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FINITE DIFFERENCE METHOD USED FOR THE BEAMS ON ELASTIC FOUNDATION – PART 1 (THEORY)

METODA KONEČNÝCH DIFERENCÍ POUŽITÁ PRO NOSNÍKY NA PRUŽNÉM PODKLADU – ČÁST 1 (TEORIE)

Abstract

This article is focused on the theory of straight and curved beams on elastic (Winkler's) foundation. For solution of these problems of mechanics, the Finite Difference Method (i.e. Central Difference Method) can be applied. The basic information about finite differences and their application are explained.

Abstrakt

Článek je zaměřen na teorii přímých a křivých nosníků na pružném (Winklerově) podkladu. Pro řešení těchto úloh mechaniky, může být použita metoda konečných diferencí (tj. metoda centrálních diferencí). Základní informace o konečných diferencích a jejich aplikacích jsou vysvětleny.

1 INTRODUCTION TO THE THEORY OF BEAMS ON ELASTIC FOUNDATION

The basic analysis of bending of beams on an elastic foundation, see references [1] to [4], is developed on the assumption that the strains are small.

In this context, an elastic foundation is defined as a support which is continuously or discontinuously distributed along the length of the beam. The reaction force

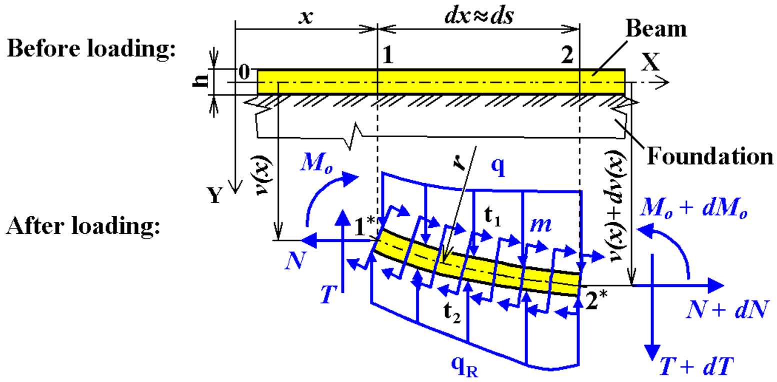

Fig. 1 Element of a straight beam on elastic foundation.

$q_R = q_R(x)$ /Nm^{-1}/ distributed in a foundation is directly proportional to the deflection $v = v(x)$ /m/ of a straight beam, see Fig. 1, or proportional to the radial displacement $u_R = u_R(\varphi)$ /m/ of a curved beam (circular arches), see Fig. 2.

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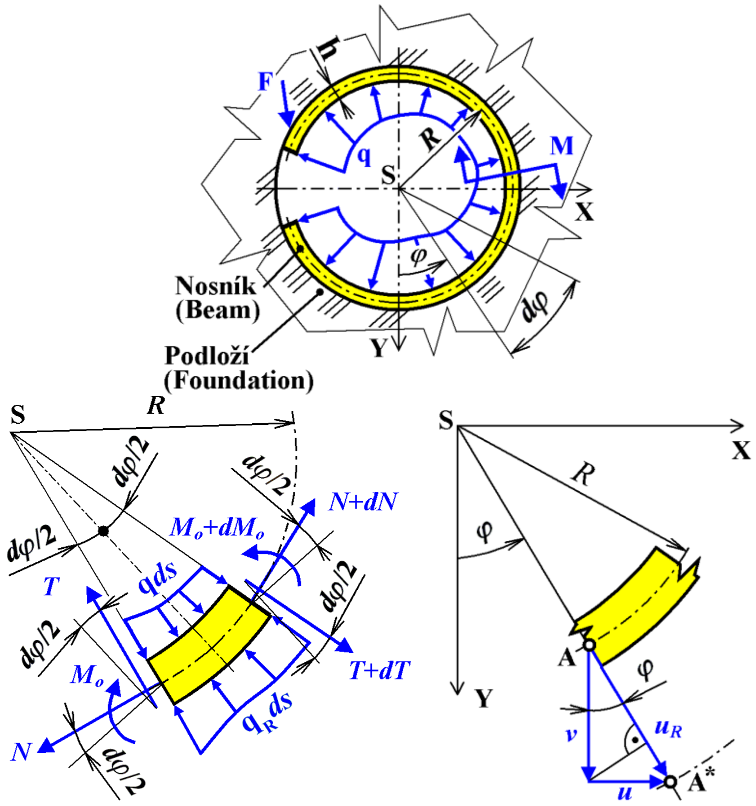

Fig. 2 Example of a curved beam on elastic foundation and its element.

This article is focused on the solution of the straight and curved beams on elastic foundation, see Fig. 1 and 2, which leads to the solution of linear differential equations via Finite Difference Method (i.e. Central Difference Method).

2 DIFFERENTIAL EQUATION FOR STRAIGHT BEAMS ON ELASTIC FOUNDATION

The bending of straight beams on elastic foundations, see Fig. 1, can be described by ordinary linear differential equation:

$$\frac{d^4 v}{dx^4} - \frac{N}{EJ_{ZT}}\frac{d^2 v}{dx^2} + \frac{\beta}{GA}\frac{d^2 \mathrm{q_R}}{dx^2} + \frac{\mathrm{q_R}}{EJ_{ZT}} = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) + \frac{\beta}{GA}\frac{d^2 \mathrm{q}}{dx^2} - \frac{\alpha_{\mathrm{t}}}{\mathrm{h}}\frac{d^2(\mathrm{t_2} - \mathrm{t_1})}{dx^2} \qquad (1)$$

where: E /Pa/ is modulus of elasticity of the beam, J_{ZT} /m^4/ is the major principal second moment of area A /m^2/ of the beam cross-section, β /1/ is shear deflection constant of the beam, G /Pa/ is shear modulus of the beam, N /N/ is normal force, $\mathrm{q} = \mathrm{q}(x)$ /Nm^{-1}/ is distributed load (intensity of force), m /N/ is distributed couple (intensity of moment), α_{t} /deg^{-1}/ is coefficient of thermal expansion of the beam, h /m/ is depth of the beam and $\mathrm{t_2} - \mathrm{t_1}$ /deg/ is transversal temperature increasing in the beam. Equation (1) is derived for the situations when input parameters E, J_{ZT}, N, β, G, A, α_{t} and h are constant. For more information about the derivation of eq. (1), see references [1] to [6].

From the Winkler's theory, see references [1] to [4], it holds that:

$$\mathrm{q_R} = kv = \mathrm{b}Kv \qquad (2)$$

where functions: $k = k(x)$ /Pa/ is stiffness of the foundation and $K = K(x)$ /Nm^{-3}/ is modulus of the foundation which can be expressed as functions of variable x /m/ (i.e. longitudinal changes in the foundation) and b /m/ is width of the beam (see Fig. 3).

Hence, from eq. (1) and (2) follows:

$$\frac{d^4v}{dx^4} - \frac{N}{EJ_{ZT}}\frac{d^2v}{dx^2} + \frac{\beta}{GA}\frac{d^2(kv)}{dx^2} + \frac{kv}{EJ_{ZT}} = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) + \frac{\beta}{GA}\frac{d^2\mathrm{q}}{dx^2} - \frac{\alpha_{\mathrm{t}}}{\mathrm{h}}\frac{d^2(\mathrm{t}_2 - \mathrm{t}_1)}{dx^2} \tag{3}$$

In the most situations, the influences of shearing force, temperature and intensity of moment can be neglected (or the beam is not exposed to them). Hence, from eq. (3) follows the simple form:

$$\frac{d^4v}{dx^4} - \frac{N}{EJ_{ZT}}\frac{d^2v}{dx^2} + \frac{kv}{EJ_{ZT}} = \frac{\mathrm{q}}{EJ_{ZT}} . \tag{4}$$

3 DIFFERENTIAL EQUATION FOR CURVED BEAMS ON ELASTIC FOUNDATION

The bending of curved beams on elastic foundations, see Fig. 2, can be described by ordinary linear differential equation:

$$\frac{d^5u_R}{d\varphi^5} + 2\frac{d^3u_R}{d\varphi^3} + \Omega^2 u_R = \frac{R^4}{EJ_{\mathrm{ZT}}}\frac{d\mathrm{q}}{d\varphi} , \tag{5}$$

where: R /m/ is radius of the beam, φ /rad/ is angle variable and parameter Ω /1/ is given by equation:

$$\Omega = \sqrt{1 + \frac{kR^4}{EJ_{\mathrm{ZT}}}} . \tag{6}$$

From the Winkler's theory, see references [1] to [4], it is evident that:

$$\mathrm{q}_{\mathrm{R}} = ku_R = \mathrm{b}Ku_R \tag{7}$$

All others parameters mentioned in equations (4) to (7) are explained in former text.

4 FINITE DIFFERENCES

Let us consider an equidistant partition of the beam with a step Δ /m/ and nodal points (nodes) "i" along its length, see Fig. 3 (unloaded beam) and Fig. 4 (loaded beam).

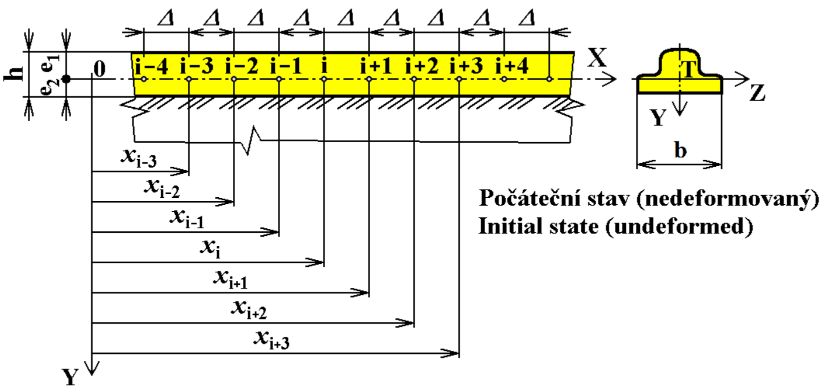

Fig. 3 Solved straight beam is divided into nodes "i".

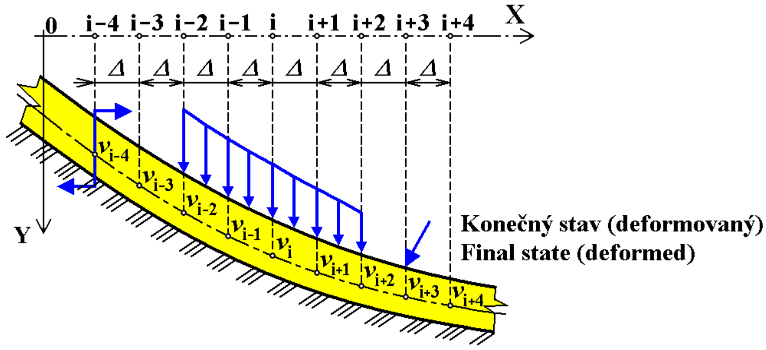

Fig. 4 Solved straight beam is divided into nodes "i".

Deflection curve $v = v(x)$ of a straight beam and its derivatives are approximated by polygon curves, see Fig. 5.

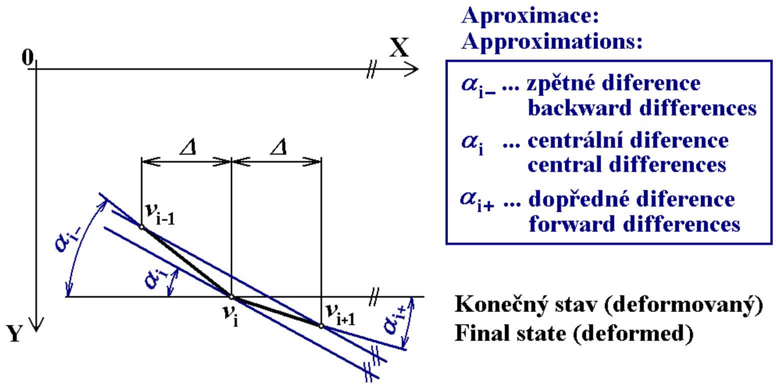

Fig. 5 Approximation of the deflections by polygon curve and approximation of first derivative.

Finite differences can be defined as an approximation of derivatives. Hence, for the value of the first derivative, three types of differences can be defined according to Fig. 5:

- Backward difference at the point "i":

$$v_{i-}^{(1)} = \frac{dv(x = x_i)}{dx} \approx \tan(\alpha_{i-}) = \frac{v_i - v_{i-1}}{\Delta}. \tag{8}$$

- Forward difference at the point "i":

$$v_{i+}^{(1)} = \frac{dv(x = x_i)}{dx} \approx \tan(\alpha_{i+}) = \frac{v_{i+1} - v_i}{\Delta}. \tag{9}$$

- Central difference at the point "i":

$$v_i^{(1)} = \frac{dv(x = x_i)}{dx} \approx \tan(\alpha_i) = \frac{v_{i+}^{(1)} + v_{i-}^{(1)}}{2} = \frac{v_{i+1} - v_{i-1}}{2\Delta}. \tag{10}$$

In some references (for example [6]) are symbols "i-", "i+" noted as "i-½" and "i+½".

Central differences (CD) are more accurate, therefore they will be applied in the following text. Similarly, the higher derivatives (at the point "i") can be approximated by the central differences as:

$$v_i^{(2)} = \frac{d^2 v(x = x_i)}{dx^2} \approx \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2}, \tag{11}$$

$$v_i^{(3)} = \frac{d^3 v(x = x_i)}{dx^3} \approx \frac{v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2}}{2\Delta^3}, \tag{12}$$

$$v_i^{(4)} = \frac{d^4 v(x = x_i)}{dx^4} \approx \frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{\Delta^4}, \tag{13}$$

$$v_i^{(5)} = \frac{d^5 v(x = x_i)}{dx^5} \approx \frac{v_{i+3} - 4v_{i+2} + 5v_{i+1} - 5v_{i-1} + 4v_{i-2} - v_{i-3}}{2\Delta^5}, \tag{14}$$

$$v_i^{(6)} = \frac{d^6 v(x = x_i)}{dx^6} \approx \frac{v_{i+3} - 6v_{i+2} + 15v_{i+1} - 20v_i + 15v_{i-1} - 6v_{i-2} + v_{i-3}}{\Delta^6}. \tag{15}$$

Similarly, for a curved beams (i.e. approximations for derivatives of function $u_R = u_R(\varphi)$), CD formulas can be derived by substitution of variables v_i (for example $u_{R\,i}^{(1)} = \frac{du_R(\varphi = \varphi_i)}{d\varphi} \approx \tan(\alpha_i) = \frac{u_{R\,i+}^{(1)} + u_{R\,i-}^{(1)}}{2} = \frac{u_{R\,i+1} - u_{R\,i-1}}{2\Delta}$ etc.).

5 CENTRAL DIFFERENCE METHOD (CDM) FOR STRAIGHT BEAMS

According the Central Difference Method (CDM), the differential equations (4) for straight beams can be approximated at the general point "i" (see eq. (11) and (13)) as:

$$v_{i+2} - \left(4 + \frac{N\Delta^2}{EJ_{ZT}}\right)v_{i+1} + \left(6 + \frac{2N\Delta^2}{EJ_{ZT}} + \frac{k_i\Delta^4}{EJ_{ZT}}\right)v_i - \left(4 + \frac{N\Delta^2}{EJ_{ZT}}\right)v_{i-1} + v_{i-2} = \frac{q_i\Delta^4}{EJ_{ZT}}. \tag{16}$$

where k_i and q_i are the stiffness of the foundation and the distributed loading at the point "i", respectively.

Equation (16) can be written for all nodes $i = 0, 1, 2, \dots, n$ (i.e. set of n+1 linear equations following from the discretization of eq. (4)), see Fig. 6. This set of equations, together with four discretized boundary conditons, lead to the solution of system of n+5 linear equations. There are always four fictitious nodes (-2, -1 and n+1, n+2) outside the ends of the beam, see Fig. 6. Hence, values of v_i at each node "i" (i.e. values of n+5 deflections) can be received, see also reference [3] and [7].

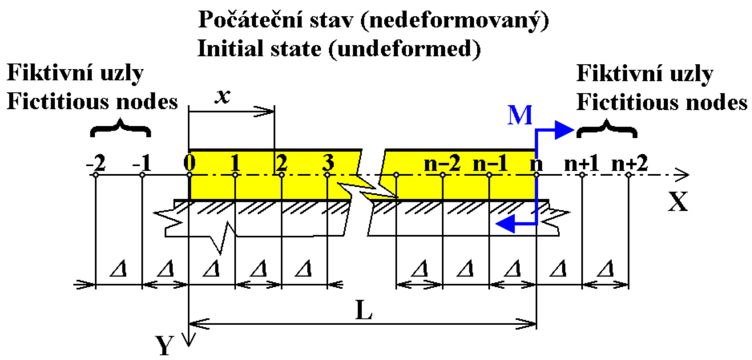

Fig. 6 Example solved in reference [3] (straight beam on elastic foundation loaded by couple M).

Note: Theoretically, if step $\Delta \to 0$ (i.e. $n \to \infty$) then the numerical solution converge to exact solution.

Fig. 7 and 8 show the sparse matrices arising from CDM for beam on elastic foundation loaded by couple M, see Fig. 6. This example is solved in reference [3].

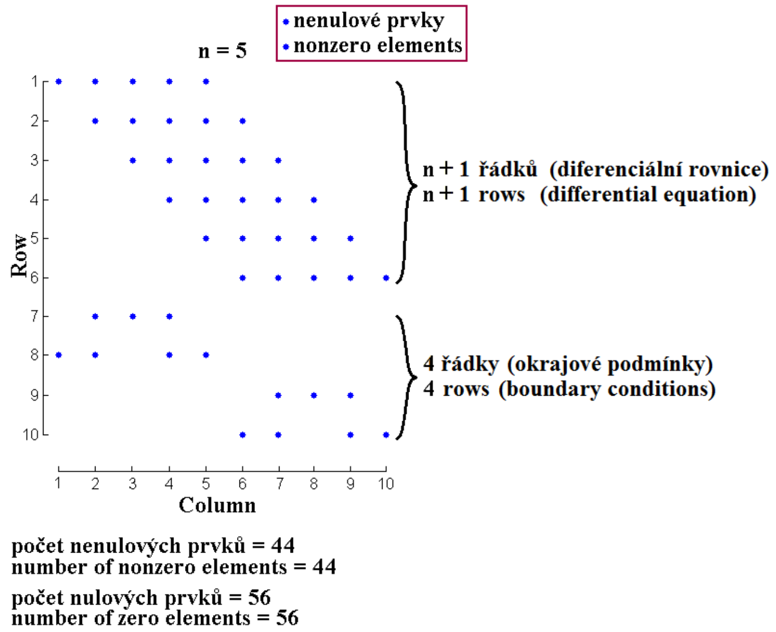

Fig. 7 Example solved in reference [3] (sparsity patterns of matrices in CDM, number of elements n = 5).

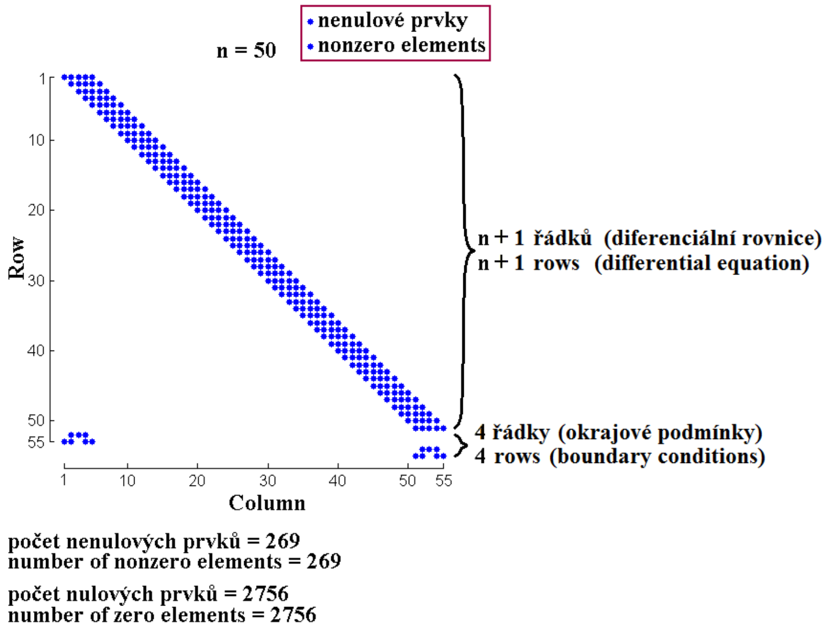

Fig. 8 Example solved in reference [3] (sparsity patterns of matrices in CDM, number of elements n = 50).

6 CENTRAL DIFFERENCE METHOD (CDM) FOR CURVED BEAMS

According the CDM, the differential equations (5) for curved beams can be approximated at the general point "i" (see modified eq. (12) and (14)) as:

$$u_{R\,\mathrm{i}+3}+\left(2\Delta^2-4\right)u_{R\,\mathrm{i}+2}+\left(5-4\Delta^2+\Omega_\mathrm{i}^2\Delta^4\right)u_{R\,\mathrm{i}+1}+ \\ +\left(-5+4\Delta^2-\Omega_\mathrm{i}^2\Delta^4\right)u_{R\,\mathrm{i}-1}+\left(-2\Delta^2+4\right)u_{R\,\mathrm{i}-2}-u_{R\,\mathrm{i}-3}=\frac{R^4\Delta^5}{EJ_{ZT}}\mathrm{q}_\mathrm{i}^{(1)}, \tag{17}$$

where k_i, $\Omega_\mathrm{i}=\sqrt{1+\frac{k_\mathrm{i}R^4}{EJ_{\mathrm{ZT}}}}$ and $\mathrm{q}_\mathrm{i}^{(1)}=\frac{d\mathrm{q}(\varphi=\varphi_\mathrm{i})}{d\varphi}$ are stiffness of the foundation, parameter and first derivative of distributed load at the point "i", respectively.

Equation (17) can be written for all nodes $\mathrm{i}=0,1,2,\ldots,\mathrm{n}$ (i.e. set of n+1 linear equations following from the discretization of eq. (5)). This set of equations, together with five discretized boundary conditons, lead to the solution of system of n+7 linear equations. Hence, values of $u_{R\mathrm{i}}$ at each node "i" (i.e. values of n+7 deflections) can be received, see also reference [3] and [7].

Note: Theoretically, if step $\Delta\to 0$ (i.e. $\mathrm{n}\to\infty$) then numerical solution converge to exact solution.

CONCLUSION

This article shows derivations and way of application of the Central Difference Method (CDM) as a numerical method suitable for the solution of the straight or curved beams on elastic foundation. For more information about applications of CDM, see [3], [6], [7] and [8]. CDM seems to be a good alternative to widely spread Finite Element Method.

Another ways of the solutions and applications of structures on elastic foundation are presented in [1] to [9].

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