Random curves, scaling limits and Loewner evolutions

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Abstract

In this paper, we provide framework of estimates for describing 2D scaling limits by Schramm's SLE curves. In particular, we show that a weak estimate on the probability of an annulus crossing implies that a random curve arising from a statistical mechanics model will have scaling limits and those will be well-described by Loewner evolutions with random driving forces. Interestingly, our proofs indicate that existence of a nondegenerate observable with a conformally-invariant scaling limit seems sufficient to deduce the required condition.

Our paper serves as an important step in establishing the convergence of Ising and FK Ising interfaces to SLE curves, moreover, the setup is adapted to branching interface trees, conjecturally describing the full interface picture by a collection of branching SLEs.

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1 Introduction

Oded Schramm's introduction of SLE as the only possible conformally invariant scaling limit of interfaces has led to much progress in our understanding of 2D lattice models at criticality. For several of them it was shown that interfaces (domain wall boundaries) indeed converge to Schramm's SLE curves as the lattice mesh tends to zero [30, 33, 31, 20, 26, 34, 7, 27].

All the existing proofs start by relating some observable to a discrete harmonic or holomorphic function with appropriate boundary values and describing its scaling limit in terms of its continuous counterpart. Conformal invariance of the latter allowed then to construct the scaling limit of the interface itself by sampling the observable as it is drawn. The major technical problem in doing so is how to deduce the strong convergence of interfaces from some weaker notions. So far two routes have been suggested: first to prove the convergence of the driving process in Loewner characterization, and then improve it to convergence of curves, cf. [20]; or first establish some sort of precompactness for laws of discrete interfaces, and then prove that any sub-sequential scaling limit is in fact an SLE, cf. [31].

We will lay framework for both approaches, showing that a rather weak hypotheses is sufficient to conclude that an interface has sub-sequential scaling limits, but also that they can be described almost surely by Loewner evolutions. We build upon an earlier work of Aizenman and Burchard [2], but draw stronger conclusions from similar conditions, and also reformulate them in several geometric as well as conformally invariant ways.

At the end we check this condition for a number of lattice models. In particular, this paper serves as an important step in establishing the convergence of Ising and FK Ising interfaces [9]. Interestingly, our proofs indicate that existence of a non-degenerate observable with a conformally-invariant scaling limit seems sufficient to deduce the required condition. These techniques also apply to interfaces in massive versions of lattice models, as in [22]. In particular, the proofs for loop-erased random walk and harmonic explorer we include below can be modified to their massive counterparts, as those have similar martingale observables [22].

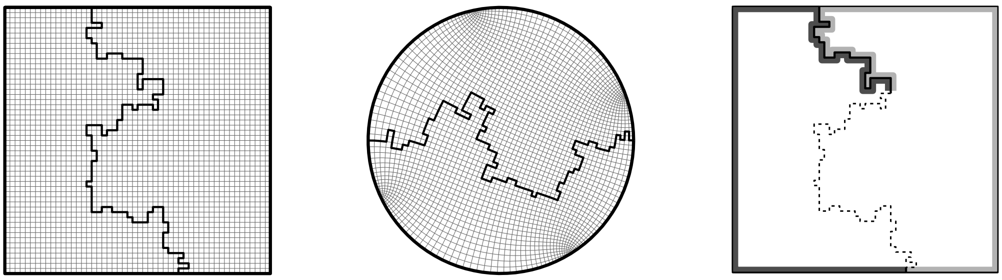
Moreover, our setup is adapted to branching interface trees, conjecturally converging to branching SLE$(\kappa, \kappa-6)$, cf [28]. We are preparing a follow-up [17], which will exploit this in the context of the critical FK Ising model. In the percolation case a construction was proposed in [6], also using the Aizenman–Burchard work.

Another approach to a single interface was proposed by Sheffield and Sun [29]. They ask for milder condition on the curve, but require simultaneous convergence of the Loewner evolution driving force when the curve is followed in two opposite directions towards generic targets. The latter property is missing in many of the important situations we have in mind, like convergence of the full interface tree.

1.1 The setup and the assumptions

Our paper is concerned with sequences of random planar curves and different conditions sufficient to establish their precompactness.

We start with a probability measure $\mathbb{P}$ on the set $X(\mathbb{C})$ of planar curves, having in mind an interface (a domain wall boundary) in some lattice model of statistical physics or a self-avoiding random trajectory on a lattice. By a *planar curve* we mean a continuous mapping $\gamma : [0,1] \to \mathbb{C}$. The resulting space $X(\mathbb{C})$ is endowed with the usual supremum metric with minimum taken over all reparameterizations, which is therefore parameterization-independent, see the section 2.1.1. Then we consider $X(\mathbb{C})$ as a measurable space with Borel σ-algebra. For any domain $V \subset \mathbb{C}$, let $X_{\text{simple}}(V)$ be the set of Jordan curves $\gamma : [0,1] \to \overline{V}$ such that $\gamma(0,1) \subset V$. Note that the end points are allowed to lie on the boundary.



(a) Typical setup: a random curve is defined on a lattice approximation of U and is connecting two boundary points a and b.

(b) The same random curve after a conformal transformation to $\mathbb{D}$ taking a and b to -1 and $+1$, respectively.

(c) Under the domain Markov property the curve conditioned on its beginning part has the same law as the one in the domain with the initial segment removed.

Figure 1: The assumptions of the main theorem are often easier to verify in the domain where the curve is originally defined (a) and the slit domains appearing as we trace the curve (c). Nevertheless, to set up the Loewner evolution we need to uniformize conformally to a fixed domain, e.g. the unit disc (b). Figure (c) illustrates the domain Markov property under which it is possible to verify the simpler "time 0" condition (presented in this section) instead of its conditional versions (see Section 2.1.3).

Typically, the random curves we want to consider connect two boundary points $a, b \in \partial U$ in a simply connected domain U. Also it is possible to assume that the random curve is (almost surely) simple, because the curve is usually defined on a lattice with small but finite lattice mesh without "transversal" self-intersections. Therefore, by slightly perturbing the lattice and the curve it is possible to remove self-intersections. The main theorem of this paper involves the Loewner equation, and consequently the curves have to be either simple or non-self-traversing, i.e., curves that are limits of sequences of simple curves.

While we work with different domains U, we still prefer to restate our conclusions for a fixed domain. Thus we encode the domain U and the curve end points $a, b \in \partial U$ by a conformal transformation ϕ from U onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The domain $U = U(\phi)$ is then the domain of definition of ϕ and the points a and b are preimages $\phi^{-1}(-1)$ and $\phi^{-1}(1)$, respectively, if necessary define these in the sense of prime ends.

Because of the above reasons the first fundamental object in our study is a *pair* $(\phi, \mathbb{P})$ where ϕ is a *conformal map* and $\mathbb{P}$ is a *probability measure on curves* with the following restrictions: Given ϕ we define the domain $U = U(\phi)$ to be the domain of definition of ϕ and we require that ϕ is a conformal map from U onto the unit disc $\mathbb{D}$. Therefore U is a simply connected domain other than $\mathbb{C}$. We require also that $\mathbb{P}$ is supported on (a closed subset of)

$$\left\{ \gamma \in X_{\text{simple}}(U) \;:\; \begin{matrix} \text{the beginning and end point of} \\ \phi(\gamma) \text{ are } -1 \text{ and } +1\text{, respectively} \end{matrix} \right\}. \tag{1}$$

The second fundamental object in our study is *some collection* Σ of pairs $(\phi, \mathbb{P})$ satisfying the above restrictions.

Because the spaces involved are metrizable, when discussing convergence we may always think of Σ as a sequence $((\phi_n, \mathbb{P}_n))_{n \in \mathbb{N}}$. In applications, we often have in mind a sequence of interfaces for the same lattice model but with varying lattice mesh $\delta_n \searrow 0$: then each $\mathbb{P}_n$ is supported on curves defined on the δ_n-mesh lattice. The main reason for working with the more abstract family compared to a sequence is to simplify the notation. If the set in (1) is non-empty, which is assumed, then there are in fact plenty of such curves, see Corollary 2.17 in [23].

We uniformize by a disk $\mathbb{D}$ to work with a bounded domain. As we show later in the paper, our conditions are conformally invariant, so the choice of a particular uniformization domain is not important.

For any $0 < r < R$ and any point $z_0 \in \mathbb{C}$, denote the annulus of radii r and R centered at z_0 by $A(z_0, r, R)$:

$$A(z_0, r, R) = \{ z \in \mathbb{C} \;:\; r < |z - z_0| < R \}. \tag{2}$$

The following definition makes speaking about crossing of annuli precise.

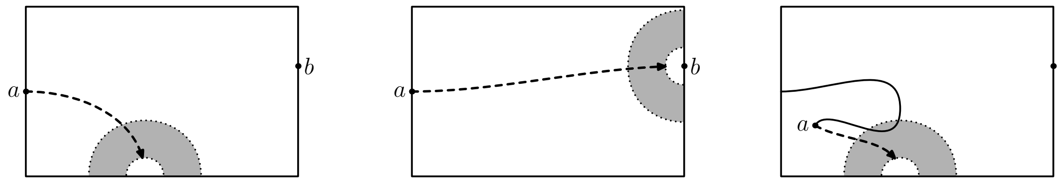
Definition 1.1. For a curve $\gamma : [T_0, T_1] \to \mathbb{C}$ and an annulus $A = A(z_0, r, R)$, γ is said to be a *crossing* of the annulus A if both $\gamma(T_0)$ and $\gamma(T_1)$ lie outside A and they are in the different components of $\mathbb{C} \setminus A$. A curve γ is said to *make a crossing* of the annulus A if there is a subcurve which is a crossing of A. A *minimal crossing* of the annulus A is a crossing which doesn't have genuine subcrossings.

We cannot require that crossing any fixed annulus has a small probability under $\mathbb{P}$: indeed, annuli centered at a or at b have to be crossed at least once. For that reason we introduce the following definition for a fixed simply connected domain U and an annulus $A = A(z_0, r, R)$ which is let to vary. If $\partial B(z_0, r) \cap \partial U = \emptyset$ define $A^u = \emptyset$, otherwise

$$A^u = \left\{ z \in U \cap A \;:\; \begin{matrix} \text{the connected component of } z \text{ in } U \cap A \\ \text{doesn't disconnect } a \text{ from } b \text{ in } U \end{matrix} \right\}. \tag{3}$$

This reflects the idea explained in Figure 2.

The main theorem is proven under a set of equivalent conditions. In this section, two simplified versions are presented. They are so called *time* 0 *conditions* which imply the stronger *conditional* versions if our random curves satisfy the domain Markov property, cf. Figure 1(c). It should be noted that even in physically interesting situations the latter might fail, so the conditions presented in the section 2.1.3 should be taken as the true assumptions of the main theorem.



(a) Unforced crossing: the component of the annulus is not disconnecting a and b. It is possible that the curve avoids the set.

(b) Forced crossing: the component of the annulus disconnects a and b and does it in the way, that every curve connecting a and b has to cross the annulus at least once.

(c) There is an ambiguous case which resembles more either one of the previous two cases depending on the geometry. In this case the component of the annulus separates a and b, but there are some curves from a to b in U which don't cross the annulus.

Figure 2: The general idea of Condition G2 is that an event of an unforced crossing has uniformly positive probability to fail. In all of the pictures the solid line is the boundary of the domain, the dotted lines are the boundaries of the annulus and the dashed lines refer to the crossing event we are considering.

Condition G1. The family Σ is said to satisfy a *geometric bound on an unforced crossing (at time zero)* if there exists $C > 1$ such that for any $(\phi, \mathbb{P}) \in \Sigma$ and for any annulus $A = A(z_0, r, R)$ with $0 < C\, r \leq R$,

$$\mathbb{P}\,(\gamma \text{ makes a crossing of } A \text{ which is contained in } A^u) \leq \frac{1}{2}. \tag{4}$$

A *topological quadrilateral* $Q = (V; S_k, k = 0, 1, 2, 3)$ consists a domain V which is homeomorphic to a square in a way that the boundary arcs S_k, $k = 0, 1, 2, 3$, are in counterclockwise order and correspond to the four edges of the square. There exists a unique positive L and a conformal map from Q onto a rectangle $[0, L] \times [0, 1]$ mapping S_k to the four edges of the rectangle with image of S_0 being $\{0\} \times [0, 1]$. The number L is called the *modulus* of (or the *extremal length the curve family* joining the opposite sides of) Q and we will denote it by $m(Q)$.

Condition C1. The family Σ is said to satisfy a *conformal bound on an unforced crossing (at time zero)* if there exists $M > 0$ such that for any $(\phi, \mathbb{P}) \in \Sigma$ and for any topological quadrilateral Q with $V(Q) \subset U$, $S_1 \cup S_3 \subset \partial U$ and $m(Q) \geq M$

$$\mathbb{P}\,(\gamma \text{ makes a crossing of } Q) \leq \frac{1}{2}. \tag{5}$$

Remark 1.2. Notice that in Condition G1 we require that the bound holds for *all* components of A^u simultaneously, whereas in Condition C1 the bound holds for *one* topological quadrilateral. On the other, the set of topological quadrilaterals is bigger than the set of topological quadrilaterals Q whose boundary arcs $S_0(Q)$ and $S_2(Q)$ are subsets of different boundary components of some annulus and $V(Q)$ is subset of that annulus. The latter set is the set of shapes relevant in Condition G1, at least naively speaking.

1.2 Main theorem

Denote by $\phi\mathbb{P}$ the pushforward of $\mathbb{P}$ by ϕ defined by

$$(\phi\mathbb{P})(A) = \mathbb{P}(\phi^{-1}(A)) \tag{6}$$

for any measurable $A \subset X_{\text{simple}}(\mathbb{D})$. In other words $\phi\mathbb{P}$ is the law of the random curve $\phi(\gamma)$. Given a family Σ as above, define the family of pushforward measures

$$\Sigma_{\mathbb{D}} = \{\phi\mathbb{P} \,:\, (\phi, \mathbb{P}) \in \Sigma\} . \tag{7}$$

The family $\Sigma_{\mathbb{D}}$ consist of measures on the curves $X_{\text{simple}}(\mathbb{D})$ connecting -1 to 1.

Fix a conformal map

$$\Phi(z) = i\frac{z+1}{1-z} \tag{8}$$

which takes $\mathbb{D}$ onto the *upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$. Note that if γ is distributed according to $\mathbb{P} \in \Sigma_{\mathbb{D}}$, then $\tilde{\gamma} = \Phi(\gamma)$ is a simple curve in the upper half-plane slightly extending the definition of $X_{\text{simple}}(\mathbb{H})$, namely, $\tilde{\gamma}$ is simple with $\tilde{\gamma}(0) = 0 \in \mathbb{R}$, $\tilde{\gamma}((0,1)) \subset \mathbb{H}$ and $|\gamma(t)| \to \infty$ as $t \to 1$. Therefore by results of Appendix A.1, if $\tilde{\gamma}$ is parametrized with the half-plane capacity, then it has a continuous driving term $W = W_\gamma : \mathbb{R}_+ \to \mathbb{R}$. As a convention the driving term or process of a curve or a random curve in $\mathbb{D}$ means the driving term or process in $\mathbb{H}$ after the transformation Φ and using the half-plane capacity parametrization.

The following theorem and its reformulation, Proposition 3.2, are the main results of this paper. Note that the following theorem concerns with $\Sigma_{\mathbb{D}}$. The proof will be presented in the section 3. See Section 2.1.3 for the exact assumptions of the theorem, namely, Condition G2.

Theorem 1.3. *If the family Σ of probability measures satisfies Condition G2, then the family $\Sigma_{\mathbb{D}}$ is tight and therefore relatively compact in the topology of the weak convergence of probability measures on $(X, \mathcal{B}_X)$. Furthermore if $\mathbb{P}_n \in \Sigma_{\mathbb{D}}$ is converging weakly and the limit is denoted by $\mathbb{P}^*$ then the following statements hold $\mathbb{P}^*$ almost surely*

(i) the point 1 is not a double point, i.e., $\gamma(t) = 1$ only if $t = 1$,

(ii) the tip $\gamma(t)$ of the curve lies on the boundary of the connected component of $\mathbb{D} \setminus \gamma[0,t]$ containing 1 (having the point 1 on its boundary), for all t,

(iii) if $\hat{K}_t$ is the hull of $\Phi(\gamma[0,t])$, then the capacity $\operatorname{cap}_{\mathbb{H}}(\hat{K}_t) \to \infty$ as $t \to 1$

(iv) for any parametrization of γ the capacity $t \mapsto \operatorname{cap}_{\mathbb{H}}(\hat{K}_t)$ is strictly increasing and if $(K_t)_{t\in\mathbb{R}_+}$ is $(\hat{K}_t)_{t\in[0,1)}$ reparametrized with capacity, then the corresponding g_t satisfies the Loewner equation with a driving process $(W_t)_{t\in\mathbb{R}_+}$ which is Hölder continuous for any exponent $\alpha < 1/2$.

Furthermore, there exists $\varepsilon > 0$ such that for any t, $\mathbb{E}^*[\exp(\varepsilon|W_t|/\sqrt{t})] < \infty$.

Remark 1.4. Note that the claims (i)–(iii) don't depend on the parameterization.

The following corollary clarifies the relation between the convergence of random curves and the convergence of their driving processes. For instance, it shows that if the driving processes of Loewner chains satisfying Condition G2 converge, also the limiting Loewner chain is generated by a curve.

Corollary 1.5. *Suppose that $(W^{(n)})_{n\in\mathbb{N}}$ is a sequence of driving processes of random Loewner chains that are generated by simple random curves $(\gamma^{(n)})_{n\in\mathbb{N}}$ in $\mathbb{H}$ satisfying Condition G2. If $(\gamma^{(n)})_{n\in\mathbb{N}}$ are parametrized by capacity, then the sequence of pairs $(\gamma^{(n)}, W^{(n)})_{n\in\mathbb{N}}$ is tight in the topology of uniform convergence on the compact intervals of $\mathbb{R}_+$ in the capacity parametrization. Furthermore, if either $(\gamma^{(n)})_{n\in\mathbb{N}}$ or $(W^{(n)})_{n\in\mathbb{N}}$ converges (weakly), also the other one converges and the limits agree in the sense that $\gamma = \lim_n \gamma_n$ is driven by $W = \lim_n W_n$.*

Here we use a bit loosely the terminology introduced above: for Condition G2 in $\mathbb{H}$ we either want to use a metric of the Riemann sphere, which makes $\mathbb{H}$ relatively compact, or we map the domain onto $\mathbb{D}$, say. It is also understood that $a = \gamma^{(n)}(0)$ and $b = \infty$ in the definition of A^u.

For the next corollary let's define the space of *open curves* by identifying in the set of continuous maps $\gamma : (0,1) \to \mathbb{C}$ different parametrizations. The topology will be given by the convergence on the compact subsets of $(0,1)$. See also Section 3.6. It is necessary to consider open curves since in rough domains nothing guarantees that there are curves starting from a given boundary point or prime end.

We say that (U_n, a_n, b_n), $n \in \mathbb{N}$, converges to (U, a, b) in the *Carathéodory sense* if there exists conformal and onto mappings $\psi_n : \mathbb{D} \to U_n$ and $\psi : \mathbb{D} \to U$ such that they satisfy $\psi_n(-1) = a_n$, $\psi_n(+1) = b_n$, $\psi(-1) = a$ and $\psi(+1) = b$ (possibly defined as prime ends) and such that ψ_n converges to ψ uniformly in the compact subsets of $\mathbb{D}$ as $n \to \infty$. Note that this the limit is not necessarily unique as a sequence (U_n, a_n, b_n) can converge to different limits for different choices of ψ_n. However if know that (U_n, a_n, b_n), $n \in \mathbb{N}$, converges to (U, a, b), then $\psi(0) \in U_n$ for large enough n and U_n converges to U in the usual sense of Carathéodory kernel convergence with respect to the point $\psi(0)$. For the definition see Section 1.4 of [23].

The next corollary shows that if we have a converging sequence of random curves in the sense of Theorem 1.3 and if they are supported on domains which converge in the Carathéodory sense, then the limiting random curve is supported on the limiting domain. Note that the Carathéodory kernel convergence allows that there are deep fjords in U_n which are "cut off" as $n \to \infty$. One can interpret the following corollary to state that with high probability the random curves don't enter any of these fjords. This is a desired property of the convergence.

Corollary 1.6. *Suppose that (U_n, a_n, b_n) converges to (U^*, a^*, b^*) in the Carathéodory sense (here a^*, b^* are possibly defined as prime ends) and suppose that $(\phi_n)_{n\geq 0}$ are conformal maps such that $U_n = U(\phi_n), a_n = a(\phi_n), b_n = b(\phi_n)$ and $\lim \phi_n = \phi^*$ for which $U^* = U(\phi^*), a^* = a(\phi^*), b = b(\phi^*)$. Let $\hat{U} = U^* \setminus (V_a \cup V_b)$ where V_a and V_b are some neighborhoods of a and b, respectively, and set $\hat{U}_n = \phi_n^{-1} \circ \phi(\hat{U})$. If $(\phi_n, \mathbb{P}_n)_{n\geq 0}$ satisfy Condition G2 and γ^n has the law $\mathbb{P}_n$, then γ^n restricted to $\hat{U}_n$ has a weakly converging subsequence in the topology of X, the laws for different $\hat{U}$ are consistent so that it is possible to define a random curve γ on the open interval $(0,1)$ such that the limit for γ^n restricted to $\hat{U}_n$ is γ restricted to the closure of $\hat{U}$.*

Especially, almost surely the limit of γ_n is supported on open curves of U^* and don't enter $(\limsup \overline{U_n}) \setminus \overline{U}^*$.

1.3 An application to the continuity of SLE

This section is devoted to an application of Theorem 1.3.

Consider SLE(κ), $\kappa \in [0, 8)$, for different values of κ. For an introduction to Schramm–Loewner evolution see Appendix A.1 below and [18]. The driving processes of the different SLEs can be given in the same probability space in the obvious way by using the same standard Brownian motion for all of them. A natural question is to ask whether or not SLE is as a random curve continuous in the parameter κ. See also [15], where it is proved that SLE is continuous in κ for small and large κ in the sense of almost sure convergence of the curves when the driving processes are coupled in the way given above. We will prove the following theorem using Corollary 1.5.

Theorem 1.7. *Let $\gamma^{[\kappa]}(t)$, $t \in [0, T]$, be SLE(κ) parametrized by capacity. Suppose that $\kappa \in [0, 8)$ and $\kappa_n \to \kappa$ as $n \to \infty$. Then as $n \to \infty$, $\gamma^{[\kappa_n]}$ converges weakly to $\gamma^{[\kappa]}$ in the topology of the supremum norm.*

We'll present the proof here since it is independent of the rest of the paper except that it relies on Corollary 1.5, Proposition 2.5 (equivalence of geometric and conformal conditions) and Remark 2.8 (on the domain Markov property). The reader can choose to read those parts before reading this proof.

Proof. Let $\kappa_0 \in [0, 8)$. First we verify that the family consisting of SLE(κ)s on $\mathbb{D}$, say, where κ runs over the interval $[0, \kappa_0]$, satisfies Condition G2. Since SLE$_\kappa$ has the conformal domain Markov property, it is enough to verify Condition C1 in $\mathbb{H}$. More specifically, it is enough to show that there exists $M > 0$ such that if $Q = (V, S_0, S_1, S_2, S_3)$ is a topological quadrilateral with $m(Q) \geq M$ such that $V \subset \mathbb{H}$, $S_k \subset \mathbb{R}_+ := [0, \infty)$ for $k = 1, 3$ and S_2 separates S_0 from ∞ in $\mathbb{H}$, then

$$\mathbb{P}(\text{SLE}(\kappa) \text{ intersects } S_0) \leq \frac{1}{2} \tag{9}$$

for any $\kappa \in [0, \kappa_0]$.

Suppose that $M > 0$ is large and Q satisfies $m(Q) \geq M$. Let $Q' = (V'; S_0', S_2')$ be the doubly connected domain where V' is the interior of the closure of $V \cup V^*$, V^* is the mirror image of V with respect to the real axis, and S_0' and S_2' are the inner and outer boundary of V', respectively. Then the modulus (or extremal length) of Q', which is defined as the extremal length of the curve family connecting S_0' and S_2' in V' (for the definition see Chapter 4 of [1]), is given by $m(Q') = m(Q)/2$.

Let $x = \min(\mathbb{R} \cap S_0') > 0$ and $r = \max\{|z - x| \,:\, z \in S_0'\} > 0$. Then Q' is a doubly connected domain which separates x and a point on $\{z \,:\, |z - x| = r\}$ from $\{0, \infty\}$. By Theorem 4.7 of [1], of all the doubly connected domains with this property, the complement of $(-\infty, 0] \cup [x, x + r]$ has the largest modulus. By the equation 4.21 of [1],

$$\exp(2\pi\, m(Q')) \leq 16 \left(\frac{x}{r} + 1 \right) \tag{10}$$

which implies that $r \leq \rho x$ where

$$\rho = \left(\frac{1}{16}\exp(\pi M) - 1\right)^{-1} \tag{11}$$

which can be as small as we like by choosing M large.

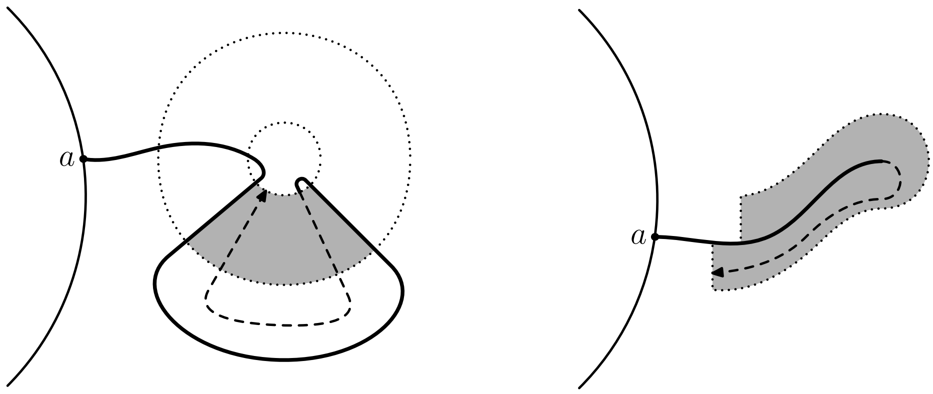
If SLE(κ) crosses Q then it necessarily intersects $\overline{B(x,r)}$. By the scale invariance of SLE(κ)

$$\mathbb{P}(\text{SLE}(\kappa) \text{ intersects } S_0) \leq \mathbb{P}\left(\text{SLE}(\kappa) \text{ intersects } \overline{B(1,\rho)}\right). \tag{12}$$

Now by standard arguments [24], the right hand side can be made less than 1/2 for $\kappa \in [0, \kappa_0]$ and $0 < \rho \leq \rho_0$ where $\rho_0 > 0$ is suitably chosen constant.

Denote the driving process of $\gamma^{[\kappa]}$ by $W^{[\kappa]}$. If $\kappa_n \to \kappa \in [0, 8)$, then obviously $W^{[\kappa_n]}$ converges weakly to $W^{[\kappa]}$. Hence by Corollary 1.5 also $\gamma^{[\kappa_n]}$ converges weakly to some $\tilde{\gamma}$ whose driving process is distributed as $W^{[\kappa]}$. That is, $\gamma^{[\kappa_n]}$ converges weakly to $\gamma^{[\kappa]}$ as $n \to \infty$ provided that $\kappa_n \to \kappa$ as $n \to \infty$. □

1.4 Structure of this paper



(a) When the radius of the inner circle goes to zero, the dashed line is no longer visible from a faraway reference point. If such an event has positive probability for the limiting measure, then the Loewner equation doesn't describe the whole curve.

(b) Longitudinal crossing of an arbitrarily thin tube of fixed length along the curve or the boundary violates the local growth needed for the continuity of the Loewner driving term.

Figure 3: In the proof of Theorem 1.3, the regularity of random curves is established by establishing a probability upper bound on multiple crossings and excluding two unwanted scenarios presented in this figure.

In Section 2, the general setup of this paper is presented. Four conditions are stated and shown to be equivalent. Any one of them can be taken as the main assumption for Theorem 1.3.

The proof of Theorem 1.3 is presented in Section 3. The proof consist of three parts: the first one is the existence of regular parametrizations of the random curves and the second and third steps are described in Figure 3. The relevant condition is verified for a list of random curves arising from statistical mechanics models in Section 4.

2 The space of curves and equivalence of conditions

2.1 The space of curves and conditions

2.1.1 The space of curves

We follow the setup of Aizenman and Burchard's paper [2]: planar curves are continuous mappings from $[0,1]$ to $\mathbb{C}$ modulo reparameterizations. Let

$$C' = \{f \in C\left([0,1],\mathbb{C}\right) \,:\, f \equiv \textit{const.} \text{ or } f \text{ not constant on any subinterval}\}.$$

It is also possible to work with the whole space $C\left([0,1],\mathbb{C}\right)$, but the next definition is easier for C'. Define an equivalence relation $\sim$ in C' so that $f_1 \sim f_2$ if they are related by an increasing homeomorphism $\psi : [0,1] \to [0,1]$ with $f_2 = f_1 \circ \psi$. The reader can check that this defines an equivalence relation. The mapping $f_1 \circ \psi$ is said to be a *reparameterization* of f_1 or that f_1 is *reparameterized* by ψ.

Note that these parameterizations are somehow arbitrary and are different from the Loewner parameterization which we are going to construct.

Denote the equivalence class of f by $[f]$. The set of all equivalence classes

$$X = \{[f] : f \in C'\}$$

is called the *space of curves*. Make X a metric space by setting

$$\mathrm{d}_X([f],[g]) = \inf\{\|f_0 - g_0\|_\infty : f_0 \in [f], g_0 \in [g]\}. \tag{13}$$

It is easy to see that this is a metric, see e.g. [2]. The space X with the metric d_X is complete and separable reflecting the same properties of $C\left([0,1],\mathbb{C}\right)$. And for the same reason as $C\left([0,1],\mathbb{C}\right)$ is not compact neither is X.

Define two subspaces, the space X_{simple} of *simple curves* and the space X_0 of curves with no self-crossings by

$$\begin{aligned} X_{\text{simple}} &= \{[f] : f \in C', f \text{ injective}\} \\ X_0 &= \overline{X_{\text{simple}}} \end{aligned}$$

Note that $X_0 \subsetneq X$ since there exists $\gamma_0 \in X \setminus X_{\text{simple}}$ with positive distance to X_{simple}. For example, such is the broken line passing through points $-1, 1, i$ and $-i$ which has a double point which is stable under small perturbations.

How do the curves in X_0 look like? Roughly speaking, they may touch themselves and have multiple points, but they can have no "transversal" self-intersections. For example, the broken line through points $-1, 1, i, 0, -1+i$, also has a double point at 0, but it can be removed by small perturbations. Also, every passage through the double point separates its neighborhood into two components, and every other passage is contained in (the closure) of one of those. See also Figure 4.

Given a domain $U \subset \mathbb{C}$ define $X(U)$ as the closure of $\{[f] : f \in C', f[0,1] \subset U\}$ in (X, d_X). Define also $X_0(U)$ as the closure of the set of simple curves in $X(U)$. The notation $X_{\text{simple}}(U)$ we preserve for

$$X_{\text{simple}}(U) = \big\{[f] : f \in C', f\big((0,1)\big) \subset U, f \text{ injective}\big\}\,,$$

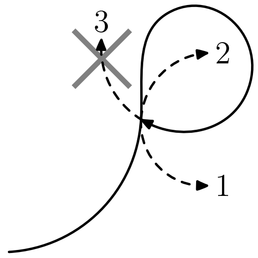


Figure 4: In this example the options 1 and 2 are possible so that the resulting curve in the class X_0. If the curve continues along 3 it doesn't lie in X_0, namely, there is no sequence of simple curves converging to that curve.

so the end points of such curves may lie on the boundary. Note that the closure of $X_{\text{simple}}(U)$ is still $X_0(U)$.

Use also notation $X_{\text{simple}}(U, a, b)$ for curves in $X_{\text{simple}}(U)$ whose end points are $\gamma(0) = a$ and $\gamma(1) = b$. We will quite often consider some reference sets as $X_{\text{simple}}(\mathbb{D}, -1, +1)$ and $X_{\text{simple}}(\mathbb{H}, 0, \infty)$ where the latter can be understood by extending the above definition to curves defined on the Riemann sphere, say.

We will often use the letter γ to denote elements of X, i.e. a curve modulo reparameterizations. Note that topological properties of the curve (such as its endpoints or passages through annuli or its *locus* $\gamma[0, 1]$) as well as metric ones (such as dimension or length) are independent of parameterization. When we want to put emphasis on the locus, we will be speaking about *Jordan* curves or arcs, usually parameterized by the open unit interval $(0, 1)$.

Denote by $\mathrm{Prob}(X)$ the space of probability measures on X equipped with the Borel σ-algebra $\mathcal{B}_X$ and the weak-$*$ topology induced by continuous functions (which we will call weak for simplicity). Suppose that $\mathbb{P}_n$ is a sequence of measures in $\mathrm{Prob}(X)$.

If for each n, $\mathbb{P}_n$ is supported on a closed subset of X_{simple} (which for discrete curves can be assumed without loss of generality) and if $\mathbb{P}_n$ converges weakly to a probability measure $\mathbb{P}$, then $1 = \limsup_n \mathbb{P}_n(X_0) \leq \mathbb{P}(X_0)$ by general properties of the weak convergence of probability measures [5]. Therefore $\mathbb{P}$ is supported on X_0 but in general it doesn't have to be supported on X_{simple}.

2.1.2 Comment on the probability structure

Suppose $\mathbb{P}$ is supported on $D \subset X(\mathbb{C})$ which is a closed subset of $X_{\text{simple}}(\mathbb{C})$. Consider some measurable map $\chi : D \to C([0, \infty), \mathbb{C})$ so that $\chi(\gamma)$ is a parametrization of γ. If necessary χ can be continued to D^c by setting $\chi = 0$ there.

Let π_t be the natural projection from $C([0, \infty), \mathbb{C})$ to $C([0, t], \mathbb{C})$. Define a σ-algebra

$$\mathcal{F}_t^{\chi,0} = \sigma(\pi_s \circ \chi, 0 \leq s \leq t) \ ,$$

and make it right continuous by setting $\mathcal{F}_t^{\chi} = \bigcap_{s>t} \mathcal{F}_s^{\chi,0}$.

For a moment denote by $(\tau, \hat{\tau})$ for given $\gamma, \hat{\gamma} \in D$ the maximal pair of times such that $\chi(\gamma)|_{[0,\tau]}$ is equal to $\chi(\hat{\gamma})|_{[0,\hat{\tau}]}$ in X, that is, equal modulo a reparametrization. We call χ a *good parametrization* of the curve family D, if for each $\gamma, \hat{\gamma} \in D$, $\tau = \hat{\tau}$ and $\chi(\gamma, t) = \chi(\hat{\gamma}, t)$ for all $0 \leq t \leq \tau$.

Each reparametrization from a good parametrization to another can be represented as stopping times T_u, $u \geq 0$. From this it follows that the set of stopping times is the same for every good parametrization. We will use simply the notation $\gamma[0,t]$ to denote the σ-algebra $\mathcal{F}_t^\chi$. The choice of a good parametrization χ is immaterial since all the events we will consider are essentially reparametrization invariant. But to ease the notation it is useful to always have some parametrization in mind.

Often there is a natural choice for the parametrization. For example, if we are considering paths on a lattice, then the probability measure is supported on polygonal curves. In particular, the curves are piecewise smooth and it is possible to use the arc length parametrization, i.e. $|\gamma'(t)| = 1$. One of the results in this article is that given the hypothesis, which is described next, it is possible to use the capacity parametrization of the Loewner equation. Both the arc length and the capacity are good parameterizations.

The following lemma follows immediately from above definitions.

Lemma 2.1. *If $A \subset \mathbb{C}$ is a non-empty, closed set, then $\tau_A = \inf\{t \geq 0 \,:\, \chi(\gamma, t) \in A\}$ is a stopping time.*

2.1.3 Four equivalent conditions

Recall the general setup: we are given a collection $(\phi, \mathbb{P}) \in \Sigma$ where the conformal map ϕ contains also the information about the domain $(U, a, b) = (U(\phi), a(\phi), b(\phi))$ and $\mathbb{P}$ is a probability measure on $X_{\text{simple}}(U, a, b)$. Furthermore, we assume that each γ which distributed as $\mathbb{P}$ has some suitable parametrization.

For given domain U and for given simple (random) curve γ on U, we always define $U_\tau = U \setminus \gamma[0, \tau]$ for each (random) time τ. We call U_τ as the domain at time τ.

Definition 2.2. For a fixed domain (U, a, b) and for fixed simple (random) curve in U starting from a, define for any annulus $A = A(z_0, r, R)$ and for any (random) time $\tau \in [0, 1]$, $A^u_\tau = \emptyset$ if $\partial B(z_0, r) \cap \partial U_\tau = \emptyset$ and

$$A^u_\tau = \left\{ z \in U_\tau \cap A \,:\, \begin{array}{c} \text{the connected component of } z \text{ in } U_\tau \cap A \\ \text{doesn't disconnect } \gamma(\tau) \text{ from } b \text{ in } U_\tau \end{array} \right\} \tag{14}$$

otherwise. A connected set C disconnects $\gamma(\tau)$ from b if it disconnects some neighbourhood of $\gamma(\tau)$ from some neighbourhood of b in U_τ. If $\gamma[\tau, 1]$ contains a crossing of A which is contained in A^u_τ, we say that γ *makes an unforced crossing* of A in U_τ (or an unforced crossing of A observed at time τ). The set A^u_τ is said to be *avoidable* at time τ.

Remark 2.3. Neighbourhoods are needed here only for incorporate the fact that $\gamma(t)$ and b are boundary points.

The first two of the four equivalent conditions are geometric, asking an unforced crossing of an annulus to be unlikely uniformly in terms of the modulus.

Condition G2. The family Σ is said to satisfy a *geometric bound on an unforced crossing* if there exists $C > 1$ such that for any $(\phi, \mathbb{P}) \in \Sigma$, for any stopping time $0 \leq \tau \leq 1$ and for any annulus $A = A(z_0, r, R)$ where $0 < C\, r \leq R$,

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ makes a crossing of } A \text{ which is contained in } A^u_\tau \,\big|\, \gamma[0, \tau]\big) < \frac{1}{2}. \tag{15}$$

Condition G3. The family Σ is said to satisfy a *geometric power-law bound on an unforced crossing* if there exist $K > 0$ and $\Delta > 0$ such that for any $(\phi, \mathbb{P}) \in \Sigma$, for any stopping time $0 \leq \tau \leq 1$ and for any annulus $A = A(z_0, r, R)$ where $0 < r \leq R$,

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ makes a crossing of } A \text{ which is contained in } A^u_\tau \,\big|\, \gamma[0,\tau]\big) \leq K \left(\frac{r}{R}\right)^\Delta . \tag{16}$$

Let $Q \subset U_t$ be a topological quadrilateral, i.e. an image of the square $(0,1)^2$ under a homeomorphism ψ. Define the "sides" $\partial_0 Q$, $\partial_1 Q$, $\partial_2 Q$, $\partial_3 Q$, as the "images" of

$$\{0\} \times (0,1), \quad (0,1) \times \{0\}\,, \quad \{1\} \times (0,1), \quad (0,1) \times \{1\}$$

under ψ. For example, we set

$$\partial_0 Q := \lim_{\epsilon \to 0} \mathrm{Clos}\,(\psi\,(\ (0,\epsilon) \times (0,1)\ ))\,.$$

We consider Q such that two opposite sides $\partial_1 Q$ and $\partial_3 Q$ are contained in ∂U_t. A *crossing* of Q is a curve in U_t connecting two opposite sides $\partial_0 Q$ and $\partial_2 Q$. The latter without loss of generality (just perturb slightly) we assume to be smooth curves of finite length inside U_t. Call Q *avoidable* if it doesn't disconnect $\gamma(t)$ and b inside U_t.

Condition C2. The family Σ is said to satisfy a *conformal bound on an unforced crossing* if there exists a constant $M > 0$ such that for any $(\phi, \mathbb{P}) \in \Sigma$, for any stopping time $0 \leq \tau \leq 1$ and any avoidable quadrilateral Q of U_τ, such that the modulus $m(Q)$ is larger than M

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ crosses } Q \,\big|\, \gamma[0,\tau]\big) \leq \frac{1}{2}. \tag{17}$$

Remark 2.4. In the condition above, the quadrilateral Q depends on $\gamma[0,\tau]$, but this does not matter, as we consider *all* such quadrilaterals. A possible dependence on $\gamma[0,\tau]$ ambiguity can be addressed by mapping U_t to a reference domain and choosing quadrilaterals there. See also Remark 2.9.

Condition C3. The family Σ is said to satisfy a *conformal power-law bound on an unforced crossing* if there exist constants K and ϵ such that for any $(\phi, \mathbb{P}) \in \Sigma$, for any stopping time $0 \leq \tau \leq 1$ and any avoidable quadrilateral Q of U_τ

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ crosses } Q \,\big|\, \gamma[0,\tau]\big) \leq K \exp(-\epsilon\, m(Q)). \tag{18}$$

Proposition 2.5. *The four conditions G2, G3, C2 and C3 are equivalent and conformally invariant.*

This proposition is proved below in Section 2.2. Equivalence of conditions immediately implies the following

Corollary 2.6. *Constant* $1/2$ *in Conditions G2 and C2 can be replaced by any other from* $(0,1)$.

2.1.4 Remarks concerning the conditions

Remark 2.7. Conditions G2 and G3 could be described as being *geometric* since they involve crossing of fixed shape. Conditions C2 and C3 are *conformally invariant* because they are formulated using the extremal length which is a conformally invariant quantity. The conformal invariance in Proposition 2.5 means for example, that if Condition G2 holds with a constant $C > 1$ for $(\mathbb{P}, \phi)$ defined in U and if $\psi : U \to U'$ is conformal and onto, then Condition G2 holds for $(\psi\mathbb{P}, \phi \circ \psi^{-1})$ with a constant $C' > 1$ which depends only on the constant C but not on $(\mathbb{P}, \phi)$ or ψ.

Remark 2.8. To formulate the *domain Markov property* let's suppose that $\Sigma = (\mathbb{P}_n^{U,a,b}, \phi_n^{U,a,b})$ where U is the domain of $\phi_n^{U,a,b}$ as usual and $(\phi_n^{U,a,b})^{-1}(-1) = a$ $(\phi_n^{U,a,b})^{-1}(+1) = b$ and $n \in \mathbb{N}$. Here $n \in \mathbb{N}$ refers to a sequence of lattice mesh which tends to zero as $n \to \infty$. If the family satisfies

$$\mathbb{P}_n^{U,a,b} \left(\gamma|_{[t,1]} \in \cdot \,\middle|\, \gamma|_{[0,t]} \right) = \mathbb{P}_n^{U \setminus \gamma[0,t], \gamma(t), b}$$

for any U, a, b, n and for a set of times t then it is said to have domain Markov property. This property could be formulated more generally so that if $\mathbb{P} \in \Sigma$, then $\mathbb{P} \left(\gamma|_{[t,1]} \in \cdot \,\middle|\, \gamma|_{[0,t]} \right)$ is equal to some measure $\mathbb{P}' \in \Sigma$.

When the domain Markov property holds, the "time zero conditions" G1 and C1 are sufficient for Conditions G2 and C2, respectively.

Remark 2.9. Our conditions impose an estimate on conditional probability, which is hence satisfied almost surely. By taking a countable dense set of round annuli (or of topological rectangles), we see that it does not matter whether we require the estimate to hold separately for *any* given annulus almost surely; or to hold almost surely for *every* annulus. The same argument applies to topological rectangles.

Remark 2.10. Suppose now that the random curve γ is an interface in a statistical physics model with two possible states at each site, say, *blue* and *red*. In that case U will be a simply connected domain formed by entire faces of some lattice, say, hexagonal lattice, $a, b \in \partial U$ are boundary points, the faces next to the arc ab are colored blue and next to the arc ba red and γ is the interface between the blue cluster of ab (connected set of blue faces) and the red cluster of ba.

In this case under *positive associativity* (e.g. observing blue faces somewhere increases the probability of observing blue sites elsewhere) the sufficient condition implying Condition G2 is uniform upper bound for the probability of the *crossing event of an annulur sector* with alternating boundary conditions (red–blue–red–blue) on the four boundary arcs (circular–radial–circular–radial) by blue faces. For more detail, see Section 4.1.6.

2.2 Equivalence of the geometric and conformal conditions

In this section we prove Proposition 2.5 about equivalence of geometric and conformal conditions. We start with recalling the notion of Beurling's extremal length and then proceed to the proof. Note that since Condition C2 is conformally invariant, conformal invariance of other conditions immediately follows.

Suppose that a curve family $\Gamma \subset X$ consist of curves that are regular enough for the purposes below. A non-negative Borel function ρ on $\mathbb{C}$ is called *admissible* if

$$\int_\gamma \rho \, \mathrm{d}\Lambda \ \geq 1 \tag{19}$$

for each $\gamma \in \Gamma$. Here $\mathrm{d}\Lambda$ is the arc-length measure.

The *extremal length* of a curve family $\Gamma \subset X$ is defined as

$$m(\Gamma) = \frac{1}{\inf_\rho \int \rho^2 \, \mathrm{d}A} \tag{20}$$

where the infimum is taken over all the admissible functions ρ. Here $\mathrm{d}A$ is the area measure (Lebesgue measure on $\mathbb{C}$). The quantity inside the infimum is called the *ρ-area* and the quantity on the left-hand side of the inequality (19) is called the *ρ-length* of γ. The following basic estimate is easy to obtain.

Lemma 2.11. *Let $A = A(z_0, r_1, r_2)$, $0 < r_1 < r_2$, be an annulus. Suppose that Γ is a curve family with the property that each curve $\gamma \in \Gamma$ contains a crossing of A. Then,*

$$m(\Gamma) \geq \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right) \tag{21}$$

and therefore

$$r_1 \geq r_2 \cdot \exp\left(-2\pi m(\Gamma)\right). \tag{22}$$

Proof. Let $\widehat{\Gamma}$ be the family of curves connecting the two boundary circles of A. If ρ is admissible for $\widehat{\Gamma}$ then it is also admissible for Γ. Hence, $m(\Gamma) \geq m(\widehat{\Gamma}) = 2\pi \log(r_2/r_1)$. □

We now proceed to showing the equivalence of four conditions by establishing the following implications:

G2⇔G3 Condition G2 directly follows from G3 by setting $C := (2K)^{1/\Delta}$.

In the opposite direction, an unforced crossing of the annulus $A(z_0, r, R)$ implies consecutive unforced crossings of the concentric annuli $A_j := A(z_0, C^{j-1}r, C^j r)$, with $j \in \{1, \ldots, n\}$, $n := \lfloor \log(R/r)/\log C \rfloor$, which have conditional (on the past) probabilities of at most $1/2$ by Condition G2. Trace the curve γ denoting by τ_j the ends of unforced crossings of A_{j-1}'s (with $\tau_1 = \tau$), and estimating

$$\begin{aligned} \mathbb{P}\big(\gamma[\tau, 1] \text{ crosses } A^u_\tau \,\big|\, \gamma[0, \tau]\big) &\leq \prod_{j=1}^n \mathbb{P}\big(\gamma[\tau_j, 1] \text{ crosses } (A_j)^u_{\tau_j} \,\big|\, \gamma[0, \tau_j]\big) \\ \leq \left(\frac{1}{2}\right)^n \leq \left(\frac{1}{2}\right)^{(\log(R/r)/\log C)-1} &= 2\left(\frac{r}{R}\right)^{\log 2/\log C}. \end{aligned}$$

We infer condition G3 with $K := 2$ and $\Delta := \log 2/\log C$.

C2⇔C3 This equivalence is proved similarly to the equivalence of the geometric conditions. The only difference is that instead of cutting an annulus into concentric ones of moduli C, we start with an avoidable quadrilateral Q, and cut from it $n = [m(Q)/M]$ quadrilaterals $Q_1, \ldots, Q_n$ of modulus M. If Q is mapped by a conformal map ϕ onto the rectangle $\{z : 0 < \mathrm{Re}\, z < m(Q),\, 0 < \mathrm{Im}\, z < 1\}$, we can set $Q_j := \phi^{-1}\{z : (j-1)M < \mathrm{Re}\, z < jM,\, 0 < \mathrm{Im}\, z < 1\}$. Then as we trace γ, all Q_j's are avoidable for its consecutive pieces.

G2⇒C2 We show that Condition G2 with constant C implies Condition C2 with $M = 4(C+1)^2$.

Let $m \geq M$ be the modulus of Q, i.e. the extremal length $m(\Gamma)$ of the family Γ of curves joining $\partial_0 Q$ to $\partial_2 Q$ inside Q. Let Γ^* be the dual family of curves joining $\partial_1 Q$ to $\partial_3 Q$ inside Q, then $m(\Gamma) = 1/m(\Gamma^*)$.

Denote by d_1 the distance between $\partial_1 Q$ and $\partial_3 Q$ in the inner Euclidean metric of Q, and let γ^* be the some curve of length $\leq 2d_1$ joining $\partial_1 Q$ to $\partial_3 Q$ inside Q. Observe that any crossing γ of Q has the diameter $d \geq 2Cd_1$. Indeed, working with the extremal length of the family Γ^*, take a metric ρ equal to 1 in the d_1-neighborhood of γ. Then its area integral $\iint \rho^2$ is at most $(d+2d_1)^2$. But every curve from Γ^* intersects γ and runs through this neighborhood for the length of at least d_1, thus having ρ-length at least d_1. Therefore $1/m = m(\Gamma^*) \geq (d_1)^2/(d+2d_1)^2$, so we conclude that $m \leq (2 + d/d_1)^2$ and hence

$$d \geq (\sqrt{m} - 2)d_1 \geq (2(C+1) - 2)\, d_1 = 2Cd_1. \tag{23}$$

Now take an annulus A centered at the middle point of γ^* with inner radius d_1 and outer radius $R := Cd_1$. It is sufficient to prove that every crossing of Q contains an unforced crossing of A.

Assume on the contrary that γ is a curve crossing Q but not A. Clearly γ has to intersect γ^*, say at w. But γ^* is entirely contained inside the inner circle of A. On the other hand by (23) the diameter of γ is bigger than $2R$. Thus γ intersects both boundary circles of A, and we deduce Condition C2.

C3⇒G2 Now we will show that Condition C3 with constants K and ϵ (equivalent to Condition C2) implies Condition G2 with constant $C = (2Ke^2)^{2\pi/\epsilon}$.

We have to show that probability of an unforced crossing of a fixed annulus $A = A(z_0, r, Cr)$ is at most $1/2$. Without loss of generality assume that we work with the crossings from the inner circle to the outer one.

For $x \in [0, \log C]$ denote by $\mathcal{I}^x$ the (at most countable) set of arcs I^x which compose $\Omega \cap \partial B(z_0, re^x)$. By $|I|$ we will denote the length of the arc I measured in radians (regardless of the circle radius). Given two arcs I^x and I^y with $y < x$, we will write $I^y \prec I^x$ if any interface γ intersecting I^x has to intersect I^y first, and can do so without intersecting any other arc from $\mathcal{I}^y$ afterwards. We denote by $I^y(I^x)$ the unique arc $I^y \in \mathcal{I}^y$ such that $I^y \prec I^x$.

By $Q(I^x)$ we denote the topological quadrilateral which is cut from Ω by the arcs I^x and $I^0(I^x)$. Denote

$$\ell\,(I^x) = \ell_0^x\,(I^x) := \int_0^x \frac{1}{|I^y(I^x)|}dy.$$

By the Beurling estimate of extremal length,

$$m(Q(I^x)) \geq \ell\left(I^x\right). \tag{24}$$

Note that if γ crosses A and intersects I^x, then it makes an unforced crossing of $Q(I^x)$, so we conclude that by Condition C3 the probability of crossing I^x is majorated by

$$K\ \exp\left(-\epsilon\ell\left(I^x\right)\right). \tag{25}$$

Denote also $|\mathcal{I}^x| := \sum |I^x|$ and $\ell\left(\mathcal{I}^x\right) := \int_0^x \frac{1}{|\mathcal{I}^y|} dy$

We call a collection of arcs $\{I_j\}$ (possibly corresponding to different x's) *separating*, if every unforced crossing γ intersects one of those. To deduce Condition G2, by (25) it is enough to find a separating collection of arcs such that

$$\sum_j \exp\left(-\epsilon\,\ell\left(I_j\right)\right) < \frac{1}{2K}. \tag{26}$$

Note that for every x the total length $|\mathcal{I}^x| \leq 2\pi$, and so by our choice of constant C we have

$$\ell\left(\mathcal{I}^{\log C}\right) \geq \frac{\log C}{2\pi} \geq \frac{2}{\epsilon},$$

as well as

$$\exp\left(2 - \epsilon\ell\left(\mathcal{I}^{\log C}\right)\right) \leq \exp\left(2 - \epsilon\frac{\log C}{2\pi}\right) \leq \exp\left(2 - \log\left(2Ke^2\right)\right) = \frac{1}{2K}.$$

Therefore it is enough to establish under the assumption $\ell\left(\mathcal{I}^w\right) \geq \frac{2}{\epsilon}$ the existence of arcs I_j separating $\mathcal{I}^w$ with the following estimate:

$$\sum_j \exp\left(-\epsilon\,\ell\left(I_j\right)\right) \leq \exp\left(2 - \epsilon\ell\left(\mathcal{I}^w\right)\right). \tag{27}$$

We will do this in an abstract setting for families of arcs. Besides properties mentioned above, we note that for any two arcs I and J the arcs $I^x(I)$ and $I^x(J)$ either coincide or are disjoint. Also without loss of generality any arc I we consider satisfies $I \prec J$ for some $J \in \mathcal{I}^w$.

By a limiting argument it is enough to prove (27) for $\mathcal{I}^w$ of finite cardinality n, and we will do this by induction in n.

If $n = 1$, then we take the only arc J in $\mathcal{I}^w$ as the separating one, and the estimate (27) readily follows:

$$\exp\left(-\epsilon\,\ell\left(J\right)\right) = \exp\left(-\epsilon\,\ell\left(\mathcal{I}^w\right)\right) < \exp\left(2 - \epsilon\,\ell\left(\mathcal{I}^w\right)\right).$$

Suppose $n > 1$. Denote by v the minimal number such that $\mathcal{I}^v$ contains more than one arc.

If

$$\ell_v^w\left(\mathcal{I}^w\right) := \int_v^w \frac{1}{|\mathcal{I}^y|} dy < \frac{2}{\epsilon},$$

then we take the only arc J in $\mathcal{I}^{v-\delta}$ as the separating one. The required estimate (27) then holds if δ is small enough:

$$\begin{aligned}\exp\left(-\epsilon\,\ell\left(J\right)\right) &= \exp\left(-\epsilon\,\ell_0^w\left(\mathcal{I}\right)+\epsilon\,\ell_{v-\delta}^w\left(\mathcal{I}\right)\right)\\ &\leq \exp\left(-\epsilon\,\ell\left(\mathcal{I}_0^w\right)+\epsilon\frac{2}{\epsilon}\right)=\exp\left(-\epsilon\,\ell\left(\mathcal{I}\right)+2\right).\end{aligned}$$

Now assume that, on the contrary,

$$\ell_v^w\left(\mathcal{I}^w\right)\geq\frac{2}{\epsilon}.$$

Suppose $\mathcal{I}^v$ is composed of the arcs J_k. For each k denote by $\mathcal{I}_k^x$ the collection of arcs $I\in\mathcal{I}^x$ such that $J_k\prec I$. Since

$$\ell_v^w\left(\mathcal{I}_k^w\right)\geq\ell_v^w\left(\mathcal{I}^w\right)\geq\frac{2}{\epsilon},\tag{28}$$

we can apply the induction assumption to each of those collections $\mathcal{I}_k^w$ on the interval $x\in[v,w]$, obtaining a set of separating arcs $\{I_{j,k}\}_j$ such that

$$\sum_j\exp\left(-\epsilon\,\ell_v\left(I_{j,k}\right)\right)\leq\exp\left(2-\epsilon\ell_v^w\left(\mathcal{I}_k\right)\right).\tag{29}$$

Then the desired estimates follows from

$$\begin{aligned}\sum_{j,k}\exp\left(-\epsilon\,\ell\left(I_{j,k}\right)\right)&\leq\exp\left(-\epsilon\,\ell_0^v\left(\mathcal{I}^v\right)\right)\ \sum_k\sum_j\exp\left(-\epsilon\,\ell_v\left(I_{j,k}\right)\right)\\ &\leq\exp\left(-\epsilon\,\ell_0^v\left(\mathcal{I}^v\right)\right)\ \sum_k\exp\left(2-\epsilon\ell_v^w\left(\mathcal{I}_k^w\right)\right)\\ &\overset{*}{\leq}\exp\left(-\epsilon\,\ell_0^v\left(\mathcal{I}^v\right)\right)\ \exp\left(2-\epsilon\ell_v^w\left(\mathcal{I}^w\right)\right)\\ &=\exp\left(2-\epsilon\ell\left(\mathcal{I}^w\right)\right),\end{aligned}\tag{30}$$

assuming we have the inequality (30$*$) above. To prove it we first observe that for $x\in[v,w]$,

$$\sum_k\left|\mathcal{I}_k^x\right|=\left|\mathcal{I}^x\right|.$$

Using Jensen's inequality for the probability measure

$$\left(\int_v^w\frac{dy}{|\mathcal{I}^y|}\right)^{-1}\frac{dy}{|\mathcal{I}^y|},$$

and the convex function x^{-1}, we write

$$\begin{aligned}\ell_v^w\left(\mathcal{I}_k\right)&=\int_v^w\frac{1}{|\mathcal{I}_k^y|}dy=\int_v^w\left(\frac{|\mathcal{I}_k^y|}{|\mathcal{I}^y|\,\ell_v^w\left(\mathcal{I}\right)}\right)^{-1}\frac{dy}{|\mathcal{I}^y|\,\ell_v^w\left(\mathcal{I}\right)}\\ &\geq\left(\int_v^w\frac{|\mathcal{I}_k^y|}{|\mathcal{I}^y|\,\ell_v^w\left(\mathcal{I}\right)}\frac{dy}{|\mathcal{I}^y|\,\ell_v^w\left(\mathcal{I}\right)}\right)^{-1}=\left(\int_v^w\frac{|\mathcal{I}_k^y|\,dy}{|\mathcal{I}^y|^2\,\ell_v^w\left(\mathcal{I}\right)^2}\right)^{-1}.\end{aligned}$$

Thus

$$\sum_k \frac{1}{\ell_v^w(\mathcal{I}_k)} \leq \sum_k \left(\int_v^w \frac{|\mathcal{I}_k^y| \, dy}{|\mathcal{I}^y|^2 \, \ell_v^w(\mathcal{I})^2} \right) = \int_v^w \frac{\sum_k |\mathcal{I}_k^y| \, dy}{|\mathcal{I}^y|^2 \, \ell_v^w(\mathcal{I})^2}$$
$$= \int_v^w \frac{dy}{|\mathcal{I}^y|} \frac{1}{\ell_v^w(\mathcal{I})^2} = \ell_v^w(\mathcal{I}) \frac{1}{\ell_v^w(\mathcal{I})^2} = \frac{1}{\ell_v^w(\mathcal{I})}. \tag{31}$$

An easy differentiation shows that the function $F(x) := \exp(-\epsilon/x)$ vanishes at 0, is increasing and convex on the interval $[0, \epsilon/2]$, and so is sublinear there. Observing that the numbers $1/\ell_v^w(\mathcal{I}_k)$ as well as their sum belong to this interval by (28) and (31), we can write

$$\sum_k \exp(-\epsilon \ell_v^w(\mathcal{I}_k)) = \sum_k F(1/\ell_v^w(\mathcal{I}_k)) \leq F\left(\sum_k 1/\ell_v^w(\mathcal{I}_k) \right)$$
$$\leq F(1/\ell_v^w(\mathcal{I})) = \exp(-\epsilon \ell_v^w(\mathcal{I})),$$

thus proving the inequality (30∗) and the desired implication.

This completes the circle of implications, thus proving Proposition 2.5.

3 Proof of the main theorem

In this section, we present the proof of Theorem 1.3. As a general strategy, we find an increasing sequence of events $E_n \subset X_{\text{simple}}(\mathbb{D})$ such that

$$\lim_{n\to\infty} \inf_{\mathbb{P}\in\Sigma_{\mathbb{D}}} \mathbb{P}(E_n) = 1$$

and the curves in E_n have some good properties which among other things guarantee that the closure of E_n is contained in the class of Loewner chains.

The structure of this section is as follows. To use the main lemma (Lemma A.5 in appendix, which constructs the Loewner chain) we need to verify its three assumptions. In the section 3.2, it is shown that with high probability the curves will have parametrizations with uniform modulus of continuity. Similarly the results in the section 3.3 guarantee that the driving processes in the capacity parametrization have uniform modulus of continuity with high probability. In the section 3.4, a uniform result on the visibility of the tip $\gamma(t)$ is proven giving the uniform modulus of continuity of the functions F of Lemma A.5. Finally in the end of this section we prove the main theorem and its corollaries.

A tool which makes many of the proofs easier is the fact that we can use always the most suitable form of the equivalent conditions. Especially, by the results of Section 2.2 if Condition G2 can be verified in the original domain then Condition G2 (or any equivalent condition) holds in any reference domain where we choose to map the random curve as long as the map is conformal. Furthermore, Condition G2 holds after we observe the curve up to a fixed time or a random time and then erase the observed initial part by conformally mapping the complement back to reference domain.

3.1 Reformulation of the main theorem

In this section we reformulate the main result so that its proof amounts to verifying four (more or less) independent properties, which are slightly technical to formulate. The basic definitions are the following, see Sections 3.2, 3.3 and 3.4 for more details. Let $\rho_n \searrow 0$, $\alpha, \alpha' > 0$, $T > 0$, $R > 0$ and $\psi : [0,\infty) \to [0,\infty)$ continuous and strictly increasing with $\psi(0) = 0$. Define the following random variables

$$N_0 = \sup\{n \geq 2 \,:\, \gamma \text{ intersects } \partial B(1,\rho_{n-1}) \text{ after intersecting } \partial B(1,\rho_n)\} \tag{32}$$

$$C_{1,\alpha} = \inf \left\{ C > 0 \,:\, \begin{array}{c} \gamma \text{ can be parametrized s.t.} \\ |\gamma(s) - \gamma(t)| \leq C|t-s|^{\alpha} \quad \forall (t,s) \in [0,1]^2 \end{array} \right\} \tag{33}$$

$$C_{2,\alpha',T} = \inf \left\{ C > 0 \,:\, |W_\gamma(s) - W_\gamma(t)| \leq C|t-s|^{\alpha'} \quad \forall (t,s) \in [0,T]^2 \right\} \tag{34}$$

$$C_{3,\psi,T,R} = \inf \left\{ C > 0 \,:\, |F_\gamma(t,y) - \hat\gamma(t)| \leq C\psi(y) \quad \forall (t,y) \in [0,T] \times [0,R] \right\} \tag{35}$$

where $\hat\gamma = \Phi(\gamma)$ and

$$F_\gamma(t,y) = g_t^{-1}(W_\gamma(t) + i\,y) \tag{36}$$

which can be called hyperbolic geodesic ending to the tip of the curve.

Definition 3.1. If Σ_0 is a collection of probability measures on a metric space X_0, then a random variable $f : X_0 \to \mathbb{R}$ is said to be *tight* or *stochastically bounded* in Σ_0 if and only if for each $\varepsilon > 0$ there is $M > 0$ such that $\mathbb{P}(|f| \leq M) \geq 1 - \varepsilon$ for all $\mathbb{P} \in \Sigma_0$.

Theorem 1.3 follows from the next proposition, which we will prove in Sections 3.2, 3.3 and 3.4.

Proposition 3.2. *If Σ satisfies Condition G2 and $\Sigma_{\mathbb{D}}$ is as in* (7)*, then the following statements hold*

- *The random curve γ is* uniformly transient *in $\Sigma_{\mathbb{D}}$: The random variable N_0 is tight in $\Sigma_{\mathbb{D}}$.*
- *The family of measures $\Sigma_{\mathbb{D}}$ is* tight *in X: There exists $\alpha > 0$ such that $C_{1,\alpha}$ is a tight random variable in $\Sigma_{\mathbb{D}}$.*
- *The family of measures $\Sigma_{\mathbb{D}}$ is* tight *in driving process convergence: There exists $\alpha' > 0$ such that $C_{2,\alpha',T}$ is a tight random variable in $\Sigma_{\mathbb{D}}$ for each $T > 0$.*
- *There exists ψ such that $C_{3,\psi,T,R}$ is a tight random variable in $\Sigma_{\mathbb{D}}$ for each $T > 0$, $R > 0$.*

3.2 Extracting weakly convergent subsequences of probability measures on curves

In this subsection, we first review the results of [2] and then we verify their assumption (which they call hypothesis H1) given that Condition G2 holds. At some point in the course of the proof, we observe that it is nicer to work with a smooth domain such as $\mathbb{D}$, hence justifying the effort needed to prove the equivalence of the conditions.

For the background in the weak convergence of probability measures the reader should see for example [5]. Recall the following definition:

Definition 3.3. Let Y be a metric space and $\mathcal{B}_Y$ the Borel σ-algebra. A collection Π of probability measures on $(Y, \mathcal{B}_Y)$ is said to be *tight* if for each $\varepsilon > 0$ there exist a compact set $K \subset Y$ so that $\mathbb{P}(K) > 1 - \varepsilon$ for any $\mathbb{P} \in \Pi$.

Prohorov's theorem states that a family of probability measures is relatively compact if it is tight, see Theorem 5.1 in [5]. Moreover, in a separable and complete metric space relative compactness and tightness are equivalent.

Define the modulus of continuity of γ by

$$w(\gamma, \delta) = \max\{ |\gamma(t) - \gamma(s)| \ : \ (s,t) \in [0,1]^2 \text{ s.t. } |s-t| \le \delta\}. \tag{37}$$

Denote

$$M(\gamma, l) = \min \left\{ n \in \mathbb{N} : \begin{array}{c} \exists \text{ partition } 0 = t_0 < t_1 < \ldots < t_n = 1 \text{ s.t.} \\ \operatorname{diam}(\gamma[t_{k-1}, t_k]) \le l \text{ for } 1 \le k \le n \end{array} \right\}. \tag{38}$$

Clearly $w(\gamma, \delta)$ depends on the parameterization of γ whereas $M(\gamma, l)$ doesn't.

By Theorem 2.3 of [2] if for a curve γ, there are constants $\alpha > 0$ and $K > 0$ such that

$$M(\gamma, l) \le K\, l^{-\alpha} \tag{39}$$

for any $0 < l < 1$, then for any $0 < \beta < \alpha^{-1}$, there is a parametrization $\hat{\gamma}$ of γ and a constant $\hat{K}$ such that

$$|\hat{\gamma}(t) - \hat{\gamma}(s)| \le \hat{K}\, |t-s|^{\beta} \tag{40}$$

for any $(s,t) \in [0,1]^2$. Here the constant $\hat{K}$ depends on the constants K, α, β but not directly on γ. Conversely, if there exist a parametrization $\hat{\gamma}$ of γ such that the inequality (40) holds for some $\hat{K}, \beta$, then for any $0 < \alpha \le \beta^{-1}$ there is K so that (39) holds. Note that in the latter case, in fact, it is possible to choose $\alpha = \beta^{-1}$ and $K = 2\,(\hat{K})^{\frac{1}{\beta}}$. Hence it is equivalent to consider either the *tortuosity bound* (39) and the *modulus of continuity bound* (40)

The compact subsets $K \subset X$ were characterized in Lemma 4.1 in [2]. A closed set $K \subset X$ is compact if and only if there exists a function $\psi : (0,1] \to (0,1]$ such that

$$M(\gamma, l) \le \frac{1}{\psi(l)}$$

for any $\gamma \in K$ and for any $0 < l \le 1$. And this is equivalent to the existence of parametrization which allow a uniform bound on the modulus of continuity.

Condition G4. A collection of measures Σ_0 on $X(\mathbb{C})$ is said to satisfy a *power-law bound on multiple crossings* if for each n, there are constants $\Delta_n \ge 0$, $K_n > 0$ such that

$$\mathbb{P}\big(\gamma \text{ makes } n \text{ crossings of } A(z_0, r, R)\big) \le K_n \left(\frac{r}{R}\right)^{\Delta_n} \tag{41}$$

for each $\mathbb{P} \in \Sigma_0$ and so that $\Delta_n \to \infty$ as $n \to \infty$.

Remark 3.4. The sequence (Δ_n) can trivially be chosen to be non-decreasing. Hence it is actually enough to check that $\Delta_{n_j} \to \infty$ along a subsequence $n_j \to \infty$. Note also that there is no restriction on the constants K_n.

In the setting where γ is uniformly bounded, i.e., there exists $R > 0$ such that $\mathbb{P}(\gamma \subset B(0, R)) = 1$ for all $\mathbb{P} \in \Sigma_0$, by the result of [2], Condition G4 is a sufficient condition guaranteeing that there exists $\alpha > 0$ such that $\sup_{0<l<1} M(\gamma, l)\, l^\alpha$ is stochastically bounded and therefore that Σ_0 is tight. The following result is the main result of this subsection.

Proposition 3.5. *If Σ satisfies Condition G3, then $\Sigma_{\mathbb{D}}$ satisfies Condition G4. Furthermore, the collection $\Sigma_{\mathbb{D}}$ is tight and there exist $\alpha > 0$ and $\beta > 0$ so that the random variables*

$$Z_\alpha(\gamma) = \sup \left\{ M(\gamma, l) \cdot l^\alpha \;:\; 0 < l < 1 \right\} \tag{42}$$

$$\hat{Z}_\beta(\gamma) = \inf_{\hat{\gamma}} \sup \left\{ w(\gamma, \delta) \cdot \delta^{-\beta} \;:\; 0 < \delta < 1 \right\} \tag{43}$$

are stochastically bounded in $\Sigma_{\mathbb{D}}$. In (43) the infimum is over all reparametrizations $\hat{\gamma}$ of γ.

Let $D_t = \mathbb{D} \setminus \gamma(0, t]$. Let $\tilde{C} > 1$. For an annulus $A = A(z_0, r, \tilde{C}^3 r)$ define three concentric subannuli $A_k = A(z_0, \tilde{C}^{k-1} r, \tilde{C}^k r)$, $k = 1, 2, 3$. Define the *index* $I(A, D_t) \in \{0, 1, 2, \ldots\}$ of γ at time t with respect to A to be the minimal number of crossings of A_2 made by $\tilde{\gamma}$ where $\tilde{\gamma}$ runs over the set of all possible futures of $\gamma[0, t]$

$$\{\tilde{\gamma} \in X_{\text{simple}}(D_t) \;:\; \tilde{\gamma} \text{ connects } \gamma(t) \text{ to } b\}.$$

Consider a sequence of stopping times $\tau_0 = 0$ and

$$\tau_{k+1} = \inf\{t > \tau_k : \gamma[\tau_k, t] \text{ crosses } A\}$$

where $k = 0, 1, 2, \ldots$ Define also $\sigma_0 = 0$ and

$$\sigma_{k+1} = \inf\{t > \sigma_k : \gamma[\sigma_k, t] \text{ crosses } A_2\}.$$

Since $\gamma(\tau_k)$ and $\gamma(\tau_{k+1})$ lie in the different components of $\mathbb{C} \setminus A$, the curve $\gamma[\tau_k, \tau_{k+1}]$ has to cross A_2 an odd number of times. Hence there are odd number of l such that $\tau_k < \sigma_{l+1} < \tau_{k+1}$. For each l, $\gamma[\sigma_l, \sigma_{l+1}]$ crosses A_2 exactly once and therefore the index changes by ± 1. From this it follows that

$$I(A, D_{\tau_{k+1}}) = I(A, D_{\tau_k}) + 2n - 1$$

with $n \in \mathbb{Z}$.

Lemma 3.6. *Let $A = A(z_0, r, R)$ be an annulus. If A is not on $\partial\mathbb{D}$, i.e. $\overline{B(z_0, r)} \cap \partial\mathbb{D} = \emptyset$, then on the event $\tau < 1$, $I(A, D_\tau) = 1$, where τ is the hitting time of $\overline{B(z_0, r)}$.*

If A is on ∂D_s and the index increases $I \mapsto I + 2n - 1$, $n \in \mathbb{N}$, during a minimal crossing $\gamma[s, t]$ of A then the total number of unforced crossings of the annuli A_k, $k = 1, 2, 3$, made by $\gamma[s, t]$ has to be at least $2n - 1$.

Proof. The first claim follows easily from connectedness of $\mathbb{D}$.

Suppose now that A is on ∂D_s. Let $m \leq m'$ be such that

$$\sigma_{m-1} < s < \sigma_m \text{ and } \sigma_{m'} < t < \sigma_{m'+1}.$$

Let $y_l = I(A, D_{\sigma_l}) - I(A, D_{\sigma_{l-1}})$. Notice that $y_l \in \{-1, 1\}$. Now

$$\sum_{l=m}^{m'} y_l = 2n - 1$$

The following properties, which are illustrated in Figure 5, are the key observations of this proof:

- If $y_{m'} = 1$ then the last crossing $\gamma[\sigma_{m'}, t]$ of an component of A_1 or A_3 has to be unforced (in the domain $D_{\sigma_{m'-1}}$).
- If $y_l = 1 = y_{l+1}$ then the latter crossing $\gamma[\sigma_l, \sigma_{l+1}]$ is an unforced crossing of A_2 (in the domain D_{σ_l}).

To prove these claims let's use the symbols in Figure 5. For both of these claims, notice that if $y_l = 1$ then the corresponding crossing of A_2 during the time between σ_{l-1} and σ_l, crosses a component V of $A_2 \cap D_{\sigma_{l-1}}$ which may have both left-hand and right-hand boundary but the two boundary segments which extend across A_2 have to be of the same type. In Figure 5, V is the dark gray area. This implies that the light gray areas, that is, components of A_k, $k = 1, 2, 3$ which are beyond V, have monochromatic boundaries of type H where H is either "left-hand" or "right-hand".

The first claim is immediate after the previous observation: after $\sigma_{m'}$, γ has to cross a component of A_k where $k = 1$ or 3, which then has to be one of the components shown above to have monochromatic boundary and the crossing is therefore unforced. The components which may be crossed in this case are denoted by $\square$ in the figure.

The second claim follows when we notice that the next crossing after σ_l has to increase the index. Hence the component to be crossed has to have the monochromatic boundary of type H. These components are denoted by $\star$ in the figure. The other option would be a crossing of the unique component with both types of boundary which can be crossed before any other component of A_2. This is marked by $\triangle$ in the figure. This option would decrease the index which is in conflict with the assumptions.

To use these properties let's divide proof in two cases $y_{m'} = -1$ and $y_{m'} = 1$.

If $y_{m'} = -1$, then

$$\max_{j=m,\dots,m'} \sum_{l=m}^{j} y_l \geq 2n.$$

Therefore there has to be at least $2n - 1$ pairs $(l, l+1)$ so that $y_l = 1 = y_{l+1}$. This can be easily proven by induction. Hence the claim holds in this case.

If $y_{m'} = 1$, then the are at least $2n - 2$ pairs $(l, l+1)$ so that $y_l = 1 = y_{l+1}$ by the same argument as in the previous case. In addition to this the last crossing $\gamma[\sigma_{m'}, t]$ is unforced crossing of A_1 or A_3. Hence the claim holds also in this case. $\square$

Now we are ready to give the proof of the main result of this section. Notice that here we need that the domain is smooth otherwise the number n_0 below wouldn't be bounded. There are of course ways to bypass this: if we want that the measures are supported on Hölder curves (including the end points on the boundary), then we need to assume that minimal number of crossing of annuli centered at a or b grows as a power of r as $r \to 0$.

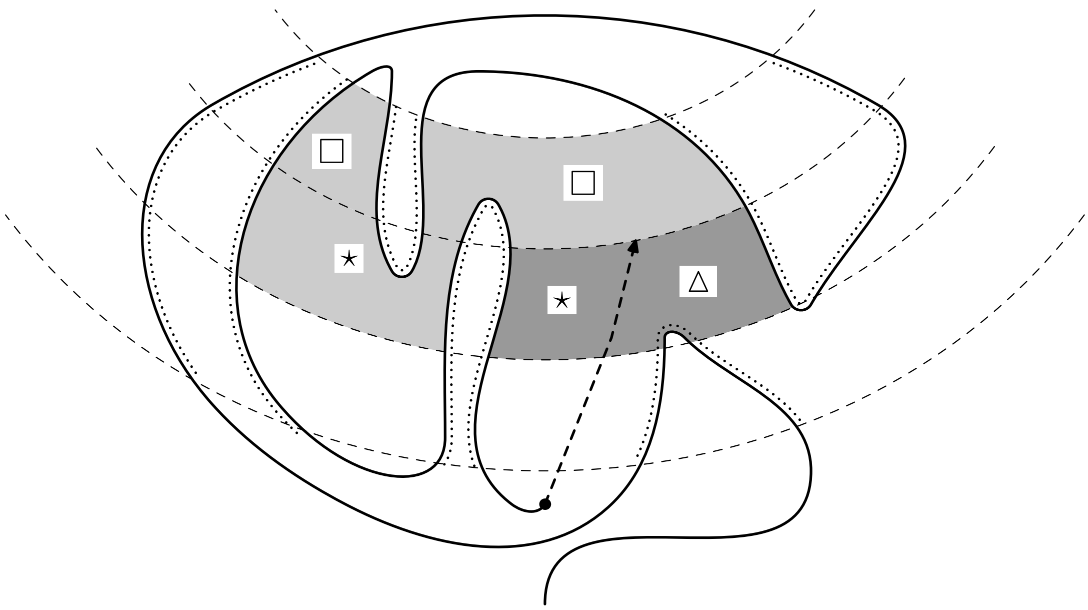


Figure 5: A sector of three concentric annuli with an initial segment of the interface. The boundaries of the annuli are the dashed circular arcs (which are only partly shown in the figure). The curve up to time s is the solid line, $\gamma(s)$ is the dot in the end of that line and the other end of the curve is connected to -1 which lies outside of the picture. Dotted lines indicate the right-hand side of the curve, which is important for the forced/unforced classification of the crossings coming after time s. The dashed arrow is the segment of γ for times between s and σ_l (or $\sigma_{m'}$).

Proof of Proposition 3.5. We will prove the first claim that if Σ satisfies Condition G3, then $\Sigma_{\mathbb{D}}$ satisfies Condition G4. The rest of the theorem follows then from Theorem 1.1 of [2].

First of all, we can concentrate to the case that the variables z_0, r, R are bounded. We can assume that $z_0 \in B(0, 3/2), r < 1/2, R < 1$. In the complementary case either the left-hand side of (41) is zero by the fact that there are no crossing of the annulus that stay inside the unit disc or the ratio r/R is uniformly bounded away from zero. In the latter case the constant K_n can be chosen so that the right-hand side of (41) is grater than one and (41) is satisfied trivially.

Denote as usual $A = A(z_0, r, R)$. By the fact that $R < 1$, at most one of the points ± 1 is in A. If either ± 1 is in A, denote the distance from that point to z_0 by ρ. Then $r < \rho < R$ and a trivial inequality shows that

$$\max\left\{\frac{\rho}{r}, \frac{R}{\rho}\right\} \geq \sqrt{\frac{R}{r}}.$$

Hence for for each annulus, it is possible choose a smaller annulus inside so that the points ± 1 are away from that annulus and the ratio of the radii is still at least square root of the original one. If we are able to show existence of the constants K_n and Δ_n for annuli A such that $\{-1, 1\} \cap A = \emptyset$ then constants $\hat{K}_n = K_n$ and $\hat{\Delta}_n = \Delta_n/2$ can be used for a general annulus.

Let A be such that $\{-1, 1\} \cap A = \emptyset$ and set n_0 and τ in the following way: if $\overline{B(z_0, r)}$ intersects the boundary, let $n_0 = 1$ when $\overline{B(z_0, r)}$ contains -1 or 1 and

$n_0 = 0$ otherwise and let $\tau = 0$. If $\overline{B(z_0, r)}$ doesn't intersect the boundary, let $n_0 = 2$ and let $\tau = \inf\{t \in [0,1] \,:\, \gamma(t) \in \overline{B(z_0, r)}\}$.

By Lemma 3.6, if there is a crossing of A that increases the index, there are unforced crossings of the annuli A_k, $k = 1, 2, 3$. We can apply this result after time τ. If the curve doesn't make any unforced crossings of the annuli A_k, $k = 1, 2, 3$, then there are at most n_0 crossing of A. This argument generalizes so that if there are $n > n_0$ crossing of A, we apply Condition G3 $(n - n_0)/2$ times in the annuli A_k, $k = 1, 2, 3$, to get the bound

$$\mathbb{P}\left(\gamma \text{ makes } n \text{ crossings of } A(z_0, r, R)\right) \leq K^{\frac{n-n_0}{2}} \cdot \left(\frac{r}{R}\right)^{\frac{\Delta}{6}(n-n_0)}$$

for any $\mathbb{P} \in \Sigma_{\mathbb{D}}$. Hence the proposition holds for $\Delta_n = \Delta \cdot (n-2)/12$. $\square$

3.3 Continuity of driving process and finite exponential moment

Let $\Phi : \mathbb{D} \to \mathbb{H}$ be a conformal mapping such that $\Phi(-1) = 0$ and $\Phi(1) = \infty$. To make the choice unique, it is also possible to fix $\Phi(z) = \frac{2i}{1-z} + \mathcal{O}(1)$ as $z \to 1$, i.e.

$$\Phi(z) = i\frac{1+z}{1-z}. \tag{44}$$

Denote by $\Phi_t = \Phi \circ g_t$.

Denote by $W(\,\cdot\,, \Phi\gamma)$ the driving process of $\Phi\gamma$ in the capacity parametrization. Our primary interest is to estimate the tails of the distribution of the increments of the driving process. Let's first study what kind of events are those when $|W(t, \Phi\gamma) - W(s, \Phi\gamma)|$ is large. Let $0 < u < L$ where u/L is small. Consider a hull K that is a subset of a rectangle $R_{L,u} = [-L, L] \times [0, u]$. If $K \cap [L, L + iu] \neq \emptyset$ then for any z in this set, $.9L \leq g_K(z) \leq 1.1L$ as proved below in Lemma A.11. On the other hand if $K \cap [-L + iu, L + iu] \neq \emptyset$ then $\mathrm{cap}_{\mathbb{H}}(K) \geq \frac{1}{4}u^2$. This is proved in Lemma A.13.

Based on this

$$\mathbb{P}\left(\left|W\left(\frac{1}{4}u^2, \Phi\gamma\right)\right| \geq 2L\right) \leq \mathbb{P}\big(\, \mathrm{Re}[(\Phi\gamma)(\tau_{R_{L,u}})] = \pm L \,\big), \tag{45}$$

where $\tau = \inf\{\, t \in [0,1] \;:\; \Phi\gamma(t) \in \mathbb{H} \cap \partial R_{L,u} \,\}$.

Proposition 3.7. *If Condition G2 holds, then there is constants $K > 0$ and $c > 0$ so that*

$$\mathbb{P}\big(\, \mathrm{Re}[(\Phi\gamma)(\tau_{R_{L,u}})] = \pm L \,\big) \leq K e^{-c\frac{L}{u}} \tag{46}$$

for any $0 < u < L$.

Proof. If Condition G2 holds then it also holds in $\mathbb{H}$ by the results of Section 2.2. Let $C > 1$ be the constant of Condition G2 in $\mathbb{H}$.

By symmetry, it is enough to consider the event E that $\Phi\gamma$ exits the rectangle $R_{L,u}$ from the right-hand side $\{L\} \times [0, u]$. Let $n = \lfloor L/(Cu) \rfloor$. Consider the lines $J_k = \{Cu \cdot k\} \times [0, u]$, $k = 1, 2, \ldots, n$. On the event E, each of the lines J_k are hit before $\tau_{R_{L,u}}$ and the hitting times are ordered

$$0 < \tau_{J_1} < \tau_{J_2} < \ldots < \tau_{J_n} \leq \tau_{R_{L,u}} < 1$$

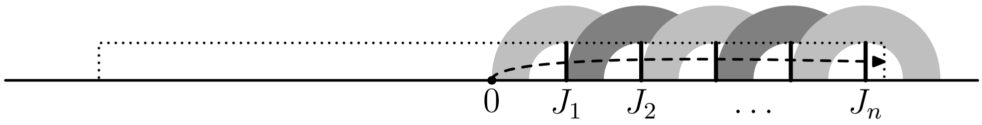


Figure 6: If $\sup\{|W_u - W_s| \ : \ u \in [s,t]\} \geq L$, then the curve $u \mapsto g_s(\gamma(u)) - W_s$, $s \leq u \leq t$, exits the rectangle $[-L, L] \times [0, 2\sqrt{t-s}]$ from one of the sides $\{\pm L\} \times [0, 2\sqrt{t-s}]$. Especially the curve has to intersect all the dashed vertical lines and make an unforced crossing of each of the annuli centered at the base points of those lines.

See Figure 6.

Let $x_k = Cu \cdot k$ which is the base point of J_k. Now especially on the event E the annulus $A(x_1, u, Cu)$ is crossed and after each τ_{J_k} the annulus $A(x_{k+1}, u, Cu)$ is crossed. Hence Condition G2 can be applied with the stopping times $0, \tau_{J_1}, \ldots, \tau_{J_{n-1}}$ and the annuli $A(x_1, u, Cu), A(x_2, u, Cu), \ldots, A(x_n, u, Cu)$. This gives the upper bound 2^{-n} for the probability of E. Hence the inequality (46) follows with suitable constants depending only on C. $\square$

This result can be used to check the Hölder continuity of the driving process for dyadic time intervals. The following properties are enough to guarantee the Hölder continuity. See Lemma 7.1.6 and the proof of Theorem 7.1.5 in [12].

Proposition 3.8. *Let $\upsilon(t) = \mathrm{cap}_{\mathbb{H}}(\Phi\gamma[0,t])/2$ for any $t \in [0,1)$ and define $\upsilon(1) = \lim_{t\to 1} \upsilon(t) \in (0, \infty]$. If Condition G2 holds, then*

1. *For all $\mathbb{P} \in \Sigma_{\mathbb{D}}$, $\mathbb{P}(\upsilon(1) = \infty) = 1$. There exists a sequence $b_n \in \mathbb{R}$ such that*

$$\mathbb{P}\left(\sup_{0 \leq t \leq n} |W_t(\hat{\gamma})| \leq b_n \right) \geq 1 - \frac{1}{n} \tag{47}$$

for any $\mathbb{P} \in \Sigma_{\mathbb{D}}$.

2. *Fix $T > 0$ and $0 < \alpha < \frac{1}{2}$. Let $X' \subset X_{\mathrm{simple}}(\mathbb{D})$ be the set of simple curves such that $\upsilon(1) > T$. Define*

$$G_n = \left\{ \gamma \in X' : \sup_{j 2^{-n} \leq u \leq (j+1) 2^{-n}} |W_u(\hat{\gamma}) - W_{j2^{-n}}(\hat{\gamma})| \leq 2^{-\alpha n} \right\}. \tag{48}$$

Then for large enough $n \geq n_0(\alpha, T, K, c)$

$$\mathbb{P}\big(G_n\big) \geq 1 - 2^{-n}.$$

Proof. 1. Let $b_n = (4/c)\sqrt{n}\log(K\,n)$. Then by (45) and (46)

$$\mathbb{P}\left(\sup_{0 \leq t \leq n} |W_t(\hat{\gamma})| > b_n \right) \leq K \exp\left(-c \frac{b_n}{4\sqrt{n}} \right) = \frac{1}{n}. \tag{49}$$

Especially, $\mathbb{P}(\upsilon(1) = \infty) = 1$.

2. Estimate the probability of the complement of G_n by the following sum

$$\begin{aligned}\mathbb{P}\left(G_n^c\right) &\leq \sum_{j=0}^{2^n} \mathbb{P}\left(\max_{u\in[T(j-1)2^{-n},Tj2^{-n}]} |W_u - W_{T(j-1)2^{-n}}| > 2^{-\alpha n}\right) \\ &\leq K 2^n e^{-(c/4)T^{-1/2}2^{(1/2-\alpha)n}} \leq 2^{-n}\end{aligned}$$

for n large enough depending on α, T, K, c. □

Theorem 3.9. *If Condition G2 holds, then each $\mathbb{P} \in \Sigma_{\mathbb{D}}$ is supported on Loewner chains which are α-Hölder continuous for any $0 < \alpha < 1/2$ and the α-Hölder norm of the driving process restricted to $[0,T]$ for $T > 0$ is stochastically bounded.*

3.4 Continuity of the hyperbolic geodesic to the tip

In the proof of the main theorem, we are going to apply Lemma A.5 of the appendix. Therefore we repeat here the following definition: for a simple curve γ in $\mathbb{H}$, let $(g_t)_{t\in\mathbb{R}_+}$ and $(W(t))_{t\in\mathbb{R}_+}$ be its Loewner chain and driving function. Then we define the *hyperbolic geodesic from ∞ to the tip $\gamma(t)$* as $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \overline{\mathbb{H}}$ by

$$F(t,y) = g_t^{-1}(W(t) + i\, y).$$

The corresponding geodesic in $\mathbb{D}$ for the curve $\Phi^{-1}\gamma$ is

$$F_{\mathbb{D}}(t,y) = \Phi^{-1} \circ F(t,y). \tag{50}$$

Consider now the collection $\Sigma_{\mathbb{D}}$ and the random curve γ in $X_{\text{simple}}(\mathbb{D}, -1, +1)$. Define F and $F_{\mathbb{D}}$ as above for the curves $\Phi\gamma$ and γ, respectively. For $\rho > 0$, let τ_ρ be the hitting time of $B(1,\rho)$, i.e., τ_ρ is the smallest t such that $|\gamma(t) - 1| \leq \rho$. The following is the main result of this subsection.

Theorem 3.10. *Suppose that Σ satisfies Condition G2. There exist a continuous increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$ and for any $\rho > 0$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\mathbb{P}\left(\begin{array}{c}\sup_{t\in[0,\tau_\rho]} |F(t,y') - F(t,y)| \leq \psi(|y-y'|) \\ \forall y,y' \in [0,L] \;\; s.t. \;\; |y-y'| \leq \delta\end{array}\right) \geq 1 - \varepsilon \tag{51}$$

for each $\mathbb{P} \in \Sigma_{\mathbb{D}}$.

The proof is postponed after an auxiliary result, which is interesting in its own right. Namely, the next proposition gives a "superuniversal" arms exponent, i.e., the property is uniform for basically all models of statistical physics: under Condition G2 a certain event involving six crossings of an annulus has small probability to occur *anywhere*. Therefore the corresponding six arms exponent, if it exists, has value always greater than 2.

Let $D_t = \mathbb{D} \setminus \gamma(0,t]$ and define the following event event $E(r,R) = E_\rho(r,R)$ on $X_{\text{simple}}(\mathbb{D})$: Define $E(r,R)$ as the event that there exists $(s,t) \in [0,\tau]^2$ with $s < t$ such that

- diam $\big(\gamma[s,t]\big) \geq R$ and
- there exists a crosscut C, diam$(C) \leq r$, that separates $\gamma(s,t]$ from $B(1,\rho)$ in $\mathbb{D} \setminus \gamma(0,s]$.

Denote the set of such pairs (s,t) by $\mathcal{T}(r,R)$.

Let's first demonstrate that the event $E(r,R)$ is equivalent to a certain six arms event (four arms if it occurs near the boundary) occurring somewhere in $\mathbb{D}$ – the converse is also true, although we don't need it here. If C is as in the definition of $E(r,R)$, then for $r < \min\{\rho, R\}/2$ at least one of the end points of C have to lie on $\gamma(0,s]$. Let $T(C) \leq s$ be the largest time such that $\gamma(T(C)) \in \overline{C}$. Then also $(T(C),t) \in \mathcal{T}(r,R)$ and we easily see that $\gamma[T(C),t]$ makes a crossing which is contained in $(A(\gamma(T(C)),r,R))^u_{T(C)}$. Therefore on the event $E(r,R)$ there is $z_0 \in \mathbb{D}$ such that $A(z_0,r,R)$ contains at least six crossing when $|z_0| < 1-r$ or four crossings when $|z_0| \geq 1-r$ and at least one of the crossings is unforced.

Proposition 3.11. *If $\Sigma_{\mathbb{D}}$ satisfies Condition G2, then as $r \to 0$*

$$\sup\{\mathbb{P}\left(E(r,R)\right) \,:\, \mathbb{P} \in \Sigma_{\mathbb{D}}\} = o(1).$$

Remark 3.12. Since $\mathbb{P}\left(E(r,R)\right)$ is decreasing in R, the bound is uniform for $R \geq R_0 > 0$.

The idea of the proof is the following: divide the curve γ into N arcs

$$J_k = \gamma[\sigma_{k-1}, \sigma_k] \tag{52}$$

$0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = 1$ such that diam$(J_k) \leq R/4$, $k = 1,2,\ldots,N$. Let $J_0 = \partial\mathbb{D}$. For the event $E(r,R)$, firstly there has to exist a fjord of depth R and the mouth of the fjord is formed by some pair (J_j, J_k), $j < k$, and the number of such pairs is less than N^2. Secondly, there has to be piece of the curve which enters the fjord, hence resulting an unforced crossing. Hence (given N^2) the probability that $E(r,R)$ occurs is less than *const.* $\cdot\, N^2 (r/R)^{\Delta}$.

Proof. It is useful to do this by defining σ_k as stopping times by setting $\sigma_k = 0$, $k \leq 0$, and then recursively

$$\sigma_k = \sup\left\{ t \in [\sigma_{k-1}, 1] : \text{diam}\left(\gamma[\sigma_{k-1}, t]\right) < \frac{R}{4} \right\}.$$

Let J_k, $k > 0$, be as in (52) and let $J_0 = \partial\mathbb{D}$. Observe that if the curve is divided into pieces that have diameter at most $R/4 - \varepsilon$, $\varepsilon > 0$, then none of these pieces can contain more than one of the $\gamma(\sigma_k)$. Therefore $N \leq \inf_{\varepsilon > 0} M(\gamma, R/4 - \varepsilon) \leq M(\gamma, R/8)$ and N is stochastically bounded.

Define also stopping times

$$\tau_{j,k} = \inf\{\, t \in [\sigma_{k-1}, \sigma_k] \;:\; \text{dist}(\gamma(t), J_j) \leq 2r\}. \tag{53}$$

for $0 \leq j < k$.

Let $z_0 \in C$. Let V be the connected component of $A(z_0,r,R) \cap D_s$ which is (first, if multiple) crossed by $\gamma[s,t]$. Then there is an unique arc $C' \subset \partial V$ of $\partial B(z_0,r)$

which disconnects V from $\gamma(s)$ (and from $+1$). Let now $j < k$ be such that the end points of C' are in J_j and J_k. Obviously this implies that $\sigma_j < \sigma_k \leq s$.

Since the diameter of C less than r, $\mathrm{dist}(J_j, J_k) \leq 2r$ and $\tau_{j,k}$ is finite. Therefore it is possible to define

$$z_{j,k} = \gamma(\tau_{j,k}). \tag{54}$$

Let $w_{j,k} \in J_j$ be any point in J_j such that $|z_{j,k} - w_{j,k}| = d(z_{j,k}, J_j) \leq 2r$. Let $C'' = [z_{j,k}, w_{j,k}] \cap D_s$. Then C' disconnects C'' from $\gamma(s)$ (and from $+1$) in D_s. Hence it is also clear that $\gamma[s,t]$ has to contain a crossing of $A_{j,k} := A(z_{j,k}, 2r, R/2)$ which is contained in $(A_{j,k})^u_{\tau_{j,k}}$ because going from D_s to $D_{\tau_{j,k}}$ doesn't change anything in that part of V which is disconnected by C'' from $\gamma(s)$ (and from $+1$). Consequently if we define $E_{j,k} = E_{j,k}(r,R)$ as

$$E_{j,k} = \left\{ \gamma \in X_{\mathrm{simple}}(\mathbb{D}, -1, +1) \ : \ \begin{matrix} \gamma[\tau_{j,k}, 1] \text{ contains a crossing of } A_{j,k} \\ \text{which is contained in } (A_{j,k})^u_{\tau_{j,k}} \end{matrix} \right\} \tag{55}$$

we have shown that $E(r,R) \subset \bigcup_{j=0}^{\infty} \bigcup_{k=j+1}^{\infty} E_{j,k}$.

Let $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\mathbb{P}(N > m) \leq \varepsilon/2$ for all $\mathbb{P} \in \Sigma_{\mathbb{D}}$. Now

$$\begin{aligned} \mathbb{P}(E(r,R)) \leq & \mathbb{P}(N > m) + \mathbb{P}\left[\bigcup_{0 \leq j < k} \{N \leq m\} \cap E_{j,k} \right] \\ \leq & \frac{\varepsilon}{2} + \mathbb{P}\left[\bigcup_{0 \leq j < k \leq m} \{N \leq m\} \cap E_{j,k} \right] \\ \leq & \frac{\varepsilon}{2} + \sum_{0 \leq j < k \leq m} \mathbb{P}\left[\{N \leq m\} \cap E_{j,k} \right] \\ \leq & \frac{\varepsilon}{2} + K m^2 \left(\frac{r}{R} \right)^{\Delta} \leq \varepsilon \end{aligned} \tag{56}$$

when r is smaller than $r_0 > 0$ which depends on R and ε. Here we used the facts that $\{N \leq m\} \cap E_{j,k} = \emptyset$ when $k > m$ and that $\mathbb{P}[\{N \leq m\} \cap E_{j,k}] \leq \mathbb{P}[E_{j,k}]$. $\square$

Proof of Theorem 3.10. In this proof, we work on the unit-disc. Fix $\rho > 0$ and let $\tau = \tau_\rho$ as above. Let $D' = \mathbb{D} \setminus \overline{B}(1, \rho)$. Since Φ and Φ^{-1} are uniformly continuous on D' and $\Phi(D')$, respectively, it is sufficient to prove the corresponding claim for $F_{\mathbb{D}}$. Furthermore it is sufficient to show that $|F_{\mathbb{D}}(t,y) - \gamma(t)| \leq \psi(y)$ for $0 < y \leq \delta$, because $y \mapsto F_{\mathbb{D}}(t,y)$, $y \in [\delta, 1]$, is equicontinuous family by Koebe distortion theorem.

Let $R_n > 0$, $n \in \mathbb{N}$, be any sequence such that $R_n \searrow 0$ as $n \to \infty$. By the previous proposition, we can choose a sequence r_n, $n \in \mathbb{N}$, such that $r_n < R_n$ and

$$\mathbb{P}(E(r_n, R_n)) \leq 2^{-n} \tag{57}$$

for all $n \in \mathbb{N}$ and for all $\mathbb{P} \in \Sigma_{\mathbb{D}}$. Therefore the random variable $N := \max\{n \in \mathbb{N} : \gamma \in E(r_n, R_n)\}$ is tight: for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\mathbb{P}(N \leq m) \geq 1 - \varepsilon \tag{58}$$

for all $\mathbb{P} \in \Sigma_{\mathbb{D}}$. Fix now $\varepsilon > 0$ and let $m \in \mathbb{N}$ be such that (58) holds.

Define $n_0(\delta)$ to be the maximal integer such that the inequality

$$\frac{2\pi}{\sqrt{|\log \delta|}} \leq r_{n_0(\delta)} \tag{59}$$

holds. For given $0 < \delta < 1$, there is a $\delta' \in [\delta, \delta^{1/2}]$ which can depend on t and γ such that the crosscut $C := \{\Phi^{-1} \circ g_t^{-1}(W(t) + i\delta' e^{i\theta}) \, : \, \theta \in (0, \pi)\}$ has length less than $2\pi/\sqrt{|\log \delta|}$, see Proposition 2.2 in [23].

Now if $N > n_0(\delta)$, then there must be a path from $w := \Phi^{-1} \circ g_t^{-1}(W(t) + i\delta')$ to $\gamma(t)$ in D_t that has diameter less than $R_{n_0(\delta)}$. By Gehring-Hayman theorem (Theorem 4.20 in [23]) the diameter of the hyperbolic geodesic $y \mapsto F_{\mathbb{D}}(t, y)$, $0 \leq y \leq \delta'$, is of the same order as the smallest possible diameter of a curve connecting w and $\gamma(t)$. Consequently there is a universal constant $c > 0$ such that

$$\operatorname{diam}\{F_{\mathbb{D}}(t, y) \, : \, y \in [0, \delta]\} \leq c R_{n_0(\delta)} \tag{60}$$

for all $t \in [0, \tau]$, for all $\delta > 0$ such that $n_0(\delta) > m$ and for all γ such that $N \leq m$. $\square$

3.5 Proof of the main theorem

Proof of Theorem 1.3 (Main theorem). Fix $\varepsilon > 0$. We will first choose four event E_k, $k = 1, 2, 3, 4$, that have large probability, namely,

$$\mathbb{P}(E_k) \geq 1 - \varepsilon/4 \tag{61}$$

for all $\mathbb{P} \in \Sigma_{\mathbb{D}}$. Then those events have large probability occurring simultaneously since

$$\mathbb{P}\left(\bigcap_{k=1}^{4} E_k \right) \geq 1 - \varepsilon. \tag{62}$$

Once we have defined E_k, denote $E = \bigcap_{k=1}^{4} E_k$.

We choose E_1 in such a way that the half-plane capacity of $\gamma[0, t]$ goes to infinity as $t \to \infty$ in a tight way on $\Sigma_{\mathbb{D}}$. We use Proposition 3.8 and choose E_1 the intersection of the events in the inequality (47) where $n = k^2$ runs from $k = m_1$ to ∞ where m_1 is chosen so that (61) holds. Then we choose E_2 and E_3 using Proposition 3.5 and Theorem 3.9 so that E_2 is the set of Hölder continuous, simple curves with Hölder exponent $\alpha_c > 0$ with a Hölder constant K_c which are chosen to satisfy (61) and E_3 is the set of simple curves with Hölder continuous driving process which has Hölder exponent $\alpha_d > 0$ with a Hölder constant K_d which are chosen to satisfy (61). Finally using Theorem 3.10 we set E_4 to be the set of simple curves that have function ψ as in Theorem 3.10 and $\delta > 0$ such that the geodesic to the tip is continuous with $|F(t, y) - F(t, y')| \leq \psi(|y - y'|)$ for $|y - y'| < \delta$. Also here ψ and $\delta > 0$ are chosen so that (61) holds. Now the rest of the claims follow from Lemma A.5 of the appendix. $\square$

3.6 The proofs of the corollaries of the main theorem

In this section we will prove Corollaries 1.5 and 1.6.

Proof of Corollary 1.5. If $\gamma^{(n)}$ satisfy Condition G2, then by (the proof of) Theorem 1.3 the sequence of pairs $(\gamma^{(n)}, W^{(n)})_{n\geq 0}$ is tight. If one of them converge, by tightness we can choose a subsequence so that the other one also converges and by Theorem 1.3 the limits agree in the sense that the limiting curve is driven by the limiting driving process. Since the limit is unique (and given by the limit of the one which originally converged), we don't need to take a subsequence as the entire sequence converges. □

For the proof of Corollary 1.6 notice first that by the proof of C3⇒G2 in Section 2.2 we have constants C_1, C_2 such that if $Q \subset U$ is a simply connected domain, whose boundary consists of a subset of ∂U and some subsets of U which are crosscuts S_0 and S_2^j, $j = 1, 2, \ldots$, (finite or infinite set), and if Q has the property that it doesn't disconnect a from b and S_0 is the "outermost" of the crosscuts (disconnecting the others from a and b), then

$$\mathbb{P}(\gamma \text{ crosses } Q) \leq C_1 \exp(-C_2 m(Q)) \tag{63}$$

where crossing means that γ intersects one of the S_2^j's and $m(Q)$ is the extremal length of the curve family connecting S_0 to $\bigcup_j S_2^j$. Use the notation $S_0(Q)$ for the outermost crosscut and $\mathcal{S}_2(Q)$ for the collection of S_2^j, $j = 1, 2, \ldots$.

Lemma 3.13. *Let $(U, a, b, \mathbb{P})$ be a domain and a measure such that* (63) *with some C_1 and C_2 is satisfied for all Q as above. Then for each $\varepsilon > 0$ and $R > 0$ there is δ which only depends on C_1, C_2, ε, R and* area(U) *such that the following holds. Let Q_j, $j \in I$ be a collection of quadrilaterals satisfying the conditions above such that* $\operatorname{diam}(S_0(Q_j)) < \delta$ *for all j and the length of the shortest path from $S_0(Q_j)$ to $\mathcal{S}_2(Q_j)$ is at least R. Then*

$$\sum_{j\in I} \mathbb{P}(\gamma \textit{ crosses } Q_j) \leq \varepsilon. \tag{64}$$

Proof. Take any δ-ball $B(z_j, \delta)$ that contains the crosscut $S_j := S_0(Q_j)$. The standard estimate of extremal length in Lemma 2.11 gives that

$$m(Q_j) \geq \frac{\log \frac{R}{\delta}}{2\pi}. \tag{65}$$

We claim also that

$$m(Q_j) \geq \frac{(R-\delta)^2}{A_j}. \tag{66}$$

To prove the second inequality fix j for the time being. Let

$$\eta(r) = \{z \in \mathbb{C} \,:\, |z - z_j| = r, z \in Q_j\}$$

Define a metric $\rho : \mathbb{C} \to \mathbb{R}_+$ by setting $\rho(z) = 1/\Lambda(\eta(r))$, if $z \in \eta(r)$, and $\rho(z) = 0$, otherwise. Then for any crossing γ of Q_j

$$\text{length}_\rho(\gamma) \geq \int_\delta^R \frac{\mathrm{d}r}{\Lambda(\eta(r))} \tag{67}$$

$$\text{area}(\rho) = \int_\delta^r \Lambda(\eta(r)) \frac{\mathrm{d}r}{\Lambda(\eta(r))^2} = \int_\delta^R \frac{\mathrm{d}r}{\Lambda(\eta(r))} \tag{68}$$

Now the claim follows from the Cauchy-Schwarz inequality

$$\int_\delta^R \frac{\mathrm{d}r}{\Lambda(\eta(r))} A_j \geq \int_\delta^R \frac{\mathrm{d}r}{\Lambda(\eta(r))} \int_\delta^R \Lambda(\eta(r))\mathrm{d}r \geq \left(\int_\delta^R \mathrm{d}r \right)^2 = (R-\delta)^2 \tag{69}$$

and the lower bound $m(Q_j) \geq \inf_\gamma \mathrm{length}_\rho(\gamma)^2/\mathrm{area}(\rho)$.

Fix some $\varepsilon > 0$. Let $I_1 \subset I$ be the set of all $j \in I$ such that $A_j \geq \delta^{\frac{C_2}{4\pi}}$. Then since Q_j are disjoint, the number of elements in I_1 is at most $\mathrm{area}(U)\delta^{-\frac{C_2}{4\pi}}$

$$\begin{aligned} \sum_{j\in I_1} \mathbb{P}(\gamma \text{ crosses } Q_j) &\leq C_1 \sum_{j \in I_1} \exp\left(-C_2 \frac{\log \frac{R}{\delta}}{2\pi} \right) \\ &= C_1 \,\mathrm{area}(U)\, R^{-\frac{C_2}{2\pi}}\, \delta^{\frac{C_2}{4\pi}} \leq \frac{\varepsilon}{2} \end{aligned}$$

when δ is small, more precisely, when $0 < \delta < \delta_1$ where δ_1 depends on C_1, C_2, $\mathrm{area}(U)$, R and ε only.

On the other hand, on $I \setminus I_1$, $A_j < \delta^{\frac{C_2}{4\pi}}$ and therefore

$$\sum_{j\in I\setminus I_1} \mathbb{P}(\gamma \text{ crosses } Q_j) \leq C_1 \sum_{j\in I\setminus I_1} \exp\left(-C_2 \frac{(R-\delta)^2}{A_j} \right) \leq C_1 \sum_{j\in I\setminus I_1} A_j^2 \tag{70}$$

$$\leq C_1\, \delta^{\frac{C_2}{4\pi}} \sum_{j\in I\setminus I_1} A_j \leq C_1 \,\mathrm{area}(U)\, \delta^{\frac{C_2}{4\pi}} \leq \frac{\varepsilon}{2} \tag{71}$$

for $0 < \delta < \delta_2$ where $\delta_2 = \delta_2(C_1, C_2, R, \mathrm{area}(U), \varepsilon)$. Here we used that $\exp(-\tilde{C}/x) < x^2$ when $0 < x < x_0(\tilde{C})$. □

Suppose now that (U_n, a_n, b_n) converges in the Carathéodory sense to (U, a, b). We call a subset V of U_n a (δ, R)-*fjord* if it is a connected component of $U_n \setminus S$ for some crosscut S of U_n such that $\mathrm{diam}(S) \leq \delta$, S disconnects V from a_n and b_n and the set of points $z \in V$ such that $\mathrm{dist}_{U_n}(z, S) \geq R$ is non-empty, where dist_{U_n} is the distance inside U_n, i.e., the length of the shortest path connecting the two sets. The crosscut S is called the *mouth* of the fjord.

Proof of Corollary 1.6. By the assumptions $U_n \subset B(0, M)$, for some $M > 0$.

The precompactness of the family of measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ when restricted outside of neighborhoods of a_n and b_n follows from the results of Section 3.2. So it sufficient to establish that the subsequential measures are supported on the curves of U (when restricted outside of the neighborhoods of a and b).

Fix $0 < \delta_1 < 1/2$. For $\delta > 0$ small enough and for all n there is a (unique) connected component of the open set

$$\phi_n^{-1} \left(\mathbb{D} \cap (B(-1, \delta_1) \cup B(1, \delta_1)) \right) \cup \{ z \,:\, \mathrm{dist}(z, \partial U_n) > \delta \} \tag{72}$$

which contains the corresponding neighborhoods of a_n and b_n. Call it $\hat{U}_n^\delta$. For $R > 0$ define

$$P(R, \delta, n) = \mathbb{P}(\exists t \in [0,1] \text{ s.t. } \mathrm{dist}_{U_n}(\gamma(t), \hat{U}_n^\delta) \geq 2R). \tag{73}$$

Suppose now that the event in (73) happens then γ has to enter one of the $(3\delta, R)$-fjords in depth R at least. By approximating the mouths of the fjords from outside by curves in 3δ-grid (either real or imaginary part of the point on the curve belongs to $3\delta\mathbb{Z}$) and by exchanging some parts of curves if they intersect, we now define a finite collection of fjords with mouths S_j on the grid which are pair-wise disjoint. And the event in (73) implies that γ enters one of these fjords to depth R at least. Denote the set of points in the fjord of S_j that are at most at distance R to S_j by Q_j.

Now by Lemma 3.13, for each $\varepsilon > 0$ and $R > 0$, there exists δ_0 which is independent of n such that for each $0 < \delta < \delta_0$,

$$P(R, \delta, n) \leq \sum_j \mathbb{P}_n(\gamma \text{ crosses } Q_j) \leq \varepsilon \tag{74}$$

Choose sequences $\varepsilon_m = 2^{-m}$, $R_m = 2^{-m}$ and $\delta_m \searrow 0$ such that this estimate is satisfied. Then we see that the sum $\sum_{m=1}^{\infty} P(R_m, \delta_m, n)$ is uniformly convergent for all n. Hence by the Borel–Cantelli lemma for any subsequent measure $\mathbb{P}^*$, the curve γ restricted outside δ_1 neighborhoods of a and b stay in the closure of

$$\bigcup_{\delta > 0} \lim_{n \to \infty} \hat{U}_n^\delta \setminus \phi_n^{-1} \left(\mathbb{D} \cap (B(-1, \delta_1) \cup B(1, \delta_1)) \right) \tag{75}$$

which gives the claim. □

4 Interfaces in statistical physics and Condition G2

In this section, prove (or in some cases survey the proof) that the interfaces in the following models satisfy Condition G2:

- Fortuin–Kasteleyn model with the parameter value $q = 2$, a.k.a. FK Ising, at criticality on the square lattice or on a isoradial graph
- Site percolation at criticality on the triangular lattice
- Harmonic explorer on the hexagonal lattice
- Loop-erased random walk on the square lattice.

We also comment why Condition G2 fails for uniform spanning tree.

4.1 Fortuin–Kasteleyn model

In Section 4.1.1 we define the FK model or random cluster model on a general graph and state the FKG inequality which is needed when verifying Condition G2. Then in Sections 4.1.2–4.1.5 we define carefully the model on the square lattice. As a consequence it is possible to define the interface as a *simple* curve and the set of domains is *stable* under growing the curve. Neither of these properties is absolutely necessary but the former was a part of the standard setup that we chose to work in and the latter makes the verification of Condition G2 slightly easier. Finally, in Section 4.1.6 we prove that Condition G2 holds for the critical FK Ising model on the square lattice.

4.1.1 FK Model on a general graph

Suppose that $G = (V(G), E(G))$ is a finite graph, which is allowed to be a multigraph, that is, more than one edge can connect a pair of vertices. For any $q > 0$ and $p \in (0, 1)$, define a probability measure on $\{0, 1\}^{E(G)}$ by

$$\mu_G^{p,q}(\omega) = \frac{1}{Z_G^{q,p}} \left(\frac{p}{1-p}\right)^{|\omega|} q^{k(\omega)} \tag{76}$$

where $|\omega| = \sum_{e \in E(G)} \omega(e)$, $k(\omega)$ is the number of connected components in the graph $(V(G), \omega)$ and $Z_G^{p,q}$ is the normalizing constant making the measure a probability measure. This random edge configuration is called the *Fortuin-Kasteleyn model* (FK) or the *random cluster model.*

Suppose that there is a given set $E_W \subset E(G)$ which is called the set of *wired edges.* Write $E_W = \bigcup_{i=1}^n E_W^{(i)}$ where $(E_W^{(i)})_{i=1,2,\ldots,n}$ are the connected components of E_W. Let P be a partition of $\{1, 2, \ldots, n\}$. In the set

$$\Omega_{E_W} = \{\{0,1\}^{E(G)} : \omega(e) = 1 \text{ for any } e \in E_W\} \tag{77}$$

define a function $k_P(\omega)$ to be the number of connected components in $(V(G), \omega)$ counted in a way that for any $\pi \in P$ all the connected components $E_W^{(i)}$, $i \in \pi$, are counted to be in the same connected component. The reader can think that for each $\pi \in P$ we add a new vertex v_π to $V(G)$ and connect v_π to a vertex in every $E_W^{(i)}$, $i \in \pi$, by an edge which we then add also to E_W and hence in the new graph there are exactly $|P|$ connected components of the wired edges and each of those components contain exactly one v_π. Call these new graph $\hat{G}$ and $\hat{E}_W$. Now the random-cluster measure with wired edges is defined on Ω_{E_W} to be

$$\mu_{G,E_W,P}^{p,q}(\omega) = \frac{1}{Z_{G,E_W,P}^{p,q}} \left(\frac{p}{1-p}\right)^{|\omega|} q^{k_P(\omega)}. \tag{78}$$

It is easy to check that if $\hat{G} \setminus \hat{E}_W$ is defined to be the graph obtained when each component of $\hat{E}_W$ is suppressed to a single vertex v_π (all the other edges going out of that set are kept and now have v_π as one of their ends) then we have the identity

$$\mu_{G,E_W,P}^{p,q}(\omega) = \mu_{\hat{G} \setminus \hat{E}_W}^{p,q}(\omega') \tag{79}$$

where ω' is the restriction of ω to $E(G) \setminus E_W$. Therefore the more complicated measure (78) with wired edges can always be returned to the simpler one (76). If E_W is connected, then there is only one partition and we can use the notation $\mu_{G,E_W}^{p,q}$. Sometimes we omit some of the subscripts if they are otherwise known.

A function $f : \{0,1\}^{E(G)} \to \mathbb{R}$ is said to be *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega(e) \leq \omega'(e)$ for each $e \in E(G)$. A function f is *decreasing* if $-f$ is increasing. An event $F \subset \{0,1\}^{E(G)}$ is increasing or decreasing if its indicator function $\mathbb{1}_F$ is increasing or decreasing, respectively.

A fundamental property of the FK models is the following inequality.

Theorem 4.1 (FKG inequality)**.** *Let $q \geq 1$ and $p \in (0,1)$ and let $G = (V(G), E(G))$ be a graph. If f and g are increasing functions on $\{0,1\}^{E(G)}$ then*

$$\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g) \tag{80}$$

where $\mathbb{E}$ is expected value with respect to $\mu_G^{p,q}$.

Remark 4.2. As explained above the measure $\mu_G^{p,q}$ can be replaced by any measure conditioned to have wired edges.

For the proof see Theorem 3.8 in [13]. The edges where $\omega(e) = 1$ are called *open* and the edges where $\omega(e) = 0$ are called *closed*. The property (80) is called positive association and it means essentially that knowing that certain edges are open increases the probability for the other edges to be open.

It is well known that the FK model with parameter q is connected to the Potts model with parameter q. Here we are interested in the model connected to the Ising model and hence we mainly focus to the case $q = 2$ which is called *FK Ising* (model).

4.1.2 Modified medial lattice

Consider the planar graph $(\mathbb{Z}^2)_{\text{even}}$ formed by the set of vertices $\{(i, j) \in \mathbb{Z}^2 : i + j \text{ even}\}$ and the set of edges so that (i, j) and (k, l) are connected by an edge if and only if $|i - k| = 1 = |j - l|$. Similarly define $(\mathbb{Z}^2)_{\text{odd}}$ which can be seen as a translation of $(\mathbb{Z}^2)_{\text{even}}$ by the vector $(1, 0)$, say. Both $(\mathbb{Z}^2)_{\text{even}}$ and $(\mathbb{Z}^2)_{\text{odd}}$ are square lattices. Figure 7(a) shows a chessboard coloring of the plane. In that figure, the vertices of $(\mathbb{Z}^2)_{\text{even}}$ are the centers of the blue squares, say, and the vertices of $(\mathbb{Z}^2)_{\text{odd}}$ are the centers of the red squares, and two vertices (of the same color) are connected by an edge if the corresponding squares touch by corners. Note also that $(\mathbb{Z}^2)_{\text{even}}$ and $(\mathbb{Z}^2)_{\text{odd}}$ are the dual graphs of each other.

Let $\hat{L} = (\mathbb{Z} + 1/2)^2$, i.e., the graph formed by the vertices and the edges of the colored squares in the chessboard coloring. It is called the *medial lattice* of $(\mathbb{Z}^2)_{\text{even}}$ and its dual $(\mathbb{Z}^2)_{\text{odd}}$. Note that vertices of $\hat{L}$ are exactly those points where an edge of $(\mathbb{Z}^2)_{\text{even}}$ and an edge of $(\mathbb{Z}^2)_{\text{odd}}$ intersect.

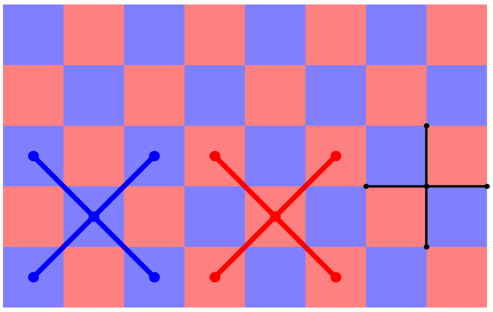
It is useful to modify the medial lattice slightly. At each vertex of $\hat{L}$ position a white square so that the corners are lying on the edges of $\hat{L}$. The size of the square can be chosen so that the resulting blue and red octagons are regular. See Figure 7(b). Denote the graph formed by the vertices and the edges of the octagons by L and call it *modified medial lattice* of $(\mathbb{Z}^2)_{\text{even}}$ (or $(\mathbb{Z}^2)_{\text{odd}}$). The dual of L, i.e. the blue and red octagons and the white squares (or rather their centers), is called the *bathroom tiling*.

Similarly, it is possible to define the modified medial lattice of a general planar graph G. For each middle point of an edge put a vertex of $\hat{L}$. Go around each vertex of G and connect any vertex of $\hat{L}$ to its successor by an edge. The resulting graph is the medial graph. Notice that each vertex has degree four and hence it is possible to replace each vertex by an quadrilateral. The result is the modified medial lattice.

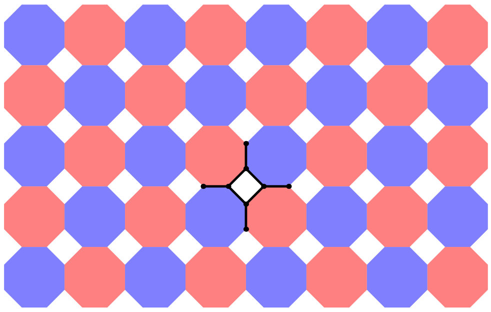
4.1.3 Admissible domains

Suppose that we are given two paths $(c_j(k))$, $j = 1, 2$, on the modified medial lattice and k runs over the values $0, 1, \ldots, n_j$, that satisfy the following properties:

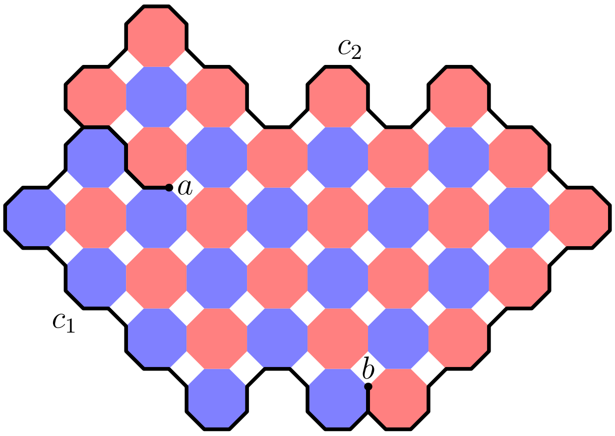
- Each c_j is simple and has only blue and white faces of the bathroom tiling on its one side and red and white faces on the other side.
- The first (directed) edges $(c_j(0), c_j(1))$ coincide and the edge is between a blue and a red face. Denote by a the common starting point of c_j, $j = 1, 2$.



(a) The chessboard coloring holds within three square lattices: $(\mathbb{Z}^2)_{\text{even}}$ (blue dots and lines), $(\mathbb{Z}^2)_{\text{odd}}$ (red dots and lines) and the medial lattice $\hat{L}$ (black dots and lines).



(b) The modified medial lattice L is formed when every vertex of $\hat{L}$ is replaced by a square. The dual lattice of L is called bathroom tiling for obvious reasons.



(c) An admissible domain: here c_1 and c_2 agree on the beginning and end and they are otherwise avoiding each other and the domain they cut from the bathroom tiling has boundary consisting of two monochromatic arcs.

Figure 7: Modified medial lattice and its admissible domain.

- The last edges $(c_j(n_j - 1), c_j(n_j))$ coincide and the edge is between a blue and a red face. Denote by b the common ending point of c_j, $j = 1, 2$.
- The paths c_j may have arbitrarily long common beginning and end parts, but otherwise they are avoiding each other.
- The unique connected component of $\mathbb{C} \setminus \bigcup \hat{c}_j$ which is bounded, has a and b on its boundary, where $\hat{c}_j$ is the locus of the polygonal line corresponding to vertices $c_j(k)$, $0 \le k \le n_j$. Denote this component by $U = U(c_1, c_2)$.

Let's call a pair (c_1, c_2) satisfying these properties an *admissible boundary* and $U = U(c_1, c_2)$ is called *admissible domain*. Let's use a shortened notation that U contains the information how c_1 and c_2 or a and b are chosen.

Suppose that $(\gamma(k))_{0 \le k \le l}$ is a path on the modified medial lattice that starts from a and possibly ends to b but is otherwise avoiding c_1 and c_2. Suppose also that γ has the property that it has only blue and white faces on one side and only red and white faces on the other side. Call this kind of path *admissible path*. If

$\gamma(k)$, $0 \leq k \leq 2m < l$ and c_j are concatenated in a natural way (they have only one common point a) as a curve from $\gamma(2m)$ to b and this curve is denoted as $c_{j,2m}$, then the pair $(c_{1,2m}, c_{2,2m})$ is an admissible boundary.

Later it will be useful to consider the following object. Define *generalized admissible domain of* $2n$ *marked boundary points* or simply $2n$*-admissible domain* as the $U(c_1, c_2, \ldots, c_{2n})$ as the bounded component of $\mathbb{C} \setminus (\hat{c}_1 \cup \ldots \cup \hat{c}_{2n})$ where c_j are simple paths on L so that c_{2k-1} and c_{2k} agree on the beginning and c_{2k} and c_{2k+1} on the end (here use cyclic order so that $c_{2n+1} = c_1$) and otherwise as above, especially the marked point $c_1(0), c_1(n_1), c_3(0), c_3(n_3), \ldots$ are on the boundary of $U(c_1, c_2, \ldots, c_{2n})$.

4.1.4 Advantages of the definitions

Now the advantages of the above definitions are the following:

- It is easier to deal with simple curves on the discrete level. This is the primary motivation of looking the *modified* medial lattice.
- As note above, if we start from an admissible boundary and follow an admissible curve, then the pair $(c_{1,2m}, c_{2,2m})$ stays as an admissible boundary. It is practical to have a *stable* class of domains in that sense.
- Let $(\pi(t)_{0 \leq t \leq l})$ be the polygonal curve corresponding to $(\gamma(k))$ so that $\pi(k) = \gamma(k)$ and the parametrization is linear on the intervals $[k, k+1]$. Then the points of π are bounded away from the boundary $\partial U = \hat{c}_1 \cup \hat{c}_2$ except near the end points, that is,

  $$\mathrm{d}(\pi(t), \partial U) \geq 2\eta \quad \text{when } 1 \leq t \leq l-1$$

  and similarly the points on

  $$\mathrm{d}(\pi(t), \pi(s)) \geq 2\eta \quad \text{when } |t-s| \geq 1$$

  here $\eta > 0$ is a constant depending on the lattice. Later, we can use this to deal with the scales smaller than η when checking the condition.

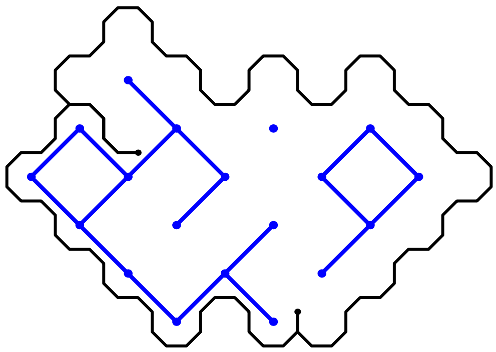
4.1.5 FK model on the square lattice

Let $U = U(c_1, c_2)$ be an admissible domain and assume that the octagons along c_1 (inside U away from the common part with c_2) are blue. Denote by $G = G(c_1, c_2) \subset (\mathbb{Z}^2)_{\text{even}}$ the graph formed by the centers of the blue octagons inside U and by E_W the blue edges along c_1. E_W is connected and it will be the set of wired edges. Let G' be the planar dual of G, that is, the graph formed by the centers of the red octagons inside U. Let G_L be the subgraph of L formed by the vertices in $U \cup \{a, b\}$ and edges which stay inside U.

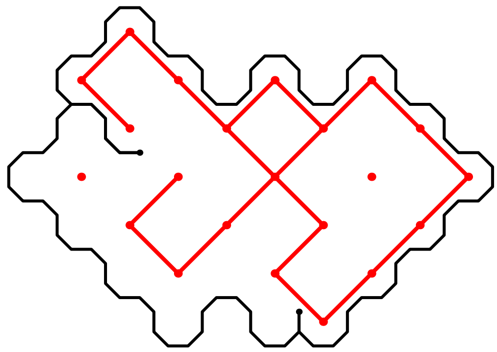
For each $0 < p < 1$, $q > 0$, define a probability measure on $\Omega = \Omega(c_1, c_2) = \{\omega \in \{0,1\}^{E(G)} : \omega = 1 \text{ on } E_W\}$ by

$$\mu_U^{p,q} = \mu_{G,E_W}^{p,q} \tag{81}$$

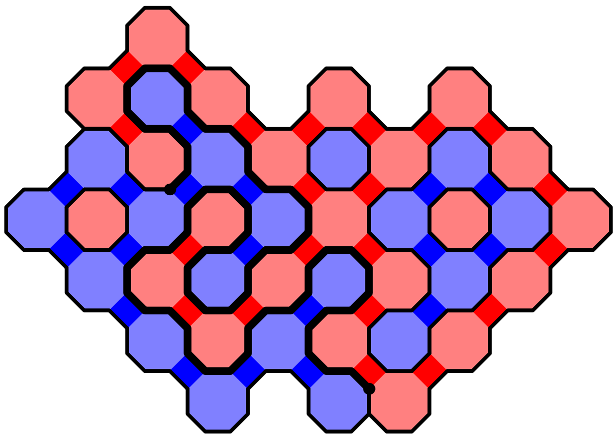
The setup is illustrated in Figure 8. There is a natural dual ω' of ω defined on $E(G')$ such that for each white square in U the edge going through that square is



(a) A configuration of open edges on G satisfying the wired boundary condition along c_1.



(b) The corresponding dual configuration of open edges on G'. Note that it is wired along c_2.



(c) Coloring of the squares with blue and red enables to define the collection of interfaces which separate the blue and red regions.

Figure 8: The correspondence between the configuration on G (a), the configuration on G' (b) and the interfaces and the coloring of the squares (c).

open in ω' if and only if the edge of $E(G)$ going through that square is closed in ω. The duality between ω and ω' is shown in Figures 8(a) and 8(b). Which of the two edges intersecting in a white square is open in ω or ω' can be represented by coloring the square with that color. The picture then looks like Figure 8(c). The essential information of that picture is encoded in the set of interfaces, that is, one interface starting from and ending to the boundary, because of the boundary conditions, and several *loops*. These interfaces are separating open cluster of ω from open cluster of ω'. Moreover, there is one-to-one correspondence between ω, ω' and the interface picture. The random curve connecting a and b in the interface picture is denoted by γ and its law by $\mathbb{P}_U$ when the values of p and q are otherwise known.

It is generally known that the probability measure $\mu_U^{p,q}$ can be written in the form

$$\mu_U^{p,q}(\omega) = \frac{1}{Z'} \left(\frac{p}{(1-p)\sqrt{q}} \right)^{|\omega|} (\sqrt{q})^{\text{number of loops}}. \tag{82}$$

From this it follows that

$$p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}} \tag{83}$$

is a self-dual value of p. When $p = p_{\mathrm{sd}}$, the quantity inside the first brackets is equal to 1, and it doesn't make difference whether the model was originally defined in G or G'. Both give the same probability for the configuration of Figure 8(c). It turns out that the self-dual value $p = p_{\mathrm{sd}}$ is also the critical value at least for $q \geq 1$, see [4].

4.1.6 Verifying Condition G2 for the critical FK Ising

For each admissible domain U (and for each choice of a and b) define a conformal and onto map $\phi_U : U \to \mathbb{D}$ such that $\phi_U(a) = -1$ and $\phi_U(b) = 1$. In this subsection, the following result will be proven.

Proposition 4.3. *Let $\mathbb{P}_U$ be the law of the critical FK Ising interface in U, i.e., $\mathbb{P}_U$ is the law of γ under $\mu_U^{p_{\mathrm{sd}},2}$. Then the collection*

$$\Sigma_{\text{FK Ising}} = \{(\phi_U, \mathbb{P}_U) \;:\; U \text{ admissible domain}\} \tag{84}$$

satisfies Condition G2.

Remark 4.4. In a typical application, a sequence U_n of admissible domains and a sequence of positive numbers h_n are chosen. Then the family

$$\Sigma = \{\delta_{n,*}\,(\phi_{U_n}, \mathbb{P}_U) \;:\; n \in \mathbb{N}\} \tag{85}$$

also satisfies Condition G2, where $\delta_{n,*}$ is the pushforward map of the scaling $z \mapsto h_n\, z$. The scaling factors h_n play no role in checking Condition G2.

We postpone the proof of Proposition 4.3 until the required tools have been presented.

Consider a 4-admissible domain $U = U(c_1, c_2, c_3, c_4)$ such that c_1 and c_3 are wired arcs. Let the marked points be a_j, $j = 1,2,3,4$, in counterclockwise direction and assume that a_1 and a_2 lie on c_1 and a_3 and a_4 lie on c_3. Then there is a unique conformal mapping ϕ_U from U to $\mathbb{H}$ such that $b_j = \phi(a_j) \in \mathbb{R}$ satisfy

$$b_1 < b_2 < b_3 < b_4, \quad b_2 - b_1 = b_4 - b_3, \quad b_2 = -1 \quad \text{and} \quad b_3 = 1.$$

A sequence of domains U_n is said to *converge in the Carathéodory sense* if the mappings $\phi_{U_n}^{-1}$ converge uniformly in the compact subsets of $\mathbb{H}$.

Denote $O(U)$ the event that there is a open crossing of a 4-admissible domain U.

Proposition 4.5. *Let $U_n = h_n \hat{U}_n$ be a sequence domain such that the sequence of reals $h_n \searrow 0$ and $\hat{U}_n$ is a sequence of 4-admissible domains. If the sequence U_n converges to a quadrilateral (U, a, b, c, d) in the Carathéodory sense as $n \to \infty$, then $\mathbb{P}_n[O(\hat{U}_n)]$ converges to a value $s \in [0,1]$. If (U, a, b, c, d) is non-degenerate then $0 < s < 1$. Here $\mathbb{P}_n$ is the probability measure $\mu_{\hat{U}_n,P}^{p_{\mathrm{sd}},2}$ where P is a fixed partitioning of the set $\{1,2\}$.*

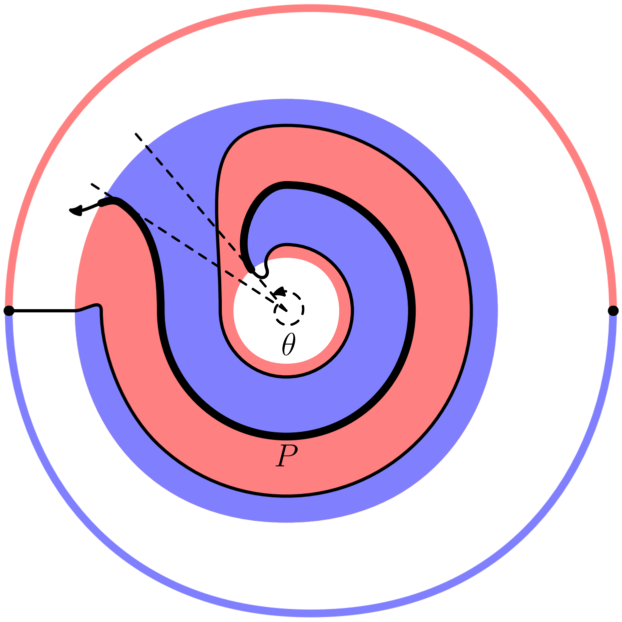


Figure 9: The boundary of any U_1 is wired and the boundary of U_2 is dual wired. As one boundary arc (P) is fixed the components are in the 2π sector starting from that curve.

This proposition is proved in [10] for general isoradial graphs. The following is a direct consequence of Proposition 4.5.

Corollary 4.6. *If (U, a, b, c, d) is non-degenerate then there are $\varepsilon > 0$ and $n_0 > 0$ so that $\varepsilon < \mathbb{P}_n[O(\hat{U}_n)] < 1 - \varepsilon$ for any $n > n_0$.*

Proof of Proposition 4.3. Note first that no information is added between $t = 2n$ and $t = 2n + 2$ and that at $t = 2n$ the domain $U_t = U \setminus \gamma[0, t]$ is an admissible domain. Hence we can assume that $t = 0$ and consider all admissible domains, that is, it is not necessary to consider stopping times, if we consider all admissible domains.

We can also assume that $r > \eta$ where η is as in the section 4.1.4. Namely, any disc $B = B(z_0, r)$ where $0 < r \leq \eta$ intersecting the boundary of the domain, can't intersect with the interior of the medial graph. Choose $C > 0$ such that there are no trivialities such as an edge crossing $A(z_0, r, R)$ for some z_0 and $r > \eta$ and $R > Cr$.

Let U be an admissible domain and $G(U) \subset (\mathbb{Z}^2)_{\text{even}}, G'(U) \subset (\mathbb{Z}^2)_{\text{odd}}, G_L(U) \subset L$ the corresponding graphs. Let $A = A(z_0, r, R)$ be an annulus such that $r > \eta$. Write $\mu_1 = \mu_U^{p_{\text{sd}},2}$. Let $A^u = U_1 \sup U_2$ where the union is disjoint such that U_k is next to c_k. See also Figure 9.

Since γ is the interface which separates the cluster of open edges connected to c_1 from the cluster of dual open edges connected to c_2

$$
\begin{aligned}
&\mathbb{P}_U(\gamma \text{ crosses } A^u) \\
&\leq \mu_1(\text{open crossing of } U_1 \text{ or dual open crossing of } U_2) \\
&\leq \mu_1(\text{open crossing of } U_1) + \mu_1(\text{dual open crossing of } U_2) \\
&\leq 2K
\end{aligned}
$$

where K maximum of the two terms on the preceding line. Therefore we have to prove that $K \leq 1/4$. By symmetry, it is enough to prove that

$$\mu_1(\text{open crossing of } U_1) \leq 1/4. \tag{86}$$

Let A_{blue} be the maximal subdomain of the annulus A has a boundary on the medial lattice and the next to the boundary inside the domain are blue octagons and white squares alternating. Similarly, let A_{red} be the maximal subdomain of the annulus A that has red boundary. Let V_- and V_+ be the connected components of the boundary vertices on A_{blue} and denote by $V_- \leftrightarrow V_+$ the event that there is a open path between V_- and V_+ in the given graph. Let $G_2 \subset G$ be the subgraph corresponding to the domain $U_1 \cap A_{\text{blue}}$. Let $E_2 \subset E(G_2)$ be the set of blue edges along the boundary. Let μ_2 be the random cluster measure on G_2 such that the edges in E_2 are wired and all the components of E_2 are counted to be separate. Then by considering $f = \mathbb{1}_{E_2 \subset \omega}$ in the FKG inequality we have that

$$\mu_1(\text{open crossing of } U_1) \leq \mu_2\left(V_- \leftrightarrow V_+\right).$$

Similarly, it is enough to prove there is a constant $s < 1$ such that

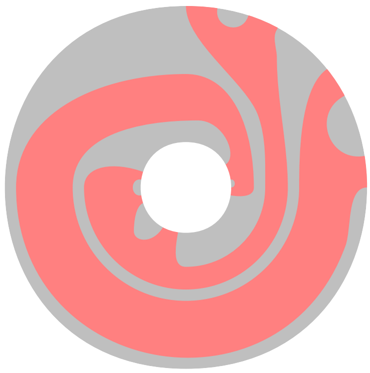
$$\mu_2\left(V_- \leftrightarrow V_+\right) \leq s \tag{87}$$

for a fixed ratio R/r since using this in several concentric annuli we get (86) for a larger annulus. Yet another similar argument shows that we can consider only annuli $A(z_0, r, R)$ where $r > C'\eta$ for any fixed $C' \geq 1$. Namely, if (86) holds for $r > C'\eta$ then for $\eta < r \leq C'\eta$ we can ignore the part below scale $C'\eta$ and only consider crossing between R and $C'\eta$ and then we notice that $R \geq Cr > (C/C') \cdot (C'\eta)$ and therefore by modifying the value C we get (86) for the whole range of r. Therefore we will prove (87) when $r > C'\eta$ when C' is suitably chosen and R/r fixed.

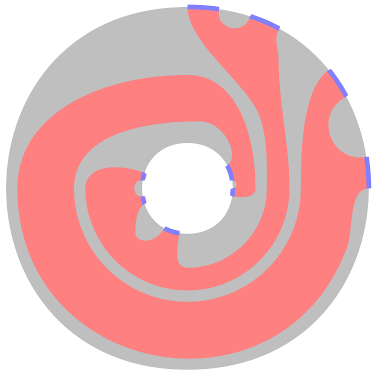
Let P be one of the boundary arcs of U_1 which cross A. Write the points $z \in P$ in polar coordinates $z = z_0 + \rho e^{i\xi}$ so that ξ is continuous along P. Denote by θ the difference between the maximum and the minimum value of ξ along P and by α the minimum value of ξ. The value of α is determined only up to additive multiple of 2π but θ is unique. Now ξ spans the interval $[\alpha, \alpha + \theta]$ along P. The rest of the proof is divided in to two cases: $\theta \leq 4$ and $\theta > 4$.

Case $\theta \leq 4$: Consider the right half-plane $\mathbb{H}_1 = \{(\rho, \xi) : \rho > 0, \xi \in \mathbb{R}\}$ as an infinite covering surface of $\mathbb{C} \setminus \{z_0\}$ such that $(\rho, \xi) \in \mathbb{H}_1$ gets projected on $z_0 + \rho\, e^{i\xi} \in \mathbb{C} \setminus \{z_0\}$. Lift the lattice L to $\mathbb{H}_1$ using this mapping locally in neighborhoods where it is a bijection and define S_{blue} as the lift of A_{blue}, that is, as the maximal subdomain of $S = S(r, R) = \{(\rho, \xi) : 0 < \rho < R, \xi \in \mathbb{R}\}$ such that the boundary is on the medial lattice and it is a blue boundary. Let G_3 be the subgraph of the lifted $(\mathbb{Z}^2)_{\text{even}}$ corresponding to the domain $S_{\text{blue}} \cap (r, R) \times (\alpha, \alpha + 6\pi)$ and denote the edges along the vertical boundary as E_3. Let μ_3 be the random-cluster measure on G_3 where E_3 is wired and the components of E_3 are counted to be separate. Now G_2 can be seen as a subgraph of G_3. If the wired edges of the dual of G_2 are denoted by E_2', then applying the FKG inequality for the decreasing event $\{\omega : E_2' \subset \omega'\}$ and for the measure μ_3 shows that

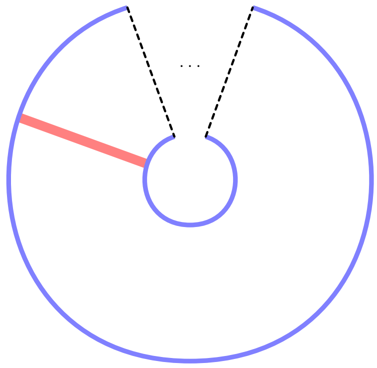
$$\mu_2\left(V_- \leftrightarrow V_+\right) \leq \mu_3\left(V_- \leftrightarrow V_+\right).$$



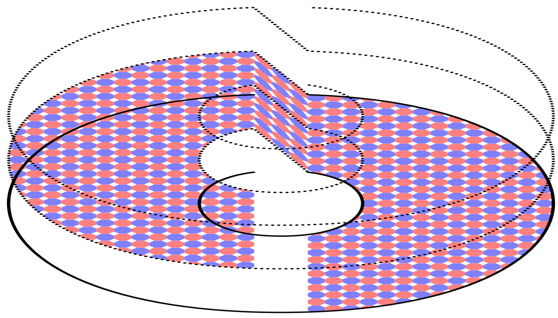
(a) The domain U_1.



(b) In the measure μ_2 the edges along the boundaries of the annulus are wired.



(c) In the case $\theta \leq 4\pi$, the final domain to be considered is a 6π opening in the annulus with wired edges along the boundaries if the annulus. The blue color indicates the wired edges and the red color the dual wired edges. The crossing is between the two blue boundary components.

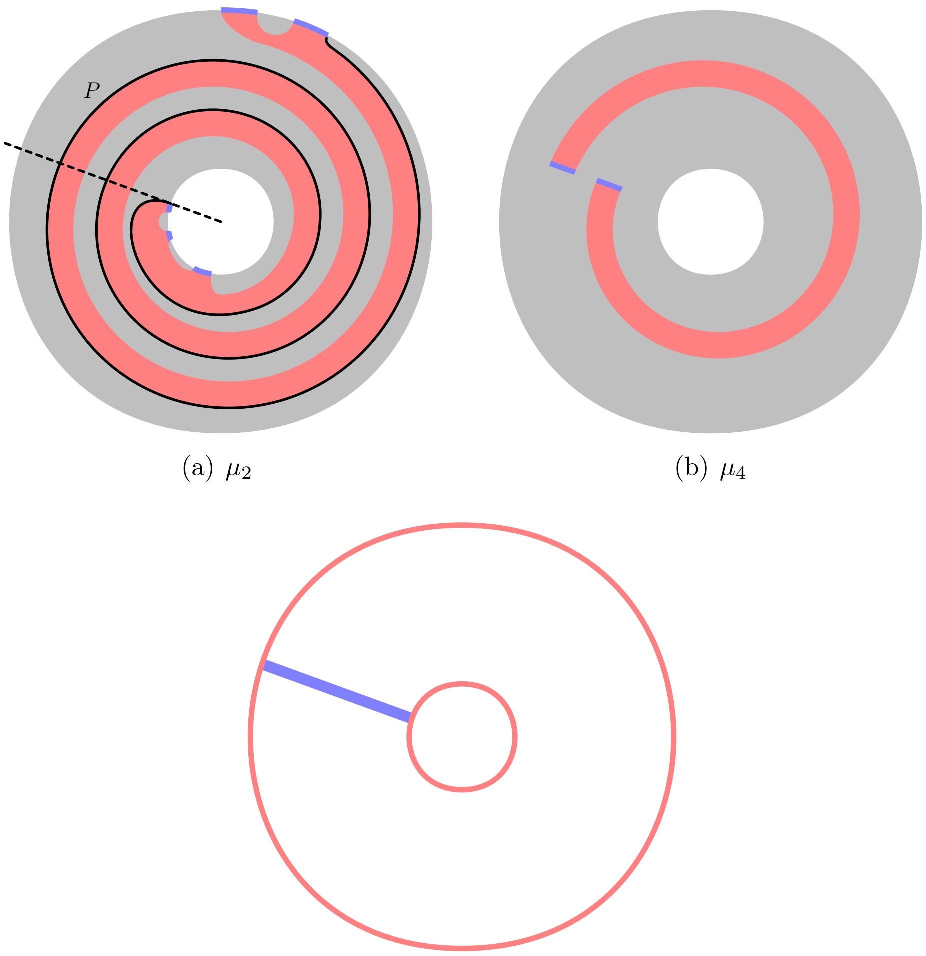


(d) A schematic illustration how to cover the annulus with infinitely many layers of the lattice.

Figure 10: Illustration how the FKG inequality is applied here in general and especially in the case $\theta \leq 4\pi$.

Now we use Corollary 4.6 to show that there are constants $C_1 \geq$ and $s_1 < 1$ such that the right land side of the previous inequality is less than s_1 uniformly for any $r \geq C_1\eta$ and $R = 3r$.

Case $\theta > 4$: Similarly as in the other case define S_{red} to be the lift of A_{red} to S. Now note that any component of G_2 (view as lifted to S) intersects the radials $\alpha + 2\pi$ and $\alpha + 4\pi$ and any open crossing has to intersect those radial. Hence in the same way as above we can add blue boundary and blue wired edges to those radials and ignore the part outside $(r, R) \times (\alpha + 2\pi, \alpha + 4\pi)$. Denote the resulting graph by G_4 and the measure by μ_4 and denote the vertices of the lifted $(\mathbb{Z}^2)_{\text{even}}$ along those two radials by $V_{2\pi}$ and $V_{4\pi}$. Call the dual wired edges of μ_4 by E_4'. Finally if G_5 is the graph corresponding to the domain $S_{\text{red}} \cap (r, R) \times (\alpha + 2\pi, \alpha + 4\pi)$ and E_5 are the boundary edges along the radials, then let μ_5 be the random-cluster measure on G_5



(a) μ_2 (b) μ_4

(c) μ_5

Figure 11: Illustration how the FKG inequality is applied in the case $\theta > 4\pi$.

with wired edges E_5. In the same way as above, we can apply the FKG inequality for μ_5 and for the decreasing event $\{\omega : E_4' \subset \omega'\}$ to get the second inequality in

$$\mu_2 (V_- \leftrightarrow V_+) \leq \mu_4 (V_{2\pi} \leftrightarrow V_{4\pi}) \leq \mu_5 (V_{2\pi} \leftrightarrow V_{4\pi}) .$$

Use again Corollary 4.6 to show that there are constants $C_2 \geq 1$ and $s_2 < 1$ such that the right land side of the previous inequality is less than s_2 uniformly for any $r \geq C_2 \eta$ and $R = 3r$. The claim follows for $s = \max\{s_1, s_2\}$ and $C' = \max\{C_1, C_2\}$. $\square$

4.2 Percolation

Here we verify that the interface of site percolation on the triangular lattice at criticality satisfies Condition G2. More generally we could work on any graph dual to a planar *trivalent graph*. The triangular lattice is denoted by $\mathbb{T}$ and it consist of the set of vertices $\{x_1 e_1 + x_2 e_2 : x_k \in \mathbb{Z}\}$ where $e_1 = 1$ and $e_2 = \exp(i\,\pi/3)$ and the set of edges such that vertices v_1, v_2 are connected by an edge if and only if $|v_1 - v_2| = 1$. The dual lattice of the triangular lattice is the hexagonal lattice $\mathbb{T}'$ consisting of vertices $\{z_\pm + x_1 e_1 + x_2 e_2 : x_k \in \mathbb{Z}\}$ where $z_\pm = (1/\sqrt{3}) \exp(\pm i\,\pi/6)$ and two vertices v_1, v_2 are neighbors if $|v_1 - v_2| = 1/\sqrt{3}$.

The percolation measure on the whole triangular lattice with a parameter $p \in [0,1]$ is the probability measure $\mu^p_{\mathbb{T}}$ on $\{\text{open}, \text{closed}\}^{\mathbb{T}}$ such that independently each vertex is open with probability p and closed with probability $1-p$. The independence property of the percolation measure gives a consistent way to define the measure on any subset of $\mathbb{T}$ by restricting the measure to that set. The well-known critical value of p is $p_c = 1/2$.

In the case of triangular lattice lattice define the set of *admissible domains* containing any domain U with boundary $\partial U = c_1 \cup c_2$ where c_1 and c_2 are

- simple paths on the the hexagonal lattice (write them as $(c_k(n))_{n=0,1,\ldots,N_k}$)
- mutually avoiding except that they have common beginning and end part: $c_1(k) = c_2(k)$, $k = 0,1,\ldots,l_1$, and $c_1(N_1 - k) = c_2(N_2 - k)$, $k = 0,1,\ldots,l_2$, where $l_1, l_2 > 0$
- such that $a = c_1(0) = c_2(0)$ and $b = c_1(N_1) = c_2(N_2)$ are contained on the boundary of the bounded component of $\mathbb{C} \setminus (c_1 \cup c_2)$ and furthermore there is at least one path from a to b staying in $U \cap \mathbb{T}'$.

The last condition is needed to guarantee that a and b are boundary points of the bounded domain and that the subgraph containing all the vertices reach from either a or b is connected. Note that the graph is in fact simply connected.

On an admissible domain U with boundary arcs c_1 and c_2, denote by V the set of vertices on $\mathbb{T}$ inside U, denote by V_1 the set of vertices on $\mathbb{T}$ next to c_1 and by V_2 the set of vertices next to c_2. Define a probability measure μ^p_U, $p \in [0,1]$, on the set $\{\text{open}, \text{closed}\}^V$ such that vertices are each chosen to be independently to be open with the probability p and closed with the probability $1-p$ and such that it satisfies the boundary conditions: the vertices are open on V_1 and closed on V_2. Now there are interfaces on $\mathbb{T}'$ separating clusters of open vertices from clusters of closed vertices. Define $\mathbb{P}_U$ be the law of the unique interface connecting a to b under the critical percolation measure $\mu^{p_c}_U$

The proof of the fact that the collection $(\mathbb{P}_U \,:\, U$ admissible) satisfies Condition G2 couldn't be easier to prove once we have the Russo–Seymour–Welsh theory (RSW). Let B_n be the set of points in the triangular lattice that are at graph-distance n or less from 0 and let $A_n = B_{3n} \setminus B_n$ and let O_n be the event that there is a open path inside A_n separating 0 from ∞. Then there exists $q > 0$ such that for any n

$$\mu^{p_c}_{\mathbb{T}}(O_n) \geq q. \tag{88}$$

Denote by O'_n the event that there is a closed path inside A_n separating 0 from ∞. By symmetry the same estimate holds for O'_n.

Let now $\tilde{A}_n = B_{9n} \setminus B_n$, i.e. $\tilde{A}_n$ is the union of the disjoint sets A_n and A_{3n}. Now probabilities that A_n contains an open path and A_{3n} contains a closed path (both separating 0 from ∞) are independent and hence the corresponding joint event has positive probability

$$\mu^{p_c}_{\mathbb{T}}(O_n \cap O'_{3n}) \geq q^2. \tag{89}$$

Proposition 4.7. *The collection of the laws of the interface of site percolation at criticality on triangular lattice*

$$\Sigma_{\text{Percolation}} = \{(U, \phi_U, \mathbb{P}_U) \,:\, U \text{ an admissible domain}\}. \tag{90}$$

satisfies Condition G2.

Remark 4.8. Exactly the same proof as for FK Ising works for percolation. However RSW provides a simpler way to prove the proposition.

Proof. As in the case of FK Ising, we don't have to consider the stopping times at all. The reason for this is that if $\gamma : [0, N] \to U \cup \{a, b\}$ is the interface parametrized such that $\gamma(k)$, $k = 0, 1, 2, \ldots, N$, are the verices along the path, then $U \setminus \gamma(0, k]$ is admissible for any $k = 0, 1, 2, \ldots, N$ and no information is added during $(k, k + 1)$. Hence after stopping we stay within the family (90). Here we also need that the law of percolation conditioned to the vertices explored up to time n is the percolation measure in the domain where $\gamma(k)$, $k = 1, 2, \ldots, n$, are erased.

For any U, we can apply a translation and consider annuli around the origin. Consider the annular region $B_{9^N n} \setminus B_n$ for any $n, N \in \mathbb{N}$. By the inequality (89) the probability that γ makes an unforced crossing is at most $(1 - q^2)^N \leq 1/2$, for large enough N. $\square$

4.3 Harmonic explorer

The result that the harmonic explorer (HE) satisfies Condition G2 appears already in [26]. We will here just recall the definitions and state the auxiliary result needed. For all the details we refer to [26].

In this section and also in Sections 4.4 and 4.5 the models are directly related to simple random walk. The next basic estimate is needed for bounds like in Conditions G3 and C3.

Lemma 4.9 (Beurling estimate of simple random walk). *Let $L = \mathbb{Z}^2$ or $L = \mathbb{T}$ and let $(X_t)_{t=0,1,2,\ldots}$ be a simple random walk on L with the law P_x such that $P_x(X_0 = x) = 1$ and let τ_B be the hitting time of a set B. For an annulus $A = A(z_0, r, R)$, denote by $E(A)$ the event that a simple random walk starting at $x \in A \cap L$ makes a non-trivial loop around z_0 before exiting A, that is, there exists $0 \leq s < t \leq \tau_{\mathbb{C} \setminus A}$ s.t. $X|_{[s,t]}$ is not nullhomotopic in A. Then there exists $K > 0$ and $\Delta > 0$ such that*

$$P_x(E(A(z_0, r, R)))) \geq 1 - K \left(\frac{r}{R}\right)^{\Delta}$$

for any annulus $A(z_0, r, R)$ with $1 \leq r \leq R$ and for any $x \in A(z_0, r, R) \cap L$ such that $\sqrt{rR} - 1 < |x - z_0| < \sqrt{rR} + 1$.

Sketch of proof. Either use the similar property of Brownian motion and the convergence of simple random walk to Brownian motion or construct the event $E(A(z_0, r, 4r)$ for $|x - z_0| \approx 2r$ from elementary events which, for $L = \mathbb{Z}^2$, are of the type that a random walk started from $(n, n) \in \mathbb{Z}^2$ will exit the rectangle $R_n = [0, \lfloor an \rfloor] \times [0, 2n]$ through the side $\{\lfloor an \rfloor\} \times [0, 2n]$. That elementary event for given $a > 1$ has positive probability uniformly over all n. $\square$

We use here the same definition as in the case of percolation for admissible domains, for c_k, for V_k etc. In the same way as above, the random curve γ will be defined on $\mathbb{T}'$. We describe here how to take the first step in the harmonic explorer. Let U be an admissible domain and choose a and b in some way. Suppose for concreteness that c_1 follows the boundary clockwise from a to b and therefore c_1 lies to the "left" from a and c_2 lies to the "right". Denote by $H_U : U \cap \mathbb{T} \to [0,1]$ the discrete harmonic function on $U \cap \mathbb{T}$ that has boundary values 1 on V_1 and 0 on V_2.

Now $\gamma(0) = a$ has either one or two neighbor vertices in U. If it has only one, then set $\gamma(1)$ equal to that vertex. If it has two neighbors, say, w_L and w_R (defined such that $w_L - a, w_R - a, c_1(1) - a$ are in the clockwise order) calculate the value of $p_0 = H_U(v_0)$ at the center v_0 of the hexagon that is lying next to all these three vertices. Then flip a biased coin and set $\gamma(1) = w_R$ with probability p_0 and $\gamma(1) = w_L$ with probability $1 - p_0$. Note that the rule followed when there is only one neighbor can be seen as a special case of the second rule.

Extend γ linearly between $\gamma(0)$ and $\gamma(1)$ and set now $U_1 = U \setminus \gamma(0,1]$ which is an admissible domain. Repeat the same procedure for U_1 to define $\gamma(2)$ using a biased coin independent from the first one so that the curve turns right with probability $p_1 = H_{U_1}(v_1)$ and left with probability $1 - p_1$ where v_1 is the center of the hexagon next to $\gamma(1)$ and its neighbors except for $\gamma(0)$. Then define $U_2 = U_1 \setminus \gamma(1,2] = U \setminus \gamma(0,2]$ and continue the construction in the same manner. This repeated procedure defines a random curve $\gamma(k)$, $k = 0,1,2,\ldots,N$, such that $\gamma(0) = a$, $\gamma(N) = b$, γ is simple and stays in U.

A special property of this model is that the values of the harmonic functions $M_n = H_{U_n}(v)$ for fixed $v \in U \cap \mathbb{T}$ but for randomly varying U_n defined as above will be a martingale with respect to the σ-algebra generated by the coin flips or equivalently by the curve or the domains (U_n).

It turns out that in this case, the harmonic "observables" $(H_{U_n}(v))_{v \in U \cap \mathbb{T}, n=0,1,\ldots,N}$, provide also a method to verify the Condition G2. This is done in Porposition 6.3 of the article [26]. We only sketch the proof here. Let U be an admissible domain and $A = A(z_0, r, R)$ an annulus. Let V_- be the set of vertices in $V_1 \cap B(z_0, 3r)$ that are disconnected from b by A^u and let the corresponding part of A^u be A^u_-. Let $\tilde{M}_n = \sum_{x \in V_-} \tilde{H}_{U_n}(x)$, where $\tilde{H}_U(x)$, $x \in V_1$ is defined to be the harmonic measure of V_2 seen from x and can be expressed in terms of H_U as the average value H_U among the neighbors of x. Now the key observation in the above proof is that $(\tilde{M}_n)$ is a martingale with $\tilde{M}_0 = \mathcal{O}((r/R)^{\Delta})$ for some $\Delta > 0$ (following from Beurling estimate of simple random walk) and on the event of crossing one of A^u_- it increases to $\mathcal{O}(1)$. A martingale stopping argument tells that the probability of the crossing event is then $\mathcal{O}((r/R)^{\Delta})$.

Proposition 4.10 (Schramm–Sheffield)**.** *The family of harmonic explorers satisfies Condition G2.*

4.4 Chordal loop-erased random walk

The loop-erased random walk is one of the random curves proved to be conformally invariant. In [20], the radial loop-erased random walk between an interior point and a boundary point was considered. We'll treat here the chordal loop-erased random walk between two boundary points. Condition G2 is slightly harder to verify in this case. Namely, the natural extension of Condition G2 to the radial case can be verified in the same way, except that Proposition 4.11 is not necessary, and it is done in [20].

Let $(X_t)_{t=0,1,\ldots}$ be a simple random walk (SRW) on the lattice $\mathbb{Z}^2$ and P_x its law so that $P_x(X_0 = x) = 1$. Consider a bounded, simply connected domain $U \subset \mathbb{C}$ whose boundary ∂U is a path in $\mathbb{Z}^2$. Call the corresponding graph G, i.e., G consists of vertices $\overline{U} \cap \mathbb{Z}^2$ and the edges which stay in U (except that the end points may be in ∂U). Let V be the set of vertices and $\partial V := V \cap \partial U$. When $X_0 = x \in \partial V$ condition SRW on $X_1 \in U$. For any $X_0 = x \in V$ define T to be the hitting time of the boundary, i.e., $T = \inf\{t \geq 1 \,:\, X_t \in \partial V\}$.

For $a \in V$ and $b \in \partial V$ define $P_{a\to b} = P^U_{a\to b}$ to be the law of $(X_t)_{t=0,1,2,\ldots,T}$ with $X_0 = a$ conditioned on $X_T = b$. If $(X_t)_{t=0,1,2,\ldots,T}$ distributed according to $P^U_{a\to b}$ then the process $(Y_t)_{t=0,1,2,\ldots,T'}$, which is obtained from (X_t) by erasing all loops in chronological order, is called *loop-erased random walk* (LERW) from a to b in U. Denote its law by $\mathbb{P}^{U,a,b}$. We will show that the collection $\{\mathbb{P}^{U,a,b} \,:\, (U,a,b)\}$ of chordal LERWs satisfies Condition C2, where U runs over all simply connected domains as above and $\{a,b\} \subset \partial U$.

Denote by τ_A the hitting time of the set A by the simple random walk $(X_t)_{t=0,1,\ldots}$ or $(X_t)_{t=0,1,\ldots,T}$. Let $\omega_U(x,A) = P^U_x(X_T \in A) = P^U_x(\tau_A \leq T)$.

Proposition 4.11. *There exists $\varepsilon_0 > 0$ such that for any $c > 0$ there exists $L_0 > 0$ such that the following holds. Let U be a discrete domain (∂U is a path in $\mathbb{Z}^2$) and let Q be a topological quadrilateral with "sides" S_0, S_1, S_2, S_3 and which lies on the boundary in the sense that $S_1, S_3 \subset \partial U$. Let $A \subset V \setminus Q$ be a set of vertices such that S_0 disconnects S_2 from A. If $m(Q) \geq L_0$, then there exists $u \in Q$ and $r > 0$ such that*

(i) $B := V \cap B(u,r) \subset Q$,

(ii) $\min_{x\in B} \omega_U(x,A) \geq c \max_{x\in S_2} \omega_U(x,A)$ *and*

(iii) $P^Q_{x\to y}(X[0,T] \cap B \neq \emptyset) \geq \varepsilon_0$ *for any* $x \in S_0$ *and* $y \in S_2$.

Proof. Cut Q into three quads (topological quadrilaterals) by transversal paths p_1 and p_2 and call these quads Q_k, $k = 1,2,3$. The sides of Q_k are denoted by S^k_j, $j = 0,1,2,3$, and we assume that $S^1_0 = S_0$, $S^1_2 = p_1 = S^2_0$, $S^2_2 = p_2 = S^3_0$ and $S^3_2 = S_2$.

We assume that $m(Q_1) = m(Q_2) = l$ and $m(Q_3) = L - 2l$ where $L = m(Q)$. Using the Beurling estimate, Lemma 4.9, it is possible to fix l so large that $\omega_{Q_1\cup Q_2}(z, S^1_0 \cup S^2_2) \leq 1/100$ for any z on the discrete path closest to $S^1_2 = S^2_0$.

Since the harmonic measure $z \mapsto \omega_{Q_1\cup Q_2}(z, S^1_1 \cup S^2_1)$ changes at most by a constant factor between neighboring sites, we can find u along the discrete path closest to $S^1_2 = S^2_0$ in such a way that $\omega_{Q_1\cup Q_2}(u, S^1_1 \cup S^2_1), \omega_{Q_1\cup Q_2}(u, S^1_3 \cup S^2_3) \geq 1/6$. Let r be equal to half of the inradius of $Q_1 \cup Q_2$ at u. Then $B := V \cap B(u,r)$ satisfies (i) by definition and (iii) for some $\varepsilon_0 > 0$ follows from Proposition 3.1 of [8].

Let $H(x) = \omega_U(x,A)$. Let $c' > 0$ be such that $H(x) \geq c'\, H(y)$ for any $x, y \in B$. The constant c' can be chosen to be universal by Harnack's lemma. Let $M = \max_{x\in S^2_2} H(x)$ and let x^* be the point where the maximum is attained. By the maximum principle there is a path π from x^* to A such that $H \geq M$ on π. Now $H(u) \geq M/6$ and hence $\min_{x\in B} H(x) \geq Mc'/6$. Finally by the Beurling estimate $\max_{x\in S^2_2} H(x) \geq \exp(\alpha m(Q_3)) \max_{x\in S_2} H(x)$ for some universal constant $\alpha > 0$. And hence we can choose L_0 so large that (ii) holds for any $L \geq L_0$. □

Theorem 4.12. *Condition C2 holds for LERW.*

Proof. Let $L_0 > 0$ and $\varepsilon_0 > 0$ be as in Proposition 4.11 for $c = 2$. Consider a quad Q with $L = m(Q) \geq L_0 > 0$ as in Proposition 4.11 for $A = \{b\}$. We will show that there is uniformly positive probability that $(X_t)_{t=0,1,\dots,T}$ conditioned on $X_T = b$ doesn't cross Q. By iterating that estimate $n \in \mathbb{N}$ times (for large enough n) we get that the probability of crossing is at most $1/2$ for $L \geq n\, L_0$.

We can assume $P_a(\tau_{S_2} < T \,|\, X_T = b) \geq 1/2$, otherwise there wouldn't be anything to prove. By the previous proposition

$$P_a(\tau_B < (\tau_{S_2} \wedge T) \,|\, X_T = b) \geq \varepsilon_0 P_a(\tau_{S_2} < T \,|\, X_T = b) \geq \frac{\varepsilon_0}{2}.$$

Now since $\max_{x \in S_2} P_x(X_T = b) \leq (1/2) \cdot \min_{y \in B} P_y(X_T = b)$ by assumption,

$$\begin{aligned} P_y(\tau_{S_2} < T \,|\, X_T = b) &= \frac{P_y(\tau_{S_2} < T,\, X_T = b)}{P_y(X_T = b)} \\ &\leq \frac{\max_{x \in S_2} P_x(X_T = b)}{P_y(X_T = b)} \leq \frac{1}{2} \end{aligned}$$

for any $y \in B$. Combine these estimates to show that

$$P_a(\tau_{S_2} < T \,|\, X_T = b) \leq 1 - P_a(\tau_B < T < \tau_{S_2}) \,|\, X_T = b) \leq 1 - \frac{\varepsilon_0}{4}$$

from which the claim follows. □

4.5 Condition G2 fails for uniform spanning tree

For a given connected graph G, a spanning tree is a subgraph T of G such that T is a tree, i.e., connected and without any cycles, and T is spanning, i.e., $V(T) = V(G)$. A *uniform spanning tree* (UST) of G is a spanning tree sampled uniformly at random from the set of all spanning trees of G. More precisely, if T is a uniform spanning tree and t is any spanning tree of G then

$$\mathbb{P}(T = t) = \frac{1}{N(G)} \tag{91}$$

where $N(G)$ is the number of spanning trees of G. The UST model can be analyzed via simple random walks and electrical networks, see [14] and references therein. The conformal invariance of UST on planar graphs was shown in [20] where Lawler, Schramm and Werner proved that the UST Peano curve (see below) converges to SLE(8). Their work partly relies on Aizenman–Burchard theorem and [3] where the relevant crossing estimate was established.

Concerning the current work, the UST Peano curve gives a counterexample: it is a curve otherwise eligible but it fails to satisfy Condition G2. For discrete random curves converging to some SLE(κ) this shows that Condition G2 is only relevant to the case $0 \leq \kappa < 8$. In some sense $0 \leq \kappa \leq 8$ is the physically relavant case. For instance, the reversibility property holds only in this range of κ. Therefore it is interesting to extend the methods of this paper to the spacial case of UST Peano curve. This shown to be possible by the first author of this paper in [16].

Consider a finite subgraph $G_\delta \subset \delta\mathbb{Z}^2$, $\delta > 0$, which is simply connected, i.e., it is a union of entire faces of $\delta\mathbb{Z}^2$ such that the corresponding domain is Jordan domain of $\mathbb{C}$. A boundary edge of G_δ is a edge e in G_δ such that there is a face in $\delta\mathbb{Z}^2$ which contains e but which doesn't belong entirely to G. Take a non-empty connected set E_W of boundary edges not equal to the entire set of boundary edges. Then E_W will be a path which we call *wired boundary.* Call its end points in the counterclockwise direction as $\tilde{a}_\delta$ and $\tilde{b}_\delta$.

Let T be a uniform spanning tree on G_δ conditioned on $E_W \subset T$. Then T can be seen as an unconditioned UST of the contracted graph G_δ/E_W. The UST Peano curve is defined to be the simple cycle γ on $\delta(1/4+\mathbb{Z}/2)^2$ which is clockwise oriented and follows T as close as possible, i.e., for each k, there is either a vertex of G_δ on the right-hand side of $(\gamma(k), \gamma(k+1))$ or there is a edge of T. We restrict this path to a part which goes from a point next to $\tilde{a}_\delta$ to a point next to $\tilde{b}_\delta$. With an appropriate choice of the domain U_δ, γ is a simple curve in U_δ connecting boundary points a_δ and b_δ and it is also a space-filling curve, i.e., γ visits all the vertices $U_\delta \cap \delta(1/4+\mathbb{Z}/2)^2$.

It is easy to see that γ doesn't satisfy Condition G2: since it is space filling it will make an unforced crossing of $A(z_0, r, R)$ with probability 1 if there are any sites which are disconnected from a_δ and b_δ by a component of $A(z_0, r, R)$. However the probability more than 2 crossings in such a component is small. This approach is taken in [16], where it is sufficient to consider the following event: let $Q \subset U_\delta$ be a topological quadrilateral such that $\partial_0 Q$ and $\partial_2 Q$ are subsets of U_δ and $\partial_1 Q$ and $\partial_3 Q$ are subsets of wired part of ∂U_δ. Then Q has the property that it doesn't disconnect a_δ from b_δ. We call a set of edges $E \subset E(G_\delta) \cap Q$ a *confining layer* if $E \cup E_W$ is a tree and the vertex set of E contains a crossing of Q from $\partial_1 Q$ to $\partial_3 Q$. This is equivalent to the fact that there can be only two vertex disjoint crossings of Q from $\partial_0 Q$ to $\partial_2 Q$ by paths in $U_\delta \cap \delta(1/4+\mathbb{Z}/2)^2$. Based on the connection to the simple random walk by Wilson's method, it is possible to establish that there exists universal constant $c > 0$ such that

$$\mathbb{P}(T \text{ contains a confining layer in } Q) \geq 1 - \frac{1}{c}\exp(-c\, m(Q, \partial_0 Q, \partial_2 Q)). \tag{92}$$

This will be sufficient for the treatment of UST Peano curve in the similar manner as in this article.

A Appendixes

A.1 Schramm–Loewner evolution

We will be interested in describing random curves in simply connected domains with boundary in the complex plane by Loewner evolutions with random driving functions. Since the setup for Loewner evolutions is conformally invariant, we can define them in some fixed domain. A standard choice is the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Another choice could be the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Consider a simple curve $\gamma : [0, T] \to \mathbb{C}$ such that $\gamma(0) \in \mathbb{R}$ and $\gamma(t) \in \mathbb{H}$ for any $t > 0$. Let $K_t = \gamma[0, t]$ and $H_t = \mathbb{H} \setminus K_t$. Note that K_t is compact and H_t is simply connected.

There is a unique conformal mapping $g_t : H_t \to \mathbb{H}$ satisfying the normalization $g_t(\infty) = \infty$ and $\lim_{z\to\infty}[g_t(z) - z] = 0$. This is called the *hydrodynamical*

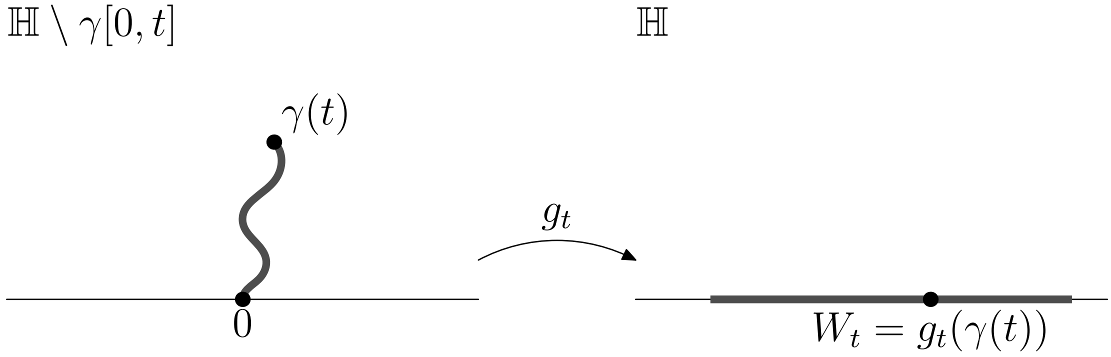


Figure 12: The mapping g_t maps the complement of $\gamma[0,t]$ onto the upper half-plane. The tip $\gamma(t)$ is mapped to a point W_t on the real line.

normalization and then around the infinity

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \dots \tag{93}$$

The coefficient $a_1(t) = \operatorname{cap}_{\mathbb{H}}(K_t)$ is called the *half-plane capacity* of K_t or shorter the *capacity*. Quite obviously, $a_1(0) = 0$, and it can be shown that $t \mapsto a_1(t)$ is strictly increasing and continuous. The curve can be reparameterized (which also changes the value of T) such that $a_1(t) = 2t$ for each t.

Assuming the above normalization and parameterization, the family of mappings $(g_t)_{t\in[0,T]}$ satisfies the upper half-plane version of the *Loewner differential equation*, that is

$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z) - W_t} \tag{94}$$

for any $t \in [0,T]$, where the "*driving function*" $t \mapsto W_t$ is continuous and real-valued. It can be proven that g_t extends continuously to the point $\gamma(t)$ and $W_t = g_t(\gamma(t))$. For the proofs of these facts see Chapter 4 of [18]. An illustration of the construction is in Figure 12. The equation or rather its version on the unit disc was introduced by Loewner in 1923 in his study of the Bieberbach conjecture [21].

Consider more general families of growing sets. Call a compact subset K of $\overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K$ is simply connected, as a *hull* . The sets K_t given by a simple curve, as above, are hulls. Also other families of hulls can be described by the Loewner equation with a continuous driving function. The necessary and sufficient condition is given in the following proposition. Also some facts about the capacity are collected there.

Proposition A.1. *Let $T > 0$ and $(K_t)_{t\in[0,T]}$ a family of hulls s.t. $K_s \subset K_t$, for any $s < t$, and let $H_t = \mathbb{H} \setminus K_t$.*

- *If $(K_t \setminus K_s) \cap \mathbb{H} \neq \emptyset$ for all $s < t$, then $t \mapsto \operatorname{cap}_{\mathbb{H}}(K_t)$ is strictly increasing*
- *If $t \mapsto H_t$ is continuous in Carathéodory kernel convergence, then $t \mapsto \operatorname{cap}_{\mathbb{H}}(K_t)$ is continuous.*
- *Assume that $\operatorname{cap}_{\mathbb{H}}(K_t) = 2t$ (under the first two assumptions there is always such a time reparameterization). Then there is a continuous driving function*

W_t so that g_t satisfies Loewner equation (94) *if and only if for each $\delta > 0$ there exists $\varepsilon > 0$ so that for any $0 \leq s < t \leq T$, $|t-s| < \delta$, a connected set $C \subset H_s$ can be chosen such that $\operatorname{diam}(C) < \varepsilon$ and C separates $K_t \setminus K_s$ from infinity.*

Two first statements are relatively simple. The second claim is almost self-evident: Carathéodory kernel convergence means that $g_s \to g_t$ as $s \to t$ in the compact subsets of H_t and then we have to use the fact that $\operatorname{cap}_{\mathbb{H}}(K)$ can be expressed as an integral $\frac{1}{2\pi}\int_0^\pi \operatorname{Re}\left(Re^{i\theta}g_K(Re^{i\theta})\right)\mathrm{d}\theta$ for R large enough. The third claim is proved in [19].

By the third claim not all continuous W_t correspond to a simple curve. One important class of $(K_t)_{t\in[0,T]}$ are the ones generated by a curve in the following sense: For a curve $\gamma : [0,T] \to \overline{\mathbb{H}}$, $\gamma(0) \in \mathbb{R}$, that is not necessarily simple, define H_t to be the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ and $K_t = \overline{\mathbb{H} \setminus H_t}$. For each t, K_t is a hull and the collection of hulls $(K_t)_{t\in[0,T]}$ is said to be *generated by the curve γ*. But even this class is not general enough: a counterexample is a spiral that winds infinitely many times around a circle in the upper half-plane and then unwinds.

A *Schramm–Loewner evolution*, SLE_κ, $\kappa > 0$, is a random $(K_t)_{t\geq 0}$ corresponding to a random driving function $W_t = \sqrt{\kappa}B_t$ where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. SLE was introduced by Schramm [25] in 1999. An important result about them is that they are curves in the following sense:

- $0 < \kappa \leq 4$: K_t is a simple curve
- $4 < \kappa < 8$: K_t is generated by a curve
- $\kappa \geq 8$: K_t is a space filling curve

This is proven $\kappa \neq 8$ in [24]. For $\kappa = 8$, it follows since SLE_8 is a scaling limit of a random planar curve in the sense explained in the current paper, see [20]. So based on this result, the above definition can be reformulated: a Schramm–Loewner evolution is a random curve in the upper half-plane whose Loewner evolution is driven by a Brownian motion.

In fact, Schramm–Loewner evolutions are characterized by the conformal Markov property [25], see [32] for an extended discussion. For this reason, if the scaling limit of a random planar curve is conformally invariant in an appropriate sense, then it has to be SLE_κ, for some $\kappa > 0$.

A.2 Equicontinuity of Loewner chains

In this section, we prove simple statements about equicontinuity of general Loewner chains. For g_t as in the previous section, denote its inverse by $f_t = g_t^{-1}$, which satisfies the corresponding Loewner equation

$$\partial_t f_t(z) = -f_t'(z)\frac{2}{z - W_t} \tag{95}$$

together with the initial condition $f_0(z) = z$. We call any of the equivalent objects $(g_t)_{t\in[0,T]}$, $(f_t)_{t\in[0,T]}$ and $(K_t)_{t\in[0,T]}$ as a *Loewner chain* (with the driving term $(W_t)_{t\in[0,T]}$).

Let $V_{T,\delta} = [0,T] \times \{z \in \mathbb{C} : \operatorname{Im} z \geq \delta\}$

Lemma A.2. *For any $T, \delta > 0$ the family*

$$\left\{ \tilde{F} : V_{T,\delta} \to \mathbb{C} \; : \; \begin{array}{c} \text{there is a Loewner chain } (f_t)_{t \in \mathbb{R}_+} \text{ s.t.} \\ \tilde{F}(t,z) = f_t(z), \, \forall (t,z) \in V_{T,\delta} \end{array} \right\} \tag{96}$$

is equicontinuous and

$$\left| \partial_t \tilde{F}(t,z) \right| \le \frac{2}{\delta} e^{8 \frac{t}{\delta^2}}, \qquad \left| \partial_z \tilde{F}(t,z) \right| \le e^{8 \frac{t}{\delta^2}} \tag{97}$$

for any $\tilde{F}$ in the set (96) *and for any* $(t,z) \in V_{T,\delta}$.

Proof. Since $V_{T,\delta}$ is convex, it is sufficient to show (97). The equicontinuity follows from that bound by integrating along a line segment in $V_{T,\delta}$

Let $\Phi_w(z) = i(\operatorname{Im} w)\frac{1+z}{1-z} + \operatorname{Re} w$ and $f : \mathbb{H} \to \mathbb{C}$ be any conformal map. Then

$$z \mapsto \frac{f \circ \Phi_w(z) - f \circ \Phi_w(0)}{\Phi_w'(0) f'(w)}$$

belongs to the class S of univalent functions, see Chapter 2 of [11], and therefore by Bieberbach's theorem

$$(\operatorname{Im} w) \left| \frac{f''(w)}{f'(w)} \right| \le 3.$$

If we apply this bound to the Loewner equation of f_t' we find that

$$|\partial_t f_t'(z)| \le \frac{8}{(\operatorname{Im} z)^2} |f_t'(z)|$$

and therefore

$$|f_t'(z)| \le e^{8 \frac{t}{(\operatorname{Im} z)^2}} \le e^{8 \frac{T}{\delta^2}}.$$

Furthermore, plugging this estimate in the Loewner equation gives

$$|\partial_t f_t(z)| \le \frac{2}{\operatorname{Im} z} e^{8 \frac{T}{\delta^2}} \le \frac{2}{\delta} e^{8 \frac{T}{\delta^2}}.$$

□

For $T, \delta > 0$ and family of hulls $(K_t)_{t \in [0,T]}$, let

$$S_K(T,\delta) = \{(t,z) \in [0,T] \times \overline{\mathbb{H}} \, : \, \operatorname{dist}(z, K_t) \ge \delta\} \tag{98}$$

Lemma A.3. *Let γ_n be a sequence of curves in $\mathbb{H}$ and let γ be a curve in $\mathbb{H}$ all parametrized with the interval $[0,T]$, $T > 0$, and let $g_{n,t}$ and g_t be the normalized conformal maps related to the hulls $K_{n,t}$ and K_t of $\gamma_n[0,t]$ and $\gamma[0,t]$, respectively. If $\gamma_n \to \gamma$ uniformly, then $g_{n,t} \to g_t$ uniformly on $S_K(T,\delta)$. Especially* $\operatorname{cap}_{\mathbb{H}} \gamma_n[0,\cdot] \to \operatorname{cap}_{\mathbb{H}} \gamma[0,\cdot]$ *uniformly.*

Proof. The lemma follows from the Carathéodory convergence theorem (Theorem 3.1 of [11] and Theorem 1.8 of [23]). Convergence is uniform in t since the interval $[0,T]$ is compact. □

Lemma A.4. *Let W_n be a sequence of continuous functions on $[0,T]$ and let W be a continuous functions on $[0,T]$ and let $g_{n,t}$ and g_t be the solutions of Loewner equation with the driving terms $W_{n,t}$ and W_t, respectively, and let K_t be the hull of g_t. If $W_n \to W$ uniformly, then $g_{n,t} \to g_t$ uniformly on $S_K(T,\delta)$ and g_t and W satisfy the Loewner equation.*

Proof. This lemma follows from the basic properties of ordinary differential equations. □

A.2.1 Main lemma

Consider a sequence $\tilde{\gamma}_n \in X_{\text{simple}}(\mathbb{D})$ with $\tilde{\gamma}_n(0) = -1$ and $\tilde{\gamma}_n(1) = +1$ which converges to some curve $\tilde{\gamma} \in X$. After choosing a parametrization and using the chosen conformal transformation from $\mathbb{D}$ to $\mathbb{H}$, it is natural to consider for some $T > 0$ a sequence of one-to-one continuous functions $\gamma_n : [0,T] \to \mathbb{C}$ with $\gamma_n(0) \in \mathbb{R}$ and $\gamma(0,T] \subset \mathbb{H}$ such that γ_n converges uniformly to a continuous function $\gamma : [0,T] \to \mathbb{C}$ which is not constant on any subinterval of $[0,T]$. In this section we present practical conditions under which γ is a Loewner chain, that is, γ can be reparametrized with the half-plane capacity

Let

$$\upsilon_n(t) = \frac{1}{2}\mathrm{cap}_{\mathbb{H}}(\gamma_n[0,t]), \qquad \upsilon(t) = \frac{1}{2}\mathrm{cap}_{\mathbb{H}}(\gamma[0,t]). \tag{99}$$

Then $t \mapsto \upsilon_n(t)$ and $t \mapsto \upsilon(t)$ are continuous and $\upsilon_n \to \upsilon$ uniformly as $n \to \infty$. Especially, $\lim_n \upsilon_n(T) = \upsilon(T)$ and by the assumptions $\upsilon(1) > 0$. Furthermore, $t \mapsto \upsilon(t)$ is non-decreasing. Let $(W_n(t))_{t\in[0,\upsilon_n(T)]}$ be the driving term of γ_n which exists since γ_n is simple.

When is it true that γ has a continuous driving term? It is a fact that if υ is strictly increasing then γ has a driving term $W((t))_{t\in[0,\upsilon(1)]}$ and that $W_n \to W$ uniformly on $[0,\upsilon(1))$. However we won't prove this auxiliary result, instead we prove a weaker result which gives a practical conditions to be verified.

Lemma A.5. *Let $T > 0$ and for each $n \in \mathbb{N}$, let $\gamma_n : [0,T] \to \mathbb{C}$ be injective continuous function such that $\gamma_n(0) \in \mathbb{R}$ and $\gamma_n(0,T] \subset \mathbb{H}$. Suppose that*

1. *$\gamma_n \to \gamma$ uniformly on $[0,T]$ and γ is not constant on any subinterval of $[0,T]$*
2. *$W_n \to W$ uniformly on $[0,\upsilon(T)]$.*
3. *$F_n \to F$ uniformly on $[0,T] \times [0,1]$, where*

$$F_n(t,y) = g^{-1}_{\gamma_n[0,t]}\big(W_n(\upsilon_n(t)) + iy\big). \tag{100}$$

Then $t \mapsto \nu$ is strictly increasing and $g_t := g_{\gamma\circ\upsilon^{-1}[0,t]}$ satisfies the Loewner equation with the driving term W. Furthermore, the sequence of mappings $(t,z) \mapsto g_{\gamma_n\circ\upsilon_n^{-1}[0,t]}$ converges to g_t uniformly on

$$S_K(T,\delta) = \{(t,z) \in [0,T] \times \overline{\mathbb{H}} \,:\, \mathrm{dist}(z,K_t) \geq \delta\} \tag{101}$$

for any $\delta > 0$. Here K_t is the hull of $\gamma[0,t]$.

Remark A.6. By applying a scaling and corresponding time change, it's enough that there exists $\varepsilon > 0$ such that $F_n \to F$ uniformly on $[0,T] \times [0,\varepsilon]$.

Proof. By Lemma A.3, $\gamma_n \to \gamma$ implies that $\upsilon_n \to \upsilon$ uniformly as $n \to \infty$. Let $f_{n,t} = g^{-1}_{\gamma_n[0,t]}$. Since

$$F_n(t,y) = f_{n,t}(W_n \circ \upsilon_n(t) + i\,y) \tag{102}$$

and since by Lemma A.2

$$|f_{n,t}(z) - f_{n,t'}(z')| \leq C(\delta,\upsilon(T))\,\left(|\upsilon_n(t) - \upsilon_n(t')| + |z - z'|\right), \tag{103}$$

it follows directly from the assumptions that if for some $s < t$, $\upsilon(s) = \upsilon(t)$, then $F(s, y) = F(u, y) = F(t, y)$ for all $u \in [s, t]$ and $y \geq 0$. Especially $\gamma(u) = F(u, 0)$ is constant on the interval $u \in [s, t]$ which contradicts with the assumptions of the lemma. Hence υ is strictly increasing. An application of Helly's selection theorem gives that υ_n^{-1} converges uniformly to υ^{-1}. Therefore $\gamma_n \circ \upsilon_n^{-1}$ converges uniformly to $\gamma \circ \upsilon^{-1}$ and hence for any $\delta > 0$ $(t, z) \mapsto g_{\gamma_n \circ \upsilon_n^{-1}[0,t]}$ converges to g_t uniformly on the set (101). The convergence of W_n together with standard results about ODE's imply that $g_{\gamma_n \circ \upsilon_n^{-1}[0,t]}$, which are the solutions of the Loewner equation with the driving terms W_n, converge uniformly to the solution of the Loewner equation with the driving term W, see Lemma A.4. Hence g_t is generated by γ and driven by W. □

Lemma A.7. *Let $\gamma : [0, T] \to \mathbb{C}$ be continuous and not constant on any subinterval of $[0, T]$. Let $\gamma_n : [0, T] \to \mathbb{C}$ be a sequence of simple parametrized curves such that $\gamma_n(0) \in \mathbb{R}$ and $\gamma_n(\,(0, T]\,) \subset \mathbb{H}$. Suppose that $\gamma_n \to \gamma$ uniformly as $n \to \infty$. If*

- *$(W_n)_{n \in \mathbb{N}}$ is equicontinuous and*
- *there exist increasing continuous $\psi : [0, \delta) \to \mathbb{R}_+$ such that $\psi(0) = 0$ and $|F_n(t, y) - \gamma_n(t)| \leq \psi(y)$ for all $0 < y < \delta$ and for all $n \in \mathbb{N}$*

then W_n converges to some continuous W, γ can be continuously reparametrized with the half-plane capacity and $\gamma \circ \upsilon^{-1}$ is driven by W.

Proof. It is clearly enough to show that $(F_n)_{n \in \mathbb{N}}$ is a equicontinuous family of functions on $[0, T] \times [0, 1]$. The claim then follows from the previous lemma after choosing by Arzelà-Ascoli theorem a subsequence n_j such that F_{n_j} and W_{n_j} converge.

Let $g_{n,t} = g_{\gamma_n[0,t]}$ and $f_{n,t} = g_{n,t}^{-1}$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|F_n(t, y) - \gamma_n(t)| \leq \frac{\varepsilon}{6} \tag{104}$$

$$|\gamma_n(t') - \gamma_n(t)| \leq \frac{\varepsilon}{6} \tag{105}$$

when $0 \leq y \leq \delta$ and $t, t' \in [0, T]$ with $|t - t'| \leq \delta$. Then by the triangle inequality

$$|F_n(t', y') - F_n(t, y)| \leq \frac{\varepsilon}{2} \tag{106}$$

for all $0 \leq y, y' \leq \delta$, for all $t, t' \in [0, T]$ with $|t - t'| \leq \delta$ and for all $n \in \mathbb{N}$.

By (102) and Lemma A.2, the family of mappings $(F_n|_{[0,T] \times [\delta, 1]})_{n \in \mathbb{N}}$ is equicontinuous. Hence we can choose $0 < \tilde{\delta} \leq \delta$ such that (106) for all $\delta \leq y, y' \leq 1$ with $|y - y'| \leq \tilde{\delta}$, for all $t, t' \in [0, T]$ with $|t - t'| \leq \tilde{\delta}$ and for all $n \in \mathbb{N}$. Hence by the triangle inequality

$$|F_n(t', y') - F_n(t, y)| \leq \varepsilon \tag{107}$$

for all $0 \leq y, y' \leq 1$ with $|y - y'| \leq \tilde{\delta}$, for all $t, t' \in [0, T]$ with $|t - t'| \leq \tilde{\delta}$ and for all $n \in \mathbb{N}$. □

A.3 Some facts about conformal mappings

In this section, a collection of simple lemmas about normalized conformal mappings is presented. Only elementary methods are used, and therefore it is advantageous to present the proofs here, even though they appear elsewhere in the literature.

Denote the inverse mapping of g_K by f_K and by $I \subset \mathbb{R}$ the image of ∂K under g_K, i.e. $I = \overline{\{x \in \mathbb{R} : \operatorname{Im} f_K(x) > 0\}}$. Now f_K can be given by integral with Poisson kernel of upper half-plane as

$$f_K(z) = z + \frac{1}{\pi} \int_I \frac{\operatorname{Im} f_K(x)}{x - z} \mathrm{d}x. \tag{108}$$

This gives a nice proof of the following fact.

Lemma A.8. *Denote $u_+ = \max I$ and $u_- = \min I$ and $x_\pm = f_K(u_\pm)$. Assume $\mathbb{H} \cap K \neq \emptyset$. Then*

$$f_K(x) < x \text{ when } x \geq u_+ \text{ and } f_K(x) > x \text{ when } x \leq u_- \tag{109}$$

and

$$g_K(x) > x \text{ when } x \geq x_+ \text{ and } g_K(x) < x \text{ when } x \leq x_-. \tag{110}$$

Proof. Note that $\operatorname{Im} f_K(x)$ is non-negative everywhere. It is positive in a set of non-zero Lebesgue measure, otherwise the equation (108) would imply that f_K is an identity which is a contradiction. Now the equation (108) implies directly the equation (109). Apply g_K on both sides to get the equation (110). □

The lemma can be used, for example, in the following way.

Lemma A.9. *Let $K \subset K'$ be two hulls. Let $x \in \mathbb{R}$ s.t. $g_K(\overline{K' \setminus K}) \cap (x, \infty) = \emptyset$, and let $z = f_K(x)$. Then $g_K(z) \leq g_{K'}(z)$.*

Proof. Apply Lemma A.8 to hull $J = g_K(\overline{K' \setminus K})$ and $u = g_J(x)$. □

Let's introduce still one more concept. Consider now $K = [-l, l] \times [0, h]$ where $l, h > 0$. The domain $\mathbb{H} \setminus K$ can be thought as a polygon with the vertices $w_1 = -l$, $w_2 = -l + ih$, $w_3 = l + ih$, $w_4 = l$ and $w_5 = \infty$. The interior angles at these vertices are $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{3\pi}{2}$, $\alpha_3 = \frac{3\pi}{2}$, $\alpha_4 = \frac{\pi}{2}$ and $\alpha_5 = 0$, respectively. For the last one, this has to be thought on the Riemann sphere.

Mappings from $\mathbb{H}$ to polygons are well-known. They are Schwarz-Christoffel mappings. In this case, when $f_K(\infty) = w_5 = \infty$ all such mappings can be written in the form

$$f_K(z) = A + C \int^z \frac{\sqrt{\zeta - z_2}\sqrt{\zeta - z_3}}{\sqrt{\zeta - z_1}\sqrt{\zeta - z_4}} \mathrm{d}\zeta. \tag{111}$$

Here $f_K(z_i) = w_i$, $i = 1, 2, 3, 4$. So in a sense $\operatorname{Re} A, \operatorname{Im} A, C$ and z_i are parameters in the problem. Two of them can be chosen freely and the rest are determined from them. The branches of the square roots are chosen so that far on the positive real axis the square root is positive and then analytic continuation is used.

In our case f_K is normalized at the infinity. This fixes $C = 1$ and $\operatorname{Re} A$ so that it cancels the constant term in the expansion of the integral. But if we are only interested in differences $f_K(z) - f_K(z')$ we don't have to care about A.

Lemma A.10. *Let $K = [-l/2, l/2] \times [0, h]$, $h, l > 0$, and let z_i be as above. Then*

$$z_3 = -z_2 = \frac{1}{2} l \left(1 + o(1)\right) \text{ and } z_4 - z_3 = z_2 - z_1 = \frac{2}{\pi} h \left(1 + o(1)\right) \text{ as } \frac{h}{l} \to 0. \quad (112)$$

Proof. Note first that by symmetry $z_1 = -z_4$ and $z_2 = -z_4$.

Denote $\lambda = z_3 - z_2$ and $\theta = z_4 - z_3$. We would like to estimate λ and θ in terms of l and h.

Calculate $h = \operatorname{Im}(w_3 - w_4)$ as an integral along the real axis

$$h = \int_{z_3}^{z_4} \sqrt{\frac{\zeta - z_2}{\zeta - z_1}} \sqrt{\frac{\zeta - z_3}{z_4 - \zeta}} \mathrm{d}\zeta.$$

Since the first factor of the integrand is a decreasing function ζ, it can be bounded with the values at the end points z_3 and z_4. After couple of variable changes, the integral of the second factor is

$$\int_{z_3}^{z_4} \sqrt{\frac{\zeta - z_3}{z_4 - \zeta}} \mathrm{d}\zeta = \frac{\pi}{2}(z_4 - z_3)$$

and therefore

$$\frac{\pi}{2} \sqrt{\frac{1}{1 + \frac{\theta}{\lambda}}}\, \theta \le h \le \frac{\pi}{2} \sqrt{\frac{1 + \frac{\theta}{\lambda}}{1 + 2\frac{\theta}{\lambda}}}\, \theta. \quad (113)$$

Calculate $l = w_3 - w_2$ as

$$l = \int_{z_2}^{z_3} \sqrt{\frac{(\zeta - z_2)(z_3 - \zeta)}{(\zeta - z_1)(z_4 - \zeta)}} \mathrm{d}\zeta = 2 \int_0^{z_3} \sqrt{\frac{z_3^2 - \zeta^2}{z_4^2 - \zeta^2}} \mathrm{d}\zeta$$

The integrand is always less or equal then one. So $l \le \lambda$. For the lower bound, note that the integrand is a decreasing function of ζ. Therefore

$$\int_0^{z_3} \sqrt{\frac{z_3^2 - \zeta^2}{z_4^2 - \zeta^2}} \mathrm{d}\zeta \ge \zeta_0 \sqrt{\frac{z_3^2 - \zeta_0^2}{z_4^2 - \zeta_0^2}}$$

Maximize this with respect to $\zeta_0 \in [0, z_3]$ to get

$$\int_0^{z_3} \sqrt{\frac{z_3^2 - \zeta^2}{z_4^2 - \zeta^2}} \mathrm{d}\zeta \ge \left(z_4 - \sqrt{z_4^2 - z_3^2} \right)$$

To conclude this

$$\left(1 + 2\frac{\theta}{\lambda} - 2\sqrt{\frac{\theta}{\lambda}} \sqrt{1 + \frac{\theta}{\lambda}} \right) \lambda \le l \le \lambda \quad (114)$$

The inequalities (113) and (114) can be combined to conclude that $\frac{\theta}{\lambda}$ is small when $\frac{h}{l}$ is small. And in this case $\theta \approx \frac{2}{\pi} h$ and $\lambda \approx l$. And all the claims follow. $\square$

Lemma A.11. *Let hull K be a subset of a rectangle $[-l, l] \times [0, h]$, $l, h > 0$. If $K \cap (\{l\} \times [0, h]) \neq \emptyset$ then uniformly for any z in this set $g_K(z) = l\big(1 + o(1)\big)$ as $\frac{h}{l} \to 0$.*

Proof. Assume that $K \cap \mathbb{H} \neq \emptyset$. Otherwise the statement is trivial since $z = l$ and g_K is identity.

Let $K' = [-l, l] \times [0, h]$. Then $K \subset K'$. Take any $z \in K \cap (\{l\} \times [0, h])$. Let $x_+ = l$, $u_+ = g_K(x_+)$ and $v_+ = g_{K'}(x_+)$.

By Lemma A.9 and Lemma A.10 $l \leq u_+ \leq v_+ = l\big(1 + o(1)\big)$. And by an length area principle, for example, Wolff's lemma (Proposition 2.2 of [23]), $0 \leq u_+ - g_K(z) = o(1)\, l$. □

Lemma A.12. *If $i \in K$ then* $\mathrm{cap}_{\mathbb{H}}(K) \geq \frac{1}{4}$.

Proof. First of all note that this is sharp. It is attained by a vertical slit extending from 0 to i.

Now assume that there is K containing i s.t. $\mathrm{cap}_{\mathbb{H}}(K) < \frac{1}{4}$. It is possible to choose $\tilde{K}$ containing K s.t. the capacities are arbitralily close and the boundary of $\tilde{K}$ is a smooth curve. This can be done by choosing a smooth, simple curve γ that separates an interval containing $g_K(K)$ from ∞ in $\mathbb{H}$. Then $\tilde{K}$ is the hull that has $f_K(\gamma)$ as the boundary. Therefore there exists now $\tilde{K}$ s.t. it contains i, $\mathrm{cap}_{\mathbb{H}}(\tilde{K}) < \frac{1}{4}$ and the boundary is a curve.

Therefore, there exists a simple curve $\gamma(t)$, $t \in [0, T]$, parameterized by the capacity so that $0 < T < \frac{1}{4}$ and γ contains some point lying on the line $i + \mathbb{R}$. Now take any point z s.t. $\mathrm{Im}\, z > 4T$, and let $Z_t = X_t + iY_t = g_t(z)$. Then by Loewner equation

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -\frac{2Y_t}{(X_t - U_t)^2 + Y_t} \geq -\frac{2}{Y_t}.$$

Therefore

$$Y_t \geq \sqrt{\big(\mathrm{Im}\, z\big)^2 - 4t} > 0.$$

Hence $z \notin \gamma[0, T]$. This leads to a contradiction: γ doesn't intersect the line $i + \mathbb{R}$. □

Lemma A.13. *Let K be a hull. If $K \cap \big(\mathbb{R} \times \{hi\}\big) \neq \emptyset$ then* $\mathrm{cap}_{\mathbb{H}}(K) \geq \frac{1}{4}h^2$. *If $K \subset [-l, l] \times [0, h]$, then* $\mathrm{cap}_{\mathbb{H}}(K) \leq \mathrm{cap}_{\mathbb{H}}\big([-l, l] \times [0, h]\big)$ *and* $\mathrm{cap}_{\mathbb{H}}\big([-l, l] \times [0, h]\big) = \frac{1}{2\pi} hl\big(1 + o(1)\big)$ *as* $\frac{h}{l} \to 0$.

Proof. The lower bound follows from Lemma A.12 and scaling.

For the upper bound let's use the Schwarz-Christoffel mapping. Write

$$\begin{aligned} \mathrm{cap}_{\mathbb{H}}(K) &= \frac{1}{8}\big(-z_1^2 - z_4^2 + z_2^2 + z_3^2\big) = \frac{1}{4}\big(z_4 - z_3\big)\big(z_4 + z_3\big) \\ &= \frac{1}{2\pi} hl\big(1 + o(1)\big) \end{aligned} \tag{115}$$

This gives the desired upper bound. □

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