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FINITE DIFFERENCE METHOD USED FOR THE BEAMS ON ELASTIC FOUNDATION – PART 2 (APPLICATIONS)

METODA KONEČNÝCH DIFERENCÍ POUŽITÁ PRO NOSNÍKY NA PRUŽNÉM PODKLADU – ČÁST 2 (APPLICATIONS)

Abstract

This article (as a continuation of the former work published in this journal), is focused on the theory and applications of beams on elastic (Winkler's) foundation. For solution of these problems of mechanics, the Finite Difference Method can be applied. Practical examples (i.e. beams with constant or variable stiffness of foundation) are explained and solved (Matlab software).

Abstrakt

Článek (jako pokračování předchozí práce publikované v tomto časopise) je zaměřen na teorii a aplikaci nosníků na pružném (Winklerově) podkladu. Pro řešení těchto úloh mechaniky, může být použita metoda konečných diferencí. Praktické příklady (tj. nosníky s konstantní nebo proměnlivou tuhostí podloží) jsou vysvětleny a řešeny (program Matlab).

1 INTRODUCTION TO THE BEAMS ON ELASTIC FOUNDATION AND CENTRAL DIFFERENCE METHOD (CDM)

This article (as a continuation of the former work published in this journal, see reference [18]), is focused on the theory and applications of beams on elastic (Winkler's) foundation.The bending of the beams on elastic foundation as well as the theory of Central Difference Method are explained in references [2], [5], [6], [8], [10], [11], [13], [14] and [18].

In the most situations, in bending, the other influences of normal forces, shearing forces, temperature and intensity of moment can be neglected (or the beam is not exposed to them), see references [2], [6], [8], [10] and [18]. Hence, we consider the following linear differential equation:

$$\frac{d^4 v}{dx^4} + \frac{kv}{EJ_{ZT}} = \frac{q}{EJ_{ZT}} \quad , \qquad (1)$$

where $v = v(x)$ /m/ is deflection of a beam, E /Pa/ is modulus of elasticity of the beam, J_{ZT} /m^4/ is the major principal second moment of area A /m^2/ of the beam cross-section, $q = q(x)$ /Nm^{-1}/ is distributed load (intensity of force) and $k = k(x)$ /Pa/ is stiffness of the foundation and x /m/ is length coordinate of the beam. Equation (1) is derived for the situations when input parameters E, J_{ZT} are constant and deformations are small.

Let us divide the beam into nodes "i" equally spaced (with step Δ /m/) along its length, see references [5], [8], [13], [14], [18]. Deflection curve of the beam and its derivatives are approximated

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** Associate Prof., MSc., Karel FRYDRÝŠEK Ph.D., ING-PAED IGIP, VŠB - Technical University of Ostrava, Faculty of Mechanical Engineering, Department of Mechanics of Materials, 17. listopadu 15, Ostrava, tel. (+420) 59 7323495, e-mail karel.frydrysek@vsb.cz

by polygon curves, see references [5], [8], [11] and [18]. Derivatives of function v at general points "i" can be approximated via central differences as:

$$v_i^{(1)} = \frac{dv(x=x_i)}{dx} \approx \frac{v_{i+1} - v_{i-1}}{2\Delta}\ ,\ v_i^{(2)} = \frac{d^2v(x=x_i)}{dx^2} \approx \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} \qquad (2)$$

$$v_i^{(3)} = \frac{d^3v(x=x_i)}{dx^3} \approx \frac{v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2}}{2\Delta^3}\ ,\ v_i^{(4)} = \frac{d^4v(x=x_i)}{dx^4} \approx \frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{\Delta^4}\ . \qquad (3)$$

According to the Central Difference Method (CDM), the eq. (1) can be approximated (by central differences eq. (2), (3)) at the point "i" as:

$$v_{i+2} - 4v_{i+1} + \left(6 + \frac{k_i\Delta^4}{EJ_{ZT}}\right)v_i - 4v_{i-1} + v_{i-2} = \frac{q_i\Delta^4}{EJ_{ZT}}\ , \qquad (4)$$

where $k_i = k(x_i)$.

Equation (4) can be written for all nodes $i = 0, 1, 2, \ldots, n$ (i.e. set of n+1 linear equations following from the discretization of eq. (1)) where n /1/ denotes number of divisions (i.e. number of elements). This set of equations together with four discretized boundary conditons lead to the solution of system of n+5 linear equations. There are always four fictitious nodes (-2, -1 and n+1, n+2) outside the ends of the beam. Hence, values of v_i at each node i (i.e. values of n+5 deflections) can be received, see also reference [5], [8] and [18].

2 FIRST EXAMPLE (BEAM ON ELASTIC FOUNDATION WITH CONSTANT STIFFNESS OF FOUNDATION)

Beam of length L /m/ is rested on an elastic foundation with the constant stiffness of foundation k. The beam is loaded by a force F /N/, see Fig. 1. This beam is not loaded by the distributed loading (i.e. q =0 Nm^{-1}).

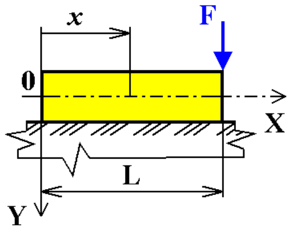


Fig. 1 Example 1 (beam on elastic foundation loaded by force).

According to the theory, two boundary conditions can be written at the point x = 0 m:

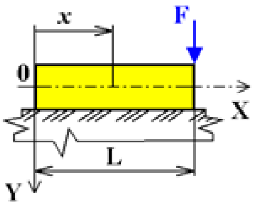
$$\left.\begin{aligned} M_o(x=0) = -EJ_{ZT}\frac{d^2v(x=0)}{dx^2} = 0 \quad &\Rightarrow \quad \frac{d^2v(x=0)}{dx^2} = 0 \\ T(x=0) = -EJ_{ZT}\frac{d^3v(x=0)}{dx^3} = 0 \quad &\Rightarrow \quad \frac{d^3v(x=0)}{dx^3} = 0 \end{aligned}\right\}\ , \qquad (5)$$

and two boundary conditions at the point x = L:

$$\left.\begin{aligned} M_o(x=L) = -EJ_{ZT}\frac{d^2v(x=L)}{dx^2} = 0 \quad &\Rightarrow \quad \frac{d^2v(x=L)}{dx^2} = 0 \\ T(x=L) = -EJ_{ZT}\frac{d^3v(x=L)}{dx^3} = F \quad &\Rightarrow \quad \frac{d^3v(x=L)}{dx^3} = -\frac{F}{EJ_{ZT}} \end{aligned}\right\}\ , \qquad (6)$$

where M_o /Nm/ is bending moment and T /N/ is shearing force.
The exact (analytical) solution is derived in reference [2], see Tab. 1.

Tab. 1 Exact solution of example 1 (beam on elastic foundation loaded by force).

<table>
<tr><td rowspan="3">
</td><td>$\omega = \sqrt[4]{\frac{k}{4EJ_{ZT}}}$</td><td>$A_1 = \bar{B}\left(\bar{C}+\bar{D}\right)$</td><td>$A_2 = \bar{B}\bar{D}$</td><td>$A_3 = \bar{B}\left(\bar{C}-\bar{D}\right)$</td></tr>
<tr><td colspan="4">$v = \left(A_1 e^{\omega x} + A_3 e^{-\omega x}\right)\cos\omega x + 2A_2 \cosh\omega x \sin\omega x$</td></tr>
<tr><td colspan="4">$\frac{d^2v}{dx^2} = 2\omega^2\left[\left(A_3 e^{-\omega x} - A_1 e^{\omega x}\right)\sin\omega x + 2A_2 \sinh\omega x \cos\omega x\right]$</td></tr>
<tr><td colspan="5">$\frac{dv}{dx} = \omega\left[A_1 e^{\omega x}\left(\cos\omega x - \sin\omega x\right) - A_3 e^{-\omega x}\left(\cos\omega x + \sin\omega x\right) + 2A_2\left(\cosh\omega x \cos\omega x + \sinh\omega x \sin\omega x\right)\right]$</td></tr>
<tr><td rowspan="2">$\bar{B} = \frac{F\omega}{2k\left(\sinh^2\omega L - \sin^2\omega L\right)}$</td><td rowspan="2" colspan="2">$\bar{C} = 2\left(\sinh\omega L \cos\omega L - \sin\omega L \cosh\omega L\right)$</td><td colspan="2">$\frac{d^4v}{dx^4} = -4\omega^4 v$</td></tr>
<tr><td colspan="2">$\bar{D} = 2\sinh\omega L \sin\omega L$</td></tr>
<tr><td rowspan="2" colspan="3">$\frac{d^3v}{dx^3} = 2\omega^3\left[A_3 e^{-\omega x}\left(\cos\omega x - \sin\omega x\right) - A_1 e^{\omega x}\left(\cos\omega x + \sin\omega x\right) + 2A_2\left(\cosh\omega x \cos\omega x - \sinh\omega x \sin\omega x\right)\right]$</td><td colspan="2">$M_o = -EJ_{ZT}\frac{d^2v}{dx^2} = \frac{-k}{4\omega^4}\frac{d^2v}{dx^2}$</td></tr>
<tr><td colspan="2">$T = \frac{dM_o}{dx} = -EJ_{ZT}\frac{d^3v}{dx^3} = \frac{-k}{4\omega^4}\frac{d^3v}{dx^3}$</td></tr>
</table>

Let the length L of the beam is equidistantly divided into n parts with the step $\Delta = L/n$, where node "0" is at the distance $x = 0$ m and node "n" is at the distance $x = L$.

Because q = 0 Nm^{-1}, the eq. (4) can be written in the form:

$$v_{i+2} - 4v_{i+1} + \left(6 + \frac{k\Delta^4}{EJ_{ZT}}\right)v_i - 4v_{i-1} + v_{i-2} = 0, \text{ for } i = 0, 1, 2, 3, \ldots, n\,. \tag{7}$$

According to eq. (2) and (3), the boundary conditions (5) to (6) can be approximated via central differences as:

$$\left.\begin{aligned} \frac{d^2v(x=0)}{dx^2} &\approx \frac{v_{-1} - 2v_0 + v_1}{\Delta^2} = 0 \quad\Rightarrow\quad v_{-1} - 2v_0 + v_1 = 0 \\ \frac{d^3v(x=0)}{dx^3} &\approx \frac{-v_{-2} + 2v_{-1} - 2v_1 + v_2}{2\Delta^3} = 0 \quad\Rightarrow\quad -v_{-2} + 2v_{-1} - 2v_1 + v_2 = 0 \end{aligned}\right\}, \tag{8}$$

$$\left.\begin{aligned} \frac{d^2v(x=L)}{dx^2} &\approx \frac{v_{n-1} - 2v_n + v_{n+1}}{\Delta^2} = 0 \quad\Rightarrow\quad v_{n-1} - 2v_n + v_{n+1} = 0 \\ \frac{d^3v(x=L)}{dx^3} &\approx \frac{-v_{n-2} + 2v_{n-1} - 2v_{n+1} + v_{n+2}}{2\Delta^3} = -\frac{F}{EJ_{zt}} \quad\Rightarrow\quad -v_{n-2} + 2v_{n-1} - 2v_{n+1} + v_{n+2} = -\frac{2\Delta^3 F}{EJ_{zt}} \end{aligned}\right\}. \tag{9}$$

Expressions (7), (8) and (9) lead to system of n+5 linear equations representing by a sparse matrix. Hence, the values of deflection v_i at each node can be calculated (i.e. v_{-2}, v_{-1}, v_0, v_1, v_2, v_3, ... , v_{n-3}, v_{n-2}, v_n, v_{n+1}, v_{n+2}). Deflections at fictitious nodes -2, -1, n+1, and n+2 (i.e. v_{-2}, v_{-1}, v_{n+1} and v_{n+2}) are defined out of the range of the beam, therefore they do not have physical meaning. However, these nodes are important for the solution.

Analytical solution (i.e. exact solution derived in reference [2]) is compared with the numerical solution acquired via CDM in Fig. 2 (example: calculated for inputs L = 20 m,

$E = 2\times10^{11}$ Pa, $J_{ZT} = 2\times10^{-3}$ m^4, $k = 2\times10^7$ Pa, $F = 10^5$ N, Matlab software). In theoretical point of view, It is evident, the numerical solution leads to the exact solution for $\Delta \to 0$ (i.e. for $n \to \infty$)

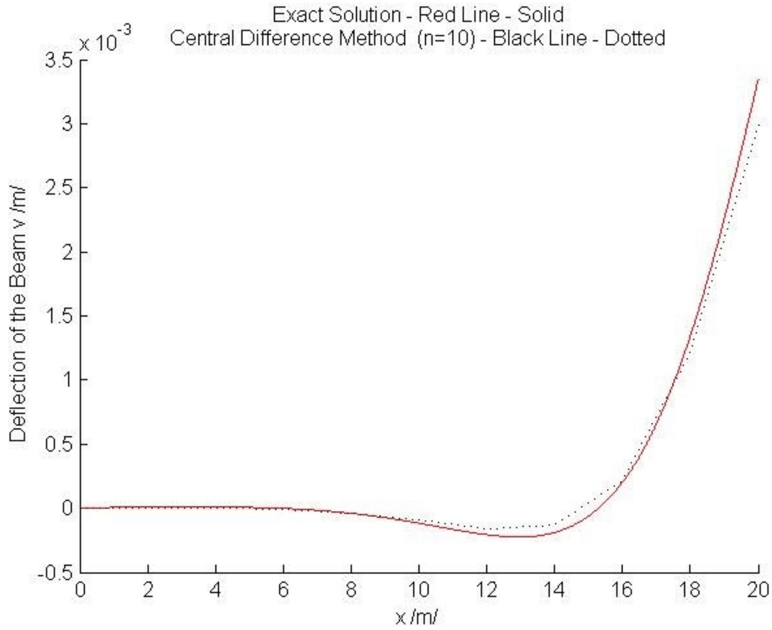


Fig. 2 Example 1(deflection of the beam, numerical and analytical approaches).

Tab. 2 Example 2 (exact solution and numerical solution).

x **/m/:**	**Exact Solution *v* /m/:**	**Numerical Solution (n=10) *v* /m/:**
0	4.3856e-006	5.2876e-006
2	8.5953e-006	6.2651e-006
4	9.9029e-006	5.1275e-006
6	-7.9842e-007	-5.2522e-006
8	-3.8946e-005	-3.6103e-005
10	-0.00011568	-9.4453e-005
12	-0.00020581	-0.00015845
14	-0.0001897	-0.00013066
16	0.00020293	0.00021307
18	0.0013442	0.0012014
20	0.0033437	0.0029927

Programming, Matlab software:

```
function BEAM_FORCE(n,L,E,Jzt,k,F)
% n ... Number of divisions (i.e. numer of elements) of the beam /1/
% L  ... Length of the beam /m/
% E  ... Modulus of elasticity /Pa/
% Jzt ... Major principal second moment of area /m4/
% k  ... Stiffness of the foundation /Pa/
% F  ... External force /N/
```

```
% EXACT SOLUTION v /m/:
omega=(k/(4*E*Jzt))^(1/4);
B=(F*omega)/(2*k*((sinh(omega*L))^2-(sin(omega*L))^2));
C=2*(sinh(omega*L)*cos(omega*L)-sin(omega*L)*cosh(omega*L));
D=2*sinh(omega*L)*sin(omega*L); A1=B*(C+D); A2=B*D; A3=B*(C-D);
x=0:0.01:L;
v=(A1*exp(omega*x)+A3*…
exp(-omega*x)).*cos(omega*x)+2*A2*cosh(omega*x).*sin(omega*x);
% FIGURE (exact displacement):
hold on, plot(x,v,'r')
% CENTRAL DIFFERENCE METHOD (CDM)
h=L/n; % Length of step (distance between nodes) /m/
x=(-2*h):h:(L+2*h); % Coordinates (for numerical solution)
V=zeros(n+5,n+5);
for i=1:n+1
  V(i,i)=1; V(i,i+1)=-4; V(i,i+2)=6+(k*h^4)/(E*Jzt);
  V(i,i+3)=-4; V(i,i+4)=1;
end
V(n+2,2)=1;V(n+2,3)=-2;V(n+2,4)=1;
V(n+3,1)=-1;V(n+3,2)=2;V(n+3,4)=-2;V(n+3,5)=1;
V(n+4,n+2)=1;V(n+4,n+3)=-2;V(n+4,n+4)=1;
V(n+5,n+1)=-1;V(n+5,n+2)=2;V(n+5,n+4)=-2;V(n+5,n+5)=1;
RightSide=zeros(n+4,1);
RightSide(n+5)=-(2*F*h^3)/(E*Jzt);
vNUM= V\RightSide;
% CMD (deleting of fictitious nodes):
x=x(3:n+3); vNUM=vNUM(3:n+3);
% FIGURE (CDM displacement):
plot(x,vNUM,'k')
% Table of values of deflections of the beam
Deflection=[x',vNUM]
```

3 SECOND EXAMPLE (BEAM ON ELASTIC FOUNDATION WITH VARIABLE STIFFNESS OF FOUNDATION)

Beam of length L /m/ is rested on elastic foundation with a variable stiffness of foundation:

$$k = k(x) = \frac{k_{MAX} + k_{MIN}}{2} + \frac{k_{MAX} - k_{MIN}}{2}\left[\sin(bx+\beta)\right]. \qquad (10)$$

where k_{MAX} and k_{MIN} /Pa/ are maximum and minimum values of stiffness of foundation, b /m^{-1}/ and β /rad/ are parameters of variability of the stiffness. The beam is exposed to force F /N/ and constant distributed load $q = q_0$, see Fig. 3.

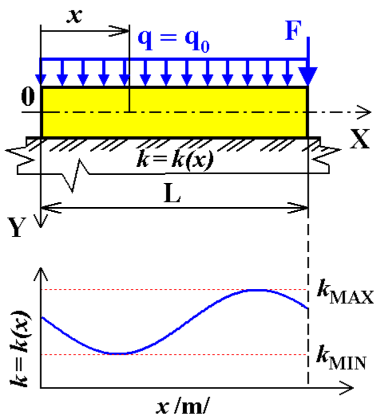


Fig. 3 Example 2 (beam on elastic foundation loaded by force F and distributed loading q).

Differential equation of this problem is given by eq. (1). There is not possible to find analytical solution for this problem. Therefore, the CDM (similarly as in the chapter 2) can be applied, see eq. (4):

$$v_{i+2} - 4v_{i+1} + \left(6 + \frac{k_i \Delta^4}{EJ_{ZT}}\right) v_i - 4v_{i-1} + v_{i-2} = \frac{q_0 \Delta^4}{EJ_{ZT}}, \quad \text{for } i = 0, 1, 2, 3, \ldots, n \cdot \tag{11}$$

where $k_i = k(x = x_i)$, $\Delta = L/n$ and n is number of division of the beam.

Similarly as in the chapter 2, for four boundary conditions can be written:

$$\left.\begin{aligned} M_o(x=0) &= -EJ_{ZT}\frac{d^2 v(x=0)}{dx^2} \Rightarrow v_{-1} - 2v_0 + v_1 = 0 \\ T(x=0) &= -EJ_{ZT}\frac{d^3 v(x=0)}{dx^3} = 0 \Rightarrow -v_{-2} + 2v_{-1} - 2v_1 + v_2 = 0 \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} M_o(x=L) &= -EJ_{ZT}\frac{d^2 v(x=L)}{dx^2} = 0 \Rightarrow v_{n-1} - 2v_n + v_{n+1} = 0 \\ T(x=L) &= -EJ_{ZT}\frac{d^3 v(x=L)}{dx^3} = F \Rightarrow -v_{n-2} + 2v_{n-1} - 2v_{n+1} + v_{n+2} = \frac{-2F\Delta^3}{EJ_{ZT}} \end{aligned}\right\} \tag{13}$$

From eq. (11) to (13) can be derived set of n+5 linear equations. Hence, the values of deflection v_i at each node $i = -2, -1, 0, \ldots, n+2$ can be calculated.

Solution by the CDM is done for input values $L = 20$ m, $E = 2\times10^{11}$ Pa, $J_{ZT} = 2\times10^{-5}$ m^4, q=10^4 Nm^{-1}, $k_{MAX} = 5\times10^7$ Pa, $k_{MIN} = 10^7$ Pa, F = 10^3 N, $b = 2$ m^{-1} and $\beta = 3$ rad, see Fig. 4.

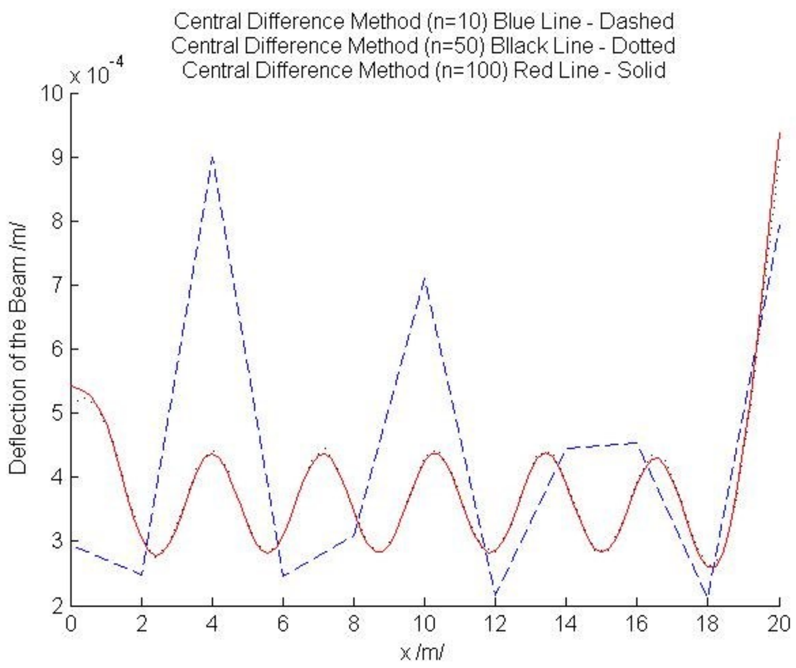


Fig. 4 Example 2 (deflection of the beam, numerical approach).

Tab. 3 Example 2 (numerical solutions).

x **/m/:**	**Numerical Solution (n=10)** ***v*** **/m/:**	**Numerical Solution (n=50)** ***v*** **/m/:**	**Numerical Solution (n=100)** ***v*** **/m/:**
0	0.00029403	0.00051469	0.00054232
2	0.00024746	0.00030531	0.00030555
4	0.00089938	0.00044347	0.00043686
6	0.00024647	0.00030706	0.00030861
8	0.00030952	0.00034736	0.00034685
10	0.0007103	0.00043131	0.00042561
12	0.00021659	0.00028283	0.00028544
14	0.00044577	0.00039378	0.00039061
16	0.00045369	0.00039325	0.00038977
18	0.00021253	0.00025967	0.00025977
20	0.00079236	0.0009008	0.00093839

Programming, Matlab software:

```
function BEAM_VARIABLE_STIFFNESS(L,n,b,beta,E,Jzt,kmin,kmax,q0,F)
% n ... Number of divisions (i.e. numer of elements) of the beam /1/
% L  ... Length of the beam /m/
% E  ... Modulus of elasticity /Pa/
% Jzt ... Major principal second moment of area /m4/
% k  ... Stiffness of the foundation /Pa/
% F  ... External force /N/
```

```
% q0 ... Constant distributed load
% kmin ... Minimum value of stiffness foundation /Pa/
% kmax ... Maximum value of stiffness foundation /Pa/
% b ... Parametr of variability of the stiffness /1/m/
% beta ... Parametr of variability of the stiffness /rad/
% CENTRAL DIFFERENCE METHOD (CDM)
h=L/n; % Length of step (distance between nodes) /m/
x=(-2*h):h:(L+2*h); % Coordinates (for numerical solution)
V=zeros(n+5,n+5);
for i=1:n+1
  V(i,i)=1; V(i,i+1)=-4;
  V(i,i+2)=6+((h^4)/(E*Jzt))*((kmax+kmin)/2+…
  ((kmax-kmin)/2)*sin(b*x(i+2)+beta));
  V(i,i+3)=-4; V(i,i+4)=1;
end
V(n+2,2)=1;V(n+2,3)=-2; V(n+2,4)=1;
V(n+3,1)=-1;V(n+3,2)=2; V(n+3,4)=-2;V(n+3,5)=1;
V(n+4,n+2)=1;V(n+4,n+3)=-2; V(n+4,n+4)=1;
V(n+5,n+1)=-1;V(n+5,n+2)=2; V(n+5,n+4)=-2;V(n+5,n+5)=1;
for i=1:n+1
  RightSide(i)=(q0*h^4)/(E*Jzt);
end
RightSide(n+2)=0;
RightSide(n+3)=0;
RightSide(n+4)=0;
RightSide(n+5)=-(2*h^3*F)/(E*Jzt);
v=V\RightSide';
% CMD (Deleting of fictitious nodes):
x=x(3:n+3); v=v(3:n+3);
% FIGURE (CDM displacement):
hold on, plot(x,v,'k')
% Table of values of deflection of the beam
Deflection=[x',v]
```

CONCLUSIONS

This article shows application of Central Difference Method (CDM) as a numerical method suitable for the solution of the beams rested on elastic foundation. Solution of two practical examples (i.e beam with constant stiffness of foundation and beam with variable stiffness of foundation, programming, Matlab software) are presented.

Another ways of the solutions and applications of structures on elastic foundation are presented in [1], [2], [3], [5], [6], [8], [9], [10], [12], [13], [14], [16], [17], [19] and [20].

For more information about applications of CDM, see [5], [8], [11], [13] and [14].

In the future, probabilistic approaches will be used in connection with CDM, see references [3] to [8], [14] and [15] (i.e. applications of Simulation-Based Reliability Method – SBRA and other methods).

ACKNOWLEDGEMENT

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