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Research Article

On the Mini-Superambitwistor Space and $\mathcal{N} = 8$ Super-Yang-Mills Theory

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We construct a new supertwistor space suited for establishing a Penrose-Ward transform between certain bundles over this space and solutions to the $\mathcal{N} = 8$ super-Yang-Mills equations in three dimensions. This mini-superambitwistor space is obtained by dimensional reduction of the superambitwistor space, the standard superextension of the ambitwistor space. We discuss in detail the construction of this space and its geometry before presenting the Penrose-Ward transform. We also comment on a further such transform for purely bosonic Yang-Mills-Higgs theory in three dimensions by considering third-order formal "subneighborhoods" of a miniambitwistor space.

1. Introduction and Results

A convenient way of describing solutions to a wide class of field equations has been developed using twistor geometry [1–3]. In this picture, solutions to nonlinear field equations are mapped bijectively via the Penrose-Ward transform to holomorphic structures on vector bundles over an appropriate twistor space. Such twistor spaces are well known for many theories including self-dual Yang-Mills (SDYM) theory and its supersymmetric extensions as well as $\mathcal{N}$-extended full super-Yang-Mills (SYM) theories. In three dimensions, there are twistor spaces suited for describing the Bogomolny equations and their supersymmetric variants. The purpose of this paper is to fill the gaps for three-dimensional $\mathcal{N} = 8$ super-Yang-Mills theory as well as for three-dimensional Yang-Mills-Higgs theory; the cases for intermediate $\mathcal{N}$ follow trivially. The idea we follow in this paper has partly been presented in [4].

Recall that the supertwistor space describing $\mathcal{N} = 3$ SDYM theory is the open subset $\mathcal{P}^{3|3} := \mathbb{C}P^{3|3} \setminus \mathbb{C}P^{1|3}$; its anti-self-dual counterpart is $\mathcal{P}_*^{3|3} \cong \mathcal{P}^{3|3}$, where the parity assignment of the appearing coordinates is simply inverted. Furthermore, we denote by $\mathcal{P}^{2|3}$

the mini-supertwistor space obtained by dimensional reduction from $\mathcal{P}^{3|3}$ and used in the description of the supersymmetric Bogomolny equations in three dimensions.

For $\mathcal{N} = 4$ SYM theory, the appropriate twistor space $\mathcal{L}^{5|6}$ is now obtained from the product $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ upon imposing a quadric condition reducing the bosonic dimensions by one (in fact, the field theory described by $\mathcal{L}^{5|6}$ is $\mathcal{N} = 3$ SYM theory in four dimensions, which is equivalent to $\mathcal{N} = 4$ SYM theory on the level of equations of motion; in three dimensions, the same relation holds between $\mathcal{N} = 6$ and $\mathcal{N} = 8$ SYM theories). We perform an analogous construction for $\mathcal{N} = 8$ SYM theory by starting from the product $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3}$ of two mini-supertwistor spaces. The dimensional reduction turning the super-self-duality equations in four dimensions into the super-Bogomolny equations in three dimensions translates into a reduction of the quadric condition, which yields a constraint only to be imposed on the diagonal $\mathbb{C}P_\Delta^1 = \mathrm{diag}(\mathbb{C}P^1 \times \mathbb{C}P_*^1)$ in the base of the vector bundle $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \to \mathbb{C}P^1 \times \mathbb{C}P_*^1$. Thus, the resulting space $\mathcal{L}^{4|6}$ is not a vector bundle but only a fibration, and the sections of this fibration form a torsion sheaf, as we will see. More explicitly, the bosonic partsof the fibers of $\mathcal{L}^{4|6}$ over $\mathbb{C}P^1 \times \mathbb{C}P_*^1$ are isomorphic to $\mathbb{C}^2$ at generic points, but over the diagonal $\mathbb{C}P_\Delta^1$, they are isomorphic to $\mathbb{C}$.

As expected, we find a twistor correspondence between points in $\mathbb{C}^{3|12}$ and holomorphic sections of $\mathcal{L}^{4|6}$ as well as between points in $\mathcal{L}^{4|6}$ and certain sub-supermanifolds in $\mathbb{C}^{3|12}$. After introducing a real structure on $\mathcal{L}^{4|6}$, one finds a nice interpretation of the spaces involved in the twistor correspondence in terms of lines with marked points in $\mathbb{R}^3$, which resembles the appearance of flag manifolds in the well-established twistor correspondences. Recalling that $\mathcal{L}^{5|6}$ is a Calabi-Yau supermanifold (the essential prerequisite for being the target space of a topological B-model), we are led to examine an analogous question for $\mathcal{L}^{4|6}$. The Calabi-Yau property essentially amounts to a vanishing of the first Chern class of $T\mathcal{L}^{5|6}$, which in turn encodes information about the degeneracy locus of a certain set of sections of the vector bundle $\mathcal{L}^{5|6} \to \mathbb{C}P^1 \times \mathbb{C}P_*^1$. We find that the degeneracy loci of $\mathcal{L}^{5|6}$ and $\mathcal{L}^{4|6}$ are equivalent (identical up to a principal divisor).

A Penrose-Ward transform for $\mathcal{N} = 8$ SYM theory can now be conveniently established. To define the analogue of a holomorphic vector bundle over the space $\mathcal{L}^{4|6}$, we have to remember that in the Čech description, a holomorphic vector bundle is completely characterized by its transition functions, which in turn form a group-valued Čech 1-cocycle. These objects are still well defined on $\mathcal{L}^{4|6}$ and we will use such a 1-cocycle to define what we will call a pseudobundle over $\mathcal{L}^{4|6}$. In performing the transition between these pseudobundles and solutions to the $\mathcal{N} = 8$ SYM equations, care must be taken when discussing these bundles over the subset $\mathcal{L}^{4|6}|_{\mathbb{C}P_\Delta^1}$ of their base. Eventually, however, one obtains a bijection between gauge equivalence classes of solutions to the $\mathcal{N} = 8$ SYM equations and equivalence classes of holomorphic pseudobundles over $\mathcal{L}^{4|6}$, which turn into holomorphically trivial vector bundles upon restriction to any holomorphic submanifold $\mathbb{C}P^1 \times \mathbb{C}P_*^1 \hookrightarrow \mathcal{L}^{4|6}$.

Considering the reduction of $\mathcal{L}^{5|6} \subset \mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ to the bodies of the involved spaces (i.e., putting the fermionic coordinates on all the spaces to zero), it is possible to find a twistor correspondence for certain formal neighborhoods of $\mathcal{L}^{5|0} \subset \mathcal{P}^{3|0} \times \mathcal{P}_*^{3|0}$ on which a Penrose-Ward transform for purely bosonic Yang-Mills theory in four dimensions can be built. To improve our understanding of the mini-superambitwistor space, it is also helpful to discuss the analogous construction with $\mathcal{L}^{4|0}$. We find that a third-order subthickening, (i.e., a thickening of the fibers which are only of dimension one) inside of $\mathcal{P}^{2|0} \times \mathcal{P}_*^{2|0}$ must be considered to describe solutions to the Yang-Mills-Higgs equations in three dimensions by using pseudobundles over $\mathcal{L}^{4|0}$.

To clarify the role of the space $\mathcal{L}^{4|6}$ in detail, it would be interesting to establish a dimensionally reduced version of the construction considered by Movshev in [5]. In this paper, the author constructs a "Chern-Simons triple" consisting of a differential graded algebra (A, d) and a d-closed trace functional on a certain space ST related to the superambitwistor space. This Chern-Simons triple on ST is then conjectured to be equivalent to $\mathcal{N} = 4$ SYM theory in four dimensions. The way the construction is performed suggests a quite straightforward dimensional reduction to the case of the mini-superambitwistor space. Besides delivering a Chern-Simons triple for $\mathcal{N} = 8$ SYM theory in three dimensions, this construction would possibly shed more light on the unusual properties of the fibration $\mathcal{L}^{4|6}$.

Following Witten's seminal paper [6], there has been growing interest in different supertwistor spaces suited as target spaces for the topological B-model (see e.g. [4, 7–14]). Although it is not clear what the topological B-model on $\mathcal{L}^{4|6}$ looks like exactly (we will present some speculations in Section 3.7), the mini-superambitwistor space might also prove to be interesting from the topological string theory point of view. In particular, the mini-superambitwistor space $\mathcal{L}^{4|6}$ is probably the mirror of the mini-supertwistor space $\mathcal{P}^{2|4}$. Maybe even the extension of infinite dimensional symmetry algebras [12] from the self-dual to the full case is easier to study in three dimensions due to the greater similarity of self-dual and full theory and the smaller number of conformal generators.

Note that we are *not* describing the space of null geodesics in three dimensions; this space has been constructed in [13].

The outline of this paper is as follows. In Section 2, we review the construction of the supertwistor spaces for SDYM theory and SYM theory. Furthermore, we present the dimensional reduction yielding the mini-supertwistor space used for capturing solutions to the super-Bogomolny equations. Section 3, the main part, is then devoted to deriving the mini-superambitwistor space in several ways and discussing in detail the associated twistor correspondence and its geometry. Moreover, we comment on a topological B-model on this space. In Section 4, the Penrose-Ward transform for three-dimensional $\mathcal{N} = 8$ SYM theory is presented. First, we review both the transform for $\mathcal{N} = 4$ SYM theory in four dimensions and aspects of $\mathcal{N} = 8$ SYM theory in three dimensions. Then, we introduce the pseudobundles over $\mathcal{L}^{4|6}$, which take over the role of vector bundles over the space $\mathcal{L}^{4|6}$. Eventually, we present the actual Penrose-Ward transform in detail. In the last section, we discuss the third-order subthickenings of $\mathcal{L}^{4|0}$ in $\mathcal{P}^{2|0} \times \mathcal{P}_*^{2|0}$, which are used in the Penrose-Ward transform for purely bosonic Yang-Mills-Higgs theory.

2. Review of Supertwistor Spaces

We will briefly review some elementary facts on supertwistor spaces and fix our conventions in this section. For a broader discussion of supertwistor and superambitwistor spaces in conventions close to the ones employed here, see [15]. For more details on the mini-supertwistor spaces, we refer to [4, 14].

2.1. Supertwistor Spaces

The supertwistor space of $\mathbb{C}^{4|2\mathcal{N}}$ is defined as the rank $2|\mathcal{N}$ holomorphic supervector bundle

$$\mathcal{P}^{3|\mathcal{N}} := \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^{\mathcal{N}} \otimes \Pi\mathcal{O}(1) \tag{2.1}$$

over the Riemann sphere $\mathbb{C}P^1$. Here, Π is the parity changing operator which inverts the parity of the fiber coordinates. The base space of this bundle is covered by the two patches $U_\pm$ on which we have the standard coordinates $\lambda_\pm \in U_\pm \cong \mathbb{C}$ with $\lambda_+ = (\lambda_-)^{-1}$ on $U_+ \cap U_-$. Over $U_\pm$, we introduce furthermore the bosonic fiber coordinates $z_\pm^\alpha$ with $\alpha = 1, 2$ and the fermionic fiber coordinates $\eta_i^\pm$ with $i = 1, \ldots, \mathcal{N}$. On the intersection $U_+ \cap U_-$, we thus have

$$z_+^\alpha = \frac{1}{z_-^3} z_-^\alpha, \qquad \eta_i^+ = \frac{1}{z_-^3}\eta_i^- \quad \text{with } z_\pm^3 = \lambda_\pm. \tag{2.2}$$

The supermanifold $\mathcal{P}^{3|\mathcal{N}}$ as a whole is covered by the two patches $\mathcal{U}_\pm := \mathcal{P}^{3|\mathcal{N}}|_{U_\pm}$ with local coordinates $(z_\pm^1, z_\pm^2, z_\pm^3, \eta_1^\pm, \ldots, \eta_\mathcal{N}^\pm)$.

Global holomorphic sections of the vector bundle $\mathcal{P}^{3|\mathcal{N}} \to \mathbb{C}P^1$ are given by polynomials of degree one, which are parameterized by moduli $(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathbb{C}^{4|2\mathcal{N}}$ via

$$z_\pm^\alpha = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm, \qquad \eta_i^\pm = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm, \tag{2.3}$$

where we introduced the simplifying spinorial notation

$$(\lambda_{\dot\alpha}^+) := \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}, \qquad (\lambda_{\dot\alpha}^-) := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}. \tag{2.4}$$

Equations (2.3), the so-called *incidence relations*, define a twistor correspondence between the spaces $\mathcal{P}^{3|\mathcal{N}}$ and $\mathbb{C}^{4|2\mathcal{N}}$, which can be depicted in the double fibration

$$\begin{array}{ccc} & \mathcal{F}^{5|2\mathcal{N}} & \\ \swarrow^{\pi_2} & & \searrow^{\pi_1} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}} \end{array} \tag{2.5}$$

Here, $\mathcal{F}^{5|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1$ and the projections are defined as

$$\begin{aligned} \pi_1\left(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \lambda_{\dot\alpha}^\pm\right) &:= \left(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}\right), \\ \pi_2\left(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \lambda_{\dot\alpha}^\pm\right) &:= \left(x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm, \lambda_\pm, \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm\right). \end{aligned} \tag{2.6}$$

We can now read off the following correspondences:

$$\begin{aligned} \left\{\text{projective lines } \mathbb{C}P_{x,\eta}^1 \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{points } (x,\eta) \text{ in } \mathbb{C}^{4|2\mathcal{N}}\right\}, \\ \left\{\text{points } p \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{null } (\beta-) \text{ superplanes } \mathbb{C}_p^{2|2\mathcal{N}} \text{ in } \mathbb{C}^{4|2\mathcal{N}}\right\}. \end{aligned} \tag{2.7}$$

While the first correspondence is rather evident, the second one deserves a brief remark. Suppose $(\hat{x}^{\alpha\dot{\alpha}}, \hat{\eta}_i^{\dot{\alpha}})$ is a solution to the incidence relations (2.3) for a fixed point $p \in \mathcal{P}^{3|\mathcal{N}}$. Then, the set of all solutions is given by

$$\left\{ (x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}) \right\} \quad \text{with } x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}} + \mu^{\alpha}\lambda_{\pm}^{\dot{\alpha}}, \quad \eta_i^{\dot{\alpha}} = \hat{\eta}_i^{\dot{\alpha}} + \varepsilon_i \lambda_{\pm}^{\dot{\alpha}}, \tag{2.8}$$

where μ^{α} is an arbitrary commuting 2-spinor and ε_i is an arbitrary vector with Graßmann-odd entries. The coordinates $\lambda_{\pm}^{\dot{\alpha}}$ are defined by (2.4) and $\lambda_{\pm}^{\dot{\alpha}} := \varepsilon^{\dot{\alpha}\dot{\beta}}\lambda^{\pm}_{\dot{\beta}}$ with $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}} = 1$. One can choose to work on any patch containing p. The sets defined in (2.8) are then called *null* or β-superplanes.

The double fibration (2.5) is the foundation of the Penrose-Ward transform between equivalence classes of certain holomorphic vector bundles over $\mathcal{P}^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended supersymmetric self-dual Yang-Mills equations on $\mathbb{C}^4$ (see e.g. [15]).

The tangent spaces to the leaves of the projection π_2 are spanned by the vector fields

$$V_{\alpha}^{\pm} := \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \qquad V_{\pm}^{i} := \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^{\dot{\alpha}}}. \tag{2.9}$$

Note furthermore that $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold. The bosonic fibers contribute each +1 to the first Chern class and the fermionic ones −1 (this is related to the fact that Berezin integration amounts to differentiating with respect to a Graßmann variable). Together with the contribution from the tangent bundle of the base space, we have in total a trivial first Chern class. This space is thus suited as the target space for a topological B-model [6].

2.2. The Superambitwistor Space

The idea leading naturally to a superambitwistor space is to "glue together" both the self-dual and anti-self-dual subsectors of $\mathcal{N} = 3$ SYM theory to the full theory. For this, we obviously need a twistor space $\mathcal{P}^{3|3}$ with coordinates $(z_{\pm}^{\alpha}, z_{\pm}^{3}, \eta_i^{\pm})$ together with a "dual" copy (the word "dual" refers to the spinor indices and not to the line bundles underlying $\mathcal{P}^{3|3}$) $\mathcal{P}_*^{3|3}$ with coordinates $(u_{\pm}^{\dot{\alpha}}, u_{\pm}^{3}, \theta_{\pm}^{i})$. The dual twistor space is considered as a holomorphic supervector bundle over the Riemann sphere $\mathbb{C}P_*^1$ covered by the patches $U_{\pm}^{*}$ with the standard local coordinates $\mu_{\pm} = u_{\pm}^{3}$. For convenience, we again introduce the spinorial notation $(\mu_{\alpha}^{+}) = (1, \mu_{+})^T$ and $(\mu_{\alpha}^{-}) = (\mu_{-}, 1)^T$. The two patches covering $\mathcal{P}_*^{3|3}$ will be denoted by $\mathcal{U}_{\pm}^{*} := \mathcal{P}_*^{3|3}|_{U_{\pm}^{*}}$, and the product space $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ of the two supertwistor spaces is thus covered by the four patches

$$\mathcal{U}_{(1)} := \mathcal{U}_{+} \times \mathcal{U}_{+}^{*}, \qquad \mathcal{U}_{(2)} := \mathcal{U}_{-} \times \mathcal{U}_{+}^{*}, \qquad \mathcal{U}_{(3)} := \mathcal{U}_{+} \times \mathcal{U}_{-}^{*}, \qquad \mathcal{U}_{(4)} := \mathcal{U}_{-} \times \mathcal{U}_{-}^{*}, \tag{2.10}$$

on which we have the coordinates $(z_{(a)}^{\alpha}, z_{(a)}^{3}, \eta_i^{(a)}; u_{(a)}^{\dot{\alpha}}, u_{(a)}^{3}, \theta_{(a)}^{i})$. This space is furthermore a rank 4|6 supervector bundle over the space $\mathbb{C}P^1 \times \mathbb{C}P_*^1$. The global sections of this bundle are parameterized by elements of $\mathbb{C}^{4|6} \times \mathbb{C}_*^{4|6}$ in the following way:

$$z_{(a)}^{\alpha} = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}^{(a)}, \qquad \eta_i^{(a)} = \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}^{(a)}; \qquad u_{(a)}^{\dot{\alpha}} = x_*^{\alpha\dot{\alpha}}\mu_{\alpha}^{(a)}, \qquad \theta_{(a)}^{i} = \theta^{\alpha i}\mu_{\alpha}^{(a)}. \tag{2.11}$$

The superambitwistor space is now the subspace $\mathcal{L}^{5|6} \subset \mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ obtained from the *quadric condition* (the "gluing condition")

$$\kappa_{(a)} := z_{(a)}^{\alpha}\mu_{\alpha}^{(a)} - u_{(a)}^{\dot{\alpha}}\lambda_{\dot{\alpha}}^{(a)} + 2\theta_{(a)}^{i}\eta_{i}^{(a)} = 0. \tag{2.12}$$

In what follows, we will denote the restrictions of $\mathcal{U}_{(a)}$ to $\mathcal{L}^{5|6}$ by $\widehat{\mathcal{U}}_{(a)}$.

Because of the quadric condition (2.12), the bosonic moduli are not independent of $\mathcal{L}^{5|6}$, but one rather has the relation

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} - \theta^{\alpha i}\eta_i^{\dot{\alpha}}, \qquad x_*^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \theta^{\alpha i}\eta_i^{\dot{\alpha}}. \tag{2.13}$$

The moduli $(x^{\alpha\dot{\alpha}})$ and $(x_*^{\alpha\dot{\alpha}})$ are, therefore, indeed antichiral and chiral coordinates on the (complex) superspace $\mathbb{C}^{4|12}$ and with this identification, one can establish the following double fibration using (2.11):

$$\begin{array}{ccc} & \mathcal{F}^{6|12} & \\ {\scriptstyle \pi_2}\swarrow & & \searrow{\scriptstyle \pi_1} \\ \mathcal{L}^{5|6} & & \mathbb{C}^{4|12} \end{array} \tag{2.14}$$

where $\mathcal{F}^{6|12} \cong \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P_*^1$ and π_1 is the trivial projection. Thus, one has the correspondences

$$\begin{gathered} \left\{\text{subspaces } \left(\mathbb{C}P^1 \times \mathbb{C}P_*^1\right)_{x_0,\eta,\theta} \text{ in } \mathcal{L}^{5|6}\right\} \longleftrightarrow \left\{\text{points } (x_0,\eta,\theta) \text{ in } \mathbb{C}^{4|12}\right\}, \\ \left\{\text{points } p \text{ in } \mathcal{L}^{5|6}\right\} \longleftrightarrow \left\{\text{null superlines in } \mathbb{C}^{4|12}\right\}. \end{gathered} \tag{2.15}$$

The above-mentioned null superlines are intersections of β-superplanes and dual α-superplanes. Given a solution $(\hat{x}_0^{\alpha\dot{\alpha}}, \hat{\eta}_i^{\dot{\alpha}}, \hat{\theta}^{\alpha i})$ to the incidence relations (2.11) for a fixed point p in $\mathcal{L}^{5|6}$, the set of points on such a null superline takes the form

$$\left\{\left(x_0^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}, \theta^{\alpha i}\right)\right\} \quad \text{with } x_0^{\alpha\dot{\alpha}} = \hat{x}_0^{\alpha\dot{\alpha}} + t\mu_{(a)}^{\alpha}\lambda_{(a)}^{\dot{\alpha}}, \quad \eta_i^{\dot{\alpha}} = \hat{\eta}_i^{\dot{\alpha}} + \varepsilon_i\lambda_{(a)}^{\dot{\alpha}}, \quad \theta^{\alpha i} = \hat{\theta}^{\alpha i} + \tilde{\varepsilon}^{i}\mu_{(a)}^{\alpha}. \tag{2.16}$$

Here, t is an arbitrary complex number and ε_i and $\tilde{\varepsilon}^i$ are both 3-vectors with Graßmann-odd components. The coordinates $\lambda_{(a)}^{\dot{\alpha}}$ and $\mu_{(a)}^{\alpha}$ are chosen from arbitrary patches on which they are both well defined. Note that these null superlines are in fact of dimension $1|6$.

The space $\mathcal{F}^{6|12}$ is covered by four patches $\widetilde{\mathcal{U}}_{(a)} := \pi_2^{-1}(\widehat{\mathcal{U}}_{(a)})$ and the tangent spaces to the $1|6$-dimensional leaves of the fibration $\pi_2 : \mathcal{F}^{6|12} \to \mathcal{L}^{5|6}$ from (2.14) are spanned by the holomorphic vector fields

$$W^{(a)} := \mu_{(a)}^{\alpha}\lambda_{(a)}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad D_{(a)}^{i} = \lambda_{(a)}^{\dot{\alpha}}D_{\dot{\alpha}}^{i}, \qquad D_i^{(a)} = \mu_{(a)}^{\alpha}D_{\alpha i}, \tag{2.17}$$

where $D_{\alpha i}$ and $D^i_{\dot\alpha}$ are the superderivatives defined by

$$D_{\alpha i} := \frac{\partial}{\partial\theta^{\alpha i}} + \eta_i^{\dot\alpha}\frac{\partial}{\partial x_0^{\alpha\dot\alpha}}, \qquad D^i_{\dot\alpha} := \frac{\partial}{\partial\eta_i^{\dot\alpha}} + \theta^{\alpha i}\frac{\partial}{\partial x_0^{\alpha\dot\alpha}}. \tag{2.18}$$

Just as the space $\mathcal{P}^{3|4}$, the superambitwistor space $\mathcal{L}^{5|6}$ is a Calabi-Yau supermanifold. To prove this, note that we can count first Chern numbers with respect to the base $\mathbb{C}P^1\times\mathbb{C}P^1_*$ of $\mathcal{L}^{5|6}$. In particular, we define the line bundle $\mathcal{O}(m,n)$ to have first Chern numbers m and n with respect to the two $\mathbb{C}P^1$s in the base. The (unconstrained) fermionic part of $\mathcal{L}^{5|6}$ which is given by $\mathbb{C}^3\otimes\Pi\mathcal{O}(1,0)\oplus\mathbb{C}^3\otimes\Pi\mathcal{O}(0,1)$ contributes $(-3,-3)$ in this counting, which has to be cancelled by the body $\mathcal{L}^5$ of $\mathcal{L}^{5|6}$. Consider, therefore, the map

$$\kappa : \left(z^{\alpha}_{(a)}, \lambda^{(a)}_{\dot\alpha}, u^{\dot\alpha}_{(a)}, \mu^{(a)}_{\alpha}\right) \longmapsto \left(\kappa_{(a)}\big|_{\eta=\theta=0}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_{\alpha}\right), \tag{2.19}$$

where $\kappa_{(a)}$ has been defined in (2.12). This map is a vector bundle morphism and gives rise to the short exact sequence

$$0 \longrightarrow \mathcal{L}^5 \longrightarrow \mathbb{C}^2\otimes\mathcal{O}(1,0)\oplus\mathbb{C}^2\otimes\mathcal{O}(0,1) \xrightarrow{\kappa} \mathcal{O}(1,1) \longrightarrow 0. \tag{2.20}$$

The first Chern classes of the bundles in this sequence are elements of $H^2(\mathbb{C}P^1\times\mathbb{C}P^1,\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$, which we denote by $\alpha_1 h_1 + \alpha_2 h_2$ with $\alpha_1,\alpha_2\in\mathbb{Z}$. Then, the short exact sequence (2.20) together with the Whitney product formula yields

$$(1+h_1)(1+h_1)(1+h_2)(1+h_2) = (1+\alpha_1 h_1+\alpha_2 h_2+\cdots)(1+h_1+h_2), \tag{2.21}$$

where (α_1,α_2) label the first Chern class of $\mathcal{L}^5$ considered as a holomorphic vector bundle over $\mathbb{C}P^1\times\mathbb{C}P^1_*$. It follows that $c_1 = (1,1)$, and taking into account the contribution of the tangent space to the base (recall that $T^{1,0}\mathbb{C}P^1 \cong \mathcal{O}(2)$) $\mathbb{C}P^1\times\mathbb{C}P^1_*$, we conclude that the contribution of the tangent space to $\mathcal{L}^5$ to the first Chern class of $\mathcal{L}^{5|6}$ is cancelled by the contribution of the fermionic fibers.

Since $\mathcal{L}^{5|6}$ is a Calabi-Yau supermanifold, this space can be used as a target space for the topological B-model. However, it is still unclear what the corresponding gauge theory action will look like. The most obvious guess would be some holomorphic BF-type theory [16–18] with B a “Lagrange multiplier (0,3)-form.”

2.3. Reality Conditions on the Superambitwistor Space

On the supertwistor spaces $\mathcal{P}^{3|\mathcal{N}}$, one can define a real structure which leads to Kleinian signature on the body of the moduli space $\mathbb{R}^{4|2\mathcal{N}}$ of real holomorphic sections of the fibration π_2 in (2.5). Furthermore, if $\mathcal{N}$ is even, one can can impose a symplectic Majorana condition which amounts to a second real structure which yields Euclidean signature. We saw above that the superambitwistor space $\mathcal{L}^{5|6}$ originates from two copies of $\mathcal{P}^{3|3}$ and, therefore, we cannot straightforwardly impose the Euclidean reality condition. However, besides the real structure leading to Kleinian signature, one can additionally define a reality condition by relating spinors of opposite helicities to each other. In this way, we obtain a Minkowski metric on the body of $\mathbb{R}^{4|12}$. In the following, we will focus on the latter.

Consider the antilinear involution τ_M which acts on the coordinates of $\mathcal{L}^{5|6}$ according to

$$\tau_M\left(z^{\alpha}_{\pm},\lambda^{\pm}_{\dot\alpha},\eta^{\pm}_i;u^{\dot\alpha},\mu^{\pm}_{\alpha},\theta^i_{\pm}\right) := \left(\overline{u^{\dot\alpha}_{\pm}},\overline{\mu^{\pm}_{\dot\alpha}},\overline{\theta^i_{\pm}};\overline{z^{\alpha}_{\pm}},\overline{\lambda^{\pm}_{\dot\alpha}},\overline{\eta^{\pm}_i}\right). \tag{2.22}$$

Sections of the bundle $\mathcal{L}^{5|6}\ \to \mathbb{C}P^1\times\mathbb{C}P^1_*$ which are τ_M-real are thus parameterized by the moduli

$$x^{\alpha\dot\beta} = \overline{x^{\dot\beta\alpha}},\qquad \eta^{\dot\alpha}_i = \overline{\theta^{\alpha i}}. \tag{2.23}$$

We extract furthermore the contained real coordinates via the identification

$$\begin{aligned} x^{1\dot1} &= x^0+x^1, &\qquad x^{1\dot2} &= x^2-\mathrm{i}x^3,\\ x^{2\dot1} &= x^2+\mathrm{i}x^3, &\qquad x^{2\dot2} &= x^0-x^1, \end{aligned} \tag{2.24}$$

and obtain a metric of signature $(1,3)$ on $\mathbb{R}^4$ from $\mathrm{d}s^2 := \det(\mathrm{d}x^{\alpha\dot\alpha})$. Note that we can also make the identification (2.24) in the complex case $(x^{\mu})\in\mathbb{C}^4$, and then even on $\mathcal{P}^{3|\mathcal{N}}$. In the subsequent discussion, we will always employ (2.24) which is consistent, because we will not be interested in the real version of $\mathcal{P}^{3|\mathcal{N}}$.

2.4. *The Mini-Supertwistor Spaces*

To capture the situation obtained by a dimensional reduction $\mathbb{C}^{4|2\mathcal{N}}\ \to\ \mathbb{C}^{3|2\mathcal{N}}$, one uses the so-called mini-supertwistor spaces. Note that the vector field

$$\mathcal{T}_3 := \frac{\partial}{\partial x^3} = \mathrm{i}\left(\frac{\partial}{\partial x^{2\dot1}}-\frac{\partial}{\partial x^{1\dot2}}\right) \tag{2.25}$$

considered on $\mathcal{F}^{5|2\mathcal{N}}$ from diagram (2.5) can be split into a holomorphic and an antiholomorphic part when restricted from $\mathcal{F}^{5|2\mathcal{N}}$ to $\mathcal{P}^{3|\mathcal{N}}$:

$$\mathcal{T}_3\big|_{\mathcal{P}^{3|\mathcal{N}}} = \mathcal{T}+\overline{\mathcal{T}},\qquad \mathcal{T}_+ = \mathrm{i}\left(\frac{\partial}{\partial z^2_+}-z^3_+\frac{\partial}{\partial z^1_+}\right),\qquad \mathcal{T}_- = \mathrm{i}\left(z^3_-\frac{\partial}{\partial z^2_-}-\frac{\partial}{\partial z^1_-}\right). \tag{2.26}$$

Let $\mathcal{G}$ be the abelian group generated by $\mathcal{T}$. Then, the orbit space $\mathcal{P}^{3|\mathcal{N}}/\mathcal{G}$ is given by the holomorphic supervector bundle

$$\mathcal{P}^{2|\mathcal{N}} := \mathcal{O}(2)\oplus\mathbb{C}^{\mathcal{N}}\otimes\Pi\mathcal{O}(1) \tag{2.27}$$

over $\mathbb{C}P^1$, and we call $\mathcal{P}^{2|\mathcal{N}}$ a *mini-supertwistor space*. We denote the patches covering $\mathcal{P}^{2|\mathcal{N}}$ by $\mathcal{V}_\pm := \mathcal{U}_\pm \cap \mathcal{P}^{2|\mathcal{N}}$. The coordinates of the base and the fermionic fibers of $\mathcal{P}^{2|\mathcal{N}}$ are the same as those of $\mathcal{P}^{3|\mathcal{N}}$. For the bosonic fibers, we define

$$w_+^1 := z_+^1 + z_+^3 z_+^2 \quad \text{on } \mathcal{V}_+, \qquad w_-^1 := z_-^2 + z_-^3 z_-^1 \quad \text{on } \mathcal{V}_-, \tag{2.28}$$

and introduce additionally $w_\pm^2 := z_\pm^3 = \lambda_\pm$ for convenience. On the intersection $\mathcal{V}_+ \cap \mathcal{V}_-$, we thus have the relation $w_+^1 = (w_-^2)^{-2} w_-^1$. This implies that $w_\pm^1$ describes global sections of the line bundle $\mathcal{O}(2)$. We parameterize these sections according to

$$w_\pm^1 = y^{\dot\alpha\dot\beta} \lambda_{\dot\alpha}^\pm \lambda_{\dot\beta}^\pm \quad \text{with } y^{\dot\alpha\dot\beta} = y^{(\dot\alpha\dot\beta)} \in \mathbb{C}^3, \tag{2.29}$$

and the new moduli $y^{\dot\alpha\dot\beta}$ are identified with the previous ones $x^{\alpha\dot\beta}$ by the equation $y^{\dot\alpha\dot\beta} = x^{(\dot\alpha\dot\beta)}$. The incidence relation (2.29) allows us to establish a double fibration

$$\begin{array}{ccc} & \mathcal{F}^{6|12} & \\ {\scriptstyle \pi_2} \swarrow & & \searrow {\scriptstyle \pi_1} \\ \mathcal{L}^{5|6} & & \mathbb{C}^{4|12} \end{array} \tag{2.30}$$

where $\mathcal{K}^{4|2\mathcal{N}} \cong \mathbb{C}^{3|2\mathcal{N}} \times \mathbb{C}P^1$. We again obtain a twistor correspondence

$$\begin{gathered} \{\text{projective lines } \mathbb{C}P^1_{x,\eta} \text{ in } \mathcal{P}^{2|\mathcal{N}}\} \longleftrightarrow \{\text{points } (y,\eta) \text{ in } \mathbb{C}^{3|2\mathcal{N}}\}, \\ \{\text{points } p \text{ in } \mathcal{P}^{2|\mathcal{N}}\} \longleftrightarrow \{2|\mathcal{N}\text{-dimensional superplanes in } \mathbb{C}^{3|2\mathcal{N}}\}. \end{gathered} \tag{2.31}$$

The $2|\mathcal{N}$-dimensional superplanes in $\mathbb{C}^{3|2\mathcal{N}}$ are given by the set

$$\{(y^{\dot\alpha\dot\beta}, \eta_i^{\dot\alpha})\} \quad \text{with } y^{\dot\alpha\dot\beta} = \hat{y}^{\dot\alpha\dot\beta} + \kappa^{(\dot\alpha}\lambda_\pm^{\dot\beta)}, \quad \eta_i^{\dot\alpha} = \hat{\eta}_i^{\dot\alpha} + \varepsilon_i \lambda_\pm^{\dot\alpha}, \tag{2.32}$$

where $\kappa^{\dot\alpha}$ and ε_i are an arbitrary complex 2-spinor and a vector with Graßmann-odd components, respectively. The point $(\hat{y}^{\dot\alpha\dot\beta}, \hat{\eta}_i^{\dot\alpha}) \in \mathbb{C}^{3|2\mathcal{N}}$ is again an initial solution to the incidence relations (2.29) for a fixed point $p \in \mathcal{P}^{2|\mathcal{N}}$. Note that although these superplanes arise from null superplanes in four dimensions via dimensional reduction, they themselves are not null.

The vector fields along the projection ν_2 are now spanned by

$$V_{\dot\alpha}^\pm := \lambda_\pm^{\dot\beta} \partial_{(\dot\alpha\dot\beta)}, \qquad V_\pm^i := \lambda_\pm^{\dot\alpha} \frac{\partial}{\partial \eta_i^{\dot\alpha}} \tag{2.33}$$

with

$$\partial_{\dot1\dot1} := \frac{\partial}{\partial y^{\dot1\dot1}}, \qquad \partial_{\dot2\dot2} := \frac{\partial}{\partial y^{\dot2\dot2}}, \qquad \partial_{(\dot1\dot2)} = \partial_{(\dot2\dot1)} := \frac{1}{2}\frac{\partial}{\partial y^{\dot1\dot2}}. \tag{2.34}$$

The mini-supertwistor space $\mathcal{P}^{2|4}$ is again a Calabi-Yau supermanifold, and the gauge theory equivalent to the topological B-model on this space is a holomorphic BF theory [4].

3. The Mini-Superambitwistor Space

In this section, we define and examine the mini-superambitwistor space $\mathcal{L}^{4|6}$, which we will use to build a Penrose-Ward transform involving solutions to $\mathcal{N} = 8$ SYM theory in three dimensions. We will first give an abstract definition of $\mathcal{L}^{4|6}$ by a short exact sequence, and present more heuristic ways of obtaining the mini-superambitwistor space later.

3.1. Abstract Definition of the Mini-Superambitwistor Space

The starting point is the product space $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3}$ of two copies of the $\mathcal{N} = 3$ mini-supertwistor space. In analogy to the space $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$, we have coordinates

$$\left(w^1_{(a)}, w^2_{(a)} = \lambda_{(a)}, \eta_i^{(a)}; v^1_{(a)}, v^2_{(a)} = \mu_{(a)}, \theta^i_{(a)}\right) \tag{3.1}$$

on the patches $\mathcal{V}_{(a)}$ which are Cartesian products of $\mathcal{V}_\pm$ and $\mathcal{V}_\pm^*$:

$$\mathcal{V}_{(1)} := \mathcal{V}_+ \times \mathcal{V}_+^*, \qquad \mathcal{V}_{(2)} := \mathcal{V}_- \times \mathcal{V}_+^*, \qquad \mathcal{V}_{(3)} := \mathcal{V}_+ \times \mathcal{V}_-^*, \qquad \mathcal{V}_{(4)} := \mathcal{V}_- \times \mathcal{V}_-^*. \tag{3.2}$$

For convenience, let us introduce the subspace $\mathbb{C}P^1_\Delta$ of the base of the fibration $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$ as

$$\mathbb{C}P^1_\Delta := \operatorname{diag}\left(\mathbb{C}P^1 \times \mathbb{C}P^1_*\right) = \left\{(\mu_\pm, \lambda_\pm) \in \mathbb{C}P^1 \times \mathbb{C}P^1_* \mid \mu_\pm = \lambda_\pm\right\}. \tag{3.3}$$

Consider now the map $\xi : \mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \to \mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$ which is defined by

$$\xi : \left(w^1_{(a)}, w^2_{(a)}, \eta_i^{(a)}; v^1_{(a)}, v^2_{(a)}, \theta^i_{(a)}\right) \longmapsto \begin{cases} \left(w^1_\pm - v^1_\pm + 2\theta^i_\pm \eta_i^\pm, w^2_\pm\right) & \text{for } w^2_\pm = v^2_\pm \\ \left(0, w^2_{(a)}\right) & \text{else,} \end{cases} \tag{3.4}$$

where $\mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$ is the line bundle $\mathcal{O}(2)$ over $\mathbb{C}P^1_\Delta$. In this definition, we used the fact that a point for which $w^2_\pm = v^2_\pm$ is at least on one of the patches $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(4)}$. Note, in particular, that the map ξ is a morphism of vector bundles. Therefore, we can define a space $\mathcal{L}^{4|6}$ via the short exact sequence

$$0 \longrightarrow \mathcal{L}^{4|6} \longrightarrow \mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \xrightarrow{\xi} \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \longrightarrow 0 \tag{3.5}$$

(cf. (2.20)). We will call this space the *mini-superambitwistor space*. Analogously to above, we will denote the pull-back of the patches $\mathcal{V}_{(a)}$ to $\mathcal{L}^{4|6}$ by $\widehat{\mathcal{V}}_{(a)}$. Obviously, the space $\mathcal{L}^{4|6}$ is a fibration, and we can switch to the corresponding short exact sequence of sheaves of local sections:

$$0 \longrightarrow \mathcal{L}^{4|6} \longrightarrow \mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \overset{\xi}{\longrightarrow} \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \longrightarrow 0. \tag{3.6}$$

Note the difference in notation: (3.5) is a sequence of vector bundles, while (3.6) is a sequence of sheaves. To analyze the geometry of the space $\mathcal{L}^{4|6}$ in more detail, we will restrict ourselves to the body of this space and put the fermionic coordinates to zero. Similarly to the case of the superambitwistor space, this is possible as the map ξ does not affect the fermionic dimensions in the exact sequence (3.5); this will become clearer in the discussion in Section 3.2.

Inspired by the sequence defining the skyscraper sheaf (a sheaf with sections supported only at the point p) $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_p \to 0$, we introduce the following short exact sequence:

$$0 \longrightarrow \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \overset{\zeta}{\longrightarrow} \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \longrightarrow \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \oplus \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \longrightarrow 0. \tag{3.7}$$

Here, we defined $\zeta : (a,b) \mapsto (a\varepsilon^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\alpha}}\mu_{\dot{\beta}}, b\varepsilon^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\alpha}}\mu_{\dot{\beta}})$, where $\lambda_{\dot{\alpha}}$ and $\mu_{\dot{\alpha}}$ are the usual homogeneous coordinates on the base space $\mathbb{C}P^1 \times \mathbb{C}P^1_*$. The sheaf $\mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$ is a torsion sheaf (sometimes sloppily referred to as a skyscraper sheaf) with sections supported only over $\mathbb{C}P^1_\Delta$. Finally, we trivially have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \overset{\alpha_1}{\longrightarrow} \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \oplus \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \overset{\alpha_2}{\longrightarrow} \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \longrightarrow 0, \tag{3.8}$$

where $\alpha_1 : (a) \mapsto (a,a)$ and $\alpha_2 : (a,b) \mapsto (a-b)$.

Using the short exact sequences (3.6), (3.7), and (3.8) as well as the nine lemma, we can establish the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(1,-1)\oplus\mathcal{O}(-1,1) & \overset{\zeta}{\longrightarrow} & \mathcal{L}^4 & \longrightarrow & \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) & \longrightarrow & 0 \\
 & & \downarrow\text{id} & & \downarrow & & \downarrow\alpha_1 & & \\
0 & \longrightarrow & \mathcal{O}(1,-1)\oplus\mathcal{O}(-1,1) & \overset{\zeta}{\longrightarrow} & \mathcal{O}(2,0)\oplus\mathcal{O}(0,2) & \longrightarrow & \mathcal{O}_{\mathbb{C}P^1_\Delta}(2)\oplus\mathcal{O}_{\mathbb{C}P^1_\Delta}(2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\alpha_2 & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) & \overset{\text{id}}{\longrightarrow} & \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.9}$$

From the horizontal lines of this diagram and the five lemma, we conclude that $\mathcal{L}^4 \oplus \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) = \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$. Thus, $\mathcal{L}^4$ is not a locally free sheaf (a more sophisticated argumentation would use the common properties of the torsion functor to establish that $\mathcal{L}^4$ is a torsion sheaf; furthermore, one can write down a further diagram using the nine lemma which shows that $\mathcal{L}^4$ is a coherent sheaf) but a torsion sheaf, whose stalks over $\mathbb{C}P^1_\Delta$ are isomorphic to the stalks of $\mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$, while the stalks over $(\mathbb{C}P^1 \times \mathbb{C}P^1_*) \setminus \mathbb{C}P^1_\Delta$ are isomorphic to the stalks of $\mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$. Therefore, $\mathcal{L}^4$ is not a vector bundle, but a fibration (the homotopy lifting property typically included in the definition of a fibration is readily derived from the definition of $\mathcal{L}^4$) with fibers $\mathbb{C}^2$ over generic points and fibers $\mathbb{C}$ over $\mathbb{C}P^1_\Delta$. In particular, the total space of $\mathcal{L}^4$ is *not* a manifold.

The fact that the total space of the bundle $\mathcal{L}^{4|6}$ is neither a supermanifold nor a supervector bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ seems at first slightly disturbing. However, we will show that once one is aware of this new aspect, it does not cause any deep difficulties as far as the twistor correspondence and the Penrose-Ward transform are concerned.

3.2. The Mini-Superambitwistor Space by Dimensional Reduction

In the following, we will motivate the abstract definition more concretely by considering explicitly the dimensional reduction of the space $\mathcal{L}^{5|6}$, we will also fix our notation in terms of coordinates and moduli of sections. For this, we will first reduce the product space $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ and then impose the appropriate reduced quadric condition. In a first step, we want to eliminate in both $\mathcal{P}^{3|3}$ and $\mathcal{P}^{3|3}_*$ the dependence on the bosonic modulus x^3. Thus, we should factorize by

$$\mathcal{T}_{(a)} = \begin{cases} \dfrac{\partial}{\partial z_+^2} - z_+^3 \dfrac{\partial}{\partial z_+^1} & \text{on } \mathcal{U}_{(1)} \\ z_-^3 \dfrac{\partial}{\partial z_-^2} - \dfrac{\partial}{\partial z_-^1} & \text{on } \mathcal{U}_{(2)} \\ \dfrac{\partial}{\partial z_+^2} - z_+^3 \dfrac{\partial}{\partial z_+^1} & \text{on } \mathcal{U}_{(3)} \\ z_-^3 \dfrac{\partial}{\partial z_-^2} - \dfrac{\partial}{\partial z_-^1} & \text{on } \mathcal{U}_{(4)} \end{cases}, \qquad \mathcal{T}^*_{(a)} = \begin{cases} \dfrac{\partial}{\partial u_+^{\dot 2}} - u_+^{\dot 3} \dfrac{\partial}{\partial u_+^{\dot 1}} & \text{on } \mathcal{U}_{(1)} \\ \dfrac{\partial}{\partial u_+^{\dot 2}} - u_+^{\dot 3} \dfrac{\partial}{\partial u_+^{\dot 1}} & \text{on } \mathcal{U}_{(2)} \\ u_-^{\dot 3} \dfrac{\partial}{\partial u_-^{\dot 2}} - \dfrac{\partial}{\partial u_-^{\dot 1}} & \text{on } \mathcal{U}_{(3)} \\ u_-^{\dot 3} \dfrac{\partial}{\partial u_-^{\dot 2}} - \dfrac{\partial}{\partial u_-^{\dot 1}} & \text{on } \mathcal{U}_{(4)}, \end{cases} \tag{3.10}$$

which leads us to the orbit space

$$\mathcal{P}^{2|3} \times \mathcal{P}^{2|3}_* = \left(\mathcal{P}^{3|3}/\mathcal{G}\right) \times \left(\mathcal{P}^{3|3}_*/\mathcal{G}^*\right), \tag{3.11}$$

where $\mathcal{G}$ and $\mathcal{G}^*$ are the abelian groups generated by $\mathcal{T}$ and $\mathcal{T}^*$, respectively. Recall that the coordinates we use on this space have been defined in (3.1). The global sections of the bundle $\mathcal{P}^{2|4} \times \mathcal{P}^{2|4}_* \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$ are captured by the parameterization

$$w^1_{(a)} = y^{\dot\alpha\dot\beta} \lambda^{(a)}_{\dot\alpha} \lambda^{(a)}_{\dot\beta}, \qquad v^{\dot 1}_{(a)} = y_*^{\dot\alpha\dot\beta} \mu^{(a)}_{\dot\alpha} \mu^{(a)}_{\dot\beta}, \qquad \theta^i_{(a)} = \theta^{\dot\alpha i} \mu^{(a)}_{\dot\alpha}, \qquad \eta^{(a)}_i = \eta^{\dot\alpha}_i \lambda^{(a)}_{\dot\alpha}, \tag{3.12}$$

where we relabel the indices of $\mu^{(a)}_\alpha \to \mu^{(a)}_{\dot\alpha}$ and the moduli $y_*^{\alpha\beta} \to y_*^{\dot\alpha\dot\beta}$, $\theta^{i\alpha} \to \theta^{i\dot\alpha}$, since there is no distinction between left- and right-handed spinors on $\mathbb{R}^3$ or its complexification $\mathbb{C}^3$.

The next step is obviously to impose the quadric condition, gluing together the self-dual and anti-self-dual parts. Note that when acting with $\mathcal{T}$ and $\mathcal{T}^*$ on $\kappa_{(a)}$ as given in (2.12), we obtain

$$\mathcal{T}_{(1)}\kappa_{(1)} = \mathcal{T}^*_{(1)}\kappa_{(1)} = \left(\mu_+ - \lambda_+\right), \qquad \mathcal{T}_{(2)}\kappa_{(2)} = \mathcal{T}^*_{(2)}\kappa_{(2)} = \left(\lambda_-\mu_+ - 1\right),$$
$$\mathcal{T}_{(3)}\kappa_{(3)} = \mathcal{T}^*_{(3)}\kappa_{(3)} = \left(1 - \lambda_+\mu_-\right), \qquad \mathcal{T}_{(4)}\kappa_{(4)} = \mathcal{T}^*_{(4)}\kappa_{(4)} = \left(\lambda_- - \mu_-\right). \tag{3.13}$$

This implies that the orbits generated by $\mathcal{T}$ and $\mathcal{T}^*$ become orthogonal to the orbits of $\partial/\partial\kappa$ only at $\mu_\pm = \lambda_\pm$. We can, therefore, safely impose the condition

$$\left(w^1_\pm - v^1_\pm + 2\theta^i_\pm \eta^\pm_i\right)\big|_{\lambda_\pm=\mu_\pm} = 0, \tag{3.14}$$

and the subset of $\mathcal{P}^{2|3} \times \mathcal{P}^{2|3}_*$ which satisfies this condition is obviously identical to the mini-superambitwistor space $\mathcal{L}^{4|6}$ defined above.

The condition (3.14) naturally fixes the parametrization of global sections of the fibration $\mathcal{L}^{4|6}$ by giving a relation between the moduli used in (3.12). This relation is completely analogous to (2.13) and reads

$$y^{\dot\alpha\dot\beta} = y_0^{\dot\alpha\dot\beta} - \theta^{(\dot\alpha i}\eta_i^{\dot\beta)}, \qquad y_*^{\dot\alpha\dot\beta} = y_0^{\dot\alpha\dot\beta} + \theta^{(\dot\alpha i}\eta_i^{\dot\beta)}. \tag{3.15}$$

We clearly see that this parameterization arises from (2.13) by dimensional reduction from $\mathbb{C}^4 \rightarrow \mathbb{C}^3$. Furthermore, even with this identification, $w^1_\pm$ and $v^1_\pm$ are independent of $\lambda_\pm \neq \mu_\pm$. Thus, indeed, imposing the condition (3.14) only at $\lambda_\pm = \mu_\pm$ is the dimensionally reduced analogue of imposing the condition (2.12) on $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$.

3.3. *Comments on Further Ways of Constructing* $\mathcal{L}^{4|6}$

Although the construction presented above seems most natural, one can imagine other approaches of defining the space $\mathcal{L}^{4|6}$. Completely evident is a second way, which uses the description of $\mathcal{L}^{5|6}$ in terms of coordinates on $\mathcal{F}^{6|12}$. Here, one factorizes the correspondence space $\mathcal{F}^{6|12}$ by the groups generated by the vector field $\mathcal{T}_3 = \mathcal{T}^*_3$ and obtains the correspondence space $\mathcal{K}^{5|12} \cong \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ together with (3.15). A subsequent projection π_2 from the dimensionally reduced correspondence space $\mathcal{K}^{5|12}$ then yields the mini-superambitwistor space $\mathcal{L}^{4|6}$ as defined above.

Furthermore, one can factorize $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ only by $\mathcal{G}$ to eliminate the dependence on one modulus. This will lead to $\mathcal{P}^{2|3} \times \mathcal{P}^{3|3}_*$ and following the above discussion of imposing the quadric condition on the appropriate subspace, one arrives again at (3.14) and the space $\mathcal{L}^{4|6}$. Here, the quadric condition already implies the remaining factorization of $\mathcal{P}^{2|3} \times \mathcal{P}^{3|3}_*$ by $\mathcal{G}^*$.

Eventually, one could anticipate the identification of moduli in (3.15) and, therefore, want to factorize by the group generated by the combination $\mathcal{T} + \mathcal{T}^*$. Acting with this sum on $\kappa_{(a)}$ will produce the sum of the results given in (3.13), and the subsequent discussion of the quadric condition follows the one presented above.

3.4. Double Fibration

Knowing the parameterization of global sections of the mini-superambitwistor space fibered over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ as defined in (3.15), we can establish a double fibration, similarly to all the other twistor spaces we encountered so far. Even more instructive is the following diagram, in which the dimensional reduction of the involved spaces becomes evident:

$$
\begin{array}{ccccc}
 & & \mathcal{F}^{6|12} & & \\
 & \swarrow^{\pi_2} & \downarrow & ^{\pi_1}\searrow & \\
\mathcal{L}^{5|6} & & & & \mathbb{C}^{4|12} \\
\downarrow & & \mathcal{K}^{5|12} & & \downarrow \\
 & \swarrow^{\nu_2} & & ^{\nu_1}\searrow & \\
\mathcal{L}^{4|6} & & & & \mathbb{C}^{3|12}
\end{array}
\qquad (3.16)
$$

The upper half is just the double fibration for the quadric (2.14), while the lower half corresponds to the dimensionally reduced case. The reduction of $\mathbb{C}^{4|12}$ to $\mathbb{C}^{3|12}$ is obviously done by factoring out the group generated by $\mathcal{T}_3$. The same is true for the reduction of $\mathcal{F}^{6|12} \cong \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ to $\mathcal{K}^{5|12} \cong \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$. The reduction from $\mathcal{L}^{5|6}$ to $\mathcal{L}^{4|6}$ was given above and the projection ν_2 from $\mathcal{K}^{5|12}$ onto $\mathcal{L}^{4|6}$ is defined by (3.12). The four patches covering $\mathcal{F}^{6|12}$ will be denoted by $\widetilde{\mathcal{U}}_{(a)} := \nu_2^{-1}(\widehat{\mathcal{U}}_{(a)})$.

The double fibration defined by the projections ν_1 and ν_2 yields the following twistor correspondences:

$$
\begin{aligned}
\left\{\text{subspaces } \left(\mathbb{C}P^1 \times \mathbb{C}P^1\right)_{y_0,\eta,\theta} \text{ in } \mathcal{L}^{4|6}\right\} &\longleftrightarrow \left\{\text{points } (y_0,\eta,\theta) \text{ in } \mathbb{C}^{3|12}\right\}, \\
\left\{\text{generic points } p \text{ in } \mathcal{L}^{4|6}\right\} &\longleftrightarrow \left\{\text{superlines in } \mathbb{C}^{3|12}\right\}, \\
\left\{\text{points } p \text{ in } \mathcal{L}^{4|6} \text{ with } \lambda_\pm = \mu_\pm\right\} &\longleftrightarrow \left\{\text{superplanes in } \mathbb{C}^{3|12}\right\}.
\end{aligned}
\qquad (3.17)
$$

The superlines and the superplanes in $\mathbb{C}^{3|12}$ are defined as the sets

$$
\begin{aligned}
&\left\{\left(y^{\dot\alpha\dot\beta},\eta_i^{\dot\alpha},\theta^{\dot\alpha i}\right)\right\} \quad \text{with } y^{\dot\alpha\dot\beta} = \hat{y}^{\dot\alpha\dot\beta} + t\lambda_{(a)}^{(\dot\alpha}\mu_{(a)}^{\dot\beta)}, \quad \eta_i^{\dot\alpha} = \hat{\eta}_i^{\dot\alpha} + \varepsilon_i\lambda_{(a)}^{\dot\alpha}, \quad \theta^{\dot\alpha i} = \hat{\theta}^{\dot\alpha i} + \tilde{\varepsilon}^i\mu_{(a)}^{\dot\alpha}, \\
&\left\{\left(y^{\dot\alpha\dot\beta},\eta_i^{\dot\alpha},\theta^{\dot\alpha i}\right)\right\} \quad \text{with } y^{\dot\alpha\dot\beta} = \hat{y}^{\dot\alpha\dot\beta} + \kappa^{(\dot\alpha}\lambda_{(a)}^{\dot\beta)}, \quad \eta_i^{\dot\alpha} = \hat{\eta}_i^{\dot\alpha} + \varepsilon_i\lambda_{(a)}^{\dot\alpha}, \quad \theta^{\dot\alpha i} = \hat{\theta}^{\dot\alpha i} + \tilde{\varepsilon}^i\lambda_{(a)}^{\dot\alpha},
\end{aligned}
\qquad (3.18)
$$

where t, $\kappa^{\dot\alpha}$, ε_i, and $\tilde{\varepsilon}^i$ are an arbitrary complex number, a complex commuting 2-spinor, and two 3-vectors with Graßmann-odd components, respectively. Note that in the last line, $\lambda_\pm^{\dot\alpha} = \mu_\pm^{\dot\alpha}$, and we could also have written

$$
\left\{\left(y^{\dot\alpha\dot\beta},\eta_i^{\dot\alpha},\theta^{\dot\alpha i}\right)\right\} \quad \text{with } y^{\dot\alpha\dot\beta} = \hat{y}^{\dot\alpha\dot\beta} + \kappa^{(\dot\alpha}\mu_{(a)}^{\dot\beta)}, \quad \eta_i^{\dot\alpha} = \hat{\eta}_i^{\dot\alpha} + \varepsilon_i\mu_{(a)}^{\dot\alpha}, \quad \theta^{\dot\alpha i} = \hat{\theta}^{\dot\alpha i} + \tilde{\varepsilon}^i\mu_{(a)}^{\dot\alpha}. \qquad (3.19)
$$

The vector fields spanning the tangent spaces to the leaves of the fibration ν_2 are for generic values of $\mu_\pm$ and $\lambda_\pm$ given by

$$W^{(a)} := \mu^{\dot\alpha}_{(a)}\lambda^{\dot\beta}_{(a)}\partial_{(\dot\alpha\dot\beta)},$$
$$\widetilde{D}^i_{(a)} := \lambda^{\dot\beta}_{(a)}\widetilde{D}^i_{\dot\beta} := \lambda^{\dot\beta}_{(a)}\left(\frac{\partial}{\partial\eta^{\dot\beta}_i} + \theta^{\dot\alpha i}\partial_{(\dot\alpha\dot\beta)}\right), \tag{3.20}$$
$$D^{(a)}_i := \mu^{\dot\alpha}_{(a)}D_{\dot\alpha i} := \mu^{\dot\alpha}_{(a)}\left(\frac{\partial}{\partial\theta^{\dot\alpha i}} + \eta^{\dot\beta}_i\partial_{(\dot\alpha\dot\beta)}\right),$$

where the derivatives $\partial_{(\dot\alpha\dot\beta)}$ have been defined in (2.34). At $\mu_\pm = \lambda_\pm$, however, the fibers of the fibration $\mathcal{L}^{4|6}$ over $\mathbb{C}P^1\times\mathbb{C}P^1_*$ loose one bosonic dimension. As the space $\mathcal{K}^{5|12}$ is a manifold, this means that this dimension has to become tangent to the projection ν_2. In fact, one finds that over $\mathbb{C}P^1_\Delta$ besides the vector fields given in (3.20), also the vector fields

$$\widetilde{W}^\pm_{\dot\beta} = \mu^{\dot\alpha}_\pm\partial_{(\dot\alpha\dot\beta)} = \lambda^{\dot\alpha}_\pm\partial_{(\dot\alpha\dot\beta)} \tag{3.21}$$

annihilate the coordinates on $\mathcal{L}^{4|6}$. Therefore, the leaves to the projection $\nu_2 : \mathcal{K}^{5|12}\to\mathcal{L}^{4|6}$ are of dimension $2|6$ for $\mu_\pm = \lambda_\pm$ and of dimension $1|6$ everywhere else.

3.5. Real Structure on $\mathcal{L}^{4|6}$

Quite evidently, a real structure on $\mathcal{L}^{4|6}$ is inherited from the one on $\mathcal{L}^{5|6}$, and we obtain directly from (2.22) the action of τ_M on $\mathcal{P}^{2|4}\times\mathcal{P}^{2|4}_*$, which is given by

$$\tau_M(w^1_\pm,\lambda^\pm_{\dot\alpha},\eta^\pm_i;v^1_\pm,\mu^\pm_{\dot\alpha},\theta^i_\pm) := \left(\overline{v^1_\pm},\overline{\mu^\pm_{\dot\alpha}},\overline{\theta^i_\pm};\overline{w^1_\pm},\overline{\lambda^\pm_{\dot\alpha}},\overline{\eta^\pm_i}\right). \tag{3.22}$$

This action descends in an obvious manner to $\mathcal{L}^{4|6}$, which leads to a real structure on the moduli space $\mathbb{C}^{3|12}$ via the double fibration (3.16). Thus, we have as the resulting reality condition

$$y^{\dot\alpha\dot\beta}_0 = \overline{y^{\dot\beta\dot\alpha}_0}, \qquad \eta^{\dot\alpha}_i = \overline{\theta^{\dot\alpha i}}, \tag{3.23}$$

and the identification of the bosonic moduli $y^{\dot\alpha\dot\beta}$ with the coordinates on $\mathbb{R}^3$ reads as

$$y^{\dot1\dot1}_0 = x^0_0 + x^1_0, \qquad y^{\dot1\dot2}_0 = y^{\dot2\dot1}_0 = x^2_0, \qquad y^{\dot2\dot2}_0 = x^0_0 - x^1_0. \tag{3.24}$$

The reality condition $\tau_M(\cdot) = \cdot$ is indeed fully compatible with the condition (3.14) which reduces $\mathcal{P}^{2|4}\times\mathcal{P}^{2|4}_*$ to $\mathcal{L}^{4|6}$. The base space $\mathbb{C}P^1\times\mathbb{C}P^1_*$ of the fibration $\mathcal{L}^{4|6}$ is reduced to a single sphere S^2 with real coordinates $(1/2)(\lambda_\pm+\mu_\pm) = (1/2)(\lambda_\pm+\overline{\lambda}_\pm)$ and $(1/2\mathrm{i})(\lambda_\pm-\mu_\pm) = (1/2\mathrm{i})(\lambda_\pm-\overline{\lambda}_\pm)$, while the diagonal $\mathbb{C}P^1_\Delta$ is reduced to a circle S^1_Δ parameterized by the real coordinates $(1/2)(\lambda_\pm+\overline{\lambda}_\pm)$. The τ_M-real sections of $\mathcal{L}^{4|6}$ have to satisfy $w^1_\pm = \tau_M(w^1_\pm) = \overline{v}^1_\pm$.

Thus, the fibers of the fibration $\mathcal{L}^{4|6} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$, which are of complex dimension 2|6 over generic points in the base and complex dimension 1|6 over $\mathbb{C}P^1_\Delta$, are reduced to fibers of real dimension 2|6 and 1|6, respectively. In particular, note that $\theta^i_\pm \eta^\pm_i = \overline{\eta}_i^{\,\pm} \overline{\theta}^i_\pm = -\overline{\theta}^i_\pm \overline{\eta}_i^{\,\pm}$ is purely imaginary and, therefore, the quadric condition (3.14) together with the real structure τ_M implies that $w^1_\pm = \overline{v}^1_\pm = \overline{w}^1_\pm + 2\overline{\theta}^i_\pm \overline{\eta}_i^{\,\pm}$ for $\lambda_\pm = \mu_\pm = \overline{\lambda}_\pm$. Thus, the body $\overset{\circ}{w}{}^1_\pm$ of $w^1_\pm$ is purely real and we have $w^1_\pm = \overset{\circ}{w}{}^1_\pm - \theta^i_\pm \eta^\pm_i$ and $v^1_\pm = \overset{\circ}{w}{}^1_\pm + \theta^i_\pm \eta^\pm_i$ on the diagonal S^1_Δ.

3.6. Interpretation of the Involved Real Geometries

For the best-known twistor correspondences (i.e., the correspondence (2.5)) (more precisely, the compactified version thereof) its dual, and the correspondence (2.14), there is a nice description in terms of flag manifolds (see e.g., [3]). For the spaces involved in the twistor correspondences including minitwistor spaces, one has a similarly nice interpretation after restricting to the real situation. For simplicity, we reduce our considerations to the bodies (i.e., drop the fermionic directions) of the involved geometries, as the extension to corresponding supermanifolds is quite straightforward.

Let us first discuss the double fibration (2.30), and assume that we have imposed a suitable reality condition on the fiber coordinates, the details of which are not important. We follow again the usual discussion of the real case and leave the coordinates on the sphere complex. As correspondence space on top of the double fibration, we have thus the space $\mathbb{R}^3 \times S^2$, which we can understand as the set of oriented lines (not only the ones through the origin) in $\mathbb{R}^3$ with one marked point. Clearly, the point of such a line is given by an element of $\mathbb{R}^3$, and the direction of this line in $\mathbb{R}^3$ is parameterized by a point on S^2. The minitwistor space $\mathcal{P}^2 \cong \mathcal{O}(2)$ now is simply the space of all lines in $\mathbb{R}^3$ [19]. Similarly to the case of flag manifolds, the projections ν_1 and ν_2 in (2.30) become, therefore, obvious. For ν_1, simply drop the line and keep the marked point. For ν_2, drop the marked point and keep the line. Equivalently, we can understand ν_2 as moving the marked point on the line to its shortest possible distance from the origin. This leads to the space $TS^2 \cong \mathcal{O}(2)$, where the S^2 parameterizes again the direction of the line, which can subsequently be still moved orthogonally to this direction, and this freedom is parameterized by the tangent planes to S^2 which are isomorphic to $\mathbb{R}^2$.

Now in the case of the fibration included in (3.16), we impose the reality condition (3.22) on the fiber coordinates of $\mathcal{L}^4$. In the real case, the correspondence space $\mathcal{K}^5$ becomes the space $\mathbb{R}^3 \times S^2 \times S^2$ and this is the space of two oriented lines in $\mathbb{R}^3$ intersecting in a point. More precisely, this is the space of two oriented lines in $\mathbb{R}^3$ each with one marked point, for which the two marked points coincide. The projections ν_1 and ν_2 in (3.16) are then interpreted as follows. For ν_1, simply drop the two lines and keep the marked point. For ν_2, fix one line and move the marked point (the intersection point) together with the second line to its shortest distance to the origin. Thus, the space $\mathcal{L}^4$ is the space of configurations in $\mathbb{R}^3$, in which a line has a common point with another line at its shortest distance to the origin.

Let us summarize all the above findings in Table 1.

3.7. Remarks Concerning a Topological B-Model on $\mathcal{L}^{4|6}$

The space $\mathcal{L}^{4|6}$ is not well suited as a target space for a topological B-model since it is not a (Calabi-Yau) manifold. However, one clearly expects that it is possible to define an analogous

Table 1

Space	Relation to $\mathbb{R}^3$
$\mathbb{R}^3$	marked points in $\mathbb{R}^3$
$\mathbb{R}^3 \times S^2$	oriented lines with a marked point in $\mathbb{R}^3$
$\mathcal{P}^2 \cong \mathcal{O}(2)$	oriented lines in $\mathbb{R}^3$ (with a marked point at shortest distance to the origin.)
$\mathbb{R}^3 \times S^2 \times S^2$	two oriented lines with a common marked point in $\mathbb{R}^3$
$\mathcal{L}^4$	two oriented lines with a common marked point at shortest distance from one of the lines to the origin in $\mathbb{R}^3$

model since, if we assume that the conjecture in [20, 21] is correct, such a model should simply be the mirror of the minitwistor string theory considered in [14]. This model would furthermore yield some holomorphic Chern-Simons type equations of motion. The latter equations would then define holomorphic pseudobundles over $\mathcal{L}^{4|6}$ by an analogue of a holomorphic structure. These bundles will be introduced in Section 4.3 and in our discussion, they substitute the holomorphic vector bundles.

Interestingly, the space $\mathcal{L}^{4|6}$ has a property which comes close to vanishing of a first Chern class. Recall that for any complex vector bundle, its Chern classes are Poincaré dual to the degeneracy cycles of certain sets of sections (this is a Gauß-Bonnet formula). More precisely, to calculate the first Chern class of a rank r vector bundle, one considers r generic sections and arranges them into an $r \times r$ matrix L. The degeneracy loci on the base space are then given by the zero locus of $\det(L)$. Clearly, this calculation can be translated directly to $\mathcal{L}^{4|6}$.

We will now show that $\mathcal{L}^{4|6}$ and $\mathcal{L}^{5|6}$ have equivalent degeneracy loci (i.e., they are equal up to a principal divisor) which, speaking about ordinary vector bundles, would not affect the first Chern class. Our discussion simplifies considerably if we restrict our attention to the bodies of the two supertwistor spaces and put all the fermionic coordinates to zero. Note that this will not affect the result, as the quadric conditions defining $\mathcal{L}^{5|6}$ and $\mathcal{L}^{4|6}$ do not affect the fermionic dimensions: the fermionic parts of the fibrations $\mathcal{L}^{5|6}$ and $\mathcal{L}^{4|6}$ are identical, which is easily seen by considering the global sections generating the total spaces of the fibrations. Instead of the ambitwistor spaces, it is also easier to consider the vector bundles $\mathcal{P}^3 \times \mathcal{P}^3_*$ and $\mathcal{P}^2 \times \mathcal{P}^2_*$ over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$, respectively, with the appropriately restricted sets of sections. Furthermore, we will stick to our inhomogeneous coordinates and perform the calculation only on the patch $\mathcal{U}_{(1)}$, but all this directly translates into homogeneous, patch-independent coordinates. The matrices to be considered are

$$L_{\mathcal{L}^5} = \begin{pmatrix} x_1^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_2^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_3^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_4^{1\dot\alpha}\lambda^+_{\dot\alpha} \\ x_1^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_2^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_3^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_4^{2\dot\alpha}\lambda^+_{\dot\alpha} \\ x_1^{\alpha1}\mu^+_{\alpha} & x_2^{\alpha1}\mu^+_{\alpha} & x_3^{\alpha1}\mu^+_{\alpha} & x_4^{\alpha1}\mu^+_{\alpha} \\ x_1^{\alpha2}\mu^+_{\alpha} & x_2^{\alpha2}\mu^+_{\alpha} & x_3^{\alpha2}\mu^+_{\alpha} & x_4^{\alpha2}\mu^+_{\alpha} \end{pmatrix}, \qquad L_{\mathcal{L}^4} = \begin{pmatrix} y_1^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta} & y_2^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta} \\ y_1^{\dot\alpha\dot\beta}\mu^+_{\dot\alpha}\mu^+_{\dot\beta} & y_2^{\dot\alpha\dot\beta}\mu^+_{\dot\alpha}\mu^+_{\dot\beta} \end{pmatrix}, \tag{3.25}$$

and one computes the degeneracy loci for generic moduli to be given by the equations

$$(\lambda_+ - \mu_+)^2 = 0, \qquad (\lambda_+ - \mu_+)(\lambda_+ - \varrho_+) = 0 \tag{3.26}$$

on the bases of $\mathcal{L}^5$ and $\mathcal{L}^4$, respectively. Here, ϱ_+ is a rational function of μ_+ and, therefore, it is obvious that both degeneracy cycles are equivalent.

When dealing with degenerated twistor spaces, one usually retreats to the correspondence space endowed with some additional symmetry conditions [22]. It is conceivable that a similar procedure will help to define the topological B-model in our case. Also, defining a suitable blowup of $\mathcal{L}^{4|6}$ over $\mathbb{C}P^1_\Delta$ could be the starting point for finding an appropriate action.

4. The Penrose-Ward Transform for the Mini-Superambitwistor Space

4.1. Review of the Penrose-Ward Transform on the Superambitwistor Space

Let $\mathcal{E}$ be a topologically trivial holomorphic vector bundle of rank n over $\mathcal{L}^{5|6}$ which becomes holomorphically trivial when restricted to any subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x_0,\eta,\theta} \hookrightarrow \mathcal{L}^{5|6}$. Due to the equivalence of the Čech and the Dolbeault descriptions of holomorphic vector bundles, we can describe $\mathcal{E}$ either by holomorphic transition functions $\{f_{ab}\}$ or by a holomorphic structure $\bar{\partial}_{\hat{\mathcal{A}}} = \bar{\partial} + \hat{\mathcal{A}}$. Starting from a transition function f_{ab}, there is a splitting

$$f_{ab} = \hat{\varphi}_a^{-1}\hat{\varphi}_b, \tag{4.1}$$

where the $\hat{\varphi}_a$ are smooth $\mathsf{GL}(n,\mathbb{C})$-valued functions (in fact, the collection $\{\hat{\varphi}_a\}$ forms a Čech 0-cochain) on $\mathcal{U}_{(a)}$ since the bundle $\mathcal{E}$ is topologically trivial. This splitting allows us to switch to the holomorphic structure $\bar{\partial} + \hat{\mathcal{A}}$ with $\hat{\mathcal{A}} = \hat{\varphi}\bar{\partial}\hat{\varphi}^{-1}$, which describes a trivial vector bundle $\hat{\mathcal{E}} \cong \mathcal{E}$. Note that the additional condition of holomorphic triviality of $\mathcal{E}$ on subspaces $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x_0,\eta,\theta}$ will restrict the explicit form of $\hat{\mathcal{A}}$.

Back at the bundle $\mathcal{E}$, consider its pull-back $\pi_2^*\mathcal{E}$ with transition functions $\{\pi_2^* f_{ab}\}$, which are constant along the fibers of $\pi_2 : \mathcal{F}^{6|12} \to \mathcal{L}^{5|6}$:

$$W^{(a)}\pi_2^* f_{ab} = D^i_{(a)}\pi_2^* f_{ab} = D_i^{(a)}\pi_2^* f_{ab} = 0. \tag{4.2}$$

The additional assumption of holomorphic triviality upon reduction onto a subspace allows for a splitting

$$\pi_2^* f_{ab} = \psi_a^{-1}\psi_b \tag{4.3}$$

into $\mathsf{GL}(n,\mathbb{C})$-valued functions $\{\psi_a\}$ which are holomorphic on $\tilde{\mathcal{U}}_{(a)}$. Evidently, there is such a splitting holomorphic in the coordinates $\lambda_{(a)}$ and $\mu_{(a)}$ on $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x_0,\eta,\theta}$ since $\mathcal{E}$ becomes holomorphically trivial when restricted to these spaces. Furthermore, these subspaces are holomorphically parameterized by the moduli $(x_0^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}, \theta^{\alpha i})$, and thus the splitting (4.3) is holomorphic in all the coordinates of $\mathcal{F}^{6|12}$. Due to (4.2), we have on the intersections $\widetilde{\mathcal{U}}_{(a)} \cap \widetilde{\mathcal{U}}_{(b)}$:

$$\begin{aligned} \psi_a D^i_{(a)} \psi_a^{-1} &= \psi_b D^i_{(a)} \psi_b^{-1} =: \lambda^{\dot\alpha}_{(a)} \mathcal{A}^i_{\dot\alpha}, \\ \psi_a D^{(a)}_i \psi_a^{-1} &= \psi_b D^{(a)}_i \psi_b^{-1} =: \mu^{\alpha}_{(a)} \mathcal{A}_{\alpha i}, \\ \psi_a W^{(a)} \psi_a^{-1} &= \psi_b W^{(a)} \psi_b^{-1} =: \mu^{\alpha}_{(a)} \lambda^{\dot\alpha}_{(a)} \mathcal{A}_{\alpha\dot\alpha}, \end{aligned} \tag{4.4}$$

where $\mathcal{A}^i_{\dot\alpha}$, $\mathcal{A}_{\alpha i}$, and $\mathcal{A}_{\alpha\dot\alpha}$ are independent of $\mu_{(a)}$ and $\lambda_{(a)}$. The introduced components of the supergauge potential $\mathcal{A}$ fit into the linear system

$$\begin{aligned} \mu^{\alpha}_{(a)} \lambda^{\dot\alpha}_{(a)} \left(\partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha}\right) \psi_a &= 0, \\ \lambda^{\dot\alpha}_{(a)} \left(D^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}\right) \psi_a &= 0, \\ \mu^{\alpha}_{(a)} \left(D_{\alpha i} + \mathcal{A}_{\alpha i}\right) \psi_a &= 0, \end{aligned} \tag{4.5}$$

whose compatibility conditions are

$$\left\{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\right\} + \left\{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\right\} = 0, \qquad \left\{\nabla_{\alpha i}, \nabla_{\beta j}\right\} + \left\{\nabla_{\beta i}, \nabla_{\alpha j}\right\} = 0, \qquad \left\{\nabla_{\alpha i}, \nabla^j_{\dot\alpha}\right\} - 2\delta^j_i \nabla_{\alpha\dot\alpha} = 0. \tag{4.6}$$

Here, we used the obvious shorthand notations $\nabla^i_{\dot\alpha} := D^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}$, $\nabla_{\alpha i} := D_{\alpha i} + \mathcal{A}_{\alpha i}$, and $\nabla_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha}$. However, (4.6) are well known to be equivalent to the equations of motion of $\mathcal{N} = 3$ SYM theory on $\mathbb{C}^4$ [23] (note that most of our considerations concern the complexified case), and, therefore, also to $\mathcal{N} = 4$ SYM theory on $\mathbb{C}^4$.

We thus showed that there is a correspondence between certain holomorphic structures on $\mathcal{L}^{5|6}$, holomorphic vector bundles over $\mathcal{L}^{5|6}$ which become holomorphically trivial when restricted to certain subspaces and solutions to the $\mathcal{N} = 4$ SYM equations on $\mathbb{C}^4$. The redundancy in each set of objects is modded out by considering gauge equivalence classes and holomorphic equivalence classes of vector bundles, which renders the above correspondences one-to-one.

4.2. $\mathcal{N} = 8$ *SYM Theory in Three Dimension*

This theory is obtained by dimensionally reducing $\mathcal{N} = 1$ SYM theory in ten dimensions to three dimensions, or, equivalently, by dimensionally reducing four-dimensional $\mathcal{N} = 4$ SYM theory to three dimensions. As a result, the 16 real supercharges are rearranged in the latter case from four spinors transforming as a $\mathbf{2}_{\mathbb{C}}$ of $\mathsf{Spin}(3,1) \cong \mathsf{SL}(2,\mathbb{C})$ into eight spinors transforming as a $\mathbf{2}_{\mathbb{R}}$ of $\mathsf{Spin}(2,1) \cong \mathsf{SL}(2,\mathbb{R})$.

The automorphism group of the supersymmetry algebra is $\mathsf{Spin}(8)$, and the little group of the remaining Lorentz group $\mathsf{SO}(2,1)$ is trivial. As massless particle content, we, therefore, expect bosons transforming in the $\mathbf{8}_v$ and fermions transforming in the $\mathbf{8}_c$ of $\mathsf{Spin}(8)$. One of the bosons will, however, appear as a dual gauge potential on $\mathbb{R}^3$ after dimensional reduction, and, therefore, only a $\mathsf{Spin}(7)$ R-symmetry group is manifest in the action and the equations of motion. In the mini-superambitwistor formulation, the manifest subgroup of the R-symmetry group is only $\mathsf{SU}(3) \times \mathsf{U}(1) \times \mathsf{SU}(3) \times \mathsf{U}(1)$. Altogether, we have a gauge potential A_μ with $\mu = 1, \dots, 3$, seven scalars ϕ^i with $i = 1, \dots, 7$, and eight spinors $\chi^j_{\dot\alpha}$ with $j = 1, \dots, 8$.

Moreover, recall that in four dimensions, $\mathcal{N} = 3$ and $\mathcal{N} = 4$ super-Yang-Mills theories are equivalent on the level of field content and corresponding equations of motion. The only difference is found in the manifest R-symmetry groups which are $\mathsf{SU}(3) \times \mathsf{U}(1)$ and $\mathsf{SU}(4)$, respectively. This equivalence obviously carries over to the three-dimensional situation. $\mathcal{N} = 6$ and $\mathcal{N} = 8$ super-Yang-Mills theories are equivalent regarding their field content and the equations of motion. Therefore, it is sufficient to construct a twistor correspondence for $\mathcal{N} = 6$ SYM theory to describe solutions to the $\mathcal{N} = 8$ SYM equations.

4.3. Pseudobundles over $\mathcal{L}^{4|6}$

Because the mini-superambitwistor space is only a fibration and not a manifold, there is no notion of holomorphic vector bundles over $\mathcal{L}^{4|6}$. However, our space is close enough to a manifold to translate all the necessary terms in a simple manner.

Let us fix the covering $\mathfrak{U}$ of the total space of the fibration $\mathcal{L}^{4|6}$ to be given by the patches $\mathcal{V}_{(a)}$ introduced above. Furthermore, define $\mathfrak{S}$ to be the sheaf of smooth $\mathsf{GL}(n, \mathbb{C})$-valued functions on $\mathcal{L}^{4|6}$ and $\mathfrak{H}$ to be its subsheaf consisting of holomorphic $\mathsf{GL}(n, \mathbb{C})$-valued functions on $\mathcal{L}^{4|6}$, i.e. smooth and holomorphic functions which depend only on the coordinates given in (3.12) and $\lambda_{(a)}$, $\mu_{(a)}$.

We define a *complex pseudo-bundle over* $\mathcal{L}^{4|6}$ of rank n by a Čech 1-cocycle $\{f_{ab}\} \in \check{Z}^1(\mathfrak{U}, \mathfrak{S})$ on $\mathcal{L}^{4|6}$ in full analogy with transition functions defining ordinary vector bundles. If the 1-cocycle is an element of $\check{Z}^1(\mathfrak{U}, \mathfrak{H})$, we speak of a *holomorphic pseudo-bundle over* $\mathcal{L}^{4|6}$. Two pseudo-bundles given by Čech 1-cocycles $\{f_{ab}\}$ and $\{f'_{ab}\}$ are called *topologically equivalent* (*holomorphically equivalent*), if there is a Čech 0-cochain $\{\psi_a\} \in \check{C}^0(\mathfrak{U}, \mathfrak{S})$ (a Čech 0-cochain $\{\psi_a\} \in \check{C}^0(\mathfrak{U}, \mathfrak{H})$) such that $f_{ab} = \psi_a^{-1} f'_{ab} \psi_b$. A pseudo-bundle is called *trivial* (*holomorphically trivial*), if it is topologically equivalent (holomorphically equivalent) to the trivial pseudo-bundle given by $\{f_{ab}\} = \{\mathbb{1}_{ab}\}$.

In the corresponding discussion of Čech cohomology on ordinary manifolds, one can achieve independence of the covering if the patches of the covering are all Stein manifolds. An analogous argument should also be applicable here, but for our purposes, it is enough to restrict to the covering $\mathfrak{U}$.

Besides the Čech description, it is also possible to introduce an equivalent Dolbeault description, which will, however, demand an extended notion of Dolbeault cohomology classes.

4.4. The Penrose-Ward Transform Using the Mini-Superambitwistor Space

With the double fibration contained in (3.16), it is not hard to establish the corresponding Penrose-Ward transform, which is essentially a dimensional reduction of the four-dimensional case presented in Section 4.1.

On $\mathcal{L}^{4|6}$, we start from a trivial rank n holomorphic pseudo-bundle over $\mathcal{L}^{4|6}$ defined by a 1-cocycle $\{f_{ab}\}$ which becomes a holomorphically trivial vector bundle upon restriction to any subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{y_0,\eta,\theta} \hookrightarrow \mathcal{L}^{4|6}$. The pull-back of the pseudo-bundle over $\mathcal{L}^{4|6}$ along ν_2 is the vector bundle $\widetilde{\mathcal{E}}$ with transition functions $\{\nu_2^* f_{ab}\}$ satisfying by definition

$$W^{(a)}\nu_2^* f_{ab} = \widetilde{D}^i_{(a)}\nu_2^* f_{ab} = D_i^{(a)}\nu_2^* f_{ab} = 0, \tag{4.7}$$

at generic points of $\mathcal{L}^{4|6}$ and for $\lambda_\pm = \mu_\pm$, we have

$$\widetilde{W}^{(a)}_{\dot\alpha}\nu_2^* f_{ab} = \widetilde{D}^i_{(a)}\nu_2^* f_{ab} = D_i^{(a)}\nu_2^* f_{ab} = 0. \tag{4.8}$$

Restricting the bundle $\widetilde{\mathcal{E}}$ to a subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{y_0,\eta,\theta} \hookrightarrow \mathcal{L}^{4|6} \subset \mathcal{F}^{5|12}$ yields a splitting of the transition function $\nu_2^* f_{ab}$

$$\nu_2^* f_{ab} = \psi_a^{-1}\psi_b, \tag{4.9}$$

where $\{\psi_a\}$ are again $\mathsf{GL}(n,\mathbb{C})$-valued functions on $\widetilde{\mathcal{U}}_{(a)}$ which are holomorphic. From this splitting together with (4.7), one obtains that

$$\begin{aligned}
\psi_a \widetilde{D}^i_{(a)}\psi_a^{-1} &= \psi_b \widetilde{D}^i_{(a)}\psi_b^{-1} =: \lambda^{\dot\alpha}_{(a)}\widetilde{\mathcal{A}}^i_{\dot\alpha},\\
\psi_a D_i^{(a)}\psi_a^{-1} &= \psi_b D_i^{(a)}\psi_b^{-1} =: \mu^{\dot\alpha}_{(a)}\mathcal{A}_{\dot\alpha i},\\
\psi_a W^{(a)}\psi_a^{-1} &= \psi_b W^{(a)}\psi_b^{-1} =: \mu^{\dot\alpha}_{(a)}\lambda^{\dot\beta}_{(a)}\mathcal{B}_{\dot\alpha\dot\beta} \quad \text{for } \lambda \neq \mu,\\
\psi_a W^{(a)}_{\dot\alpha}\psi_a^{-1} &= \psi_b W^{(a)}_{\dot\alpha}\psi_b^{-1} =: \lambda^{\dot\beta}_{(a)}\widetilde{\mathcal{B}}_{\dot\alpha\dot\beta} \quad \text{for } \lambda = \mu.
\end{aligned} \tag{4.10}$$

These equations are due to a generalized Liouville theorem, and continuity yields that $\widetilde{\mathcal{B}}_{\dot\alpha\dot\beta} = \mathcal{B}_{\dot\alpha\dot\beta}$. Furthermore, one immediately notes that a transition function $\nu_2^* f_{ab}$, which satisfies (4.7) is of the form

$$f_{ab} = f_{ab}\left(y^{\dot\alpha\dot\beta}\lambda^{(a)}_{\dot\alpha}\lambda^{(a)}_{\dot\beta}, y^{\dot\alpha\dot\beta}\mu^{(a)}_{\dot\alpha}\mu^{(a)}_{\dot\beta}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_{\dot\alpha}\right), \tag{4.11}$$

and thus condition (4.8) is obviously fulfilled at points with $\lambda_\pm = \mu_\pm$. Altogether, since we neither loose any information on the gauge potential nor do we loose any constraints on it, we can restrict our discussion to generic points with $\lambda \neq \mu$, which simplifies the presentation.

The superfield $\mathcal{B}_{\dot\alpha\dot\beta}$ decomposes into a gauge potential and a Higgs field Φ:

$$\mathcal{B}_{\dot\alpha\dot\beta} := \mathcal{A}_{(\dot\alpha\dot\beta)} + \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi. \tag{4.12}$$

The zeroth-order component in the superfield expansion of Φ will be the seventh real scalar joining the six scalars of $\mathcal{N} = 4$ SYM in four dimensions, which are the zeroth component of the superfield Φ_{ij} defined in

$$\left\{D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i}, D_{\dot\beta j} + \mathcal{A}_{\dot\beta j}\right\} =: -2\varepsilon_{\dot\alpha\dot\beta}\Phi_{ij}. \tag{4.13}$$

Thus, as mentioned above, the $\mathsf{Spin}(7)$ R-symmetry group of $\mathcal{N} = 8$ SYM theory in three dimensions will not be manifest in this description.

Equations (4.10) are equivalent to the linear system

$$\begin{aligned} \mu^{\dot\alpha}_{(a)}\lambda^{\dot\beta}_{(a)}\left(\partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}\right)\psi_a &= 0, \\ \lambda^{\dot\alpha}_{(a)}\left(\tilde D^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha}\right)\psi_a &= 0, \\ \mu^{\dot\alpha}_{(a)}\left(D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i}\right)\psi_a &= 0. \end{aligned} \tag{4.14}$$

To discuss the corresponding compatibility conditions, we introduce the following differential operators:

$$\tilde\nabla^i_{\dot\alpha} := \tilde D^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha}, \qquad \nabla_{\dot\alpha i} := D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i}, \qquad \nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}. \tag{4.15}$$

We thus arrive at

$$\left\{\tilde\nabla^i_{\dot\alpha}, \tilde\nabla^j_{\dot\beta}\right\} + \left\{\tilde\nabla^i_{\dot\beta}, \tilde\nabla^j_{\dot\alpha}\right\} = 0, \qquad \left\{\nabla_{\dot\alpha i}, \nabla_{\dot\beta j}\right\} + \left\{\nabla_{\dot\beta i}, \nabla_{\dot\alpha j}\right\} = 0, \qquad \left\{\nabla_{\dot\alpha i}, \tilde\nabla^j_{\dot\beta}\right\} - 2\delta^j_i \nabla_{\dot\alpha\dot\beta} = 0, \tag{4.16}$$

and one clearly sees that (4.16) are indeed (4.6) after a dimensional reduction $\mathbb{C}^4 \to \mathbb{C}^3$ and defining $\Phi := A_4$. As it is well known, the supersymmetry (and the R-symmetry) of $\mathcal{N} = 4$ SYM theory are enlarged by this dimensional reduction, and we, therefore, obtained indeed $\mathcal{N} = 8$ SYM theory on $\mathbb{C}^3$.

To sum up, we obtained a correspondence between holomorphic pseudobundles over $\mathcal{L}^{4|6}$ which become holomorphically trivial vector bundles upon reduction to any subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{y_0,\eta,\theta} \hookrightarrow \mathcal{L}^{4|6}$ and solutions to the three-dimensional $\mathcal{N} = 8$ SYM equations. As this correspondence arises by a dimensional reduction of a correspondence which is one-to-one, it is rather evident that also in this case, we have a bijection between both the holomorphic pseudobundles over $\mathcal{L}^{4|6}$ and the solutions after factoring out holomorphic equivalence and gauge equivalence, respectively.

5. Purely Bosonic Yang-Mills-Higgs Theory from Third-Order Subneighborhoods

In this section, we want to turn to the purely bosonic situation (in other words, all the superspaces used up to now loose their Graßmann-odd dimensions) and describe solutions to the three-dimensional Yang-Mills-Higgs equations using a mini-ambitwistor space (when speaking about Yang-Mills-Higgs theory, we mean a theory without quartic interaction term). That is, we will consider the dimensional reduction of the purely bosonic case discussed in [23, 24] from $d = 4$ to $d = 3$. In these papers, it has been shown that solutions to the Yang-Mills field equations are equivalent to holomorphic vector bundles over a third-order thickening of the ambitwistor space $\mathcal{L}^5$ in $\mathcal{P}^3 \times \mathcal{P}^3_*$.

5.1. Thickenings of Complex Manifolds

Given a complex manifold Y of dimension d, a thickening [25, 26] of a submanifold $X \subset Y$ with codimension 1 is an infinitesimal neighborhood of X in Y described by an additional Graßmann-even but nilpotent coordinate. More precisely, the mth order thickening of X is denoted by $X_{(m)}$ and defined as the manifold X together with the extended structure sheaf

$$\mathcal{O}_{(m)} = \frac{\mathcal{O}_Y}{\mathcal{J}^{m+1}}, \tag{5.1}$$

where $\mathcal{O}_Y$ is the structure sheaf of Y, and $\mathcal{J}$ the ideal of functions on Y which vanish on X. We can choose local coordinates $(x^1, \dots, x^{d-1}, y)$ on Y such that X is given by $y = 0$. The mth order thickening $X_{(m)}$, given by the scheme $(X, \mathcal{O}_{(m)})$ is then described by the coordinates $(x^1, \dots, x^{d-1}, y)$ together with the relation $y^{m+1} \sim 0$ (for more details on infinitesimal neighborhoods, see e.g., [9] and references therein).

Note that it is easily possible to map $\mathcal{L}^{5|6}$ to a third-order thickening of $\mathcal{L}^5 \subset \mathcal{P}^3 \times \mathcal{P}^3_*$ by identifying the nilpotent even coordinate y with $2\theta^i \eta_i$ (cf. [27]). However, we will not follow this approach for two reasons. First, the situation is more subtle in the case of $\mathcal{L}^{4|6}$ since $\mathcal{L}^4$ only allows for a nilpotent even direction inside $\mathcal{P}^2 \times \mathcal{P}^2_*$ for $\lambda_\pm = \mu_\pm$. Second, this description has several drawbacks when the discussion of the Penrose-Ward transform reaches the correspondence space, where the concepts of thickenings (and the extended fattenings) are not sufficient (see [27]).

5.2. Third-Order Thickenings and $d = 4$ Yang-Mills Theory

Consider a vector bundle E over the space $\mathbb{C}^4 \times \mathbb{C}^4$ with coordinates $r^{\alpha\dot\alpha}$ and $s^{\alpha\dot\alpha}$. On E, we assume a gauge potential $A = A^r_{\alpha\dot\alpha} \mathrm{d}r^{\alpha\dot\alpha} + A^s_{\beta\dot\beta} \mathrm{d}s^{\beta\dot\beta}$. Furthermore, we introduce the coordinates

$$x^{\alpha\dot\alpha} = \frac{1}{2}\left(r^{\alpha\dot\alpha} + s^{\alpha\dot\alpha}\right), \qquad k^{\alpha\dot\alpha} = \frac{1}{2}\left(r^{\alpha\dot\alpha} - s^{\alpha\dot\alpha}\right) \tag{5.2}$$

on the base of E. We claim that the Yang-Mills equations $\nabla^{\alpha\dot\alpha} F_{\alpha\dot\alpha\beta\dot\beta} = 0$ are then equivalent to

$$\begin{aligned}
\left[\nabla^r_{\alpha\dot\alpha}, \nabla^r_{\beta\dot\beta}\right] &= *\left[\nabla^r_{\alpha\dot\alpha}, \nabla^r_{\beta\dot\beta}\right] + \mathcal{O}\left(k^3\right), \\
\left[\nabla^s_{\alpha\dot\alpha}, \nabla^s_{\beta\dot\beta}\right] &= -*\left[\nabla^s_{\alpha\dot\alpha}, \nabla^s_{\beta\dot\beta}\right] + \mathcal{O}\left(k^3\right), \\
\left[\nabla^r_{\alpha\dot\alpha}, \nabla^s_{\beta\dot\beta}\right] &= \mathcal{O}\left(k^3\right),
\end{aligned} \tag{5.3}$$

where we define $*F^{r,s}_{\alpha\dot\alpha\beta\dot\beta} := (1/2)\varepsilon^{r,s}_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta} F^{\gamma\dot\gamma\delta\dot\delta}_{r,s}$ separately on each $\mathbb{C}^4$(one could also insert an i into this definition but on $\mathbb{C}^4$, this is not natural).

To understand this statement, note that (5.3) are equivalent to

$$\begin{aligned}
\left[\nabla^x_{\alpha\dot\alpha}, \nabla^x_{\beta\dot\beta}\right] &= \left[\nabla^k_{\alpha\dot\alpha}, \nabla^k_{\beta\dot\beta}\right] + \mathcal{O}\left(k^3\right), \\
\left[\nabla^k_{\alpha\dot\alpha}, \nabla^x_{\beta\dot\beta}\right] &= *\left[\nabla^k_{\alpha\dot\alpha}, \nabla^k_{\beta\dot\beta}\right] + \mathcal{O}\left(k^3\right),
\end{aligned} \tag{5.4}$$

which is easily seen by performing the coordinate change from (r,s) to (x,k). These equations are solved by the expansion [23, 24]

$$A^{k}_{\alpha\dot{\alpha}} = -\frac{1}{2}F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}k^{\beta\dot{\beta}} - \frac{1}{3}k^{\gamma\dot{\gamma}}\nabla^{x,0}_{\gamma\dot{\gamma}}\left(*F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}\right)k^{\beta\dot{\beta}},$$
$$A^{x}_{\alpha\dot{\alpha}} = A^{x,0}_{\alpha\dot{\alpha}} - *F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}k^{\beta\dot{\beta}} - \frac{1}{2}k^{\gamma\dot{\gamma}}\nabla^{x,0}_{\gamma\dot{\gamma}}\left(F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}\right)k^{\beta\dot{\beta}}, \tag{5.5}$$

if and only if $\nabla^{\alpha\dot{\alpha}}_{x,0}F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0$ is satisfied. Here, a superscript 0 always denotes an object evaluated at $k^{\alpha\dot{\alpha}} = 0$. Thus, we saw that a solution to the Yang-Mills equations corresponds to a solution to (5.3) on $\mathbb{C}^4 \times \mathbb{C}^4$.

As discussed before, the self-dual and anti-self-dual field strengths solving the first and second equations of (5.3) can be mapped to certain holomorphic vector bundles over $\mathcal{P}^3$ and $\mathcal{P}^3_*$, respectively. On the other hand, the potentials given in (5.5) are now defined on a second-order infinitesimal neighborhood (not a thickening) of the diagonal in $\mathbb{C}^4 \times \mathbb{C}^4$ for which $\mathcal{O}(k^3) = 0$. In the twistor description, this potential corresponds to a transition function $f_{+-} \sim \psi_+^{-1}\psi_-$, where the Čech 0-cochain $\{\psi_\pm\}$ is a solution to the equations

$$\lambda^{\dot{\alpha}}_{\pm}\left(\frac{\partial}{\partial r^{\alpha\dot{\alpha}}} + A^{r}_{\alpha\dot{\alpha}}\right)\psi_\pm = \mathcal{O}(k^4),$$
$$\mu^{\alpha}_{\pm}\left(\frac{\partial}{\partial s^{\alpha\dot{\alpha}}} + A^{s}_{\alpha\dot{\alpha}}\right)\psi_\pm = \mathcal{O}(k^4). \tag{5.6}$$

Roughly speaking, since the gauge potentials are defined to order k^2 and since $\partial/\partial r^{\alpha\dot{\alpha}}$ and $\partial/\partial s^{\alpha\dot{\alpha}}$ contain derivatives with respect to k, the above equations can indeed be rendered exact to order k^3. The exact definition of the transition function is given by

$$f_{+-,i} := \sum_{j=0}^{i} \psi^{-1}_{+,j}\psi_{-,i-j}, \tag{5.7}$$

where the additional indices label the order in k. On the twistor space side, a third-order neighborhood in k corresponds to a third-order thickening in

$$\mathring{\kappa}_{(a)} := z^{\alpha}_{(a)}\mu^{(a)}_{\alpha} - u^{\dot{\alpha}}_{(a)}\lambda^{(a)}_{\dot{\alpha}}. \tag{5.8}$$

Altogether, we see that a solution to the Yang-Mills equations corresponds to a topologically trivial holomorphic vector bundle over a third-order thickening of $\mathcal{L}^5$ in $\mathcal{P}^3 \times \mathcal{P}^3_*$, which becomes holomorphically trivial, when restricted to any $\mathbb{C}P^1 \times \mathbb{C}P^1_* \hookrightarrow \mathcal{L}^5$.

5.3. Third-Order Subthickenings and $d = 3$ *Yang-Mills-Higgs Theory*

Let us now translate the above discussion to the three-dimensional situation. First of all, the appropriate Yang-Mills-Higgs equations obtained by dimensional reduction are

$$\nabla^{(\dot{\alpha}\dot{\beta})} F_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})} = [\phi, \nabla_{(\dot{\gamma}\dot{\delta})}\phi], \qquad \Delta\phi := \nabla^{(\dot{\alpha}\dot{\beta})}\nabla_{(\dot{\alpha}\dot{\beta})}\phi = 0, \tag{5.9}$$

while the self-dual and anti-self-dual Yang-Mills equations correspond after the dimensional reduction to two Bogomolny equations which read as

$$F_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})} = \varepsilon_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})(\dot{\varepsilon}\dot{\zeta})}\nabla^{(\dot{\varepsilon}\dot{\zeta})}\phi, \qquad F_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})} = -\varepsilon_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})(\dot{\varepsilon}\dot{\zeta})}\nabla^{(\dot{\varepsilon}\dot{\zeta})}\phi, \tag{5.10}$$

respectively. Using the decomposition $F_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})} = \varepsilon_{\dot{\alpha}\dot{\gamma}} f_{\dot{\beta}\dot{\delta}} + \varepsilon_{\dot{\beta}\dot{\delta}} f_{\dot{\alpha}\dot{\gamma}}$, the above two equations can be simplified to

$$f_{\dot{\alpha}\dot{\beta}} = \frac{\mathrm{i}}{2}\nabla_{(\dot{\alpha}\dot{\beta})}\phi, \qquad f_{\dot{\alpha}\dot{\beta}} = -\frac{\mathrm{i}}{2}\nabla_{(\dot{\alpha}\dot{\beta})}\phi. \tag{5.11}$$

Analogously to the four dimensional case, we start from a vector bundle E over the space $\mathbb{C}^3 \times \mathbb{C}^3$ with coordinates $p^{(\dot{\alpha}\dot{\beta})}$ and $q^{(\dot{\alpha}\dot{\beta})}$, additionally we introduce the coordinates

$$y^{(\dot{\alpha}\dot{\beta})} = \frac{1}{2}\left(p^{(\dot{\alpha}\dot{\beta})} + q^{(\dot{\alpha}\dot{\beta})}\right), \qquad h^{(\dot{\alpha}\dot{\beta})} = \frac{1}{2}\left(p^{(\dot{\alpha}\dot{\beta})} - q^{(\dot{\alpha}\dot{\beta})}\right), \tag{5.12}$$

and a gauge potential

$$A = A^{p}_{(\dot{\alpha}\dot{\beta})}\mathrm{d}p^{(\dot{\alpha}\dot{\beta})} + A^{q}_{(\dot{\alpha}\dot{\beta})}\mathrm{d}q^{(\dot{\alpha}\dot{\beta})} = A^{y}_{(\dot{\alpha}\dot{\beta})}\mathrm{d}y^{(\dot{\alpha}\dot{\beta})} + A^{h}_{(\dot{\alpha}\dot{\beta})}\mathrm{d}h^{(\dot{\alpha}\dot{\beta})} \tag{5.13}$$

on E. The differential operators we will consider in the following are obtained from covariant derivatives by dimensional reduction and take, for example, the shape

$$\nabla^{y}_{\dot{\alpha}\dot{\beta}} = \frac{\partial}{\partial y^{(\dot{\alpha}\dot{\beta})}} + \left[A^{y}_{(\dot{\alpha}\dot{\beta})} + \frac{\mathrm{i}}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\phi^{y}, \cdot\right]. \tag{5.14}$$

We now claim that the Yang-Mills-Higgs equations (5.9) are equivalent to

$$\begin{aligned}
\left[\nabla^{p}_{\dot{\alpha}\dot{\beta}}, \nabla^{p}_{\dot{\gamma}\dot{\delta}}\right] &= *\left[\nabla^{p}_{\dot{\alpha}\dot{\beta}}, \nabla^{p}_{\dot{\gamma}\dot{\delta}}\right] + \mathcal{O}(h^3), \\
\left[\nabla^{q}_{\dot{\alpha}\dot{\beta}}, \nabla^{q}_{\dot{\gamma}\dot{\delta}}\right] &= -*\left[\nabla^{q}_{\dot{\alpha}\dot{\beta}}, \nabla^{q}_{\dot{\gamma}\dot{\delta}}\right] + \mathcal{O}(h^3), \\
\left[\nabla^{p}_{\dot{\alpha}\dot{\beta}}, \nabla^{q}_{\dot{\gamma}\dot{\delta}}\right] &= \mathcal{O}(h^3),
\end{aligned} \tag{5.15}$$

where we can use $*[\nabla^{p,q}_{\dot\alpha\dot\beta}, \nabla^{p,q}_{\dot\gamma\dot\delta}] = \varepsilon_{\dot\alpha\dot\beta\dot\gamma\dot\delta\dot\varepsilon\dot\zeta}\nabla^{\dot\varepsilon\dot\zeta}_{p,q}\phi^{p,q}$. These equations can be simplified in the coordinates (y, h) to equations similar to (5.3), which are solved by the field expansion

$$\begin{aligned}
A^{h}_{(\dot\alpha\dot\beta)} &= -\frac{1}{2}F^{y,0}_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)}h^{(\dot\gamma\dot\delta)} - \frac{1}{3}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\varepsilon_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}\left(\nabla^{(\dot\sigma\dot\tau)}_{y,0}\phi\right)h^{(\dot\varepsilon\dot\zeta)}, \\
\phi^{h} &= \frac{1}{2}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\phi^{y,0}h^{(\dot\gamma\dot\delta)} + \frac{1}{6}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\varepsilon^{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}F^{y,0}_{(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}h_{(\dot\alpha\dot\beta)}, \\
A^{y}_{(\dot\alpha\dot\beta)} &= A^{y,0}_{(\dot\alpha\dot\beta)} - \varepsilon_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}\left(\nabla^{(\dot\sigma\dot\tau)}_{y,0}\phi^{y,0}\right)h^{(\dot\varepsilon\dot\zeta)} - \frac{1}{2}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\left(F^{y,0}_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)}\right)h^{(\dot\varepsilon\dot\zeta)}, \\
\phi^{y} &= \phi^{y,0} + \frac{1}{2}\varepsilon^{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}F^{y,0}_{(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}h_{(\dot\alpha\dot\beta)} + \frac{1}{2}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\left(\nabla_{(\dot\alpha\dot\beta)}\phi^{y,0}\right)h^{(\dot\alpha\dot\beta)},
\end{aligned} \tag{5.16}$$

if and only if the Yang-Mills-Higgs equations (5.9) are satisfied.

Thus, solutions to the Yang-Mills-Higgs equations (5.9) correspond to solutions to (5.15) on $\mathbb{C}^3 \times \mathbb{C}^3$. Recall that solutions to the first two equations of (5.15) correspond in the twistor description to holomorphic vector bundles over $\mathcal{P}^2 \times \mathcal{P}^2_*$. Furthermore, the expansion of the gauge potential (5.16) is an expansion in a second-order infinitesimal neighborhood of $\mathrm{diag}(\mathbb{C}^3 \times \mathbb{C}^3)$. As we saw in the construction of the mini-superambitwistor space $\mathcal{L}^{4|6}$, the diagonal for which $h^{(\dot\alpha\dot\beta)} = 0$ corresponds to $\mathcal{L}^4 \subset \mathcal{P}^2 \times \mathcal{P}^2_*$. The neighborhoods of this diagonal will then correspond to *subthickenings* of $\mathcal{L}^4$ inside $\mathcal{P}^2 \times \mathcal{P}^2_*$, that is, for $\mu_\pm = \lambda_\pm$, we have the additional nilpotent coordinate ξ. In other words, the subthickening of $\mathcal{L}^4$ in $\mathcal{P}^2 \times \mathcal{P}^2_*$ is obtained by turning one of the fiber coordinates of $\mathcal{P}^2 \times \mathcal{P}^2$ over $\mathbb{C}P^1_\Delta$ into a nilpotent even coordinate (in a suitable basis). Then, we can finally state the following:

Gauge equivalence classes of solutions to the three-dimensional Yang-Mills-Higgs equations are in one-to-one correspondence with gauge equivalence classes of holomorphic pseudobundles over a third-order subthickening of $\mathcal{L}^4$, which become holomorphically trivial vector bundles when restricted to a $\mathbb{C}P^1 \times \mathbb{C}P^1$ holomorphically embedded into $\mathcal{L}^4$.

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