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WILLIAMS EXPANSION TERMS AND THEIR IMPORTANCE FOR ACCURATE STRESS FIELD DESCRIPTION IN SPECIMENS WITH A CRACK

ČLENY WILLIAMSOVA ROZVOJE A JEJICH VÝZNAM PRO PŘESNÝ POPIS POLE NAPĚTÍ V TĚLESECH S TRHLINOU

Abstract

The paper deals with an analysis of the stress and displacement fields in various cracked specimens (loaded in mode I as well as in mode II). The main goal is to determine an accurate approximation of the near-crack-tip fields which can be subsequently used e.g. in the case of quasi-brittle materials for estimation of the extent of a nonlinear zone at the crack tip in general, the fracture process zone in particular. Values of coefficients of the so-called higher-order terms of the power expansion by which the fields can be expressed are determined via regression from results of numerical simulations. The analysis is conducted using 2D numerical models and ANSYS FE computational system is employed. Various aspects regarding the description of the stress/displacement fields by means of the Williams expansion are discussed; especially the convergence of the coefficients of the first several terms of the series expansion, their absolute values and importance for an accurate stress approximation.

Abstrakt

Článek se zabývá analýzou polí napětí a posuvů u několika těles s trhlinou (zatížených módem I i II). Hlavním cílem je přesně aproximovat zmiňovaná pole v blízkosti vrcholu trhliny, což může být následně využito např. pro odhad rozsahu obecně nelineární zóny u čela trhliny, konkrétně lomové procesní zóny v případě kvazi-křehkých materiálů. Hodnoty koeficientů tzv. členů vyšších řádů mocninného rozvoje, pomocí kterého mohou být ona pole vyjadřována, jsou určovány pomocí regrese na základě výsledků numerických simulací. Analýza je prováděna na 2D numerických modelech realizovaných ve výpočtovém systému ANSYS. Diskutovány jsou různé aspekty popisu polí napětí/posuvů pomocí Williamsova rozvoje; především konvergence koeficientů prvních několika členů řady, jejich absolutní hodnoty a význam pro přesnou aproximaci napětí.

Keywords

Williams series, over-deterministic method, higher-order terms, crack-tip stress field, FEM.

1 INTRODUCTION

Accurate knowledge of stress/displacement fields in structures with cracks plays a key role when their fracture behavior is investigated. While only one parameter (i.e. the stress intensity factor) is sufficient in the case of brittle materials, materials exhibiting quasi-brittle fracture are more difficult to assess. The reason is that there exists a relatively large zone near the crack tip where the fracture process occurs and where the material behaves nonlinearly. Therefore, it is necessary to

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know the stress/displacement field also in farther distances from the crack tip than it is usual for brittle and even elasto-plastic fracture. However, this is incompatible with the theory of the classical linear elastic fracture mechanics considering that the (singular) stress behavior is fully described by means of the stress intensity factor. Nevertheless, this idea is not more valid if the distance (r) from the crack tip increases. Therefore, the so-called Williams expansion [1] for the stress/displacement distribution near a crack tip is used in this work and several more (non-singular) terms of the series are considered. A regression method based on the knowledge of the displacement field around the crack tip (calculated by means of conventional finite elements software) is used for estimation of the coefficients of the mentioned higher-order terms. Different cracked geometries are analyzed and discussion from different points of view is presented.

2 THEORETICAL BACKGROUND

In order to explain clearly the procedure of the approximation of the stress/displacement distribution in quasi-brittle materials, several basic terms and definitions should be introduced.

2.1 Williams expansion for near-crack tip stress/displacement field

In 1957, Williams derived a series expansion enabling description of the stress/displacement field near a crack tip. The form of the infinite power expansion defined for a 2D homogeneous isotropic linear-elastic body with a crack subjected to arbitrary remote loading (combination of mode I and II) can be written as:

$$\sigma_{ij} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n f^{\sigma}_{\mathrm{I},ij}(\theta, n) + \sum_{m=1}^{\infty} \frac{m}{2} r^{\frac{m}{2}-1} b_m f^{\sigma}_{\mathrm{II},ij}(\theta, m) \ , \tag{1}$$

$$u_i = \sum_{n=0}^{\infty} \frac{n}{2} r^{\frac{n}{2}} a_n f^{u}_{\mathrm{I},ij}(\theta, n, E, \nu) + \sum_{m=0}^{\infty} \frac{m}{2} r^{\frac{m}{2}} b_m f^{u}_{\mathrm{II},ij}(\theta, m, E, \nu) \ , \tag{2}$$

where:

σ_{ij} – stress tensor components [Pa],

u_i – displacement vector components [m],

r, θ – polar coordinates with the origin of the coordinate system positioned to the crack tip and positive x axis pointing ahead of the crack tip [m, rad],

E, ν – Young's modulus and Poisson's ratio, [Pa] and [-], respectively,

n, m – index of the mode I and mode II power expansion terms, respectively [-],

a_n, b_m – coefficient of the nth mode I term and of the mth mode II term of the Williams series, respectively (depend on the relative crack length $\alpha = a/W$, where a [m] is the crack length and W [m] is the specimen width) [Pa/m$^{n/2-1}$] and [Pa/m$^{m/2-1}$], respectively,

$f^{\sigma}_{\mathrm{I},ij}$, $f^{\sigma}_{\mathrm{II},ij}$ – known functions corresponding to the stress distribution for loading mode I and II, respectively [-],

$f^{u}_{\mathrm{I},ij}$, $f^{u}_{\mathrm{II},ij}$ – known functions corresponding to the displacement distribution for loading mode I and II, respectively [-].

In order to avoid difficulties connected to using physical units of the coefficients of the individual higher-order terms, the dimensionless shape functions $g_{\mathrm{I},n}$ and $g_{\mathrm{II},m}$ normalized through the nominal stress σ (defined in the central plane of the specimen and caused by the applied load) and through the specimen width W are introduced as:

$$g_{\mathrm{I},n}(\alpha) = \frac{a_n(\alpha)}{\sigma} W^{\frac{n}{2}-1} \quad \text{for} \quad n = 1,3,4,..., N \quad \text{and} \quad g_{\mathrm{I},2}(\alpha) = t_{\mathrm{I}}(\alpha) = \frac{4a_2(\alpha)}{\sigma} \ , \tag{3}$$

$$g_{\mathrm{II},m}(\alpha) = \frac{b_m(\alpha)}{\sigma} W^{\frac{m}{2}-1} \quad \text{for} \quad m = 1,3,4,..., M \quad \text{and} \quad g_{\mathrm{II},2}(\alpha) = t_{\mathrm{II}}(\alpha) = \frac{4b_2(\alpha)}{\sigma} \ . \tag{4}$$

2.2 Over-deterministic method

Because the above introduced coefficients of the higher-order terms have to be determined numerically, the so-called over-deterministic method (ODM) based on knowledge of the displacement field near the crack tip has been chosen for their estimation. The method utilizes directly the definition of the displacement field, see Eq. (2). The displacements u, v in a set of nodes around the crack tip are calculated by means of the finite element (FE) method in arbitrary FE software and together with the polar coordinates of the nodes investigated used as inputs for a system of equations, where the only variables are the coefficients of the Williams expansion. In order to fulfill the basic idea of the method (an over-determined system of equation), the number of equations corresponding to the number of nodes selected for the method application has to be greater than number of unknowns, i.e. number of the expansion terms that shall be calculated. This condition can be written mathematically in the form:

$$2k > N + M + 2 \ , \qquad (5)$$

where:

k – number of nodes selected around the crack tip for investigation of their displacements [-],

N – total number of the mode I expansion terms considered in stress/displacement approximation, see Eq. (2) where the terms from a_0 until a_N are summarized [-],

M – total number of the mode II expansion terms considered in stress/displacement approximation, see Eq. (2) where the terms from b_0 until b_M are summarized [-].

Note that Eq. (2) for application of the ODM can be reduced if only pure mode I or mode II of loading is present. More details about the ODM can be found for instance in [2] or in some of the authors' works devoted to parametric studies on ODM accuracy, convergence, or mesh sensitivity etc., see e.g. [3-8].

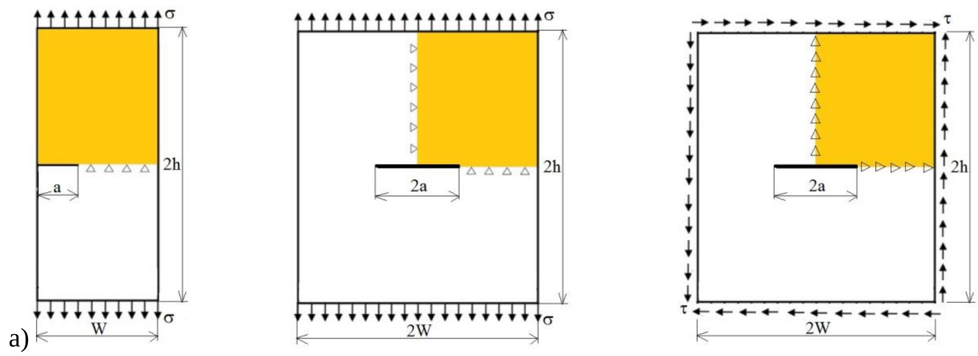
3 NUMERICAL MODELS

In order to study and assess the importance of the coefficients of the higher-order terms of the Williams expansion in a wider range of fracture mechanics problems, various geometries (with various loading/boundary conditions) has been investigated. In this paper, studies on three different specimens are presented. Following configurations, see Fig. 1, have been modeled by means of FEM and then further subjected to the ODM application which was implemented in a commercial mathematical package (Wolfram Mathematica [9]):

a) single edge cracked plate under uniaxial tension (SECT),

b) central cracked plate under uniaxial tension (CCT),

c) central cracked plate under pure shear (CCS).



a) a) SECT b) CCT c) CCS

Fig. 1 Cracked geometries investigated: a) single edge cracked plate under uniaxial tension; b) central cracked plate under uniaxial tension; c) central cracked plate under pure shear

Note that whereas the first two configurations represent the loading mode I examples, the last one represents the pure mode II of loading. Numerical models for all the specimens studied have been created in ANSYS FE computational system [10] (due to symmetry only one half/quarter of the cracked body could be modeled, see Fig. 1). Eight-node iso-parametric finite elements (PLANE 82) have been used for modelling; the crack-tip singularity was taken into account via using triangular crack elements with shifted mid-side nodes. The model dimensions and values of applied stress were considered as: $W = h = 1$ mm, $a = 0.4$ mm, $\sigma = \tau = 1$ MPa. Because of the independence of the coefficients of the Williams expansion on material properties, values of Young's modulus and Poisson's ratio were chosen as $E = 1$ MPa and $\nu = 0.25$. Note that for application of the ODM 50 nodes of the fifth and sixth ring of elements around the crack tip have been used.

4 RESULTS

In this chapter several various aspects of the stress/displacement fields approximation by means of Williams expansion are presented and discussed. First, the convergence of the coefficients of the higher-order terms for the three selected geometries (SECT, CCT and CCS, see Fig. 1) is assessed. In Fig. 2, dependences of the first eight coefficients of the Williams expansion on the number of terms considered are plotted. The results have been obtained on the SECT geometry. However, analogical results obtained on the other studied geometries show the same trend, i.e. that only first five/six coefficients converge clearly with increasing N or M, respectively.

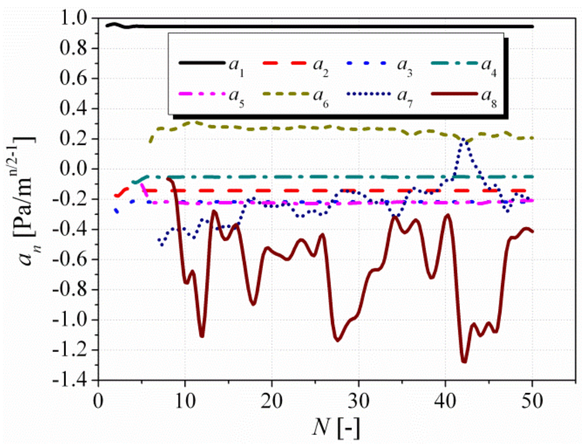


Fig. 2 Convergence of the coefficients of Williams expansion with increasing number of the terms considered during ODM application for a single edge cracked plate under uniaxial tension

Note that numerous authors' experience with the ODM application have been applied during the analysis (corresponding for instance to the number of nodes used for the ODM evaluation, or FE mesh parameters etc., see e.g. [3-8]). However, although a relatively high number of nodes has been used for investigations ($k = 50$) and their distance from the crack tip was large enough, it has not led to better convergence. It may be the reason why there are always published only first five terms of the Williams expansion in available fracture mechanics works of other authors, see e.g. [2, 11-16].

Second aspect that should be discussed are the absolute values of the coefficients and their dimensionless shape functions $g_{I,n}$ and $g_{II,m}$ respectively. They are introduced in Tab. 1.

Tab. 1 Dimensionless shape functions $g_{I,n}$ and $g_{II,m}$ calculated for three different geometries

	CCT	SECT		CCS
$g_{I,1}$ [-]	5.44E−01	9.45E−01	$g_{II,1}$ [-]	−5.06E−01
$g_{I,2}$ [-]	−1.26E+00	−5.71E−01	$g_{II,2}$ [-]	3.50E+00
$g_{I,3}$ [-]	2.88E−01	−2.17E−01	$g_{II,3}$ [-]	−3.06E−01
$g_{I,4}$ [-]	4.15E−02	−5.34E−02	$g_{II,4}$ [-]	−5.45E−02
$g_{I,5}$ [-]	−2.05E−01	−2.22E−01	$g_{II,5}$ [-]	1.14E−01
$g_{I,6}$ [-]	4.94E−02	2.79E−01	$g_{II,6}$ [-]	−4.80E−02

It can be seen from the table presented that the values of the normalized (dimensionless) values of coefficients of the terms of the Williams expansion are more or less of the same order, i.e. the values do not decrease with the increasing order as one could expect. Therefore, it can be concluded that for larger distances from the crack tip (where the singularity does not prevail anymore) also the higher-order terms of the Williams expansion seem to be relevant and can play an important role for the stress/displacement field description and for the crack behavior assessment, respectively.

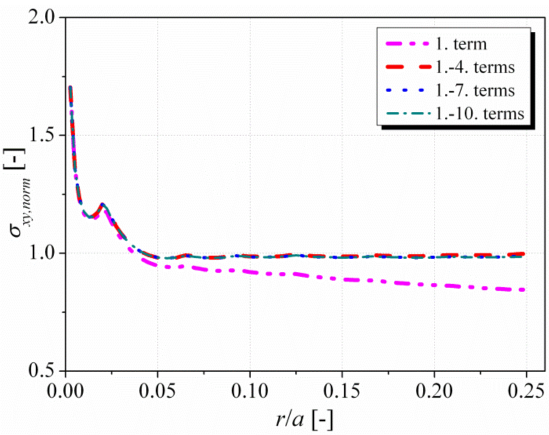


Fig. 3 Distribution of the normalized shear stress $\sigma_{xy,norm}$ in a CCS specimen calculated by means of Williams power expansion assuming various numbers of the higher-order terms

A kind of proof of the previous statement is depicted in Fig. 3 where the normalized stress distribution ahead of the crack tip in a CCS specimen under consideration of various numbers of the Williams expansion terms is presented. Note that the shear stress calculated from the Williams expansion is normalized (divided) through the values obtained from the FE analysis and thus, directly the deviations can be seen (note that $\sigma_{xy,norm}$ = 1 would mean that FE results equal exactly the results calculated from the Williams expansion).

Fig. 3 shows that if only the first singular term of the Williams expansion is considered for approximation of the stress field ahead of the crack tip, the stress values calculated can differ significantly from those estimated numerically by means of FEM. It is also evident that the larger is the distance from the crack tip the greater is the difference in the selected stress component. On the other hand, Fig. 3 shows that using more terms brings very good (and very similar) results. Therefore, if the stress shall be known precisely even in larger distances from the crack tip (which is the case of quasi-brittle materials) it is to recommend to take into account more than only one or two terms of the Williams expansion in order to obtain more reliable results.

5 CONCLUSIONS

A fracture mechanics analysis has been carried out on three various cracked specimens loaded in mode I as well as in mode II. The Williams expansion has been used for description of the stress/displacement distribution near the crack tip and the over-deterministic method has been applied for estimation of the coefficients of the expansion. For the configurations studied it has been shown that only first five/six terms converge unambiguously with increasing number of the terms considered for the stress approximation. This result has been commented in connection to authors' experience and results published by other researchers. It has been also discussed and shown that higher-order terms of the Williams power series can play a key role if a knowledge of accurate stress/displacement fields not only very close to the crack tip is required. The studies show that probably more than one or two terms (utilized within the well-known one- or two-parameter fracture mechanics) should be taken into account. This can be important for instance in the case of quasi-brittle materials, where the stress distribution has to be known also farther from the crack tip in order to perform a reliable fracture analysis.

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