

SPECTRA OF LINEAR FRACTIONAL COMPOSITION OPERATORS
AND PROPERTIES OF UNIVERSAL OPERATORS

RIIKKA SCHRODERUS

Academic dissertation

*To be presented for public examination
with the permission of the Faculty of Science of the University of Helsinki
in Auditorium CK112, Exactum, on 5th May 2017 at 12 o'clock noon.*

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
2017

Supervisors

Hans-Olav Tylli and Pekka Nieminen, University of Helsinki, Finland

Pre-examiners

Isabelle Chalendar, Université Paris Est Marne la Vallée, France

Pamela Gorkin, Bucknell University, Pennsylvania, USA

Opponent

Mikael Lindström, Åbo Akademi University, Finland

Custos

Kari Astala, University of Helsinki, Finland

Cover Design: Joonas Jukka

ISBN 978-951-51-3096-9 (paperback)

ISBN 978-951-51-3097-6 (PDF)

<http://ethesis.helsinki.fi/>

Unigrafia Oy
Helsinki 2017

ABSTRACT. The topics of this thesis in mathematics belong to the area of operator theory which, in general, studies linear transformations between complete normed vector spaces. Here, all operators considered are bounded and act on complex separable infinite-dimensional Hilbert space. A prototypical example of Hilbert spaces is formed by the square summable sequences of complex numbers. Other common Hilbert spaces consist of functions which are analytic on some open domain of the complex plane. The characteristic property of analytic functions is that they are locally given by a convergent power series and so the behaviour of such functions is rather rigid.

This thesis consists of the introductory part and three research articles, the first and the third being co-authored with, respectively, E. A. Gallardo-Gutiérrez and H.-O. Tylli. Our focus in the first two articles is in the spectral properties of composition operators which are induced by linear fractional transformations (also known as Möbius maps). As the name suggests, a composition operator composes a function with a fixed mapping called the inducing map. In studying these operators we can take advantage of function theoretic tools, and it is not surprising that the properties of composition operator depend intricately on the inducing map. The spectrum of an operator acting on an infinite-dimensional space generalizes the concept of eigenvalues of a finite matrix. In general, determining the spectrum of a given operator is not an easy task.

In the first article we compute the spectra of composition operators induced by certain linear fractional self-maps of the unit disc. Here the operators act on the whole range of weighted Dirichlet spaces which are Hilbert spaces of analytic functions on the unit disc. Earlier results in this context cover e.g. the classical Hardy space, the weighted Bergman spaces and the classical Dirichlet space. Our results complete the spectral picture of linear fractional composition operators on the weighted Dirichlet spaces. In particular, we found a way to compute the spectra of parabolic and invertible hyperbolic composition operators on the spaces that are contained in the classical Dirichlet space.

The second article continues the study of the spectral properties of linear fractional composition operators in the corresponding setting on the upper half-plane. The analogous spaces of the half-plane differ significantly from their counterparts in the unit disc; for instance, not all linear fractional self-maps of the half-plane induce bounded composition operators. We were able to compute the spectra of all bounded linear fractional composition operators acting on the Hardy, and the weighted Bergman spaces of the upper half-plane. Moreover, we show that the essential spectra coincide with the spectra on all cases. One of the main tools is the Paley-Wiener theorem along with its generalizations which allow us to transfer the computations to certain Hilbert spaces defined on the positive real line.

An operator is called universal if it has such a rich lattice of invariant subspaces that it models any given operator (up to a constant multiple) when restricted to some of its invariant subspaces. The definition of universal operators can be generalized to commuting n -tuples of operators. In the third article we consider general properties of universal operators on separable infinite-dimensional Hilbert spaces. The task of finding and analysing concrete universal operators have gained a lot of attention since they could help in answering the question whether every operator on a separable infinite-dimensional Hilbert space has a non-trivial invariant subspace. We study the properties of the whole class of universal operators and as one of our main results we prove that universal operators have a big essential spectrum containing the origin. In addition, we give new examples of universal operators and universal commuting pairs.

ACKNOWLEDGEMENTS

As a little girl I wanted to become a magician just because I wanted to *know* how magic tricks are done. I guess mathematics has lured me for the same reason, my curiosity for finding out the hidden secrets behind fascinating phenomena. The journey to this dissertation has included many unexpected turns, moments of both magical enlightenment and deep despair, but most importantly many great people who made it possible.

First of all I wish to thank my supervisors Hans-Olav Tylli and Pekka Nieminen for their time and dedication to my thesis. I am deeply grateful to you, Hans, for believing in me and in this project also — and especially — during the times I felt hopelessness. Pekka, I appreciate your perceptive comments to my work and I have learned a lot from your lucid way of presenting mathematics.

I can not express my gratitude enough to my collaborator, Professor Eva A. Gallardo-Gutiérrez, for treating me as one of her own students, always encouraging and inspiring me. My research visits to Madrid turned out to be much more than I could have ever wished for. ¡Eva, *muchísimas gracias por todo lo que has hecho por mí!*

I was very delighted by the encouraging reports from the pre-examiners, Professors Isabelle Chalendar and Pamela Gorkin. I wish to convey my sincerest thanks to them for taking the time to read my thesis. I am honoured for having Professor Mikael Lindström acting as the Opponent of the public examination of my thesis. The very first introduction to linear fractional transformations and composition operators was given to me by Professor Kari Astala, the advisor of my Master and Licentiate Theses, and it is my pleasure to have him acting as the Custos of the public examination of my thesis.

The financial support from the Finnish National Doctoral Programme in Mathematics and its Applications, the Alfred Kordelin Foundation and the Magnus Ehrnrooth Foundation was crucial to this work and is greatly acknowledged. For research visits and conference trips I have received funding from the Doctoral Programme in Mathematics and Statistics of University of Helsinki.

In this tough and still quite male-dominated world of mathematics every single supportive person and word counts. I still remember very well the heartening praises from my lower secondary and high school science teachers Pekka Hakkarainen and Liisa Karppinen which gave me courage to follow my passion for mathematics and science. During the years of my graduate studies — which were numerous — I have had the pleasure of getting to know many people of the staff of the Department of Mathematics and Statistics. Many thanks to Professor Juha Oikkonen for his ongoing support and trust particularly in what comes to teaching mathematics. Thanks to Jani Virtanen for having interest in my work and writing recommendation letters at my early steps towards PhD. To all mathematicians who happen to be females and who I have met along the way, I wish to say that your example has been vital to me.

I consider myself very fortunate by having such awesome friends. I thank every one of you for balancing these often lonely and stressful workdays with laughter and late nights. Special thanks to Laura Sutinen for the heartfelt friendship that started 15 years ago when we both were first-year math students. Laura, your wisdom and boldness have had a great influence on me. Thanks to Anna-Kaisa Harju, Minna Nyrhinen-Blázquez and Emilia Savolainen I got to know the best of Madrid. We all live in Helsinki at the moment and I am overjoyed by the fact that the spirit of Spain is present every time we meet.

I am grateful to my parents for trusting me to search my very own way in life yet letting me know that they are there for me. There is no better way to build up *sisu* than being a little sister and for that I owe thanks to my big brother, Ilkka. Playing card and board games with him throughout my childhood taught me many things, but primarily to try harder and never to give up.

The last and the greatest thanks I reserve to my partner, Mikko. Having been able to share the high-lights as well as the feelings of frustration with him has brightened the views and smoothened the curves of this roller coaster ride. Mikko, your love and understanding have been invaluable. \diamond

Helsinki, April 2017

Riikka Schroderus

LIST OF INCLUDED ARTICLES

This dissertation consists of the introductory part and the following three articles:

- [A] Eva A. Gallardo-Gutiérrez and Riikka Schroderus. The spectra of linear fractional composition operators on the weighted Dirichlet spaces. *Journal of Functional Analysis*, Volume 271, Issue 3, 720–745, (2016).
- [B] Riikka Schroderus. Spectra of linear fractional composition operators on the Hardy and weighted Bergman spaces of the half-plane. *Journal of Mathematical Analysis and Applications*, Volume 447, Issue 2, 817–833, (2017).
- [C] Riikka Schroderus and Hans-Olav Tylli. On universal operators and universal pairs. Preprint: arXiv: 1702.05276 (2017).

The articles listed above are henceforth referred to by [A], [B] and [C] whereas other references are numbered [1], [2], [3], ...

The joint articles [A] and [C] contain significant contribution by the candidate.

Articles [A] and [B] are reprinted with the permission of their respective copyright holders.

CONTENTS

Acknowledgements	v
List of included articles	vii
1. Overview	1
2. Preliminaries	2
2.1. Linear fractional composition operators	2
2.2. Hilbert spaces of analytic functions	3
2.3. Weighted Dirichlet spaces of \mathbb{D}	4
2.4. Spectra of linear operators	5
3. Spectra in the unit disc setting	7
3.1. Background	8
3.2. Main results of Article [A]	10
3.3. Supplementary results regarding the essential spectra	12
3.4. Related developments	14
4. Spectra in the half-plane setting	14
4.1. Hardy, weighted Bergman, and weighted Dirichlet spaces of Π^+	14
4.2. Parabolic and hyperbolic composition operators on Π^+	15
4.3. Results of Article [B]	17
4.4. Main tools for finding the spectra	17
4.5. Further questions	19
5. Essential spectra of the non-invertible parabolic composition operators	19
6. Universal operators	24
6.1. Main results of Article [C]	25
6.2. Further questions	29
Closing words	29
References	30

1. OVERVIEW

The primary theme of this dissertation falls into the area known as function-theoretic operator theory—a branch of mathematics that studies (bounded) linear transformations, usually called *operators*, on analytic function spaces. The defining property of analytic functions is that they are locally given as convergent power series. More precisely, any analytic function $f : \Omega \rightarrow \mathbb{C}$, where Ω is an open subset of the complex plane \mathbb{C} , can be written as $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all z close enough to any given $z_0 \in \Omega$. Typically some of the fundamental questions when studying operators are to determine if the operator is bounded, or moreover compact, and to find its *spectrum* on a given space. The spectrum of an operator acting on an infinite dimensional space generalizes the concept of eigenvalues of a finite matrix. Nonetheless, determining the spectrum of an operator is a challenging and complicated task in general.

Composition operators $f \mapsto C_\phi f = f \circ \phi$ with analytic symbols $\phi : \Omega \rightarrow \Omega$ offer an intriguing example of a concrete operator class where the behaviour of the operator C_ϕ depends critically on the properties of the inducing map ϕ . The research of composition operators on analytic function spaces has its roots in the late 19th century. Namely, the solutions to the famous *Schröder's equation* [54],

$$f \circ \phi = \lambda f,$$

for a fixed ϕ , can be interpreted as the eigenfunctions f of C_ϕ corresponding to the eigenvalue λ . Here, the domain of the functions is the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In 1884, Königs [34] showed that Schröder's equation has a non-trivial solution when ϕ is a self-map of the unit disc, i.e. $\phi(\mathbb{D}) \subseteq \mathbb{D}$, such that $\phi(a) = a$ for some (unique) $a \in \mathbb{D}$ and $0 < |\phi'(a)| < 1$. It took a few more decades of intense work by great mathematicians until the actual research of (bounded) composition operators—first on Hardy spaces $H^p(\mathbb{D})$ for $1 \leq p < \infty$ —started around the 1960s. Even though the ongoing interest in composition operators has already led to a very rich theory (to get a glimpse, see [55, 16, 23], for instance) there are still numerous questions waiting to be solved.

In this thesis we study composition operators induced by *linear fractional self-maps* of \mathbb{D} and the upper half-plane $\Pi^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$. In particular, we study the spectral properties of these operators on certain Hilbert spaces of analytic functions. Composition operators arising from the linear fractional transformations are one of the few operator classes for which the spectra can be explicitly computed and, therefore, they have diverse applications. This is valuable since the linear fractional composition operators also serve as models for variety of operators. The behaviour of linear fractional composition operators depends, however, strongly on the space which is seen already by considering the Hardy spaces $H^2(\mathbb{D})$ and $H^2(\Pi^+)$. The systematic analysis of spectral properties of composition operators was initiated in the 1960s when Nordgren [46] computed the spectra for invertible composition operators in $H^2(\mathbb{D})$. A natural continuation was to consider the same question in $H^p(\mathbb{D})$, for $1 \leq p < \infty$, and in the disc algebra, which was done by Kamowitz [32, 33]. The abovementioned Königs' solution to Schröder's equation also paved the way to the spectral theory of compact composition operators studied by Caughran and Schwartz [5].

To this date, the research on the spectra of composition operators on analytic function spaces has remained very active and numerous authors have contributed to this subject, see e.g. [10, 16, 29, 27, 50].

Another topic of this thesis is the *universality of operators*. The universal operators (or universal *models*) on Hilbert spaces were introduced in 1959 by Rota [52]. Motivated by the *Invariant Subspace Problem*, the aim was to find an operator having such a rich lattice of invariant subspaces that it could represent *any* given (bounded) operator when restricted to some of its invariant subspaces. Rather surprisingly, such operators do exist as Rota showed by giving a concrete example. At the moment of writing we have a large amount of concrete examples of universal operators from different operator classes including weighted shift operators, composition operators and (adjoints of) Toeplitz operators, see [47, 49, 51, 11, 14]. In this thesis we analyse the properties of the class of universal operators and consider universal *pairs* defined recently in [44]. The universal operators on Hilbert spaces are also linked to the invertible *hyperbolic* composition operators. Indeed, as one of the main results of this thesis, we compute the spectra and the point spectra of invertible hyperbolic composition operators on the weighted Dirichlet spaces contained in the disc algebra. This allows us to give yet another concrete example of a universal operator.

The rest of this introductory part is organized as follows: In Section 2 we introduce the central concepts of this thesis at a fairly general level. In Sections 3, 4 and 6 we will give the explicit definitions needed in each context and present the main results of Articles [A], [B] and [C], respectively. We will also provide additional information about the essential spectra of parabolic and invertible hyperbolic composition operators supplementing the results in [A]; see Propositions 3.1 and 3.2 for, respectively, the invertible hyperbolic and parabolic composition operators and Section 5 for the non-invertible parabolic ones.

2. PRELIMINARIES

2.1. Linear fractional composition operators. We will consider composition operators C_ϕ ,

$$f \mapsto C_\phi f = f \circ \phi,$$

where $\phi : \Omega \rightarrow \Omega$ and $f : \Omega \rightarrow \mathbb{D}$ are analytic in an open subset $\Omega \subset \mathbb{C}$. In what follows, Ω is usually the unit disc \mathbb{D} or the upper half-plane Π^+ . Our interest lies in the special case where the symbol ϕ is a linear fractional transformation and so, unless otherwise stated, ϕ is assumed to be such.

We first consider the linear fractional transformations (abbreviated henceforth as LFTs) in general. For more information on this topic, see e.g. [55, Chapter 0] or [31, Chapter 2]. The LFTs, i.e. the Möbius maps or Möbius transformations, are mappings of the form

$$(2.1) \quad \phi(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

With the convention

$$(2.2) \quad \phi(\infty) = a/c \quad \text{and} \quad \phi(-d/c) = \infty$$

these mappings become meromorphic bijections from $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ onto itself. In addition, the LFTs are conformal, i.e. angle-preserving. In the sequel, $\phi(z) \not\equiv z$. We say that z_0 is a *fixed point* of ϕ if $\phi(z_0) = z_0$. The LFTs have exactly two fixed points in $\overline{\mathbb{C}}$ counting multiplicity.

Conjugation is an important tool in working with LFTs. We say that ϕ and $\tilde{\phi}$ are conjugate to each other, or that ϕ and $\tilde{\phi}$ belong to the same *conjugacy class*, if there exists a LFT g such that

$$(2.3) \quad \phi = g^{-1} \circ \tilde{\phi} \circ g.$$

Note that g necessarily maps the fixed points of ϕ into the fixed points of $\tilde{\phi}$. Moreover, if z_0 and w_0 are fixed points of ϕ and $\tilde{\phi}$, respectively, satisfying $g(z_0) = w_0$ and $z_0, w_0 \neq \infty$, then it holds that

$$\phi'(z_0) = \tilde{\phi}'(w_0).$$

If ϕ has only one fixed point (of multiplicity two) we say that ϕ is *parabolic*. From (2.1) and (2.2) it can be deduced that in this case we can conjugate ϕ to a LFT $\tilde{\phi}$ having ∞ as its only fixed point, that is,

$$\tilde{\phi}(z) = z + b, \text{ where } b \neq 0.$$

For the fixed point $p \in \overline{\mathbb{C}}$ of a parabolic LFT ϕ , we have that

$$(2.4) \quad \phi_n(z) := \phi \circ \phi \cdots \circ \phi(z) \longrightarrow p, \text{ as } n \longrightarrow \infty,$$

for all $z \in \overline{\mathbb{C}}$. That is, p is an *attractive* fixed point.

If ϕ has two fixed points p and q , we can choose the conjugating map g in (2.3) such that $g(\{p, q\}) = \{0, \infty\}$. It follows that $\tilde{\phi}$ fixes 0 and ∞ and so

$$\tilde{\phi}(z) = \mu_\phi z,$$

where $\mu_\phi \in \mathbb{C} \setminus \{0, 1\}$ is called the *multiplier* (or *characteristic constant*) of ϕ . The type of a LFT having two fixed points depends on this multiplier which is unique up to its reciprocal $1/\mu_\phi$. Hence, we can assume that $|\mu_\phi| \leq 1$. The LFTs having the same multiplier belong to the same conjugacy class and, moreover, $\mu_\phi = \tilde{\phi}'(0) = \phi'(p)$.

If $|\mu_\phi| = 1$, then ϕ is called *elliptic*. If $\mu_\phi \in (0, 1)$, then ϕ is *hyperbolic*, and if $\mu_\phi \in \mathbb{D} \setminus [0, 1)$, we call ϕ *loxodromic*. The fixed points of elliptic maps are *neutral* in the sense that the formula in (2.4) is not satisfied for any $z \in \overline{\mathbb{C}}$ that is not a fixed point. Hyperbolic and loxodromic maps, however, have one attractive fixed point, say p , and the formula (2.4) holds for all $z \in \overline{\mathbb{C}} \setminus \{q\}$. The other fixed point q is *repulsive*, that is, for all $z \in \overline{\mathbb{C}} \setminus \{p\}$,

$$\phi_{-n}(z) := \phi^{-1} \circ \phi^{-1} \cdots \circ \phi^{-1}(z) \longrightarrow q, \text{ as } n \longrightarrow \infty.$$

2.2. Hilbert spaces of analytic functions. We first recall some very basic facts on Hilbert spaces and operators acting on them in general, mainly to fix the notation. Denote by \mathcal{H} a complex separable infinite-dimensional Hilbert space. Recall that the norm in \mathcal{H} is induced by the inner product, that is $\|f\|_{\mathcal{H}}^2 = \langle f, f \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$. Also, when \mathcal{H} is a Hilbert space of functions $f : \Omega \longrightarrow \mathbb{C}$, the point evaluation $f \longmapsto f(w)$ is a bounded

linear functional $\mathcal{H} \rightarrow \mathbb{C}$ for all $w \in \Omega$. Hence, for all $w \in \Omega$, there exists a function $\kappa_w \in \mathcal{H}$ such that

$$f(w) = \langle f, \kappa_w \rangle_{\mathcal{H}}.$$

The function κ_w is called the *reproducing kernel* of \mathcal{H} at the point w .

The algebra of bounded linear operators $T : \mathcal{H} \rightarrow \mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. We say that an operator T is invertible if there exists an operator $T^{-1} \in \mathcal{L}(\mathcal{H})$, called the inverse of T , such that $T^{-1}T = I = TT^{-1}$. The adjoint of $T \in \mathcal{L}(\mathcal{H})$, denoted by T^* , is a bounded operator $\mathcal{H} \rightarrow \mathcal{H}$ satisfying, for all $f, g \in \mathcal{H}$,

$$\langle Tf, g \rangle_{\mathcal{H}} = \langle f, T^*g \rangle_{\mathcal{H}}.$$

The spaces under consideration in this thesis will be Hilbert spaces of analytic functions containing some very well-known ones such as the Dirichlet, Hardy, and Bergman spaces of \mathbb{D} or Π^+ . In the most classical setting one considers operators in the Hardy space $H^2(\mathbb{D})$ which consists of the analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, having Taylor coefficients $a_n = f^{(n)}(0)/n!$ belonging to the sequence space $\ell^2(\mathbb{N})$,

$$\ell^2(\mathbb{N}) = \left\{ a = (a_n) : \|a\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} < \infty \right\}.$$

Hence, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\|f\|_{H^2(\mathbb{D})} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

The norm in the Hardy space $H^2(\mathbb{D})$ also has an integral presentation,

$$\|f\|_{H^2(\mathbb{D})} = \lim_{r \rightarrow 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2}.$$

In the next subsection we define the weighted Dirichlet spaces of \mathbb{D} and consider briefly the boundedness of composition operators acting on them. The corresponding spaces of the upper half-plane are defined in Section 4.

2.3. Weighted Dirichlet spaces of \mathbb{D} . The weighted Dirichlet spaces \mathcal{D}_β , for $\beta \in \mathbb{R}$, consist of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\beta} \right)^{1/2} < \infty.$$

The spaces \mathcal{D}_β , for $\beta \in \mathbb{R}$, are Hilbert spaces with orthonormal basis $(z^n/(n+1)^\beta)$.

In our parametrization, the classical Dirichlet space $\mathcal{D}^2(\mathbb{D}) = \mathcal{D}_{1/2}$, the Hardy space $H^2(\mathbb{D}) = \mathcal{D}_0$ and the Bergman space $A^2(\mathbb{D}) = \mathcal{D}_{-1/2}$. Moreover, from the definition we see that the spaces are continuously embedded in each other, that is, $\mathcal{D}_\beta \subsetneq \mathcal{D}_\gamma$ for all $\gamma < \beta$.

The spaces \mathcal{D}_β , for $\beta < 0$, are also (and perhaps, *better*) known as the weighted Bergman spaces $A_\alpha^2(\mathbb{D})$, for $\alpha = -1 - 2\beta > -1$, with equivalent norm

$$\|f\|_{A_\alpha^2(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) \right)^{1/2}, \quad z = x + iy,$$

where $dA(z) = \frac{dx dy}{\pi}$ (for more information on Bergman spaces, see e.g. [25]).

Denote by $H^\infty(\mathbb{D})$ the algebra of bounded analytic functions $\mathbb{D} \rightarrow \mathbb{C}$. It is well known that, for $\beta \leq 0$, $H^\infty(\mathbb{D}) \subset \mathcal{D}_\beta$ and, moreover, that $fg \in \mathcal{D}_\beta$ for $f \in H^\infty(\mathbb{D})$ and $g \in \mathcal{D}_\beta$. This does not hold e.g. for the classical Dirichlet space, in fact, there are functions in $H^\infty(\mathbb{D})$ that do not belong to \mathcal{D}_β for $\beta \geq 1/4$ (see [23, Prop. 3.10]).

Recall that the *disc algebra* $\mathcal{A}(\mathbb{D})$ consists of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ which are continuous on $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. Here, and in the sequel, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The space $\mathcal{A}(\mathbb{D})$ is a Banach algebra endowed with the sup-norm, $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. Using the Cauchy-Schwartz inequality we see that the small spaces—by which we mean the spaces \mathcal{D}_β for $\beta > 1/2$ —are contained $\mathcal{A}(\mathbb{D})$. Indeed, for all $z \in \overline{\mathbb{D}}$, we have that

$$\begin{aligned} |f(z)| &= \left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \sum_{n=0}^{\infty} |a_n| \leq \left(\sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\beta} \right)^{1/2} \left(\sum_{n=0}^{\infty} (n+1)^{-2\beta} \right)^{1/2} \\ &= m_\beta \cdot \|f\|_\beta, \end{aligned}$$

where $m_\beta = \left(\sum_{n=0}^{\infty} (n+1)^{-2\beta} \right)^{1/2} < \infty$ whenever $\beta > 1/2$. It follows that, for all $\beta > 1/2$, $\mathcal{D}_\beta \subset \mathcal{A}(\mathbb{D})$.

On $H^2(\mathbb{D}) = \mathcal{D}_0$ the boundedness of composition operators is well understood. Actually, due to Littlewood Subordination Theorem (which gives the boundedness for C_ϕ in a special case $\phi(0) = 0$) together with the automorphism-invariance of $H^2(\mathbb{D})$, we have that C_ϕ is bounded when ϕ is *any* analytic self-map of \mathbb{D} , in particular, any LFT. The same argument holds also on the weighted Bergman spaces of \mathbb{D} , i.e. on \mathcal{D}_β for $\beta < 0$, which all contain the Hardy space.

On \mathcal{D}_β for $\beta > 0$ —for instance, on the classical Dirichlet space—it is no longer true that all composition operators are bounded. One reason is that since $C_\phi z = \phi$ and $z \in \mathcal{D}_\beta$ for all $\beta \in \mathbb{R}$, a necessary condition for C_ϕ to be bounded is that ϕ itself belongs to \mathcal{D}_β in question. However, the linear fractional composition operators are bounded on \mathcal{D}_β , for $\beta > 0$ as well (see [37, Thm. D'] which extends [29, Thm. 5]).

2.4. Spectra of linear operators. In what follows, we will state the definitions, *inter alia*, regarding the spectrum (of a bounded linear operator) only in Hilbert space even though many of them make sense for any Banach space as well.

Let $T \in \mathcal{L}(\mathcal{H})$. The spectrum of T is defined by

$$\sigma(T) = \sigma(T; \mathcal{H}) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible on } \mathcal{H}\}.$$

By the Open Mapping Theorem, the operator $T - \lambda I : \mathcal{H} \rightarrow \mathcal{H}$ fails to be invertible if it is not bounded below or it is not onto. If $T - \lambda I$ is not bounded below on \mathcal{H} , we say that λ belongs to the *approximate point spectrum*, denoted by $\sigma_a(T; \mathcal{H})$. The set of eigenvalues

of T , i.e. those $\lambda \in \mathbb{C}$ such that $Tf = \lambda f$ for some non-zero $f \in \mathcal{H}$, is called the *point spectrum* and denoted by $\sigma_p(T; \mathcal{H})$. Clearly $\sigma_p(T) \subset \sigma_a(T)$. The spectrum of $T \in \mathcal{L}(\mathcal{H})$ is always a non-empty, compact subset of \mathbb{C} .

The spectral radius $\rho(T; \mathcal{H}) = \max\{|\lambda| : \lambda \in \sigma(T; \mathcal{H})\}$ satisfies the equation

$$(2.5) \quad \rho(T; \mathcal{H}) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

An important subset of the spectrum is the *essential spectrum* where the *Fredholm operators* come into play. In the sequel, we use the notation $\text{Ran}(T; \mathcal{H})$ and $\text{Ker}(T; \mathcal{H})$ (or $\text{Ran} T$ and $\text{Ker} T$ if there is no ambiguity on the space) for the range and kernel of $T \in \mathcal{L}(\mathcal{H})$, respectively. The codimension of the range is defined by $\text{codim Ran}(T; \mathcal{H}) = \dim \mathcal{H}/T(\mathcal{H})$. Recall that if $\text{codim Ran}(T; \mathcal{H})$ is finite, then the range $T(\mathcal{H})$ is closed (see [43, Lemma III.16.2], for instance). An operator $T \in \mathcal{L}(\mathcal{H})$ is called Fredholm if $\dim \text{Ker} T$ and $\text{codim Ran} T$ are finite. The *index* of a Fredholm operator T is defined by

$$\text{ind } T = \dim \text{Ker } T - \text{codim Ran } T.$$

The essential spectrum is defined by

$$\sigma_{\text{ess}}(T; \mathcal{H}) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm on } \mathcal{H}\}.$$

The essential spectrum of $T \in \mathcal{L}(\mathcal{H})$ is always a closed set (and non-empty if $\dim \mathcal{H} = \infty$). From the definition of Fredholm operators it is easy to see that, for instance, the eigenvalues of T that are of infinite multiplicity are contained in the essential spectrum. The essential spectrum plays an important role also in recognizing universal operators; we will discuss this in Section 6.

Atkinson's Theorem (see [45, Thm. 1.4.16], for instance) states that the Fredholmness of $T \in \mathcal{L}(\mathcal{H})$ is equivalent to the invertibility of the quotient element $T + \mathcal{K}(\mathcal{H})$ in the Calkin algebra $\mathcal{C}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the closed ideal of the compact operators on \mathcal{H} . Hence, we have an alternative definition for the essential spectrum:

$$\sigma_{\text{ess}}(T; \mathcal{H}) = \{\lambda \in \mathbb{C} : (T - \lambda I) + \mathcal{K}(\mathcal{H}) \text{ is not invertible in } \mathcal{C}(\mathcal{H})\}.$$

For the essential spectral radius $\rho_{\text{ess}}(T; \mathcal{H}) = \max\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T; \mathcal{H})\}$ we have an analogous formula to (2.5), namely,

$$\rho_{\text{ess}}(T; \mathcal{H}) = \lim_{n \rightarrow \infty} \|T^n\|_{\text{ess}}^{1/n},$$

where the *essential operator norm* of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\|T\|_{\text{ess}} = \inf \{\|T - K\| : K \text{ is a compact operator on } \mathcal{H}\}.$$

Recall that the operators $S \in \mathcal{L}(\mathcal{H}_1)$ and $T \in \mathcal{L}(\mathcal{H}_2)$ are *similar* if there exists a linear isomorphism $J : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$S = J^{-1}TJ.$$

We will constantly use the fact that the (approximative, point, essential) spectra of similar operators coincide. Also, if T is invertible with the inverse T^{-1} , then

$$\sigma(T^{-1}) = \{1/\lambda : \lambda \in \sigma(T)\}.$$

The above equation holds if we replace σ by the point spectrum σ_p . For the adjoint T^* it holds that

$$\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}.$$

A similar identity is true for the essential spectrum $\sigma_{ess}(T^*)$ as well.

The LFTs possess some nice properties which are useful when determining the spectra of linear fractional composition operators. Firstly, the LFTs in the same conjugacy class induce similar composition operators whenever the composition operator induced by the conjugating map is bounded and invertible: If $\phi = g^{-1} \circ \tilde{\phi} \circ g$, then

$$C_\phi = C_g C_{\tilde{\phi}} C_{g^{-1}} = C_g C_{\tilde{\phi}}^{-1} C_g^{-1}$$

provided that the operators C_g and C_g^{-1} are bounded. Consequently, to compute the spectra of composition operators induced by LFTs of a certain conjugacy class, it is enough to compute the spectrum for any representative of the conjugacy class. Usually, the representative is chosen to be a *normalized* one, that is, having fixed points in the set $\{-1, 0, 1, \infty\}$. Secondly, by considering the multiplier of ϕ_n (or in the parabolic case the iterates of $z \mapsto z + b$), where

$$\phi_n := \phi \circ \phi \circ \dots \circ \phi \quad (n \text{ times}),$$

we see that the n th iterate, for all $n \in \mathbb{N}$, of any non-loxodromic LFT ϕ is of the same type (elliptic/hyperbolic/parabolic, automorphism/non-automorphism) as ϕ . This property is often beneficial in determining the (essential) spectral radius since $C_\phi^n = C_{\phi_n}$. Note however, that the conjugacy class is not invariant under iteration. Also, C_ϕ is invertible if and only if ϕ is an automorphism of the domain of the space. As we will see, the derivative of ϕ at one of its fixed point plays a major role in the spectrum of C_ϕ on the spaces considered.

3. SPECTRA IN THE UNIT DISC SETTING

In this section we consider the composition operators C_φ acting on the weighted Dirichlet spaces \mathcal{D}_β , for all $\beta \in \mathbb{R}$, where φ is a linear fractional self-map of \mathbb{D} . There are altogether 8 essentially different kinds of linear fractional self-maps of \mathbb{D} (3 automorphisms and 5 non-automorphisms) which can be distinguished by their type and fixed point configuration. In Article [A] our aim was to find the spectra of parabolic and invertible hyperbolic composition operators on certain weighted Dirichlet spaces, that is, to fill the empty boxes in Table 1 below.

φ	$\beta \leq 0$	$\beta \in (0, 1/2)$	$\beta = 1/2$	$\beta \in (1/2, 1)$	$\beta \geq 1$
Parabolic autom.	\mathbb{T} , [16]	\mathbb{T} , [50]	\mathbb{T} , [27], [23]	\mathbb{T} , [50]	
Parab. non-autom.	$\mathcal{S} := \overline{\{e^{-\alpha t} : t \geq 0\}}$, [10]	\mathcal{S} , [50]	\mathcal{S} , [27]	\mathcal{S} , [50]	
Hyperbolic autom.	$\{\varphi'(p)^\gamma \leq \lambda \leq \varphi'(p)^{-\gamma}\}$, [16]		\mathbb{T} , [27]		
Hyperb. non-autom. $p \in \mathbb{T}$	$\{ \lambda \leq \varphi'(p)^{-\gamma}\} \cup \{\varphi'(p)^k : k = 0, 1, \dots\}$, [29]				
Hyperb. non-autom. $p \in \mathbb{D}$	$\{ \lambda \leq \varphi'(p)^\gamma\} \cup \{\varphi'(p)^k : k = 0, 1, \dots\}$, [29]				
C_φ compact	$\{\varphi'(p)^k : k = 0, 1, 2, \dots\} \cup \{0\}$				
Elliptic autom.	$\overline{\{\varphi'(p)^k : k = 0, 1, 2, \dots\}}$, [16]				

TABLE 1. The previously known spectra of linear fractional composition operators. Here, p denotes the attractive fixed point of φ , when it exists, and otherwise (φ elliptic) p is the fixed point in \mathbb{D} . Above $\gamma = (1 - 2\beta)/2$ and $\alpha = \varphi''(p)$.

3.1. Background. To get an overview, let us briefly take a look at the earlier results regarding the spectra of linear fractional composition operators in this setting (for a short summary, see Table 1 above). Notice that the constant functions belong to \mathcal{D}_β , for all $\beta \in \mathbb{R}$, and $C_\varphi 1 = 1$. Therefore, any $C_\varphi : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ has at least the (trivial) eigenvalue 1.

The simplest examples are LFTs (that can be conjugated to a mapping) of the form

$$\varphi_s(z) = sz, \quad \text{where } s \notin \{0, 1\}, \quad |s| \leq 1,$$

having 0 and ∞ as its fixed points. Depending on the constant $s = \varphi'_s(0)$, the map φ_s is an elliptic automorphism of \mathbb{D} (if $|s| = 1$), a hyperbolic non-automorphism (if $s \in (0, 1)$) or a loxodromic mapping (if $s \notin (0, 1)$ and $|s| < 1$). Since $C_{\varphi_s} z^k = s^k z^k$, the operator $C_{\varphi_s} : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$, for all $\beta \in \mathbb{R}$, is a diagonal operator with respect to the orthonormal basis $(z^n / (n+1)^\beta)$, i.e. we can write

$$C_{\varphi_s} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & s & 0 & 0 & \dots \\ 0 & 0 & s^2 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

In the elliptic case the spectrum of C_{φ_s} is either the unit circle \mathbb{T} or a finite subgroup $\{s^k : k = 0, 1, 2, \dots\}$ of \mathbb{T} if s is a root of unity. When φ_s is a hyperbolic or loxodromic LFT having 0 as the attractive fixed point, the diagonal entries of C_{φ_s} converge to zero and so, C_{φ_s} is compact. It follows that $\sigma(C_{\varphi_s}; \mathcal{D}_\beta)$ consists of 0 and the discrete eigenvalues

of C_{φ_s} , which in this case are $\{s^k : k = 0, 1, 2, \dots\}$. The compactness of C_{φ_s} , where φ_s is a hyperbolic or a loxodromic mapping having the attractive fixed point in \mathbb{D} , can also be seen from the fact that $\overline{\varphi_s(\mathbb{D})} \subset \mathbb{D}$, see [16, Thm. 7.20]. Note that these hyperbolic and loxodromic maps are special cases of Königs' mappings for which Schröder's equation has non-trivial solutions!

Hurst [29] computed the spectra for C_φ on \mathcal{D}_β , for all $\beta \in \mathbb{R}$, when φ is a hyperbolic non-automorphism having exactly one fixed point in \mathbb{T} . If the attractive fixed point $p \in \mathbb{T}$, then $\sigma(C_\varphi; \mathcal{D}_\beta)$ is a closed, origin-centered disc with radius $\varphi'(p)^{(2\beta-1)/2}$ together with eigenvalues of the form $\{\varphi'(p)^k : k = 0, 1, \dots\}$. Note that since $\varphi'(p) \in (0, 1)$, the radius decreases as the spaces get smaller and so, depending on β , the points $\varphi'(p)^k$ fall inside or outside the closed disc. If $p \in \mathbb{D}$, then $\sigma(C_\varphi; \mathcal{D}_\beta)$ is again a closed, origin-centered disc together with eigenvalues $\{\varphi'(p)^k : k = 0, 1, \dots\}$. In this case the radius of the disc $\varphi'(p)^{(1-2\beta)/2}$ increases as the spaces get smaller. The strategy in his proofs is to compute the spectra first on \mathcal{D}_β , for $\beta \leq 0$ or $\beta < 0$, and then use a certain duality argument [29, Thm. 5] to obtain the spectra on small spaces. (In fact, the result in [29, Thm. 5] is of great value for us as well and it will be re-encountered in the next subsection.)

Any hyperbolic automorphism of \mathbb{D} can be conjugated to a mapping of the form

$$\varphi_r(z) = \frac{z+r}{1+rz}, \quad r \in (0, 1),$$

having -1 and 1 (the attractive one) as its fixed points. The multiplier of φ_r is $\lambda_{\varphi_r} = \frac{1-r}{1+r} \in (0, 1)$. The spectra of C_{φ_r} on \mathcal{D}_β , for $\beta \leq 0$, (see [16]) are annuli centered at the origin and radii depending on β and the multiplier of φ_r . Moreover, each interior point of the spectrum is an eigenvalue of infinite multiplicity so that the essential spectrum coincides with the spectrum on all spaces. In the classical Dirichlet space $\mathcal{D}^2 = \mathcal{D}_{1/2}$ the spectrum of an invertible hyperbolic composition operator is the unit circle as Higdon [27] proved by taking the computation via similarity to the Dirichlet space of the (upper) half-plane $\mathcal{D}_0(\Pi^+)$ and, further, to $L^2([0, \infty), t \frac{dt}{2\pi})$.

In the parabolic case it is enough to consider the composition operators induced by the LFTs of the form

$$\varphi_\alpha(z) = \frac{(2-\alpha)z + \alpha}{-\alpha z + 2 + \alpha}, \quad \operatorname{Re} \alpha \geq 0.$$

The fixed point of φ_α is 1. By considering $\tau_\alpha : \Pi^+ \rightarrow \Pi^+$ satisfying $\tau_\alpha = h \circ \varphi \circ h^{-1}$, where $h(z) = i \frac{1+z}{1-z}$ is a LFT mapping \mathbb{D} onto Π^+ , it is easy to see that φ_α is an automorphism of \mathbb{D} if and only if $\operatorname{Re} \alpha = 0$. In the automorphism case, the spectrum $\sigma(C_{\varphi_\alpha}; \mathcal{D}_\beta) = \mathbb{T}$ for $\beta < 1$. When φ_α is a parabolic non-automorphism of \mathbb{D} , the spectrum of C_{φ_α} on \mathcal{D}_β , for $\beta < 1$, is the spiral $\{e^{-\alpha t} : t \in [0, \infty)\} \cup \{0\}$. The above spectra of C_{φ_α} were first computed in the Hardy space (for $\operatorname{Re} \alpha = 0$ by Nordgren [46] and for $\operatorname{Re} \alpha > 0$ by Cowen [10]), and then extended to the weighted Bergman spaces by Cowen and MacCluer (see [16]). In the automorphism case each point in the spectrum is an eigenvalue of infinite multiplicity and so the spectrum coincides with the essential spectrum on \mathcal{D}_β for $\beta \leq 0$.

(In fact, this is true for all $\beta \in \mathbb{R}$, see Proposition 3.2.) In the classical Dirichlet space $\mathcal{D}^2 = \mathcal{D}_{1/2}$ the spectra for invertible and non-invertible parabolic composition operators were computed by Higdon [27] (see also an alternative proof for the automorphism case in [23]). Subsequently, Pons [50] was able to extend the results further for $\beta < 1$ by interpolation.

In Article [A] we completed the spectral picture in this context, i.e. computed the spectra in the hyperbolic automorphism case for $\beta \in (0, 1/2) \cup (1/2, \infty)$ and in the parabolic (automorphism and non-automorphism) case for $\beta \geq 1$. In particular, we found a way to compute the spectra of parabolic or invertible hyperbolic composition operators on small spaces, where neither the techniques used in spaces that contain $H^\infty(\mathbb{D})$ nor interpolation are possible.

3.2. Main results of Article [A]. When considering the spectra of linear fractional composition operators, the most interesting case seems to be when the operator is induced by the hyperbolic automorphism of \mathbb{D} . The surprise is in the small spaces as we will see shortly. First, let us look upon the spectra on the scale $\beta \in (0, 1/2)$. As one might expect, the spectrum of an invertible hyperbolic composition operator on \mathcal{D}_β for $\beta \in (0, 1/2)$ is an annulus that shrinks towards the unit circle as $\beta \rightarrow 1/2$:

Theorem ([A, Thm. 3.1]). *Let $\varphi_r(z) = \frac{z+r}{1+rz}$, where $r \in (0, 1)$. For $\beta \in (0, 1/2)$ the spectrum of $C_{\varphi_r} : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ is the annulus*

$$\sigma(C_{\varphi_r}; \mathcal{D}_\beta) = \left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^\gamma \leq |\lambda| \leq \left(\frac{1+r}{1-r} \right)^\gamma \right\},$$

where $\gamma = (1 - 2\beta)/2$. Moreover, the point spectrum of C_{φ_r} contains the open annulus

$$\left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^\gamma < |\lambda| < \left(\frac{1+r}{1-r} \right)^\gamma \right\},$$

where all eigenvalues are of infinite multiplicity.

The vital part in the proof of [A, Thm. 3.1] is to show that, for $-(1 - 2\beta)/2 < \operatorname{Re} w < (1 - 2\beta)/2$, the functions

$$g_w(z) = \left(\frac{1+z}{1-z} \right)^w$$

satisfying Schröder's eigenvalue equation

$$C_{\varphi_r} g_w = \left(\frac{1+r}{1-r} \right)^w g_w,$$

belong to \mathcal{D}_β for $\beta \in (0, 1/2)$ as they do for $\beta \leq 0$. Also, since every interior point of the spectrum is an eigenvalue of infinite multiplicity, we have that $\sigma(C_{\varphi_r}; \mathcal{D}_\beta) = \sigma_{\text{ess}}(C_{\varphi_r}; \mathcal{D}_\beta)$ for $\beta \in (0, 1/2)$. It should be remarked that in Article [A] there is a minor misprint appearing three times at the bottom of p. 730: the sequence in the exponent of the eigenvalue and the corresponding eigenfunctions should be

$$\frac{i2\pi n}{\ln \left(\frac{1-r}{1+r} \right)}$$

instead of $i2\pi n$. This does not have effect on the computations or results.

In order to find the spectra of the parabolic and invertible hyperbolic composition operators C_φ on \mathcal{D}_β for $\beta > 1/2$, the key observation is the following reflection principle (see also [A, Cor. 3.6]):

Lemma ([A, Lemma 3.7]). *Suppose φ , $\varphi(z) = (az + b)(cz + d)^{-1}$, is a linear fractional self-map of \mathbb{D} and let $\psi(z) = (\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})^{-1}$. Then for all $\beta > 1/2$,*

$$\sigma(C_\varphi^*; \mathcal{D}_\beta) = \sigma(C_\psi; \mathcal{D}_{-\beta+1}).$$

The proof of the lemma above uses the similarity of the operators $C_\varphi^*|_{z\mathcal{D}_\beta} : z\mathcal{D}_\beta \rightarrow z\mathcal{D}_\beta$ and $\tilde{C}_\psi : z\mathcal{D}_{-\beta+1} \rightarrow z\mathcal{D}_{-\beta+1}$, where $\tilde{C}_\psi f = C_\psi f - f(\psi(0))$ (see [A, Cor. 3.6]). This similarity is obtained by applying [29, Thm. 5] (the upper half of the commutative diagram below) and by noting that differentiation is an isomorphism $z\mathcal{D}_{-\beta+1} \rightarrow \mathcal{D}_{-\beta}$.

$$\begin{array}{ccc} z\mathcal{D}_\beta & \xrightarrow{C_\varphi^*} & z\mathcal{D}_\beta \\ \downarrow & & \downarrow \\ \mathcal{D}_{-\beta} & \xrightarrow{M_\psi/C_\psi} & \mathcal{D}_{-\beta} \\ \downarrow & & \downarrow \\ z\mathcal{D}_{-\beta+1} & \xrightarrow{\tilde{C}_\psi} & z\mathcal{D}_{-\beta+1} \end{array}$$

When φ in the above lemma is of the form $\varphi_r(z) = \frac{z+r}{1+rz}$, for $r \in (0, 1)$, or $\varphi_\alpha(z) = \frac{(2-\alpha)z+\alpha}{-\alpha z+2+\alpha}$, for $\operatorname{Re} \alpha \geq 0$, then ψ is $\varphi_{-r} = \varphi_r^{-1}$ or $\varphi_{\bar{\alpha}}$, respectively. Note that $\varphi_{\bar{\alpha}} = \varphi_\alpha^{-1}$ when $\operatorname{Re} \alpha = 0$.

In the hyperbolic case (by applying the reflection principle for φ_r and extending the result by similarity to all hyperbolic automorphisms) we have the following theorem:

Theorem ([A, Thm. 3.9]). *Let φ be a hyperbolic automorphism of \mathbb{D} with characteristic constant $\lambda_\varphi \in (0, 1)$. For $\beta > 1/2$ the spectrum of $C_\varphi : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ is the annulus*

$$\sigma(C_\varphi; \mathcal{D}_\beta) = \left\{ \lambda \in \mathbb{C} : \lambda_\varphi^{(2\beta-1)/2} \leq |\lambda| \leq \lambda_\varphi^{(1-2\beta)/2} \right\}.$$

Moreover,

$$\sigma_p(C_\varphi; \mathcal{D}_\beta) = \{1\}.$$

Recall above that the characteristic constant satisfies $\lambda_\varphi = \varphi'(p)$. The phenomenon that the spectrum starts to grow again after the Dirichlet space as the spaces get smaller is rather peculiar. Summarizing the results, we have obtained that, for all $\beta \in \mathbb{R}$,

$$\sigma(C_{\varphi_r}; \mathcal{D}_\beta) = \left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^{|1-2\beta|/2} \leq |\lambda| \leq \left(\frac{1-r}{1+r} \right)^{-|1-2\beta|/2} \right\}.$$

The spaces \mathcal{D}_β , for $\beta > 1/2$, are too small to possess eigenfunctions for C_{φ_r} , except the trivial one. This can be shown by using the fact that $\mathcal{D}_\beta \subset \mathcal{A}(\mathbb{D})$ for $\beta > 1/2$. Instead, in the small spaces it is the adjoint $C_{\varphi_r}^*$ that has big point spectrum. We can improve the

results this far by showing that the essential spectrum coincides with the spectrum also on \mathcal{D}_β , for $\beta \geq 1/2$ (see Proposition 3.1 in the next subsection).

In the parabolic case, there are no surprises and the spectra in the small spaces are the same as they are in the big spaces. By the same argument as for the hyperbolic, the invertible parabolic composition operators on \mathcal{D}_β can not have any other eigenvalues than the trivial one when $\beta > 1/2$:

Theorem ([A, Thm. 3.10]). *Let φ be a parabolic automorphism of \mathbb{D} . Then for any $\beta \geq 1$ the spectrum of $C_\varphi : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ is the unit circle. Moreover, for $\beta > 1/2$*

$$\sigma_p(C_\varphi; \mathcal{D}_\beta) = \{1\}.$$

We stress that the functions $e_t \in H^\infty(\mathbb{D})$, for $t \geq 0$, defined by

$$e_t(z) = \exp\left(t \frac{z+1}{z-1}\right),$$

are *potential* eigenfunctions for C_{φ_α} corresponding to the eigenvalue $e^{-\alpha t}$, independently of whether φ_α is an automorphism ($\operatorname{Re} \alpha = 0$) or a non-automorphism ($\operatorname{Re} \alpha > 0$)¹. While $t = 0$ corresponds the trivial case, the functions e_t for $t > 0$ belong to \mathcal{D}_β *if and only if* $\beta < 1/4$ (see [23, Prop. 3.10]). When φ_α is an automorphism, the eigenvalues of C_{φ_α} are of infinite multiplicity and so the spectra coincides with the essential spectra on \mathcal{D}_β , for $\beta < 1/4$. We can extend this to hold on all other weighted Dirichlet spaces as well (see Proposition 3.2 in the next subsection).

Let \mathbb{C}_+ denote the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The last piece in our spectral puzzle is the parabolic non-automorphism case on small spaces for which we have the following theorem:

Theorem ([A, Thm. 4.1]). *Let $\varphi_\alpha : \mathbb{D} \rightarrow \mathbb{D}$, $\varphi_\alpha(\mathbb{D}) \subsetneq \mathbb{D}$, be a parabolic LFT of the form $\varphi_\alpha(z) = \frac{(2-\alpha)z+\alpha}{-\alpha z+2+\alpha}$, where $\alpha \in \mathbb{C}_+$. Then, for all $\beta \geq 1$,*

$$\sigma(C_{\varphi_\alpha}; \mathcal{D}_\beta) = \{e^{-\alpha t} : t \in [0, \infty)\} \cup \{0\}.$$

Moreover, for all $\beta < 1/4$,

$$\sigma_p(C_{\varphi_\alpha}; \mathcal{D}_\beta) = \{e^{-\alpha t} : t \in [0, \infty)\}.$$

Also in this case we are now able to prove that the essential spectrum coincides with the spectrum on all spaces \mathcal{D}_β . Since the essential spectra of non-invertible parabolic composition operators is not known on any of the spaces \mathcal{D}_β , more work is required and therefore, we will postpone the proof to Section 5.

3.3. Supplementary results regarding the essential spectra. We prove here that the essential spectra of invertible hyperbolic and parabolic composition operators coincide with the spectra on \mathcal{D}_β for, respectively, $\beta \geq 1/2$ and $\beta \geq 1/4$. We first consider the hyperbolic case.

¹The eigenfunctions e_t , for $t \geq 0$, were first discovered in [46, Thm. 5], see also [16, Thm. 7.5].

Proposition 3.1. *Let φ_r be a hyperbolic automorphism of \mathbb{D} of the form $\varphi_r(z) = \frac{z+r}{1+rz}$, where $r \in (0, 1)$. Then, for all $\beta \geq 1/2$,*

$$(3.1) \quad \sigma_{ess}(C_{\varphi_r}; \mathcal{D}_\beta) = \left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^{(2\beta-1)/2} \leq |\lambda| \leq \left(\frac{1-r}{1+r} \right)^{(1-2\beta)/2} \right\}.$$

Proof. In the classical Dirichlet space ($\beta = 1/2$) this is true by [B, Thm. 5.3] since the operators $C_{\varphi_r} : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ and $C_\tau : \mathcal{D}_0(\Pi^+) \rightarrow \mathcal{D}_0(\Pi^+)$ for $\tau : \Pi^+ \rightarrow \Pi^+$ satisfying $\tau = h \circ \varphi_r \circ h^{-1}$ are unitarily equivalent. The crux of the matter here is to choose the function sequence (F_c) in [27, Thm. 3.2] slightly more delicately as in the proof of [B, Thm. 4.1].

Assume then that $\beta > 1/2$ and recall that Equation (3.1) holds for all $\beta' = -\beta + 1 < 1/2$ by [16, Thm. 7.4] and [A, Thm. 3.1]. Even though the reflection principle [A, Cor. 3.6 & Lemma 3.7] does *not* give a similarity between the operators $C_{\varphi_r}^*$ on \mathcal{D}_β and $C_{\varphi_r}^{-1}$ on $\mathcal{D}_{-\beta+1}$, their essential spectra coincide, too. The reason is that the operators $C_{\varphi_r}^* - \lambda I : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ and $C_{\varphi_r}^{-1} - \lambda I : \mathcal{D}_{-\beta+1} \rightarrow \mathcal{D}_{-\beta+1}$ are compact perturbations of, respectively, the operators $C_{\varphi_r}^* - \lambda I : z\mathcal{D}_\beta \rightarrow z\mathcal{D}_\beta$ and $\tilde{C}_{\varphi_r}^{-1} - \lambda I : z\mathcal{D}_{-\beta+1} \rightarrow z\mathcal{D}_{-\beta+1}$ which are similar by [A, Cor. 3.6]. For the details, see *Step 2* in the proof of Theorem 5.2 and notice that the reasoning there can be adopted as such to this case as well. The final conclusion follows by the fact that $T \in \mathcal{L}(\mathcal{H})$ is Fredholm if and only if T^* is and by noting that the annulus in question is invariant under complex conjugation as well as taking reciprocals. We have that $\sigma(C_{\varphi_r}; \mathcal{D}_\beta) = \sigma_{ess}(C_{\varphi_r}; \mathcal{D}_\beta)$, for $\beta > 1/2$. \square

As discussed earlier, the spectrum of C_φ on \mathcal{D}_β coincides with the essential spectrum for $\beta < 1/4$ when φ is a parabolic automorphism. This holds for $\beta \geq 1/4$ as well:

Proposition 3.2. *Let φ_α be a parabolic automorphism of \mathbb{D} , of the form $\varphi_\alpha(z) = \frac{(2-\alpha)z+\alpha}{-\alpha z+2+\alpha}$, where $\operatorname{Re} \alpha = 0$. Then, for all $\beta \geq 1/4$,*

$$(3.2) \quad \sigma_{ess}(C_{\varphi_\alpha}; \mathcal{D}_\beta) = \mathbb{T}.$$

Proof. In a similar manner as in Proposition 3.1, we can use the reflection principle [A, Cor. 3.6] to show that Equation (3.2) is true on \mathcal{D}_β for $\beta > 3/4$ since for $\beta' = -\beta + 1 < 1/4$ this is known (again, for details, see *Step 2* in the proof of Theorem 5.2). Also, in the classical Dirichlet space we have $\sigma_{ess}(C_{\varphi_\alpha}; \mathcal{D}_{1/2}) = \mathbb{T}$ by [B, Thm. 5.2].

Assume then that $\beta \in (1/2, 3/4]$. We want to show that $C_{\varphi_\alpha} - \lambda I$ is not Fredholm on \mathcal{D}_β for any $\lambda \in \mathbb{T}$. Assume on the contrary that there exists $\lambda_0 \in \mathbb{T}$ such that $C_{\varphi_\alpha} - \lambda_0 I$ is Fredholm on \mathcal{D}_β . Since the set of Fredholm operators is open in $\mathcal{L}(\mathcal{H})$ (see [43, Thm. III.16.17] or [45, Thm. 1.4.17], for instance) we actually have that $C_{\varphi_\alpha} - \mu I$ is Fredholm for all $\mu \in J \subset \mathbb{T}$, where J is some small arc containing λ_0 . Moreover, the Fredholm index of $C_{\varphi_\alpha} - \mu I$ would have to be 0 since $C_{\varphi_\alpha} - \lambda I$ is invertible for all $\lambda \in \mathbb{C} \setminus \mathbb{T}$ and the Fredholm index is locally constant. By [A, Thm. 3.10], we have that $\operatorname{Ker}(C_{\varphi_\alpha} - \lambda I) = 0$ for any $\lambda \neq 1$. Hence $C_{\varphi_\alpha} - \mu I$ would have to be invertible for all $\mu \in J \setminus \{1\}$. This is a contradiction since $\mu \in \sigma(C_{\varphi_\alpha}; \mathcal{D}_\beta)$. It follows that $C_{\varphi_\alpha} - \lambda I : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$, for $\beta \in (1/2, 3/4]$, is not Fredholm for any $\lambda \in \mathbb{T}$.

Again, using the reflection principle [A, Cor. 3.6], we can deduce that $\sigma_{ess}(C_{\varphi_\alpha}; \mathcal{D}_\beta) = \sigma(C_{\varphi_\alpha}; \mathcal{D}_\beta)$ also when $\beta \in [1/4, 1/2)$. □

3.4. Related developments. The spectral results of linear fractional composition operators on analytic Hilbert spaces, especially on $H^2(\mathbb{D})$, has naturally been a stepping stone for extending the results to more general cases. Nowadays, the research has lead to spectral results on analytic Banach spaces such as $H^p(\mathbb{D})$ for $1 \leq p < \infty$, Bloch or BMOA space, on Hardy spaces of the ball $H^2(B_N)$ for $N > 1$, or on weighted Hardy spaces $H^2(\beta)$ of the unit disc (see e.g. [36, 15, 56]). Another direction is to relinquish the one-to-one property of inducing functions by considering e.g. (non-automorphic) inner functions and try to obtain the spectra for such composition operators (see [10] or [16]).

Recently, the interest regarding the spectra has turned into weighted composition operators $W_{\varphi, \psi}$, $W_{\varphi, \psi}f = \psi(f \circ \varphi)$, acting on spaces of analytic functions. This far, the results are mainly for the invertible weighted composition operators, i.e. the ones having ψ bounded away from zero and φ an automorphism of \mathbb{D} , see [24, 30, 6].

4. SPECTRA IN THE HALF-PLANE SETTING

In Article [B] the spectra as well as the essential spectra are computed for bounded linear fractional composition operators acting on the Hardy, the weighted Bergman, and the weighted Dirichlet spaces of the upper half-plane. Besides the obvious case, i.e. the (unweighted) Dirichlet space, only the spectra (and the essential spectra) of the invertible or self-adjoint parabolic and invertible hyperbolic composition operators on the Hardy space $H^2(\Pi^+)$ have been computed earlier by Matache [41].

4.1. Hardy, weighted Bergman, and weighted Dirichlet spaces of Π^+ . We start by giving the explicit definitions of the spaces in question and consider briefly the boundedness of composition operators acting on them in general.

The Hardy space of Π^+ is defined as follows

$$H^2(\Pi^+) = \left\{ F : \Pi^+ \rightarrow \mathbb{C} \text{ analytic, } \|F\|_{H^2(\Pi^+)} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |F(x + iy)|^2 dx \right)^{1/2} < \infty \right\}.$$

The space $H^2(\Pi^+)$ is isometrically embedded into $L^2(\mathbb{R})$ via the mapping $F \mapsto F^*$, where $F^*(x) = \lim_{y \rightarrow 0^+} F(x + iy)$ (for more information on Hardy spaces of the half-plane, see [17, Chapter 11], for instance).

For $\alpha > -1$, the weighted Bergman spaces $\mathcal{A}_\alpha^2(\Pi^+)$ are

$$\mathcal{A}_\alpha^2(\Pi^+) = \left\{ F : \Pi^+ \rightarrow \mathbb{C} \text{ analytic, } \|F\|_{\mathcal{A}_\alpha^2(\Pi^+)} = \left(\int_{\Pi^+} |F(x + iy)|^2 y^\alpha dx dy \right)^{1/2} < \infty \right\}.$$

The Hardy space $H^2(\Pi^+)$ can, for our purposes, formally be interpreted as the “limit case” of the weighted Bergman spaces as $\alpha \rightarrow -1$ and, for convenience, we will write $H^2(\Pi^+) = \mathcal{A}_{-1}^2(\Pi^+)$.

Following [18], we define the weighted Dirichlet spaces $\mathcal{D}_\alpha^2(\Pi^+)$, for $\alpha > -1$, by setting

$$\mathcal{D}_\alpha^2(\Pi^+) = \{F : \Pi^+ \rightarrow \mathbb{C} \text{ analytic, } F' \in \mathcal{A}_\alpha^2(\Pi^+)\}.$$

Note that the spaces $\mathcal{D}_\alpha^2(\Pi^+)$, for $\alpha > -1$, actually consist of unique members of each equivalence class of functions that differ by a constant. We set the norm to be

$$\|F\|_{\mathcal{D}_\alpha^2(\Pi^+)} = \left(\int_{\Pi^+} |F'(x + iy)|^2 y^\alpha dx dy \right)^{1/2}.$$

In a sense (compare Theorems 1 and 3 in [18]), as the spectral results will also show, the weighted Dirichlet space $\mathcal{D}_\alpha^2(\Pi^+)$, for any $\alpha > -1$, could be understood as “the weighted Bergman space $\mathcal{A}_{\alpha-2}(\Pi^+)$ ”. Note that the spaces $\mathcal{A}_\alpha^2(\Pi^+)$, for $\alpha \geq -1$, or $\mathcal{D}_\alpha^2(\Pi^+)$, for $\alpha > -1$, are not nested as in the case of the weighted Dirichlet spaces of \mathbb{D} .

The boundedness of composition operators C_ϕ on the Hardy and the weighted Bergman spaces defined on Π^+ is more delicate than in the unit disc setting. Namely, for (any) analytic $\phi : \Pi^+ \rightarrow \Pi^+$, the operator C_ϕ is bounded if and only if the symbol ϕ fixes infinity and the angular derivative at infinity, defined by

$$\phi'(\infty) = \lim_{w \rightarrow \infty} \frac{w}{\phi(w)} \text{ (non-tangentially),}$$

is bounded and positive (for this, see [40] or [19] for the Hardy, and [20] for the weighted Bergman spaces). The proofs in [19, 20] are based on showing that the abovementioned condition is equivalent with certain kernel functions being positive from which it follows that the adjoint C_ϕ^* is bounded on a dense linear span of reproducing kernels. Moreover, by [19, 20], we have that for bounded composition operators C_ϕ on $\mathcal{A}_\alpha^2(\Pi^+)$, for all $\alpha \geq -1$, the operator norm coincides with the essential operator norm, the spectral radius and the essential spectral radius and is obtained from the quantity $\phi'(\infty)$ in the following manner:

$$\|C_\phi\| = \|C_\phi\|_{ess} = \rho(C_\phi; \mathcal{A}_\alpha^2(\Pi^+)) = \rho_{ess}(C_\phi; \mathcal{A}_\alpha^2(\Pi^+)) = (\phi'(\infty))^{(\alpha+2)/2}.$$

4.2. Parabolic and hyperbolic composition operators on Π^+ . Let τ be a LF self-map of Π^+ . The condition that τ has to fix ∞ for C_τ to be bounded on $\mathcal{A}_\alpha^2(\Pi^+)$, for $\alpha \geq -1$, restricts our consideration for LFTs to parabolic and hyperbolic composition operators: It is a straightforward computation to show that the LF self-maps of Π^+ that fix ∞ are precisely of the form

$$(4.1) \quad \tau(w) = \mu w + w_0, \text{ where } \mu > 0 \text{ and } \text{Im } w_0 \geq 0.$$

In this case, $\tau'(\infty) = \mu^{-1} \in (0, \infty)$ so that C_τ is bounded on $\mathcal{A}_\alpha^2(\Pi^+)$, for all $\alpha \geq -1$. On the weighted Dirichlet spaces, the boundedness of these particular C_τ (as well as the spectral results) follows by [B, Lemma 5.1].

If $\mu = 1$ (and necessarily $w_0 \neq 0$) in Equation (4.1), then τ has only one fixed point and so it is parabolic. Otherwise τ is hyperbolic. Moreover, τ is an automorphism of Π^+ if and only if $w_0 \in \mathbb{R}$. In the hyperbolic case the similarity of composition operators (obtained via conjugation of inducing maps) allows us to reduce the consideration into three different

forms of τ : Any hyperbolic automorphism $w \mapsto \mu w + w_0$, where $\mu \in (0, 1) \cup (1, \infty)$ and $w_0 \in \mathbb{R}$, can be conjugated to a LFT of the form

$$\tau(w) = \mu w.$$

If τ is a non-automorphism having ∞ as the attractive fixed point, then it can be conjugated to a mapping of the form

$$(4.2) \quad \tau_1(w) = \mu^{-1}w + i\mu^{-1}(1 - \mu), \text{ where } \mu \in (0, 1).$$

If ∞ is the repulsive fixed point of a non-automorphism τ , then we choose a mapping of the form

$$(4.3) \quad \tau_2(w) = \mu w + i(1 - \mu), \text{ where } \mu \in (0, 1),$$

to be the representative of each conjugacy class.

It would be tempting to think that the spectral results for C_τ would follow by similarity from those in the unit disc setting for C_φ , where $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ satisfies $\varphi = h^{-1} \circ \tau \circ h$ for some conformal map $h : \mathbb{D} \rightarrow \Pi^+$. Indeed, this is the case in the unweighted Dirichlet space, where *all* linear fractional self-maps of Π^+ induce bounded composition operators since $C_h : \mathcal{D}_0^2(\Pi^+) \rightarrow \mathcal{D}_{1/2}/\mathbb{C}$ is unitary. However, the composition operator $C_h : \mathcal{A}_\alpha^2(\Pi^+) \rightarrow \mathcal{A}_\alpha^2(\mathbb{D})$, although bounded, does not have a bounded inverse for any $\alpha \geq -1$. The boundedness of C_h as well as unboundedness of C_h^{-1} can be seen by computing the norms: It is enough to consider the usual case, where $h(z) = i\frac{1+z}{1-z}$ having $h^{-1}(w) = \frac{w-i}{w+i}$ as its inverse map. By the change of variable $z \mapsto h^{-1}(w)$, we have, for all $\alpha > -1$ and $F \in \mathcal{A}_\alpha^2(\Pi^+)$, that

$$\begin{aligned} \|C_h F\|_{\mathcal{A}_\alpha^2(\mathbb{D})}^2 &= \int_{\mathbb{D}} |F \circ h(z)|^2 (1 - |z|^2)^\alpha dA(z) = \int_{\Pi^+} |F(w)|^2 \left(1 - \left|\frac{w-i}{w+i}\right|\right)^\alpha \frac{4 dA(w)}{|w+i|^4} \\ &= \int_{\Pi^+} |F(w)|^2 (\operatorname{Im} w)^\alpha \frac{4^{\alpha+1}}{|w+i|^{2\alpha+4}} dA(w) \leq 4^{\alpha+1} \|F\|_{\mathcal{A}_\alpha^2(\Pi^+)}^2, \end{aligned}$$

since $|w+i|^{2\alpha+4} > 1$ for all $w \in \Pi^+$ and $\alpha > -1$. On the other hand, for $f \in \mathcal{A}_\alpha^2(\mathbb{D})$, it holds that

$$\begin{aligned} \|C_h^{-1} f\|_{\mathcal{A}_\alpha^2(\Pi^+)}^2 &= \int_{\Pi^+} |f \circ h^{-1}(w)|^2 (\operatorname{Im} w)^\alpha dA(w) = \int_{\mathbb{D}} |f(z)|^2 \left(\operatorname{Im} \frac{i(1+z)}{1-z}\right)^\alpha \frac{4 dA(z)}{|1-z|^4} \\ &= \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \frac{4}{|1-z|^{2\alpha+4}} dA(z), \end{aligned}$$

where the factor $|1-z|^{-2\alpha-4}$ is not bounded and, therefore, neither is the operator C_h^{-1} . In the Hardy space this can be seen similarly.

Before describing the main tools and ideas that are used to compute the (essential) spectra in the half-plane setting, let us unveil the results.

4.3. Results of Article [B]. In what follows, τ is always of the form (4.1) so that C_τ is bounded on all spaces in question. In the parabolic case the spectra are similar as in the unit disc setting, namely, on the Hardy and weighted Bergman spaces of Π^+ we have the following theorem:

Theorem ([B, Thm. 1.1]). *Let τ be a parabolic self-map of Π^+ , that is, $\tau(w) = w + w_0$, where $\text{Im } w_0 \geq 0$ and $w_0 \neq 0$. Then the spectrum and the essential spectrum of C_τ acting on the Hardy or the weighted Bergman spaces of the upper half-plane equals*

$$\sigma(C_\tau) = \sigma_{\text{ess}}(C_\tau) = \overline{\{e^{iw_0 t} : t \in [0, \infty)\}} = \begin{cases} \mathbb{T}, & \text{when } w_0 \in \mathbb{R}, \\ \{e^{iw_0 t} : t \in [0, \infty)\} \cup \{0\}, & \text{when } w_0 \in \Pi^+. \end{cases}$$

The spectra of hyperbolic composition operators on Π^+ are, however, very different than in \mathbb{D} :

Theorem ([B, Thm. 1.2]). *Let τ be a hyperbolic self-map of Π^+ , that is, $\tau(w) = \mu w + w_0$, where $\mu \in (0, 1) \cup (1, \infty)$ and $\text{Im } w_0 \geq 0$. Then, for all $\alpha \geq -1$, the spectrum and the essential spectrum of C_τ acting on $\mathcal{A}_\alpha^2(\Pi^+)$ is*

$$\begin{aligned} \text{i) } \sigma(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)) &= \sigma_{\text{ess}}(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)) = \{\lambda \in \mathbb{C} : |\lambda| = \mu^{-(\alpha+2)/2}\}, \text{ when } w_0 \in \mathbb{R}, \\ \text{ii) } \sigma(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)) &= \sigma_{\text{ess}}(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)) = \{\lambda \in \mathbb{C} : |\lambda| \leq \mu^{-(\alpha+2)/2}\}, \text{ when } w_0 \in \Pi^+. \end{aligned}$$

From these results we are also able to compute the spectra and the essential spectra of the parabolic and the hyperbolic composition operators on the weighted Dirichlet spaces $\mathcal{D}_\alpha^2(\Pi^+)$, for $\alpha > -1$. This is due to the following lemma, where the similarity is obtained via differentiation:

Lemma 4.1 ([B, Lemma 5.1]). *Let $\tau : \Pi^+ \rightarrow \Pi^+$ be a hyperbolic or parabolic map of the form $\tau(w) = \mu w + w_0$, where $\mu > 0$ and $\text{Im } w_0 \geq 0$ (and if $\mu = 1$, then $w_0 \neq 0$). Then, for all $\alpha > -1$, the operators $C_\tau : \mathcal{D}_\alpha^2(\Pi^+) \rightarrow \mathcal{D}_\alpha^2(\Pi^+)$ and $\mu C_\tau : \mathcal{A}_\alpha^2(\Pi^+) \rightarrow \mathcal{A}_\alpha^2(\Pi^+)$ are similar.*

It follows that (see [B, Thms. 5.2 & 5.3]), for all $\alpha > -1$,

$$\sigma(C_\tau; \mathcal{D}_\alpha^2(\Pi^+)) = \mu \sigma(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)).$$

4.4. Main tools for finding the spectra. In some sense the half-plane setting is more straightforward since the mappings we are considering have rather simple forms. Indeed, since we are dealing (mostly) with composition operators induced by *translations* (parabolic maps) and *dilations* (hyperbolic automorphisms) of Π^+ for which the behaviour of inverse Fourier transforms \mathcal{F}^{-1} are known, the Paley-Wiener theorem in $H^2(\Pi^+)$ ([48], see also [53]) and its generalization to the weighted Bergman spaces turn out to be very useful for our purposes.

Denote by L_β^2 , for all $\beta \geq 0$, the Hilbert space consisting of measurable functions $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ satisfying

$$\int_0^\infty |f(t)|^2 t^{-\beta} dt < \infty.$$

Theorem (Paley-Wiener Theorem(s)). *Let $\alpha \geq -1$. Then $F \in \mathcal{A}_\alpha^2(\Pi^+)$ if and only if there exists a function $f \in L_{\alpha+1}^2$ such that*

$$F(w) = (\mathcal{F}f)(w) = \int_0^\infty f(t)e^{iwt}, \quad w \in \Pi^+.$$

Moreover,

$$\|F\|_{\mathcal{A}_\alpha^2(\Pi^+)}^2 = \|f\|_{L_{\alpha+1}^2}^2 = b_\alpha \int_0^\infty |f(t)|^2 t^{-(\alpha+1)} dt,$$

where b_α is a constant depending only on α .

(For a detailed proof in the general case, see [18].) This is to say that the Fourier transform provides a similarity between the operators $C_\tau : \mathcal{A}_\alpha^2(\Pi^+) \rightarrow \mathcal{A}_\alpha^2(\Pi^+)$ and $\widehat{C}_\tau : L_{\alpha+1}^2 \rightarrow L_{\alpha+1}^2$, where \widehat{C}_τ satisfies

$$\widehat{C}_\tau(\mathcal{F}^{-1}F) = \mathcal{F}^{-1}(F \circ \tau)$$

for all $F \in \mathcal{A}_\alpha^2(\Pi^+)$. Hence, we get the commuting diagram

$$\begin{array}{ccc} \mathcal{A}_\alpha^2(\Pi^+) & \xrightarrow{C_\tau} & \mathcal{A}_\alpha^2(\Pi^+) \\ \downarrow \mathcal{F}^{-1} & & \downarrow \mathcal{F}^{-1} \\ L_{\alpha+1}^2 & \xrightarrow{\widehat{C}_\tau} & L_{\alpha+1}^2. \end{array}$$

Let us denote $f = \mathcal{F}^{-1}F$. It is well-known (see [53], for instance) that

$$\mathcal{F}^{-1}(F(w + w_0))(t) = e^{iw_0 t} f(t)$$

and that

$$\mathcal{F}^{-1}(F(\mu w))(t) = \frac{1}{\mu} f(t/\mu).$$

It follows that for a parabolic map τ , where $\tau(w) = w + w_0$, the operator \widehat{C}_τ on $L_{\alpha+1}^2$ is multiplication by the function $g(t) = e^{iw_0 t}$. If τ is a normalized hyperbolic automorphism of Π^+ of the form $\tau(w) = \mu w$, then $\widehat{C}_\tau f(t) = \frac{1}{\mu} f(t/\mu)$. In these particular cases of τ , the essential spectra of \widehat{C}_τ , hence of C_τ , can be found by using the following result which is specific for Hilbert space in this form:

Theorem ([9, Thm. XI.2.3]). *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator on a Hilbert space \mathcal{H} . If there exists an orthonormal sequence $(f_n) \subset \mathcal{H}$ such that*

$$\|(T - \lambda)f_n\|_{\mathcal{H}} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then $\lambda \in \sigma_{\text{ess}}(T; \mathcal{H})$.

Therefore, when τ is (or, rather, can be conjugated to) a translation or dilation, the main work is to find, for each λ in the expected spectrum, an orthonormal sequence of functions $(g_n) \in L_{\alpha+1}^2$ such that

$$\|(\widehat{C}_\tau - \lambda)g_n\|_{L_{\alpha+1}^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

When τ is a hyperbolic non-automorphism, we can not use the similarity given by the Fourier transform or the above theorem since we do not know the explicit form of \widehat{C}_τ . Fortunately, in the case of τ_1 (as defined in (4.2)), we actually *can* take advantage of the spectral results in the unit disc setting: Let J be the isometric isomorphism $\mathcal{A}_\alpha^2(\mathbb{D}) \rightarrow \mathcal{A}_\alpha^2(\Pi^+)$, for $\alpha \geq -1$, defined by

$$(4.4) \quad (Jf)(w) = f\left(\frac{w-i}{w+i}\right) \frac{c_\alpha}{(w+i)^{\alpha+2}},$$

where the constant c_α depends only on α (for the proof, see e.g. [28] for the Hardy, and [18] for the weighted Bergman spaces; here we also write $\mathcal{A}_{-1}^2(\mathbb{D}) = H^2(\mathbb{D})$). If $\varphi = h^{-1} \circ \tau \circ h$, where the conformal bijection $h : \mathbb{D} \rightarrow \Pi^+$ is defined by $h(z) = i \frac{1+z}{1-z}$, then (4.4) gives a similarity between the operator $C_\varphi : \mathcal{A}_\alpha^2(\mathbb{D}) \rightarrow \mathcal{A}_\alpha^2(\mathbb{D})$ and the weighted composition operator

$$\left(\frac{\tau(w)+i}{w+i}\right)^{\alpha+2} C_\tau : \mathcal{A}_\alpha^2(\Pi^+) \rightarrow \mathcal{A}_\alpha^2(\Pi^+).$$

When $\tau = \tau_1$, the weight $\left(\frac{\tau(w)+i}{w+i}\right)^{\alpha+2}$ reduces to a constant μ^{-1} .

In the remaining case, that is, when $\tau = \tau_2$ (as defined in (4.3)), a crucial observation is that on $\mathcal{A}_\alpha^2(\Pi^+)$, for all $\alpha \geq -1$,

$$C_{\tau_1} = \mu^{\alpha+2} C_{\tau_2}^*.$$

4.5. Further questions. In the (unweighted) Dirichlet space of the upper half-plane all linear fractional mappings induce bounded composition operators. As we have seen, this is not the case in the Hardy, or weighted Bergman spaces. Therefore, it would be interesting to find out if the (unweighted) Dirichlet space is the only space where LF self-maps of Π^+ induce bounded composition operators also when ∞ is not a fixed point. Or, if there are other such spaces, then where in the scale do the aforementioned operators lose boundedness?

Another obvious question arising from the results of Article [B] is that are the parabolic composition operators in \mathbb{D} and Π^+ similar? If yes, then what is the operator between $\mathcal{A}_\alpha^2(\mathbb{D})$ and $\mathcal{A}_\alpha^2(\Pi^+)$, at least for some $\alpha \geq -1$, that would provide the similarity since the composition C_h does not.

5. ESSENTIAL SPECTRA OF THE NON-INVERTIBLE PARABOLIC COMPOSITION OPERATORS

We will provide here a Banach algebraic computation of the essential spectrum of non-invertible parabolic composition operators acting on \mathcal{D}_β for $\beta \in \mathbb{R}$ (Theorem 5.2), which supplements the results in [A]. In addition, this technique can be used in the half-plane setting (Theorem 5.4) where it gives a different approach to proving [B, Thm. 1.1.ii)]. Suitable references for more information of the spectral theory of Banach algebras are e.g. [45, Chapter 1] or [9, Chapter VII].

Let \mathcal{H} be a separable infinite-dimensional Hilbert space. Recall the notation $\mathcal{C}(\mathcal{H})$ for the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and \mathbb{C}_+ for the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also,

we remind the reader that the family $(T_c)_{c \in \mathbb{C}_+} \subset \mathcal{L}(\mathcal{H})$ is a holomorphic semi-group of operators if

- i) $T_c T_d = T_{c+d}$ for $c, d \in \mathbb{C}_+$,
- ii) $(f, c) \mapsto T_c f$ is jointly continuous, and
- iii) $c \mapsto T_c f$ is holomorphic $\mathbb{C}_+ \rightarrow \mathcal{L}(\mathcal{H})$ for each $f \in \mathcal{H}$.

For the rest of this section, we denote by \mathcal{S}_c , for any $c \in \mathbb{C}_+$, the spiral

$$\mathcal{S}_c := \{0\} \cup \{e^{-ct} : t \geq 0\}.$$

Proposition 5.1. *Let $(T_c)_{c \in \mathbb{C}_+} \subset \mathcal{L}(\mathcal{H})$ be a holomorphic semi-group of operators. Assume that $\sigma(T_c) = \mathcal{S}_c$ for all $c \in \mathbb{C}_+$. If, in addition, $T_c - T_d$ is non-compact for all $c, d \in \mathbb{C}_+$ such that $c \neq d$, then $\sigma_{\text{ess}}(T_c) = \mathcal{S}_c$.*

Proof. Since $\sigma_{\text{ess}}(T_c) \subset \sigma(T_c) = \mathcal{S}_c$ we have to show that each point in \mathcal{S}_c belongs to the essential spectrum of T_c .

Let $q : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ be the linear quotient map, that is, $q(T) = T + \mathcal{K}(\mathcal{H})$. Since $(T_c)_{c \in \mathbb{C}_+}$ is a holomorphic semi-group of operators in $\mathcal{L}(\mathcal{H})$, it follows that $(q(T_c))_{c \in \mathbb{C}_+}$ is a holomorphic semi-group taking values in $\mathcal{C}(\mathcal{H})$. Let \mathcal{B} be the commutative unital subalgebra of $\mathcal{C}(\mathcal{H})$ generated by

$$\{q(I)\} \cup \{q(T_c) : c \in \mathbb{C}_+\}.$$

Recall that, by Atkinson's theorem, we have

$$\sigma_{\text{ess}}(T_c; \mathcal{H}) = \sigma(q(T_c); \mathcal{C}(\mathcal{H})).$$

Moreover, by classical Gelfand theory (see [45, Thm. 1.3.4], for instance), the spectrum in the commutative subalgebra \mathcal{B} is

$$\sigma(q(T_c); \mathcal{B}) = \{\lambda(q(T_c)) : \lambda \text{ is a non-zero multiplicative linear functional } \mathcal{B} \rightarrow \mathbb{C}\}.$$

We claim next that

$$\sigma(q(T_c); \mathcal{B}) = \mathcal{S}_c.$$

The proof of the inclusion $\sigma(q(T_c); \mathcal{B}) \subset \{0\} \cup \{e^{-ct} : t \geq 0\}$ uses a standard idea (originating from [10, Thm. 6.6] and used later on in [27, Thm. 3.3] and [A, Thm. 4.1], for instance) in proving that the spectrum of non-invertible parabolic composition operator acting on $H^2(\mathbb{D})$ or \mathcal{D}_β , for $\beta \leq 1/2$, is contained in the spiral \mathcal{S}_c . The difference is that we must consider $\lambda(q(T_c))$, instead of $\lambda(T_c)$. First of all, for any multiplicative linear functional $\lambda : \mathcal{B} \rightarrow \mathbb{C}$, the function $c \mapsto \gamma(c) := \lambda(q(T_c))$ defines a holomorphic map $\mathbb{C}_+ \rightarrow \mathbb{C}$. Moreover,

$$\gamma(c+d) = \lambda(q(T_{c+d})) = \lambda(q(T_c)) \cdot \lambda(q(T_d)) = \gamma(c) \cdot \gamma(d),$$

for any $c, d \in \mathbb{C}_+$, since $(q(T_c))_{c \in \mathbb{C}_+}$ is a holomorphic semi-group. Consequently, either $\gamma \equiv 0$ or else there is $t \in \mathbb{C}$ such that $\gamma(c) = e^{-ct}$ for $c \in \mathbb{C}_+$. In addition,

$$\begin{aligned} |e^{-ct}| &= \lim_{n \rightarrow \infty} |e^{-nct}|^{1/n} \leq \lim_{n \rightarrow \infty} |\lambda(q(T_{nc}))|^{1/n} \leq \lim_{n \rightarrow \infty} \|(q(T_c))^n\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|T_c^n\|_{\text{ess}}^{1/n} = \rho_{\text{ess}}(T_c) \leq \rho(T_c) = 1, \end{aligned}$$

where ρ and ρ_{ess} denote, respectively, the spectral and the essential spectral radii. It follows that $t \in [0, \infty)$ and so $\sigma(q(T_c); \mathcal{B}) \subset \{0\} \cup \{e^{-ct} : t \geq 0\}$.

To prove the reverse inclusion $\{0\} \cup \{e^{-ct} : t \geq 0\} \subset \sigma(q(T_c); \mathcal{B})$ define, for any fixed $t \geq 0$, the map λ by setting

$$\lambda(q(T_c)) = e^{-ct}, \quad c \in \mathbb{C}_+,$$

and $\lambda(q(I)) = 1$. Since $T_c - T_d$, where $c \neq d$, is *not* compact on \mathcal{H} by assumption, we have that $c \in \mathbb{C}_+$ uniquely determines the quotient class $q(T_c) = T_c + \mathcal{K}(\mathcal{H})$. Therefore, λ is well-defined. Moreover,

$$\lambda(q(T_c) \cdot q(T_d)) = \lambda(q(T_{c+d})) = e^{-(c+d)t} = \lambda(q(T_c)) \cdot \lambda(q(T_d)),$$

for all $c, d \in \mathbb{C}_+$, so that λ is a multiplicative linear functional on the generators of \mathcal{B} and, therefore, admits an extension to whole \mathcal{B} . Altogether, we have that $\mathcal{S}_c = \sigma(q(T_c); \mathcal{B})$.

Finally, by general spectral theory of Banach algebras (see, [9, Thm. VII.5.4.(a)], for instance)

$$\mathcal{S}_c = \partial\sigma(q(T_c); \mathcal{B}) \subset \partial\sigma(q(T_c); \mathcal{C}(\mathcal{H})) = \partial\sigma_{ess}(T_c; \mathcal{H}) \subset \mathcal{S}_c,$$

since the spiral \mathcal{S}_c is its own boundary. \square

Using the above proposition we are able to determine the essential spectrum for non-invertible parabolic composition operators acting on \mathcal{D}_β , for $\beta \in \mathbb{R}$, as well as on $H^2(\Pi^+)$, $\mathcal{A}_\alpha^2(\Pi^+)$ and $\mathcal{D}_\alpha^2(\Pi^+)$, for $\alpha > -1$.

Theorem 5.2. *Let φ be a parabolic linear fractional transformation such that $\varphi(\mathbb{D}) \subsetneq \mathbb{D}$ with the fixed point $p \in \mathbb{T}$. Then, for all $\beta \in \mathbb{R}$,*

$$\sigma(C_\varphi; \mathcal{D}_\beta) = \sigma_{ess}(C_\varphi; \mathcal{D}_\beta) = \mathcal{S}_c,$$

where $c = \varphi''(p) \in \mathbb{C}_+$.

Proof. Recall that it is enough to consider the parabolic LFTs of the form $\varphi_c(z) = \frac{(2-c)z+c}{-cz+2+c}$, where $c \in \mathbb{C}_+$, fixing the point 1. The essential spectra for all non-invertible parabolic composition operators are obtained via similarity.

Step 1. Denote $C_{\varphi_c} = C_c$ for simplicity. Recall that $\sigma(C_c; \mathcal{D}_\beta) = \mathcal{S}_c$ for all $\beta \in \mathbb{R}$ (see [10, 27, 50] and [A]). The operators $(C_c)_{c \in \mathbb{C}_+}$ form a holomorphic semi-group on $\mathcal{L}(\mathcal{D}_\beta)$ for all $\beta \in \mathbb{R}$, since the inducing functions $(\varphi_c)_{c \in \mathbb{C}_+}$ form a holomorphic semi-group (the definition of holomorphic semi-group of functions is analogous to that of operators). Moreover, by [1, Cor. 3.7] we know that $C_c - C_d : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$ is *not* compact for any $\beta \leq 1$ when $c \neq d$ (see also Remark 5.3 for more detailed background of this result). By Proposition 5.1, we get that

$$\sigma(C_c; \mathcal{D}_\beta) = \sigma_{ess}(C_c; \mathcal{D}_\beta) = \mathcal{S}_c,$$

for all $\beta \leq 1$.

Step 2. To show that this is also true on \mathcal{D}_β for all $\beta > 1$, we use the reflection principle [A, Cor. 3.6]. Let $\lambda \in \mathcal{S}_c$ and fix $\beta > 1$. We claim that $C_c - \lambda I$ is *not* Fredholm on \mathcal{D}_β , i.e. that $\mathcal{S}_c \subset \sigma_{ess}(C_c; \mathcal{D}_\beta)$, which is enough since $\sigma_{ess}(C_c; \mathcal{D}_\beta) \subset \sigma(C_c; \mathcal{D}_\beta) = \mathcal{S}_c$.

Let $\beta' = -\beta + 1 < 0$. As seen above, $C_c - \lambda I : \mathcal{D}_{\beta'} \rightarrow \mathcal{D}_{\beta'}$ is not a Fredholm operator. Let us write $C_c - \lambda I$ in terms of an operator matrix acting on $z\mathcal{D}_{\beta'} \oplus \mathbb{C}$. Now

$$C_c - \lambda I = \begin{bmatrix} \tilde{C}_c - \lambda I & 0 \\ P_{\mathbb{C}}C_c|_{z\mathcal{D}_{\beta'}} & (1 - \lambda) \end{bmatrix},$$

where $\tilde{C}_c : z\mathcal{D}_{\beta'} \rightarrow z\mathcal{D}_{\beta'}$ is defined by $\tilde{C}_c f = C_c f - (f \circ \varphi_c)(0)$. Moreover, we can write

$$\begin{bmatrix} \tilde{C}_c - \lambda I & 0 \\ P_{\mathbb{C}}C_c|_{z\mathcal{D}_{\beta'}} & (1 - \lambda) \end{bmatrix} = \begin{bmatrix} \tilde{C}_c - \lambda I & 0 \\ 0 & 0 \end{bmatrix} + K_1,$$

where $K_1 = \begin{bmatrix} 0 & 0 \\ P_{\mathbb{C}}C_c|_{z\mathcal{D}_{\beta'}} & (1 - \lambda) \end{bmatrix}$ is a compact operator on $z\mathcal{D}_{\beta'} \oplus \mathbb{C}$. It follows that $\tilde{C}_c - \lambda I$ is not Fredholm on $z\mathcal{D}_{\beta'}$ (see [43, Thm. III.16.9], for instance). By [A, Cor 3.6], the operators $\tilde{C}_c : z\mathcal{D}_{\beta'} \rightarrow z\mathcal{D}_{\beta'}$ and $C_{\bar{c}}^* : z\mathcal{D}_{\beta} \rightarrow z\mathcal{D}_{\beta}$ are similar and hence, $C_{\bar{c}}^* - \lambda I$ is not Fredholm on $z\mathcal{D}_{\beta}$. Similarly as above, the operator $C_{\bar{c}}^* - \lambda I : \mathcal{D}_{\beta} \rightarrow \mathcal{D}_{\beta}$ can be written as an operator matrix acting on $z\mathcal{D}_{\beta} \oplus \mathbb{C}$, namely

$$C_{\bar{c}}^* - \lambda I = \begin{bmatrix} C_{\bar{c}}^* - \lambda I & (\kappa_{\varphi_c(0)} - 1) \\ 0 & (1 - \lambda) \end{bmatrix},$$

where κ_w is the reproducing kernel of \mathcal{D}_{β} at a point $w \in \mathbb{D}$. Now, the non-Fredholmness of $C_{\bar{c}}^* - \lambda I$ on \mathcal{D}_{β} follows from that of $C_{\bar{c}}^* - \lambda I$ on $z\mathcal{D}_{\beta}$ since

$$\begin{bmatrix} C_{\bar{c}}^* - \lambda I & (\kappa_{\varphi_c(0)} - 1) \\ 0 & (1 - \lambda) \end{bmatrix} = \begin{bmatrix} C_{\bar{c}}^* - \lambda I & 0 \\ 0 & 0 \end{bmatrix} + K_2,$$

where $K_2 = \begin{bmatrix} 0 & (\kappa_{\varphi_c(0)} - 1) \\ 0 & (1 - \lambda) \end{bmatrix}$ is a compact operator on $z\mathcal{D}_{\beta} \oplus \mathbb{C}$. Since an operator is Fredholm if and only its adjoint has this property (see [43, Thm. III.16.4], for instance), we obtain that the operator $C_{\bar{c}} - \bar{\lambda} I : \mathcal{D}_{\beta} \rightarrow \mathcal{D}_{\beta}$ is not Fredholm. Finally, to obtain the desired result, note that $\bar{c} \in \mathbb{C}_+$ whenever $c \in \mathbb{C}_+$ and that $\bar{\lambda} \in \mathcal{S}_{\bar{c}}$ whenever $\lambda \in \mathcal{S}_c$. \square

Remark 5.3. [1, Cor. 3.7] extends some earlier non-compactness results by using interpolation. In fact, on the Hardy space $H^2(\mathbb{D})$ ($\beta = 0$) the non-compactness of $C_c - C_d$ follows from a result of Bourdon [3, Thm. 4.3]. Indeed, since $\varphi_c(1) = \varphi'_c(1) = 1$ for all $c \in \mathbb{C}_+$, but $\varphi''_c(1) = c \neq d = \varphi''_d(1)$, we have that the difference $C_c - C_d$ is non-compact on $H^2(\mathbb{D})$. In the weighted Dirichlet spaces \mathcal{D}_{β} , for $\beta < 1/2$ —which include e.g. the weighted Bergman spaces $\mathcal{A}_{\alpha}^2(\mathbb{D})$ for $\alpha > -1$ —the non-compactness follows from a result of Moorhouse [42, Thm. 4] (note that the parametrization in [42] differs from ours). Namely, the necessary condition

$$(5.1) \quad \lim_{|z| \rightarrow 1^-} \left| \frac{\varphi_c(z) - \varphi_d(z)}{1 - \overline{\varphi_c(z)}\varphi_d(z)} \right| \left(\frac{1 - |z|^2}{1 - |\varphi_c(z)|^2} + \frac{1 - |z|^2}{1 - |\varphi_d(z)|^2} \right) = 0,$$

for compactness of $C_c - C_d : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta$, for $\beta > 1/2$, is *not* satisfied. To see this, consider any sequence (z_a) of points in \mathbb{D} such that

$$\frac{1 - |z_a|^2}{|1 - z_a|^2} = A,$$

where $A > 0$ is a constant. The first factor, the *pseudo-hyperbolic distance* of $\varphi_c(z)$ and $\varphi_d(z)$, in (5.1) is bounded below by a positive constant since, by inspection,

$$\left| \frac{\varphi_c(z_a) - \varphi_d(z_a)}{1 - \overline{\varphi_c(z_a)}\varphi_d(z_a)} \right| = \frac{|c - d||1 - z_a|^2}{|2(1 - |z_a|^2) + (\bar{c} + d)|1 - z_a|^2|} = \frac{|c - d|}{|2A + \bar{c} + d|} > 0.$$

To show that also the second factor converges to a positive constant, note first that $\frac{1 - |z_a|^2}{|1 - z_a|^2} = \operatorname{Re} \frac{1 + z_a}{1 - z_a}$, where $\frac{1 + z_a}{1 - z_a} \in \mathbb{C}_+$. Since $g(z) = \frac{1 + z}{1 - z}$ is a linear fractional transformation from \mathbb{D} onto \mathbb{C}_+ , it is easy to see that $(z_a) \subset \mathbb{D}$ and $z_a \rightarrow 1$ *non-tangentially*. Now we can use the Julia-Carathéodory Theorem (see [16, Thm. 2.44], for instance) which guarantees that, for any $c \in \mathbb{C}_+$,

$$\frac{1 - |z_a|}{1 - |\varphi_c(z_a)|} \rightarrow \frac{1}{\varphi'_c(1)} = 1, \quad \text{as } z_a \rightarrow 1 \text{ (n.t.)}.$$

Since $\frac{1}{2} \frac{1 - |z_a|}{1 - |\varphi_c(z_a)|} \leq \frac{1 - |z_a|^2}{1 - |\varphi_c(z_a)|^2} \leq 2 \frac{1 - |z_a|}{1 - |\varphi_c(z_a)|}$ for all z_a , it follows that the second factor in (5.1) converges to a positive constant as $z_a \rightarrow 1$.

Later, Kriete and Moorhouse [35] studied the non-compactness of linear combinations on the Hardy, and the weighted Bergman spaces.

Indeed, these results would have been enough for our purposes since the reflection principle takes care of the scale $\beta > 1/2$ and the non-compactness of $C_c - C_d$ on the Dirichlet space $\mathcal{D}_{1/2}$ can be seen by taking the consideration to the half-plane via unitary equivalence and proceeding as in Theorem 5.4 by using a version of Paley-Wiener Theorem for the Dirichlet space (see [18, Thm. 3]).

We can use Proposition 5.1 to give an alternative proof to [B, Thm. 1.1.ii)], i.e. the essential spectrum of non-invertible parabolic composition operators in the upper half-plane setting. Recall our convention $H^2(\Pi^+) = \mathcal{A}_1^2(\Pi^+)$.

Theorem 5.4. *Let $\tau(w) = \tau_c(w) = w + ic$, where $c \in \mathbb{C}_+$, be a parabolic self-map of Π^+ such that $\tau(\Pi^+) \subsetneq \Pi^+$. Then, for all $\alpha \geq -1$,*

$$\sigma_{\text{ess}}(C_{\tau_c}; \mathcal{A}_\alpha^2(\Pi^+)) = \mathcal{S}.$$

Moreover, equality holds on $\mathcal{D}_\alpha(\Pi^+)$, for $\alpha > -1$, as well.

Proof. It is easy to check that $(C_{\tau_c})_{c \in \mathbb{C}_+}$ forms a holomorphic semi-group of operators. Observe that *Step 1* in the proof of [B, Thm. 1.1] alone gives actually a formula for the spectrum of C_c (but not the essential spectrum!). This follows from the fact that the spectrum of a multiplication operator is just the closure of the image of the symbol and, as we already know very well, in this case $C_c : \mathcal{A}_\alpha^2(\Pi^+) \rightarrow \mathcal{A}_\alpha^2(\Pi^+)$ is similar to

the multiplication by e^{-ct} on $L^2_{\alpha+1} = L^2_{\alpha+1}(\mathbb{R}_+)$ by the Paley-Wiener theorem and its generalization. So, $\sigma(C_{\tau_c}; \mathcal{A}^2_{\alpha}(\Pi^+)) = S_c$.

We will also use the suitable versions of Paley-Wiener theorem to show that $C_{\tau_c} - C_{\tau_d}$ is non-compact on $\mathcal{A}^2_{\alpha}(\Pi^+)$, for all $\alpha \geq -1$ and $c, d \in \mathbb{C}_+$ such that $c \neq d$. Indeed, the operator $C_{\tau_c} - C_{\tau_d} : \mathcal{A}^2_{\alpha}(\Pi^+) \rightarrow \mathcal{A}^2_{\alpha}(\Pi^+)$, for $\alpha \geq -1$, is similar to $M_g : L^2_{\alpha+1} \rightarrow L^2_{\alpha+1}$, where M_g is the multiplication by the function

$$g(t) = e^{-ct} - e^{-dt}, \quad t > 0.$$

Note that $g(t) = e^{-ct} - e^{-dt} \neq 0$ since $c \neq d$. Moreover, since g is continuous, there exists an open interval $I \subset \mathbb{R}_+$ such that $0 < A \leq |g(t)| \leq B < \infty$ for all $t \in I$ and suitable constants A, B . It follows that

$$(5.2) \quad A_1 \int_I |f(t)|^2 t^{-(\alpha+1)} dt \leq \int_I |(gf)(t)|^2 t^{-(\alpha+1)} dt \leq B_1 \int_I |f(t)|^2 t^{-(\alpha+1)} dt,$$

for some constants $A_1, B_1 > 0$.

Denote by $K \subset L^2_{\alpha+1}$ the space of functions belonging to $L^2_{\alpha+1}$ and being supported on I . By (5.2), the subspace $M_g(K)$ is isomorphic to K which is isomorphic to $L^2_{\alpha+1}$. Therefore, M_g is not compact on $L^2_{\alpha+1}$ and neither is $C_{\tau_c} - C_{\tau_d}$ on $\mathcal{A}^2_{\alpha}(\Pi^+)$, for any $\alpha \geq -1$, by similarity.

Since the assumptions of Proposition 5.1 are satisfied, we have that, for all $\alpha > -1$,

$$\sigma(C_{\tau_c}; \mathcal{A}^2_{\alpha}(\Pi^+)) = \sigma_{ess}(C_{\tau_c}; \mathcal{A}^2_{\alpha}(\Pi^+)) = S_c.$$

By similarity (see [B, Lemma 5.1] and note that $\mu = 1$ in this case) the above holds also on $\mathcal{D}^2_{\alpha}(\Pi^+)$, for $\alpha > -1$. \square

6. UNIVERSAL OPERATORS

An operator U acting on a separable infinite-dimensional Hilbert space \mathcal{H} is called *universal* if for any $T \in \mathcal{L}(\mathcal{H})$ there exists a constant $c \neq 0$ and a closed U -invariant subspace $\mathcal{M} \subset \mathcal{H}$, i.e. $U(\mathcal{M}) \subset \mathcal{M}$, such that the operators $cT : \mathcal{H} \rightarrow \mathcal{H}$ and $U|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ are similar.

Rota's model, a backward shift of infinite multiplicity B_{∞} ,

$$B_{\infty} : (\oplus_n \ell^2)_2 \rightarrow (\oplus_n \ell^2)_2, \quad B_{\infty}(x_1, x_2, \dots) = (x_2, x_3, \dots),$$

is a generic example of universal operators. The main tool for finding new concrete examples of universal operators is a very elegant sufficient condition—that also Rota's model satisfy—given by Caradus in 1969:

Theorem (Caradus' Theorem [4]). *An operator $U \in \mathcal{L}(\mathcal{H})$ is universal if $U(\mathcal{H}) = \mathcal{H}$ and $\dim \text{Ker } U = \infty$.*

Note that the condition $\dim \text{Ker } U = \infty$ is necessary for U to model the zero-operator; sometimes the zero-operator is excluded from the consideration but a universal operator has to model an operator with infinite-dimensional kernel in any case. For an overview of

universal operators, see Chapter 8 in the recent monograph [7] by Chalendar and Partington and the survey [12] by Cowen and Gallardo-Gutiérrez.

The idea of universal operators was motivated by the Invariant Subspace Problem (ISP) on Hilbert space which continues to be one of the major open problems in operator theory. The question is whether every operator $T \in \mathcal{L}(\mathcal{H})$ has a closed invariant subspace that is *non-trivial*, i.e. different from the trivial subspaces $\{0\}$ and \mathcal{H} . The ISP was stated explicitly in the 1950s after Beurling's work on invariant subspaces of shifts and the proof by Aronszajn and Smith [2] that all compact operators on Hilbert spaces with dimension at least 2 have a non-trivial invariant subspace (a result that also von Neumann had discovered without publishing it). Since then it has attracted the attention of many mathematicians in the field. For the numerous different approaches to this (in)famous problem, we refer to [7].

Universal operators provide one point of view for trying to find an answer to ISP. Namely, the answer is positive *if and only if* all *minimal* invariant subspaces of some (and hence any) universal operator are 1-dimensional. Here, invariant subspace is called minimal if it does not contain any other invariant subspaces except for $\{0\}$. Indeed, if every operator $T \in \mathcal{L}(\mathcal{H})$ has a non-trivial invariant subspace and $U \in \mathcal{L}(\mathcal{H})$ is a universal operator having an invariant subspace \mathcal{M} such that, for some non-zero $c \in \mathbb{C}$, $\mu T : \mathcal{H} \rightarrow \mathcal{H}$ and $U|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ are similar, then \mathcal{M} is a non-minimal infinite-dimensional subspace since it is isomorphic to \mathcal{H} . On the other hand, a finite-dimensional U -invariant subspace always contains an eigenspace which is 1-dimensional and U -invariant. The other direction follows likewise since $T : \mathcal{H} \rightarrow \mathcal{H}$ is similar, for some choices of μ and \mathcal{M} , to $\mu^{-1}U|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$, where \mathcal{H} and \mathcal{M} are isomorphic.

An old tale took a new twist in 1987, when Nordgren, Rosenthal and Wintrobe [47] showed that $C_{\varphi} - \lambda I : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is universal when φ is a hyperbolic automorphism of \mathbb{D} and λ is an interior point of the spectrum $\sigma(C_{\varphi}; H^2(\mathbb{D}))$. In fact, they showed that $C_{\varphi} - \lambda I$ on $H^2(\mathbb{D})$ satisfies Caradus' condition. Recently, Cowen and Gallardo-Gutiérrez [13] gave another, more straightforward proof for the onto-ness of $C_{\varphi} - \lambda I$. Since the invariant subspaces of $C_{\varphi} - \lambda I$ are exactly the invariant subspaces of C_{φ} , this result reduces the ISP to be a question of the lattice of invariant subspaces of C_{φ} on $H^2(\mathbb{D})$. The potential benefit of this is that the (non-universal) operator C_{φ} on $H^2(\mathbb{D})$ is very concrete and this brings the function-theoretic tools to one's hand. Therefore, the problem of whether every invariant subspace of C_{φ} contains an eigenspace, has aroused interest; see e.g. [38, 8, 39, 21, 22].

6.1. Main results of Article [C]. Article [C] is a smorgasbord of universality. We consider the class of universal operators, their properties and provide new concrete examples of universal operators. Moreover, we construct examples of commuting *universal pairs*, which is a concept introduced by Müller [44].

The class of universal operators. In Article [C] one of our aims was to find some conditions which would help to recognize universal (or non-universal) operators. Caradus' sufficient condition—or even its generalization by Pozzi [51, Thm. 3.8] to operators whose range

has finite codimension—are far from being necessary. Indeed, we construct some simple examples of universal operators which have infinite codimensional or non-closed range. Nevertheless, it was observed in [12, p. 44] that an operator $U \in \mathcal{L}(\mathcal{H})$ is universal if and only if it has an invariant subspace \mathcal{M} such that $U|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ satisfies Caradus' condition.

Denote by $\Phi_+(\mathcal{H})$ the class of *upper semi-Fredholm operators* on \mathcal{H} , that is,

$$\Phi_+(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \text{Ran } T \text{ is closed and } \dim \text{Ker } T = n < \infty\}.$$

In the theorem below, $\sigma_e^+(T)$ denotes the essential approximative point spectrum,

$$\sigma_e^+(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\}.$$

The main observation in [C] concerning the full class of universal operators is that an universal operator must have big essential spectrum.

Theorem ([C, Thm. 2.2]). *Suppose that $U \in \mathcal{L}(\mathcal{H})$ is an arbitrary universal operator. Then the following hold:*

- (i) *There is $r > 0$ such that the open disk*

$$B(0, r) \subset \sigma_p(U) \cap \sigma_e^+(U) \subset \sigma_e(U) \subset \sigma(U),$$

and, moreover, any $\lambda \in B(0, r)$ is an eigenvalue of U having infinite multiplicity. In particular, if $U \in \mathcal{L}(\mathcal{H})$ then 0 is an interior point of $\sigma_e^+(U)$, $\sigma_e(U)$, $\sigma_p(U)$ as well as $\sigma(U)$.

- (ii) *There is $r > 0$ and a vector-valued holomorphic map $z \mapsto y_z : B(0, r) \rightarrow \mathcal{H}$ consisting of eigenvectors for U , for which*

$$Uy_z = zy_z, \quad z \in B(0, r).$$

The proof relies on the properties of Rota's model. The results above are useful in excluding many concrete operators from being universal. For instance, an operator of the form $T - \lambda I$, where $\lambda \in \sigma(T)$, can not be universal if $\lambda \in \partial\sigma(T)$.

Denote by $\Phi_-(\mathcal{H})$ the class of *lower semi-Fredholm operators* on \mathcal{H} , that is,

$$\Phi_-(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \text{codim Ran } T < \infty\}.$$

The subclass of universal operators satisfying the Caradus-Pozzi condition is exactly

$$\Phi_-(\mathcal{H}) \setminus \Phi(\mathcal{H}).$$

It follows from the theory of semi-Fredholm operators that the Caradus-Pozzi class is preserved by small and compact perturbations. On the contrary, the class of all universal operators is neither open nor preserved by compact perturbations as our next example shows (see [C, Example 2.2]): Consider the operator

$$V = \begin{pmatrix} U & I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix},$$

where $U \in \mathcal{L}(\mathcal{H})$ is universal and $I_{\mathcal{H}}$ is the identity map of \mathcal{H} . Then V is universal on $\mathcal{H} \oplus \mathcal{H}$. Define a sequence (V_n) of injective, hence non-universal, operators on $\mathcal{H} \oplus \mathcal{H}$ by setting

$$V_n = \begin{pmatrix} U & I_{\mathcal{H}} \\ \frac{1}{n}I_{\mathcal{H}} & 0 \end{pmatrix}.$$

Since $\|V_n - V\| = \frac{1}{n}$ we have that $V \in U(\mathcal{H} \oplus \mathcal{H})$ is not an interior point of the full class of universal operators on $\mathcal{H} \oplus \mathcal{H}$.

Furthermore, let $K : \mathcal{H} \rightarrow \mathcal{H}$ be the compact diagonal map defined by $Ke_n = \frac{1}{n}e_n$ for $n \in \mathbb{N}$, where (e_n) is some fixed orthonormal basis of H . We consider the operator

$$W = \begin{pmatrix} U & I_{\mathcal{H}} \\ K & 0 \end{pmatrix} = V + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix},$$

on $\mathcal{H} \oplus \mathcal{H}$, that is, $W(x, y) = (Ux + y, Kx)$ for $(x, y) \in \mathcal{H} \oplus \mathcal{H}$. Thus W is a compact perturbation of $V \in \mathcal{U}(\mathcal{H} \oplus \mathcal{H})$, but W is not a universal operator, since $\text{Ker}(W) = \{(0, 0)\}$.

Universality of $C_{\varphi} - \lambda I$. In what follows, φ is always a hyperbolic automorphism of \mathbb{D} . Recall that the spectrum of C_{φ} on \mathcal{D}_{β} , for all $\beta \in \mathbb{R}$, is

$$\sigma(C_{\varphi}; \mathcal{D}_{\beta}) = \{\lambda \in \mathbb{C} : \lambda_{\varphi}^{|1-2\beta|/2} \leq |\lambda| \leq \lambda_{\varphi}^{-|1-2\beta|/2}\},$$

where $\lambda_{\varphi} \in (0, 1)$ is the multiplier of φ (see Section 3 for the spectral results). Moreover, when $\beta < 1/2$, all interior points of the spectrum are eigenvalues of C_{φ} of infinite multiplicity and when $\beta > 1/2$, we have that $\sigma_p(C_{\varphi}; \mathcal{D}_{\beta}) = \{1\}$.

The main result of [C] points to an interesting phenomenon regarding the universality of $C_{\varphi} - \lambda I$ on \mathcal{D}_{β} . Firstly, the operator $C_{\varphi} - \lambda I : \mathcal{D}_{\beta} \rightarrow \mathcal{D}_{\beta}$ is not universal for any $\lambda \in \mathbb{C}$ or $\beta \geq 1/2$. In the classical Dirichlet space $\mathcal{D}_{1/2}$ the answer follows from [C, Thm. 2.2] (see also [C, Cor. 2.4]) since $\sigma(C_{\varphi}; \mathcal{D}_{1/2}) = \mathbb{T}$. For $\beta > 1/2$, just notice that $\dim \text{Ker}(C_{\varphi} - \lambda I; \mathcal{D}_{\beta})$ is 0 (when $\lambda \neq 1$) or 1 (when $\lambda = 1$).

Recall that $H^2(\mathbb{D}) = \mathcal{D}_0$ and that

$$\sigma(C_{\varphi}; \mathcal{D}_0) = \sigma(C_{\varphi}; \mathcal{D}_1).$$

Since $C_{\varphi} - \lambda I : \mathcal{D}_0 \rightarrow \mathcal{D}_0$, where λ is an interior point of $\sigma(C_{\varphi}; \mathcal{D}_0)$, satisfies Caradus' condition by [47], we are able to show that the adjoint $C_{\varphi}^* - \bar{\lambda}I$ is universal on \mathcal{D}_1 (denoted by $S^2(\mathbb{D})$ in [C]).

Theorem ([C, Thm. 3.1]). *Let φ be a hyperbolic automorphism of \mathbb{D} . The operator $C_{\varphi}^* - \lambda I$ is universal on \mathcal{D}_1 when λ is an interior point of $\sigma(C_{\varphi}; \mathcal{D}_1)$.*

The proof starts from the NRW-result and it uses the similarity given by the reflection principle [A, Cor. 3.6]. Indeed, the operator $C_{\varphi}^* - \lambda I : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ satisfies Caradus' condition for $\lambda \neq 1$ and for $C_{\varphi}^* - I : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ it holds that $\text{codim Ran}(C_{\varphi}^* - I) = 1$.

Moreover, by using Heller's formula for the adjoint C_{φ}^* (see [26, Thm. 6.5]) and the fact that Caradus-Pozzi class is preserved by compact perturbations, we can conclude that the

operator

$$\frac{1+r^2}{1-r^2}C_{\varphi_r} - \frac{r}{1-r^2}(M_z^* + M_z)C_{\varphi_r} - \lambda I : \mathcal{D}_1 \longrightarrow \mathcal{D}_1$$

is universal as well. Above, M_z on \mathcal{D}_1 denotes the multiplication by the function $f(z) = z$ and its adjoint M_z^* is defined by

$$M_z^* \left(\sum_{n=0}^{\infty} a_n z^n \right) = a_1 + \sum_{n=1}^{\infty} a_{n+1} \left(\frac{n+1}{n} \right)^n z^n.$$

Universal pairs. Müller [44] generalized the concept of universality for commuting n -tuples $(T_1, T_2, \dots, T_n) \in \mathcal{L}(\mathcal{H})^n$, where $n \in \mathbb{N}$. The commuting n -tuple (U_1, U_2, \dots, U_n) is universal, if for each $(T_1, T_2, \dots, T_n) \in \mathcal{L}(\mathcal{H})^n$ there exists a constant $c \neq 0$ and a subspace $\mathcal{M} \subset \mathcal{H}$ invariant for all U_i , $i = 1, \dots, n$, such that $U_i|_{\mathcal{M}}$ and cT_i are similar for all $i = 1, \dots, n$. From the definition it is clear that each operator U_i , for all $i = 1, \dots, n$, has to be universal itself but much more is required and even constructing universal pairs ($n = 2$) is rather difficult. The observation below shows that the operators can not depend on each other in algebraic sense to form a universal pair:

Proposition ([C, Prop. 4.1]). *Let $T \in \mathcal{L}(\mathcal{H})$.*

- (i) *If $S \in \mathcal{L}(\mathcal{H})$ commutes with T , then (T, ST) is not a universal commuting pair. In particular, $(T, p(T)T)$ is not a universal commuting pair for any complex polynomial $p(z) = a_1 z + \dots + a_n z^n$ satisfying $p(0) = 0$, where $p(T) = a_1 T + \dots + a_n T^n$.*
- (ii) *(T^m, T^n) is not a universal commuting pair for any $m, n \in \mathbb{N}$.*

There is a sufficient condition—analogueous to that of Caradus—for a commuting n -tuple to be universal which can be formulated for pairs as follows:

Proposition ([44, Cor. 8]). *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be commuting surjective operators satisfying*

- (i) $\dim \text{Ker}(T_1) \cap \text{Ker}(T_2) = \infty$,
- (ii) $\text{Ker}(T_1 T_2) = \text{Ker}(T_1) + \text{Ker}(T_2)$.

Then the pair (T_1, T_2) is universal for all commuting pairs.

Using the above condition we were able to find new examples of universal pairs: Denote by $\mathcal{C}_2(\mathcal{H})$ the space of Hilbert-Schmidt operators on \mathcal{H} . Equipped with the Hilbert-Schmidt norm

$$\|T\|_{\mathcal{C}_2} = \left(\sum_{n=1}^{\infty} \|T e_n\|^2 \right)^{1/2},$$

which is independent of the choice of the orthonormal basis $(e_n) \subset \mathcal{H}$, the space $\mathcal{C}_2(\mathcal{H})$ is a separable Hilbert space. Moreover, for all $T \in \mathcal{C}_2(\mathcal{H})$ and $S, U \in \mathcal{L}(\mathcal{H})$ it holds that

$$\|STU\|_{\mathcal{C}_2} \leq \|S\| \|U\| \|T\|_{\mathcal{C}_2}.$$

This implies that the left multiplication $L_U : \mathcal{C}_2(\mathcal{H}) \longrightarrow \mathcal{C}_2(\mathcal{H})$, defined by $L_U T = UT$ for any $U \in \mathcal{L}(\mathcal{H})$, is a bounded linear operator. Similarly, the right multiplication $R_U : T \longmapsto TU$, for any $U \in \mathcal{L}(\mathcal{H})$, is bounded on $\mathcal{C}_2(\mathcal{H})$. Moreover, $(L_U, R_U) \in \mathcal{L}(\mathcal{C}_2(\mathcal{H}))^2$ is a commuting pair.

Let $B_\infty : (\oplus_n \ell^2)_2 \longrightarrow (\oplus_n \ell^2)_2$ be Rota's universal model and denote by B_∞^* its adjoint, which is the forward shift of infinite multiplicity,

$$B_\infty^* : (\oplus_n \ell^2)_2 \longrightarrow (\oplus_n \ell^2)_2, \quad B_\infty^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

It is shown in [C, Example 4.2.(iii)] that the commuting pair

$$(L_{B_\infty}, R_{B_\infty^*}) \in \mathcal{L}(\mathcal{C}_2((\oplus_n \ell^2)_2))^2$$

is universal.

It appears to be an intriguing question whether it is possible to find a universal pair constructed from the NRW-operators $C_\varphi - \lambda I$ on $H^2(\mathbb{D})$. While our search for such a pair was eventually unsuccessful, we managed to exclude many candidates (see [C, Example 4.3]). For instance,

- (i) $(C_{\varphi_r} - \lambda I, C_{\varphi_r} - \mu I)$ is not a universal pair for any $\lambda \neq \mu$.
- (ii) $(C_{\varphi_r} - \lambda I, C_{\varphi_u} - \lambda^2 I) \in (\mathcal{L}(H^2(\mathbb{D})))^2$ is not a universal pair for any $0 < r < 1$, $u = \frac{2r}{1+r^2}$ and any interior point λ of the spectrum $\sigma(C_{\varphi_r})$. More generally, replacing $C_{\varphi_u} - \lambda^2$ above by $C_{\varphi_r^n} - \lambda^n$, for $n \geq 3$, we have yet other non-universal pairs.

6.2. Further questions. The characterization of universal operators in terms of its restriction to some of its invariant subspace satisfying Caradus' condition is perhaps not very useful since universal operators have a lot of invariant subspaces. Therefore, finding another sufficient *and* necessary condition for an operator to be universal would be valuable.

The obvious question regarding the operator $C_\varphi - \lambda I$ is that is it universal on some other space than on $H^2(\mathbb{D})$ on the scale of \mathcal{D}_β , $\beta < 1/2$, for instance, on $A^2(\mathbb{D})$? There are good reasons to believe that the answer is yes. Namely, the spectra of $C_\varphi - \lambda I$ on all \mathcal{D}_β for $\beta < 1/2$ is an annulus (radii depending on the space) with all its interior points belonging to the essential spectrum. Also, using the reflection principle and the fact that $\dim \text{Ker}(C_\varphi - \lambda I; \mathcal{D}_\beta) = 0$ for all $\lambda \neq 1$ and $\beta > 1/2$, it can be showed that $\overline{\text{Ran}}(C_\varphi - \lambda I; \mathcal{D}_{\beta'}) = \mathcal{D}_{\beta'}$, for all $\beta' = -\beta + 1 < 1/2$. However, proving the ontoneess of $C_\varphi - \lambda I$ on \mathcal{D}_β , for $\beta < 1/2$, has turned out to be very challenging.

CLOSING WORDS

The question whether $C_\varphi - \lambda I$ is universal on $A^2(\mathbb{D})$ was, in fact, a starting point of this thesis. Although this particular problem remains unsolved, it has still provided a continuous inspiration for the work presented here. Despite its long history, it seems clear that the story of composition operators is nowhere near to its end.

REFERENCES

- [1] Robert F. Allen, Katherine C. Heller, and Matthew A. Pons. Compact differences of composition operators on weighted Dirichlet spaces. *Cent. Eur. J. Math.*, 12(7):1040–1051, 2014.
- [2] Nachman Aronszajn and Kennan T. Smith. Invariant subspaces of completely continuous operators. *Ann. of Math. (2)*, 60:345–350, 1954.
- [3] Paul S. Bourdon. Components of linear-fractional composition operators. *J. Math. Anal. Appl.*, 279(1):228–245, 2003.
- [4] S. R. Caradus. Universal operators and invariant subspaces. *Proc. Amer. Math. Soc.*, 23:526–527, 1969.
- [5] James G. Caughran and Howard J. Schwartz. Spectra of compact composition operators. *Proc. Amer. Math. Soc.*, 51:127–130, 1975.
- [6] Isabelle Chalendar, Eva A. Gallardo-Gutiérrez, and Jonathan R. Partington. Weighted composition operators on the Dirichlet space: boundedness and spectral properties. *Math. Ann.*, 363(3-4):1265–1279, 2015.
- [7] Isabelle Chalendar and Jonathan R. Partington. *Modern approaches to the invariant-subspace problem*, volume 188 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011.
- [8] Vitali Chkliar. Eigenfunctions of the hyperbolic composition operator. *Integral Equations Operator Theory*, 29(3):364–367, 1997.
- [9] John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [10] Carl C. Cowen. Composition operators on H^2 . *J. Operator Theory*, 9(1):77–106, 1983.
- [11] Carl C. Cowen and Eva A. Gallardo-Gutiérrez. Consequences of universality among Toeplitz operators. *J. Math. Anal. Appl.*, 432(1):484–503, 2015.
- [12] Carl C. Cowen and Eva A. Gallardo-Gutiérrez. An introduction to Rota’s universal operators: properties, old and new examples and future issues. *Concr. Oper.*, 3:43–51, 2016.
- [13] Carl C. Cowen and Eva A. Gallardo-Gutiérrez. A new proof of a Nordgren, Rosenthal and Wintrobe theorem on universal operators. *Preprint*, 2016.
- [14] Carl C. Cowen and Eva A. Gallardo-Gutiérrez. Rota’s universal operators and invariant subspaces in Hilbert spaces. *J. Funct. Anal.*, 271(5):1130–1149, 2016.

- [15] Carl C. Cowen and Barbara D. MacCluer. Spectra of some composition operators. *J. Funct. Anal.*, 125(1):223–251, 1994.
- [16] Carl C. Cowen and Barbara D. MacCluer. *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.
- [17] Peter L. Duren. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
- [18] Peter L. Duren, Eva A. Gallardo-Gutiérrez, and Alfonso Montes-Rodríguez. A Paley-Wiener theorem for Bergman spaces with application to invariant subspaces. *Bull. Lond. Math. Soc.*, 39(3):459–466, 2007.
- [19] Sam J. Elliott and Michael T. Jury. Composition operators on Hardy spaces of a half-plane. *Bull. Lond. Math. Soc.*, 44(3):489–495, 2012.
- [20] Sam J. Elliott and Andrew Wynn. Composition operators on weighted Bergman spaces of a half-plane. *Proc. Edinb. Math. Soc. (2)*, 54(2):373–379, 2011.
- [21] Eva A. Gallardo-Gutiérrez and Pamela Gorkin. Cyclic Blaschke products for composition operators. *Rev. Mat. Iberoam.*, 25(2):447–470, 2009.
- [22] Eva A. Gallardo-Gutiérrez and Pamela Gorkin. Minimal invariant subspaces for composition operators. *J. Math. Pures Appl. (9)*, 95(3):245–259, 2011.
- [23] Eva A. Gallardo-Gutiérrez and Alfonso Montes-Rodríguez. The role of the spectrum in the cyclic behavior of composition operators. *Mem. Amer. Math. Soc.*, 167(791):x+81, 2004.
- [24] Gajath Gunatillake. Invertible weighted composition operators. *J. Funct. Anal.*, 261(3):831–860, 2011.
- [25] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu. *Theory of Bergman spaces*, volume 199 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
- [26] Katherine Heller. Adjoints of linear fractional composition operators on $S^2(\mathbb{D})$. *J. Math. Anal. Appl.*, 394(2):724–737, 2012.
- [27] William M. Higdon. The spectra of composition operators from linear fractional maps acting upon the Dirichlet space. *J. Funct. Anal.*, 220(1):55–75, 2005.
- [28] Kenneth Hoffman. *Banach spaces of analytic functions*. Dover Publications, Inc., New York, 1988. Reprint of the 1962 original.
- [29] Paul R. Hurst. Relating composition operators on different weighted Hardy spaces. *Arch. Math. (Basel)*, 68(6):503–513, 1997.

- [30] Olli Hyvärinen, Mikael Lindström, Ilmari Nieminen, and Erno Saukko. Spectra of weighted composition operators with automorphic symbols. *J. Funct. Anal.*, 265(8):1749–1777, 2013.
- [31] Gareth A. Jones and David Singerman. *Complex functions*. Cambridge University Press, Cambridge, 1987. An algebraic and geometric viewpoint.
- [32] Herbert Kamowitz. The spectra of composition operators on H^p . *J. Functional Analysis*, 18:132–150, 1975.
- [33] Herbert Kamowitz. The spectra of a class of operators on the disc algebra. *Indiana Univ. Math. J.*, 27(4):581–610, 1978.
- [34] G. Koenigs. Recherches sur les intégrales de certaines équations fonctionnelles. *Ann. Sci. École Norm. Sup. (3)*, 1:3–41, 1884.
- [35] Thomas Kriete and Jennifer Moorhouse. Linear relations in the Calkin algebra for composition operators. *Trans. Amer. Math. Soc.*, 359(6):2915–2944 (electronic), 2007.
- [36] Barbara D. MacCluer and Karen Saxe. Spectra of composition operators on the Bloch and Bergman spaces. *Israel J. Math.*, 128:325–354, 2002.
- [37] Barbara D. MacCluer, Xiangfei Zeng, and Nina Zorboska. Composition operators on small weighted Hardy spaces. *Illinois J. Math.*, 40(4):662–677, 1996.
- [38] Valentin Matache. On the minimal invariant subspaces of the hyperbolic composition operator. *Proc. Amer. Math. Soc.*, 119(3):837–841, 1993.
- [39] Valentin Matache. The eigenfunctions of a certain composition operator. In *Studies on composition operators (Laramie, WY, 1996)*, volume 213 of *Contemp. Math.*, pages 121–136. Amer. Math. Soc., Providence, RI, 1998.
- [40] Valentin Matache. Weighted composition operators on H^2 and applications. *Complex Anal. Oper. Theory*, 2(1):169–197, 2008.
- [41] Valentin Matache. Invertible and normal composition operators on the Hilbert Hardy space of a half-plane. *Concr. Oper.*, 3:77–84, 2016.
- [42] Jennifer Moorhouse. Compact differences of composition operators. *J. Funct. Anal.*, 219(1):70–92, 2005.
- [43] Vladimir Müller. *Spectral theory of linear operators and spectral systems in Banach algebras*, volume 139 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2003.

- [44] Vladimir Müller. Universal n -tuples of operators. *Math. Proc. R. Ir. Acad.*, 113A(2):143–150, 2013.
- [45] Gerard J. Murphy. *C^* -algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.
- [46] Eric A. Nordgren. Composition operators. *Canad. J. Math.*, 20:442–449, 1968.
- [47] Eric A. Nordgren, Peter Rosenthal, and F. S. Wintrobe. Invertible composition operators on H^p . *J. Funct. Anal.*, 73(2):324–344, 1987.
- [48] Raymond E. A. C. Paley and Norbert Wiener. *Fourier transforms in the complex domain*, volume 19 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.
- [49] Jonathan R. Partington and Elodie Pozzi. Universal shifts and composition operators. *Oper. Matrices*, 5(3):455–467, 2011.
- [50] Matthew A. Pons. The spectrum of a composition operator and Calderón’s complex interpolation. In *Topics in operator theory. Volume 1. Operators, matrices and analytic functions*, volume 202 of *Oper. Theory Adv. Appl.*, pages 451–467. Birkhäuser Verlag, Basel, 2010.
- [51] Elodie Pozzi. Universality of weighted composition operators on $L^2([0, 1])$ and Sobolev spaces. *Acta Sci. Math. (Szeged)*, 78(3-4):609–642, 2012.
- [52] G. C. Rota. Note on the invariant subspaces of linear operators. *Rend. Circ. Mat. Palermo (2)*, 8:182–184, 1959.
- [53] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [54] Ernst Schröder. Ueber iterirte Functionen. *Math. Ann.*, 3(2):296–322, 1870.
- [55] Joel H. Shapiro. *Composition operators and classical function theory*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.
- [56] Nina Zorboska. Compact composition operators on some weighted Hardy spaces. *J. Operator Theory*, 22(2):233–241, 1989.

