM \otimes \mathbf{R}^{p-1} = TM \times_M \cdots \times_M TM$ ($(p-1)$ times of TM) $= F(T)M$, where $F : \mathcal{V}_m \to \mathcal{V}$, $FV = V \times \cdots \times V$ ($(p-1)$ times of V), $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). So, we have the following corollary of Theorem 1 (and Lemmas 3 and 4).

Corollary 2. *Let A be a Weil algebra. There exists a $\mathcal{M}f_m$-natural operator $\tilde{J} : Q \rightsquigarrow (ACS)T^A$ sending classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $T^A M$ if and only if A is even dimensional.*

Remark 1. If m is odd and the dimension of A is odd, then $T^A M$ is odd dimensional and there is no $\mathcal{M}f_m$-natural operator in question because of the clear argument. But if m is even and A is odd dimensional, then $T^A M$ is even dimensional and theoretically there may exist $\mathcal{M}f_m$-natural operators in question. By Corollary 2, such $\mathcal{M}f_m$-operators do not exist. For example, there is no $\mathcal{M}f_m$-natural operator $\tilde{J} : Q \rightsquigarrow (ACS)T^2$ (even if m is even), where $T^2 M = J^2_0(\mathbf{R}, M)$ is the second order tangent bundle.

The bundle LM of linear frames of M is open in $TM \times_M \cdots \times_M TM$ (m times of TM). By Corollary 2, if m is odd, there is an $\mathcal{M}f_m$-natural operator $\tilde{J}$ sending classical linear connections ∇ into almost complex structures $\tilde{J}(\nabla)$ on $TM \times_M \cdots \times_M TM$ (m times of TM). So, we have the following corollary.

Corollary 3. *If m is odd, there is a $\mathcal{M}f_m$-natural operator $\tilde{J}_1$ sending classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}_1(\nabla)$ on LM such that $\tilde{J}_1(\nabla)$ is the restriction of $\tilde{J}(\nabla)$.*

Remark 2. For any positive integer m there are many $\mathcal{M}f_m$-natural operators $\tilde{J}_2 : Q \rightsquigarrow (ACS)L$. However, by Corollary 2, any such $\tilde{J}_2(\nabla)$ cannot be extended canonically to an almost complex structure $\tilde{J}(\nabla)$ on the whole $TM \times_M \cdots \times_M TM$ (m times of TM) if m is even.

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