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Research Article

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Lyapunov function for cosmological dynamical system

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Abstract: We prove the asymptotic global stability of the de Sitter solution in the Friedmann-Robertson-Walker conservative and dissipative cosmology. In the proof we construct a Lyapunov function in an exact form and establish its relationship with the first integral of dynamical system determining evolution of the flat Universe. Our result is that de-Sitter solution is asymptotically stable solution for general form of equation of state $p = p(\rho, H)$, where dependence on the Hubble function H means that the effect of dissipation are included.

Keywords: Lyapunov function, First integral, Dynamical system, Mathematical cosmology

MSC: 34D23, 83F05

1 Introduction

We assume a cosmological model with topology $\mathbb{R} \times \mathcal{M}^3$ where $\mathcal{M}^3$ is maximally symmetric 3-space; the metric of spacetime is Lorenzian $(-, +, +, +)$ and assumes the following form

$$ds^2 = -dt^2 = a(t)^2 \quad \frac{dr^2}{1 - \frac{1}{4}kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \Big) \tag{1}$$

where (t, r, θ, ϕ) are the pseudo-spherical coordinates [1], $a(t) > 0$ is the function of the time coordinate which is called the scale factor, and $k = -1, 0, 1$ is the spacial curvature. The 3-space of constant curvature is spatially open for $k = -1$, spatially closed for $k = 1$ and spatially flat for $k = 0$.

The dynamics of metric tensor $g_{\mu\nu}$: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is determined from the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu} R_{\mu\nu}$, $(x^1, x^2, x^3, x^4) = (t, r, \theta, \phi)$, is the Ricci scalar; we use natural units such that $8\pi G = c = 1$.

Matter is assumed to be in the form of perfect fluid

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu \tag{3}$$

where $u^\mu = (-1, 0, 0, 0)$ denotes the four-velocity of an observer comoving with the fluid. The functions $p(t)$, $\rho(t)$ are pressure and energy density of the matter fluid, respectively.

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Tensor energy-momentum can be generalised for non-perfect, viscous fluid [2], which for metric (1) assumes the form

$$T^0{}_0 = -\rho, \qquad T^i{}_k = \begin{cases} p - 3\xi\frac{\dot{a}}{a} & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \tag{4}$$

where $i, k = \{1, 2, 3\}$, ξ is the viscosity coefficient and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. Formally inclusion of viscous fluid is equivalent to replace pressure p by $p - 3\xi\frac{\dot{a}}{a}$ in the energy-momentum tensor.

Such a cosmological model with the imperfect fluid ($\xi =$ const) has been considered since Heller et al. [3] and Belinskii et al. [4]. The Einstein field equation (2) for this model reduces to [5]

$$\rho = -\Lambda + \frac{k}{a^2} + 3\frac{\dot{a}^2}{a^2} \tag{5}$$

$$p = \Lambda - 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \tag{6}$$

where Λ is the cosmological constant parameter. The dependence of pressure p on $H = \frac{\dot{a}}{a}$ means that we considered viscous effects with the viscosity coefficient $\xi = -\frac{1}{3}\frac{\partial p}{\partial H}$.

Equations (5)-(6) can be rewritten to the form of three-dimensional autonomous dynamical system

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{7}$$

$$\dot{\rho} = -3H(\rho + p) \tag{8}$$

$$\dot{a} = Ha \tag{9}$$

where $p = p(H, \rho)$ is the general form of assumed equation of state.

The dynamical system (7)-(9) is analyzed in the present paper. Let us study the asymptotic stability of the solution ($H_0 = \sqrt{\frac{\Lambda}{3}}, \rho_0 = 0$) (future de Sitter solution).

Definition 1.1. *A critical point* $\mathbf{x}_0$ *of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$*, is a (Lyapunov) stable point if for all neighbourhoods* U *of* $\mathbf{x}_0$ *there exists a neighbourhood* U_* *of* $\mathbf{x}_0$ *such that if* $\mathbf{x}_0 \in U_*$ *at* $t = t_0$ *then* $\phi_t(\mathbf{x}_0) \in U$ *for all* $t > t_0$*, where* ϕ_t *is the flow of a dynamical system. If the critical point* $\mathbf{x}_0$ *is stable for all* $x \in U_*$, $\lim_{t\to\infty} ||\phi_t(\mathbf{x} - \mathbf{x}_0|| = 0$.

To determine the asymptotic stability of a solution of the dynamical system considered we construct the Lyapunov function [6].

2 Lyapunov function

Let us consider shortly some basic definitions of first integrals [7].

Let $M \subset \mathbb{K}^n$ be an open subset in $\mathbb{K}^n$ where the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{X}(M)$ and $\mathcal{F}(M)$ the algebra of vector fields and functions on M, respectively. For simplicity, we assume that all objects are of the class C^∞. Let us consider a system of ordinary differential equations on M

$$\frac{dx}{dt} = X_F(x) = F(x), \quad x = (x^1, \dots, x^n) \in M \subset \mathbb{K}^n \tag{10}$$

where the vector field $X_F = \mathcal{X}(M)$ is given by

$$X_F = \sum_{i=1}^{n} f^i(x)\frac{\partial}{\partial x^i} = \sum_{i=1}^{n} f^i(x)\partial_i = f^i(x)\partial_i \tag{11}$$

where $(f^1(x), \dots, f^n(x))$ are components of the vector field X_F.

We are looking for a solution or a class of solutions of system (10). This is the motivation of the following definition.

Definition 2.1. *We say that subset $W \subset M$ is invariant with respect to system (10) if W consists only of the system's phase curves.*

It seems that it is extremely difficult to check if a given set W is invariant with respect to (10) because, in general, we do not know its solutions. However, for checking the invariance, it is enough to know if, for all $x \in W$, the vector of phase velocity in this point is tangent to M, i.e. if $F(x) \in T_x W$ for all $x \in W$.

The most important invariant sets are those allowing to reduce the dimension of the system. For this purpose one invariant set is not enough, we need a one parameter family W_c of $(n-1)$-dimensional invariant sets that gives a foliation of M. Such a foliation arises naturally when we know a first integral.

Definition 2.2. *Function $G \in \mathcal{F}(M)$ is called the first integral of system (10) if it is constant on all solutions of the system. It is equivalent to the condition*

$$X_F(G)(x) = \partial_i G(x) f^i(x) = 0, \quad \text{for} \quad x \in M. \tag{12}$$

It is well known that a level $W_c = \{x \in M | G(x) = c\}$ of a first integral is invariant, and $c \to W_c$ gives the foliation mentioned earlier.

When we cannot find a first integral of the system, it is sometimes possible to find a function whose one level is invariant.

Definition 2.3 (Lyapunov function)**.** *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in X \subset \mathbb{R}^n$ be a smooth autonomous system of equations with fixed point x_0. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous function in a neighbourhood U of $\mathbf{x}_0$, the V is called a Lyapunov function for the point x_0 if*

1. *V is differentiable in $U \setminus \{\mathbf{x}_0\}$*
2. *$V(\mathbf{x}) > V(\mathbf{x}_0)$*
3. *$\dot{V} \leq 0 \quad \forall \mathbf{x} \in U \setminus \{\mathbf{x}_0\}$.*

Theorem 2.4 (Lyapunov stability)**.** *Let $\mathbf{x}_0$ be a critical point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and let U be a domain containing $\mathbf{x}_0$. If there exists a Lyapunov function $V(\mathbf{x})$ for which $\dot{V} \leq 0$, then $\mathbf{x}_0$ is a stable fixed point. If there exists a Lyapunov function $V(\mathbf{x})$ for which $\dot{V} < 0$, then $\mathbf{x}_0$ is a asymptotically stable fixed point.*

Furthermore, if $||x|| \to \infty$ and $V(\mathbf{x}) \to \infty$ for all $\mathbf{x}$, then $\mathbf{x}_0$ is said to be globally stable or globally asymptotically stable, respectively.

Let us return to our system (7)-(9).

Theorem 2.5. *The system (7)-(9) has first integral in the form*

$$\rho - 3H^2 + \Lambda = 3\frac{k}{a^2}. \tag{13}$$

Proof. After differentiation of both sides of (13) over time t and substitution of right-hand sides of (7)-(9) we obtain the form (13) of the first integral. □

In the three-dimensional phase space the first integral (13) defines surfaces for different values of the parameter Λ. The dimension of the dynamical system (7)-(9) can be lowered over one due to this first integral

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{14}$$

$$\dot{\rho} = -3H(\rho + p) \tag{15}$$

where $p = p(H, \rho)$ in generally.

For the de Sitter fixed point of (14)-(15), we have $p = -\rho$, from equation (15). Then from equation (14) and using the first integral (13) we obtain that $k = 0$. It means that fixed point is an intersection of the trajectory of the flat model and the line $\rho + p(H, \rho) = 0$ in the phase space (H, ρ).

Theorem 2.6. *The de Sitter solution* $H_0 = \pm\sqrt{\frac{\Lambda}{3}}$ *is asymptotically stable for* $H_0 > 0$ *and asymptotically unstable for* $H_0 < 0$.

Proof. Let us propose the following Lyapunov function

$$V(H,\rho) \equiv \begin{cases} \rho - 3H^2 + \Lambda & \text{for } k = 0, 1 \\ -\left(\rho - 3H^2 + \Lambda\right) & \text{for } k = -1 \end{cases} \tag{16}$$

which can be obtained from (13) by putting $k = 0$. The surface $\{(H,\rho)\colon \rho - 3H^2 + \Lambda = 0\}$ divides the phase space into two domains occupied by the trajectories with $k = 1$ and $k = -1$, respectively.

Let us consider the first case of non-negative Lyapunov function $V(t)$ for $k = 0, 1$ in (16).

$$\begin{aligned} \dot{V}(t) &= \dot{\rho} - 6H\dot{H} = -3H(\rho + p) - 6H\left(-H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}\right) \\ &= -3H\left\{\rho + p + 2\left[-H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}\right]\right\} \\ &= -2H\left(\rho - 3H^2 + \Lambda\right) = -2H\frac{3k}{a^2} \le 0, \quad \text{if } H > 0. \end{aligned} \tag{17}$$

Analogously, we choose the second case of Lyapunov function $V(t)$ for $k = -1$ in (16) to have the function $V(t)$ to be non-negative.

Finally, we obtain that at both critical points ($H = \pm\sqrt{\frac{\Lambda}{3}}, \rho_0 = 0$) the Lyapunov function (16) vanishes. So, the conditions of Lyapunov stability Theorem 2.4 are satisfied. □

We conclude that while the stable de Sitter solution is asymptotically stable, the unstable de Sitter solution is unstable. This result was obtained by using global methods of dynamics investigations instead of the standard local stability analysis.

The choice of the Lyapunov function in the form of a first integral is suitable for proving the asymptotic stability of the stable de Sitter solution of the model. This methodological result has also the clear cosmological interpretation: the stable de Sitter universe has no hair like a black hole.

3 Dynamics of standard cosmological model

Let us consider a homogeneous and isotropic Friedmann-Robertson-Walker model with metric (1) and matter with energy density ρ and the cosmological constant Λ. We choose two phase space variables: the Hubble parameter $H = x$ and energy density $\rho = y$ and define the dynamical system

$$\dot{x} = -x^2 - \frac{1}{6}y + \frac{\Lambda}{3} \tag{18}$$

$$\dot{y} = -3xy \tag{19}$$

where the dot denotes derivative with respect to time t and $\Lambda > 0$ is the cosmological constant. It is a special case of system (14)-(15).

Let us apply the local stability analysis for system (18)-(19).

Remark 3.1. *System (18)-(19) has three critical points: stable node (*$x = -\sqrt{\frac{\Lambda}{3}}, y = 0$*), unstable node (*$x = \sqrt{\frac{\Lambda}{3}}, y = 0$*), and a saddle (*$x = 0, y = 2\Lambda$*).*

From the characteristic equation $\det(A - \lambda I) = 0$, where the linearization matrix of system (18)-(19) is

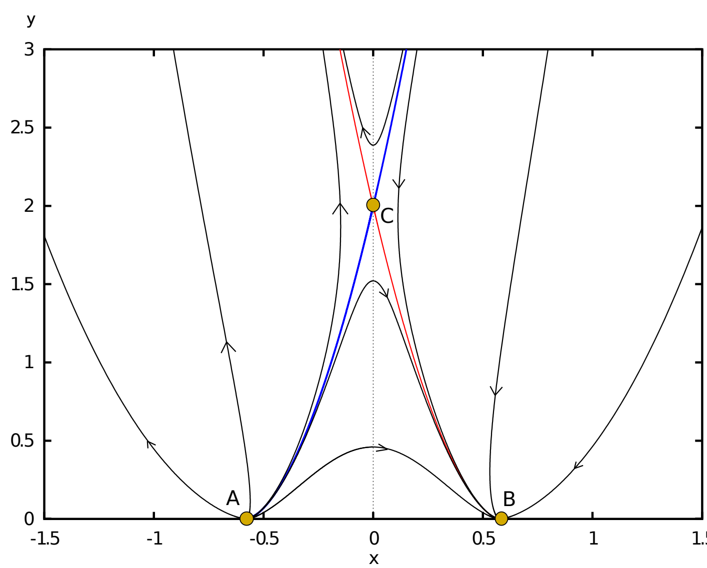
$$A = \begin{bmatrix} -2x_0 & \frac{1}{6} \\ -3y_0 & -3x_0 \end{bmatrix} \tag{20}$$

we have that the determinant and the trace of the linearization matrix A and the discriminant of the characteristic equation are $\det A = 6x_0^2 - \frac{1}{2}y_0$, $\operatorname{tr} A = -5x_0$ and $\Delta = x_0^2 + 2y_0$, respectively, where (x_0, y_0) is a critical point. Therefore,

1. the critical point $(x_0 = -\sqrt{\frac{\Lambda}{3}}, y_0 = 0)$ is an unstable node as $\det A = 2\Lambda > 0$, $\operatorname{tr} A = \frac{5}{3}\sqrt{\Lambda} > 0$, and $\Delta = \frac{\Lambda}{3} > 0$;
2. the critical point $(x_0 = \sqrt{\frac{\Lambda}{3}}, y_0 = 0)$ is a stable node as $\det A = -\Lambda < 0$, $\operatorname{tr} A = -\frac{5}{3}\sqrt{\Lambda} < 0$, and $\Delta = \frac{\Lambda}{3} > 0$;
3. the critical point $(x_0 = 0, y_0 = 2\Lambda)$ is a saddle as $\det A = 2\Lambda > 0$, $\operatorname{tr} A = 0$, and $\Delta = 4\Lambda > 0$.

The phase portrait of system (18)-(19) is presented in Figure 1.

Fig. 1. The phase portrait of system (18)-(19). There three critical points: point A represents the unstable de Sitter universe, point B represents the stable de Sitter universe, and point C represents the Einstein-de Sitter universe. The red and blue trajectories lie on unstable and stable invariant manifolds, respectively. It is assumed Λ is positive (for illustration $\Lambda = 1$).



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