TRIMMED STABLE AR(1) PROCESSES

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ABSTRACT. In this paper we investigate the distribution of trimmed sums of dependent observations with heavy tails. We consider the case of autoregressive processes of order one with independent innovations in the domain of attraction of a stable law. We show if the d largest (in magnitude) terms are removed from the sample, then the sum of the remaining elements satisfies a functional central limit theorem with random centering provided $d = d(n) \geq n^\gamma$ (for some $\gamma > 0$), $d(n)/n \to 0$. This result is used to get asymptotics for the widely used CUSUM process in case of dependent heavy tailed observations.

1. Introduction and results

Let $X_1, X_2, \ldots,$ be independent, identically distributed random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$. Lévy (1935) and Darling (1952) noted that the order of magnitude of the sum $S_n = \sum_{k=1}^n X_k$ is the same as that of its largest term and the contribution of a fixed, but large number of extremal terms is essentially responsible for the distribution of S_n. The asymptotic distribution of the trimmed sum $S_n^{(d)}$ obtained from S_n by discarding the d smallest and d largest summands was determined by LePage et al. (1981) and Csörgő et al. (1986) proved that for $d(n) \to \infty$, $d(n)/n \to 0$ the trimmed sum $S_n^{(d)}$ satisfies the central limit theorem. Arov and Bobrov (1960), Mori (1984), Hall (1978), Teugels (1981), Griffin and Pruitt (1987, 1989) and Kesten (1993) considered a different type of trimming of the sample. Let $\eta_{n,d}$ denote the d–th largest element of $|X_1|, \ldots, |X_n|$. These authors were interested in the asymptotic behavior of the modulus trimmed sum ${}^{(d)}S_n = \sum_{k=1}^n X_k I\{|X_k| \leq \eta_{n,d}\}$, i.e. when from the sum we remove the d elements with the largest absolute values. Griffin and Pruitt (1987) proved that the trimmed central limit theorem of Csörgő et al. (1986) remains valid for modulus trimmed sums provided the distribution of X_1 is symmetric, but it generally fails for nonsymmetric variables and it can happen that ${}^{(d)}S_n$ is asymptotically normal for some $d(n)$, but not for another $d'(n) \geq d(n)$. This is somewhat unexpected, since removing more large elements from the sample should result in better behavior. Sufficient conditions for the asymptotic normality of ${}^{(d)}S_n$ in the nonsymmetric case were given by Berkes and Horváth (2012). On the other hand, Berkes et

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al. (2011a) showed that if $d(n) \to \infty$, $d(n)/n \to 0$, a functional central limit theorem always holds for ${}^{(d)}S_n$ with a random centering factor.

Trimming also has important applications in statistics. As an example, we consider the detection of possible changes in the location model

$$X_j = c_j + e_j, \quad 1 \le j \le n, \tag{1.1}$$

where $e_1, \ldots, e_n$ are random errors. Under the null hypothesis of stability, the location parameter is constant, i.e.

$$H_0: \quad c_1 = c_2 = \ldots = c_n.$$

If H_0 holds, then

$$X_j = c + e_j, \quad 1 \le j \le n, \tag{1.2}$$

with some constant c. Under the alternative there are r changes:

$$\begin{aligned} H_A: \ & \text{there is } \ r \ge 1 \text{ and } 1 < k_1 < k_2 < \ldots < k_r < n \text{ such that} \\ & c_1 = \ldots = c_{k_1-1} \ne c_{k_1} = c_{k_1+1} = \ldots = c_{k_2-1} \ne c_{k_2} = c_{k_2+1} = \ldots \\ & = c_{k_r-1} \ne c_{k_r} = \ldots = c_n. \end{aligned}$$

The most popular methods to test H_0 against H_A (cf. Csörgő and Horváth (1998) and Aue and Horváth (2012)) are based on the CUSUM process

$$U_n(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i, \tag{1.3}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Clearly, if H_0 is true, then $U_n(t)$ does not depend on the common but unknown location parameter c_1. It is well known if $X_1, \ldots, X_n$ are independent and identically distributed random variables with a finite second moment, then

$$\frac{1}{(n\text{var}(X_1))^{1/2}} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where $B(x)$ is a Brownian bridge and $\xrightarrow{\mathcal{D}[0,1]}$ means weak convergence in the space $\mathcal{D}[0,1]$ of cadlag functions equipped with the Skorokhod J_1 topology (cf. Billingsley (1968)). Assuming that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables in the domain of attraction of a stable law of index $\alpha \in (0,2)$, Aue et al. (2008) showed that

$$\frac{1}{n^{1/\alpha}\hat{L}(n)} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B_\alpha(x),$$

where $\hat{L}$ is a slowly varying function at ∞ and $B_\alpha(x)$ is an α–stable bridge. (The α–stable bridge is defined as $B_\alpha(x) = W_\alpha(x) - xW_\alpha(1)$, where W_α is a Lévy α–stable motion.) Since nothing is known on the distributions of the functionals of α–stable bridges, Berkes et al. (2011a) suggested the trimmed CUSUM process

$$T_{n,d}(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i I\{|X_i| \leq \eta_{n,d}\} - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i I\{|X_i| \leq \eta_{n,d}\}. \tag{1.4}$$

Assuming that the X_i's are independent and identically distributed and are in the domain of attraction of a stable law, they proved

$$\frac{1}{\sigma_n} T_{n,d}(x) \xrightarrow{\mathcal{D}[0,1]} B(x), \tag{1.5}$$

where

$$\sigma_n^2 = \frac{\alpha}{2-\alpha} (H^{-1}(d/n))^2 d,$$

$B(t)$ is a Brownian bridge and H^{-1} denotes the generalized inverse of H, the survival function of X_1. The CUSUM process has also been widely used in case of dependent variables but it is nearly always assumed that the observations have high moments and the dependence in the sequence is weak. For a review we refer to Aue and Horváth (2012). However, very few papers consider the instability of time series models with heavy tails.

Fama (1965) and Mandelbrot (1963, 1967) pointed out that the distributions of commodity and stock returns are often heavy tailed with possible infinite variance and their research started the investigation of time series models where the marginal distributions have regularly varying tails. Davis and Resnick (1985, 1986) investigated the properties of moving averages with regularly varying tails and obtained non–Gaussian limits for the sample covariances and correlations. Their results were extended to heavy tailed ARCH by Davis and Mikosch (1998). The empirical periodogram was studied by Mikosch et al. (2000). Andrews et al. (2009) estimated the parameters of autoregressive processes with stable innovations.

In this paper we study trimmed sums of AR(1) sequences with heavy tails. Let e_i be a $\sigma(\varepsilon_j, j \leq i)$ measurable solution of

$$e_i = \rho e_{i-1} + \varepsilon_i \quad -\infty < i < \infty. \tag{1.6}$$

We assume throughout this paper that

$$\varepsilon_j, -\infty < j < \infty \text{ are independent and identically distributed,} \tag{1.7}$$

$$\begin{aligned} &\varepsilon_0 \text{ belongs to the domain of attraction of a stable} \\ &\text{random variable } \xi^{(\alpha)} \text{ with parameter } 0 < \alpha < 2, \end{aligned} \tag{1.8}$$

and

$$\varepsilon_0 \text{ is symmetric when } \alpha = 1. \tag{1.9}$$

Assumption (1.8) means that

$$\left(\sum_{j=1}^{n} \varepsilon_j - a_n \right) \Big/ b_n \xrightarrow{\mathcal{D}} \xi^{(\alpha)} \tag{1.10}$$

for some numerical sequences a_n and b_n. The necessary and sufficient condition for this is

$$\lim_{t\to\infty}\frac{P\{\varepsilon_0>t\}}{L_*(t)t^{-\alpha}}=p \quad\text{and}\quad \lim_{t\to\infty}\frac{P\{\varepsilon_0\le -t\}}{L_*(t)t^{-\alpha}}=q \tag{1.11}$$

for some numbers $p\ge 0$, $q\ge 0$, $p+q=1$, where L_* is a slowly varying function at ∞. It is known that (1.6) has a unique stationary non–anticipative solution if and only if

$$-1<\rho<1. \tag{1.12}$$

Under assumptions (1.7)–(1.12), $\{e_j\}$ is a stationary sequence and $E|e_0|^\kappa<\infty$ for all $0<\kappa<\alpha$ but $E|e_0|^\kappa=\infty$ for all $\kappa>\alpha$. AR(1) processes with stable innovations were considered by Chan and Tran (1989), Chan (1990), Aue and Horváth (2007) and Zhang and Chan (2010) who investigated the case when ρ is close to 1 and provided estimates for ρ and and other the parameters when the observations do not have finite variances.

The convergence of the finite dimensional distributions of $U_n(x)$ in the AR(1) case is an immediate consequence of Phillips and Solo (1992) representation. Let $\stackrel{fdd}{\longrightarrow}$ denote the convergence of the finite dimensional distributions. If (1.2)–(1.9) and (1.12) hold, then we have

$$\frac{1-\rho}{n^{1/\alpha}L_*(n)}U_n(x)\stackrel{\text{fdd}}{\longrightarrow}B_\alpha(x), \tag{1.13}$$

where $B_\alpha(x), 0\le x\le 1$ is an α–stable bridge and L_* is defined in (1.11). It has been pointed out by Avram and Taqqu (1986, 1992) that the fdd convergence in (1.13) cannot be replaced with weak convergence in $\mathcal{D}[0,1]$. However, Avram and Taqqu (1992) proved that $U_n(x)$ converges in the weak–M_1 sense under some additional regularity conditions. Some of their regularity conditions were removed by Tyran–Kamińska (2010). For further results on the weak convergence of dependent sequence with infinite variance in the M_1 topology we refer to Basrak et al. (2012).

We formulate now our main results. On the truncation parameter $d=d(n)$ we will assume

$$\lim_{n\to\infty} d(n)/n=0 \tag{1.14}$$

and

$$d(n)\ge n^\delta \quad\text{with some}\quad 0<\delta<1. \tag{1.15}$$

Let $F(x)=P\{X_0\le x\}$, $H(x)=P\{|X_0|>x\}$ and let $H^{-1}(t)$ be the (generalized) inverse of H. Our last condition will be used to establish the weak law of large numbers for $\eta_{n,d}$. We assume that ε_0 has a density function $p(t)$ which satisfies

$$\int_{-\infty}^{\infty}|p(t+s)-p(t)|dt\le C|s| \quad\text{with some}\quad C. \tag{1.16}$$

Let

$$A_n=d^{1/2}H^{-1}(d/n) \tag{1.17}$$

and

$$m(t) = EX_1 I\{|X_1| \le t\}. \tag{1.18}$$

Theorem 1.1. *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2}\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\frac{1}{A_n}\sum_{k=1}^{n}[X_k I\{|X_k| \le \eta_{n,d}\} - m(\eta_{n,d})] \stackrel{\mathcal{D}[0,1]}{\longrightarrow} W(x),$$

where $W(x)$ is a Wiener process.

The result in Theorem 1.1 uses the the random centering factor $m(\eta_{n,d})$. This factor is characteristic for the asymptotic distribution of the modulus trimmed partial sums process, as first observed in Berkes et al. (2011a). Since a random translation of the terms in the CUSUM process cancels out, the next result is an immediate consequence of Theorem 1.1.

Theorem 1.2. *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2}\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\frac{T_{n,d}(x)}{A_n} \stackrel{\mathcal{D}[0,1]}{\longrightarrow} B(x),$$

where $B(x)$ is a Brownian bridge.

Statistical applications of Theorem 1.2 require the estimation of the norming factor from the observations. The estimation of this term will be studied in a subsequent paper.

2. Preliminary results

The proofs of Theorems 1.1 and 1.2 are based on several technical lemmas.

We can and will assume without loss of generality that

$$E\varepsilon_0 = 0, \text{ if } 1 < \alpha < 2. \tag{2.1}$$

Under these conditions, in (1.10) we can choose $a_n = 0$ and b_n can be chosen any sequence satisfying

$$\frac{n}{b_n^{\alpha}} L_*(b_n) \to 1. \tag{2.2}$$

According to the result of Cline (1983) (cf. also Davis and Resnick (1986)), $H(x)$, the survival function of $|X_0|$ satisfies

$$H(x) = x^{-\alpha} L(x), \tag{2.3}$$

where $L(x)$ is a slowly varying function at ∞ and

$$\lim_{x\to\infty}\frac{H(x)}{P\{|\varepsilon_0| > x\}} = \lim_{x\to\infty}\frac{L(x)}{L_*(x)} = \frac{1}{1-|\rho|^{\alpha}}. \tag{2.4}$$

Let

$$u_{k,n}(t) = X_k I\{|X_k| \le tH^{-1}(d/n)\} \quad \text{and} \quad m_n(t) = E[X_0 I\{|X_0| \le tH^{-1}(d/n)\}].$$

The main goal of this section is to get bounds for $Eu_0(t)u_k(s)$ and $\operatorname{cov}(u_0(t), u_k(s))$.

Lemma 2.1. *We assume that (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold. Let* $\mathbf{Y}^{(k)} = (X_0, X_k)$ *and let* $\mathbf{Y}_i^{(k)}, i = 1, 2, \ldots$ *be independent and identically distributed copies of* $\mathbf{Y}^{(k)}$. *Then*

$$\frac{\mathbf{Y}_1^{(k)} + \ldots + \mathbf{Y}_n^{(k)}}{n^{1/\alpha} L_*(n)} \xrightarrow{\mathcal{D}} \mathbf{Z}^{(k)} \quad \text{as} \quad n \to \infty,$$

where $\mathbf{Z}^{(k)} = (Z_1^{(k)}, Z_2^{(k)})$ *with*

$$Z_1^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{-\ell}^{(\alpha)} \quad \text{and} \quad Z_2^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{k-\ell}^{(\alpha)}$$

and $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ *are independent and identically distributed copies of* $\xi^{(\alpha)}$.

Proof. It follows from (1.6) that

$$X_k - c = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell} = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} + \rho^k X_0, \quad 1 \le k < \infty. \tag{2.5}$$

Let $\varepsilon_\ell^{(i)}, -\infty < \ell < \infty, i = 1, 2, \ldots$ be independent and identically distributed copies of ε_0. Clearly

$$\mathbf{Y}_i^{(k)} = (Y_{i,1}^{(k)}, Y_{i,2}^{(k)}) \quad \text{with} \quad Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad Y_{i,2}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell}^{(i)}$$

are independent and identically distributed copies of $\mathbf{Y}^{(k)}$. Elementary algebra gives

$$\sum_{i=1}^n Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^n \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad \sum_{i=1}^n Y_{i,2}^{(k)} = \sum_{\ell=0}^{k-1} \rho^\ell \sum_{i=1}^n \varepsilon_{k-\ell}^{(i)} + \rho^k \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^n \varepsilon_{-\ell}^{(i)}.$$

For every $L \ge 0$ by (1.10) we have that (recall that under our conditions the centering factors a_n in (1.10) can be chosen 0)

$$\frac{1}{b_n} \left(\sum_{i=1}^n \varepsilon_\ell^{(i)}, -L \le \ell \le L \right) \xrightarrow{\mathcal{D}} \left(\xi_\ell^{(\alpha)}, -L \le \ell \le L \right),$$

where $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ are independent and identically distributed copies of $\xi^{(\alpha)}$. Let $0 < \kappa < \alpha$. It follows from de Acosta and Giné (1979) that

$$E \left| \frac{1}{b_n} \sum_{i=1}^n \varepsilon_\ell^{(i)} \right|^\kappa \le C_1,$$

and therefore for every $x > 0$ we have that

$$\lim_{L\to\infty}\limsup_{n\to\infty} P\left\{\sum_{\ell=L+1}^{\infty}\rho^{\ell}\left|\frac{1}{b_n}\sum_{i=1}^{n}\varepsilon_{\ell}^{(i)}\right| > x\right\} = 0$$

and similarly

$$\lim_{L\to\infty} P\left\{\sum_{\ell=L+1}^{\infty}\rho^{\ell}|\xi_{\ell}^{(\alpha)}| > x\right\} = 0.$$

This completes the proof of the lemma. □

Let $\mathbf{i}$ denote the imaginary unit.

Lemma 2.2. *Let* $\mathbf{Y}$ *be a stable vector variable with characteristic function* $\psi(s,t)$*. Then there exists a measure* ν *on the Borel sets of* R^2 *such that for some* $\mathcal{C}_1, \mathcal{C}_2$ *and any* $\gamma > 0$

$$\psi(s,t) = \exp\left\{\mathbf{i}(\mathcal{C}_1 s + \mathcal{C}_2 t) + \int_{|\mathbf{u}|>\gamma}(e^{\mathbf{i}(su_1+tu_2)} - 1)\nu(du_1, du_2)\right.$$
$$\left. + \int_{0<|\mathbf{u}|\le\gamma}(e^{\mathbf{i}(su_1+tu_2)} - 1 - \mathbf{i}(su_1 + tu_2))\nu(du_1, du_2)\right\},$$

where $\mathbf{u} = (u_1, u_2)$*.*

The result can be found, for example, in Gikhman and Skorohod (1969, Chapter 5). ν is called the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Y}$. The stable vectors in our paper will be centered, i.e. $c_1 = c_2 = 0$.

Lemma 2.3. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then we have*

$$\lim_{T\to\infty}\frac{T^{\alpha-2}}{L_*(T)}EX_0 I\{|X_0| \le vT\}X_k I\{|X_k| \le wT\} = \frac{\alpha}{2-\alpha}\frac{\rho^k}{1-|\rho|^{\alpha}}(\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

Proof. It follows from Theorem 4 of Resnick and Greenwood (1979) that

$$\lim_{n\to\infty} nP\left\{\frac{(X_0, X_k)}{b_n} \in A\right\} = \nu(A), \tag{2.6}$$

where b_n is defined in (2.2) and A is any Borel set of R^2, not containing $(0,0)$, $\nu(A) < \infty$ and the ν–measure of the boundary of A is 0. Since $nL_*(b_n)/b_n^{\alpha} \to 1$, with the choice of $n = \lfloor T^{\alpha}/L_*(T)\rfloor$ we get from (2.6) that

$$\lim_{T\to\infty}\frac{T^{\alpha}}{L_*(T)}P\{(X_0, X_k)/T \in \mathbf{A}\} = \nu(\mathbf{A}), \tag{2.7}$$

where ν is the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Z}^{(k)}$. By elementary arguments we conclude from (2.7)

$$\lim_{T\to\infty}\frac{T^{\alpha-2}}{L_*(T)}EX_0 I\{|X_0| \le vT\}X_k I\{|X_k| \le wT\} = \int_{-v}^{v}\int_{-w}^{w} xy\nu(dx, dy).$$

Since $\xi^{(\alpha)}$ is a stable random variable, its characteristic function can be written as $\exp(-\psi(t))$ and with this notation we get

$$E\exp(\mathbf{i}(sZ_1^{(k)}+tZ_2^{(k)}))=\exp\left(-\sum_{\ell=0}^{\infty}\psi(s\rho^\ell+t\rho^{k+\ell})-\sum_{\ell=0}^{k-1}\psi(t\rho^\ell)\right).$$

If $\hat{\nu}_\ell$ denotes the Lévy measure associated with the characteristic function $\exp(-\psi(s\rho^\ell+t\rho^{k+\ell}))$ and $\tilde{\nu}_\ell$ corresponds to $\exp(-\psi(t\rho^\ell))$, then we have

$$\nu(\mathbf{A})=\sum_{\ell=0}^{\infty}\hat{\nu}_\ell(\mathbf{A})+\sum_{\ell=0}^{k-1}\tilde{\nu}_\ell(\mathbf{A}).$$

Hence

$$\int_{-v}^{v}\int_{-w}^{w}xy\nu(dx,dy)=\sum_{\ell=0}^{\infty}\int_{-v}^{v}\int_{-w}^{w}xy\hat{\nu}_\ell(dx,dy).$$

Next we note that there is a positive constant A^* such that

$$\lim_{x\to\infty}\frac{P\{|\xi^{(\alpha)}|>x\}}{x^{-\alpha}}=A^*$$

and therefore by Bingham et al (1989, p. 346) we obtain that

$$\lim_{x\to\infty}\frac{E(\xi^{(\alpha)})^2I\{|\xi^{(\alpha)}|\le x\}}{x^2P\{|\xi^{(\alpha)}|>x\}}=\frac{\alpha}{2-\alpha}$$

resulting in

$$\lim_{x\to\infty}\frac{E(\xi^{(\alpha)})^2I\{|\xi^{(\alpha)}|\le x\}}{x^{2-\alpha}}=A^*\frac{\alpha}{2-\alpha}.$$

The last relation implies

$$\begin{aligned}\lim_{T\to\infty}&T^{\alpha-2}E\left[\rho^{2\ell+k}(\xi^{(\alpha)})^2I\{|\xi^{(\alpha)}|\le T\min(v|\rho|^{-\ell},w|\rho|^{-(\ell+k)})\}\right]\\&=A^*\frac{\alpha}{2-\alpha}\rho^{2\ell+k}(\min(v|\rho|^{-\ell},w|\rho|^{-(\ell+k)}))^{2-\alpha}.\end{aligned}$$

We note that $\exp(-\psi(s\rho^\ell+t\rho^{k+\ell}))$ is the characteristic function of the vector $(\rho^\ell\xi^{(\alpha)},\rho^{k+\ell}\xi^{(\alpha)})$, so repeating the arguments leading to (2.6) and (2.7) for this vector instead of (X_0,X_k) we get

$$\lim_{T\to\infty}\rho^{k+2\ell}\frac{T^{\alpha-2}}{A^*}E\xi^{(\alpha)}I\{|\rho^\ell\xi^{(\alpha)}|\le vT\}\xi^{(\alpha)}I\{|\rho^{\ell+k}\xi^{(\alpha)}|\le wT\}=\int_{-v}^{v}\int_{-w}^{w}xy\hat{\nu}_\ell(dx,dy),$$

and therefore

$$\int_{-v}^{v}\int_{-w}^{w}xy\hat{\nu}_\ell(dx,dy)=\frac{\alpha}{2-\alpha}\rho^k|\rho|^{\alpha\ell}(\min(v,w|\rho|^{-k}))^{2-\alpha}.$$

Summing for $\ell=0,1,\ldots$, we get Lemma 2.3. □

Lemma 2.4. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then for every* $k = 0, 1, 2, \dots$

$$\lim_{n\to\infty} \frac{nE(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))}{A_n^2} = \frac{\alpha}{2-\alpha}\rho^k(\min(s, t|\rho|^{-k}))^{2-\alpha}. \tag{2.8}$$

Proof. If $1 < \alpha < 2$, then

$$\lim_{n\to\infty} m_n(t) = EX_0 \quad \text{for any} \quad t > 0. \tag{2.9}$$

If $0 < \alpha < 1$, then

$$\begin{aligned} |m_n(t)| &\le \int_{-tH^{-1}(d/n)}^{tH^{-1}(d/n)} |x| dF(x) \\ &= -\int_0^{tH^{-1}(d/n)} x dH(x) = -xH(x)\Big|_0^{tH^{-1}(d/n)} + \int_0^{tH^{-1}(d/n)} H(x)dx. \end{aligned} \tag{2.10}$$

By (2.3) and Bingham et al. (1989, p. 26) we have for $0 < \alpha < 1$

$$\lim_{y\to\infty} \frac{\displaystyle\int_0^y H(x)dx}{yH(y)/(1-\alpha)} = 1, \tag{2.11}$$

and therefore

$$m_n(t) = O\left(H^{-1}(d/n)\frac{d}{n}\right). \tag{2.12}$$

If $\alpha = 1$, by assumption e_0 is symmetric, so under (1.2) we have that $X_1 = e_1 + c_1$ and therefore

$$\begin{aligned} m_n(t) &= O(1) + E[e_0 I\{|X_0| \le tH^{-1}(d/n)\}] \\ &= O(1) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} x dP\{e_1 \le x\} \\ &= O\left(H^{-1}(d/n)\frac{d}{n}\right) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} P\{e_1 \le x\}dx \\ &= O\left(H^{-1}(d/n)\frac{d}{n}\log H^{-1}(d/n)\right). \end{aligned} \tag{2.13}$$

Thus we get from (2.9)–(2.13) for all $0 < \alpha < 2$ that

$$\frac{nm_n(s)m_n(t)}{A_n^2} \to 0. \tag{2.14}$$

Lemma 2.3 yields

$$\lim_{n\to\infty} \frac{n}{A_n^2}\frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} EX_0 I\{|X_0| \le sH^{-1}(d/n)\} X_k I\{|X_k| \le tH^{-1}(d/n)\}$$

$$= \frac{\alpha}{2-\alpha}\frac{\rho^k}{1-|\rho|^\alpha}(\min(s, t|\rho|^{-k}))^{2-\alpha}.$$

By (2.4) we have

$$\lim_{n\to\infty}\frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} = \frac{1}{1-|\rho|^\alpha},$$

which completes the proof of the lemma. □

Lemma 2.5. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, we have for all* $1/2 \le s \le t \le 3/2$ *and* $0 \le x \le 1$ *that*

$$\lim_{n\to\infty}\frac{1}{A_n^2}E\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(s)-m_n(s))\right)\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(t)-m_n(t))\right)$$
$$= x\frac{\alpha}{2-\alpha}\left(s^{2-\alpha}+\sum_{k=1}^{\infty}\rho^k[(\min(s,t|\rho|^{-k})^{2-\alpha}+\min(t,s|\rho|^{-k})^{2-\alpha}]\right).$$

Proof. We note that

$$E\Bigg(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(s)-m_n(s))\Bigg)\Bigg(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(t)-m_n(t))\Bigg)$$
$$= \lfloor nx\rfloor E(u_{0,n}(s)-m_n(s))(u_{0,n}(t)-m_n(t))$$
$$+\sum_{k=1}^{\lfloor nx\rfloor-1}(\lfloor nx\rfloor-k)E(u_{0,n}(s)-m_n(s))(u_{k,n}(t)-m_n(t))$$
$$+\sum_{k=1}^{\lfloor nx\rfloor-1}(\lfloor nx\rfloor-k)E(u_{0,n}(t)-m_n(t))(u_{k,n}(s)-m_n(s)).$$

Let

$$e_k^* = \sum_{\ell=0}^{k-1}\rho^\ell\varepsilon_{k-\ell} \quad\text{and}\quad X_k^* = c_1 + e_k^*. \tag{2.15}$$

It follows from Cline(1983) that there is a constant C_1 such that

$$P\{|X_k^*| > x\} \le C_1 x^{-\alpha}L(x) \quad\text{for all } k \text{ and } 0 \le x < \infty. \tag{2.16}$$

Clearly as in (2.5),

$$X_k - X_k^* = e_k - e_k^* = \sum_{\ell=k}^{\infty}\rho^\ell\varepsilon_{k-\ell} = \sum_{j=0}^{\infty}\rho^{k+j}\varepsilon_{-j} = \rho^k(X_0 - c_1). \tag{2.17}$$

Next we write

$$|E(u_{0,n}(s)-m_n(s))(u_{k,n}(t)-m_n(t))|$$

$$
\begin{aligned}
&= |Eu_{0,n}(t)u_{0,n}(s) - m_n(t)m_n(s)| \\
&\le |E(X_{0,n}(X_k - X_k^*)I\{|X_0| \le sH^{-1}(d/n)\}I\{|X_k| \le tH^{-1}(d/n)\}| \\
&\quad + |E(X_0X_k^*I\{|X_0| \le sH^{-1}(d/n)\}I\{|X_k| \le tH^{-1}(d/n)\} - m_n(s)m_n(t)| \\
&\le A_{1,k,n} + A_{2,k,n} + A_{3,k,n}
\end{aligned}
$$

with

$$
\begin{aligned}
A_{1,k,n} &= E|X_0(X_k - X_k^*)I\{|X_0| \le sH^{-1}(d/n)\}I\{|X_k| \le tH^{-1}(d/n)\}|, \\
A_{2,k,n} &= E\big[|X_0X_k^*|I\{|X_0| \le sH^{-1}(d/n)\} \\
&\qquad \times \big|I\{|X_k| \le tH^{-1}(d/n)\} - I\{|X_k^*| \le tH^{-1}(d/n)\}\big|\,\big]
\end{aligned}
$$

and

$$
A_{3,k,n} = |E(X_0X_k^*)I\{|X_0| \le sH^{-1}(d/n)\}I\{|X_k^*| \le tH^{-1}(d/n)\} - m_n(s)m_n(t)|.
$$

Using (2.14) and (2.17) we conclude

$$
\begin{aligned}
A_{1,k,n} &\le |\rho|^k E|X_0||X_0 - c_1|I\{|X_0| \le sH^{-1}(d/n)\} \\
&\le C_2|\rho|^k (H^{-1}(d/n))^2 d/n
\end{aligned}
\tag{2.18}
$$

with some constant C_2. Next we note that

$$
\begin{aligned}
A_{2,k,n} &\le E\big[|X_0X_k^*|I\{|X_0| \le sH^{-1}(d/n)\} \\
&\qquad \times I\{tH^{-1}(d/n) - |\rho|^k|X_0| \le |X_k^*| \le tH^{-1}(d/n)\}\big] \\
&\quad + E\big[|X_0X_k^*|I\{|X_0| \le sH^{-1}(d/n)\} \\
&\qquad \times I\{tH^{-1}(d/n) \le |X_k^*| \le tH^{-1}(d/n) + |\rho|^k|X_0|\}\big] \\
&= A_{2,k,n}^{(1)} + A_{2,k,n}^{(2)}.
\end{aligned}
\tag{2.19}
$$

Using the independence of X_0 and X_k^* we get

$$
\begin{aligned}
A_{2,k,n}^{(1)} &\le E|X_0|I\{|X_0| \le sH^{-1}(d/n)\} \\
&\qquad \times E|X_k^*|I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \le |X_k^*| \le tH^{-1}(d/n)\}.
\end{aligned}
$$

By (2.16) we have that

$$
\begin{aligned}
&E|X_k^*|I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \le |X_k^*| \le tH^{-1}(d/n)\} \\
&\quad = -xP\{|X_k^*| > x\}\Big|_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} + \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} P\{|X_k^*| > x\}dx \\
&\quad \le \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} P\{|X_k^*| > x\}dx \\
&\quad \le C_3|\rho|^k H^{-1}(d/n)d/n,
\end{aligned}
\tag{2.20}
$$

where C_3 is a constant. Hence, on account of (2.9), (2.12) and (2.13) we obtain that with some constant C_4

$$A^{(1)}_{2,k,n} \le C_4 \rho^k (H^{-1}(d/n))^2 d/n$$

and similarly

$$A^{(2)}_{2,k,n} \le C_4 \rho^k (H^{-1}(d/n))^2 d/n,$$

resulting in

$$A_{2,k,n} \le C_5 \rho^k (H^{-1}(d/n))^2 d/n. \tag{2.21}$$

Using again the independence of X_0 and X^*_k we get

$$A_{3,k,n} = |m_n(s)||EX^*_k I\{|X^*_k| \le tH^{-1}(d/n)\} - m_n(t)|.$$

It is easy to see that

$$\begin{aligned} &EX^*_k I\{|X^*_k| \le tH^{-1}(d/n)\} \\ &\quad = EX^*_k I\{|X^*_k| \le tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\} \\ &\qquad + EX^*_k I\{|X^*_k| \le tH^{-1}(d/n)\} I\{|X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\} \end{aligned}$$

and by the independence of X_0 and X^*_k and (2.16) we have

$$\begin{aligned} |EX^*_k I\{|X^*_k| \le tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\}| &\le C_5 |m_n(t)| H(|\rho|^{-k/2} H^{-1}(d/n)) \\ &\le C_6 |m_n(t)| |\rho|^{k\alpha/2} d/n. \end{aligned}$$

Next we note that

$$\begin{aligned} &\big| E\big[X^*_k I\{|X^*_k| \le tH^{-1}(d/n), |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}\big] \\ &\qquad - E\big[(X^*_k + \rho^k (X_0 - c_1)) I\{|X^*_k + \rho^k (X_0 - c_1))| \le tH^{-1}(d/n), \\ &\qquad\qquad |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}\big]\big| \\ &\quad \le |\rho|^k E\big[|X_0 - c_1| I\{|X^*_k + \rho^k (X_0 - c_1))| \le tH^{-1}(d/n), \\ &\qquad\qquad |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}\big] \\ &\quad + E\big[|X^*_k||I\{|X^*_k| \le tH^{-1}(d/n), |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\} \\ &\qquad - I\{|X^*_k + \rho^k (X_0 - c_1))| \le tH^{-1}(d/n), |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}|\big] \\ &\quad \le |\rho|^k (|\rho|^{-k/2} H^{-1}(d/n) + |c_1|) \\ &\qquad + E|X^*_k| I\{(t - |\rho|^{k/2}) H^{-1}(d/n) - |c_1||\rho|^k \le |X^*_k| \le tH^{-1}(d/n)\} \\ &\qquad + E|X^*_k| I\{tH^{-1}(d/n) \le |X^*_k| \le (t + |\rho|^{-k/2}) H^{-1}(d/n) + |c_1||\rho|^k\} \\ &\quad \le C_7 (|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n) \end{aligned}$$

by (2.20). Similarly

$$\begin{aligned} &|EX_k I\{|X_k| \le tH^{-1}(d/n)\} - EX_k I\{|X_k| \le tH^{-1}(d/n), |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}| \\ &\quad \le C_8 (|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n). \end{aligned}$$

Hence

$$A_{3,k,n} \le C_9|\rho|^{\tau k}(H^{-1}(d/n))^2 d/n, \quad \text{where} \quad \tau = \min\{1,\alpha\}/2. \tag{2.22}$$

Putting together (2.18), (2.21) and (2.22) we get that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{A_n^2} \sum_{k=K}^{\lfloor nx\rfloor -1} |(\lfloor nx\rfloor - k)E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))| = 0. \tag{2.23}$$

The lemma now follows from Lemma 2.4 and (2.23). □

3. A weak convergence result

Define the two–parameter process

$$L_n(t,x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx\rfloor} (X_i I\{|X_i| \le tH^{-1}(d/n)\} - m_n(t)),$$

for $0 \le x \le 1, 1/2 \le t \le 3/2$. First we show the tightness of $L_n(t)$. The proof is based on a generalization of Lemma 6 in Berkes et al (2011b). We introduce

$$X_{i,1} = \max(X_i, 0), \qquad X_{i,2} = \min(X_i, 0)$$

and

$$m_{n,1}(t) = EX_{0,1}I\{|X_0| \le tH^{-1}(d/n)\}, \qquad m_{n,2}(t) = EX_{0,2}I\{|X_0| \le tH^{-1}(d/n)\}.$$

Similarly to $L_n(t,x)$, we define

$$L_{n,1}(t,x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx\rfloor} (X_{i,1} I\{|X_i| \le tH^{-1}(d/n)\} - m_{n,1}(t)),$$

and $L_{n,2}(t,x)$ is defined in a similar fashion. Clearly, if both $L_{n,1}$ and $L_{n,2}$ are tight, then $L_n(t,x)$ is tight as well. We prove only tightness of $L_{n,1}$, the same argument can be used in case of $L_{n,2}$. Let

$$g_n = \frac{1}{d^{1/2}\log\log n}.$$

Lemma 3.1. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then*

$$m_{n,1}(t) \text{ is a non-decreasing function on } [1/2, 3/2], \tag{3.1}$$

$$\frac{n}{A_n} \sup_{|t_2 - t_1| \le g_n} |m_{n,1}(t_2) - m_{n,1}(t_1)| \to 0, \quad n \to \infty, \tag{3.2}$$

$$E|L_{n,1}(t_2,x) - L_{n,1}(t_1,x)|^6 \le C_1|t_2 - t_1|^\tau, \quad \text{if } |t_2 - t_1| \ge g_n, \tag{3.3}$$

and

$$E|L_{n,1}(t,x_2) - L_{n,1}(t,x_1)|^6 \le C_1|x_2 - x_1|^\tau, \quad \text{if } |x_2 - x_1| \ge g_n, \tag{3.4}$$

with some $\tau > 2$ and constant C_1.

Proof. The definition of $m_{n,1}(t)$ implies immediately (3.1).

By the definition of $m_{n,1}(t)$ we have for all $1/2 \le t_1 \le t_2 \le 3/2$ that

$$
\begin{aligned}
0 \le m_{n,1}(t_2) - m_{n,1}(t_1) &= EX_{0,1}(I\{t_1 H^{-1}(d/n) < |X_0| \le t_2 H^{-1}(d/n)\}) \\
&\le \int_{t_1 H^{-1}(d/n)}^{t_2 H^{-1}(d/n)} x dH(x) \\
&\le C_2 \Big(|t_2 H^{-1}(d/n) H(t_2 H^{-1}(d/n)) - t_1 H^{-1}(d/n) H(t_1 H^{-1}(d/n))| \\
&\qquad + |t_2 - t_1| H^{-1}(d/n) H(t_1 H^{-1}(d/n)) \Big) \\
&\le C_3 |t_2 - t_1| \frac{d}{n} H^{-1}(d/n)
\end{aligned}
$$

on account of integration by parts and (2.3), establishing (3.2).

Next we introduce

$$
Y_i = \sum_{k=0}^{\lfloor K \log n \rfloor} \rho^k \varepsilon_{i-k} + c_1, \qquad Y_{i,1} = \max(Y_i, 0) \tag{3.5}
$$

and $\xi_i = \eta_i - E\eta_i$ with

$$
\eta_i = \eta_i(t_1, t_2) = Y_{i,1} I\{t_1 H^{-1}(d/n) < |Y_i| \le t_2 H^{-1}(d/n)\}.
$$

Since $E|\varepsilon_0|^{\alpha/2} < \infty$, using Markov's inequality we see that for every $\beta > 0$ there is a constant $K = K(\beta)$ such that

$$
\left| E(L_{n,1}(t_2, x) - L_{n,1}(t_1, x))^6 - \frac{1}{A_n^6} \sum_{1 \le i_1, \ldots, i_6 \le \lfloor nx \rfloor} E\xi_{i_1} \ldots \xi_{i_6} \right| \le C_5 n^{-\beta}. \tag{3.6}
$$

We note that by definition, $\{\xi_i\}$ is a stationary, $\lfloor K \log n \rfloor$–dependent sequence with zero mean. Let us divide the indices $i_1, \ldots, i_6$ into groups so that the difference between the indices within a group are less than $\lfloor K \log n \rfloor$ and between groups is larger than $\lfloor K \log n \rfloor$. Clearly $E\xi_{i_1} \ldots \xi_{i_6} = 0$, if there is at least one group containing a single element. So it suffices to consider the cases when all groups contain at least two elements. This allows the cases of one single group with 6 elements (D_1), two groups with 3+3 (D_2) or 4+2 (D_3) elements and finally 3 groups with 2 elements in each (D_4). If there is only one group, then via Hölder's inequality we have

$$
|E\xi_{i_1} \ldots \xi_{i_6}| \le E|\xi_0|^6 \le 2^6 (E|\eta_0|^6 + |E\eta_0|^6)
$$

Since the cardinality of D_1 is bounded by constant times $n(\log n)^5$ we conclude

$$\begin{aligned}
&\left|\frac{1}{A_n^6}\sum_{D_1} E\xi_{i_1}\dots\xi_{i_6}\right| \\
&\quad\le C_6\biggl(\frac{n(\log n)^5}{A_n^6}[EX_0^6 I\{t_1H^{-1}(d/n)\le |X_0|\le t_2H^{-1}(d/n)\} \\
&\qquad\qquad + (EX_0 I\{t_1H^{-1}(d/n)\le |X_0|\le t_2H^{-1}(d/n)\})^6] + n^{-\beta}\biggr).
\end{aligned}$$

Integration by parts and (2.3) yield

$$EX_0^6 I\{t_1H^{-1}(d/n)\le |X_0|\le t_2H^{-1}(d/n)\} \le C_7|t_2-t_1|\frac{d}{n}(H^{-1}(d/n))^6,$$

resulting in

$$\left|\frac{1}{A_n^6}\sum_{D_1} E\xi_{i_1}\dots\xi_{i_6}\right| \le C_8\left(\frac{(\log n)^5}{d^2}|t_2-t_1| + n^{-\beta}\right).$$

Using again the $\lfloor K\log n\rfloor$ dependence of $\{\xi_i\}$ and the fact that the cardinality of D_2 is constant times $n^2(\log n)^4$ we conclude via Hölder's inequality

$$\begin{aligned}
&\left|\frac{1}{A_n^6}\sum_{D_2} E\xi_{i_1}\dots\xi_{i_6}\right| \\
&= \left|\frac{1}{A_n^6}\sum_{D_2} E\xi_{i_1}\xi_{i_2}\xi_{i_3}E\xi_{i_4}\xi_{i_5}\xi_{i_6}\right| \\
&\le C_8\biggl(\frac{n^2(\log n)^4}{A_n^6}[EX_0^3 I\{t_1H^{-1}(d/n)\le |X_0|\le t_2H^{-1}(d/n)\} \\
&\qquad\qquad + (EX_0 I\{t_1H^{-1}(d/n)\le |X_0|\le t_2H^{-1}(d/n)\})^3]^2 + n^{-\beta}\biggr) \\
&\le C_9\left(\frac{(\log n)^4}{d}(t_2-t_1)^2 + n^{-\beta}\right).
\end{aligned}$$

Similar arguments give

$$\left|\frac{1}{A_n^6}\sum_{D_3} E\xi_{i_1}\dots\xi_{i_6}\right| \le C_{10}\left(\frac{(\log n)^4}{d}(t_2-t_1)^2 + n^{-\beta}\right).$$

Following the proof of Lemma 2.5 we obtain

$$\begin{aligned}
\left|\frac{1}{A_n^6}\sum_{D_4} E\xi_{i_1}\dots\xi_{i_6}\right| &\le C_{11}\left(\frac{1}{A_n^6}\left(n\sum_{i=0}^{\infty}\xi_0\xi_i\right)^3 + n^{-\beta}\right) \\
&\le C_{11}\left(|t_2-t_1|^3 + n^{-\beta}\right).
\end{aligned}$$

Putting together our estimates and using the choice of g_n we conclude for all $|t_2 - t_1| \geq g_n$

$$\begin{aligned}E(L_{n,1}(t_2,x) - L_{n,1}(t_2,x))^6 &\leq C_{12}\left(\frac{(\log n)^5}{d^2}|t_2 - t_1| + \frac{(\log n)^4}{d}|t_2 - t_1|^2 + |t_2 - t_1|^3 + n^{-\beta}\right)\\ &\leq C_{13}|t_2 - t_1|^\tau\end{aligned}$$

with any $2 < \tau \leq 3$ on account of assumption (1.15). Hence the proof of (3.3) is complete. The proof of (3.4) goes along the lines of the arguments used to establish (3.3) and therefore it is omitted. □

Lemma 3.2. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then* $L_n(t,x)$ *is tight in* $\mathcal{D}([1/2, 3/2] \times [0,1])$.

Proof. It follows from a minor modification of Lemma 6 in Berkes et al (2011b) that both $L_{n,1}$ and $L_{n,2}$ are tight. Since $L_n = L_{n,1} + L_{n,2}$, the result is proven. □

Next we consider the convergence of the finite dimensional distributions. It is based in the following lemma:

Lemma 3.3. *We assume that (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold. Let* $N = \lfloor (\log n)^\gamma \rfloor$ *with some* $\gamma > 0$. *Then*

$$\begin{aligned}E\left(\sum_{i=1}^{N}(X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}]\right)^4 \\ \leq C_{13}\left(N(\log N)^3(H^{-1}(d/n))^4\frac{d}{n} + N^2(H^{-1}(d/n))^4\left(\frac{d}{n}\right)^2\right)\end{aligned} \tag{3.7}$$

with some constant C_{13} *and*

$$\begin{aligned}\lim_{n\to\infty}\frac{Nn}{A_n^2}E\left(\sum_{k=1}^{N}(u_{k,n}(s) - m_n(s))\right)\left(\sum_{k=1}^{N}(u_{k,n}(t) - m_n(t))\right) \\ = \frac{\alpha}{2-\alpha}\left(s^{2-\alpha} + \sum_{k=1}^{\infty}\rho^k[(\min(s,t|\rho|^{-k})^{2-\alpha} + \min(t,s|\rho|^{-k})^{2-\alpha}]\right).\end{aligned} \tag{3.8}$$

Proof. We recall the definition of ξ_i from the proof of Lemma 3.1. For any $\beta > 0$, choosing K in the definition of Y_i in (3.5) we get that

$$E\left(\sum_{i=1}^{N}(X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}]\right)^4 \leq C_{14}\left(E\left(\sum_{i=1}^{N}\xi_i\right)^4 + n^{-\beta}\right).$$

We write

$$E\left(\sum_{i=1}^{N}\xi_i\right)^4 = \sum_{i_1,\ldots,i_4}^{N} E\xi_{i_1}\ldots\xi_{i_4}.$$

We note again that the $\{\xi_i\}$ is a stationary $K\log n$ dependent sequence with 0 mean. Let us divide the indices $i_1,\ldots,i_4$ into blocks so that the difference between the indices within a block is less than $K\log n$ and between blocks is larger than $K\log n$. Clearly $E\xi_{i_1}\ldots\xi_{i_4}=0$, if there is at least one block containing only a single element. So we need to consider the cases of one single block with 4 elements (D_1) and two blocks with 2+2 elements (D_2). The number of the elements in D_1 is not greater than constant times $N(\log N)^3$ and as we showed in the proof of Lemma 3.1

$$E\xi_0^4\le C_{14}\left((H^{-1}(d/n))^4\frac{d}{n}+n^{-\beta}\right),$$

assuming that K in (3.5) is sufficiently large. Hence

$$\left|\sum_{D_1}^{N}E\xi_{i_1}\ldots\xi_{i_4}\right|\le C_{15}\left(N(\log N)^3(H^{-1}(d/n))^4\frac{d}{n}+n^{-\beta}\right).$$

As in the proof of Lemma 3.1 we get that

$$\left|\sum_{D_2}^{N}E\xi_{i_1}\ldots\xi_{i_4}\right|\le C_{16}N^2\left(\sum_{i=0}|E\xi_0\xi_i|\right)^2$$

and

$$\sum_{i=0}^{\infty}|E\xi_0\xi_i|\le\left(C_{17}(H^{-1}(d/n))^2\frac{d}{n}+n^{-\beta}\right),$$

completing the proof of (3.7). The proof of (3.8) goes along the lines of the arguments used to establish Lemma 2.5. □

Lemma 3.4. *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then*

$$L_n(t,x)\ \longrightarrow\ \Gamma(t,x)\quad\textit{weakly in}\quad\mathcal{D}([1/2,3/2])\times[0,1]),$$

where $\Gamma(t,x)$ is a Gaussian process with $E\Gamma(t,x)=0$ and

$$E\Gamma(t,x)\Gamma(s,y)$$
$$=\min(x,y)\frac{\alpha}{2-\alpha}\left((\min(s,t))^{2-\alpha}+\sum_{k=1}^{\infty}\rho^k[(\min(s,t|\rho|^{-k})^{2-\alpha}+\min(t,s|\rho|^{-k})^{2-\alpha}]\right).$$

Proof. By Lemma 3.2, the process $L_n(t,x)$ is tight, so we need only to show the convergence of the finite dimensional distributions. By the Cramér–Wold device it is sufficient to prove the asymptotic normality of

$$Q_n=\sum_{j=1}^{J}\sum_{\ell=0}^{L}\mu_{j,\ell}(L_n(t_j,x_{\ell+1})-L_n(t_j,x_\ell))$$

for all J, L, real coefficients $\mu_{j,\ell}, 1/2 \le t_j \le 3/2, 1 \le j \le J$, and $0 = x_0 < x_1 < \ldots < x_L < x_{L+1} = 1$. We recall the definition of X_k^* from the proof of Lemma 2.5 (cf. (2.17)) and define

$$\bar{L}_n(t,x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_k^* I\{|X_k^*| \le tH^{-1}(d/n)\} - EX_k^* I\{|X_k^*| \le tH^{-1}(d/n)\}).$$

Choosing K large enough in the definition of X_k^*, we get from the arguments used in the proof of Lemmas 2.5, 3.1 and 3.3 that

$$E(L_n(t,x) - \bar{L}_n(t,x))^2 \to 0.$$

So we need to establish only the asymptotic normality of

$$\bar{Q}_n = \sum_{\ell=0}^{L} \sum_{j=1}^{J} \mu_{j,\ell} (\bar{L}_n(t_j, x_{\ell+1}) - \bar{L}_n(t_j, x_\ell)).$$

Let

$$z_{k,\ell} = \sum_{j=1}^{J} \mu_{j,\ell} (X_k^* I\{|X_k^*| \le t_j H^{-1}(d/n)\} - E[X_k^* I\{|X_k^*| \le t_j H^{-1}(d/n)\}]).$$

Since for all ℓ

$$E\left(\frac{1}{A_n} \sum_{k=1}^{\lfloor K \log n \rfloor} z_{k,\ell} \right)^2 \to 0,$$

by stationarity and the $\lfloor K \log n \rfloor$–dependence of $z_{k,\ell}$ for any ℓ we get that the variables

$$\frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor + 1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell}, \quad 1 \le \ell \le L \ \text{ are asymprotically independent.}$$

By stationarity we have

$$\frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor + 1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell} \stackrel{\mathcal{D}}{=} \frac{1}{A_n} \sum_{k=1}^{\lfloor nx_{\ell+1} \rfloor - \rfloor nx_\ell \rfloor} z_{k,\ell}.$$

Let us divide the integers of $[1, \lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor]$ into consecutive blocks $R_1, V_1, R_2, V_2, \ldots, R_s, V_s$ such that for $1 \le i \le s-1$, R_i contains $\lfloor (\log n)^\gamma \rfloor$ integers, V_i contains $\lfloor K \log n \rfloor$ integers, the last two blocks might contain less elements. Let

$$\zeta_{i,1} = \sum_{k \in R_i} z_{k,\ell} \quad \text{and} \quad \zeta_{i,2} = \sum_{k \in V_i} z_{k,\ell}.$$

Due to the $\lfloor K \log n \rfloor$ dependence and stationarity, the variables $\zeta_{i,2}, 1 \le i < s$ are independent and identically distributed and the proof of Lemma 2.5 shows that

$$E\left(\frac{1}{A_n} \sum_{i=1}^{s} \zeta_{i,2} \right)^2 \to 0.$$

Using Lemma 3.3 we get that

$$E\zeta_{i,1}^2 \geq C_{18}(\log n)^\gamma (H^{-1}(d/n))^2 d/n$$

and

$$E\zeta_{i,1}^2 \leq C_{19}\left((\log n)^\gamma (\log\log n)^3 (H^{-1}(d/n))^4 \frac{d}{n} + (\log n)^{2\gamma}(H^{-1}(d/n))^4 \left(\frac{d}{n}\right)^2\right).$$

Since s is proportional to $n/(\log n)^\gamma$, a simple calculation yields

$$\frac{\sum_{i=1}^{s} E\zeta_{i,1}^4}{\left(\sum_{i=1}^{s} E\zeta_{i,1}^2\right)^2} \to 0,$$

Thus the central limit theorem with Lyapunov's remainder term (cf. Petrov 1995, p. 154) implies the asymptotic normality of
$\sum_{1\leq k\leq \lfloor nx_{\ell+1}\rfloor - \lfloor nx_\ell\rfloor} z_{k,\ell}$. This completes the proof of Lemma 3.4. □

4. Proof of Theorems 1.1 and 1.2

We need the weak law of large numbers for $\eta_{d,n}$.

Lemma 4.1. *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have*

$$\frac{\eta_{d,n}}{H^{-1}(d/n)} \xrightarrow{P} 1$$

Proof. Using Gorodetskii (1977) and Withers (1981) we get that X_k is a strongly mixing stationary sequence with mixing rate $\alpha(k) \leq C_1 \exp(-\lambda k)$ for some $C_1 > 0$ and $\lambda > 0$. Fix $1/2 < t < 2$ and let $T_k = I\{|X_k| \geq tH^{-1}(d/n)\}, 1 \leq k \leq n$. Clearly, $ET_k = P\{|X_k| \geq tH^{-1}(d/n)\} = H(tH^{-1}(d/n))$ and due to the the regular variation of H, $ET_k/(d/n) \to t^{-\alpha}$, as $n \to \infty$. On the other hand, by the correlation inequality of Davydov (1968) we get for any $p > 2$ that

$$\begin{aligned}|ET_0T_k - ET_0ET_k| &\leq (\alpha(k))^{(p-1)/p}(ET_0^p)^{1/p}(ET_k^p)^{1/p}\\ &\leq C_1\exp(-\lambda k(p-1)/p)(ET_0^p)^{2/p}\\ &= C_1\exp(-\lambda k(p-1)/p)(ET_0)^{2/p}\\ &\leq C_2\exp(-\lambda k(p-1)/p)(d/n)^{2/p}.\end{aligned}$$

Hence setting $\bar{T}_k = T_k - ET_k$ we conclude that

$$E\left(\sum_{k=1}^{n} \bar{T}_k\right)^2 = nE\bar{T}_0^2 + 2\sum_{k=1}^{n-1}(n-k)E\bar{T}_0\bar{T}_k$$

$$
\begin{aligned}
&\leq n\left(E\bar{T}_k^2 + 2\sum_{k=1}^{n-1}|E\bar{T}_0\bar{T}_k|\right)\\
&\leq n\left(ET_0^2 + C_3\sum_{k=1}^{n-1}\exp(-\lambda k(p-1)/p)(d/n)^{2/p}\right)\\
&\leq n\left(ET_0 + C_5(d/n)^{2/p}\right)\\
&\leq n(d/n)^{2/p}.
\end{aligned}
$$

Thus by Markov's inequality we have that

$$P\left\{\sum_{k=1}^{n}\bar{T}_k \geq d^{2/p}\right\} \leq C_6 n^{(p-2)/p}/d^{2/p} \to 0,$$

provided that $d/n^{(2-p)/p} \to 0$. Since $d \geq n^{\delta}$, choosing p near 2, it follows that

$$\sum_{k=1}^{n} T_k = t^{-\alpha}d(1+o_P(1)) + o_P(d^{2/p}) = t^{-\alpha}d(1+o_P(1)).$$

In other words,

$$\frac{1}{d}\#\{k \leq n:\ |X_k| \geq tH^{-1}(d/n)\} \xrightarrow{P} t^{-\alpha}, \quad \text{as} \quad n \to \infty.$$

This shows that

$$\lim_{n\to\infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 1 \quad \text{for} \quad t < 1$$

and

$$\lim_{n\to\infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 0 \quad \text{for} \quad t > 1,$$

completing the proof of Lemma 4.1. □

Proof of Theorem 1.1. We note that $\Gamma(t,x)$ is a continuous process. Hence combining Lemmas 3.4 and 4.1 we conclude

$$L_n(\eta_{d,n}/H^{-1}(d/n), x) \quad \xrightarrow{\mathcal{D}[0,1]} \quad \Gamma(1,x).$$

It is easy to see that

$$\{\Gamma(1,x), 0 \leq x \leq 1\} \stackrel{\mathcal{D}}{=} \left\{\left(\frac{\alpha}{2-\alpha}\frac{1+\rho}{1-\rho}\right)^{1/2} W(x), 0 \leq x \leq 1\right\},$$

where $W(x)$ is a Wiener process, which completes the proof. □

Proof of Theorem 1.2. Since

$$\frac{1}{A_n}T_{n,d}(x) = L_n(\eta_{d,n}/H^{-1}(d/n), x) - \frac{\lfloor nx \rfloor}{n}L_n(\eta_{d,n}/H^{-1}(d/n), 1),$$

Theorem 1.1 yields

$$\frac{1}{A_n}T_{n,d}(x) \quad \stackrel{\mathcal{D}[0,1]}{\longrightarrow} \quad \left(\frac{\alpha}{2-\alpha}\frac{1+\rho}{1-\rho}\right)^{1/2}(W(x)-xW(1)).$$

By definition, $B(x) = W(x) - xW(1), 0 \leq x \leq 1$ is a Brownian bridge, so the proof of Theorem 1.2 is complete. □

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