Control Design of Mixed Sensitivity Problem for Educational Model of Helicopter

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Abstract. *The paper deals with the design of H-∞ robust controller, particularly with mixed sensitivity problem for elevation control. It briefly introduces basic mathematical background concerning robust control approach, which is then applied for typical example of MIMO system, that is a helicopter model. The obtained results are verified on real educational physical model CE 150 by Humusoft, ltd.*

Keywords

Algorithms and software, simulation of dynamic systems, robust control of nonlinear systems.

1. Introduction

The objective of the robust control is to design a dynamic control system operating in a real environment. The changes to the surrounding conditions can be caused by the following factors according to [15]:

- component aging,
- temperature effect,
- effect of the working environment.

The control system must not only be resistant to the aforementioned factors but it also must eliminate inaccuracy of the model, i.e. robustness is the relevant ability of the control system to accept changes. The required output value will be reached even when the changes in the properties of the controlled system are limited and constant disturbance signals are operating. From the mathematical point of view, the robust controller is not only suitable for one particular system but for a set of systems [15].

In other words, robustness plays a significant role in the design of control systems as real systems are prone to external disturbances and measurement noise. Moreover, there are often differences between the proposed mathematical models and actual real systems. A typical example is the design of a controller that will stabilize the system even if it is originally unstable and accept a particular level of performance at the presence of disturbance signals, noises, hard-to-model process dynamic characteristics or process parameter variables. Such tasks are best solved by a feedback control mechanism as they bring along a whole range of problems according to [4]:

- high price (e.g. use of sensors),
- system complexity (e.g. possibilities of implementation and reliability),
- system stability (e.g. requirement for internal stability and stabilizing controllers).

The need and significance of robustness as a part of control system designs have been developing since 1980s. Robustness in the standard SISO control is provided by a suitable gain margin and phase margin. When the first design techniques for multi-variable systems developed in 1960s, emphasis was laid on the achievement of good performance, not robustness. The methods that use multivariables were based on the linear–quadratic criterion and Gaussian disturbances. It was demonstrated that they can be successfully used in a whole range of aviation applications where it is possible to set up precise mathematical models, including the descriptions of external disturbance signals or noises. However, the application of these methods, called LQG methods (linear-quadratic Gaussian control), in other industrial branches evidently showed their bad properties from the point of robustness which led to the effort to develop a theory that would explicitly deal with the issue of robustness in the control feedback design. The pioneering work on the development of the theory, today known as the theory of optimal control in H-∞, was introduced at the beginning of 1980s by Georg Zames and Bruce A. Francis. The H-∞ approach first specified the model of system uncertainty, i.e. additive perturbation and/or output disturbances. In most cases, it is enough to find a suitable controller so that the closed loop achieves some robust stability. The performance is also a part of the optimization loss (objective) function. The elegant formulations of the solution are based on the solutions of Riccati equations, e.g. in MATLAB [4].

If we design a controller for a particular interval of parameters, then the control circuit is robustly stable. Another important parameter of robust controllers is their performance, meeting the requirements for parameters according to [15]:

- control,
- disturbance,
- speed of response (settling time).

The problem of the design of the robust controller is based on the feedback circuit (closed loop) which enables working with the sensitivity and elimination of the disturbance. On one side, the feedback of a non-stable system stabilizes it; on the other side, it may destabilize a stable system [15]. Consider the standard control diagram according to Fig. 1.

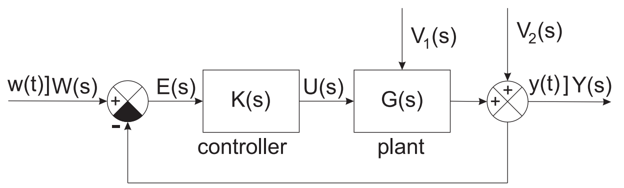

Fig. 1: Classical control scheme with the definition of circuit signals.

Description of the signals in the control circuit:

- $W(s)$: Laplace transform of the the reference signal,
- $E(s)$: Laplace transform of the control error signal,
- $U(s)$: Laplace transform of the manipulated value signal,
- $V_1(s)$: Laplace transform of the disturbance: low-frequency known and unknown disturbances; the system must eliminate them,
- $V_2(s)$: Laplace transform of the disturbance: sensors or measurement, high-frequency character with insignificant effect.

We use the rules of block algebra to define the mathematical relations within the control circuit, i.e. according to [15], [9]:

- for the open-loop transfer function:

$$L(s) = K(s)\,G(s)\,, \tag{1}$$

- for the transfer function of the control error - the sensitivity function:

$$G_E(s) = \frac{E(s)}{W(s)} = \frac{1}{1+L(s)} = S(s)\,, \tag{2}$$

- for the closed-loop transfer function - the complementary sensitivity function:

$$G_W(s) = \frac{Y(s)}{W(s)} = \frac{L(s)}{1+L(s)} = T(s)\,, \tag{3}$$

- limiting condition applies to the sum of the sensitivity function and complementary sensitivity function:

$$S(s) + T(s) = \frac{1}{1+L(s)} + \frac{L(s)}{1+L(s)} = 1. \tag{4}$$

Consider the requirements for the sensitivity function and complementary function according to [15]:

- at $v_1(t) = v_2(t) = 0$, the effect of the control action predominates and so the sensitivity function $\hat{G}_E(s) = S(s)$ must be small and the complementary function $G_W(s) = T(s)$ will be large,
- the entire control circuit carries out the elimination of the low-frequency noise $v_1(t)$, and so again, the effect of the control action predominates: the sensitivity function $G_E(s) = S(s)$ must be small and the complementary function $G_W(s) = T(s)$ will be large,
- the entire control circuit must not affect the elimination of the high-frequency noise $v_2(t)$ and so we eliminate the effects of the control action and control error, i.e. the sensitivity function $G_E(s) = S(s)$ must be small and the complementary sensitivity function $G_W(s) = T(s)$ will also be small.

As the aforementioned opposing requirements cannot be met by one controller, it is necessary to find a compromise between the sizes of the sensitivity function and the complementary sensitivity function [15], see Fig. 2.

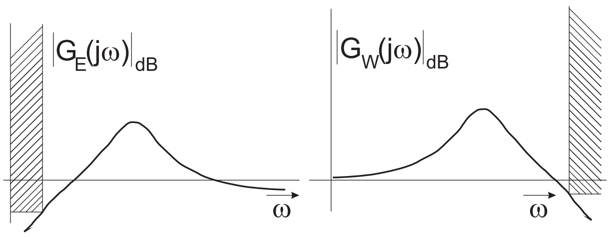

Fig. 2: Logarithmic amplitude frequency characteristics for the sensitivity function (on the left) and complementary sensitivity function (on the right).

2. Robust Controller Design

2.1. Robust Design Methods

The methods of the state space in the time domain allowed to avoid the problems with transfer function matrices and also provided means of the analysis and design of MIMO systems with more inputs and outputs. Approximately at the same time when the methods of optimal control were being developed, research focused on the extensions of means of MIMO system standard control was conducted. The robust design is based on the finding of such a controller so that the resulting system in the closed loop is also robust. Robustness became the main standpoint in the field of control, therefore specifications and methods followed shortly, i.e. according to [15], [5]:

- H_∞ method,
- H_2 method,
- LTR method (loop transfer recovery),
- μ: synthesis,
- QFT method (quantitative feedback theory),
- Kharitonov theorem for the examination of robust stability,
- specification of the small-gain theorem,
- specification of structured singular values.

The following presentation will only focus on the H_∞ method.

2.2. H_∞ Method

A control system is robust if it stays stable and meets particular behavioral criteria at the presence of possible uncertainties. The H_∞ optimization method, developed since 1980s, has proved to be a very efficient and potent design method for robust control in the field of linear, time-invariant control systems [15], [11], [3].

H_∞ controllers have their own terminology, notation and conception. This method leads to a set of suitable stable transfer functions that are physically viable. Similarly to LQR and LQG controllers, we expect optimization of the objective function that will compare different transfer functions and select the most suitable one from the set. The requirements for the closed loop are the following, i.e. according to [15]:

- Physical viability: The order of the transfer function denominator must be higher or equal to the order of the transfer numerator.
- Stability: The transfer poles must lie in the left half plane of the Gaussian plane or in the area of the Laplace transform convergence (provided that the control straight line and imaginary axis are identical).

The basic prerequisite of the H_∞ method is the knowledge of the transfer function of the given system, evaluating ∞-norm according Fig. 5 according to [15], [1]:

$$\|G\|_\infty = sup_\omega \left\{ |G(j\omega)| \right\}. \tag{5}$$

The norm can be graphically represented as the maximum of the Bode diagram provided that the transfer function is definite and has no imaginary poles, while its objective is to minimize it in the ∞-norm. It decreases the apex of the Bode diagram, which increases the robust stability margin [15].

2.3. Mixed Sensitivity Problem

Usually, practical industrial applications do not only use one objective function but a combination of several functions like that, e.g. accomplishment of the good performance of tracking the reference signal at limited energy of the reference signal. Then we solve the mixed sensitivity task, or the 'S over KS' problem defined by a general relation (for the SISO system) according to [8], [12]:

$$\begin{aligned} &\min_{K_{st}} \left\| \begin{matrix} S(s) \\ K(s)\,S(s) \end{matrix} \right\|_\infty = \\ &= \min_{K_{st}} \left\| \begin{matrix} [1+L(s)]^{-1} \\ K(s)\,[1+L(s)]^{-1} \end{matrix} \right\|_\infty . \end{aligned} \tag{6}$$

The Eq. 6 can also be expressed by the requirements of the design concerning the additive perturbation, e.g. nominal behavior, good performance of tracking the reference signal or the elimination of disturbance signals and robust stability [4].

Figure 3 shows the standard block diagram of the H_∞ configuration using the linear fractal transformation (LFT) with specification of the individual external

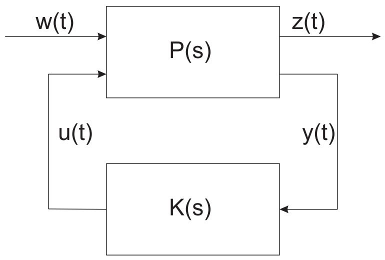

Fig. 3: Standard block diagram at the H_∞ configuration.

inputs, external outputs, inputs into the controller and its outputs, see [1] (p. 438). The control circuit contains a robust controller with transfer function $K(s)$ and a perturbed (also extended or generalized) system with transfer function $P(s)$ that has two inputs and two outputs according to [15]:

- $\vec{w}(t)$ input reference signal vector; external input signals,
- $\vec{v}(t)$ output manipulated signal vector; output control signals from the controller.

The main difference between the vectors is that the controller does not affect the inputs. The input reference signal vector $\vec{w}(t)$ includes an external noise, noise from the sensors and tracking (reference) signals. To the contrary, the outputs from the system are divided into two groups according to [15]:

- $\vec{y}(t)$ output signal vector; measured outputs,
- $\vec{z}(t)$ controlled outputs; minimized or penalized outputs.

The task is then defined so that the internally stabilizing controller $K(s)$ is searched for in the control circuit of the robust control for the given generalized system $\mathbf{P}(s)$ that minimizes or penalizes the controlled output vector $\vec{z}(t)$. In other words, we minimize the maximal norm of the transfer function between $\vec{w}(t)$ and $\vec{z}(t)$ by the given relation according to [4]:

$$\vec{z} = \left\{P_{11}(s) + P_{12}(s)K(s)I_{PK}^{-1}P_{21}(s)\right\}\vec{w}, \tag{7}$$

where:

$$I_{PK} = I - P_{22}(s)K(s), \tag{8}$$

we get a linear fractal transformation after the adjustment:

$$\vec{z} = F_l[\mathbf{P}(s), K(s)]\vec{w}. \tag{9}$$

Then, the H_∞ optimization problem can be expressed by a relation, i.e. according to [4]:

$$\min_{K_{st}} \|F_l[\mathbf{P}(s), K(s)]\|_\infty . \tag{10}$$

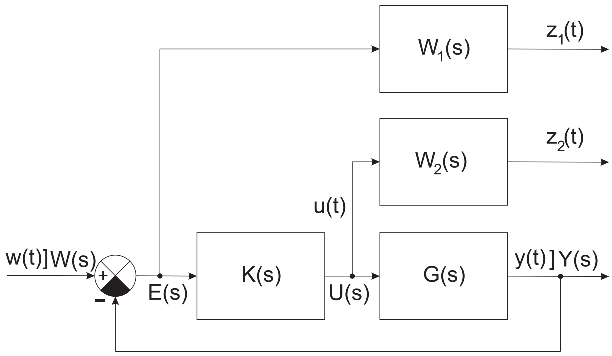

Fig. 4: Standard block diagram of the mixed sensitivity problem: controller and perturbed system containing nominal system, control error and manipulated value weighting filters.

Figure 4 shows the standard block diagram of the mixed sensitivity problem and it is basically a more detailed illustration of Fig. 3, where it is easy to deduce the following relations, i.e.:

- for external input signals:

$$\vec{w}(t) = r(t), \tag{11}$$

- for output control signals from the controller:

$$\vec{u}(t) = u(t), \tag{12}$$

- for measured outputs:

$$\vec{y}(t) = e(t), \tag{13}$$

- for minimized or penalized outputs:

$$\begin{aligned}\vec{z} &= [z_1(t),\ z_2(t)]^T = \\ &= [W_1(s)e(t),\ W_2(s)u(t)]^T.\end{aligned} \tag{14}$$

The following applies to the generalized system containing weighting filters according to [4]:

$$\mathbf{P}(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}, \tag{15}$$

while:

$$P_{11}(s) = W_1(s)[I,\ 0]^T = [W_1(s), 0]^T, \tag{16}$$

$$P_{12}(s) = [-W_1(s)G(s), W_2(s)I]^T, \tag{17}$$

$$P_{21}(s) = [I]^T = I, \tag{18}$$

$$P_{22}(s) = [-G(s)]^T = -G(s). \tag{19}$$

The weighting filters $W_1(s)$ and $W_2(s)$ are commonly used in practice; in such case, the Eg. 6 can be formally adjusted into the form describing a objective function, i.e. (for the SISO system) according to [4]:

$$\min_{K_{st}} \left\| \begin{matrix} W_1(s)S(s) \\ W_2(s)K(s)S(s) \end{matrix} \right\|_\infty . \tag{20}$$

3. Controller Design for Helicopter Elevation

On the basis of the aforementioned theoretical findings from the field of the H_∞ robust control, we can design a robust controller according to the selected conception when we deal with the question of the autonomy of the control and the problem of mixed sensitivity and also consider the input signals. Derived mathematical models in elevation and azimuth are crucial for the design of controllers, when we design the controller for both the mathematical model and real model. However, this paper only focuses on elevation, due to the extent of the task. Finally, both responses are compared to one another.

The mathematical description of helicopter model is taken from official manual, see [6]. The design of elevation controller particularly comes out from Figs. 2.1, 2.2 stated on pages 8, 15 of this manual.

3.1. Autonomy Requirement

The main problem of this MIMO system namely includes the elimination of the elevation-azimuth coupling. All in all, we naturally want to eliminate both relations but we know that the azimuth-elevation relation is not as significant. The conception is thus based on the assumption of designing two independent controllers, i.e. an elevation controller and an azimuth controller.

The definition of autonomy says that the reference signal $\vec{w}(t)$ must only affect just one corresponding output signal $\vec{y}(t)$, [7].

Autonomy also requires that the transfer matrix of the open loop is diagonal, the elements of the matrix are only located on the main diagonal. A priori, diagonality is required in the control matrix. With respect to the explicitly written sign, the transfer function of the correction term is given by the general relation based on the theory of matrix determinants (Laplace complement) according to [2]:

$$R_{i,j}(s) = (-1)^{j+1} R_{j,j}(s) \frac{\left|G^*_{j,i}(s)\right|}{\left|G^*_{j,j}(s)\right|}, \tag{21}$$

where i is matrix line index, j is matrix column index, $(-1)^{j+1}$ is Laplace (algebraic) complement, $\left|G^*_{j,i}\right|$ is matrix determinant $G^*_{j,i}(s)$ and $\left|G^*_{j,j}\right|$ is matrix determinant $G^*_{j,j}(s)$.

The relation Eq. (21) rather has a theoretical character. Directly derived conditions of autonomy pay off at a small amount of regulated signals (in our case there are two signals) rather than having to exactly remember its content and avoid making a mistake in the algebraic complement. The situation is depicted in Fig. 5, directly modified for the helicopter model.

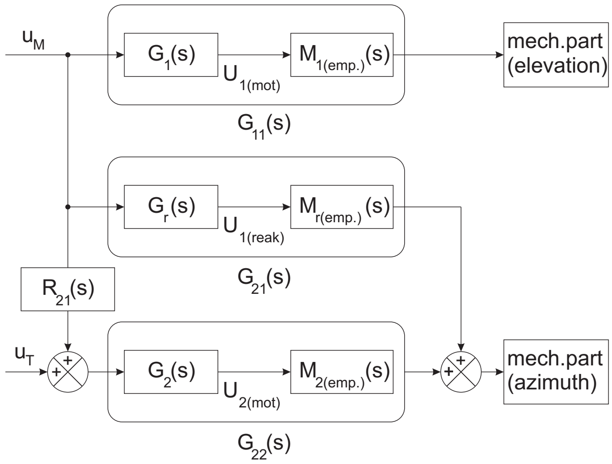

Fig. 5: Block diagram for the elimination of the elevation-azimuth coupling; the azimuth-elevation coupling is neglected.

If the manipulated value expressed by voltage U_M induces an undesirable response in the output of the azimuth mechanical part described by the Laplace transform $G_{21}(s)\,U_M(s)$, then it can be completely compensated by the correction term $R_{21}(s)$ under the condition according to [2]:

$$\begin{array}{l} G_{21}(s)\,U_M(s) + G_{22}(s)\,R_{21}(s)\,U_M(s) = 0 \\ \Rightarrow R_{21}(s) = -\frac{G_{21}(s)}{G_{22}(s)} \end{array}. \tag{22}$$

The transfer function of the second correction term can be either written directly with the use of the principle of cyclical substitution of indexes or a conditional equation can be set up again Eq. (23), i.e. according to [2]:

$$\begin{array}{l} G_{12}(s)\,U_T(s) + G_{11}(s)\,R_{12}(s)\,U_T(s) = 0 \\ \Rightarrow R_{12}(s) = -\frac{G_{12}(s)}{\hat{G}_{11}(s)} \end{array}. \tag{23}$$

The relation Eq. (23) describes the azimuth-elevation coupling that, however, is not significant, and thus can be written according to [2]:

$$R_{12}(s) = G_{12}(s) = 0. \tag{24}$$

The diagram in Fig. 5 can also be interpreted in the following way: the internal physical coupling between the elevation and azimuth in the form of transfer function $G_{21}(s)$ cannot be eliminated without the basic (constructive) interference into the system. However, we can quite easily implement an external connection between the inputs of the helicopter set by the correction term $R_{21}(s)$ that will ensure the same as the unfeasible elimination of the cross-coupling inside the MIMO system transfer matrix, decomposition of the model into two models that seemingly do not influence one another. Their control can then be ensured by two control loops independent of one another [2].

If we substitute in the relation Eq. (23), we will receive transfer function of the correction term, Eq. (25):

$$R_{21}(s) = -\frac{T_{r_2}s^2 + T_{r_1}s + 1}{B(s)} \frac{a_r U_M s + b_r}{a_2 U_T s + b_2} A(s), \quad (25)$$

where $A(s) = \left(T_2 s^2 + 1\right)^2$, $B(s) = \left(T_1 s^2 + 1\right)^2$. The relation (Eq. 25) clearly shows that the transfer function in this form does not meet the condition of physical viability as the numerator order is higher than the denominator order. Thus we define two inertias with conditions $\xi_1 \ll T_{r_1}$ and $\xi_2 \ll T_{r_2}$. The physically feasible correction transfer function eliminating the elevation-azimuth coupling is given by the relation Eq. (26):

$$R_{21}(s) = -\frac{T_{r_2}s^2 + T_{r_1}s + 1}{\xi_2 s^2 + \xi_1 s + 1} \frac{a_r 0.55 s + b_r}{a_2 0.2 s + b_2} \frac{A(s)}{B(s)}. \quad (26)$$

3.2. Controller Conception

The conception of the helicopter model robust control design was partially explained in the requirements for the autonomy of the elevation and azimuth control. The second part concerns the H_∞ robust control, namely the modification of the mixed sensitivity problem (MSP) where we also penalize the group of external output signals in addition to the control error $\vec{e}(t)$ and manipulated value $\vec{u}(t)$:

- $\vec{d}_1(t) = r(t) = \vec{w}(t)$ reference or external control signal,
- $\vec{d}_2(t)$ low-frequency signal (disturbance),
- $\vec{d}_3(t)$ high-frequency disturbance signal (noise).

We use weighting transfer functions or also called weighting filters to penalize signals incoming and outgoing from the extended system in the elevation or azimuth. The whole block diagram of the control circuit with extended system is shown in Fig. 6.

There are many different ways for the extension of the nominal system. However, the more external inputs and penalized (error) outputs there are, the more difficult it is to select the weighting filters. The weighting filters are generally stable transfer functions (not necessarily proper rational functions) of a particular order. Thus, the more we add, the higher the order the resulting system will have. Such an interconnected system can be then used to express the state description, or the transfer function of the H_∞ optimal, or

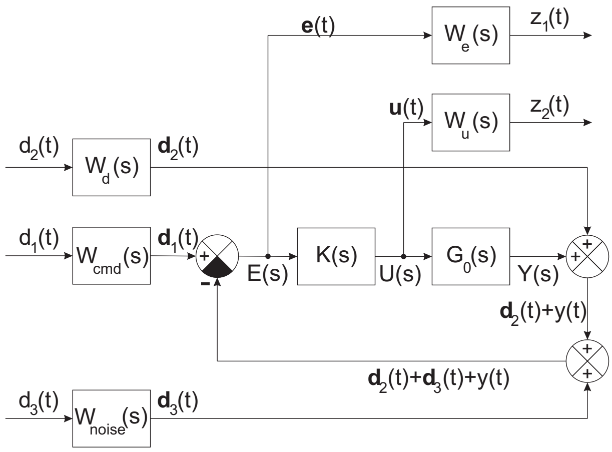

Fig. 6: Block diagram of the connection of the H_∞ robust controller, nominal system and weighting filters for the helicopter model.

suboptimal robust controller $K(s)$ with one degree of freedom (1DOF configuration).

The meaning of the individual blocks is as follows according to [10]:

- $W_{cmd}(s)$: this weighting transfer function takes care of the reference tracking. A normalized signal appears at the input and the signal at the output is in relevant physical units,
- $W_d(s)$: this weighting transfer function adjusts the frequency and amplitude characteristics of external low-frequency disturbance signals affecting the nominal system,
- $W_{noise}(s)$: this weighting transfer function represents the models of noises of sensors in the frequency domain. It tries to detect a particular piece of information in the control derived from laboratory experiments or production measurements. Naturally, the noise shows a high-frequency character,
- $W_e(s)$: this weighting transfer function penalizes the control signal from the robust controller and thus the control error. It determines the inverted value of the expected form of the output signal. The signal that appears at the input of the filter is in relevant physical units and it is normalized at the output,
- $W_u(s)$: this weighting transfer function penalizes the control signal from the robust controller and thus the manipulated value signal. It determines the inverted value of the expected form of the output signal. The signal that appears at the input of the filter is in relevant physical units and it is normalized at the output,

The complete calculation of both the transfer function of the H_∞ robust controller and the key H_∞ norm is executed in MATLAB. The calculation algorithm is based on the correct interconnection of the nominal system with the weighting filters in correspondence with Fig. 6. The program solution in the M-file appears as follows, i.e.:

```
systemnames = 'G Wcmd Wd Wnoise We Wu';
inputvar = '[d1; d2; d3; u]';
outputvar = '[We; Wu; Wcmd-G-Wnoise]';
input_to_Wcmd = '[d1]';
input_to_Wd = '[d2]';
input_to_Wnoise = '[d3]';
input_to_G = '[u]';
input_to_We = '[Wcmd-G-Wnoise]';
input_to_Wu = '[u]';
cleanupsysic = 'yes';
P = sysic
NControl = 1;
NMeasure = 1;
r = [NControl NMeasure];
[K,CL,gopt] = hinfsyn(P,NMeasure,NControl);
```

First of all, we define what systems we will interconnect, the variable `systemnames`. Then we define the input signal vector (external control signals and control signal from the controller), the variable `inputvar`. The output is represented by the variable `outputvar` containing the penalization of the control error (`We`), manipulated value (`Wu`) and the measured output (`Wcmd-G-Wnoise`). Subsequently, we connect all the inputs to the weighting filters. By the `cleanupsysic` command with the attribute value set to *yes* we confirm that we want to remove the variables `systemnames`, `inputvar` and `outputvar` from the MATLAB work environment (Workspace) immediately after the creation of the system interconnection.

The variable `P` represents the extended system or system interconnection (`sysic`, System Interconnection). To complete the enumeration of parameters for the calculation, we have to define the number of control outputs from the control (`NControl`) and the number of measured outputs (`NMeasure`). We will obtain the calculation of the H_∞ controller (`K`), closed loop transfer (`CL`) and maximum closed loop transfer norm (`gopt`) by activating the *hinfsyn* function with the following parameters: `P`, `NMeasure` and `NControl`.

The activation of the *hinfsyn* function can also be extended by more input and output parameters; in this actual case according to [10]:

- two algebraic Riccati equations are solved,
- $\gamma \in (0, +\infty)$,
- the closed-loop transfer functions is calculated with the use of the linear fractal transformation $CL = F\{\mathbf{P}(s), K(s)\}$,
- $\gamma_0 = \|CL\|_\infty = \|F\{\mathbf{P}(s), K(s)\}\|_\infty$.

MATLAB, namely the *Robust Control Toolbox*, contains other functions that can be used to solve the issue of the design of a continuous or discrete H_∞ robust controller. For completeness, we only give the prototype of the function focused on the standard mixed sensitivity problem:

```
[K, CL, gopt, INFO] = mixsyn(G, W1, W2, W3)
```

The problem of the *mixsyn* function is the number of the weighting filters and their character as they only penalize the control error (`W1`), manipulated value (`W2`) and measured output (`W3`). With regard to the selected design conception, it would not be possible to penalize inputs with this function.

3.3. Elevation Controller for Mathematical Model

The transfer function of the dynamics of the mathematical model in elevation is given by the relation according to the relation Eq. (27):

$$G_\psi(s) = \frac{\Psi(s)}{U_M(s)} = \\ = \frac{7.3315s + 1.1883}{3s^4+23s^3+116s^2+519s+1000}. \tag{27}$$

The amplitude and phase frequency characteristics are shown in Fig. 7 which also clearly show that it contains the highest value under the following conditions:

- frequency: $\omega_{\text{MAX}} = 4.9448\ \left[\text{rad}\cdot\text{s}^{-1}\right]$,
- transfer function module:

$$|G_\psi(j\omega)|_{MAX} = \\ = -16.05\ [dB] \cong 0.1576\ [-]. \tag{28}$$

The system is of the fourth order and contains four stable poles, out of which two are complex conjugate and one is a double pole:

$$p_1 = -0.2105 + j4.9448, \tag{29}$$

$$p_2 = \overline{p_1} = -0.2105 - j4.9448, \tag{30}$$

$$p_3 = p_4 = -4. \tag{31}$$

The maximum norm of the given system is (in accordance with the maximum value of the transfer function module):

$$\|G_\psi(s)\|_\infty = 0.1577\ [-]. \tag{32}$$

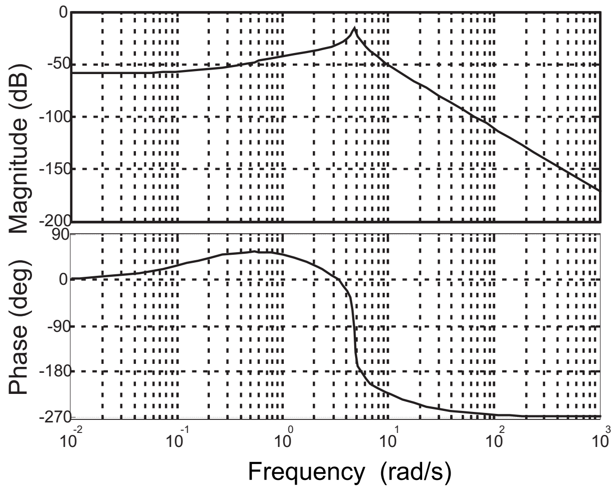

Fig. 7: Amplitude frequency characteristic and phase frequency characteristic of the given transfer function of the mathematical model in elevation.

The forms of the weighting filters for the given system are as follows, i.e.:

- weighting transfer function for reference signal:

$$W_{cmd}(s) = \frac{1}{0.25s+1}, \tag{33}$$

- weighting transfer function for low-frequency disturbance signal:

$$W_d(s) = \frac{0.5}{0.1s+1}, \tag{34}$$

- weighting transfer function for high-frequency disturbance signal (noise):

$$W_{noise}(s) = \frac{0.01s+1}{s+1}, \tag{35}$$

- weighting transfer function for control error signal:

$$W_e(s) = K_e \frac{\frac{1}{M_e}s + \omega_{be}}{s + \omega_{be}\varepsilon_e} = 0.001\frac{s+0.5}{s+0.0005}, \tag{36}$$

- weighting transfer function for manipulated value signal:

$$\hat{W}_u(s) = K_u \frac{s + \frac{\omega_{bu}}{M_u}}{\varepsilon_u s + \omega_{bu}} = 10^{-7}\frac{s+1}{0.01s+2}. \tag{37}$$

The weighting filters for the control error and manipulated signal have a prescribed transfer function form according to [14], the transfer always contains the same numerator and denominator order as to ensure the stability of the inverted transfer functions. The filter for the control error is low-pass and for the manipulated value it is high-pass. The graphic dependences of the sensitivity functions and inverted transfer functions for

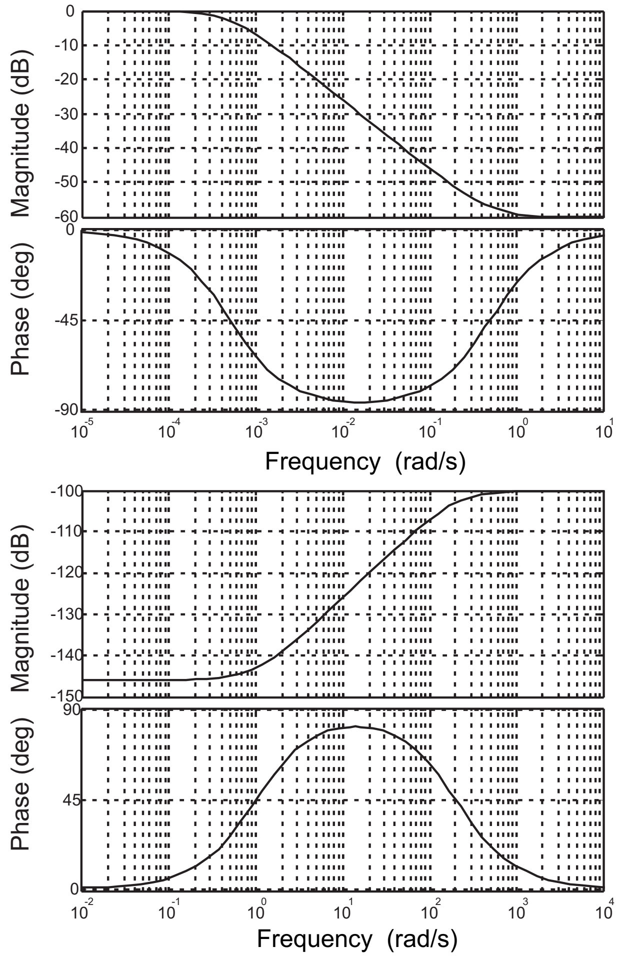

Fig. 8: Amplitude and frequency characteristics of the low-pass weighting filter $W_e(s)$ (upper part) and high-pass weighting filter $W_u(s)$ (lower part): mathematical model in elevation.

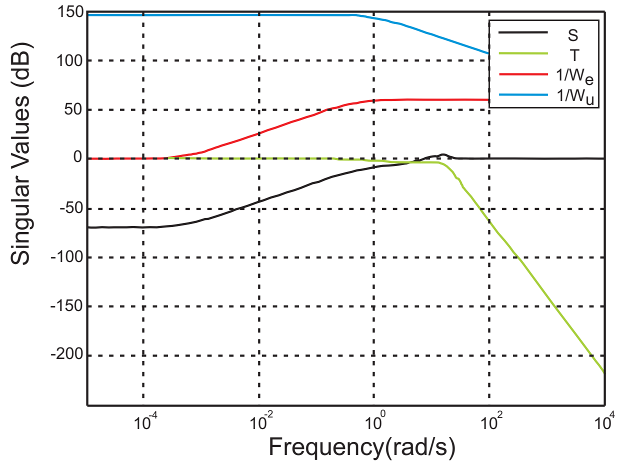

Fig. 9: Amplitude frequency characteristic of the sensitivity function $S(s)$, complementary sensitivity function $T(s)$, inverted transfer functions $1/W_e(s)$ and $1/W_u(s)$: mathematical model in elevation.

the mathematical model in elevation are as stated in Fig. 8.

One of the general requirements for the amplitude frequency characteristics of the sensitivity function $S(s)$ and the inverted transfer function $1/W_e(s)$ defines robust behavior:

$$\forall\, \omega \in \mathbb{R}: \; |S(j\omega)| \leq |1/W_e(j\omega)| = \\ = 1/|W_e(j\omega)| \Leftrightarrow \|W_e(s)\, S(s)\|_\infty \leq 1. \tag{38}$$

Figure 9 implies that the relation Eq. (38) is fully met. Similarly, it is also possible to define the amplitude frequency characteristics for the inverted transfer function $1/W_u(s)$ and the product of the controller transfer and the sensitivity function $K(s)\, S(s)$:

$$\forall\, \omega \in \mathbb{R}: \; |K(j\omega)\, S(j\omega)| \leq |1/W_u(j\omega)| \Leftrightarrow \\ \Leftrightarrow \|W_u(s)\, K(s)\, S(s)\|_\infty \leq 1. \tag{39}$$

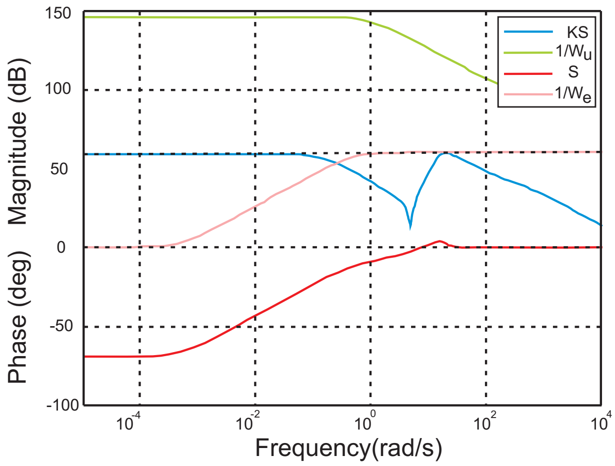

Fig. 10: Amplitude frequency characteristics of the sensitivity function $S(s)$, product of transfer functions $K(s)\, S(s)$, inverted transfer functions $1/W_e(s)$ and $1/W_u(s)$: mathematical model in elevation.

Figure 10 implies that the relation Eq. (39) is fully met. For completeness, we give the values of the key H_∞ norms, i.e.:

- optimal H_∞ norm:

$$\gamma = 5.752010^{-4}, \tag{40}$$

- closed loop H_∞ norm:

$$\|F\{\mathbf{P}(s), K(s)\}\|_\infty = 4.783710^{-4} < \gamma, \tag{41}$$

- sensitivity function H_∞ norm:

$$\|S(s)\|_\infty = 1.5357, \tag{42}$$

- complementary sensitivity function H_∞ norm:

$$\|T(s)\|_\infty = 0.9997. \tag{43}$$

The order of the designed H_∞ controller for the mathematical elevation model corresponds with the total of the orders of the individual elements of the extended system, the nominal system is of the fourth order at the most and all five weighting systems are in the first order at the most. The aforementioned implies that the controller will be a system of the ninth order at the most. Its frequency characteristics are shown in Fig. 11.

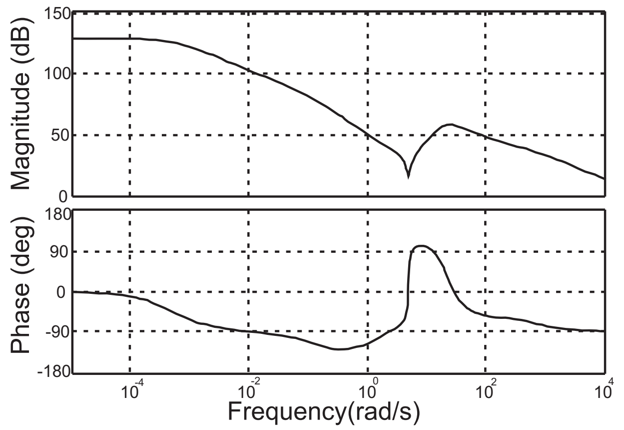

Fig. 11: Amplitude frequency characteristic and phase frequency characteristic of the H_∞ controller for mathematical model in elevation.

According to Fig. 6, a model of the H_∞ controller and the extended system was created in Simulink. The configuration of the group of external input signals is as follows:

- reference:

$$d_1(t=0) = r(t=0) = \; -0.25 \;\; [-] \\ \text{for } t \in \langle 0; 20 \rangle, \tag{44}$$

$$d_1(t) = r(t=20) = -0.05 \;\; [-] \\ \text{for } t \in \langle 20; 100 \rangle, \tag{45}$$

- LF disturbance: not included in the model,
- HF disturbance: band-limited white noise with power of 0.00001 [W] .

The response of the modeled elevation system to reference $\vec{d_1}(t)$ is as shown in Fig. 12. Thanks to the balanced ratio of the sensitivity function and the complementary sensitivity function, the elimination of the noise and disturbance will be effective.

3.4. Elevation Controller for Real Model

We will use the transfer function from the mathematical model for the design of the controller for a real

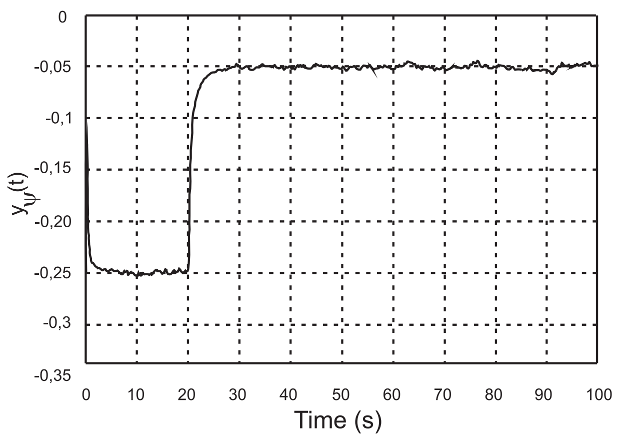

Fig. 12: Dependence of the output elevation angle y_ψ on time t in the mathematical elevation model; $y_\psi = f(t)$.

model as the controller is created on the basis of the transfer functions of the nominal system and weighting filters. The forms of the weighting filters for the given system are as follows, i.e.:

- weighting transfer function for reference signal:

$$W_{cmd}(s) = \frac{1}{0.25s+1}, \tag{46}$$

- weighting transfer function for low-frequency disturbance signal:

$$W_d(s) = \frac{0.5}{0.1s+1}, \tag{47}$$

- weighting transfer function for high-frequency disturbance signal (noise):

$$W_{noise}(s) = \frac{0.01s+1}{s+1}, \tag{48}$$

- weighting transfer function for control controller:

$$W_e(s) = 0.003\frac{0.1s+0.75}{2.25s+0.065}, \tag{49}$$

- weight transfer for manipulated value:

$$W_u(s) = \frac{0.75s+1}{s+2500} = \frac{0.75s+1}{s+\frac{1}{4\cdot 10^{-4}}}. \tag{50}$$

When compared to the mathematical elevation model, the same penalization of the group of external input signals is used but the weighting filters for the penalization of error outputs were adjusted to the real model. However, their character remained unchanged.

The graphic dependences of the sensitivity functions and inverted transfer functions for the real elevation model are as shown in Fig. 13 and Fig. 14.

Figure 15 clearly implies that the amplitude frequency characteristic of the sensitivity function $S(s)$

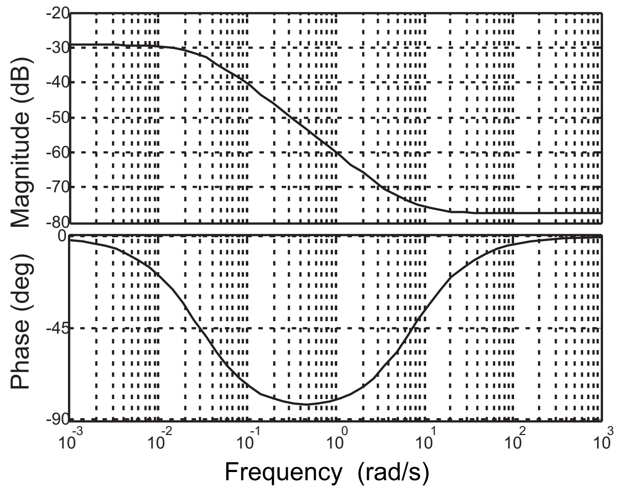

Fig. 13: Amplitude and phase frequency characteristics of low-pass filter $W_e(s)$, real model in elevation.

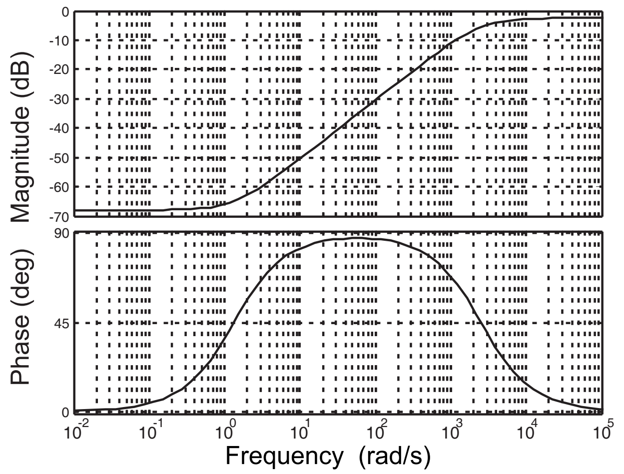

Fig. 14: Amplitude and phase frequency characteristics of high-pass weighting filter $1/W_u(s)$, real model in elevation.

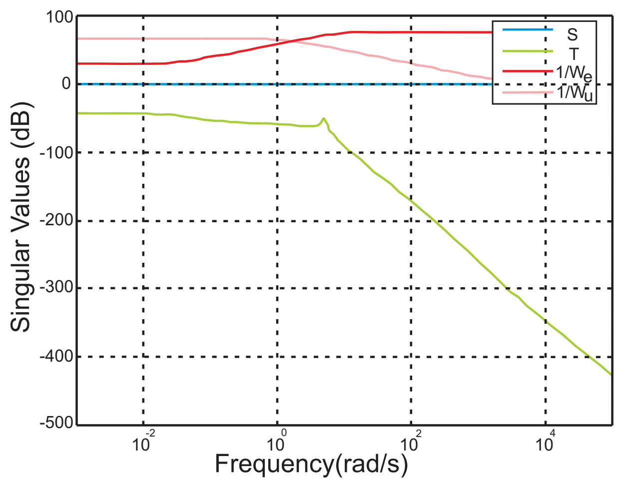

Fig. 15: Amplitude frequency characteristics of the sensitivity function $S(s)$, complementary sensitivity function $T(s)$, inverted transfer functions $1/W_e(s)$ and $1/W_u(s)$, real Mmodel in elevation.

and the inverted transfer function $1/W_e(s)$ meet the condition stated in the Eq. (38).

Figure 16 clearly shows that the amplitude frequency characteristic of the product of Laplace transforms $K(s)S(s)$ and the inverted transfer function $1/W_u(s)$ meet the condition stated in the Eq. (39). For completeness, we give the values of the key H_∞ norms, i.e.:

- optimal H_∞ norm:

$$\gamma = 0.0512, \tag{51}$$

- closed loop H_∞ norm:

$$\left\| F\{\mathbf{P}(s), K(s)\} \right\|_\infty = 0.0487 < \gamma, \tag{52}$$

- matrix H_∞ norm:

$$\left\| \begin{matrix} W_e(s)S(s) \\ W_u(s)K(s)S(s) \end{matrix} \right\|_\infty = 0.0344 < \gamma, \tag{53}$$

- sensitivity function H_∞ norm:

$$\left\| S(s) \right\|_\infty = 1.0028, \tag{54}$$

- complementary sens. fun. H_∞ norm:

$$\left\| T(s) \right\|_\infty = 0.0073, \tag{55}$$

- control sensitivity H_∞ norm:

$$\left\| R(s) \right\|_\infty = 6.6069, \tag{56}$$

- open loop H_∞ norm:

$$\left\| L(s) \right\|_\infty = 0.0074, \tag{57}$$

- controller H_∞ norm:

$$\left\| K(s) \right\|_\infty = 6.6557. \tag{58}$$

The numerical interpretation of H_∞ norms shows that it is a suboptimal solution. The controller is in the ninth order at the most, just like in the mathematical model. Its frequency characteristics are shown in Fig. 17.

The block diagram in Simulink is formally the same as in the mathematical model in elevation; the only difference is thus the nominal system represented by the real model of a helicopter. The configuration of the inputs also does not differ from the mathematical model as we want to compare the responses. The response of the modelled elevation system to reference $d_1(t)$ in Fig. 18. Figure 18 implies that the elimination of noise will not be sufficiently effective due to the prevailing sensitivity function.

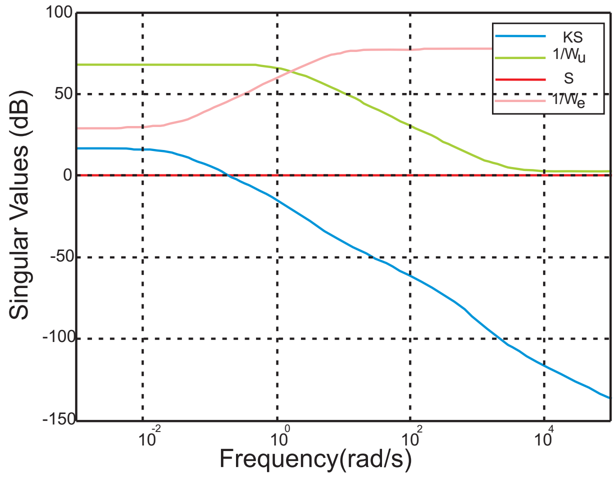

Fig. 16: Amplitude frequency characteristic of the sensitivity function $S(s)$, product of transfer functions $K(s)S(s)$, inverted transfer functions $1/W_e(s)$ and $1/W_u(s)$, real model in elevation.

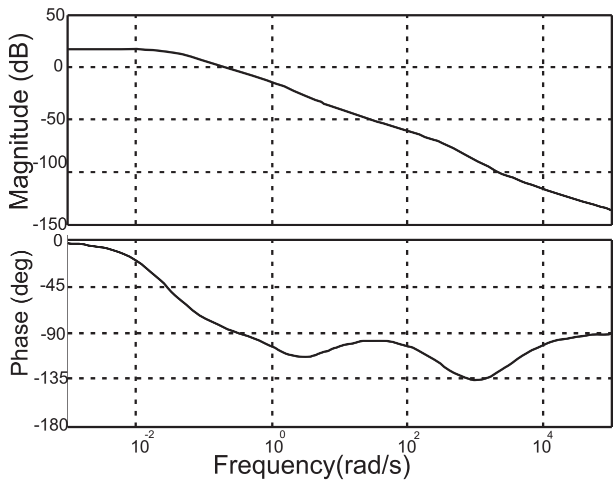

Fig. 17: Amplitude frequency characteristic and phase frequency characteristic of the H-∞ controller for real model in elevation.

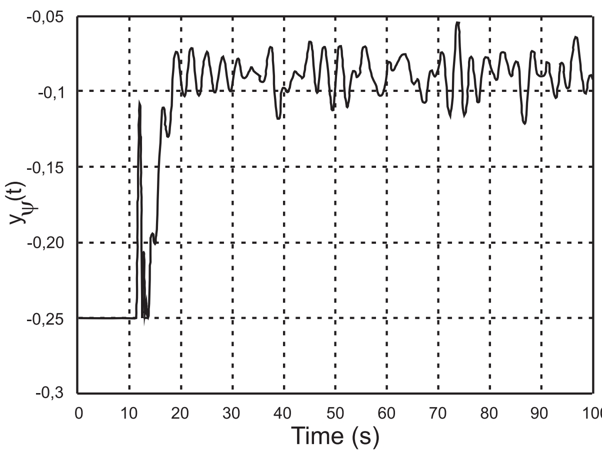

Fig. 18: Dependence of the output elevation angle y_ψ on time t in real elevation model; $y_\psi = f(t)$.

4. Conclusion

The paper focused on the problem of design of mixed-sensitivity H-∞ robust controller, the simulation of control circuit and verification of the results on educational model CE150, representing physical model of helicopter. It gives an analysis and synthesis of the robust controller of elevation of helicopter model, both for its mathematical model and for real system.

The results were verified by implementing the algorithms on two different platforms:

- Matlab&Simulink + Real Time Toolbox + MF624: Real Time Toolbox together with measuring card MF624 provide a very elegant way of fast rapid prototyping for real physical educational models, using the powerful computational environment of Matlab&Simulink.
- REX Control System + WinPAC-8000: The REX control system is an advanced tool for design and implementation of complex algorithms for automatic control. The algorithms are composed from individual function blocks, which are available in extensive function block libraries [13]. It supports a wide variety of hardware platforms, including programmable automation controllers (PAC) by ICPDAS, particularly WinPAC-8000 which was used for implementation of design control algorithms.

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