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# Boundary integral equations for optimal control problems with partial Dirichlet control

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**Abstract**

In this paper we study an optimal control problem where the Dirichlet control is considered on a part $\Gamma_D$ of the boundary $\Gamma$, while on the remaining part $\Gamma \backslash \overline{\Gamma}_D$ Neumann boundary conditions are given. Boundary integral operators are used to describe the Steklov–Poincaré operator to realize the Dirichlet to Neumann map which is involved in both the primal and adjoint boundary value problem. In the case of box constraints on the control we have to solve a variational inequality in the Sobolev trace space $H^{1/2}(\Gamma_D)$. For the related Galerkin boundary element discretisation we present stability and error estimates, and we give some numerical examples.

## 1 Introduction

Optimal control problems of partial differential equations play an important role in many applications, see, e.g., [1]; for a rigorous mathematical treatment see [2]. In particular when considering boundary control problems, the use of boundary integral equations seems to be a favourable choice. In [3] we have considered boundary element methods to solve a tracking type Dirichlet boundary control problem, where the cost or regularisation term is considered in the energy space $H^{1/2}(\Gamma)$. Since the state enters the adjoint problem as a volume density, we used the bi–harmonic boundary integral operators to rewrite the Laplace Newton potential by means of boundary integral operators. In the case of box constraints on the control we have to solve a first kind variational inequality in the energy space $H^{1/2}(\Gamma)$. Stability and error estimates for a related Galerkin boundary element method result from a rather general theory [8], in combination with Strang lemma type estimates.

In this paper we consider the case when the Dirichlet control acts only on a part $\Gamma_D$ of the boundary $\Gamma$, while on the remaining part $\Gamma \backslash \overline{\Gamma}_D$ some Neumann boundary conditions are given. For the solution of the primal and of the adjoint boundary value problems with boundary conditions of mixed type we use a symmetric boundary integral equation approach to describe the Steklov–Poincaré operator as used in the Dirichlet to Neumann map [6] to end up with an equivalent boundary integral equation formulation. In the case of box constraints on the control we finally have to solve a first kind variational inequality in $H^{1/2}(\Gamma_D)$, so that we can apply a general stability and error analysis of a related Galerkin boundary element method. However, for the discretisation of the composed boundary integral operator we have to introduce suitable boundary element approximations. Finally we present some numerical examples, and we give a comparison with the more common approach when the control is considered in $L_2(\Gamma_D)$.

# 2 Optimal Dirichlet boundary control problem

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ which is decomposed into two nonintersecting parts $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. As a model problem, we consider the Dirichlet boundary control problem to minimize

$$\mathcal{J}(u, z) = \frac{1}{2} \int_\Omega [u(x) - \overline{u}(x)]^2 \, dx + \frac{\varrho}{2} \langle Sz, z \rangle_{\Gamma_D}, \tag{2.1}$$

where $u \in H^1(\Omega)$ is the weak solution of the Laplace equation with boundary conditions of mixed type,

$$-\Delta u = 0 \quad \text{in } \Omega, \quad u = z \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} u = f \quad \text{on } \Gamma_N, \tag{2.2}$$

and where the Dirichlet control $z$ satisfies the pointwise constraints

$$z \in \mathcal{U}_{ad} := \Big\{ w \in H^{1/2}(\Gamma_D) : a(x) \le w(x) \le b(x) \text{ for } x \in \Gamma_D \Big\}. \tag{2.3}$$

In (2.1), $\overline{u} \in L_2(\Omega)$ is a given target function, $\varrho \in \mathbb{R}_+$ is a fixed cost or penalty parameter; and $f \in H^{-1/2}(\Gamma_N)$ is a given Neumann datum. Moreover, $a, b \in H^{1/2}(\Gamma_D)$ are given barrier functions satisfying $a < b$ on $\Gamma_D$. In (2.1), the cost or regularisation is described by using a $H^{1/2}(\Gamma_D)$–semi–elliptic operator $S : H^{1/2}(\Gamma_D) \to \widetilde{H}^{-1/2}(\Gamma_D)$ which is specified later.

The solution of the mixed boundary value problem (2.2) is given by $u = u_z + u_f$, where $u_f \in H^1(\Omega)$ is the unique weak solution of the mixed boundary value problem

$$-\Delta u_f = 0 \quad \text{in } \Omega, \quad u_f = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} u_f = f \quad \text{on } \Gamma_N,$$

and $u_z \in H^1(\Omega)$ solves the homogeneous mixed boundary value problem

$$-\Delta u_z = 0 \quad \text{in } \Omega, \quad u_z = z \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} u_z = 0 \quad \text{on } \Gamma_N. \tag{2.4}$$

By using Green's first formula we have, for any $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u_z(x) \cdot \nabla v(x)\, dx = \int_{\Gamma_D} \frac{\partial}{\partial n_x} u_z(x) v(x)\, ds_x =: \langle Sz, v_{|\Gamma_D} \rangle_{\Gamma_D}.$$

The Steklov–Poincaré operator $S : H^{1/2}(\Gamma_D) \to \widetilde{H}^{-1/2}(\Gamma_D)$ maps the Dirichlet control $z \in H^{1/2}(\Gamma_D)$ to the related Neumann datum $\partial_n u_z$ on $\Gamma_D$ of the solution $u_z$ of the mixed boundary value problem (2.4). The cost or regularisation term in (2.1) is therefore equivalent to the Dirichlet energy

$$\langle Sz, z \rangle_{\Gamma_D} = \int_\Omega |\nabla u_z(x)|^2\, dx. \tag{2.5}$$

As a consequence of its definition, we conclude that $S$ is self–adjoint and $H^{1/2}(\Gamma_D)$–semi–elliptic. In particular for $z \equiv 1$ we find $u_z \equiv 1$ in $\Omega$ and hence $Sz \equiv 0$ on $\Gamma$.

The solution of the mixed boundary value problem (2.4) defines a linear map $u_z = \mathcal{H}z$, where $\mathcal{H} : H^{1/2}(\Gamma_D) \to H^1(\Omega) \subset L_2(\Omega)$ is compact. Then, by using $u = \mathcal{H}z + u_f$, we consider the problem to find the minimizer $z \in \mathcal{U}_{ad}$ of the reduced cost functional

$$\begin{aligned}
\widetilde{\mathcal{J}}(z) &= \frac{1}{2} \int_\Omega \Big[ (\mathcal{H}z)(x) + u_f(x) - \overline{u}(x) \Big]^2 dx + \frac{\varrho}{2} \langle Sz, z \rangle_{\Gamma_D} \\
&= \frac{1}{2} \langle \mathcal{H}z + u_f - \overline{u}, \mathcal{H}z + u_f - \overline{u} \rangle_{L_2(\Omega)} + \frac{\varrho}{2} \langle Sz, z \rangle_{\Gamma_D} \\
&= \frac{1}{2} \langle \mathcal{H}^*\mathcal{H}z, z \rangle_{\Gamma_D} + \langle \mathcal{H}^*(u_f - \overline{u}), z \rangle_{\Gamma_D} + \frac{1}{2} \|u_f - \overline{u}\|^2_{L_2(\Omega)} + \frac{\varrho}{2} \langle Sz, z \rangle_{\Gamma_D},
\end{aligned}$$

where $\mathcal{H}^* : L_2(\Omega) \to \widetilde{H}^{-1/2}(\Gamma_D)$ is the adjoint operator of $\mathcal{H} : H^{1/2}(\Gamma_D) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{H}^*\psi, \varphi \rangle_{\Gamma_D} = \langle \psi, \mathcal{H}\varphi \rangle_{L_2(\Omega)} \quad \text{for all } \varphi \in H^{1/2}(\Gamma_D),\ \psi \in L_2(\Omega).$$

It turns out that the application of the adjoint operator $\mathcal{H}^*$ is characterized by the Neumann datum

$$(\mathcal{H}^*\psi)(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma_D,$$

where $p$ is the unique solution of the adjoint mixed boundary value problem

$$-\Delta p = \psi \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} p = 0 \quad \text{on } \Gamma_N.$$

Since the reduced cost functional $\widetilde{J}$ is convex, the minimizer $z \in \mathcal{U}_{ad}$ can be found from the variational inequality

$$\langle \varrho Sz + \mathcal{H}^*\mathcal{H}z + \mathcal{H}^*(u_f - \overline{u}), w - z \rangle_{\Gamma_D} \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad}. \tag{2.6}$$

The operator

$$T_\varrho := \varrho S + \mathcal{H}^*\mathcal{H} : H^{1/2}(\Gamma_D) \to \widetilde{H}^{-1/2}(\Gamma_D) \tag{2.7}$$

is bounded, self–adjoint, and $H^{1/2}(\Gamma_D)$–elliptic. In fact, for $z \in H^{1/2}(\Gamma_D)$ we have, by using (2.5) and $u_z = \mathcal{H}z \in L_2(\Omega)$,

$$\begin{aligned} \langle T_\varrho z, z\rangle_{\Gamma_D} &= \varrho\langle Sz, z\rangle_{\Gamma_D} + \langle \mathcal{H}^*\mathcal{H}z, z\rangle_{\Gamma_D} \\ &= \varrho\,\|\nabla u_z\|^2_{L_2(\Omega)} + \|u_z\|^2_{L_2(\Omega)} =: \|u_z\|^2_{H^1(\Omega),\varrho} \geq c_1^T\,\|z\|^2_{H^{1/2}(\Gamma_D),\varrho} \end{aligned}$$

when using a weighted Sobolev norm. Hence, the elliptic variational inequality of the first kind (2.6) admits a unique solution $z \in H^{1/2}(\Gamma_D)$, see, e.g., [2]. Moreover, we can rewrite the variational inequality (2.6) as

$$\langle \varrho Sz - \partial_n p, w - z\rangle_{\Gamma_D} \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad}, \tag{2.8}$$

where $p$ is the unique solution of the adjoint mixed boundary value problem

$$-\Delta p = u - \overline{u} \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} p = 0 \quad \text{on } \Gamma_N. \tag{2.9}$$

In what follows we will use boundary integral equation techniques [7] to describe the solutions of the primal and the adjoint mixed boundary value problems (2.2) and (2.9), respectively.

# 3 Boundary integral equations

To find the control $z \in \mathcal{U}_{ad}$ we have to solve a coupled problem of the primal and the adjoint mixed boundary value problems (2.2) and (2.9), respectively, and of the variational inequality (2.8) representing the optimality condition. In what follows we will use boundary integral operators to describe the involved Dirichlet to Neumann maps, see, e.g., [7].

The solution of the primal boundary value problem (2.2) is given by the representation formula, for $x \in \Omega$,

$$u(x) = \int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y, \tag{3.1}$$

where $U^*(x,y)$ is the fundamental solution of the Laplace operator, and from which we conclude the boundary integral equation

$$\int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y = \frac{1}{2}u(x) + \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y \quad \text{for } x \in \Gamma,$$

i.e.

$$(V\partial_n u)(x) = (\frac{1}{2}I + K)u(x) \quad \text{for } x \in \Gamma.$$

Recall that $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the Laplace single layer boundary integral operator, and $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the Laplace double layer boundary integral operator, see, e.g., [7]. Since the single layer boundary integral operator $V$ is $H^{-1/2}(\Gamma)$–elliptic, for $n = 2$

we assume $\operatorname{diam}\Omega < 1$, and therefore invertible, we conclude the Dirichlet to Neumann map

$$\partial_n u(x) = V^{-1}(\frac{1}{2}I + K)u(x) =: (Su)(x) \quad \text{for } x \in \Gamma \tag{3.2}$$

with a first representation of the Steklov–Poincaré operator $S = V^{-1}(\frac{1}{2}I + K)$. When considering the normal derivative of the solution $u$ as given by the representation formula (3.1) we obtain, for $x \in \Gamma$,

$$\frac{\partial}{\partial n_x}u(x) = \frac{1}{2}\frac{\partial}{\partial n_x}u(x) + \int_\Gamma \frac{\partial}{\partial n_x}U^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y - \frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y,$$

i.e.

$$\partial_n u(x) = (\frac{1}{2}I + K')\partial_n u(x) + (Du)(x) \quad \text{for } x \in \Gamma,$$

where $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the adjoint double layer boundary integral operator, and $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the Laplace hypersingular boundary integral operator, see [7]. Hence, by using (3.2), we conclude a second representation of the Steklov–Poincaré operator,

$$\partial_n u(x) = \Big[D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\Big]u(x) = (Su)(x) \quad \text{for } x \in \Gamma. \tag{3.3}$$

The Steklov–Poincaré operator variational problem of the primal mixed boundary value problem (2.2) is to find $u \in H^{1/2}(\Gamma)$, $u = z$ on $\Gamma_D$, such that

$$\langle Su, v\rangle_{\Gamma_N} = \langle f, v\rangle_{\Gamma_N} \tag{3.4}$$

is satisfied for all $v \in H^{1/2}(\Gamma)$, $v = 0$ on $\Gamma_D$. Let $\widetilde{z} \in H^{1/2}(\Gamma)$ be some bounded extension of $z \in H^{1/2}(\Gamma_D)$. Then it remains to find $\widetilde{u} \in \widetilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle \widetilde{S}\widetilde{u}, v\rangle_{\Gamma_N} = \langle f - S\widetilde{z}, v\rangle_{\Gamma_N} \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma_N), \tag{3.5}$$

where the Steklov–Poincaré operator $\widetilde{S} : \widetilde{H}^{1/2}(\Gamma_N) \to H^{-1/2}(\Gamma_N)$ is bounded and elliptic, and therefore invertible. Moreover, $u = \widetilde{u} + \widetilde{z}$ is uniquely determined, independent from the chosen extension $\widetilde{z}$. Hence we find

$$u = \widetilde{S}^{-1}[f - S\widetilde{z}] + \widetilde{z} \quad \text{on } \Gamma_N. \tag{3.6}$$

Next we consider the adjoint mixed boundary value problem (2.9) for which we obtain the representation formula, for $x \in \Omega$,

$$p(x) = \int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}p(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)p(y)ds_y + \int_\Omega U^*(x,y)[u(y) - \overline{u}(y)]dy.$$

Since the state $u$ enters the above representation formula as a volume density of the Newton potential, we apply integration by parts, see, e.g., [3, 5], to obtain, for $x \in \Omega$,

$$\begin{aligned} p(x) &= \int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}p(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)p(y)ds_y - \int_\Omega U^*(x,y)\overline{u}(y)dy \\ &\quad + \int_\Gamma \frac{\partial}{\partial n_y}V^*(x,y)u(y)ds_y - \int_\Gamma V^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y, \end{aligned} \tag{3.7}$$

where $V^*(x,y)$ is the fundamental solutions of the Bi–Laplace partial differential operator. When taking the Dirichlet and the Neumann traces, the representation formula (3.7) results in two boundary integral equations on $\Gamma$,

$$p = V\partial_n p + (\frac{1}{2}I - K)p + K_1 u - V_1 \partial_n u - N_0 \overline{u}, \tag{3.8}$$

$$\partial_n p = (\frac{1}{2}I + K')\partial_n p + Dp - D_1 u - K_1' \partial_n u - N_1 \overline{u}, \tag{3.9}$$

where $V_1, K_1, K_1'$, and $D_1$ are the Bi–Laplace boundary integral operators, and $N_0, N_1$ are the Laplace Newton potentials, see, e.g., [3]. When solving (3.8) for $\partial_n p$, and by inserting $\partial_n u$ from (3.2), we obtain

$$\partial_n p = V^{-1}(\frac{1}{2}I + K)p - V^{-1}K_1 u + V^{-1}V_1 V^{-1}(\frac{1}{2}I + K)u + V^{-1}N_0\overline{u}.$$

From (3.9) we then conclude

$$\partial_n p = Sp - Tu + g \quad \text{on } \Gamma \tag{3.10}$$

with

$$T := D_1 + K_1' V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 - (\frac{1}{2}I + K')V^{-1}V_1 V^{-1}(\frac{1}{2}I + K),$$

and

$$g := (\frac{1}{2}I + K')V^{-1}N_0\overline{u} - N_1\overline{u}.$$

By using the boundary condition $\partial_n p = 0$ on $\Gamma_N$ we find $p \in \widetilde{H}^{1/2}(\Gamma_N)$ from the boundary integral equation

$$\widetilde{S}p = Tu - g \quad \text{on } \Gamma_N.$$

As in (3.6) we find the representation

$$p = \widetilde{S}^{-1}[Tu - g] \quad \text{on } \Gamma_N. \tag{3.11}$$

By using (3.6) and (3.11) we obtain from (3.10)

$$\partial_n p = -(I - S\widetilde{S}^{-1})T(I - \widetilde{S}^{-1}S)\widetilde{z} + (I - S\widetilde{S}^{-1})(g - T\widetilde{S}^{-1}f) \quad \text{on } \Gamma_D.$$

From (2.8) we finally conclude the variational inequality to find $z \in \mathcal{U}_{ad}$

$$\langle \varrho S z + (I - S\widetilde{S}^{-1})T(I - \widetilde{S}^{-1}S)\widetilde{z} - (I - S\widetilde{S}^{-1})(g - T\widetilde{S}^{-1}f), w - z\rangle_{\Gamma_D} \geq 0 \tag{3.12}$$

for all $w \in \mathcal{U}_{ad}$. Recall that $\widetilde{z} \in H^{1/2}(\Gamma)$ denotes some arbitrary but fixed extension of $z \in H^{1/2}(\Gamma_D)$.

**Theorem 3.1** *The composed boundary integral operator*

$$T_\varrho := \varrho S + (I - S\widetilde{S}^{-1})T(I - \widetilde{S}^{-1}S) : H^{1/2}(\Gamma_D) \to \widetilde{H}^{-1/2}(\Gamma_D) \tag{3.13}$$

*is bounded, self–adjoint, and $H^{1/2}(\Gamma_D)$–elliptic.*

**Proof.** The boundedness and the self–ajointness of $T_\varrho$ follows from the properties of all boundary integral operators involved. For $z \in H^{1/2}(\Gamma_D)$ let $\widetilde{z} \in H^{1/2}(\Gamma)$ be some arbitrary but fixed extension. Then,

$$\langle T_\varrho z, z\rangle_{\Gamma_D} = \varrho\langle Sz, z\rangle_{\Gamma_D} + \langle T(I - \widetilde{S}^{-1}S)\widetilde{z}, (I - \widetilde{S}^{-1}S)\widetilde{z}\rangle_\Gamma,$$

and by using the single layer potential $\widetilde{V}$, see [3],

$$\langle T(I - \widetilde{S}^{-1}S)\widetilde{z}, (I - \widetilde{S}^{-1}S)\widetilde{z}\rangle_\Gamma = \|\widetilde{V}(I - \widetilde{S}^{-1}S)\widetilde{z}\|^2_{L_2(\Omega)},$$

we conclude

$$\langle T_\varrho z, z\rangle_{\Gamma_D} \geq \varrho\langle Sz, z\rangle_{\Gamma_D},$$

i.e. semi–ellipticity. For $z \equiv 1$ we have $S1 = 0$ and therefore

$$\langle T_\varrho 1, 1\rangle_{\Gamma_D} = \|\widetilde{V}1\|^2_{L_2(\Omega)} > 0,$$

i.e. $T_\varrho$ induces an equivalent norm in $H^{1/2}(\Gamma_D)$. ∎

In fact, the properties of the operator $T_\varrho$ as defined in (3.13) reflect the properties of the operator $T_\varrho = \varrho S + \mathcal{H}^*\mathcal{H}$ as given in (2.7). In fact, we can conclude the unique solvability of the first kind variational inequality (3.12). In what follows, we will consider a Galerkin boundary element discretization of the variational inequality (3.12).

# 4 Symmetric Galerkin boundary element method

Let

$$S_h^1(\Gamma_D) := S_h^1(\Gamma) \cap H^{1/2}(\Gamma_D) = \mathrm{span}\{\varphi_i\}_{i=1}^{M_D}$$

be the boundary element space of piecewise linear and continuous basis functions $\varphi_i$, which is defined with respect to a globally quasi–uniform and shape regular boundary element mesh of mesh size $h$. For continuous barrier functions $a$ and $b$, we define the discrete convex set

$$\mathcal{U}_h := \{w_h \in S_h^1(\Gamma_D) \colon a(x_i) \leq w_h(x_i) \leq b(x_i) \quad \text{for all nodes } x_i \in \overline{\Gamma}_D\}.$$

Then the Galerkin discretization of the variational inequality (3.12) reads to find $z_h \in \mathcal{U}_h$ such that

$$\langle T_\varrho z_h, w_h - z_h\rangle_{\Gamma_D} \geq \langle F, w_h - z_h\rangle_{\Gamma_D} \quad \text{for all } w_h \in \mathcal{U}_h, \tag{4.1}$$

where

$$F := (I - S\widetilde{S}^{-1})(g - T\widetilde{S}^{-1}f) \in \widetilde{H}^{-1/2}(\Gamma_D).$$

Since (4.1) is the Galerkin discretisation of a first kind variational inequality with a bounded and $H^{1/2}(\Gamma_D)$–elliptic operator $T_\varrho$, we can apply standard arguments to state the unique solvability of (4.1), and to derive the following error estimate, see [3, 8].

**Theorem 4.1** *Let $z \in \mathcal{U}_{ad}$ and $z_h \in \mathcal{U}_h$ be the unique solutions of the variational inequalities* (3.12) *and* (4.1)*, respectively. If we assume $z, a, b \in H^s(\Gamma_D)$ and $T_\varrho z - F \in \widetilde{H}^{s-1}(\Gamma_D)$ for some $s \in [\frac{1}{2}, 2]$, then there holds the error estimate*

$$\|z - z_h\|_{H^{1/2}(\Gamma_D)} \leq c\, h^{s-\frac{1}{2}} \|z\|_{H^s(\Gamma_D)}. \tag{4.2}$$

The error estimate (4.2) seems to be optimal. However, the composed boundary integral operator $T_\varrho$ as defined in (3.13) includes several inverse operators, such as the inverse single layer boundary integral operator $V^{-1}$ and the inverse Steklov–Poincaré operator $\widetilde{S}^{-1}$, and therefore, $T_\varrho$ does not allow a practical implementation in the general case. Hence, instead of (4.1) we need to consider a perturbed variational inequality to find $\widehat{z}_h \in \mathcal{U}_h$ such that

$$\langle \widehat{T}_\varrho \widehat{z}_h, w_h - \widehat{z}_h\rangle_{\Gamma_D} \geq \langle \widehat{F}, w_h - \widehat{z}_h\rangle_{\Gamma_D} \quad \text{for all } w_h \in \mathcal{U}_h, \tag{4.3}$$

where $\widehat{T}_\varrho$ and $\widehat{F}$ are appropriate approximations of $T_\varrho$ and $F$, respectively. The following theorem presents an abstract consistency result, see [4].

**Theorem 4.2** *Let $\widehat{T}_\varrho : H^{1/2}(\Gamma_D) \to \widetilde{H}^{-1/2}(\Gamma_D)$ be a bounded and $S_h^1(\Gamma_D)$–elliptic approximation of $T_\varrho$ satisfying*

$$\|\widehat{T}_\varrho v\|_{\widetilde{H}^{-1/2}(\Gamma_D)} \leq c_2^{\widehat{T}_\varrho} \|v\|_{H^{1/2}(\Gamma_D)} \quad \textit{for all } v \in H^{1/2}(\Gamma_D)$$

*and*

$$\langle \widehat{T}_\varrho v_h, v_h\rangle_{\Gamma_D} \geq c_1^{\widehat{T}_\varrho} \|v_h\|^2_{H^{1/2}(\Gamma_D)} \quad \textit{for all } v_h \in S_h^1(\Gamma_D).$$

*Let $\widehat{F} \in \widetilde{H}^{-1/2}(\Gamma_D)$ be some approximation of $F$. For the unique solution $\widehat{z}_h \in \mathcal{U}_h$ of the perturbed variational inequality* (4.3) *the error estimate*

$$\|z - \widehat{z}_h\|_{H^{1/2}(\Gamma_D)} \leq c_1 \|z - z_h\|_{H^{1/2}(\Gamma_D)} + c_2 \|(T_\varrho - \widehat{T}_\varrho)z\|_{\widetilde{H}^{-1/2}(\Gamma_D)} + c_3 \|F - \widehat{F}\|_{\widetilde{H}^{-1/2}(\Gamma_D)} \tag{4.4}$$

*holds, where $z_h \in \mathcal{U}_h$ is the unique solution of the discrete variational inequality* (4.1).

It remains to define suitable approximations $\widehat{T}_\varrho$ and $\widehat{F}$ of the operator $T_\varrho$ and of the right hand side $F$, respectively. To do so, let us first consider the Galerkin boundary element approximation of the Steklov–Poincaré operator $S$ as given in (3.3). For $u \in H^{1/2}(\Gamma)$ we have

$$Su = Du + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)u = Du + (\frac{1}{2}I + K')w,$$

where $w \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$Vw = (\frac{1}{2}I + K)u \quad \text{on } \Gamma.$$

By using the piecewise constant approximation $w_h \in S_h^0(\Gamma) = \text{span}\{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$ satisfying

$$\langle Vw_h, \tau_h\rangle_\Gamma = \langle(\frac{1}{2}I + K)u, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma)$$

we define the approximate Steklov–Poincaré operator

$$\widehat{S}u = Du + (\frac{1}{2}I + K')w_h\,. \tag{4.5}$$

Using standard arguments, see, e.g., [6], we find the stability estimate

$$\|\widehat{S}u\|_{H^{-1/2}(\Gamma)} \le c_2^{\widehat{S}}\,\|u\|_{H^{1/2}(\Gamma)} \quad \text{for all } u \in H^{1/2}(\Gamma)$$

and the error estimate

$$\|(S - \widehat{S})u\|_{H^{-1/2}(\Gamma)} \le c\,h^{s+\frac{1}{2}}\|Su\|_{H^s_{\text{pw}}(\Gamma)} \tag{4.6}$$

when assuming $Su \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Moreover,

$$\langle\widehat{S}u, u\rangle_\Gamma = \langle Du, u\rangle_\Gamma + \langle Vw_h, w_h\rangle_\Gamma \ge \langle Du, u\rangle_\Gamma$$

implies the $H^{1/2}(\Gamma_D)$–semi–ellipticity of $\widehat{S}$. The Galerkin discretisation of the approximate Steklov–Poincaré operator $\widehat{S}$ is then given by

$$\widehat{S}_h = D_h + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h),$$

where

$$\begin{array}{rclcrcl} D_h[j,i] &=& \langle D\varphi_i, \varphi_j\rangle_{\Gamma_D}, && V_h[\ell,k] &=& \langle V\psi_k, \psi_\ell\rangle_\Gamma, \\ K_h[\ell,i] &=& \langle K\varphi_i, \psi_\ell\rangle_\Gamma, && M_h[\ell,i] &=& \langle\varphi_i, \psi_\ell\rangle_\Gamma \end{array}$$

for $i,j = 1,\ldots,M_D$, $k,\ell = 1,\ldots,N$.

In the same way as above we can define an approximate operator $\widehat{T}$, see [3, Sect. 6.1], its Galerkin discretisation is given by

$$\begin{aligned} \widehat{T}_h \;=\; & D_{1,h} + K_{1,h}^\top V_h^{-1}(\frac{1}{2}M_h + K_h) + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}K_{1,h} \\ & -(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h), \end{aligned}$$

where in addition to above we used the Bi–Laplace boundary element discretisations

$$D_{1,h}[j,i] = \langle D_1\varphi_i, \varphi_j\rangle_{\Gamma_D}, \quad V_{1,h}[\ell,k] = \langle V_1\psi_k, \psi_\ell\rangle_\Gamma, \quad K_{1,h}[\ell,i] = \langle K_1\varphi_i, \psi_\ell\rangle_\Gamma.$$

Next we describe a boundary element approximation of $u = (I - \widetilde{S}^{-1}S)\widetilde{z} \in H^{1/2}(\Gamma)$, when $z \in H^{1/2}(\Gamma_D)$ is given, and $\widetilde{z} \in H^{1/2}(\Gamma)$ is an arbitrary but fixed extension. By using the symmetric representation (3.3) of the Steklov–Poincaré operator $S$ we can rewrite the variational formulation (3.4) to find $(u,w) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $u = z$ on $\Gamma_D$, such that

$$\begin{aligned} \langle Du, v\rangle_{\Gamma_N} + \langle(\frac{1}{2}I + K')w, v\rangle_{\Gamma_N} &= 0 \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma_N), \\ \langle Vw, \tau\rangle_\Gamma - \langle(\frac{1}{2}I + K)u, \tau\rangle_\Gamma &= 0 \quad \text{for all } \tau \in H^{-1/2}(\Gamma). \end{aligned}$$

Again we can introduce the Galerkin solution $(u_h, w_h) \in S_h^1(\Gamma) \times S_h^0(\Gamma)$, $u_h = Q_h z$ on $\Gamma_D$, such that

$$\langle Du_h, v_h\rangle_{\Gamma_N} + \langle(\frac{1}{2}I + K')w_h, v_h\rangle_{\Gamma_N} = 0 \quad \text{for all } v_h \in S_h^1(\Gamma) \cap \widetilde{H}^{1/2}(\Gamma_N), \tag{4.7}$$

$$\langle Vw_h, \tau_h\rangle_\Gamma - \langle(\frac{1}{2}I + K)u_h, \tau_h\rangle_\Gamma = 0 \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{4.8}$$

Note that $Q_h : L_2(\Gamma_D) \to S_h^1(\Gamma_D) \subset H^{1/2}(\Gamma_D)$ is the $L_2(\Gamma_D)$ projection which is stable in $H^{1/2}(\Gamma_D)$. Since the associated bilinear form

$$a(u,w;v,\tau) = \langle Du, v\rangle_{\Gamma_N} + \langle(\frac{1}{2}I + K')w, v\rangle_{\Gamma_N} - \langle(\frac{1}{2}I + K)u, \tau\rangle_\Gamma + \langle Vw, \tau\rangle_\Gamma$$

is $\widetilde{H}^{1/2}(\Gamma_N) \times H^{-1/2}(\Gamma)$–elliptic, we can conclude stability and error estimates by using standard arguments, i.e.

$$\|u - u_h\|_{\widetilde{H}^{1/2}(\Gamma_N)} \le c\, h^{s-1/2} \left[ \|u\|_{H^s(\Gamma)} + \|Su\|_{H^{s-1}_{\text{pw}}(\Gamma)} \right] \tag{4.9}$$

when assuming $u \in H^s(\Gamma)$ and $w = Su \in H^{s-1}_{\text{pw}}(\Gamma)$ for some $s \in [\frac{1}{2}, 2]$. Now we are in a position to define the Galerkin boundary element approximation

$$(\widehat{I - \widetilde{S}^{-1}S})\widetilde{z} := u_h.$$

The Galerkin variational formulation (4.7)–(4.8) is equivalent to a linear system,

$$\begin{pmatrix} \widetilde{D}_h & \frac{1}{2}\widetilde{M}_h^\top + \widetilde{K}_h^\top \\ -\frac{1}{2}\widetilde{M}_h - \widetilde{K}_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} -\overline{D}_h\underline{z} \\ (\frac{1}{2}\overline{M}_h + \overline{K}_h)\underline{z} \end{pmatrix},$$

where the Galerkin matrices $\widetilde{D}_h$, $\widetilde{K}_h$, and $\widetilde{M}_h$ correspond to piecewise linear and continuous basis functions to approximate $u - \widetilde{z} \in \widetilde{H}^{1/2}(\Gamma_N)$, while the matrices $\overline{D}_h$, $\overline{K}_h$, and $\overline{M}_h$

correspond to the approximation of the control $\widetilde{z} \in H^{1/2}(\Gamma)$, tested with appropriate basis functions. From the second equation we conclude

$$\underline{w} = V_h^{-1}(\frac{1}{2}\widetilde{M}_h + \widetilde{K}_h)\underline{u} + V_h^{-1}(\frac{1}{2}\overline{M}_h + \overline{K}_h)\underline{z},$$

and hence we have to solve the Schur complement system

$$\Big[\widetilde{D}_h + (\frac{1}{2}\widetilde{M}_h^\top + \widetilde{K}_h^\top)V_h^{-1}(\frac{1}{2}\widetilde{M}_h + \widetilde{K}_h)\Big]\underline{u} = -\Big[\overline{D}_h + (\frac{1}{2}\widetilde{M}_h^\top + \widetilde{K}_h^\top)V_h^{-1}(\frac{1}{2}\overline{M}_h + \overline{K}_h)\Big]\underline{z}.$$

By using

$$\widetilde{S}_h := \widetilde{D}_h + (\frac{1}{2}\widetilde{M}_h^\top + \widetilde{K}_h^\top)V_h^{-1}(\frac{1}{2}\widetilde{M}_h + \widetilde{K}_h), \quad \overline{S}_h := \overline{D}_h + (\frac{1}{2}\widetilde{M}_h^\top + \widetilde{K}_h^\top)V_h^{-1}(\frac{1}{2}\overline{M}_h + \overline{K}_h)$$

we finally obtain

$$\underline{u} = -\widetilde{S}_h^{-1}\overline{S}_h\underline{z}\,.$$

If we define the approximate operator

$$\widehat{T}_\varrho := \varrho\widehat{S} + (\widehat{I - S\widetilde{S}^{-1}})\widehat{T}(\widehat{I - \widetilde{S}^{-1}S}), \tag{4.10}$$

then we conclude its Galerkin discretisation as

$$\widehat{T}_{\varrho,h} = \varrho\widehat{S}_h + \overline{S}_h^\top\widetilde{S}_h^{-1}\widehat{T}_h\widetilde{S}_h^{-1}\overline{S}_h.$$

Note that the stiffness matrix $\widehat{T}_{\varrho,h}$ is symmetric and positive definite. Similar to above we can define an approximate evaluation of $\widehat{F}$, which gives

$$\underline{\widehat{F}} = -\overline{S}_h^\top\widetilde{S}_h^{-1}\left[(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}\underline{N_0\overline{u}} - \underline{N_1\overline{u}} - \widehat{T}_h\widetilde{S}_h^{-1}\underline{f}\right].$$

To determine the control $z_h \in \mathcal{U}_h \leftrightarrow \underline{z} \in \mathbb{R}^{M_D}$ we have to solve the discrete variational inequality

$$(\widehat{T}_{\varrho,h}\underline{z}, \underline{w} - \underline{z}) \geq (\underline{\widehat{F}}, \underline{w} - \underline{z}) \quad \text{for all } \underline{w} \in \mathbb{R}^{M_D} \leftrightarrow w_h \in \mathcal{U}_h.$$

If we introduce the discrete Lagrange multiplier $\underline{\lambda} = \widehat{T}_{\varrho,h}\underline{z} - \underline{\widehat{F}} \in \mathbb{R}^{M_D}$ we can rewrite the discrete variational inequality in terms of related complementarity conditions, i.e. in the case of the lower barrier function $a$ this gives

$$z_i \geq a(x_i), \quad \lambda_i \geq 0, \quad \lambda_i[z_i - a(x_i)] = 0,$$

while in the case of the upper barrier function we have

$$z_i \leq b(x_i), \quad \lambda_i \leq 0, \quad \lambda_i[z_i - b(x_i)] = 0.$$

Instead of the complementarity conditions we can also write a nonlinear equation, e.g., for the upper barriwr function an equivalent formulation is given by

$$\lambda_i = \max\Big\{0, \lambda_i + c[b(x_i) - z_i]\Big\} \quad \text{for } i = 1, \ldots, M_D, \quad c > 0.$$

For the solution of this nonlinear system we use a semi–smooth Newton method, which turns out to be an active set strategy, see, e.g., [8].

It turns out that the approximation errors as used in the error estimate (4.4) are consequences of the above shown error estimates (4.6) and (4.9). For the approximate solution $\widehat{z}_h$ of the perturbed variational inequality (4.3) we then conclude the error estimate

$$\|z - \widehat{z}_h\|_{H^{1/2}(\Gamma_D)} \leq c(z, f, \overline{u})\, h^{s-\frac{1}{2}} \tag{4.11}$$

when assuming $z \in H^s_{\mathrm{pw}}(\Gamma_D)$ for some $s \in [\frac{1}{2}, 2]$, i.e., when assuming sufficient regularity on the given data. Moreover, by applying the Aubin–Nitsche trick [8] we are able to derive an error estimate in $L_2(\Gamma_D)$, i.e.,

$$\|z - \widehat{z}_h\|_{L_2(\Gamma_D)} \leq c(z, f, \overline{u})\, h^s. \tag{4.12}$$

# 5 Numerical results

As numerical example we consider the mixed boundary control problem (2.1) and (2.2) in the case of a two–dimensional square domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$. The boundary $\Gamma = \partial\Omega$ consists of two parts $\Gamma_D$ and $\Gamma_N$ where

$$\Gamma_D = \Big\{(x_1, 0) : 0 < x_1 < 0.5\Big\} \cup \Big\{(0, x_2) : 0 < x_2 < 0.5\Big\}, \quad \Gamma_N = \Gamma \setminus \overline{\Gamma}_D.$$

For the cost parameter we consider $\varrho = 0.1$ and the data are chosen as

$$\overline{u}(x) = (x_1^2 + x_2^2)^{-\frac{1}{3}}, \quad f(x) = \frac{\partial}{\partial n_x}\overline{u}(x)|_{\Gamma_N}.$$

For the boundary element discretization we introduce uniform boundary meshes of the boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ on several levels $L$ where the mesh size is $h_L = 2^{-(L+1)}$. Note that the minimizer of (2.1) is not known in this example, we use the boundary element solution $\widehat{z}_{h_9}$ on the 9th level as reference solution.

In Table 1 we present the errors for the control $z$ and for the unknown Dirichlet datum $u$, and the estimated order of convergence (eoc). These results correspond to the error estimates (4.11) and (4.12).

As a second example, we consider the additional constraint $z \leq 2.6$. In Figure 1 we give a comparison of the unconstrained and constrained solutions, and in Figure 2 we plot the related controls for $x_1 \in (0, 0.5)$, $x_2 = 0$.

Moreover, we plot in Figure 3 the states $u$ of the boundary control problem (2.1)–(2.2) for $\varrho = 10^{-2}$ and $\varrho = 10^{-4}$. The singularity of the state at the origin appears clearly for small $\varrho$, see also [5, Figure 3.5] for the Dirichlet boundary control problem. Note that also the target function $\overline{u}$ has a singularity at the origin.

| L | $\|\widehat{z}_{h_L} - \widehat{z}_{h_9}\|_{L_2(\Gamma_D)}$ | eoc | $\|\widehat{z}_{h_L} - \widehat{z}_{h_9}\|_{H^{1/2}(\Gamma_D)}$ | eoc | $\|\widehat{u}_{h_L} - \widehat{u}_{h_9}\|_{L_2(\Gamma_N)}$ | eoc |
|---|---|---|---|---|---|---|
| 2 | 1.8041e-2 | - | 2.1236e-1 | - | 2.6788e-2 | - |
| 3 | 4.8635e-3 | 1.891 | 8.2073e-2 | 1.372 | 8.4929e-3 | 1.657 |
| 4 | 1.4322e-3 | 1.764 | 3.4331e-2 | 1.257 | 2.7877e-3 | 1.607 |
| 5 | 4.5382e-4 | 1.658 | 1.4228e-2 | 1.271 | 9.3811e-4 | 1.571 |
| 6 | 1.5562e-4 | 1.544 | 5.7832e-3 | 1.299 | 3.2225e-4 | 1.542 |
| 7 | 5.4047e-5 | 1.526 | 2.2475e-3 | 1.364 | 1.1217e-4 | 1.522 |
| 8 | 1.5723e-5 | 1.781 | 7.4669e-4 | 1.590 | 3.9334e-5 | 1.512 |

Table 1: The results of mixed boundary control problems without control constraints.

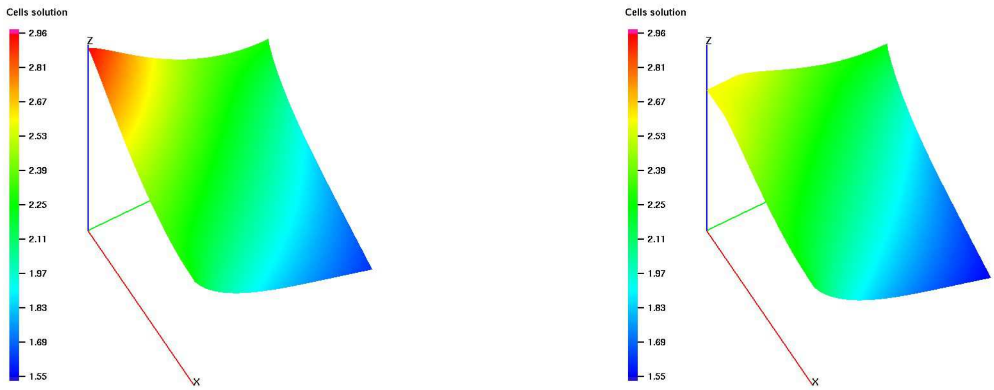


Figure 1: Comparison of unconstrained (left) and constrained (right) optimal solutions.

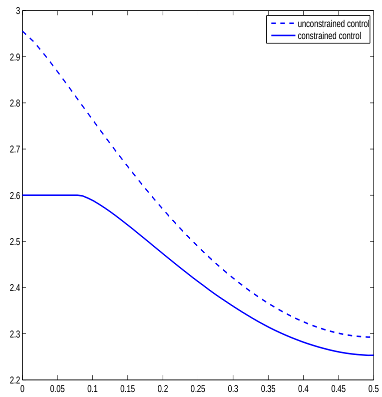


Figure 2: Optimal control of the unconstrained and constrained problems, $x_2 = 0$.

For comparison we also consider the mixed boundary control problem (2.1)-(2.2) where the control $z$ is in $L_2(\Gamma_D)$ with $\varrho = 0.1$. In Figure 4 we plot the state $u$ for the $L_2(\Gamma_D)$

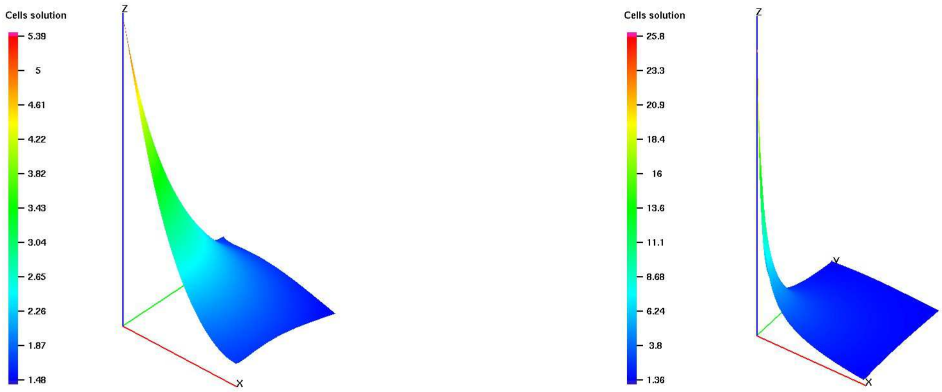


Figure 3: The states $u$ with $\varrho = 10^{-2}$ (left) and $\varrho = 10^{-4}$ (right).

setting and the related control for $x_2 = 0$, and in Figure 5 we plot the related controls for $x_1 \in (0, 0.05)$, $x_2 = 0$ and for $x_1 \in (0.45, 0.5)$, $x_2 = 0$. We see that the control is zero at all corner points, see also the discussion in [4].

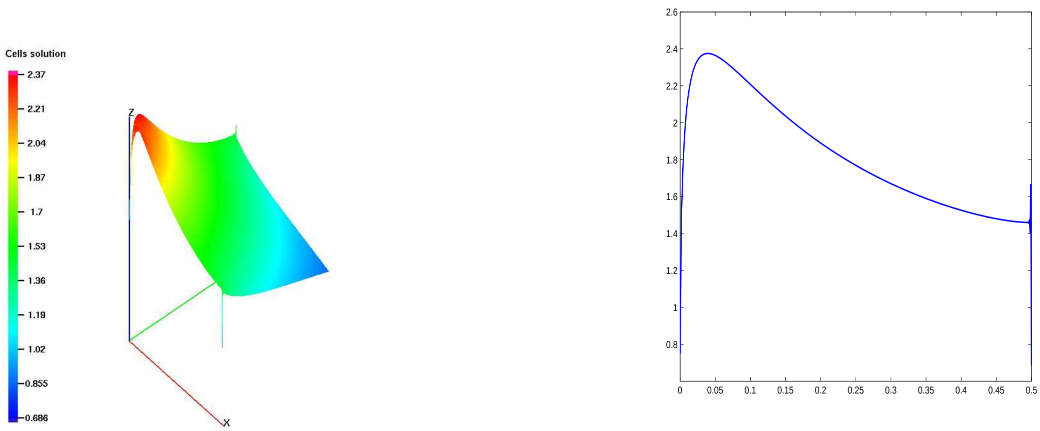


Figure 4: The state $u$ for the $L_2(\Gamma_D)$ setting (left) and the related control for $x_2 = 0$ (right).

# Acknowledgements

This work has been supported by the Austrian Science Fund (FWF) under the grant SFB Mathematical Optimization and Applications in Biomedical Sciences, Subproject Fast Finite and Boundary Element Methods for Optimality Systems, by the Asean–European University Network, and by the Vietnam Ministry of Education and Training under grant number B2013-27-09.

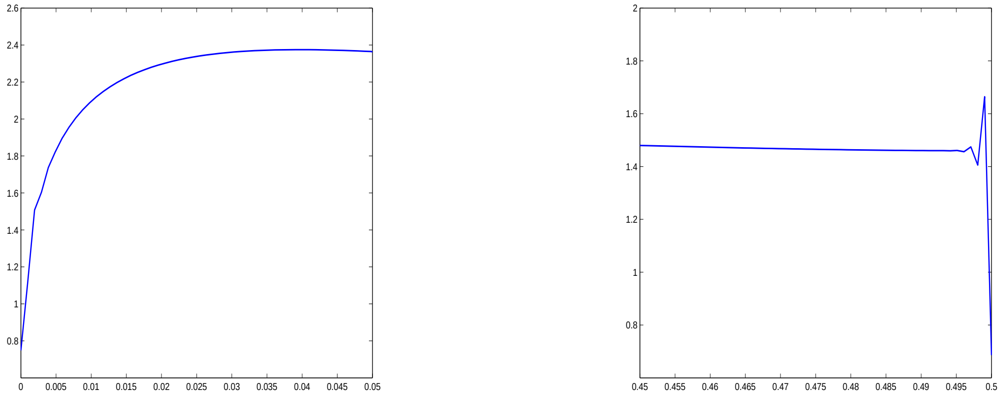


Figure 5: The related controls for $x_1 \in (0, 0.05)$, $x_2 = 0$ (left) and for $x_1 \in (0.45, 0.5)$, $x_2 = 0$ (right).

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