On trigonometric sums with random frequencies

Alina Bazarova[1], István Berkes[2] and Marko Raseta[3]

Abstract

We prove that if I_k are disjoint blocks of positive integers and n_k are independent random variables with uniform distribution on I_k, then

$$N^{-1/2}\sum_{k=1}^{N}(\sin 2\pi n_k x - \mathbb{E}(\sin 2\pi n_k x))$$

has, with probability 1, a mixed Gaussian limit distribution relative to the interval $(0,1)$ equipped with Lebesgue measure. We also investigate the case when n_k have continuous uniform distribution on disjoint intervals I_k on the positive axis.

1 Introduction

Salem and Zygmund [7] proved that if (n_k) is a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1 \qquad (k = 1, 2, \ldots) \tag{1.1}$$

then the sequence $\sin 2\pi n_k x$, $k \ge 1$ obeys the central limit theorem, i.e.

$$N^{-1/2}\sum_{k=1}^{N}\sin 2\pi n_k x \xrightarrow{d} N(0, 1/2) \tag{1.2}$$

with respect the the probability space $(0,1)$ equipped with Borel sets and Lebesgue measure. Here the exponential growth condition (1.1) can be weakened, but as Erdős

1) Warwick Biology Centre, University of Warwick, GB. Email: a.bazarova@warwick.ac.uk.
2) Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13–15, 1053 Budapest, Hungary. Email: berkes.istvan@renyi.mta.hu. Research supported by Austrian Science Fund (FWF) Grant P24302–N18 and NKFIH Grant K 108615.
3) Department of Mathematics, Keele University, GB. Email: m.raseta@keele.ac.uk.

[3] showed, there exists a sequence (n_k) growing faster than $e^{\sqrt{k}}$ such that the CLT (1.2) fails. On the other hand, using random constructions one can find slowly growing sequences (n_k) satisfying (1.2). Salem and Zygmund [8] proved that if $\xi_1, \xi_2, \ldots$ are independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking the values 0 and 1 with probability $1/2 - 1/2$ and (n_k) denotes the set of indices j such that $\xi_j = 1$, then with $\mathbb{P}$-probability 1, the CLT (1.2) holds. For this sequence (n_k) we have $n_k \sim 2k$ and by the theorem of "pure heads" we have $n_{k+1} - n_k = O(\log k)$. Berkes [1] showed that if $\mathbb{N} = \cup_{k=1}^{\infty} I_k$ where $I_1, I_2, \ldots$ are disjoint intervals of positive integers such that $|I_k| \to \infty$, and $n_1, n_2, \ldots$ are independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that n_k is uniformly distributed on I_k, then with $\mathbb{P}$-probability 1, $\sin 2\pi n_k x$ satisfies the CLT (1.2). Thus, given any positive sequence $\omega_k \to \infty$, there exists an increasing sequence (n_k) of positive integers such that $n_{k+1} - n_k = O(\omega_k)$ and $\sin 2\pi n_k x$ satisfies (1.2). In [1] the question was raised if the CLT (1.2) can hold for any sequence (n_k) with $n_{k+1} - n_k = O(1)$. Bobkov and Götze [2] showed that the answer to this question is negative, and in particular, if in the construction in [1] we choose $|I_k| = d$ for $k = 1, 2, \ldots$, then with probability 1, the limit distribution of $N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x$ is mixed normal. On the other hand, Fukuyama [4] showed, using another type of random construction, that for any $0 < \sigma^2 < 1/2$ there exists a sequence (n_k) of integers with bounded gaps $n_{k+1} - n_k$ such that (1.2) holds with a limiting normal distribution with variance σ^2. The purpose of the present paper is to return to the random models in [1], [2] and investigate the case of constant block sizes $|I_k| = d$, allowing arbitrary gaps between the blocks. We will prove the following result.

Theorem 1. *Let $I_1, I_2, \ldots$ be disjoint blocks of consecutive positive integers with size d and let $n_1, n_2, \ldots$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that n_k is uniformly distributed over I_k. Let $\lambda_k(x) = \mathbb{E}(\sin 2\pi n_k x)$. Then $\mathbb{P}$-almost surely*

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\sin 2\pi n_k x - \lambda_k(x)) \xrightarrow{d} N(0, g) \tag{1.3}$$

over the probability space $((0,1), \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra in $(0,1)$, λ is the Lebesgue measure,

$$g(x) = \frac{1}{2}\left(1 - \frac{\sin^2 d\pi x}{d^2 \sin^2 \pi x}\right) \tag{1.4}$$

and $N(0, g)$ denotes the distribution with characteristic function $\int_0^1 e^{-g(x)t^2/2} dx$.

Here $g \geq 0$ and $N(0, g)$ is the distribution of $\sqrt{g}\zeta$, where ζ is a standard normal random variable on $(0,1)$, independent of g. Clearly, $N(0, g)$ is a variance mixture of zero mean Gaussian distributions.

Note that $\sum_{k=1}^{N} \lambda_k(x) = \mathbb{E}(\sum_{k=1}^{N} \sin 2\pi n_k x)$ is the averaged version of $\sum_{k=1}^{N} \sin 2\pi n_k x$, a nonrandom trigonometric sum and Theorem 1 states that the fluctuations of the random trigonometric sum $\sum_{k=1}^N \sin 2\pi n_k x$ around its nonrandom average part always have a mixed normal limit distribution. If $\cup_{k=1}^\infty |I_k| = \mathbb{N}$, i.e. there are no gaps between the blocks I_k, then $\sum_{k=1}^n \lambda_k(x) = O(1)$ for any fixed x and thus (1.3) holds without the $\lambda_k(x)$, yielding the result of Bobkov and Götze [2]. Letting Δ_k denote the number of integers between I_k and I_{k+1} (the "gaps"), we will see that the CLT (1.3) also holds with $\lambda_k(x) = 0$ if Δ_k is nondecreasing and $\Delta_k = O(k^\gamma)$ for some $\gamma < 1/4$. If Δ_k grows at least exponentially, then so does the sequence (A_k), where A_k denotes the smallest integer of I_k. Now

$$\lambda_k(x) = \frac{\sin d\pi x}{d \sin \pi x} \sin 2\pi(A_k + d/2 - 1/2)x \tag{1.5}$$

and from the CLT of Salem and Zygmund [7] it follows that the limit distribution of $N^{-1/2} \sum_{k=1}^N \lambda_k(x)$ is $N(0, g^*)$, where

$$g^*(x) = \frac{\sin^2 d\pi x}{2d^2 \sin^2 \pi x}. \tag{1.6}$$

By Theorem 1, the limit distribution of $N^{-1/2} \sum_{k=1}^N (\sin 2\pi n_k x - \lambda_k(x))$ is $N(0, g)$ with g in (1.4) and the convolution of these two mixed Gaussian laws is $N(0, 1/2)$, which is exactly the limit distribution of $N^{-1/2} \sum_{k=1}^N \sin 2\pi n_k x$ by the theorem of Salem and Zygmund, since (n_k) grows exponentially. Thus the pure Gaussian limit distribution of $N^{-1/2} \sum_{k=1}^N \sin 2\pi n_k x$ is obtained as the combination of two mixed Gaussian distributions $N(0, g)$ with g in (1.4) and $N(0, g^*)$ with g^* in (1.6).

It is worth noting that for any fixed $x \in (0, 1)$, $\sin 2\pi n_k x - \lambda_k(x)$ are independent, uniformly bounded mean zero random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and

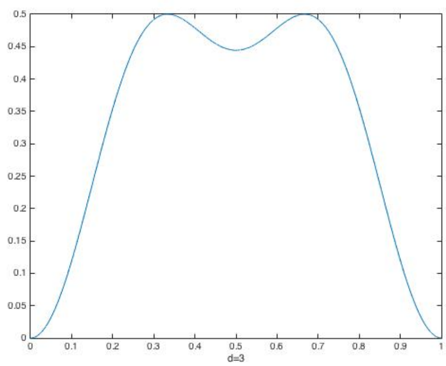
$$\mathbb{E}(\sin 2\pi n_k x - \lambda_k(x))^2 = \mathbb{E}(\sin^2 2\pi n_k x) - \lambda_k^2(x)$$
$$= \frac{1}{d} \sum_{j \in I_k} \sin^2 2\pi j x - \left(\frac{1}{d} \sum_{j \in I_k} \sin 2\pi j x \right)^2 = g(x)$$

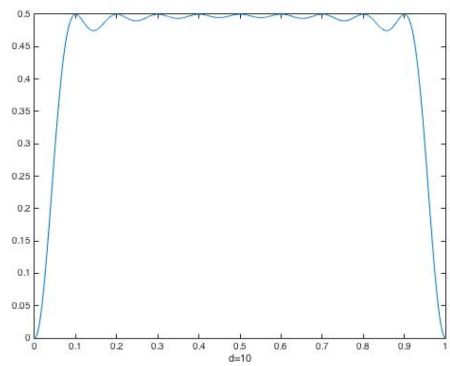
by elementary calculations. Thus by the law of the iterated logarithm we have for any fixed $x \in (0, 1)$ with $\mathbb{P}$-probability 1

$$\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^N (\sin 2\pi n_k x - \lambda_k(x)) = \sqrt{g(x)}. \tag{1.7}$$

By Fubini's theorem, with $\mathbb{P}$-probability 1 relation (1.7) holds for almost every $x \in (0, 1)$ with respect to Lebesgue measure, yielding the LIL corresponding to (1.3). Actually, the previous argument also shows that for any fixed $x \in (0, 1)$ we have (1.3) over the probability space $(\Omega, \mathbb{A}, \mathbb{P})$, with $N(0, g)$ replaced by $N(0, g(x))$. However, Fubini's theorem does not work for distributional results and thus we cannot interchange the role of $x \in (0, 1)$ and $\omega \in \Omega$ and we will need an elaborate argument in Section 2 to prove Theorem 1.

Formula (1.4) shows that for any $0 < x < 1$ we have $\lim_{x \to \infty} g(x) = 1/2$ and thus for large d the sequence $\sin 2\pi n_k x - \lambda_k(x)$ nearly satisfies the ordinary CLT and LIL with limit distribution $N(0,1/2)$ and limsup $= 1/2$, just as lacunary trigonometric series with exponential gaps. Formally, this is not surprising since for large d the expected gaps $\mathbb{E}(n_{k+1} - n_k)$ in our sequence are large. As the pictures of g for $d = 3$ and $d = 10$ below show, however, the near CLT and LIL actually hold for relatively small values of d such as $d = 10$. Thus the reason of the near CLT and LIL is not solely large gaps in the the sequence (n_k) but the random fluctuations of the sequence (n_k) as well.





The analogue of Theorem 1 is valid also in the case when $n_1, n_2, \ldots$ have continuous uniform distribution over the intervals $I_1, I_2, \ldots$. To formulate the result, define the probability measure μ on the Borel sets of $\mathbb{R}$ by

$$\mu(A) = \frac{1}{\pi} \int_A \left(\frac{\sin x}{x} \right)^2 dx, \qquad A \subset \mathbb{R}.$$

Theorem 2. *Let $n_1, n_2, \ldots$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that n_k has continuous uniform distribution on the interval $[A_k, A_k + B]$, where $A_{k+1} - A_k \geq B + 2$, $k = 1, 2, \ldots$. Let $\lambda_k(x) = \mathbb{E}(\sin n_k x)$. Then $\mathbb{P}$-almost surely*

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\sin n_k x - \lambda_k(x)) \xrightarrow{d} F \tag{1.8}$$

with respect to the probability space $(\mathbb{R}, \mathcal{B}, \mu)$, where the characteristic function of F is

$$\phi(\lambda) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\lambda^2}{4} \left(1 - \frac{4 \sin^2(Bx/2)}{B^2 x^2} \right) \right) d\mu(x). \tag{1.9}$$

2 Proofs

We will give the proof of Theorem 2, where the calculations are slightly simpler. Let

$$\varphi_k(x) = \sin n_k x - \mathbb{E}(\sin n_k x)$$

and

$$T_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(x).$$

By $A_{k+1} - A_k \geq B + 2$ and the fact that

$$\int_{-\infty}^{+\infty} \cos \alpha x \left(\frac{\sin x}{x} \right)^2 dx = 0 \quad \text{for} \quad |\alpha| > 2 \tag{2.10}$$

(see e.g. Hartman [5]) it follows that for every fixed $\omega \in \Omega$ the functions φ_k are orthogonal over $L^2_\mu(\mathbb{R})$ and thus elementary algebra shows that the $L^2_\mu(\mathbb{R})$ norm of $|T_M - T_{N^3}|$ is at most $C/\sqrt{N}$ for $N^3 \leq M \leq (N+1)^3$ with an absolute constant C. Hence to prove (1.8) it suffices to show that $T_{N^3} \xrightarrow{d} F$ $\mathbb{P}$-a.s.

A simple calculation shows that

$$\begin{aligned} \lambda_k(x) = \mathbb{E}(\sin n_k x) &= \frac{1}{B} \int_{A_k}^{A_k+B} \sin txdt = \frac{1}{Bx}(\cos A_k x - \cos(A_k + B)x) \\ &= \frac{2\sin(Bx/2)}{Bx} \sin\left(A_k + B/2\right)x \end{aligned} \tag{2.11}$$

and

$$\mathbb{E}(\cos 2n_k x) = \frac{1}{B} \int_{A_k}^{A_k+B} \cos 2txdt = \frac{\sin Bx}{Bx} \cos(2A_k + B)x.$$

Thus

$$\begin{aligned} \mathbb{E}\varphi_k^2(x) &= \mathbb{E}(\sin^2 n_k x) - \lambda_k^2(x) = \frac{1}{2}(1 - \mathbb{E}(\cos 2n_k x)) - \lambda_k^2(x) \\ &= \frac{1}{2} - \frac{\sin Bx}{2Bx} \cos(2A_k + B)x - \frac{4\sin^2(Bx/2)}{B^2x^2} \sin^2(A_k + B/2)x \\ &= \left(\frac{1}{2} - \frac{2\sin^2(Bx/2)}{B^2x^2} \right) + \left(\frac{2\sin^2(Bx/2)}{B^2x^2} - \frac{\sin Bx}{2Bx} - \right) \cos(2A_k + B)x. \end{aligned}$$

From (2.10), $A_{k+1} - A_k \geq B + 2$ and elementary trigonometric identities it follows that the functions $\cos(2A_k + B)x$ are orthogonal in $L^2_\mu(\mathbb{R})$ and thus the Rademacher-Menshov convergence theorem implies that $\sum_{k=1}^\infty k^{-1} \cos(2A_k + B)x$ converges μ-almost everywhere. Consequently, the Kronecker lemma implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \cos(2A_k + B)x = 0 \qquad \mu - \text{a.e.}$$

and thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\varphi_k^2(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right) \qquad \mu - \text{a.e.}$$

Since for fixed x $\varphi_k^2(x) - \mathbb{E}\varphi_k^2(x)$, $k = 1, 2, \ldots$ are independent, uniformly bounded, zero mean random variables, the strong law of large numbers yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (\varphi_k^2(x) - \mathbb{E}\varphi_k^2(x)) = 0 \qquad \mathbb{P} - \text{a.s.}$$

and thus we conclude that for μ-a.e. x we have $\mathbb{P}$-almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \varphi_k^2(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right). \tag{2.12}$$

By Fubini's theorem, $\mathbb{P}$-almost surely the last relation holds for μ-almost all $x \in \mathbb{R}$. Fix $\lambda \in \mathbb{R}$. Using $|\varphi_k(x)| \le 2$ and

$$\exp(z) = (1+z)\exp\left(\frac{z^2}{2} + o(z^2)\right) \qquad z \to 0$$

we get

$$\exp\left(\frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) = \left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\exp\left(-\frac{\lambda^2\varphi_k^2(x)}{2N} + o\left(\frac{\lambda^2\varphi_k^2(x)}{N}\right)\right)$$

as $N \to \infty$, uniformly in x and the implicit variable $\omega \in \Omega$. Thus the characteristic function

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{\infty} \exp\left(\frac{i\lambda}{\sqrt{N}}\sum_{k=1}^{N}\varphi_k(x)\right) d\mu(x) = \int_{-\infty}^{\infty} \exp\left(\frac{i\lambda}{\sqrt{N}}\sum_{k=1}^{N}\varphi_k(x,\omega)\right) d\mu(x)$$

of T_N with respect to the probability space $(\mathbb{R}, \mathcal{B}, \mu)$ can be written as

$$\begin{aligned}\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} &\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) \\ &\times \exp\left(-(1+o(1))\frac{\lambda^2}{2N}\sum_{k=1}^{N}{\varphi_k}^2(x)\right)\frac{1}{\pi}\left(\frac{\sin x}{x}\right)^2 dx.\end{aligned}$$

For simplicity let

$$\hat{g}(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right).$$

Using $1 + x \le e^x$ and $|\varphi_k(x)| \le 2$ we get

$$\begin{aligned}\left|\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\right| &= \prod_{k=1}^{N}\left(1 + \frac{\lambda^2}{N}{\varphi_k}^2(x)\right)^{1/2} \\ &\le \exp\left(\frac{\lambda^2}{2N}\sum_{k=1}^{N}{\varphi_k}^2(x)\right) \le e^{2\lambda^2}\end{aligned} \tag{2.13}$$

and thus the dominated convergence theorem and (2.12) imply $\mathbb{P}$-almost surely

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\exp\left(-\lambda^2\hat{g}(x)/2\right)\frac{1}{\pi}\left(\frac{\sin x}{x}\right)^2 dx + o(1).$$

Since the characteristic function $\phi(\lambda)$ of F in (1.8) is given by (1.9), to prove that $T_{N^3} \xrightarrow{d} F$ $\mathbb{P}$-a.s., it remains to show that letting

$$\Gamma_N = \int_{-\infty}^{+\infty} \left[\prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \exp\left(-\lambda^2 g(x)/2\right) \frac{1}{\pi} \left(\frac{\sin x}{x} \right)^2 dx,$$

we have

$$\Gamma_{N^3} \xrightarrow{\mathbb{P}\text{-a.s.}} 0.$$

Clearly

$$\mathbb{E}|\Gamma_N|^2 = \mathbb{E} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \left[\prod_{k=1}^{N} \left(1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) - 1 \right]$$
$$\times \exp\left(-\lambda^2 g(x)/2\right) \exp\left(-\lambda^2 g(y)/2\right) d\mu(x) d\mu(y). \tag{2.14}$$

Now using the independence of the φ_k and $\mathbb{E}\varphi_k(x) = \mathbb{E}\varphi_k(y) = 0$ we get

$$\begin{aligned}
&\mathbb{E}\left[\prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \left[\prod_{k=1}^{N} \left(1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) - 1 \right] \\
&= \mathbb{E}\left[\prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) \left(1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) \right] - 1 \\
&= \mathbb{E}\left[\prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) + \frac{\lambda^2}{N} \varphi_k(x)\varphi_k(y) \right) \right] - 1 \\
&= \prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1,
\end{aligned}$$

where $\Psi_k(x,y) = \mathbb{E}\varphi_k(x)\varphi_k(y)$. Thus interchanging the expectation with the double integral in (2.14) we get

$$\begin{aligned}
\mathbb{E}|\Gamma_N|^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1 \right] \times \\
&\quad \times \exp\left(-\lambda^2 g(x)/2 - \lambda^2 g(y)/2\right) d\mu(x) d\mu(y) \\
&\le \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1 \right| d\mu(x) d\mu(y).
\end{aligned}$$

Using $|\Psi_k(x,y)| \le 4$ and $|\log(1+x) - x| \le Cx^2$ for all $|x| \le 1$ and some constant $C > 0$, one deduces for all sufficiently large N,

$$\left| \log \prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - \sum_{k=1}^{N} \frac{\lambda^2}{N} \Psi_k(x,y) \right| \le \frac{16C\lambda^4}{N}.$$

Thus letting

$$G_N(x,y) := \sum_{k=1}^{N} \frac{\lambda^2}{N} \Psi_k(x,y)$$

we get, using $G_N(x,y) \le 4\lambda^2$, that

$$\prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N}\Psi_k(x,y)\right) = \exp\left\{G_N(x,y) + O(\lambda^4/N)\right\} = 1 + O(|G_N(x,y)|) + O(1/N).$$

Thus

$$\mathbb{E}|\Gamma_N|^2 \le C_1 \left(\frac{1}{N} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |G_N(x,y)|\, d\mu(x) d\mu(y) \right) \tag{2.15}$$

for some constant C_1. In view of $A_{k+1} - A_k \ge B + 2$ and (2.10), for any $\lambda_1 \in [A_k, A_k + B]$, $\lambda_2 \in [A_l, A_l + B]$, $k \ne l$, $\sin \lambda_1 x$ and $\sin \lambda_2 x$ are orthogonal in $L^2_\mu(\mathbb{R})$, which implies that φ_k and φ_ℓ are also orthogonal in $L^2_\mu(\mathbb{R})$. Since $\Psi_k(x,y)\Psi_l(x,y) = \mathbb{E}\varphi_k(x)\varphi_l(x)\varphi_k(y)\varphi_l(y)$, it follows that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_k(x,y)\Psi_l(x,y) d\mu(x) d\mu(y) = 0 \qquad \text{for } k \ne l$$

and thus by the Cauchy-Schwarz inequality the last integral in (2.15) is $O(N^{-1/2})$. Hence $\mathbb{E}|\Gamma_N|^2 = O(N^{-1/2})$ and thus $\sum_{N\in\mathbb{N}} \mathbb{E}|\Gamma_{N^3}|^2 < \infty$, implying $\sum_{N\in\mathbb{N}} |\Gamma_{N^3}|^2 < \infty$ and $\Gamma_{N^3} \to 0$ $\mathbb{P}$-a.s., completing the proof of (1.8).

In conclusion we prove the claim made after Theorem 1, namely that if the size of the gaps Δ_k between the blocks I_k is nondecreasing and satisfies

$$\Delta_k = O(k^\gamma), \quad \gamma < 1/4 \tag{2.16}$$

then

$$N^{-1/2} \sum_{k=1}^{N} \lambda_k(x) \longrightarrow 0 \qquad \text{a.s.}$$

and thus (1.3) holds with $\lambda_k(x) = 0$. Since we proved our main limit theorem in the continuous case of Theorem 2, we prove our claim also in the context of Theorem 2 in which case we also assume that the intervals $[A_k, A_k + B]$ have integer endpoints. In view of (2.11) it suffices to show that

$$N^{-1/2} \sum_{k=1}^{N} e^{iA_k x} \longrightarrow 0 \qquad \text{a.s.} \tag{2.17}$$

and here nothing changes if we replace x by $2\pi x$. In the case of constant Δ_k we have $A_k = Dk + D^*$ for some constants $D > 0$ and D^* and (2.17) is obvious by an explicit computation of the sum. Thus we can assume $\Delta_k \uparrow \infty$, and then also $A_{k+1} - A_k \uparrow \infty$. Recalling that the A_k are integers, let us break the sum $\sum_{k=1}^N e^{2\pi i A_k x}$ into subsums

$$Z_{N,r} = \sum_{k \le N,\, A_{k+1}-A_k = r} e^{2\pi i A_k x}, \qquad r = 1, 2, \dots . \tag{2.18}$$

Clearly $Z_{N,r}$ consists of M_r consecutive terms of $\sum_{k=1}^N e^{2\pi i A_k x}$ for some $M_r \ge 0$ and thus in the case $M_r \ge 1$ we have for some integer $P_r \ge 0$,

$$|Z_{N,r}| = \left| \sum_{j=0}^{M_r - 1} e^{2\pi i (P_r + jr)x} \right| = \left| \sum_{j=0}^{M_r - 1} e^{2\pi i j r x} \right| \le \frac{1}{|e^{2\pi i r x} - 1|} \le \frac{C}{\langle rx \rangle},$$

except when rx is an integer, where C is an absolute constant and $\langle t \rangle$ denotes the distance of t from the nearest integer. From a well known result in Diophantine approximation theory (see e.g. Kuipers and Niederreiter [6], Definition 3.3. on p. 121 and Exercise 3.5 on page 130), for every $\varepsilon > 0$ and almost all x in the sense of Lebesgue measure we have $\langle nx \rangle \ge cn^{-(1+\varepsilon)}$ for some constant $c = c(x) > 0$ and all $n \ge 1$. This shows that $Z_{N,r} = O(r^{1+\varepsilon})$ a.e. and since by (2.16) the largest r actually occurring in breaking $\sum_{k=1}^N e^{2\pi i A_k x}$ into a sum of $Z_{N,r}$'s is at most $C_1 N^\gamma$, we have

$$\left| \sum_{k=1}^N e^{2\pi i A_k x} \right| \le C_2 \sum_{r \le C_1 N^\gamma} r^{1+\varepsilon} = o(\sqrt{N}) \qquad \text{a.e.}$$

by $\gamma < 1/4$, upon choosing ε small enough.

References

[1] I. Berkes. A central limit theorem for trigonometric series with small gaps. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **47** (1979), 157–161.

[2] S. Bobkov and F. Götze, Concentration inequalities and limit theorems for randomized sums. *Probab. Theory Related Fields* **137** (2007), 49–81.

[3] P. Erdős, On trigonometric sums with gaps. *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **7** (1962), 37–42.

[4] K. Fukuyama. A central limit theorem for trigonometric series with bounded gaps. *Prob. Theory Rel. Fields* **149** (2011), 139–148.

[5] P. Hartman. The divergence of non-harmonic gap series, Duke Math. J. 9 (1942), 404–405.

[6] L. Kuipers and H. Niederreiter. Uniform distribution of sequences. Wiley, New York.

[7] R. Salem and A. Zygmund. On lacunary trigonometric series, *Proc. Nat. Acad. Sci. USA* **33** (1947), 333–338.

[8] R. Salem and A. Zygmund, Trigonometric series whose terms have random signs. *Acta Math.* **91** (1954), 245–301.