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The lower bounds on nominal interest rates

Teemu Heikki August Pekkarinen

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Tiivistelmä – Referat – Abstract

This thesis examines the lower bounds on nominal interest rates by constructing models for an agent's portfolio optimisation problem of risk-free assets. The question "is there a lower bound on nominal interest rates, and is it zero" is approached theoretically by considering the situation where a depositor decides whether to keep her money in a bank account or turn the deposits into cash or some other cash-equivalent. The two-period and multi-period models presented in this paper are applicable for examining the lower bounds on commercial bank deposit rates and central bank policy rates. The multi-period model with a stochastic cash holding cost function provides a new framework and tool for the yields of risk-free assets. By using this model, it is possible to simulate asset returns and thereby approach the lower bounds on deposit rates. The results indicate that the lower bound on nominal interest rates is negative.

Tämä tutkielma tarkastelee nimelliskorkojen alarajoja muodostaen malleja portfolio-optimointiin riskittömille omaisuuserille. Tutkimus vastaa kysymykseen "onko nimelliskoroilla alaraja ja onko se nolla" teoreettisesti pohtien tilannetta, jossa tallettaja päättää pitääkö hän rahansa talletustilillä vai nostaako hän rahansa käteiseksi tai joksikin muuksi pankkisaamiseksi. Tässä tutkimuksessa esitetyt kahden ja useamman periodin mallit soveltuvat niin liikepankkien, kuin keskuspankkienkin korkojen alarajojen tarkasteluun. Useamman periodin malli stokastisella käteisen hallussapitokustannusfunktiolla tarjoaa uudenlaisen kehikon ja työkalun riskittömien omaisuuserien tuotoille. Tätä mallia käyttäen on mahdollista simuloida eri omaisuuserien kehitystä ja siten lähestyä talletuskorkojen alarajoja. Tulokset osoittavat, että nimelliskoron alaraja on negatiivinen.

Avainsanat – Nyckelord – Keywords
The lower bounds on nominal interest rates, cash holding costs, the theory of interest

Ohjaaja tai ohjaajat – Handledare – Supervisor or supervisors
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Contents

List of Figures

List of Tables

1 Introduction

A number of central banks continue to find their interest rate policy in dire straits as a consequences of the 2008 financial crisis. Recently, the Bank of Japan has joined ranks with the European Central Bank (ECB), the Swiss National Bank, the Swedish Riksbank, and Danmarks Nationalbank in adopting a policy of negative interest rates. This has given rise to debate on the following questions: how low can central banks go and what are the lower bounds on nominal interest rates.

Several studies approach these questions from the perspective of a bank, or central bank, who sets an interest rate (see, for instance, Buiter (2009a) and Brunnermeier and Koby (2016)). This thesis, in turn, considers the lower bounds on nominal interest rates from the perspective of an agent who holds deposits and faces a deposit rate. This approach relies on examining risk-free assets – deposits and cash or other cash equivalents – and considers how an agent chooses between these stores of values. Hence, the analyses are based on the basic microeconomic theory of consumer choice, but still are applicable to examine not only the lower bound on commercial bank deposit rates but also the lower bound on central bank policy rates.

Empirically the existence of the zero lower bound (ZLB) on nominal interest rates has been broken by a quarter. The nominal interest rates on firms' deposit accounts with commercial banks as well as the deposit rates on accounts with the central banks are negative in certain countries. In addition, many nominal interest rates on financial market instruments have gone below zero as a consequence of sinking short-term interest rates of the above-mentioned central banks. Nevertheless, many economic theories still rely on the ZLB on nominal interest rates.

What are the ways to overcome the lower bound on nominal interest rates? How are the different short-term interest rates related to each other? How low can central banks go? At first this research considers these questions and then focuses more generally on the question *is there a lower bound on nominal interest rates, and is it zero*. This is answered theoretically by considering the situation where a depositor decides whether to keep her money in a bank account or turn the deposits into cash or other cash-equivalent.

The structure of this paper is as follows: Chapter 2 presents three practical ways to overcome the zero lower bound and Chapter 3 examines the relations between different assets and their interest rates. Chapter 4 provides microeconomic models for the lower bounds, which are based on the costs and benefits of risk-free assets. The fourth chapter also presents a framework which can be simulated and used to investigate the lower bounds on deposit rates. Chapter 5 considers practical applications, and the final chapter presents a conclusion of the results.

2 Practical ways to overcome the zero lower bound

The earlier studies of Irving Fisher in 1896 and 1930 point out that if money were stored without cost over time, then the rate of interest could never fall below zero. It states that no one will lend money at negative nominal interest if cash is costless to hold. Under the same argument, people would hold their money themselves if a bank were to collect negative interest on their deposits. Roughly speaking, this is the definition of the *zero* lower bound on nominal interest rates. This clarifies the importance of cash, because bank notes and coins that yield zero interest are the only instruments that restrict the sinking of interest rates nowadays.

This chapter gives a starting point for this research by considering what creates the lower bound on nominal interest rates and how to overcome it. The following sections present three practical ways to overcome the ZLB on nominal interest rates, namely abolishing currency, taxing currency, and unbundling the numéraire and the medium of exchange. These methods are mainly based on the earlier studies of Gesell (1916), Eisler (1933), Goodfriend (2000), Buiter and Panigirtzoglou (2003), and Buiter (2009a). As the next sections will show, overcoming the ZLB is challenging. Therefore the examination of the lower bounds on nominal interest rates under the existing circumstances is still relevant and important.

2.1 Abolishing currency

Hoarding cash may be costly and risky, but if interest rates become too negative, it might be profitable to do so. Abolishing currency would probably be the simplest way to get rid of the floor on nominal interest rates. If all bank liabilities were electronic, paying a negative interest on reserves would be as trivial as paying positive interest; this would amount to just charging a fee. This would not, however, be completely costless to carry out and would indeed present many important conundrums to solve. For instance, a huge amount of cash are held abroad, outside of the home country of any given currencies.

Buiter (2009a) argues that the majority of US dollar and euro currency notes are held abroad for legitimate reasons by the citizens of countries where the authorities do not have a strong reputation for low and stable inflation, or are held for illegitimate reasons, both at home and abroad (see also e.g. Feige (2012b) and Feige (2012a)). Furthermore, Rogoff (2014) explores the costs and benefits to phasing out paper currency, and emphasises that cash makes it difficult for central banks to set policy interest rates much below zero; cash is a limitation that seems to have become increasingly relevant during this century.

According to Buiter (2009a), cash and deposits are perfect substitutes from a government and private sector financing perspective, but from the point of view of the provision of liquidity services, currency and deposits may not be such. However, this is not an obstacle to removing the lower bound by abolishing currency, it only affects the welfare consequences of the loss of currency. Many reports by the central banks (e.g. the ECB or the Bank of Finland) indicate that in advanced economies with modern financial markets and payment systems, currency has lost its importance as a means of payment for transactions. This implies that people prefer deposits over cash when they are making transactions. Even if there are no significant interest rate benefits on bank accounts, the trend towards the cashless economies is still proceeding.

However, Buiter (2009a) concludes the following: authorities can remove the lower bound on nominal interest rates by abolishing currency and introducing a close substitute on which interest (negative or positive) can be paid.

2.2 Taxing currency

Charging a fee or taxing the usage of assets would work akin to a negative interest rate. Hence, if it were possible to tax currency, the lower bound on nominal interest rates would be removed. Taxing currency is, in fact, a very early invention first introduced by Gesell (1916) and then supported by Fisher (1933). Recently, the studies of Goodfriend (2000), Buiter and Panigirtzoglou (2003), and Buiter (2009a) have updated and presented this idea further.

Taxing currency is simple but laborious to implement. It requires only that individual notes and coins would be possible to identify as being up-to-date on interest due or owed. This means, when the negative interest is due, the notes must be marked somehow thereafter. Gesell (1916) proposed stamping, or in the case of conventional bearer bonds, physically clipping the coupons from the bond document or certificate. Buiter (2009a) proposes a further variant where the author gives the note an expiration date and if the note is not stamped before the expiration date, it loses its legal tender status. The negative interest could then be collected after the notes are stamped.

In addition to these proposals, there are other creative ways to tax currency. For instance, Mankiw (2009) presents that the central bank could raffle a digit from zero to nine, and then decide that the notes with a serial number ending in that digit would no longer be legal tender.[1] That would change the expected return of cash and have an effect similar to that of a negative rate on currency.

Of course, these proposals have shortcomings. These kind of acts, or even speculation against the legal tender status of currency, makes people hesitate and changes their payment behaviour; Gresham's law holds true and *bad* money displaces the *good*. Therefore Buiter (2009a) argues that there has to be a penalty on those caught with expired currency, which would insure that expired currency does not continue to circulate on a par with current currency. This would demand supervision and controlling, and as Buiter (2009a) states, new penalties and inconvenience would be anything but popular. Nevertheless, however hypothetical and unpopular, it would still be one of the solutions of getting rid of the floor on nominal interest rates.

[1]This idea was actually presented by a graduate student of Mankiw, whose name he wants to omit.

2.3 Unbundling the numéraire and the medium of exchange

The last shortly introduced way to remove the lower bound on nominal interest rates was first suggested by Eisler (1933). The key to the problem is that the function of numéraire has to be decoupled from the medium of exchange role. The way to implement this decoupling is as follows: The old government-issued fiat currency has to be replaced by a new government-issued fiat currency, which would be only for payments, and the old currency would remain in the role of the numéraire. The government then sets the sequence of spot and forward exchange rates between the old numéraire currency and the new medium of exchange currency; setting a shadow exchange rate effectively works as it were the interest rate on currency and thereby the lower bound on nominal interest rate becomes fully adjustable. (Buiter (2009a).)

In practice, to implement this kind of policy would require withdrawing all the old currency from the circulation and then replacing this with the new one. This operation faces the same problems as abolishing and taxing currency; even withdrawing all the cash held domestically would be a painstaking process, let alone the cash held abroad. Buiter (2007), Buiter (2009a), and Agarwal and Kimball (2015) provide closer discussion on this topic.

All these three practical ways to overcome the lower bound on nominal interest rates have shown their potential, but also their frailties and rigidities. The two latter ways have met criticism because of their retrospection, and the former is not supported because of its radicalism, but it still might be the most probable one. Hence, it is well-founded to examine the lower bounds on nominal interest rates in a world where cash still plays a significant role.

3 The relation between different assets and their interest rates

The first section of this chapter examines how different assets and their interest rates are related. This lays the foundation for the upcoming analyses and clarifies the problem of the lower bound in monetary policy implementation. The second section considers an effective lower bound on monetary policy. Brunnermeier and Koby (2016) argue that there is a policy rate plateau where accommodative monetary policy reverses its effect and becomes contractionary for lending. This approach clarifies why the deposit rates on the accounts with commercial banks restrict policy rates, too.

3.1 The nominal interest rates on different assets

Nowadays there are many different short-term nominal rates on assets of a different kind and these interest rates are usually closely related to each other. These instruments have their own risks, costs, and benefits which make their returns comparable. Buiter (2009b) defines the most important set of six one-period nominal interest rates as:

i: interest rate on one-period Treasury Bills.
i^M: interest rate on base money held for one period.
i^N: interest rate on currency notes and coins held for one period.
i^D: one-period interest rate on bank (excess) reserves held with the central bank.
i^B: one-period interest rate on collateralised commercial bank borrowing from the central bank.
i^{CB}: one-period official policy rate.

In addition, most central banks have a discount window where banks can borrow against a wider range of collateral. This is not, however, considered here and, in addition, the many temporary facilities created by central banks have omitted from analysis. Hence, i^B is considered as the interest rate at which commercial banks can borrow from the central bank against high-grade collateral.

The official policy rates of central banks differ from each other and the variety of arrangements is large. The Federal Funds target rate is a guide for the overnight rate in the Federal Funds market which is for unsecured interbank borrowing and lending of excess reserves with the Federal Reserve System (Fed). Since July 2008, in the euro area the official policy rate is the main refinancing operations fixed rate, which is a repo rate against high-grade collateral. This is for a range of maturities from overnight to a year. Before July 2008, the official policy rate of the euro area was the Main Refinancing Operations variable rate tenders' minimum bid rate. In Japan, the Uncollateralized Overnight Call Rate is the lending rate charged for uncollateralised loans in the Japanese interbank market. In the UK, the official policy rate is Bank Rate which is the overnight interest rate paid on commercial bank reserves with the Bank of England. However, from 1997 until early 2006, the Bank of England lent for a two-week maturity to commercial banks against high grade collateral, and thus the official policy rate in the UK was the two-week repo rate. In Sweden, since 1994, the Riksbank's policy rate has been repo rate, i.e. the rate of interest at which banks can borrow or deposit funds at the Riksbank for a period of seven days. In Switzerland, the official policy rate is a target for 3-month Libor, the 3-month unsecured interbank borrowing rate.

The deposits of commercial banks with central banks are among the safest assets a bank can hold and central banks have different implementations for the deposit rates on required reserves and excess reserves or other partitions of total reserves. For instance, the Fed did not pay an interest rate on reserves until 2008. The Board of Governors of the Federal Reserve System amended Regulation D, Reserve Requirements of Depository Institutions, in 2008 to allow Reserve Banks to pay interest on balances maintained to satisfy reserve balance requirements and excess balances. Both types of balances currently earn interest at the rate of 100 basis points in the Fed, whereas in the ECB the deposit rate for the reserves that exceed the minimum reserve requirements is -40 basis points. The ECB introduced a negative deposit facility interest rate on 11 June 2014. This kind of interest rate policy on excess reserves is to encourage lending to the private sector by making it costly for commercial banks to hold their excess reserves with central banks (see, for example, Bindseil et al. (2006) and Armenter and Lester (2017)).

After many asset purchasing programmes within the last nine years, the amount of excess reserves has enormously risen. Therefore, the monetary policy deposit rates strongly affect the profits, equity, and credit growth of banks. For instance, in 2016 the Bank of Finland earned over 261 million euro interest yields from the excess reserve deposits with the Bank of Finland, because the over night deposit interest rate was -0.4 percent during the year. The amount of over night deposits were 18 600 million euro by the end of 2016. (The Bank of Finland (2016).) However, this is just a small part of the total excess reserves in the accounts with the ECB. Figure 1 illustrates the increased amount of excess reserves of credit institutions in the USA and in the euro area.

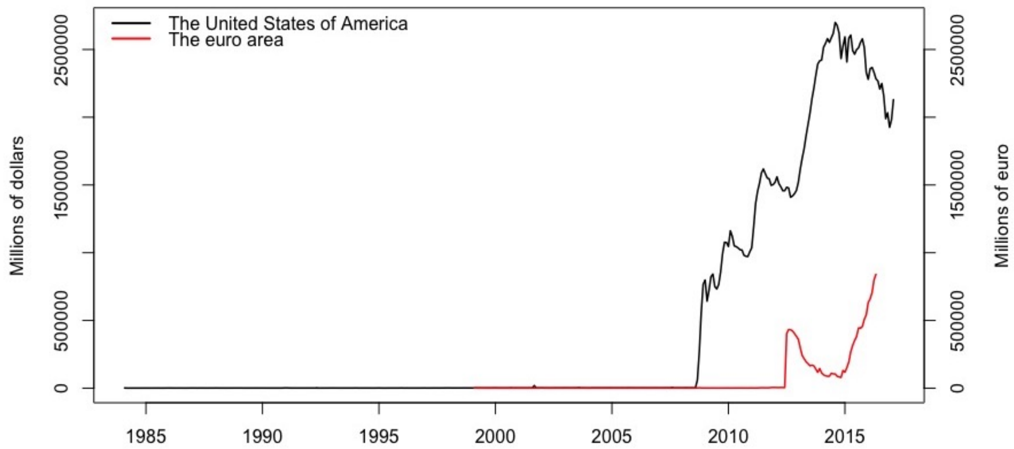


Figure 1: The excess reserves of depository institutions. Source: Federal Reserve Bank of St. Louis and European Central Bank, March 27, 2017.

The recent financial crisis has also shown that Treasury Bills are not risk-free assets. Nor is the unsecured or non-collateralised bank borrowing from the central bank, because banks can default on their obligations. However, the loan from the central bank against collateral is considered as risk-free. In addition to the interest rates, the financial instruments discussed above have all different *marginal* carry costs, i.e. the cost of storing, safekeeping (including insurance) and using these securities. Buiter (2009b) denotes these as:

k: carry cost on one-period Treasury Bills.

k^M: carry cost on base money held for one period.

k^N: carry cost on currency, i.e. notes and coins, held for one period.

k^D: one-period carry cost on bank reserves held with the central bank.

k^B: one-period carry cost on collateralised bank borrowing from the central bank.

The risks, costs, and functions of different assets make the restrictions on the relative magnitudes of these short rates and their associated carry costs possible. Buiter (2009b) states the following

$$i - k \geq i^M - k^M. \tag{3.1}$$

Further, because the base money is defined as $M = N + D$, where N denotes currency and D bank reserves held with the central bank, it follows

$$i - k \geq i^N - k^N \tag{3.2}$$

$$i - k \geq i^D - k^D \tag{3.3}$$

$$i^B - k^B \geq i^D - k^D. \tag{3.4}$$

This says that the risk-free one-period interest rate on non-monetary assets (the net of carry costs) cannot be below the rate on base money (the net of carry costs) and further, it cannot be below the rate on the components of base money. This follows from the assumption that base money, say cash, has the highest carrying costs and the lowest interest rate. Also, a central bank does not want to offer commercial banks pure arbitrage opportunity, which means that the interest rate at which banks borrow against collateral from the central bank cannot be below the rate the banks earn when depositing funds with the central bank (both the nets of carry costs). (Buiter (2009b).)

Later Buiter and Rahbari (2015) formalise these derivations by taking the benefits of different assets into account. Let the non-pecuniary benefits for the currency and non-currency store of value be denoted by b^N and b, respectively. Then, to prevent an openended shift into currency by portfolio managers, the following has to hold:

$$i + b - c \geq i^N + b^N - k^N \tag{3.5}$$

$$\Leftrightarrow$$

$$i \geq i^N + (b^N - b) - (k^N - k). \tag{3.6}$$

Now, if it is assumed that the interest rate on currency, i^N, and the carry cost of non-monetary instruments, c , are zero, the inequality above simplifies to $i \geq b^N - b - k^N$. Further Buiter and Rahbari (2015) argue that in a liquidity trap people are indifferent between the non-pecuniary benefits of currency and non-currency store of value, i.e. $b^M = b$, which leads to $i \geq -k^N$.[2] This states that the lower bound on short nominal interest rates is negative and it is defined by the marginal cost of carrying currency.

However, even if there are theoretical floors on the policy rates, this does not say it is always stimulative to cut the interest rates. If the negative or too low interest rates hurt an economy more than it boosts, then the policy becomes ineffective. The rate at which this change happens is called an *effective* lower bound. This is considered next.

3.2 An effective lower bound on monetary policy

How does an interest rate cut by the central bank affect the profits, equity, and credit growth of banks? How are policy rates linked with the deposit rates on the accounts with commercial banks? This section answers these questions by looking at the study of Brunnermeier and Koby (2016), which approaches the lower bound on a policy rate from the perspective of the balance sheets of commercial banks. They argue that the effective lower bound on monetary policy is given by the *reversal rate*, which is the rate at which accommodative monetary policy reverses its effect and becomes harmful for banking business. This means, if the policy rate were set below the reversal interest rate, then it would depress rather than stimulate lending. This would have detrimental influence on the recovery of an economy. The effects of an interest rate cut are discussed next.

[2]The idea of the liquidity trap is first represented by Hicks (1937).

First, if banks hold long-term legacy assets with fixed interest payments, then a cut in the interest rate allows banks to refinance their long-term assets at a cheaper rate which increases the value of their equity. Hence, if the banks are better capitalised, then their solidity also increases and the regulatory pressure relaxes. However, if the deposit rate on the reserve accounts with a central bank is negative, there is also a negative effect. For instance, if the central bank buys long term bonds from a private bank (Quantitative Easing), it affects the balance sheet of the bank as a drop in bond holdings and a rise in reserves on the asset side. Now, if the central bank sets a negative deposit rate on the reserve holdings with its accounts, it cuts the reserves of banks and thereby the equity of banks depreciates. This is one of the reasons why central banks have been willing to keep their deposit rate positive; decreasing rates weaken the balance sheets of banks. (Brunnermeier and Koby (2016).)

Second, a low deposit policy rate accelerates the credit expansion so that banks substitute their reserves and safe asset holdings with new loans to firms and households. If the financial sector was perfectly competitive without frictions, then any monetary policy rate cut would be fully passed through to the deposit and loan rates, which would increase investment spending. However, because this is not the case in the real world and banks have market power, the net interest rate margins (NIM) of the banks are not competed away and the banks charge a mark-up. The NIM is a ratio that measures how successful a bank is at investing its funds in comparison to the expenses on the same investments.[3] A positive NIM is also called as a *credit friction* in an economy, i.e. there is a spread between the interest rate available to savers and borrowers (see Brunnermeier and Sannikov (2014) and Cúrdia and Woodford (2016)). Hence, a policy rate cut reduces the spread and therefore lowers the profits of banks. (Brunnermeier and Koby (2016).)

A decrease in interest rates necessarily hurts profits if the loss in profit from NIMs is greater than the returns from the long-term asset holdings of banks. This negative wealth effect leads to tightening in the equity constraint of banks and thereby banks cut back

[3] The NIM can be simply calculated by taking the difference between interest returns and interest expenses and dividing it by average earning assets.

on their credit extension and are forced to scale up their safe-asset holdings. New safety assets yield a lower interest rate and therefore the profits of banks actually decline faster which lowers the value of their equity further. After the lowering in equity, banks are again forced to substitute out of risky loans into safe-assets, which in turn lowers their profit, and so on and so forth.[4] (Brunnermeier and Koby (2016).)

There are also some empirical papers that support the interest rate effects via the above-discussed channels and advocate the later theories introduced in this paper. For instance, Bech and Malkhozov (2016) argue that the decrease in reserve rates below zero has transmitted through risk-free assets, but it has been imperfect for retail deposit rates. This implies that the NIMs in retail lending have not become as narrow as in other sectors. Their data reveal that the mortgage rates in certain countries have even increased recently. However, according to Drechler et al. (2015) profit margins on deposits tend to decrease with the reserve deposit rates. The papers by Abad et al. (2016) and Saunders and Schumacher (2000) provide a new angle to measure the net interest margin of banks, and, for instance, Petersen and Rajan (1995) and Sharpe (1997) examine the structures of the banking industries. Maudos and Fernandez de Guevara (2004), Saunders and Schumacher (2000), and Drechler et al. (2015) argue that imperfect competition in financial markets affects the pass-through of policy rates. Finally, the results of Drechler et al. (2015), Rognlie (2016), and Brunnermeier and Koby (2016) show that there is no effective *zero* lower bound on policy rates.

Based on these above-mentioned channels, the effective lower bound on monetary policy depends on the interest rate exposure of the assets of banks, the tightness of financial regulation, and the market structure of the banking sector. If banks hold much long-term bonds and mortgages with fixed interest, the *stealth recapitalization* effect due to an interest rate cut is great, and the reversal interest rate is lower. Whereas the lower bound on policy rates rises with the strictness of capital requirements and competitiveness within the financial sector.

[4] The examination of the balance sheet of banks by Brunnermeier and Sannikov (2016) provides closer consideration of the equity of banks and the risks of *diabolic loops* that could drive banks to insolvency.

The NIMs are the connective element between policy rates and commercial bank deposit rates. If the commercial banks cut their deposit rates with the policy rate decreases, the NIMs would not shrink. This means, actually, that the reversal rate gets lower with the fall in the commercial bank deposit rates. The next chapter provides models for commercial bank deposit rates as well as central bank deposit rates. This makes it possible to apply the results when considering the effective lower bound on monetary policy as well.

This chapter has shown that the ultimate lower bound on nominal interest rates is the deposit rate set by central banks. If the central bank policy rate were lower than its deposit rate on reserves, any rational bank would borrow an infinite amount by issuing outside money, in order to build up infinite reserve holdings. Therefore, the rest of this paper mainly examines the lower bounds on *deposit* rates.

4 Microeconomic approach to the lower bounds

Solow (1956) writes "All theory depends on assumptions which are not quite true. That is what makes it theory." In this chapter, I introduce the theory of the lower bounds on nominal interest rates where the presented results are based on certain assumptions. I represent models which are based on the basic microeconomic theory where an agent forms her best responses to different possible actions and hence maximises her utility. It is important to keep in mind through these analyses that the research subject is the lower bounds, indeed; *the lower bound is one of the extremes of nominal interest rates, of which the other one is infinity*. The upper bounds have never been reached at any rate and, thus, this might also be the case with the lower bounds.

In this chapter, the focus is on a single agent which optimises her portfolio of risk-free assets. The agent can be thought as a consumer who hold deposits with a commercial bank or a bank that has reserves on the accounts with the central bank. The earlier studies of Buiter and Panigirtzoglou (2003), Buiter (2009a), Buiter and Rahbari (2015), and Brunnermeier and Koby (2016) support the analyses presented in this chapter, and many ideas stand on the shoulders of the classic money and interest rate theories of Jevons (1876), Fisher (1896), Fisher (1930), Keynes (1936), and Hicks (1937). Monti (1972) and Klein (1971) provide closer examination of the microeconomics of the banks.

The first section in this chapter provides models for the lower bounds and the second section examines the equilibria of the models.

4.1 Models for the lower bounds

In this section, I present two-period and multi-period models for the lower bound on nominal deposit account interest rates. Generally, these models are based on a portfolio optimisation problem where an agent decides whether to keep her risk-free assets in a non-currency store of value or turn all or part of these assets into currency. The first model considers in particular the situation where an agent has money in a bank account and ponders whether or not to withdraw the money into cash. This kind of model is quite easy to follow and hence it is a good starting point for analysis. Furthermore, the model can be generalised to concern whatever party among credit institutions with preferences of a different kind. Lastly in this section, I specify a cash holding cost function, which can be simulated and thereby approach the lower bounds on deposit rates. This is a multi-period model where the variables are considered as stochastic processes.

However, first, it is important to understand and define the costs of holding money in its different forms, which was shortly touched on in Chapter 3. The scrutinies in Table 6 and Table 7 represent some of the most common personal customer costs of using money settled by commercial banks in Finland in 2016. The data are gathered from the price lists of the banks which are freely available from the websites of the banks.

Table 6 indicates that in a modern monetary economy, such as in Finland, the costs of keeping money in a deposit account for personal customers are *fixed*. Within the limits of competition, a bank can control these costs quite easily, for instance, by assessing credit card fees and the maintenance costs of bank accounts and online bank services. The basic deposit account costs are often in monthly terms, which does not change while the amount of deposits change. Thus, the only varying parameter for the depositor is the interest rate. Deposit rates might vary between the types of accounts and usually they are tied up with the deposit period and the amount of money. However, for the sake of simplicity, deposit rates are assumed to be the same for all amounts of money and for all agents in this paper. This kind of simplification makes it possible to generalise the models for the central bank deposit rates, because those do not vary between the depositors.

In this paper, the *cash holding costs* are divided into two parts: the costs that a bank can control and the *personal cash holding costs*. It is plausible to assume that both of these costs depend on the amount of the cash; the more cash in one's possession, the more arduous it is to operate everyday chores or make new investments. Table 7 represents the costs that banks can control. For instance, a bank can set a price for the cash withdrawals using automated teller machine (ATM) and for the withdrawals that are done at a branch. Also a bank can set service fees for a transaction settled at a branch. Based on Table 7, a typical cost for the cash payments settled at a branch is approximately 6.00 € / bill in the most common banks in Finland. However, these costs might be minutiae compared with the personal cash holding costs, which can be divided into transportation, storage, security, and insurance costs. These costs depend especially on time, distances, common safety, and many other social and personal circumstances. Also, these costs might be surprisingly high if considering a huge amount of money such as that which banks have in their reserve accounts with central banks. Of course, in the case of commercial banks and their reserve holdings, cash might not be even an option as long as there are government bonds or any other low-risk instruments in the markets.

The following analyses use mainly the above-mentioned costs of holding money, but take also into consideration the benefits of different kinds of stores of value in more advanced and general models. The next section examines the lower bounds in a simple framework, where the utilities of cash and deposits are restricted to contain only the costs. This gives the intuition for the extended models which are presented in the later sections.

4.1.1 A simplified two-period model for the lower bounds

In the next presented model, all the variables are considered in one-period magnitudes. An agent optimises her portfolio at the beginning of a period and chooses the best-yielding assets. The agent repeats this in every period and thereby maximises her utility over time.

Let the structure and assumptions of the model be the following. An agent makes her portfolio optimisation between two periods, t and $t+1$. Her portfolio $p_t = a_t + \delta_t$ consists of risk-free deposits δ_t and other assets a_t, which yield $1 + i$ and $R_t = 1 + i_a$ in one period, respectively.[5] The amount of deposits is fixed, which means that the agent does not optimise the portfolio weights between δ_t and a_t. However, at the beginning of each period, the agent decides whether to withdraw some amount ψ_t of the deposits into cash

[5] Notice that the assets a_t can also contain the cash holdings from the earlier periods.

(or other cash-equivalent). The agent values deposits and cash as perfect substitutes, and therefore she chooses the best yielding asset between these two.

Let the *fixed* costs of keeping the money in the deposit account over one period be denoted by $D_t > 0$, and the cash holding costs over one period by $C_t(\psi_t) > 0$, $\forall \psi_t > 0$, which depends positively on the amount of the cash, i.e. $\frac{\partial C_t(\psi_t)}{\partial \psi_t} = C_t'(\psi_t) > 0$. For the simplicity, it is assumed that all the accounts have similar fees and all the deposit accounts pay the same interest rate, i, while the interest rate on cash (or other cash equivalent) is zero. The interest rate payments and costs are remitted at the end of a period. In addition, let the expected inflation rate during the period t be denoted by π_t^e.[6]

Next, the agent makes a decision whether to withdraw her deposits or not to maximise her utility, in other words, she chooses between two portfolios, $p_t^{(1)}$ and $p_t^{(2)}$. Let the agent's utility function of the portfolio p_t be $u(p_t)$, such that $\frac{\partial u(p_t)}{\partial p_t} > 0 \ \forall p_t > 0$. Hence, if the agent keeps all her money in the deposit account, her utility in the next period, $t+1$, will be

$$u(p_{t+1}^{(1)}) = u\Big(\frac{R_t a_t + (1+i)\delta_t - D_t}{1+\pi_t^e}\Big), \tag{4.1}$$

whereas, if she turns some amount of her deposits into cash, $\psi_t \in [0, \delta_t]$, her next period utility will be

$$u(p_{t+1}^{(2)}) = u\Big(\frac{R_t a_t + (1+i)(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t)}{1+\pi_t^e}\Big). \tag{4.2}$$

Thus the agent will make her decision between these two portfolios depending on the utility that she will gain. The agent chooses to keep all her money in the deposit account

[6]It is also possible to consider the deposits δ_t and the cash ψ_t as the amounts of money that are not consumed during in the period t, i.e. as savings. In this case, the consumption has to remain the same and to be independent of the portfolio allocation changes.

if and only if the following inequality holds

$$u(p_{t+1}^{(1)}) \geq u(p_{t+1}^{(2)}) \tag{4.3}$$

$$\Leftrightarrow$$

$$u\Big(\frac{R_t a_t + (1+i)\delta_t - D_t}{1+\pi_t^e}\Big) \geq u\Big(\frac{R_t a_t + (1+i)(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t)}{1+\pi_t^e}\Big) \tag{4.4}$$

$$\Leftrightarrow$$

$$\frac{R_t a_t + (1+i)\delta_t - D_t}{1+\pi_t^e} \geq \frac{R_t a_t + (1+i)(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t)}{1+\pi_t^e} \tag{4.5}$$

$$\Leftrightarrow$$

$$(1+i)\delta_t - D_t \geq (1+i)(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t), \tag{4.6}$$

which reduces to form

$$i \geq \frac{-C_t(\psi_t)}{\psi_t} \quad (< 0). \tag{4.7}$$

Equation (4.7) reveals that the lower bound on the deposit rate, i_{LB}, equals $\frac{-C_t(\psi_t)}{\psi_t}$ for all $\psi_t \in (0, \delta_t]$, which is, in the light of the above-mentioned assumptions, strictly negative. This derivation also states that the agent is only interest in the nominal values of portfolios, because the utility function is the same for both portfolios, the inflation similarly affects all the assets, and the returns of others assets remains unchanged. Hence, these factors are omitted from later analyses to ease calculations.

In the simplest case, if the cash holding cost function is assumed to be linear and the costs are zero if there is no cash in an agent's possession, the costs can be written as $C_t(\psi_t) = P\,\psi_t$, where $P \in \mathbb{R}_+$ is the marginal cost of cash, i.e. $\frac{\partial C_t(\psi_t)}{\partial \psi_t} = P$. Thus the lower bound becomes the same as in the second chapter

$$i \geq -C_t'(\psi_t) \quad (= -P < 0). \tag{4.8}$$

This more understandable and intuitive form says, if the expenses of turning one unit of deposits into cash is greater than paying negative interest, then it will be more profitable to keep all the money in a bank account. This interpretation sounds reasonable under the earlier presented assumptions. A positive interest rate can be defined as the marginal revenue of the money and negative interest rate as the marginal cost of deposits, thus, the intuition is clear: an agent maximises her utility by choosing the option that has the lowest marginal costs.

However, there is also a case where an agent withdraws all her deposits and suppresses her account. In this situation, an agent gets rid of her bank account costs, i.e. $D_t = 0$ on the right-hand side of Equation (4.6). Therefore the lower bound on nominal interest rates becomes

$$i \geq \frac{D_t - C_t(\psi_t)}{\psi_t} \quad \Big(> -\frac{C_t(\psi_t)}{\psi_t}, \quad \forall D_t > 0\Big) \tag{4.9}$$

which is negative if and only if $D_t < C_t(\psi_t)$. As one can see, this lower bound is higher than the previous one for all positive deposit account fees. In modern society, where payment systems are highly developed and financial markets are fairly competitive, it is reasonable to claim that $D_t < C_t(\psi_t)$ holds with large ψ_t. Imagine if an agent took care all of his wage and transaction chores without a banking system, it would take a huge amount of time and effort, but also include some insurance and storage costs; the costs of holding money in a deposit account are paltry if it is compared with the situation where an agent takes care of all her financial matters using only cash. Moreover, if $D_t > C_t(\psi_t) \forall \psi_t$ were true, why would anyone ever use bank services with zero or low interest rates. Furthermore, $D_t < C_t(\psi_t)$ holds true for the reserve accounts with central banks as well, because some of the central banks do not charge deposits costs at all. In this case, the lower bound of this special case is exactly the same as before in Equation (4.7).

Hence, this also suggests that the theoretical lower bound on nominal interest rates is, in all probability, negative. As one can see from the lower bound in Equation (4.9), it depends on only two variables that banks can partly control. This means that banks can manipulate the lower bound on nominal interest rates quite effectively and easily within the limits of competition.

One notable thing in these simple analyses is that, if a central bank were able to change the costs of holding currency, the lower bound would be completely controllable. This is, as a matter of fact, what Gesell (1916), Goodfriend (2000), Buiter (2009a), and Mankiw (2009) propose. It is important to separate the variables that affect the returns of assets, because it is only thereby possible to examine more thoroughly the lower bounds and the ways to overcome the *zero* lower bound as well.

4.1.2 A general two-period model for the lower bounds

Based on the framework of the previous section, the theory can be applied to concern whatever party among credit institutions which has deposits in an account. Only the structure of the costs would be different, but the model remains the same. Next, to make the analyses more general the benefits of holding different assets are reckoned in.

Formally, keeping the notation the same as before, but now let δ_t denote a non-currency store of value, ψ_t currency, i the nominal interest rate on the non-currency store of value, and i_m the nominal interest rate on currency. Let the agent's utility function be denoted by $v(\delta_t, \psi_t)$, which contains the costs and benefits of both assets.

Now, alike in the earlier section, the agent decides to keep all her money in the non-currency store of value if the following inequality holds

$$(1+i)\delta_t + v(\delta_t, 0) \geq (1+i)(\delta_t - \psi_t) + (1+i_m)\psi_t + v(\delta_t - \psi_t, \psi_t), \tag{4.10}$$

where the left-hand side represents the situation where the agent keeps all the money in the non-currency store of value, and the right-hand side where the agent turns some amount ψ_t of the non-currency store of value δ_t into currency. As one can see, this situation is almost identical to the earlier analyses: now the costs are more generally presented as the utilities $v(\delta_t, 0)$ and $v(\delta_t - \psi_t, \psi_t)$ and the nominal interest rate on currency, i_m, has added to the model. Equation (4.10) reduces to

$$i \geq i_m + \frac{v(\delta_t - \psi_t, \psi_t) - v(\delta_t, 0)}{\psi_t}, \tag{4.11}$$

which is the general form of the lower bound on nominal interest rates. Appendix A provides considerations of a simple multi-period model with a fixed interest rate.

If an agent values the non-currency store of value and currency as perfect substitutes, the utility function of risk-free assets can be written as follows

$$v(\delta_t, \psi_t) = \beta_1\delta_t + \beta_2\psi_t. \tag{4.12}$$

The costs and benefits of the assets defines the weights β_1 and β_2 which could include, for instance, the utility that an agent gets from the anonymous transactions of cash payments or the rapidity of deposits to do investments. The agent is fully neutral between these two assets if $\beta_1 = \beta_2 = 1$. The partial derivatives of the utility function with respect to δ_t and ψ_t are

$$\frac{\partial v(\delta_t, \psi_t)}{\partial \delta_t} = \beta_1 \equiv b - c \tag{4.13}$$

$$\frac{\partial v(\delta_t, \psi_t)}{\partial \psi_t} = \beta_2 \equiv b_m - c_m, \tag{4.14}$$

where the marginal utility of the non-currency and currency, β_1 and β_2, are separated into two parts: the marginal benefits, b and b_m, and the marginal costs, c and c_m, where the subscript m expresses the currency.

Now, by substituting the utility function of perfect substitutes given in Equation (4.12) into Equation (4.11) it follows

$$i \geq i_m + \frac{\beta_1(\delta_t - \psi_t) + \beta_2\psi_t - \beta_1\delta_t}{\psi_t} \tag{4.15}$$

$$i \geq i_m + \beta_2 - \beta_1 \tag{4.16}$$

$$i \geq i_m + (b_m - b) - (c_m - c), \tag{4.17}$$

which is *exactly* the same result as presented in Chapter 3. Therefore this general framework with the utility function of perfects substitutes is precisely possible to associate with Buiter (2009b) and Buiter and Rahbari (2015).

Moreover, the special case, where an agent chooses to keep all her wealth in either stores of value, can be examined by using Equation (4.11) and by setting $\delta_t = \psi_t$. In this case, the lower bound becomes

$$i \geq i_m + \frac{v(0, \psi_t) - v(\delta_t, 0)}{\psi_t}. \tag{4.18}$$

From this equation it is easy to get into Equation (4.9) by setting $i_m = 0, v(0, \psi_t) = -C_t(\psi_t)$, and $v(\delta_t, 0) = -D_t$.

This generalisation reveals one very special nature of the lower bounds what was left aside in Goodfriend (2000), Buiter (2009a), and Mankiw (2009), and also in Buiter and Rahbari (2015), namely the marginal benefits of a non-currency store of value b. As the marginal cost of currency, the marginal benefit of non-currency has a negative effect on the lower bound. This means, instead of manipulating the marginal cost of cash, e.g. taxing currency, the central bank or the government could control the benefits. One simple way to do this is to give tax reliefs for card payments or transactions made using deposited money, which is conceivably a more popular way to overcome the zero lower bound than taxing currency with more supervision and controlling. Actually, setting the shadow exchange rate between the numéraire and the medium of exchange is this kind of act which expressly manipulates the benefits.

In the next section I will specify a cash holding cost function and extend the simplified two-period model for multiple periods. When examining the multi-period models of the lower bounds, it is sensible to take uncertainty into consideration. This complicates the derivations, but it turns out that it is still possible to approach the lower bounds on nominal interest rates by simulating the possible outcomes.

4.1.3 A multi-period model with a stochastic cash holding cost function

Markov chains are widely used processes which have many applications as statistical models of real-world processes. One important example of Markov processes is a Poisson process, which is a simple stochastic process for modelling the times at which "arrivals" enter a system. Hence, random processes can be useful when considering the cash holding costs, too. There is a huge number of economic literature on costs that are affected by random variables. Especially the Ss models and price adjustment examinations provide many noteworthy sources, for instance, by Scarf (1960), Barro (1972), Hamermesh (1989), and Gertler and Leahy (2008). The Ss models have been used to explain inertia in investment, money demand, and cash management, and to provide microfoundations for price stickiness and the real effects of money. Cash management models were originally presented in Baumol (1952) and thereafter adjusted by Tobin (1956). Miller and Orr (1966) analyse the cash balance as having a random variable with an irregular fluctuation and proposed a stochastic model for managing the cash balance. Furthermore, stochastic cash flow management models are examined, for instance, by Whalen (1966), Liu and Xin (2008), Mierzejewski (2010), and Melo and Bilich (2011).

Next I will specify a certain structure for the cash holding function C_t and therefore approach the lower bounds on nominal interest rates. The cost function is based on a modern banking sector with certain assumptions, and hence it cannot be *directly* related to the earlier specifications of cost functions. However, the kernel of the next presented cash holding cost function stands on the shoulders of the above-mentioned studies. This section provides a framework that can be simulated and thereby gives a useful tool for lower bound analyses. Unlike the earlier sections, the model presented in this section is tailored for multi-period examinations.

First, let an agent's money holdings at the beginning of period t be denoted by m_t. The money can be deposited in a deposit account or held in cash. When the money is in the form of cash, it has the earlier-discussed personal cash holding costs (see Section 4.1). These transportation, storage, security, insurance, and other administrative costs are denoted by $P_m\, m_t$, where P_m belongs to $[0, 1]$. In addition to these costs, suppose that the agent faces non-recurring costs from transfers and losses that cash usage causes, too. For instance, Baumol (1952) and Miller and Orr (1966) specify the costs of holding cash by the sum of trading costs and opportunity cost of holding cash. Next I will define these parts more precisely.

Let $N^i \sim \text{Poisson}(\lambda_i)$ be a homogeneous Poisson point process with intensity parameter $\lambda_i > 0$ for $i = \{T, L\}$. Suppose extra transfers that cash usage causes occur as a Poisson process N_t^T with rate λ_T. These transfers have to be settled at a bank branch or at the central bank. Thereby N_t^T can be thought as the number of visits to a bank during one period. Let the cost per transfer be denoted by P_T. In addition to the extra transfers, suppose some number of notes and coins will be lost, destroyed, and robbed during a period. Assume this kind of an accident happens also as a Poisson process N_t^L with intensity λ_L. Let a unit cost of the accident be P_L, which can be just a denomination of the (average) note or a maintenance cost of a damaged note. Hence, the transfers and losses cost the agent $P_T\, N_t^T + P_L\, N_t^L$ within the period t.

As said before, the agent can hold money either in deposits or cash. Hence, if the agent decides to hold cash, she faces the opportunity cost of holding cash, which is the deposit rate i_t that a bank offers. Thus, this has to be taken into account in the cash holding cost function, too. Also, making withdrawals and deposits has costs. Turning deposits into cash costs the fixed share $\alpha \in [0, 1]$ of the funds that the agent withdraws. Whereas returning *all* the cash holdings into deposit account costs the share $R \in [0, 1]$ of the deposited cash.

Suppose that within each period an agent consumes or uses some random amount of her money ξ_t. In other words, the agent faces the demand for her money holdings, which does not depend on the allocation of the assets. Also assume that the agent receives new cash flows $w_t \geq 0$ to her bank account at the beginning of each period. Those deposits can be thought as wages or new excess reserves.

These above-mentioned costs and random variables structure the total cash holding costs. Now, as in the earlier models, the agent holds her money in cash if the interest rate i_t is lower than the lower bound i_{LB}, whereas if the interest rate is greater or equal to the lower bound, the agent holds deposits. Therefore, the cash holding costs can be written as follows

$$\begin{aligned} C_t = \big(1 - \mathbb{1}_{LB}(i_t)\big) \Big[& P_m\, m_t + P_T\, N_t^T + P_L\, N_t^L + i_t\,(m_t - \xi_t) \\ & + \mathbb{1}_{LB}(i_{t-1})\, \alpha\, m_t + \big(1 - \mathbb{1}_{LB}(i_{t-1})\big)\, \alpha\, w_t \Big] \\ & + \mathbb{1}_{LB}(i_t)\, \big(1 - \mathbb{1}_{LB}(i_{t-1})\big)\, R\, m_t, \end{aligned} \tag{4.19}$$

where $i_t\,(m_t - \xi_t)$ is the opportunity cost of holding cash.[7] The indicator function $\mathbb{1}_{LB}(i_t)$

[7] Notice that with negative interest rates the opportunity cost is also negative and, hence, it restrains the cash holding costs.

is defined as

$$\mathbb{1}_{LB}(i_t) = \begin{cases} 1 & \text{if } i_t \geq i_{LB} \\ 0 & \text{if } i_t < i_{LB}, \end{cases}$$

which determines the parts of the costs that will be realised during the period. Before going further, the indicator function usages and some timeline behaviour could be useful to articulate. When the agent enters the period t, she gets first the income w_t and then faces its period interest rate i_t. After that the agent decides whether to keep the allocation of the safe-assets unchanged or not.

First, consider the case where the interest rate sinks below the lower bound, i.e. $i_t < i_{LB}$. This implies to $\mathbb{1}_{LB}(i_t) = 0$ and thereby $\big(1 - \mathbb{1}_{LB}(i_t)\big) = 1$, which means that the first part of the costs realises, but the returning cost $R\, m_t$ does not, i.e.

$$\begin{aligned} C_t = & P_m\, m_t + P_T\, N_t^T + P_L\, N_t^L + i_t\,(m_t - \xi_t) \\ & + \mathbb{1}_{LB}(i_{t-1})\,\alpha\, m_t + \big(1 - \mathbb{1}_{LB}(i_{t-1})\big)\,\alpha\, w_t. \end{aligned}$$

Depending on the deposit rate in the previous period, the agent withdraws either all the money holdings m_t or just the new deposits w_t. If the deposit rate were higher than the lower bound in last period, the agent would turn all the earlier money holdings into cash. This means $\mathbb{1}_{LB}(i_{t-1}) = 1$ and therefore the total costs are

$$C_t = P_m\, m_t + P_T\, N_t^T + P_L\, N_t^L + i_t\,(m_t - \xi_t) + \alpha\, m_t.$$

Whereas if the deposit rate in the previous period was already below the lower bound, the agent has already withdrawn her deposits in the last period and therefore she has to withdraw only the new deposits w_t. In terms of the indicator function $\mathbb{1}_{LB}(i_{t-1}) = 1$ and thereby the cash holding costs become

$$C_t = P_m\, m_t + P_T\, N_t^T + P_L\, N_t^L + i_t\,(m_t - \xi_t) + \alpha\, w_t.$$

Second, if $i_t \geq i_{LB}$, then the agent prefers deposits over cash. This means $\mathbb{1}_{LB}(i_t) = 1$ and thereby the cash holding costs reduce to

$$C_t = \big(1 - \mathbb{1}_{LB}(i_{t-1})\big)\, R\, m_t.$$

Now, if the previous period's interest rate was below the lower bound, the indicator function is $\mathbb{1}_{LB}(i_{t-1}) = 0$ and therefore the returning costs realise, i.e. $C_t = R\,m_t$. However, if the interest rate in the last period was already higher than the lower bound, the agent has already deposited her funds in the period $t-1$. Therefore there is no cash holding costs in the period t, i.e. $\mathbb{1}_{LB}(i_{t-1}) = 1 \Rightarrow C_t = 0$.

Based on the money holdings and the expected cash holding costs, at agent optimises her portfolio by choosing the assets at the beginning of a period. This is the same framework and idea as in the examinations of the earlier sections. However, now with the certain cash holding cost function, it is possible test is this profitable behaviour with a longer horizon, when an agent repeats this optimisation t times.

If the agent decides to turn her money into cash always when the interest rate sinks below the lower bound, her money holdings at the beginning of t will be

$$m_t = w_t + (1 + i_{t-1}\,\mathbb{1}_{LB}(i_{t-1}))(m_{t-1} - \xi_{t-1} - C_{t-1}). \tag{4.20}$$

Iterating backwards t times and assuming that $m_0 = w_0$ it follows

$$m_t = w_t + \sum_{k=0}^{t-1} \Big[\prod_{j=0}^{k} \Omega_{t-j-1}\Big] (w_{t-s-1} - \xi_{t-s-1} - C_{t-s-1}) \quad t \geq 1, \tag{4.21}$$

where Ω_t is the interest rate operator $1 + i_t\,\mathbb{1}_{LB}(i_t)$. This states that the total amount of money at the beginning of period t equals the sum of net incomes from the period 0 to t.

Whereas if the agent does not turn her deposits into cash in any period, her money holdings, m'_t, will be

$$m'_t = w_t + (1 + i_{t-1})\,(m'_{t-1} - \xi_{t-1}), \tag{4.22}$$

which can be iterated to

$$m'_t = w_t + \sum_{k=0}^{t-1} \Big[\prod_{j=0}^{k} I_{t-j-1}\Big] (w_{t-s-1} - \xi_{t-s-1}), \tag{4.23}$$

where $I_t = 1 + i_t$.

Now, by giving certain parameters for the Poisson processes N^T and N^L and defining the marginal costs P_m, P_T, P_L, α, and R it is possible to test the certain levels of the lower bounds on nominal interest rates by simulating the quantities of the following inequality

$$m'_t \geq m_t \quad \Leftrightarrow \tag{4.24}$$

$$\sum_{k=0}^{t-1} \left[\prod_{j=0}^{k} (I_{t-j-1} - \Omega_{t-j-1}) \right] (w_{t-s-1} - \xi_{t-s-1}) \geq - \sum_{k=0}^{t-1} \left[\prod_{j=0}^{k} \Omega_{t-j-1} \right] C_{t-s-1}, \tag{4.25}$$

where $I_t - \Omega_t = \big(1 - \mathbb{1}_{LB}(i_t)\big)\, i_t$. As a matter of fact, this inequality just compares the yields of two assets. The difference between the earlier models and this one is that now there is stochasticity in the variables and the cash holding cost function is specified. If the movements of the variables i_t, w_t, and ξ_t were known or predictable, the comparison would be easy by using the expected values of the Poisson processes and calculating the equations.[8] However, because at least the interest movements are somehow unpredictable, the expected values of the money holdings are easier to do by using Monte Carlo integration. The following lemma describes the Monte Carlo integration, which is used in later simulations.

Lemma 1. *Let* $y_t^{(s)} = (i_t^{(s)}, w_t^{(s)}, \xi_t^{(s)})'$ *for* $s \in \{1, ..., S\}$ *be a random sample from the conditional density function* $p(y_t \,|\, \theta)$ *for all* $t \geq 1$*, where the set of the simulation parameters is denoted by* $\theta = (P_m, P_T, P_L, \lambda_T, \lambda_L, \alpha, R, i_0, i_{LB})$*. Let the money holding functions be denoted as:*

$$\begin{aligned} m(y_t^{(s)}) &= w_t^{(s)} + \big(1 + i_{t-1}^{(s)}\, \mathbb{1}_{LB}(i_{t-1}^{(s)})\big)(m_{t-1}^{(s)} - \xi_{t-1}^{(s)} - C_{t-1}^{(s)}) \\ &= w_t^{(s)} + \sum_{k=0}^{t-1} \left[\prod_{j=0}^{k} \Omega_{t-j-1}^{(s)} \right] (w_{t-s-1}^{(s)} - \xi_{t-s-1}^{(s)} - C_{t-s-1}^{(s)}) \end{aligned}$$

and

$$\begin{aligned} m'(y_t^{(s)}) &= w_t^{(s)} + (1 + i_{t-1}^{(s)})\,(m_{t-1}'^{(s)} - \xi_{t-1}^{(s)}) \\ &= w_t^{(s)} + \sum_{k=0}^{t-1} \left[\prod_{j=0}^{k} I_{t-j-1}^{(s)} \right] (w_{t-s-1}^{(s)} - \xi_{t-s-1}^{(s)}). \end{aligned}$$

[8] The expected value of a homogeneous Poisson process is defined by its intensity parameter.

Hence, based on the law of large numbers

$$m_S = \frac{1}{S}\sum_{s=0}^{S} m(y_t^{(s)}) \quad \textit{and} \quad m'_S = \frac{1}{S}\sum_{s=0}^{S} m'(y_t^{(s)})$$

converge to $\mathbb{E}_0(m_t \,|\, \theta)$ *and* $\mathbb{E}_0(m'_t \,|\, \theta)$*, respectively, as* S *goes to infinity.*

Next I will show two examples of the simulations. The first considers the excess reserves of a credit institution with the central bank and the second the deposits of a consumer with a commercial bank. In both examples, the interest rate is modelled as a discrete-time Markov process with a countable state space and a certain transition matrix. Also wages and the demand for money holdings are simulated as random walks. Formally, all the random processes $\{N_t^T, N_t^L, i_t, w_t, \xi_t\}$ follow the properties of a (discrete-time) Markov chain as follows: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t,\ t \in \mathbb{N})$, for index set $T = 0, 1, 2, ...$ and let $(S, \mathcal{S})$ be a measure space. A S-valued stochastic process $(X_t,\ t \in T)$ is adapted to the filtration with respect to the $\{\mathcal{F}_t\}$ for each $A \in \mathcal{S}$ and each $s, t \in T$ with $s < t$, $\mathbb{P}(X_t \in A \,|\, \mathcal{F}_s) = \mathbb{P}(X_t \in A \,|\, X_s)$.

The examples are purely hypothetical, but try to catch some real-world features. The first example is based on the data of the ECB depicted in Figure 1 and the second uses the personal customers' cost data given in Table 7.

Example 1. (The lower bound on a central bank deposit rate) *Let the transition matrix and the state space be as in Table 3 for the 13-dimensional discrete Markov Chain, which presents a central bank deposit rate in per annum basis. The periods are quarters and the simulation horizon is 3 years. In other words, the interest rates and the lower bounds are presented as per annum rates, but the simulations use quarterly rates. The simulations with the different lower bounds and initial rates are calibrated with parameters*

$$\theta = (P_m, P_T, P_L, \lambda_T, \lambda_L, \alpha, R, i_0, i_{LB}) = (0.003, 0, 0, 0, 0, 0.001, 0.001, i_0, i_{LB}).$$

That is, the money spent on transportation, storing, securing, and other administrative costs are 0.3 % *of the total amount of cash holdings per quarter. Withdrawing and depositing money both cost* 0.1 % *share of the transfered money. The Poisson processes* *that represent the random transfers and losses are omitted because of their minute effect on the total costs. Adding those variables with rather large rates does not change the next presented results significantly.*

An augmented-Dickey-Fuller (ADF) unit root test for the excess reserves of the depository institutions in the euro area revealed that the reserves are a unit root process (the data same as in Figure 1).[9] *Hence, the new excess reserves and the demand for those are modelled as positive random walks, such that* $\mathbb{E}(w_t - \xi_t) = 0$. *The standard deviations of the processes are based on the same data, too. The initial value for the money holdings,* m_0, *is 180 millions of euro, which is a (very) rough approximation of the average possession of excess reserves of a European credit institution in the second quarter of 2016.*

The following table summarises the results of the simulations with the different lower bounds on the deposit rate and the initial values of the interest rate.

Table 1: Simulations with $S = 10000$, $i_0 = -0.005$ (left), and $i_0 = -0.0075$ (right).

i_{LB}	$\#\{m_t'^{(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m'_S	m_S
0	9,891	109	223,233,089	221,561,852
−0.0025	9,884	116	222,579,649	221,758,401
−0.004	9,909	91	223,690,261	222,866,041
−0.005	1,638	281	222,584,595	222,546,640
−0.0075	215	140	224,174,746	224,183,460
−0.01	5	77	223,741,449	223,748,342

i_{LB}	$\#\{m_t'^{(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m'_S	m_S
0	9,381	619	220,648,714	219,358,477
−0.0025	9,072	928	223,106,760	222,556,770
−0.004	9,066	934	222,444,881	221,893,883
−0.005	8,549	1,451	221,970,768	221,775,918
−0.0075	1,154	807	221,933,855	221,988,778
−0.01	22	319	222,549,504	222,575,945

The table on the left represents the simulations with the initial deposit rate -0.5 *% and the table on the right with* -0.75 *%. The lower bounds are in the first column, the number of simulations where* m_t' *was higher than* m_t *in the second column, and the number of simulations where* m_t *was higher than* m_t' *in the third column. In the last two columns are the simulated expected values of different assets (see Lemma 1). The results indicate that the lower bound on central bank deposit rates is between* -0.5 *% and* -0.75 *%. This is lower than the ECB's current deposit rate* -0.4 *%. However, the simulations are based*

[9] The ADF test results are based on a 99 % significance level.

on the harsh assumptions and simplifications and therefore the simulated lower bound is only suggestive.

The next example considers a consumer and her deposits. The simulations are based on the Finnish data given in Table 7 and the transition matrix of the interest rate in Table 9. The periods are quarters and the simulation horizon is the same as in the first example (36 months).

Example 2. (The lower bound on a bank deposit rate) *Let the set of the simulation parameters be defined as:*

$$\theta = (P_m, P_T, P_L, \lambda_T, \lambda_L, \alpha, R, i_0, i_{LB}) = (0, 6, 1, 3, 1, \alpha, 0, i_0, i_{LB}).$$

That is, one transfer or payment settled by using cash equals 6 *euro and the expected value of the number of these transfers is 3. As Table 7 presents, cash payment done at a branch in Finland in 2016 was approximately* 6 *euro per bill, and typical number of bills in one months could be 3, consisting of, for instance, rent, phone bill, and electricity bill. The losses have been set relatively small; on the average one euro will be lost in a month.*

Let the other costs be zero at first, i.e. there are no withdrawal or depositing fees and the carrying costs are zero as well. The simulations are based on a situation where a consumer has 10000 *euro savings and receives* 2000 *euro salary at the beginning of each period. In every period, the consumer spends some random amount between* 1500 *and* 2000 *(euro) of her money holdings, i.e. she randomly saves from each payroll.*

The table below summarises the results of the simulations.

Table 2: Simulations with $S = 10000$, $\alpha = 0$, $i_0 = -0.0075$ (left), and $i_0 = -0.01$ (right).

i_{LB}	$\#\{m_t'^{(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m_S'	m_S	i_{LB}	$\#\{m_t'^{(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m_S'	m_S
0	$9,570$	430	$24,715$	$24,612$	0	$8,891$	$1,109$	$24,644$	$24,561$
-0.0025	$8,953$	$1,047$	$24,701$	$24,655$	-0.0025	$7,855$	$2,145$	$24,632$	$24,603$
-0.004	$8,913$	$1,087$	$24,719$	$24,675$	-0.004	$7,867$	$2,133$	$24,651$	$24,622$
-0.005	$7,234$	$2,766$	$24,710$	$24,700$	-0.005	$6,145$	$3,855$	$24,648$	$24,651$
-0.0075	$1,033$	$2,193$	$24,724$	$24,732$	-0.0075	$4,501$	$5,499$	$24,640$	$24,656$
-0.01	104	711	$24,739$	$24,743$	-0.01	458	$2,196$	$24,654$	$24,666$

The left-hand table presents the results of the simulations with the initial deposit rate -0.75 *% and the right-hand table with* $i_0 = -1.0$ *%. The initial values are radical because of the surprising results. Even if there are no other costs except the random bill payments done at a branch and diminutive losses, the costs are still higher than the negative interest expenses with relatively low rates. Based on the simulations, the lower bound on commercial bank deposit rates is approximately* -0.5 *%.*

When the withdrawal fee $\alpha = -0.5\%$ *has been taken into account, the results become more dramatic. This cost arrangement is based on Danske Bank's withdrawal cost at a branch (see Table 7). The simulation results are presented in the tables below, where the left one is for* $i_0 = -1.5$ *% and the right one for* $i_0 = -1.75$ *%.*

Table 3: Simulations with $S = 10000$, $\alpha = 0.005$, $i_0 = -0.015$ (left), and $i_0 = -0.0175$ (right).

i_{LB}	$\#\{m_t^{\prime(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m'_S	m_S	i_{LB}	$\#\{m_t^{\prime(s)}\}_>$	$\#\{m_t^{(s)}\}_>$	m'_S	m_S
-0.005	$9,459$	541	$24,509$	$24,359$	-0.005	$8,533$	$1,467$	$24,426$	$24,324$
-0.0075	$9,316$	684	$24,499$	$24,398$	-0.0075	$8,248$	$1,752$	$24,423$	$24,361$
-0.01	$9,364$	636	$24,508$	$24,434$	-0.01	$8,050$	$1,950$	$24,430$	$24,393$
-0.0125	$9,512$	488	$24,509$	$24,449$	-0.0125	$8,217$	$1,783$	$24,423$	$24,391$
-0.015	$2,189$	344	$24,502$	$24,494$	-0.015	$8,922$	$1,078$	$24,426$	$24,388$
-0.0175	495	154	$24,504$	$24,504$	-0.0175	$1,969$	566	$24,427$	$24,423$
-0.02	90	51	$24,490$	$24,490$	-0.02	412	242	$24,419$	$24,419$

The simulations indicate that the lower bound falls somewhere between -1.5 % and -1.75 %. However, again, both of the results are based on the above-mentioned assumptions and hence are indicative.

The two examples above have shown that it is possible to approach the lower bound on deposit rates by simulating the possible outcomes of different assets. The simulation results are based on the chosen parameters, the predestined transition matrices, and features of the random processes. During the simulations, it turned out that if the amount of money is huge, say like in Example 1, the results are highly volatile for the changes in the parameters P_m, α, and R. This is, of course, obvious, because they are the shares of money holdings and consequently have a direct impact on the total amount of the money. If the amount of money holdings are rather small, like in Example 2, the Poisson processes of transfers and losses and their unit costs play important roles in the total costs. In addition, new cash flows have the same robustness features as the total amount of money and therefore the withdrawal fee appears more important with large w_t. The results were, however, adequate and robust with the changes of the lower bounds and the initial values.

Nothing has been said of the transition matrices so far. The roles of the transition probabilities are important, because they indicate the expectations of interest rate movements.

For instance, the steady state probability distributions of the Markov chain deposit rates used in the examples are shown in the tables below.

Table 4: The steady state probability distribution of the central bank deposit rate used in Example 1.

0.01	0.0075	0.005	0.0025	0	-0.0025	-0.005	-0.0075	-0.01	-0.0125	-0.015	-0.0175	-0.02
0.751	0.188	0.047	0.012	0.002	0.0005	0.0001	0.00002	0.00000	0.00000	0.00000	0	0

Table 5: The steady state probability distribution of the commercial bank deposit rate used in Example 2.

0.01	0.0075	0.005	0.0025	0	-0.0025	-0.005	-0.0075	-0.01	-0.0125	-0.015	-0.0175	-0.02	-0.0225	-0.025
0.533	0.267	0.133	0.044	0.015	0.005	0.002	0.001	0.0001	0.00003	0.00001	0.00000	0.00000	0.00000	0.00000

A steady state distribution represent equilibria in the chain's behavior in the long run with the given state space. Now, because the used state spaces are rather small, the Markov processes of the deposit rates are in state 1 % most of the time and the negative rates are very rare. This is, in fact, a well-defined structure of the processes if the negative rates are thought only as temporary policy acts.

Tables 8 and 9 describes the movements in the short run. The transition probabilities are chosen such that they take into account the length of the period and the tendency of economic developments. Based on the transition matrix in Table 8, the deposit rate hikes more likely in every period, because the periods are quarters. Table 9 indicates that the Markov chain varies less and thereby it is not very likely that the deposit rate increases in every month.

The simulations were fairly robust with the small changes in the transition probabilities. Nevertheless, the size of the state spaces and bigger structural changes in the transition matrices make the results more volatile. Thus, the results done by the simulations should be called into question if the parameters are not well thought out. Overall, the model represented in this section seems to behave and work adequately for both lower bounds on central bank deposit rates and commercial bank deposit rates. Based on Lemma 1, the simulations converge to the expected values of the assets, which provides comparable results for the lower bounds and, hence, this framework could be a useful tool for policy acts.

Section 4.1 has examined the different possibilities of the lower bounds with different assumptions, providing several models. The models have shown that the lower bounds on nominal interest rates depend on the costs and benefits of the risk-free assets. After these derivations it is possible to examine the equilibria of the models and the agent's behaviour in different situations. The following sections take a look at these issues and reveal that an agent is indifferent between risk-free assets at the lower bound only if the cash holding cost function is linear.

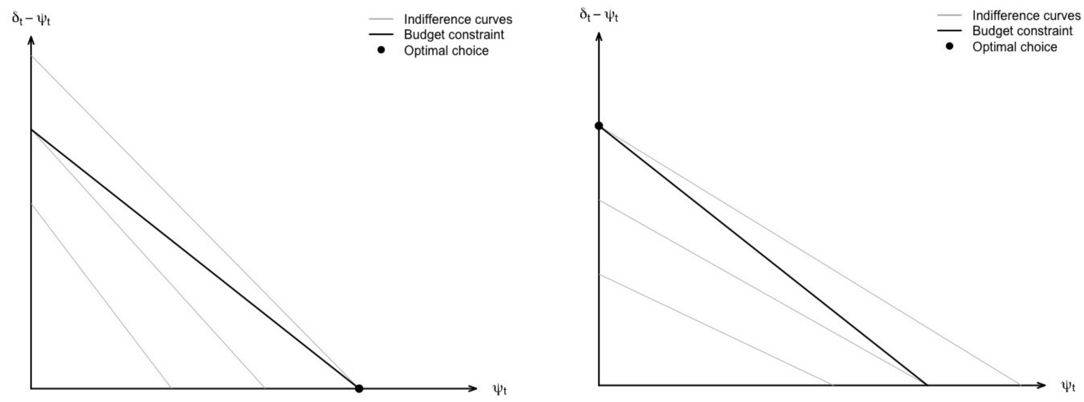
4.2 Equilibria of the models

Recalling the agent's utility function $u(p_t)$ and using the preferences of form $u(p_t) = f(a_t) + g(\delta_t, \psi_t)$ which separates the risk part and the risk-free part from each other, it is possible to focus solely on the utility function of the risk-free assets $g(\delta_t, \psi_t)$. This preference structure is similar to the earlier works of Arrow (1971), Sandmo (1968), and Aura et al. (2002). In this way it is possible to find equilibria for the earlier models of an agent's portfolio optimisation problem. Assume that the utility function of risk-free assets in period $t + 1$ is form

$$g(\delta_t, \psi_t) = (1 + i)(\delta_t - \psi_t) - D_t + (1 + i_m)\psi_t - C_t(\psi_t), \tag{4.26}$$

where the parameters and variables are the same as in the earlier sections. This form of the utility function is reasonable, because it states that the agent is risk-neutral between these two assets, which is obvious when it includes only two *risk-free* assets. Deposits and cash are risk-free assets in terms of store of value, but in terms of medium of exchange, cash has risk of getting lost or being destroyed. These risks are included in the cash holding cost function, which makes an agent's utility of the assets similar to the risk neutral preferences. Actually, deposits have a risk of getting lost by, for instance, cyber attack; however, because of its minute probability, it is omitted for simplicity. Thus, by choosing the optimal choice of ψ_t that maximises the utility function $g(\delta_t, \psi_t)$, the agent also maximises her aggregate utility of portfolio $u(p_t)$ with respect to the risk-free assets.

The results in the previous sections indicate two obvious equilibria: if the nominal interest rate is greater than the lower bound, an agent keeps all the risk-free assets as deposits (or in other non-currency store of value), and if the nominal interest rate is less than the lower bound, she turns all the deposits (non-currency store of value) into cash (currency). This does not say that an agent would not keep cash holdings at all, because she could have these included in other assets a_t, but she does not turn the *already existing* deposits into cash if she earns enough interest payments. However, if the agent has no earlier cash holding, i.e. she has not preferred cash earlier, then this leads to an outcome which states it is not efficient to hold cash at all if it is possible to deposit those in a bank account where the interest rate is greater than the lower bound. If the interest rate is below the lower bound, then an agent does not hold deposits at all. These optimal corner choices are depicted in the figures below. The slopes of the indifferent curves of the utility function of risk-free assets, $g(\delta_t, \psi_t)$, are formed of the interest rates and the costs of different stores of value, where the "budget constraint" is just $\delta_t - \psi_t$, with given δ_t. By assuming that the cash holding cost function is linear, the indifference curves are linear as well, which is the case in the figures. The choices would be the same even if the linearity of the cash holding cost function is relaxed and the benefits taken into account. In that case, the indifference curves would be convex.



(a) An agent's optimal choice if $i < i_{LB}$.

(b) An agent's optimal choice if $i > i_{LB}$.

Figure 2: An agent's optimal corner choices.

The figures, and the main parts of this section as well, use the same notation as in Section 4.1.1 where δ_t denotes deposits and ψ_t cash. During the next derivations, one can think more generally δ_t and ψ_t as a non-currency store of value and currency, respectively; the math is analogous to them, but some economic interpretations are not necessary in the line with deposits and cash.

Before going to the closer calculations, there remains the most interesting equilibrium: the situation where the nominal interest rate equals its lower bound, i.e. an agent is indifferent between deposits and cash. Figure 3 illustrates this kind of phenomenon, from where one can obtain that any choice of ψ_t is optimal and the slopes of indifferent curves and the budget constraint are the same.

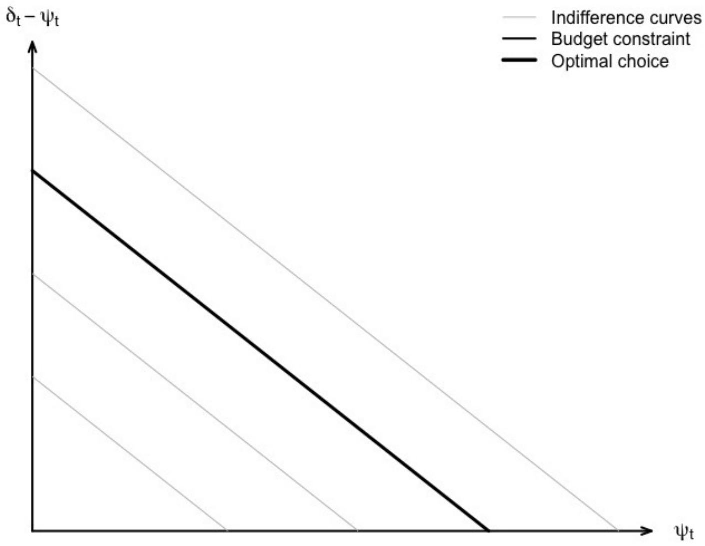


Figure 3: An agent's optimal choice at the lower bound.

The derivations in the next section take closer look on the equilibria at the lower bound. It turns out that the shape of the cash holding cost function will define the nature of the equilibria.

4.2.1 The optimisation problem

The following analyses use the simple lower bound $i_{LB} = -\frac{C_t(\psi_t)}{\psi_t}$ derived from Equation (4.6). Because the lower bound is defined in terms of the cash holding cost function, it is necessary to make three economic assumptions for the costs:

- The costs are zero, if there is no cash in one's possession, otherwise positive, i.e. $C_t(0) = 0$ and $C_t(\psi_t) \geq 0$ for all $\psi_t \geq 0$
- The cash holding cost function is an increasing function of cash, i.e. $C_t'(\psi_t) \geq 0, \forall \psi_t$
- The cash holding cost function is concave in its domain, i.e. $C_t''(\psi_t) \leq 0$.[10]

[10]A linear function is concave because it satisfies the following definition. Let $A \subset \mathbb{R}^n$ be a convex set. A function $f : A \to \mathbb{R}$ is (in set A) concave if $\forall x_1, x_2 \in A$ and $\forall t \in (0, 1)$ satisfies $f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2)$.

The first two assumptions are rather reasonable, but the third one might not be so plain. The best way to think of this is to consider the transportation or storing costs of cash; a rise in bank notes in one's possession (say in a wallet) probably do not increase the transportation or storing costs much, but it definitely increases the risk of losing money. However, these assumptions are relaxed when considering more general models later.

The agent's portfolio optimisation problem can be written as follows

$$\begin{aligned}\underset{\psi_t \in (0,\delta_t]}{max} \quad \Big\{ g(\delta_t, \psi_t) &= (1 + i_{LB})(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t) \\ &= \Big(1 - \frac{C_t(\psi_t)}{\psi_t}\Big)(\delta_t - \psi_t) - D_t + \psi_t - C_t(\psi_t) \\ &= \Big(1 - \frac{C_t(\psi_t)}{\psi_t}\Big)\delta_t - D_t \Big\} \quad s.t. \quad \delta_t \geq \psi_t,\end{aligned} \tag{4.27}$$

where the parameters are defined and interpreted in common with Section 4.1.1. As one can see, the utility function reduces to the same form as where an agent keeps all her money in the bank account and suffers the costs of negative interest rate at the lower bound, which is not a coincidence because the lower bound is derived this way. This simple utility maximising problem can be solved by using the following *Lagrangian*

$$\mathcal{L}(\delta_t, \psi_t, \lambda) = (1 - \frac{C_t(\psi_t)}{\psi_t})\delta_t - D_t + \lambda(\delta_t - \psi_t). \tag{4.28}$$

In this framework, the variable λ can be interpreted as the marginal utility of the leftover deposits which includes the benefits of both deposits and cash. This means that the higher λ, the higher the marginal benefit of deposits, and the lower λ, the higher the marginal benefits of cash. On the grounds of that interpretation, the following analyses assume that $\lambda \geq 0$; the benefits of money in the deposit account are greater or equal to the benefits of cash. This argument is suitable when considering the situation where both δ_t and ψ_t are great and the benefit comes from feasibility and rapidity to make large transactions or investments. In addition, it is important to observe that the amount of deposits, δ_t, is constant in this framework. This means that the amount of leftover deposits over the period t is equal to $\delta_t - \psi_t$ and thus the constraint $\delta_t \geq \psi_t$ is needed when considering the optimum choice. Also, in this framework turning money into some other store of value or the destruction of money is not allowed.

The next section concentrates on the first and second order conditions of the utility maximising problem. These two conditions reveal that there are multiple equilibria at the lower bound if the cash holding cost function is linear, and there is only a corner solution if the cost function is *strictly* concave.

4.2.2 The first and second order conditions

To simplify the notation, the cash holding cost function is denoted by $C_t \equiv C_t(\psi_t)$ from now on. Using the utility function (4.27), the Kuhn-Tucker conditions, which are necessary but not sufficient, are

$$\frac{\partial \mathcal{L}}{\partial \psi_t} = \frac{C_t - \psi_t C_t'}{\psi_t^2}\delta_t - \lambda = 0 \tag{4.29}$$

$$\delta_t \geq \psi_t. \tag{4.30}$$

This means, at the point $(\delta_t - \psi_t^*, \psi_t^*)$ which satisfies the first order conditions and is proposed as an maximising point, it is not possible to get more utility by changing the variable a bit. One can immediately see from the second Kuhn-Tucker condition that in the optimum there are two possibilities: (1) $\psi_t^* = \delta_t$ or (2) $\lambda = 0$. These are considered and interpreted after the derivations of the first Kuhn-Tucker condition. Equation (4.29) formulates a first-order ordinary differential equation of the cost function

$$C_t' - \psi_t^{-1}\, C_t = -\frac{\lambda\, \psi_t}{\delta_t}, \tag{4.31}$$

which can be solved to

$$C_t(\psi_t) = \Big(P - \frac{\lambda}{\delta_t}\, \psi_t\Big)\psi_t, \tag{4.32}$$

where P is the constant of integral. The more detailed derivation can be found in Appendix B. The constant of integral, P, can be interpreted as the combination of the agent's personal costs and the costs that a bank can control. For example, P can depend on bank service costs, the distances between the agent and the markets (this can be included in the transportation costs), the personal risk of holding cash, or any other matter that could affect the costs. Because the costs that the bank can control may be quite small and might vary widely depending on the country and bank, it would be possible to separate these from the personal costs on a case-by-case basis. However, these costs are just included in the concept of personal costs for simplicity. Hence P is considered to be the *personal cash holding cost* from now on. The intuitive nature of P is the following: the higher personal cash holding costs, the higher the total cash holding costs.

Furthermore, the optimal cost function depends exogenously on the amount of deposits, δ_t, such that as the deposits get larger, the cost function gets more linear. Therefore, for instance, a linear cash holding cost function using for the reserves with a central bank could be reasonable.

Next, the fundamental assumptions discussed earlier are needed when defining and using the cost function. Recalling, first, the costs must be zero if the amount of cash is zero, i.e. $C_t(0) = 0$, and second, the cost function has to be a real valued function $C : \mathbb{R}_+ \to \mathbb{R}_+$, such that $C_t(\psi_t) \geq 0, \forall \psi_t \geq 0$. The cash holding cost function in Equation (4.32) satisfies these properties if and only if $P - \frac{\lambda}{\delta_t}\, \psi_t \geq 0 \Leftrightarrow P \geq \frac{\lambda}{\delta_t}\, \psi_t$. In addition to this, the optimal cash holding cost function is increasing and concave in its domain if and only if the first and second derivatives of it with respect to ψ_t satisfy the following properties:

$$C_t'(\psi_t) = P - \frac{2\lambda}{\delta_t}\, \psi_t \geq 0 \;\Leftrightarrow\; P \geq \frac{2\lambda}{\delta_t}\, \psi_t \quad \Big(> \frac{\lambda}{\delta_t}\, \psi_t \Big) \tag{4.33}$$

$$C_t''(\psi_t) = -\frac{2\lambda}{\delta_t} \leq 0 \quad \forall\, P,\, \psi_t,\, \delta_t,\, \lambda \in \mathbb{R}_+. \tag{4.34}$$

This reveals that P has only one necessary condition: $P \geq \frac{2\lambda}{\delta_t}\psi_t \geq 2\lambda$ which is a linear function of cash with a positive slope. This is a hoped-for result, because it is quite credible to assume that personal cash holdings increase with cash and, moreover, with the marginal utility of leftover deposits λ. In addition, this restriction reveals that the costs increase with the marginal utility of leftover deposits λ, because then the personal cash holding costs P increase twice as much. This sounds plausible property, because the marginal utility of leftover deposits is like an opportunity cost for the cash in this case.

Figure 4 below represents the cash holding cost function with the above-mentioned assumptions and the properties of the personal cash holding cost P.

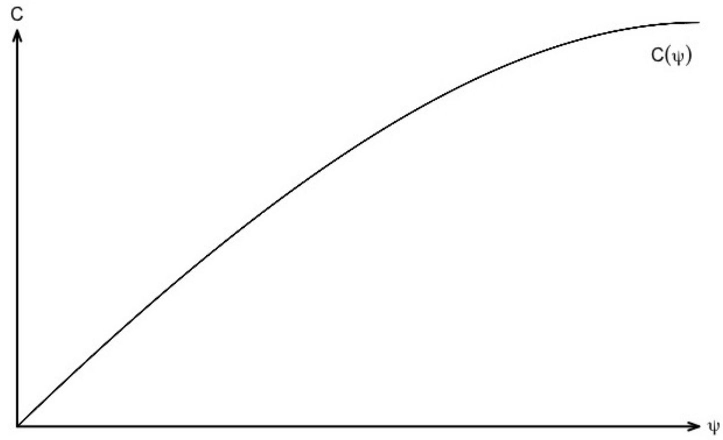


Figure 4: The cash holding cost function.

However, if λ were negative, the cash holding cost function would be convex. This could be the case if hoarding cash increasingly got more expensive when an agent raised the amount of notes and coins in her possession. Furthermore, it would mean that the marginal utility of leftover deposits would be negative, i.e. an agent would prefer cash over deposits. This is however a special case, which is possible, but probably not of concern the majority of cases.

The most important and interesting result is that the optimal cash holding cost function can be plugged into the lower bound formula and thereby it is possible to derive the lower bound function of the nominal interest rate. This gives the exact form for the lower bound which is possible to interpret, apply, and estimate in certain conditions. Hence, recalling the lower bound on the nominal interest rate $i_{LB} = -\frac{C_t(\psi_t)}{\psi_t}$ and using the derived optimal cost function, the lower bound can be written as follows

$$i_{LB}(\psi_t, \delta_t) = \frac{\lambda}{\delta_t}\,\psi_t - P \tag{4.35}$$

which is a linear function of cash with an intercept term. This function has the following properties for all $\lambda,\ \delta_t,\ \psi_t \in \mathbb{R}_+$ and $P \geq \frac{2\lambda}{\delta_t}\,\psi_t$

$$
\begin{aligned}
&1.) \quad \lim_{\psi_t \to 0} i_{LB}(\psi_t, \delta_t) = -P < 0, \\
&2.) \quad \lim_{\delta_t \to \infty} i_{LB}(\psi_t, \delta_t) = -P < 0, \\
&3.) \quad \lim_{P \to \infty} i_{LB}(\psi_t, \delta_t) = -\infty < 0, \\
&4.) \quad \lim_{P \to \frac{2\lambda}{\delta_t}\,\psi_t} i_{LB}(\psi_t, \delta_t) = -\frac{\lambda}{\delta_t}\,\psi_t < 0, \\
&5.) \quad \lim_{\psi_t \to \delta_t} i_{LB}(\psi_t, \delta_t) = \lambda - P \leq -\lambda < 0,
\end{aligned}
$$

which implicate to $i_{LB} \in (-\infty, 0)$. The property 1.) is the case where an agent keeps all her money in the deposit account and thereby the lower bound depends only on the personal costs. Notice also that for the sake of the reduction, the lower bound is now defined with $\psi_t = 0$, i.e. $i_{LB} = \frac{\lambda}{\delta_t}\,\psi_t - P$ for all $\psi_t \in [0, \delta_t]$. The properties 2.) and 3.) are intuitive: the greater the deposits or the personal cash holding cost, the lower the lower bound is. If one has a huge amount of deposits, then turning those into cash might be very costly and therefore she is not willing to withdraw her money very easily. The interpretation is the same with extremely high personal cash holding costs; the higher the personal costs, the lower the lower bound. The property 4.) states that if the personal costs are the smallest possible ones, the lower bound gets closer to zero. If the size of the marginal cost of leftover deposits is very small, then the lower bound is close to zero, too. Whereas the property 5.) reveals that if an agent is intending to withdraw all the deposits, then the lower bound is the marginal utility of leftover deposits minus the personal cash holding costs, which is also negative for all positive personal costs and marginal utility of deposits. Because of the limits of P, the properties 4.) and 5.) imply that the higher the marginal utility of leftover deposits, the smaller the lower bound, which is intuitive; if an agent values the leftover deposits more, she is less disposal to turn those into cash.

Finally, it is possible to summarise the results and take the equilibria into consideration. The second Kuhn-Tucker condition (4.30) states

$$\delta_t - \psi_t = 0, \quad \text{or} \quad \lambda = 0,$$

which states that in the optimal choice has to be $\psi_t^* = \delta_t$ with $\lambda > 0$ (or $\lambda < 0$, which is the case with convex cash holding cost function). However, this is the corner solution where an agent turns all her deposits into cash. In this case the optimal cash holding cost function and the lower bound function have the above-treated properties, and the utility of the portfolio is

$$u(p_t) = f(a_t) + g(\delta_t, \psi_t^*) = f(a_t) + \Big(1 - \frac{C_t(\psi_t^*)}{\psi_t^*}\Big)\delta_t - D_t \tag{4.36}$$

$$= f(a_t) + (1 - P + \lambda)\,\delta_t - D_t. \tag{4.37}$$

Furthermore, the agent's optimal cash holding cost function is concave and thereby has diminishing marginal costs with respect to cash and the optimal interest rate at the lower bound is increasing the function of cash. Then the agent will choose the corner solution where the interest rate is the smallest one and the cash holding costs are relatively the smallest. In this way the agent will maximise her utility by turning all the deposits into cash. This solution reveals that if an agent has strictly concave cash holding costs, then there is a jump at the lower bound where the agent turns all the risk-free deposits into cash. The nominal interest rate hence belongs to $(-\lambda, \infty)$, because $\psi_t^* = \delta_t$ and $P \geq \frac{2\lambda}{\delta_t}\psi_t^* = 2\lambda$.

Otherwise, if $\psi_t^* \in [0, \delta_t]$, then the Kuhn-Tucker conditions are satisfied if and only if $\lambda = 0$ at the lower bound. Therefore the optimal cash holding cost function and the lower bound function become

$$C_t(\psi_t) = P\psi_t \tag{4.38}$$

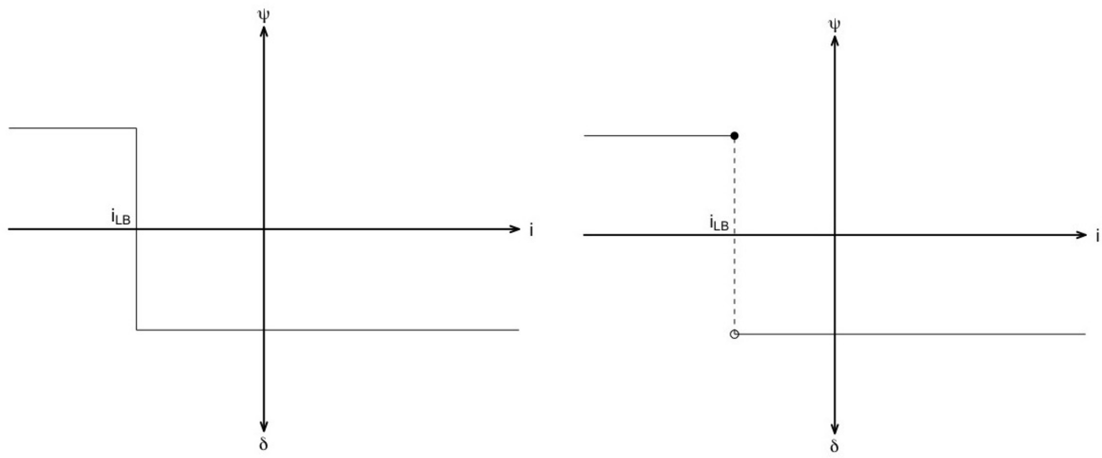
$$i_{LB}(\delta_t, \psi_t) = -P \tag{4.39}$$

for all $\psi_t \in [0, \delta_t]$ and $\lambda = 0$. This says that at the lower bound the cash holding cost function has to be linear and the lower bound equals the personal cash holding costs which is a constant. In this framework the absolute limit on the nominal interest rate is formed by the marginal cost of cash holdings, i.e. $i_{LB} = -C_t'(\psi_t) = -P$, which was also the result in many derivation in earlier sections. In addition, the utility level is

$$u(p_t) = f(a_t) + (1 - P)\,\delta_t - D_t. \tag{4.40}$$

Compared with the utility with the non-linear cash holding cost function, an agent is worse off with the linear cost function if $\lambda > 0$ and better off if $\lambda < 0$. With a linear cash holding cost function there is no a jump at the lower bound and thereby the agent is indifferent between deposits and cash. This leads to multiple equilibria solution at the lower bound.

Figure 5 illustrates the equilibrium differences between the non-linear and the linear cash holding cost function cases.



(a) The lower bound if $\lambda = 0$.   (b) The lower bound if $\lambda \neq 0$.

Figure 5: The equilibria at the lower bound.

In light of these assumptions, the theoretical lower bound on nominal interest rates is always negative, which can easily be seen from the cash holding costs that were assumed to be positive and increasing in $\mathbb{R}_+$. However, the constraints and the assumptions leave the position of the lower bound function free. The figure below depicts the lower bounds on nominal interest rates with the linear and the non-linear cash holding cost function.

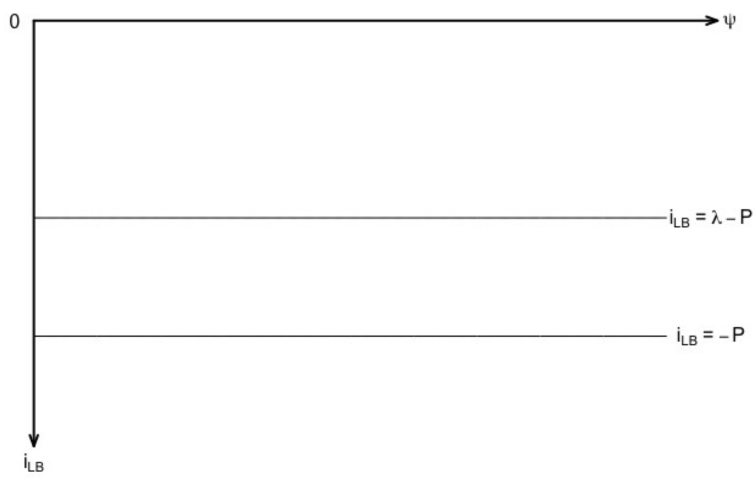


Figure 6: The lower bounds on nominal interest rates.

The second order condition for the utility maximising problem is unnecessary because the cash holding cost function is defined so that the first order conditions of the utility maximisation problem are always satisfied. This means that the second derivative of the Lagrange function with respect to ψ_t is always zero and the utility is maximised with all choices of ψ_t. This can be easily seen by plugging the optimal cash holding cost function to the Lagrangian

$$\mathcal{L}(\delta_t, \psi_t, \lambda) = (1 - \frac{C_t(\psi_t)}{\psi_t})\delta_t - D_t + \lambda(\delta_t - \psi_t) \tag{4.41}$$

$$= (1 - P + \lambda)\delta_t - D_t, \tag{4.42}$$

which is constant for all ψ_t and therefore $\frac{\partial^2 \mathcal{L}}{\partial \psi_t \partial \psi_t} = 0$. This was a desired result because the utility function is assumed to be for perfect substitutes and hence the utility level should be a constant at the lower bound. Appendix B provides more detailed derivations for the second order condition, which are suitable for the next presented equilibria of the general model as well.

The first and second order conditions have shown that there is a corner solution where an agent turns all her deposits into cash if the marginal utility of leftover deposits is assumed to be non-zero. In this case, the cash holding cost function can be a non-linear function.

However, there are multiple equilibria at the lower bound if $\lambda = 0$, which means that the cash holding cost function is linear. Then, an agent is neutral between the two risk-free assets and therefore she does not get any positive (or negative) utility from the leftover deposits.

In the next section, I will apply these derivations into the more general model where costs and benefits have been set freely.

4.2.3 Equilibria at the lower bound using the general model

In the general model an agent chooses between a non-currency store of value and currency such that she does not value just the costs but also the benefits. At the lower bound this kind of utility function can be as:

$$g(\delta_t, \psi_t) = (1 + i_{LB})(\delta_t - \psi_t) + (1 + i_m)\psi_t + v(\delta_t - \psi_t, \psi_t), \tag{4.43}$$

where, again, δ_t denotes a non-currency store of value, ψ_t currency or other cash equivalent, i_{LB} is the lower bound on the nominal interest rate on the non-currency and i_m for currency (see Section 4.1.2). Now the costs have been replaced with utility functions v. In Section 4.1.2 derived lower bound can be written as follows

$$i_{LB} = \frac{i_m \psi_t + v(\delta_t - \psi_t, \psi_t) - v(\delta_t, 0)}{\psi_t}, \tag{4.44}$$

and by plugging this into the agent's utility function (4.43) above, it reduces to form

$$g(\delta_t, \psi_t) = (1 + i_{LB})\delta_t + u(\delta_t, 0). \tag{4.45}$$

Thus, in common with the previous section this can be presented in the utility maximising problem at the lower bound as follows

$$\begin{aligned} \max_{\psi_t \in (0, \delta_t]} \quad & \Big\{ g(\delta_t, \psi_t) = \Big(1 + \frac{i_m \psi_t + v(\delta_t - \psi_t, \psi_t) - v(\delta_t, 0)}{\psi_t}\Big)\delta_t + v(\delta_t, 0) \\ & \equiv \Big(1 - \frac{h_t(\psi_t)}{\psi_t}\Big)\delta_t + v(\delta_t, 0) \Big\} \\ & s.t. \quad \delta_t \geq \psi_t, \end{aligned} \tag{4.46}$$

where $h_t(\psi_t) \equiv v(\delta_t, 0) - i_m\psi_t - v(\delta_t - \psi_t, \psi_t)$. Hence, the Lagrange function is

$$\mathcal{L}(\delta_t, \psi_t, \lambda) = (1 - \frac{h_t(\psi_t)}{\psi_t})\delta_t + v(\delta_t, 0) + \lambda(\delta_t - \psi_t), \tag{4.47}$$

and because the term $v(\delta_t, 0)$ is a constant (because ψ_t equals zero and δ_t is fixed) it follows that this general representation is, as a matter of fact, exactly the same situation as before in Equation (4.28). Thereby the Kuhn-Tucker conditions and the derivations result in the same outcomes

$$h_t(\psi_t) = \Big(P_g - \frac{\lambda}{\delta_t}\,\psi_t\Big)\psi_t \tag{4.48}$$

and furthermore the same lower bound function

$$i_{LB} = \frac{\lambda}{\delta_t}\,\psi_t - P_g, \tag{4.49}$$

where P_g is the constant of integral. Now, there is no restriction for P_g, because there is no simple economic interpretation for the function $h_t(\psi_t)$, which was defined by the utility functions and the interest rate returns of currency. As the earlier sections revealed, there is only a corner solution with non-zero λ and $\psi_t^* = \delta_t$, and multiple equilibria solution if $\lambda = 0$ and $\psi_t^* \in [0, \delta_t]$. However, because the function $h_t(\psi_t)$ can be positive or negative in this framework, this representation does not guarantee the negativity of the lower bound on nominal interest rates. To get more thorough results of this general model one should examine the function $v(\delta_t, \psi_t)$ closer.

Section 4.2 has proved that at the lower bound agents will choose cash instead of deposits if their preferences between the assets are non-linear. However, with linear preferences, as in Section 4.1.3, agents are indifferent between the risk-free assets at the lower bound. This was assumed property in the indicator functions of the stochastic cash holding function (4.19), but now it is also justified. Hence, these findings could be useful when specifying the structures and behaviour of agents at the lower bound.

5 Applications

It turns out that the idea of the costs and benefits of risk-free assets is relevant and applicable in many cases. For instance, the kernel of the frameworks introduced in the previous chapters can be plugged into the constraints of a basic consumer's utility maximisation problem, a *cash-in-advance* model, a *money-in-the-utility-function* model or any other shopping time model that has risk-free assets of a different kind in its constraints. However, in this chapter, I will show some of the most popular and easiest ones: first a two-period model for an agent's intertemporal utility maximisation problem, and second, the same model with infinite horizon.

5.1 A two-period model of consumption

In this model, a consumer occupies only two periods and she maximises her life-cycle utility by maximising her consumption c_t and c_{t+1}. Her utility function is denoted by $u(c_t)$, which has the following properties: $\frac{\partial u(c_t)}{\partial c_i} > 0$ for $i \in \{t, t+1\}$, $\frac{\partial^2 u(c_t)}{\partial c_i \partial c_j} < 0$ for $i, j \in \{t, t+1\}$, $\lim_{c \to 0} u'(c) = \infty$, and $\lim_{c \to \infty} u'(c) = 0$.[11] The consumer can save by holding her wealth in the deposit account or by holding cash. Hence, the consumer's savings are $s_t = \delta_t + \psi_t$, where δ_t denotes deposits and ψ_t cash. In spite of the earlier chapter, the deposits δ_t are not constant in these analyses. The agent's utility maximising problem can be presented as follows

$$\max_{c_t, c_{t+1}} \quad U = u(c_t) + \beta\, u(c_{t+1}) \quad \text{subject to} \tag{5.1}$$

$$P_t\, c_t = w_t - \delta_t - \psi_t \tag{5.2}$$

$$P^e_{t+1}\, c_{t+1} = w_{t+1} + (1 + i_t)\delta_t - D_t + \psi_t - C_t(\psi_t) \tag{5.3}$$

where i_t is the nominal interest rate, β is a time preference discounting parameter, w_t is a nominal wage, the costs of deposits are D_t, and the cash holdings costs $C_t(\psi_t)$. P_t denotes the price level in the period t and $\mathbb{E}_t(P_{t+1}) \equiv P^e_{t+1}$ is the expected price level in the next

[11]The last two properties are the Inada conditions (see Inada (1963)).

period $t+1$.

Plugging the first and second flow budget constraints into the utility function, the utility maximising problem becomes

$$\max_{\psi_t,\delta_t} \quad U = u\Big(\frac{w_t - \delta_t - \psi_t}{P_t}\Big) + \beta\, u\Big(\frac{w_{t+1} + (1+i_t)\delta_t - D_t + \psi_t - C_t(\psi_t)}{P^e_{t+1}}\Big) \tag{5.4}$$

which can be solved by taking the partial derivatives of U with respect to δ_t and ψ_t and setting those equal to zero:

$$\frac{\partial U}{\partial \delta_t} = -\frac{u'(c_t)}{P_t} + \frac{1+i_t}{P^e_{t+1}}\,\beta\, u'(c_{t+1}) = 0 \tag{5.5}$$

$$\frac{\partial U}{\partial \psi_t} = -\frac{u'(c_t)}{P_t} + \frac{1-C'_t(\psi_t)}{P^e_{t+1}}\,\beta\, u'(c_{t+1}) = 0, \tag{5.6}$$

from where one obtain that in the optimum it has to hold

$$i_t = -C'_t(\psi_t). \tag{5.7}$$

This is exactly the same result as in the earlier chapters. Furthermore, it is easy to see that if $i_t > -C'_t(\psi_t)$, then the consumer saves using the deposit account and does not turn cash at all, and if $i_t < -C'_t(\psi_t)$, then she turns all the deposits into cash and thereby chooses her optimal consumptions based on this savings behaviour. Hence, the most important result here is, again, that the lower bound on the nominal interest rate is $i_{LB} = -C'_t(\psi_t)$. If the interest rate is greater than or equal to the lower bound, consumers will not turn their deposits into cash and do not cause bank runs.

5.2 An infinite-horizon model of consumption

Next I will extend the previous consumer's utility maximisation problem for the infinite horizon. I will furthermore generalise the model by taking into account savings as a portfolio where the risk-free assets are separated into deposits and cash. In this framework, a consumer might be thought of as a household which maximises the utility of a dynasty or a firm which exists forever. The notation will be the same as in the previous section.

First, let the agent's optimisation problem be the following

$$\max_{c_t, c_{t+1}} \quad U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) \tag{5.8}$$

subject to the flow budget constraints

$$\begin{aligned} P_{t+s}\, c_{t+s} + p_{t+s+1} = w_{t+s} + R_{t+s}\, a_{t+s} + (1+i_{t+s})\, \delta_{t+s} - D_{t+s} \\ + \psi_{t+s} - C_{t+s}(\psi_{t+s}), \quad s = 0, 1, 2, ... \end{aligned} \tag{5.9}$$

where p_t is a portfolio of the assets a_t, deposits d_t, and cash ψ_t at the beginning of the period t, i.e. $p_t = a_t + \delta_t + \psi_t$. The return of the assets a_t are denoted by R_t and i_t is the nominal interest rate on the deposits. The other parameters are the same as in the previous section. To simplify the notation, let $\Omega_{t+s}\, p_{t+s} \equiv R_{t+s}\, a_{t+s} + (1+i_{t+s})\, \delta_{t+s} - D_{t+s} + \psi_{t+s} - C_{t+s}(\psi_{t+s})$, where Ω_t is a portfolio return operator. Then the solution for the utility maximising problem can be obtained by the Lagrange multiplier method

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \Big(\beta^s u(c_{t+s}) + \lambda_{t+s}\big(w_{t+s} + \Omega_{t+s}\, p_{t+s} - P_{t+s}\, c_{t+s} - p_{t+s+1}\big) \Big). \tag{5.10}$$

The optimality conditions are the following

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \mathbb{E}_t\Big(\beta^s u'(c_{t+s}) - \lambda_{t+s}\, P_{t+s}\Big) = 0 \quad s \geq 0 \tag{5.11}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+s+1}} = \mathbb{E}_t\Big(- \lambda_{t+s} + \lambda_{t+s+1} R_{t+s+1}\Big) = 0 \quad s \geq 0 \tag{5.12}$$

$$\frac{\partial \mathcal{L}}{\partial \delta_{t+s+1}} = \mathbb{E}_t\Big(- \lambda_{t+s} + \lambda_{t+s+1}(1 + i_{t+s+1})\Big) = 0 \quad s \geq 0 \tag{5.13}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_{t+s+1}} = \mathbb{E}_t\Big(- \lambda_{t+s} + \lambda_{t+s+1}\big(1 - C'_{t+s+1}(\psi_{t+s+1})\big)\Big) = 0 \quad s \geq 0, \tag{5.14}$$

from where one can directly see similar results as before; the consumer will weigh her portfolio fully to the asset which has the highest (expected) yield. Thus the consumer is indifferent between all the three assets if and only if $R_r = 1 + i_t = 1 - C'_t(\psi_t)$. However, if the consumer cannot, or is not willing to, turn her risk-free assets δ_t or ψ_t into the riskier assets a_t, then the lower bound on the nominal interest rate in the period t is the same as in the previous section $i_t = -C'_t(\psi_t)$.

6 Conclusion

This paper examines the lower bounds on nominal interest rates from a microeconomic perspective. This approach provides new models and frameworks which may be used for interpretation, application, estimation, and simulation. The models can be applied to examine the lower bounds on commercial bank deposit rates as well as central banks policy rates.

The main question, *is there a lower bound on nominal interest rates, and is it zero*, is answered, first, by considering the factors that create the floors on assets and their interest rates and, second, by specifying the portfolio optimisation models. Based on these models, it turns out there really are lower bounds on nominal interest rates, which are in all probability negative. The two-period and multi-period models presented in this research are suitable for risk-free assets and hence are able to derive the lower bounds on nominal interest rates. The multi-period model with a stochastic cash holding cost function provides a new framework and tool for the yields of risk-free assets. By using this model, it is possible to simulate asset returns and thereby approach the lower bounds on deposit rates.

In addition, this paper shows that an agent prefers cash over deposits at the lower bound if the cash holding cost function is assumed to be non-linear. However, if the cash holding costs are linear, the agent is indifferent between the assets at the lower bound. This result is useful for the lower bound models. For instance, the multi-period model with the stochastic cash holding function in Section 4.1.3 uses this information in its indicator functions.

The microeconomic models for the deposit rates presented in this paper can be applied to net interest margin examinations. Hence, it could be possible to derive the boundaries for the NIMs and thereby extend the reversal policy rate model of Brunnermeier and Koby (2016). By cutting commercial bank deposit rates, it is possible to push down the *effective* lower bound on policy rates. However, the equilibria examinations show that bank run is possible with too low interest rates, which probably makes banks cautious in their actions regarding the deposit rates.

Even if getting rid of cash would instantly eliminate the lower bounds on nominal interest rates, it does not remove the facts of agents' welfare presented in this paper. If an economy had its currency abolished and bank-assessed interest rates were below the lower bound that cash creates, agents would nevertheless be better off in an economy where cash would still be in use. This observation makes the results derived in this paper relevant irrespectively of the monetary system of an economy.

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Appendix A

The lower bounds for the multi-period models can be derived using the same framework as in Chapter 4.1. Let δ_T denote the non-currency store of value and ψ_T currency in the period T. Assume that i_m is the interest rate on currency and i is the interest rate on the non-currency store of value, and the utility function is the form of $v(\delta_T - \psi_T, \psi_T)$ which includes the costs and benefits of δ_T and ψ_T kept in agent's possession over the period T. Recalling the framework where the left-hand side is the situation where an agent keeps all her wealth in the non-currency and the right-hand side the situation where she turns some amount of the non-currencies into currency

$$(1+i)\delta_T + v(\delta_T, 0) \geq (1+i)(\delta_T - \psi_T) + (1+i_m)\psi_t + v(\delta_T - \psi_T, \psi_T) \Leftrightarrow \tag{6.1}$$

$$(1+i)\delta_T + v(\delta_T, 0) \geq (1+i)\delta_T + (i_m - i)\psi_T + v(\delta_T - \psi_T, \psi_T). \tag{6.2}$$

If this inequality holds, an agent keeps all her money in the form of the non-currency store of value. This equation can be solved by iterating T times to capture the over time effects. First, using iteration to the left-hand side it follows

$$\begin{aligned}
\delta_{T+1} &= (1+i)\delta_T + v(\delta_T, 0) && (6.3)\\
&= (1+i)((1+i)\delta_{T-1} + v(\delta_{T-1}, 0)) + v(\delta_T, 0) && (6.4)\\
&= (1+i)^2\delta_{T-1} + (1+i)v(\delta_{T-1}, 0) + v(\delta_T, 0) && (6.5)\\
&= (1+i)^2((1+i)\delta_{T-2} + v(\delta_{T-2}, 0)) + (1+i)v(\delta_{T-1}, 0) + v(\delta_T, 0) && (6.6)\\
&= (1+i)^3\delta_{T-2} + (1+i)^2 v(\delta_{T-2}, 0) + (1+i)v(\delta_{T-1}, 0) + v(\delta_T, 0) && (6.7)\\
&\vdots\\
&= (1+i)^T\delta_1 + \sum_{t=0}^{T-1}(1+i)^t v(\delta_{T-t}, 0), && (6.8)
\end{aligned}$$

and by denoting $A_T \equiv (i_m - i)\psi_T + v(\delta_T - \psi_T, \psi_T)$ the right-hand side gets a form

$$
\begin{aligned}
\delta_{t+1} &= (1+i)\delta_T + (i_m - i)\psi_T + u(\delta_T - \psi_T, \psi_T) \equiv (1+i)\delta_T + A_T && (6.9)\\
&= (1+i)((1+i)\delta_{T-1} + A_{T-1}) + A_T && (6.10)\\
&= (1+i)^2\delta_{T-1} + (1+i)A_{T-1} + A_T && (6.11)\\
&\vdots\\
&= (1+i)^T\delta_1 + \sum_{t=0}^{T-1}(1+i)^t A_{T-t}. && (6.12)
\end{aligned}
$$

Hence, Equation (6.1) holds if and only if

$$
\begin{aligned}
(1+i)^T\delta_1 + \sum_{t=0}^{T-1}(1+i)^t v(\delta_{T-t}, 0) \geq (1+i)^T\delta_1 + \sum_{t=0}^{T-1}(1+i)^t A_{T-t} \;\Leftrightarrow && (6.13)\\
\sum_{t=0}^{T-1}(1+i)^t v(\delta_{T-t}, 0) \geq \sum_{t=0}^{T-1}(1+i)^t A_{T-t} \;\Rightarrow && (6.14)\\
\sum_{t=0}^{T-1} v(\delta_{T-t}, 0) \geq \sum_{t=0}^{T-1} A_{T-t} && (6.15)
\end{aligned}
$$

and plugging $A_{T-t} \equiv (i_m - i)\psi_{T-t} + v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t})$ into the equation above it follows

$$
\begin{aligned}
\sum_{t=0}^{T-1} v(\delta_{T-t}, 0) \geq \sum_{t=0}^{T-1}((i_m - i)\psi_{T-t} + v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t})) \;\Leftrightarrow && (6.16)\\
\sum_{t=0}^{T-1} v(\delta_{T-t}, 0) \geq (i_m - i)\sum_{t=0}^{T-1}\psi_{T-t} + \sum_{t=0}^{T-1} v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t}) \;\Leftrightarrow && (6.17)\\
i\sum_{t=0}^{T-1}\psi_{T-t} \geq i_m\sum_{t=0}^{T-1}\psi_{T-t} + \sum_{t=0}^{T-1}(v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t}) - v(\delta_{T-t}, 0)) \;\Leftrightarrow && (6.18)\\
i \geq i_m + \sum_{t=0}^{T-1}\Big(\frac{v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t}) - v(\delta_{T-t}, 0)}{\psi_{T-t}}\Big). && (6.19)
\end{aligned}
$$

This expression is the lower bound on nominal interest rates in the multi-period model. Now, if the utility function is a form of perfect substitutes $v(\delta_t, \psi_t) = \beta_1\delta_t + \beta_2\psi_t$, the lower bound takes a form

$$i \geq i_m + \sum_{t=0}^{T-1} \Big(\frac{v(\delta_{T-t} - \psi_{T-t}, \psi_{T-t}) - v(\delta_{T-t}, 0)}{\psi_{T-t}} \Big) \;\Leftrightarrow \tag{6.20}$$

$$i \geq i_m + \sum_{t=0}^{T-1} \Big(\frac{\beta_1(\delta_{T-t} - \psi_{T-t}) + \beta_2 \psi_{T-t} - \beta_1 \delta_{T-t}}{\psi_{T-t}} \Big) \;\Leftrightarrow \tag{6.21}$$

$$i \geq i_m + \sum_{t=0}^{T-1} \Big(\frac{(\beta_2 - \beta_1)\psi_{T-t}}{\psi_{T-t}} \Big) = i_m + \sum_{t=0}^{T-1} \big(\beta_2 - \beta_1\big) \;\Leftrightarrow \tag{6.22}$$

$$i \geq i_m + (\beta_2 - \beta_1)T \equiv i_m + (b_m - b)T - (c_m - c)T, \tag{6.23}$$

where b and c denote marginal benefits and costs, respectively, and the subscript m refers to currency. This would be exactly the same result as in Section 4.1 if T were unity.

These solutions are only for the *fixed* interest rate i. Section 4.1.3 provides guiding derivations for the multi-period models where the interest rate is changing between the periods.

Appendix B

Equation (4.29) formulates the following first-order ordinary differential equation of the cost function

$$C_t' - \psi_t^{-1} C_t = -\frac{\lambda \psi_t}{\delta_t}, \tag{6.24}$$

which can be represented in more general form using the following notation

$$C_t'(\psi_t) + p(\psi_t) C_t(\psi_t) = h(\psi_t). \tag{6.25}$$

First, one can solve the integrating factor, $\mu(\psi_t)$, using the following procedure

$$\mu(\psi_t) = e^{\int p(\psi_t) d\psi_t} = e^{\int -\delta_t^{-1} d\psi_t} = e^{-ln(\psi_t)} = \psi_t^{-1} \tag{6.26}$$

and therefore using this integrating factor to working out the optimal cash holding cost function

$$\begin{aligned}
C_t &= \frac{\int \mu(\psi_t) h(\psi_t) d\psi_t \ \ (+P)}{\mu(\psi_t)} && (6.27)\\
&= \psi_t \int \psi_t^{-1} \Big(- \frac{\lambda \psi_t}{\delta_t} \Big) \, d\psi_t && (6.28)\\
&= \psi_t \int \Big(- \frac{\lambda}{\delta_t} \Big) \, d\psi_t && (6.29)\\
&= \psi_t \Big(- \frac{\lambda}{\delta_t} \psi_t + P \Big), && (6.30)
\end{aligned}$$

which gives functional form for the optimal cash holding cost function

$$C_t(\psi_t) = \Big(P - \frac{\lambda}{\delta_t} \psi_t \Big) \psi_t, \tag{6.31}$$

where P is the constant of integral. This derivation is adequate also for the general model in Section 4.2.3.

Next, I show that the second derivative of the Lagrange function presented in Section 4.2.2 is zero for all ψ_t and P. This derivation is suitable also for the general model presented in Section 4.2.3, where the restrictions of personal costs of P have relaxed. Therefore this can be proved the other way using cash holding cost functions. The second order derivative of the Lagrangian $\mathcal{L}(\delta_t, \psi_t, \lambda) = (1 - \frac{C_t}{\psi_t})\delta_t - D_t + \lambda(\delta_t - \psi_t)$ with respect to ψ_t is

$$\frac{\partial^2 \mathcal{L}}{\partial \psi_t \partial \psi_t} = \delta_t \Big(- \frac{C_t'(\psi_t)}{\psi_t} + \frac{2C_t'(\psi_t)}{\psi_t^2} - \frac{2C_t(\psi_t)}{\psi_t^3} \Big) \tag{6.32}$$

and recalling the cost function and the first and second order derivatives of it

$$C_t(\psi_t) = \Big(P - \frac{\lambda}{\delta_t}\, \psi_t \Big) \psi_t \tag{6.33}$$

$$C_t'(\psi_t) = P - \frac{2\lambda}{\delta_t}\, \psi_t \tag{6.34}$$

$$C_t'(\psi_t) = - \frac{2\lambda}{\delta_t} \tag{6.35}$$

Using these equations it is possible to solve the second order condition

$$\frac{\partial^2 \mathcal{L}}{\partial \psi_t \partial \psi_t} = \delta_t \Big(- \frac{C_t'(\psi_t)}{\psi_t} + \frac{2C_t'(\psi_t)}{\psi_t^2} - \frac{2C_t(\psi_t)}{\psi_t^3} \Big) \tag{6.36}$$

$$= \delta_t \Big(- \frac{-\frac{2\lambda}{\delta_t}}{\psi_t} + \frac{2\Big(P - \frac{2\lambda}{\delta_t}\, \psi_t\Big)}{\psi_t^2} - \frac{2\Big(P - \frac{\lambda}{\delta_t}\, \psi_t\Big)\psi_t}{\psi_t^3} \Big) \tag{6.37}$$

$$= \frac{2\lambda}{\psi_t} + \frac{\frac{2P\delta_t}{\psi_t} - 4\lambda}{\psi_t} - \frac{2P\delta_t - 2\lambda\, \psi_t}{\psi_t^2} \tag{6.38}$$

$$= \frac{2\lambda + \frac{2P\delta_t}{\psi_t} - 4\lambda - \frac{2P\delta_t}{\psi_t} + 2\lambda}{\psi_t} = 0 \quad \forall \psi_t, \delta_t, \lambda, P \in \mathbb{R}_+. \tag{6.39}$$

Tables

Table 6: The costs for personal customers of using bank accounts and credit cards in Finland in 2016.

Cost	**Aktia**	**Danske Bank**	**Handels-banken**	**OP**	**Nordea**	**S-pankki**
Monthly account fee	2.00 € / month	2.00 € / month	6.00 € / month	—	0.00 € / month	2.00 € / month
Monthly online bank fee	2.00 € / month	3.00 € / month	8.00 € / month	2.70 € / month	3.00 € / month	2.50 € / month
Fees for the most common credit cards	4.00 € / month	3.00 € / month	2.50–3.00 € / month	2.00 € / month	2.00–2.50 € / month	2.50 € / month

Table 7: The costs for personal customers of using cash via commercial bank services in Finland in 2016.

Cost	**Aktia**	**Danske Bank**	**Handelsbanken**	**OP**	**Nordea**	**S-pankki**
Withdrawals using Otto.ATM in Finland	2 withdrawals / month 0.00 €, 3 or more withdrawals 0.50 € / withdrawal	0.00 €	5 withdrawals / month 0.00 €, 6 or more withdrawals 0.50 € / withdrawal	0.00 €	0.00 €	5 withdrawals / month 0.00 €, 6 or more withdrawals 0.50 € / withdrawal
Withdrawals using other ATM in Finland	1.00 € + 2.0 % of the amount of withdrawal	0.30 € / withdrawal (from supermarket)	1.00 € + 2.0 % of the amount of withdrawal	0.75 €	—	1.00 € + 2.0 % of the amount of withdrawal
Withdrawals at a branch	0.00 €	Order 5.00 € + 0.50 % of the amount of withdrawal	—	0.00 €	0.00 €	0.00 € at a branch of S-Group
Cash payment at a branch	6.00 € / bill	6.00 € / bill	5.00–7.00 € / bill	7.00 € / bill	6.00 € / bill	—
Deposit to own account	0.00 €	0.00 €	0.00 €	0.00 €	0.00 €	0.00 € at a branch of S-Group

Table 8: The transition matrix for a central bank deposit rate.

	0.01	0.0075	0.005	0.0025	0	-0.0025	-0.005	-0.0075	-0.01	-0.0125	-0.015	-0.0175	-0.02
0.01	0.900	0.100	0	0	0	0	0	0	0	0	0	0	0
0.0075	0.400	0.500	0.100	0	0	0	0	0	0	0	0	0	0
0.005	0	0.400	0.500	0.100	0	0	0	0	0	0	0	0	0
0.0025	0	0	0.400	0.500	0.100	0	0	0	0	0	0	0	0
0	0	0	0	0.500	0.400	0.100	0	0	0	0	0	0	0
-0.0025	0	0	0	0	0.500	0.400	0.100	0	0	0	0	0	0
-0.005	0	0	0	0	0	0.500	0.400	0.100	0	0	0	0	0
-0.0075	0	0	0	0	0	0	0.500	0.400	0.100	0	0	0	0
-0.01	0	0	0	0	0	0	0	0.600	0.300	0.100	0	0	0
-0.0125	0	0	0	0	0	0	0	0	0.600	0.300	0.100	0	0
-0.015	0	0	0	0	0	0	0	0	0	0.600	0.300	0.100	0
-0.0175	0	0	0	0	0	0	0	0	0	0	0.600	0.300	0.100
-0.02	0	0	0	0	0	0	0	0	0	0	0	0.700	0.300

Table 9: The transition matrix for a commercial bank deposit rate.

	0.01	0.0075	0.005	0.0025	0	-0.0025	-0.005	-0.0075	-0.01	-0.0125	-0.015	-0.0175	-0.02	-0.0225	-0.025
0.01	0.900	0.100	0	0	0	0	0	0	0	0	0	0	0	0	0
0.0075	0.200	0.700	0.100	0	0	0	0	0	0	0	0	0	0	0	0
0.005	0	0.200	0.700	0.100	0	0	0	0	0	0	0	0	0	0	0
0.0025	0	0	0.300	0.600	0.100	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.300	0.600	0.100	0	0	0	0	0	0	0	0	0
-0.0025	0	0	0	0	0.300	0.600	0.100	0	0	0	0	0	0	0	0
-0.005	0	0	0	0	0	0.300	0.600	0.100	0	0	0	0	0	0	0
-0.0075	0	0	0	0	0	0	0.300	0.600	0.100	0	0	0	0	0	0
-0.01	0	0	0	0	0	0	0	0.400	0.500	0.100	0	0	0	0	0
-0.0125	0	0	0	0	0	0	0	0	0.400	0.500	0.100	0	0	0	0
-0.015	0	0	0	0	0	0	0	0	0	0.400	0.500	0.100	0	0	0
-0.0175	0	0	0	0	0	0	0	0	0	0	0.400	0.500	0.100	0	0
-0.02	0	0	0	0	0	0	0	0	0	0	0	0.400	0.500	0.100	0
-0.0225	0	0	0	0	0	0	0	0	0	0	0	0	0.400	0.500	0.100
-0.025	0	0	0	0	0	0	0	0	0	0	0	0	0	0.500	0.500