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THE SCALAR, VECTOR AND TENSOR FIELDS IN THEORY OF ELASTICITY AND PLASTICITY

SKALÁRNÍ, VEKTOROVÁ A TENZOROVÁ POLE V TEORII PRUŽNOSTI A PLASTICITY

Abstract

This article is devoted to an analysis of scalar, vector and tensor fields, which occur in the loaded and deformed bodies. The aim of this article is to clarify and simplify the creation of an understandable idea of some elementary concepts and quantities in field theories, such as, for example equiscalar levels, scalar field gradient, Hamilton operator, divergence, rotation and gradient of vector or tensor and others. Applications of those mathematical terms are shown in simple elasticity and plasticity tasks. We hope that content of our article might help technicians to make their studies of necessary mathematical chapters of vector and tensor analysis and field theories easier.

Abstrakt

Příspěvek je věnován analýze skalárních, vektorových a tenzorových polí, které vznikají v zatížených a deformovaných tělesech. Hlavním cílem příspěvku je usnadnit vytvoření názorné představy o některých základních pojmech a veličinách z teorie polí, jako jsou např. ekviskalární hladiny, gradient skalárního pole, Hamiltonův operátor, divergence, rotace a gradient vektoru a tenzoru aj. Použití těchto matematických prostředků je ukázáno při řešení základních úloh teorie pružnosti a plasticity. Obsah článku snad přispěje technikům k usnadnění studia potřebných matematických kapitol z vektorové a tenzorové analýzy a teorie polí.

Keywords

scalar, vector and tensor fields, tensor analysis, gradient of vector or tensor, field theories

1 INTRODUCTION

Quantities, which are functionally dependent on their position in a space, create fields. There are, for example temperature, gravitational, speed, electric, magnetic, electromagnetic fields and others. Those fields can intersect each other. From mathematical point of view, we can divide the field according to, whether the functionally dependent quantity is a scalar, vector or tensor. In this contribution, we are going to briefly discuss only fields, which occur in mechanics in solution to a problem of deformable bodies. There are following functionally dependent quantities in those fields:

a) temperature – scalar, the zero–order tensor,

b) displacement – vector, the first–order tensor,

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c) state of strain – the second–order tensor,

d) state of stress – the second–order tensor.

The scalar, vector and tensor fields describe mechanical state of a loaded body. Below, we introduce brief description and properties of those fields.

2 ELEMENTARY CONCEPTS IN FIELD THEORY

2.1 Scalar, vector and tensor fields

2.1.1 Scalar field

Scalar field is the most simple field type, because functional quantity dependent on its position in space is a scalar. It is defined only by one numerical value in every point in a space. Scalar field is then definite by certain scalar position function:

$$\varphi = f(x, y, z) \tag{1}$$

An example of scalar field in the elasticity and plasticity theory is a temperature field. Results of a scalar field are very well illustrated by a chart. That can be done by equiscalar field levels, given by equations:

$$c = f(x, y, z) \tag{2}$$

In the equation above $\underline{c}$ are consequently called constants and them correspond curves joining geometrical point locations with the same value of scalar function. Great examples of equiscalar surface are contours on maps.

During an investigation of the scalar fields, we are interested in speed dependence on a change of scalar field in a certain direction or in which direction the change is fastest. In a solution to these problems there were derived and defined two important terms as is a gradient of scalar φ:

$$\frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j} + \frac{\partial \varphi}{\partial z}\boldsymbol{k} = grad\ \varphi = \nabla \varphi \tag{3}$$

and a Hamilton operator nabla ∇ [1]

$$\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} = \nabla \tag{4}$$

Gradient of scalar field φ has geometrical and also physical meaning. It is a vector which is perpendicular to the equiscalar level and in the direction of gradient is the steepest descent of scalar φ and the rate of change is equal to an absolute value of this scalar.

Hamilton operator nabla is understood as a symbolic vector operator, it expresses that is used firstly as a vector, secondly as a differential operator. It means that in counting with this operator we use firstly rules for vector counting, secondly rules for differential counting. Usage of Hamilton operator is explained further bellow.

2.1.2 Vector field

To an every point in space can be by certain rule given as a functional quantity a vector, and then we obtain a vector field. In the theory of elasticity and plasticity there is an example of vector field a field of body point's displacement during its load. Geometrical representation of a vector field is more complicated then scalar field. Nevertheless, every vector field can be distributed into three sections in the firm coordinate system, and those sections are also position functions in the coordinate system. Then we can graphically represent a vector field with sets of three equiscalar levels.

2.1.3 Tensor field

The tensor field is even more complicated than previous two fields. Here is a functional quantity of point position in space a second order tensor. In the theory of elasticity and plasticity there are those functional quantities a strain state and stress state tensor. Geometrical idea and physical meaning of a scalar or vector is quite simple and clear, however, with second order tensor it is far more complicated and in work [1] is even written, that simple and clear image of such quantities can't be done (page 85). However, we took the challenge and tried to obtain clear representation of second order tensor on an example of stress state tensor [2]. We used definition of second order tensor by means of n dyads sum. A dyad is defined as an ordered pair of two vectors, those vectors make dyadic product. Those dyads are, at a stress state tensor, formed by dyadic products of a stress result vector on a single plane and single vector perpendicular to the plane, more fig. 1. Thus stress state tensor is given by product of three dyads:

$$\begin{aligned} \boldsymbol{T}_\sigma &= \boldsymbol{p}_i\boldsymbol{i} + \boldsymbol{p}_j\boldsymbol{j} + \boldsymbol{p}_k\boldsymbol{k} = (\sigma_x\boldsymbol{i} + \tau_{xy}\boldsymbol{j} + \tau_{xz}\boldsymbol{k})\boldsymbol{i} + (\tau_{yx}\boldsymbol{i} + \sigma_y\boldsymbol{j} + \tau_{yz}\boldsymbol{k})\boldsymbol{j} + (\tau_{zx}\boldsymbol{i} + \tau_{zy}\boldsymbol{j} + \sigma_z\boldsymbol{k})\boldsymbol{k} = \\ &= \sigma_x\boldsymbol{ii} + \tau_{yx}\boldsymbol{ij} + \tau_{zx}\boldsymbol{ik} + \\ &+ \tau_{xy}\boldsymbol{ji} + \sigma_y\boldsymbol{jj} + \tau_{zy}\boldsymbol{jk} + \\ &+ \tau_{xz}\boldsymbol{ki} + \tau_{yz}\boldsymbol{kj} + \sigma_z\boldsymbol{kk} \end{aligned} \tag{5}$$

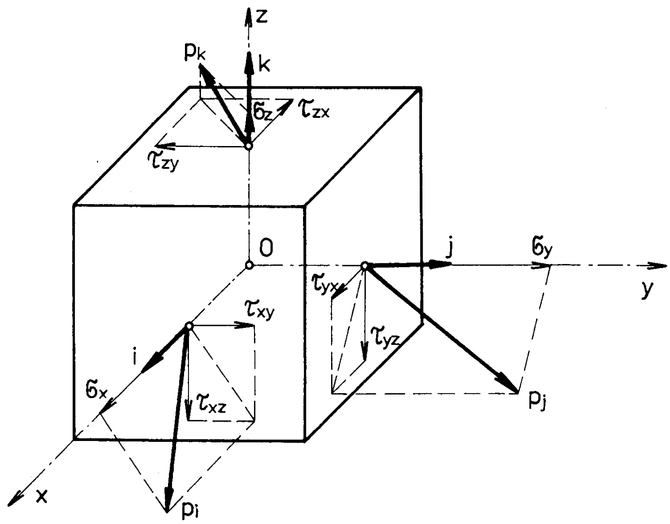

Fig. 1 Dyads, vector and scalar coordinates of stress state tensor

Vectors $\boldsymbol{p}_i$, $\boldsymbol{p}_j$ and $\boldsymbol{p}_k$ are called left vector coordinates of stress state tensor. Notation of second order tensor according to equation (5) is rather laborious. Therefore it is registered of form a matrix (6), which is an image of the stress state tensor:

$$\boldsymbol{T}_\sigma = \begin{Vmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{Vmatrix} \tag{6}$$

Elements of the matrix (6) form component of stress state tensor. They are also functionally dependent on a point position in the space, thus the tensor field can be represented by group of nine equiscalar levels.

In the theory of elasticity and mainly plasticity there are also other quantities of great importance, as is a stress state deviator and its invariants and stress intensity as well as strain state deviator and its invariants and strain intensity. Every second order tensor can be expressed as a sum of spherical tensor ***K*** and deviator ***D***. Thus for stress state and stain state tensor holds:

$$\boldsymbol{T_\sigma} = \boldsymbol{K_\sigma} + \boldsymbol{D_\sigma} = \begin{Vmatrix} \sigma_s & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_s \end{Vmatrix} + \begin{Vmatrix} (\sigma_1 - \sigma_s) & 0 & 0 \\ 0 & (\sigma_2 - \sigma_s) & 0 \\ 0 & 0 & (\sigma_3 - \sigma_s) \end{Vmatrix} \; ; \; \sigma_s = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad (7)$$

$$\boldsymbol{T_\varepsilon} = \boldsymbol{K_\varepsilon} + \boldsymbol{D_\varepsilon} = \begin{Vmatrix} \varepsilon_s & 0 & 0 \\ 0 & \varepsilon_s & 0 \\ 0 & 0 & \varepsilon_s \end{Vmatrix} + \begin{Vmatrix} (\varepsilon_1 - \varepsilon_s) & 0 & 0 \\ 0 & (\varepsilon_2 - \varepsilon_s) & 0 \\ 0 & 0 & (\varepsilon_3 - \varepsilon_s) \end{Vmatrix} \; ; \; \varepsilon_s = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} \quad (8)$$

Deviators have, like others tensors, their own invariants. The second invariants of stress state and strain state deviators are of the greatest importance among them and they are given by expressions:

$$I_2(\boldsymbol{D_\sigma}) = -\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \quad (9)$$

$$I_2(\boldsymbol{D_\varepsilon}) = -\frac{1}{6}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right] \quad (10)$$

By means of these invariants, shear stress intensity and shear strain intensity is defined then:

$$S_\tau = \sqrt{-I_2(\boldsymbol{D}_\sigma)} = \sqrt{\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \quad (11)$$

$$S\gamma = 2 \cdot \sqrt{-I_2(\boldsymbol{D}_\varepsilon)} = \sqrt{\frac{2}{3}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right]} \quad (12)$$

Except these quantities there are also defined stress intensity and strain intensity, where holds:

$$S_\sigma = S_\tau \sqrt{3} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \quad (13)$$

$$S_\varepsilon = S\gamma \frac{\sqrt{2}}{3} = \frac{2}{3\sqrt{3}} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} \quad (14)$$

Quantities mentioned above play an important role in the theory of elasticity and mainly plasticity, as it will be shown below.

2.2 Divergence, rotation and gradient of vector and tensor

In this article above there has been defined the scalar field gradient as a product of a nabla operator and scalar. We can also similarly define a vector field gradient. Beside the gradient there are, for the vector field solution, defined other important terms as a vector divergence and vector rotation.

2.2.1 Divergence of a vector

Divergence of a vector is defined as a scalar product of operator nabla ∇ and vector ***u***.

$$div\ \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot \left(u_x \boldsymbol{i} + u_y \boldsymbol{j} + u_z \boldsymbol{k}\right) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad (15)$$

Equation (15) represents a rule, where to ***u*** vectors of the vector function, are assigned scalar quantities *div* ***u***. Divergence definition was applied for a vector function of displacements ***u*** in a deformed body.

2.2.2 Rotation of a vector

Vector rotation is defined as a vector product of operator nabla ∇ and vector ***u***:

$$rot\,\boldsymbol{u} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times \left(u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}\right) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \tag{16}$$

The vector product results of operator ∇ and vector ***u*** is a vector.

2.2.3 Gradient of a vector

It is defined as a dyadic product of an operator nabla ∇ and vector ***u***, the result of the dyadic product is a second order tensor and its image is a matrix in equation (17):

$$\begin{aligned} grad\,\mathrm{u} &= \left(\frac{\partial}{\partial x}\mathrm{i} + \frac{\partial}{\partial y}\mathrm{j} + \frac{\partial}{\partial z}\mathrm{k}\right)\left(u_x\mathrm{i} + u_y\mathrm{j} + u_z\mathrm{k}\right) = \\ &= \begin{matrix} \frac{\partial u_x}{\partial x}\mathrm{ii} + & \frac{\partial u_y}{\partial x}\mathrm{ij} + & \frac{\partial u_z}{\partial x}\mathrm{ik} + \\ \frac{\partial u_x}{\partial y}\mathrm{ji} + & \frac{\partial u_y}{\partial y}\mathrm{jj} + & \frac{\partial u_z}{\partial y}\mathrm{jk} + \\ \frac{\partial u_x}{\partial z}\mathrm{ki} + & \frac{\partial u_y}{\partial z}\mathrm{kj} + & \frac{\partial u_z}{\partial z}\mathrm{kk} \end{matrix} \approx \begin{Vmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{Vmatrix} \end{aligned} \tag{17}$$

2.2.4 Divergence of second order tensor

Similarly as in vector field, we can also in a tensor field define terms as a divergence, rotation and gradient of a tensor. The tensor divergence is then defined as a scalar product of operator nabla ∇ and tensor $\boldsymbol{T_\sigma}$.

$$\begin{aligned} div\,\boldsymbol{T_\sigma} &= \nabla\cdot\boldsymbol{T_\sigma} = \nabla\cdot\left(\boldsymbol{p}_i\boldsymbol{i} + \boldsymbol{p}_j\boldsymbol{j} + \boldsymbol{p}_k\boldsymbol{k}\right) = \nabla\cdot\left(\boldsymbol{p}_i\boldsymbol{i}\right) + \nabla\cdot\left(\boldsymbol{p}_j\boldsymbol{j}\right) + \nabla\cdot\left(\boldsymbol{p}_k\boldsymbol{k}\right) = \\ &= \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_i \\ p_i\end{matrix}\right)\boldsymbol{i} + \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_i \\ i\end{matrix}\right)\boldsymbol{i} + \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_j \\ p_j\end{matrix}\right)\boldsymbol{j} + \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_j \\ j\end{matrix}\right)\boldsymbol{j} + \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_k \\ p_k\end{matrix}\right)\boldsymbol{k} + \left(\begin{matrix}\nabla\cdot\boldsymbol{p}_k \\ k\end{matrix}\right)\boldsymbol{k} = \\ &= \boldsymbol{i}\;div\;\boldsymbol{p}_i + \boldsymbol{p}_i\;\;grad\;\boldsymbol{i} + \boldsymbol{j}\;div\;\boldsymbol{p}_j + \boldsymbol{p}_j\;\;grad\;\boldsymbol{j} + \boldsymbol{k}\;div\;\boldsymbol{p}_k + \boldsymbol{p}_k\;\;grad\;\boldsymbol{k} = \\ &= \left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\right)\boldsymbol{i} + \\ &+ \left(\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}\right)\boldsymbol{j} + \\ &+ \left(\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\sigma_z}{\partial z}\right)\boldsymbol{k} \end{aligned} \tag{18}$$

In the equation (18), there were used the rules for scalar and dyadic multiplication of vectors, computing with nabla operator and the fact that gradients of constant quantity unit vectors ***i***, ***j*** and ***k*** are zero. Divergence of second order tensor is a vector and scalar coordinates of this vector are given by vector coordinate divergences of the original second order tensor.

3 ELEMENTARY SYSTEMS OF EQUATIONS IN ELASTICITY AND PLASTICITY THEORY

3.1 Static conditions of equilibrium

In order to keep the loaded body in equilibrium, there has to be for each its point, shown in form of infinitely small cube (fig. 1), fulfilled static conditions of equilibrium. For this cube, we can write three componential equations for force and three for moment equilibrium, and obtain following [3]:

$$\left.\begin{array}{l}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{yx}}{\partial y}+\dfrac{\partial\tau_{zx}}{\partial z}=0\\[2mm]\dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{zy}}{\partial z}=0\\[2mm]\dfrac{\partial\tau_{xz}}{\partial x}+\dfrac{\partial\tau_{yz}}{\partial y}+\dfrac{\partial\sigma_z}{\partial z}=0\end{array}\right\}\qquad(19)$$

$$\tau_{xy}=\tau_{yx}\ ;\ \tau_{yz}=\tau_{zy}\ ;\ \tau_{zx}=\tau_{xz}\qquad(20)$$

Relations in the equation (20), following from the moment conditions of equilibrium, represent a law about conjugation of shear stresses and their result is that stress state tensor is a symmetric tensor.

Static conditions of equilibrium (19) can be written also by means of quantity, defined and derived above in the description of tensor field. If we multiply the equation (18) by quantity surface of cube side according to fig. 1, we obtain as a result three forces in the directions of unit vectors ***i***, ***j*** and ***k***. Vector sum of those three forces is a resulting force, working on an elementary cube. If there should be the elementary cube in equilibrium, the resulting force working on the cube must be zero. Hence zero will be only if their sections are zero in directions of the unit vectors ***i***, ***j***, and ***k***. It means that scalar coordinates at the unit vectors in equation (18) must be zero. Therefore there must be valid:

$$\left.\begin{array}{l}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}+\dfrac{\partial\tau_{xz}}{\partial z}=0\\[2mm]\dfrac{\partial\tau_{yx}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{yz}}{\partial z}=0\\[2mm]\dfrac{\partial\tau_{zx}}{\partial x}+\dfrac{\partial\tau_{zy}}{\partial y}+\dfrac{\partial\sigma_z}{\partial z}=0\end{array}\right\}\qquad(21)$$

Considering equations (20), we can see that equation (21) expresses the same as equation (19) and therefore we can also write the static conditions of equilibrium in a form like this:

$$div\boldsymbol{T}_{\boldsymbol{\sigma}}=0\qquad(22)$$

3.2 Geometric equations

These equations give dependence between components of the displacement vector and strain state tensor. Based on a geometrical analysis of the deformed body, between those components are relations [3], [4]:

$$\left.\begin{array}{lll}\varepsilon_x = \dfrac{\partial u_x}{\partial x} & ; & \gamma_{xy} = \dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y} \\ \varepsilon_y = \dfrac{\partial u_y}{\partial y} & ; & \gamma_{xz} = \dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z} \\ \varepsilon_z = \dfrac{\partial u_z}{\partial z} & ; & \gamma_{yz} = \dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}\end{array}\right\} \tag{23}$$

Strain state tensor is given by a sum of three dyads and its reflection is a matrix (24):

$$\boldsymbol{T_\varepsilon} = \boldsymbol{u}_i \boldsymbol{i} + \boldsymbol{u}_j \boldsymbol{j} + \boldsymbol{u}_k \boldsymbol{k} \approx \left\| \begin{array}{ccc} \varepsilon_x & \dfrac{\gamma_{yx}}{2} & \dfrac{\gamma_{zx}}{2} \\ \dfrac{\gamma_{xy}}{2} & \varepsilon_y & \dfrac{\gamma_{zy}}{2} \\ \dfrac{\gamma_{xz}}{2} & \dfrac{\gamma_{yz}}{2} & \varepsilon_z \end{array} \right\| \tag{24}$$

We are going to show that strain state tensor can be written also by means of above defined concept, more precisely by means of displacement vector gradient in a form:

$$\boldsymbol{T_\varepsilon} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})_c\right] = \frac{1}{2}\left[grad\, \boldsymbol{u} + (grad\, \boldsymbol{u})_c\right] \tag{25}$$

The tensor $(\nabla \boldsymbol{u})_c$ above is tensor conjugated toward a tensor $\nabla \boldsymbol{u}$.

If we express gradient of a displacement vector by means of matrix as in equation (17) and similarly also tensor conjugated, we can write:

$$\boldsymbol{T_\varepsilon} = \frac{1}{2}\left[\left\| \begin{array}{ccc} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & \dfrac{\partial u_z}{\partial z} \end{array} \right\| + \left\| \begin{array}{ccc} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{array} \right\|\right] =$$

$$= \left\| \begin{array}{ccc} \dfrac{\partial u_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right) & \dfrac{\partial u_y}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) & \dfrac{\partial u_z}{\partial z} \end{array} \right\| \tag{26}$$

Strain state tensor is also symmetric tensor. If we compare scalar coordinates of the strain state tensor in the matrix according to equations (24) and (26), then we obtain geometric equations, given by equation (23).

3.3 Physical equations

Physical equations determine the dependence between components of strain state tensor and stress state tensor. In the area of elastic deformations there is this dependence linear and in the area of plastic deformations nonlinear. Theory of elastic and small elastic – plastic deformations is based on an elementary assumption that stress state deviator $\boldsymbol{D_\sigma}$ and strain state deviator $\boldsymbol{D_\varepsilon}$ are similar and coaxial, or also that second invariant of the stress state deviator is a function of second invariant of strain state deviator [5]:

$$\boldsymbol{D}_{\boldsymbol{\sigma}} = \psi \cdot \boldsymbol{D}_{\boldsymbol{\varepsilon}} \; ; \qquad I_2(\boldsymbol{D}_{\boldsymbol{\sigma}}) = f\left[I_2(\boldsymbol{D}_{\boldsymbol{\varepsilon}})\right] \tag{27}$$

In the area of elastic deformations, when Hook's general law is valid, is possible to simply derive from equation (27) that quantity ψ is a constant and is equal to:

$$\psi = 2G = \frac{E}{1+\mu} \tag{28}$$

In the area of plastic deformations the quantity ψ is not constant and can be express as functionally dependent on the above defined intensities of shear stresses S_τ and shear strains S_γ [6]:

$$\psi = \frac{2S_\tau}{S_\gamma} \tag{29}$$

For dependence between the main stresses and the main relative deformations is possible to derive for a case of plane strain relation [7]:

$$(\sigma_1 - \sigma_2) = \frac{2S_\tau}{S_\gamma}(\varepsilon_1 - \varepsilon_2) \tag{30}$$

We can see that quantities intensities of stress and strain play in the theory of plasticity great importance. Equation (30) is a basis for experimental stress analysis in the area of plastic deformations by means of photoplasticity methods. We can experimentally find out courses of directions and differences of the main relative deformations and separation of the main stresses has to be calculated by means of equation (30) and numerical methods.

4 CONCLUSION

This article analyzed scalar, vector, and tensor fields, which occur in elastically or plastically deformed bodies during their load. A solution to these fields can be done either in components, or by means of vector and tensor calculus. Mathematical notation in a componential solution is very time-consuming and demanding. Solution and notation by means of tensor calculus is significantly quicker and clearer. To understand the tensor calculus, there is necessary to spend a certain effort and mainly acquire clear idea about quantities and concepts, which are used in tensor calculus. We have tried to give clear idea about strain state and stress state tensor earlier in works [2] and [4]. In this paper we made an attempt to show practical applications of divergence, rotation and gradient of scalar, vector and tensor in solution of elementary equations in theory of elasticity and plasticity.

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