Convergence of asymptotic systems of non-autonomous neural network models with infinite distributed delays

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Abstract

In this paper we investigate the global convergence of solutions of non-autonomous Hopfield neural network models with discrete time-varying delays, infinite distributed delays, and possible unbounded coefficient functions. Instead of using Lyapunov functionals, we explore intrinsic features between the non-autonomous systems and their asymptotic systems to ensure the boundedness and global convergence of the solutions of the studied models. Our results are new and complement known results in the literature. The theoretical analysis is illustrated with some examples and numerical simulations.

Keywords: Neural networks, Unbounded coefficients, Bounded coefficients, Infinite distributed delays, Boundedness; Global convergence, Asymptotic systems.

Mathematics Subject Classification: 34K20, 34K25, 34K60, 92B20.

1 Introduction

In the last decades, retarded functional differential equations have attracted the attention of an increasing number of scientists due to their potential application in different sciences. Differential equations with delays have served as models in population dynamics, ecology, epidemiology, disease evolution, neural networks. Neural network models possess good potential applications in areas such as pattern recognition, signal and image processing, and optimization (see [2, 17, 18], and the references therein). Thus, in order to describe their dynamics, it is highly desirable to establish criteria for boundedness, existence of invariant sets, global convergence, and asymptotic behavior of solutions of neural network models in several settings (see [3, 4, 12, 13, 17, 19, 20, 21, 22, 23, 24] and the references therein).

In a classic study of neural network dynamics, Hopfield [11] proposed, in 1984, the following neural network model

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + I_i, \quad t \geq 0,\ i = 1, \dots, n, \tag{1.1}$$

where $n \in \mathbb{N}$ is the number of neurons, $x_i(t)$ is the state of the ith neuron at time t, $b_i(\cdot)$ is the charging function for the ith neuron, $f_j(\cdot)$ are the activation functions, a_{ij} denotes the strengths of connectivity between neurons j and i, and I_i is the input to the ith neuron.

In order to be more realistic, differential equations describing neural networks should incorporate time delays to take into account the synaptic transmission time among neurons, or, in artificial neural networks, the communication time among amplifiers. In 1989, Marcus and Westervelt [14] introduced for the first time a discrete delay in the Hopfield model (1.1), and they observed that the delay can destabilize the system. In fact, the delays can affect the dynamic behavior of neural network models [1] and, for this reason, stability of delayed neural networks has been investigated extensively ([2, 4, 5, 12, 15, 17, 18, 19, 20, 21, 22, 23, 24], and the references therein). Another relevant fact to take into account is that the neuron charging time, the interconnection weights, and the external inputs often change as time proceeds. Thus, the neural network models with temporal structure of neural activities should be introduced and investigated (see [3, 4, 18]). In this paper, we consider non-autonomous Hopfield neural network models with unbounded discrete and distributed delays.

In [6, 19, 20, 21, 23], the definition of asymptotic system (see Definition 2.1) of a non-autonomous system was introduced and the authors remarked that, in general, dynamic behavior of an asymptotic system is not available to characterize the dynamic behavior of the original system. For example, it easy to verify that the following equation, presented in [21],

$$x'(t) = -x(t) + \frac{2+t}{(t+1)^2}x^2(t), \qquad t \geq 0, \tag{1.2}$$

has an unbounded solution, $x(t) = t+1$, however, the zero solution of the linear equation

$$x'(t) = -x(t), \qquad t \geq 0, \tag{1.3}$$

is globally asymptotically stable and (1.3) is an asymptotic equation of (1.2). But, if we identify situations where the dynamic behavior of a system is characterized by the dynamic of one of its asymptotic systems, then it is extremely relevant to study the dynamic of asymptotic system because, in these cases, it is possible to obtain behavior properties of the original system from known properties of one of its asymptotic systems. In fact, an asymptotic system may be autonomous, periodic, almost periodic or other special non-autonomous system, i.e. easier to study than the original system, and by exploring the intrinsic features of the asymptotic system, we can obtain significant properties of the original system. The purpose of this paper is to present some sufficient conditions for boundedness and global convergence of solutions of non-autonomous Hopfield neural network systems and its asymptotic systems with both bounded and unbounded coefficient functions.

Moreover, the models studied here have infinite delays and, when we are dealing with functional differential equations with infinite delays, the choice of an admissible Banach phase space requires special attention in order to have well-posedness of the initial value problem and standard results on existence, uniqueness, and continuation of solutions (see [7, 9, 10]). We note that, many papers, dealing with neural networks with unbounded delays, do not provide an explicit phase space.

After the introduction, the present paper is divided into four sections. In Section 2, the models and its phase space are presented. In Section 3, we consider models with bounded coefficient functions and a criterion for boundedness of solutions and a criterion for global convergence of the models are derived. In Section 4, we consider neural network models with unbounded coefficient functions and similar results are given. Finally, in last section, illustrative numerical simulations are presented to show the effectiveness of the theoretical results.

2 Notations and model description

We denote by $BC = BC((-\infty, 0]; \mathbb{R}^n)$ the space of bounded and continuous functions, $\phi : (-\infty, 0] \to \mathbb{R}^n$, equipped with the norm $||\phi|| = \sup_{s\leq 0} |\phi(s)|$, where $|\cdot|$ is the maximum norm in $\mathbb{R}^n$, i.e. $|x| = \max\{|x_i| : i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $a \in \mathbb{R}^n$, we also use a to denote the constant function $\varphi(s) = a$ in BC. A vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ is said to be positive if $c_i > 0$ for all $i \in \{1, \dots, n\}$ and in this case we write $c > 0$. A function $\xi : [a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, is said to be eventually monotone if there exists $t^* > a$ such that ξ is non-decreasing (or non-increasing) on $[t^*, +\infty)$. For a real sequence $(u_n)_{n\in\mathbb{N}}$, we write $u_n \nearrow +\infty$ to say that $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence such that $\lim_{n\to+\infty} u_n = +\infty$.

Given a continuous function $f : [0, +\infty) \to \mathbb{R}$, we say that:

1. f is *periodic* if the exists $\omega > 0$ such that

$$f(t + \omega) = f(t), \qquad \forall t \geq 0;$$

2. f is *almost periodic* if, for any $\varepsilon > 0$ there exists $\omega = \omega(\varepsilon) > 0$ such that every interval $[a, a + \omega] \subseteq [0, +\infty)$ contains at least one point of α such that

$$|f(t + \alpha) - f(t)| < \varepsilon, \qquad \forall t \geq 0;$$

3. f is *pseudo almost periodic* if it can be expressed as

$$f = f_1 + \varphi$$

where f_1 is an almost periodic function and $\varphi : [0, +\infty) \to \mathbb{R}$ is a bounded continuous function such that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t |\varphi(s)|ds = 0.$$

It is well-known that a periodic, an almost periodic, or a pseudo almost periodic function f is bounded and we denote $\overline{f} := \sup_{t\geq 0} |f(t)|$ and $\underline{f} := \inf_{t\geq 0} |f(t)|$.

For an open set $D \subseteq BC$ and $f : [0, +\infty) \times D \to \mathbb{R}^n$ a continuous function, consider the functional differential equation (FDE) given in general setting by

$$x'(t) = f(t, x_t), \ \ t \geq 0, \tag{2.1}$$

where, as usual, x_t denotes the function $x_t : (-\infty, 0] \to \mathbb{R}^n$ defined by $x_t(s) = x(t + s)$ for $s \leq 0$. By a solution of (2.1) on an interval $I \subseteq \mathbb{R}$, we mean a function $x : (-\infty, \sup I) \to \mathbb{R}^n$ such that $x_t \in D$, $x(t)$ is continuous differentiable, and (2.1) holds for all $t \in I$ (see [9]).

It is well-known that the Banach space BC is not an admissible phase space for (2.1), in the sense of [7], thus the standard existence, uniqueness, continuous dependence type results are not available. Instead of BC, we consider the admissible Banach space

$$UC_g = \left\{\phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s\leq 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ is uniformly continuous on } (-\infty, 0]\right\},$$

equipped with the norm $||\phi||_g = \sup_{s\leq 0} \frac{|\phi(s)|}{g(s)}$, where $g : (-\infty, 0] \to [1, \infty)$ is a function satisfying:

(g1) g is a non-increasing continuous function and $g(0) = 1$;

(g2) $\lim_{u\to 0^-} \frac{g(s + u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

(g3) $g(s) \to +\infty$ as $s \to -\infty$.

See [9] for more details.

As $BC \subseteq UC_g$, then BC is a subspace of UC_g, and we denote by BC_g the space BC with the norm $|| \cdot ||_g$.

As UC_g is an admissible Banach space, we consider the FDE (2.1) in the phase space UC_g, for a convenient function g, and we assume that f has enough smooth properties to ensure the existence and uniqueness of solution for the initial value problem (see [9]). The solution of (2.1) with initial condition $x_{t_0} = \varphi$, for $t_0 \geq 0$ and $\varphi \in UC_g$, is denoted by $x(t, t_0, \varphi)$. Moreover, from [9] again, if f takes closed bounded subsets of its domain into bounded sets of $\mathbb{R}^n$, then the solution $x(t, t_0, \varphi)$ is extensible to $(-\infty, a]$, with $a > t_0$, whenever it is bounded. It is relevant to emphasize that, from [9], the solution $x(t, t_0, \varphi)$ is differentiable, once the differentiability of solutions plays an important role in the proof of main results.

In this paper, we consider the generalized Hopfield neural network model with both discrete time-varying and continuous distributed infinite delays given by

$$\begin{aligned} x_i'(t) = -b_i(t, x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} \Bigg(& a_{ijp}(t)h_{ijp}(x_j(t - \tau_{ijp}(t))) \\ & + c_{ijp}(t)f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t + s))d\eta_{ijp}(s)\right)\Bigg) + I_i(t), \qquad t \geq 0, \end{aligned} \tag{2.2}$$

$i = 1, \dots, n$, where $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $a_{ijp}, c_{ijp}, I_i : [0, +\infty) \to \mathbb{R}$, $h_{ijp}, f_{ijp}, g_{ijp} : \mathbb{R} \to \mathbb{R}$, and $\tau_{ijp} : [0, \infty) \to [0, \infty)$ are continuous functions, and $\eta_{ijp} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded functions, normalized so that $\eta_{ijp}(0) - \eta_{ijp}(-\infty) = 1$, for all $i, j \in \{1, \dots, n\}$, $p \in \{1, \dots, P\}$. We remark that the model (2.2) is general enough to include, as particular situations, the Hopfield neural network models studied in [19, 21, 22, 23] (see systems (3.6), (3.10), and (3.15) below). In [5], a function $g : (-\infty, 0] \to [1, +\infty)$ was defined by

(i) $g(s) = 1$ on $[-r_1, 0]$;

(ii) $g(-r_n) = n$, $n \in \mathbb{N}$;

(iii) g is continuous and piecewise linear (linear on intervals $[-r_{n+1}, -r_n]$),

where $r_n \nearrow +\infty$ is a suitable sequence of positive numbers, in such a way that conditions **(g1)**, **(g2)**, and **(g3)** hold, and

$$\int_{-\infty}^{0} g(s) d\eta_{ijp}(s) < +\infty, \qquad i, j = 1, \dots, n, \; p = 1, \dots, P.$$

See more details in [5, Lemma 4.1]. Thus, we may consider the differential system (2.2) in the phase space UC_g. As we are dealing with neural network systems, we restrict our attention to initial bounded conditions, i.e.,

$$x_{t_0} = \varphi, \qquad \text{with} \qquad \varphi \in BC, \tag{2.3}$$

for some $t_0 \geq 0$.

In the sequel, for (2.2) the following hypotheses will be considered:

(A1) for each $i \in \{1, \dots, n\}$, there exists a function $\beta_i : [0, +\infty) \to [0, +\infty)$ such that

$$\frac{b_i(t, u) - b_i(t, v)}{u - v} \geq \beta_i(t), \quad \forall t \geq 0, \, \forall u, v \in \mathbb{R}, u \neq v;$$

(A2) For each $i, j \in \{1, \dots, n\}$ and $p \in \{1, \dots, P\}$, $h_{ijp}, f_{ijp}, g_{ijp} : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_{ijp}, μ_{ijp}, and σ_{ijp}, respectively;

(A3) For each $i, j \in \{1, \dots, n\}$ and $p \in \{1, \dots, P\}$, $t - \tau_{ijp}(t) \to \infty$ as $t \to \infty$;

(A4) There exists $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$,

$$\limsup_{t \to +\infty} \left(-d_i \beta_i(t) + \sum_{j=1}^{n} \sum_{p=1}^{P} d_j (\gamma_{ijp} |a_{ijp}(t)| + \mu_{ijp} \sigma_{ijp} |c_{ijp}(t)|) \right) < 0.$$

We note that the hypothesis set **(A1)-(A4)** does not imply the boundedness of solutions of (2.2), as it is demonstrated by the next simple example.

Example 2.1. It is easy to verify that the delay scalar equation

$$x'(t) = -t x(t) + \frac{t}{4 + 2 \sin t} \sin x(t - 1) + \frac{t}{2 + \sin t} + t^2 + \frac{t}{2} + 1, \qquad t \geq 0, \tag{2.4}$$

has an unbounded solution $x(t) = t + 1$ and the hypotheses **(A1)-(A4)** hold. We remark that the coefficient functions are unbounded.

As we shall see in Theorem 3.1, if all coefficient functions are bounded then all solutions of (2.2) are bounded. Without assuming bounded coefficient functions in (2.2), we shall conclude, from Theorem 4.1, that either all solutions are bounded or all solutions are unbounded. Consequently, we can conclude that all solutions of (2.4) are unbounded.

In this paper, we study the relationship between the solutions of system (2.2) and the solutions of its asymptotic systems. Here, we use the usual concept of asymptotic system in the literature [19, 20, 21, 23].

Definition 2.1. *The system*

$$x_i'(t) = -\hat{b}_i(t, x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(\hat{a}_{ijp}(t)h_{ijp}(x_j(t-\hat{\tau}_{ijp}(t))) +$$
$$+\hat{c}_{ijp}(t)f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))d\eta_{ijp}(s)\right)\Bigg) + \hat{I}_i(t), \qquad t \geq 0, \quad (2.5)$$

$i = 1, \ldots, n$, *is said to be an asymptotic system of system* (2.2) *if* $\hat{b}_i(t,u)$, $\hat{a}_{ijp}(t)$, $\hat{c}_{ijp}(t)$, $\hat{\tau}_{ijp}(t)$, *and* $\hat{I}_i(t)$ *are continuous real functions such that* $\hat{b}_i$ *satisfies (A1) for some non-negative function* $\hat{\beta}_i$ *and*

$$\begin{aligned}\lim_{t\to+\infty}(\beta_i(t) - \hat{\beta}_i(t)) &= \lim_{t\to+\infty}(b_i(t,u(t)) - \hat{b}_i(t,u(t))) = \lim_{t\to+\infty}(a_{ijp}(t) - \hat{a}_{ijp}(t)) \\ &= \lim_{t\to+\infty}(c_{ijp}(t) - \hat{c}_{ijp}(t)) = \lim_{t\to+\infty}(\tau_{ijp}(t) - \hat{\tau}_{ijp}(t)) \\ &= \lim_{t\to+\infty}(I_i(t) - \hat{I}_i(t)) = 0,\end{aligned} \qquad (2.6)$$

for every bounded continuous function $u : \mathbb{R} \to \mathbb{R}$.

From (2.6), it is obvious that hypothesis **(A4)** is equivalent to

$$\limsup_{t\to+\infty}\left(-\hat{\beta}_i(t) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}(\gamma_{ijp}|\hat{a}_{ijp}(t)| + \mu_{ijp}\sigma_{ijp}|\hat{c}_{ijp}(t)|)\right) < 0, \qquad 1, \ldots, n. \qquad (2.7)$$

Before we consider the global convergence of the models, we need to show that all solutions of (2.2), with bounded initial condition, are defined on $\mathbb{R}$.

Lemma 2.1. *Assume (A1) and (A2) hold.*
Then, each solution $x(t) = x(t, t_0, \varphi)$ *(with* $t_0 \geq 0$ *and* $\varphi \in BC$*) of* (2.2) *is defined on* $\mathbb{R}$.

Proof. Let $x(t) = (x_1(t), \ldots, x_n(t))$ be the maximal solution of the initial value problem (2.2)-(2.3) and define $z(t) = (z_1(t), \ldots, z_n(t)) := (|x_1(t)|, \ldots, |x_n(t)|)$. For each $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned}z_i'(t) &= \operatorname{sign}(x_i(t))x_i'(t) \\ &= \operatorname{sign}(x_i(t))\Bigg[-b_i(t,x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(a_{ijp}(t)h_{ijp}(x_j(t-\tau_{ijp}(t))) \\ &\qquad +c_{ijp}(t)f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))d\eta_{ijp}(s)\right)\Bigg) + I_i(t)\Bigg],\end{aligned}$$

by integration, we obtain

$$\begin{aligned}z_i(t) &\leq z_i(t_0) - \int_{t_0}^{t}\operatorname{sign}(x_i(u))\Big(b_i(u,x_i(u)) - b_i(u,0)\Big)du + \int_{t_0}^{t}|b_i(u,0)|du + \int_{t_0}^{t}|I_i(u)|du \\ &+ \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg[\int_{t_0}^{t}\Big|a_{ijp}(u)\Big(h_{ijp}(x_j(u-\tau_{ijp}(u))) - h_{ijp}(0)\Big) + a_{ijp}(u)h_{ijp}(0)\Big|du \\ &+ \int_{t_0}^{t}\left|c_{ijp}(u)\left(f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(u+s))d\eta_{ijp}(s)\right) - f_{ijp}(g_{ijp}(0))\right) + c_{ijp}(u)f_{ijp}(g_{ijp}(0))\right|du\Bigg]\end{aligned}$$

and, from hypotheses **(A1)-(A2)**, we conclude that

$$\begin{aligned} z_i(t) &\le \|\varphi\| - \int_{t_0}^t \beta_i(u) z_i(u)du + \int_{t_0}^t \overline{b}(u)du + \int_{t_0}^t \overline{I}(u)du \\ &+ \int_{t_0}^t \sum_{j=1}^n \sum_{p=1}^P \Big(\overline{a}(u)\overline{\gamma}\|x_u\| + \overline{a}(u)|h_{ijp}(0)| + \overline{c}(u)\overline{\mu}\,\overline{\sigma}\|x_u\| + \overline{c}(u)|f_{ijp}(g_{ijp}(0))| \Big) du \\ &\le \|\varphi\| + \int_{t_0}^t \overline{b}(u) + \overline{I}(u) + nP(\overline{a}(u)\overline{h}(0) + \overline{c}(u)\overline{F}(0))du + \int_{t_0}^t nP(\overline{a}(u)\overline{\gamma} + \overline{c}(u)\overline{\mu}\,\overline{\sigma})\|x_u\|du, \end{aligned}$$

where $\overline{h}(0) = \max_{i,j,p} |h_{ijp}(0)|$, $\overline{F}(0) = \max_{i,j,p} |f_{ijp}(g_{ijp}(0))|$, $\overline{\gamma} = \max_{i,j,p} \gamma_{ijp}$, $\overline{\mu} = \max_{i,j,p} \mu_{ijp}$, $\overline{\sigma} = \max_{i,j,p} \sigma_{ijp}$, $\overline{a}(u) = \max_{i,j,p} |a_{ijp}(u)|$, $\overline{b}(u) = \max_i |b_i(u,0)|$, $\overline{c}(u) = \max_{i,j,p} |c_{ijp}(u)|$, and $\overline{I}(u) = \max_i |I_i(u)|$. Defining the continuous functions $\upsilon, \nu : [t_0, +\infty) \to [0, +\infty)$ by

$$\upsilon(t) = \|\varphi\| + \int_{t_0}^t \overline{b}(u) + \overline{I}(u) + nP(\overline{a}(u)\overline{h}(0) + \overline{c}(u)\overline{F}(0))du \;\text{ and }\; \nu(t) = nP(\overline{a}(t)\overline{\gamma} + \overline{c}(t)\overline{\mu}\,\overline{\sigma})$$

respectively, we get, for $t \ge t_0$,

$$\|z_t\| \le \upsilon(t) + \int_{t_0}^t \nu(u)\|z_u\|du$$

and, by the generalized Gronwall's inequality, see [8], we have

$$\|z_t\| \le \upsilon(t) + \int_{t_0}^t \nu(u)\upsilon(u)e^{\int_u^t \nu(v)dv}du,$$

and the conclusion follows from the Continuation Theorem (see [9]). □

3 Bounded coefficient functions

In this section, we address the boundedness and global convergence of solutions of (2.2) and of its asymptotic systems, assuming that all coefficient functions are bounded, that is

(B) for each $i, j \in \{1, \dots, n\}$ and $p \in \{1, \dots, P\}$, the functions $a_{ijp}, c_{ijp}, I_i : [0, +\infty) \to \mathbb{R}$ and $b_i(\cdot, u) : [0, +\infty) \to \mathbb{R}$ are continuously bounded for all $u \in \mathbb{R}$.

In the first theorem we establish sufficient conditions ensuring the boundedness of solutions.

Theorem 3.1. *Assume (A1), (A2), (A4), and (B) hold.*
Then all solutions of (2.2) *with initial bounded condition are bounded.*

Proof. As $b_i(t,0)$, $a_{ijp}(t)$, $c_{ijp}(t)$, and $I_i(t)$ are bounded, there exists $M > 0$ such that

$$M \ge |b_i(t,0)| + |I_i(t)| + \sum_{j=1}^n \sum_{p=1}^P \Big(|a_{ijp}(t)|\,|h_{ijp}(0)| + |c_{ijp}(t)|\,|f_{ijp}(g_{ijp}(0))| \Big),$$

for all $t \ge 0$ and $i \in \{1, \dots, n\}$.

From **(A4)**, there exist $T > 0$ and $l < 0$ such that

$$-\beta_i(t) + \sum_{j=1}^n \sum_{p=1}^P \frac{d_j}{d_i} \Big(\gamma_{ijp}|a_{ijp}(t)| + \mu_{ijp}\sigma_{ijp}|c_{ijp}(t)| \Big) < l, \qquad \forall t \ge T. \tag{3.1}$$

Let $x(t, t_0, \varphi) = x(t) = (x_1(t), \dots, x_n(t))$ be a maximal solution of (2.2), for some $t_0 \ge 0$ and $\varphi \in BC$, and define $z(t) = (d_1^{-1}|x_1(t)|, \dots, d_n^{-1}|x_n(t)|)$. By contradiction, assume that $x(t)$ is unbounded.

From Lemma 2.1 $x(t)$ is defined on $\mathbb{R}$ and consequently there exist $i \in \{1, \dots, n\}$ and a positive sequence $(t_k)_{k\in\mathbb{N}}$ such that $T < t_k \nearrow +\infty$, $0 < z_i(t_k) \nearrow +\infty$,

$$z_i(t_k) = \|z_{t_k}\| \geq \|z_t\|, \quad \text{and} \quad z_i'(t_k) \geq 0, \quad \forall k \in \mathbb{N},\ \forall t \leq t_k. \tag{3.2}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned}
z_i'(t_k) &= \operatorname{sign}(x_i(t_k))d_i^{-1}x_i'(t_k) \\
&= -d_i^{-1}\operatorname{sign}(x_i(t_k))\Big(b_i(t_k, x_i(t_k)) - b_i(t_k, 0)\Big) + \operatorname{sign}(x_i(t_k))d_i^{-1}\Big(- b_i(t_k, 0) + I_i(t_k)\Big) \\
&\quad +\operatorname{sign}(x_i(t_k))d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Big(a_{ijp}(t_k)h_{ijp}(x_j(t_k - \tau_{ijp}(t_k))) - a_{ijp}(t_k)h_{ijp}(0)+ \\
&\quad +c_{ijp}(t_k)f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t_k + s))d\eta_{ijp}(s)\right) - c_{ijp}(t_k)f_{ijp}(g_{ijp}(0))\Big)+ \\
&\quad +\operatorname{sign}(x_i(t_k))d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Big(a_{ijp}(t_k)h_{ijp}(0) + c_{ijp}(t_k)f_{ijp}(g_{ijp}(0))\Big),
\end{aligned}$$

and from **(A1)-(A2)** we obtain

$$\begin{aligned}
z_i'(t_k) &\leq -\beta_i(t_k)z_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|a_{ijp}(t_k)|\gamma_{ijp}z_j(t_k - \tau_{ijp}(t_k)) + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\|z_{j,t_k}\|\Big) + d_i^{-1}M \\
&\leq -\beta_i(t_k)z_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big)\|z_{t_k}\| + d_i^{-1}M,
\end{aligned}$$

and (3.2) implies

$$z_i'(t_k) \leq \left(-\beta_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big)\right) z_i(t_k) + d_i^{-1}M, \quad \forall k \in \mathbb{N}.$$

From (3.1) and (3.2) we conclude that $z_i'(t_k) \leq l z_i(t_k) + d_i^{-1}M \to -\infty$, as $k \to +\infty$, which is a contradiction. □

Now, we state sufficient conditions ensuring the global attractivity of solutions of system (2.2) and of its asymptotic systems (2.5).

Theorem 3.2. *Assume (A1)-(A4) and (B) hold.*
Then

$$\lim_{t\to+\infty} |x_i(t) - \hat{x}_i(t)| = 0, \quad i = 1, \dots, n,$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ *and* $\hat{x}(t) = (\hat{x}_1(t), \dots, \hat{x}_n(t))$ *solutions of systems* (2.2) *and* (2.5) *respectively, with bounded initial conditions.*

Proof. Let $x(t)$ and $\hat{x}(t)$ solutions of (2.2) and (2.5) respectively, with bounded initial conditions, and define $y(t) = (d_1^{-1}|x_1(t) - \hat{x}_1(t)|, \dots, d_n^{-1}|x_n(t) - \hat{x}_n(t)|)$.

From Lemma 2.1 and Theorem 3.1, we know that $x(t)$ and $\hat{x}(t)$ are bounded on $\mathbb{R}$. It follows that $y(t)$ is a non-negative bounded function on $\mathbb{R}$ and it is possible to define $Y_0 := \sup_{t\in\mathbb{R}} |y(t)|$, the limits

$$u_i := \limsup_{t\to+\infty} y_i(t), \quad i = 1, \dots, n,$$

and

$$u := \max_i\{u_i\} \in [0, +\infty).$$

It remains to prove that $u = 0$.

Let $i \in \{1, \ldots, n\}$ be such that $u_i = u$. It is easy to prove that there is a positive sequence $(t_k)_{k\in\mathbb{N}}$ such that

$$t_k \nearrow +\infty, \qquad y_i(t_k) \to u, \qquad \text{and} \qquad y_i'(t_k) \to 0, \qquad \text{as} \qquad k \to +\infty, \tag{3.3}$$

in fact:

Case 1. If $y_i(t)$ is eventually monotone, then $\lim\limits_{t\to+\infty} y_i(t) = u$ and, for large t, $y_i(t)$ is a differentiable, monotone and bounded real function. Hence there is a positive sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow +\infty$ and $y_i'(t_k) \to 0$, as $k \to +\infty$;

Case 2. If $y_i(t)$ is not eventually monotone, then there is a positive sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow +\infty$, $y_i'(t_k) = 0$, and $y_i(t_k) \to u$, as $k \to +\infty$. Thus (3.3) holds.

For the sake of contradiction, assume that $u > 0$.

Fix $\delta > 0$ and let $T = T(\delta) > 0$ be such that $\delta < u$, $|y(t)| < u_\delta := u + \delta$ for $t \geq T$ and

$$\int_{-\infty}^{-T} d\eta_{ijp}(s) < \frac{\delta}{Y_0}, \qquad \forall j \in \{1, \ldots, n\},\, p \in \{1, \ldots, P\}.$$

Since $t - \tau_{ijp}(t) \to \infty$ and $\tau_{ijp}(t) - \hat{\tau}_{ijp}(t) \to 0$ as $t \to \infty$, and $y_i(t_k) \to u$ as $k \to +\infty$, then there is $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $t_k - \hat{\tau}_{ijp}(t_k) > T$, $t_k > 2T$, and $y_i(t_k) > u_{-\delta} := u - \delta > 0$. From the hypotheses, we conclude that, for all $k > k_0$,

$$\begin{aligned}
y_i'(t_k) &= \operatorname{sign}(x_i(t_k) - \hat{x}_i(t_k))d_i^{-1}(x_i'(t_k) - \hat{x}_i'(t_k)) \\
&= \operatorname{sign}(x_i(t_k) - \hat{x}_i(t_k))d_i^{-1}\Big(- (b_i(t_k, x_i(t_k)) - \hat{b}_i(t_k, x_i(t_k))) - (\hat{b}_i(t_k, x_i(t_k)) - \hat{b}_i(t_k, \hat{x}_i(t_k)))\Big) \\
&\quad +\operatorname{sign}(x_i(t_k) - \hat{x}_i(t_k))\sum_{j=1}^{n}\sum_{p=1}^{P} d_i^{-1}\Big[(a_{ijp}(t_k) - \hat{a}_{ijp}(t_k))h_{ijp}(x_j(t_k - \tau_{ijp}(t_k))) \\
&\quad +\hat{a}_{ijp}(t_k)\Big(h_{ijp}(x_j(t_k - \tau_{ijp}(t_k))) - h_{ijp}(x_j(t_k - \hat{\tau}_{ijp}(t_k)))\Big) \\
&\quad +\hat{a}_{ijp}(t_k)\Big(h_{ijp}(x_j(t_k - \hat{\tau}_{ijp}(t_k))) - h_{ijp}(\hat{x}_j(t_k - \hat{\tau}_{ijp}(t_k)))\Big) \\
&\quad +(c_{ijp}(t_k) - \hat{c}_{ijp}(t_k))f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t_k + s))d\eta_{ijp}(s)\right) \\
&\quad +\hat{c}_{ijp}(t_k)\left(f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t_k + s))d\eta_{ijp}(s)\right) - f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t_k + s))d\eta_{ijp}(s)\right)\right)\Big] \\
&\quad +\operatorname{sign}(x_i(t_k) - \hat{x}_i(t_k))(I_i(t_k) - \hat{I}_i(t_k))d_i^{-1} \\
&\leq -\hat{\beta}_i(t_k)y_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}y_j(t_k - \hat{\tau}_{ijp}(t_k)) \\
&\quad +|\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\int_{-\infty}^{0} y_j(t_k + s)d\eta_{ijp}(s)\Big] + \varepsilon_{i1}(t_k),
\end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
\varepsilon_{i1}(t) \quad := \quad & d_i^{-1}|b_i(t,x_i(t)) - \hat{b}_i(t,x_i(t))| + \sum_{j=1}^{n}\sum_{p=1}^{P} d_i^{-1}\Big[|a_{ijp}(t) - \hat{a}_{ijp}(t)|\,|h_{ijp}(x_j(t-\tau_{ijp}(t)))| + \\
& + |\hat{a}_{ijp}(t)|\gamma_{ijp}|x_j(t-\tau_{ijp}(t)) - x_j(t-\hat{\tau}_{ijp}(t))| + \\
& + |c_{ijp}(t) - \hat{c}_{ijp}(t)|\left|f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))d\eta_{ijp}(s)\right)\right|\Big] + d_i^{-1}|I_i(t) - \hat{I}_i(t)|.
\end{aligned}
$$

As $t \mapsto b_i(t,w)$ and $x_i(t)$ are bounded on $[0,+\infty)$, for all $w \in \mathbb{R}$, and, from **(A1)**, $w \mapsto b_i(t,w)$ is non-decreasing for all $t \geq 0$, then $t \mapsto b_i(t,x_i(t))$ is a bounded function on $[0,+\infty)$. As $a_{ijp}(t), c_{ijp}(t), I_i(t)$, and $x_j(t)$ are also bounded on $[0,+\infty)$ for all i, j, p, it follows from (2.2) that $x'_j(t)$ are bounded on $(0+\infty)$. Thus $x_j(t)$ are uniformly continuous and consequently, from (2.6),

$$
\lim_{t\to+\infty} \varepsilon_{i1}(t) = 0. \tag{3.5}
$$

It follows from (3.4) that, for all $k \geq k_0$,

$$
\begin{aligned}
y'_i(t_k) \quad \leq \quad & -\hat{\beta}_i(t_k)y_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}y_j(t_k - \hat{\tau}_{ijp}(t_k)) \\
& + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\int_{-\infty}^{0} y_j(t_k+s)d\eta_{ijp}(s)\Big] + \varepsilon_{i1}(t_k) \\
\leq \quad & -\hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}u_\delta \\
& + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big(\int_{-\infty}^{-T} y_j(t_k+s)d\eta_{ijp}(s) + \int_{-T}^{0} y_j(t_k+s)d\eta_{ijp}(s)\Big)\Big] + \varepsilon_{i1}(t_k) \\
\leq \quad & -\hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\left[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}u_\delta + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\left(\delta + u_\delta\int_{-T}^{0} d\eta_{ijp}(s)\right)\right] \\
& + \varepsilon_{i1}(t_k) \\
\leq \quad & -\hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big)u_{2\delta} + \varepsilon_{i1}(t_k).
\end{aligned}
$$

Since $y'_i(t_k) \to 0$ as $k \to +\infty$, then letting $\delta \to 0$ and $k \to +\infty$ it follows from (2.7) and (3.5) that

$$
0 \leq \left(\limsup_{k\to+\infty}\left[-\hat{\beta}_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big)\right]\right)u < 0,
$$

which is a contradiction. Consequently, $u = 0$ and the proof is concluded. □

Obviously, system (2.2) can be regarded as an asymptotic system of itself, thus we have the following corollary.

Corollary 3.3. *Assume (A1)-(A4) and (B) hold.*

If $x(t) = (x_1(t),\dots,x_n(t))$ and $x^*(t) = (x_1^*(t),\dots,x_n^*(t))$ are solutions of (2.2) with bounded initial conditions, then

$$
\lim_{t\to+\infty} |x_i(t) - x_i^*(t)| = 0, \qquad i = 1,\dots,n.
$$

With the following examples, we show the effectiveness of presented results and a comparison with some stability criteria in the literature is given.

Example 3.1. For systems (2.2) and (2.5) with the restrictions $f_{ijp}(x) = 0$, $P = 2$, $h_{ij1} = h_{ij2} = h_j$, $\gamma_{ij1} = \gamma_{ij2} = \gamma_j$, $\tau_{ij1}(t) = 0$, $\tau_{ij2}(t) = \tau_{ij}(t)$, $b_i(t,u) = \beta_i(t)u$, and $\hat{b}_i(t,u) = \hat{\beta}_i(t)u$, i.e., for the models

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} a_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + I_i(t), \qquad t \geq 0, \tag{3.6}$$

and

$$x_i'(t) = -\hat{\beta}_i(t)x_i(t) + \sum_{j=1}^{n} \hat{a}_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} \hat{a}_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + \hat{I}_i(t), \qquad t \geq 0, \tag{3.7}$$

where

$$\begin{aligned} \lim_{t\to+\infty}(\beta_i(t) - \hat{\beta}_i(t)) &= \lim_{t\to+\infty}(a_{ij1}(t) - \hat{a}_{ij1}(t)) = \lim_{t\to+\infty}(a_{ij2}(t) - \hat{a}_{ij2}(t)) \\ &= \lim_{t\to+\infty}(I_i(t) - \hat{I}_i(t)) = 0, \end{aligned} \tag{3.8}$$

the global convergence of the models was already studied in [19, 20, 21]. In [23], the slight general situation $b_i(t,x) = \beta_i(t)g_i(x)$, with $g_i : \mathbb{R} \to \mathbb{R}$ satisfying **(A1)**, was considered. Clearly, (3.6) is a special case of model (2.2) and its asymptotic system (3.7) is a special case of model (2.5), thus from Theorem 3.2 we obtain the following result:

Corollary 3.4. *Assume that $h_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constant γ_j, the functions $\tau_{ij} : [0,+\infty) \to [0,+\infty)$ are continuous such that $t - \tau_{ij}(t) \to +\infty$ as $t \to +\infty$, and $\beta_i, \hat{\beta}_i : [0,+\infty) \to [0,+\infty)$, $a_{ij1}, \hat{a}_{ij1}, a_{ij2}, \hat{a}_{ij2}, I_i, \hat{I}_i : [0,+\infty) \to \mathbb{R}$ are continuous bounded functions such that* (3.8) *holds.*

If there exists $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$,

$$\limsup_{t\to+\infty} \left(-d_i\beta_i(t) + \sum_{j=1}^{n} d_j\gamma_j(|a_{ij1}(t)| + |a_{ij2}(t)|) \right) < 0, \tag{3.9}$$

then

$$\lim_{t\to+\infty} |x_i(t) - \hat{x}_i(t)| = 0, \qquad i = 1, \dots, n,$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ and $\hat{x}(t) = (\hat{x}_1(t), \dots, \hat{x}_n(t))$ solutions of systems (3.6) *and* (3.7) *respectively, with bounded initial conditions.*

Remark 3.1. Note that Corollary 3.4 improves [19, Theorem 2.1] and [23, Theorem 2.1] because the authors assume that system (3.6) has a periodic asymptotic system, i.e., (3.7) is a periodic system, and they also assume the stronger condition

$$\exists d = (d_1, \dots, d_n) > 0,\ \exists \eta > 0 : -d_i\beta_i(t) + \sum_{j=1}^{n} d_j\gamma_j(|a_{ij1}(t)| + |a_{ij2}(t)|) < -\eta,\ \forall t \geq 0,\ i \in \{1, \dots, n\},$$

instead of (3.9).

In [21], instead of condition (3.9), the hypothesis

(H) There exists $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$

$$\limsup_{t\to+\infty} \left(\sum_{j=1}^{n} \frac{d_j\gamma_i(|a_{ji1}(t)| + |a_{ji2}(t)|)}{d_i\beta_j(t)} \right) < 1,$$

with $\liminf\limits_{t\to+\infty} \beta_i(t) > 0$, is assumed and, as it is illustrated in Section 5 with the model (5.1), the conditions (3.9) and **(H)** are different. Consequently Theorem 3.2 gives a new global convergence criterion.

Example 3.2. Consider the following Hopfield neural network model

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)h_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)f_j\left(\sigma_j\int_{-\infty}^{0} G_{ij}(-s)x_j(t+s)ds\right) + I_i(t), \quad (3.10)$$

for $t \geq 0$ and $i = 1,\dots,n$, where $\sigma_j \geq 0$, $h_j, f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_j and μ_j respectively, the delay kernel functions $G_{ij} : [0,+\infty) \to [0,+\infty)$ are piecewise continuous and integrable such that

$$\int_0^{+\infty} G_{ij}(u)du = 1, \quad (3.11)$$

and $\beta_i, a_{ij}, c_{ij}, I_i : [0,+\infty) \to \mathbb{R}$ are continuous functions such that

$$\begin{aligned}\lim_{t\to+\infty}(\beta_i(t) - \hat{\beta}_i(t)) &= \lim_{t\to+\infty}(a_{ij}(t) - \hat{a}_{ij}(t)) = \lim_{t\to+\infty}(c_{ij}(t) - \hat{c}_{ij}(t)) \\ &= \lim_{t\to+\infty}(I_i(t) - \hat{I}_i(t)) = 0, \end{aligned} \quad (3.12)$$

for some almost periodic continuous functions $\hat{\beta}_i, \hat{a}_{ij}, \hat{c}_{ij}, \hat{I}_i : [0,+\infty) \to \mathbb{R}$, $i,j = 1,\dots,n$. Thus the following system is an asymptotic system of (3.10)

$$x_i'(t) = -\hat{\beta}_i(t)x_i(t) + \sum_{j=1}^{n} \hat{a}_{ij}(t)h_j(x_j(t)) + \sum_{j=1}^{n} \hat{c}_{ij}(t)f_j\left(\sigma_j\int_{-\infty}^{0} G_{ij}(-s)x_j(t+s)ds\right) + \hat{I}_i(t). \quad (3.13)$$

System (3.10) arises as another special case of model (2.2), when we consider $P = 1$, $b_i(t,u) = \beta_i(t)u$, $a_{ij1}(t) = a_{ij}(t)$, $h_{ij1}(u) = h_j(u)$, $\tau_{ij1}(t) = 0$, $c_{ij1}(t) = c_{ij}(t)$, $f_{ij1}(u) = f_j(u)$, $g_{ij1}(u) = \sigma u$ for all $t \geq 0$, $u \in \mathbb{R}$, $i,j = 1,\dots,n$, and the functions η_{ij1} are defined by

$$\eta_{ij1}(s) = \int_{-\infty}^{s} G_{ij}(-u)du, \quad s \in (-\infty, 0], \; i,j = 1,\dots,n.$$

In [22], the following result for the existence of an almost periodic solution of (3.13) was established.

Theorem 3.5. *[22] Assume that, for each $i,j = 1,\dots,n$,*

(i) *the functions $\hat{\beta}_i, \hat{a}_{ij}, \hat{c}_{ij}, \hat{I}_i : [0,+\infty) \to \mathbb{R}$ are continuous almost periodic such that*

$$\underline{\hat{\beta}_i} = \inf_{t\geq 0} \hat{\beta}_i(t) > 0;$$

(ii) *the functions $h_j, f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_j and μ_j respectively;*

(iii) *the delay kernel function $G_{ij} : [0,+\infty) \to [0,+\infty)$ is piecewise continuous and integrable such that (3.11) holds.*

If there exists $d = (d_1,\dots,d_n) > 0$ such that, for each $i \in \{1,\dots,n\}$,

$$\left(\underline{\hat{\beta}_i}d_i\right)^{-1}\left[\sum_{j=1}^{n} d_j\left(\gamma_j\overline{\hat{a}_{ij}} + \mu_j\sigma_j\overline{\hat{c}_{ij}}\right)\right] < 1, \quad (3.14)$$

where $\overline{\hat{a}_{ij}} = \sup\limits_{t\geq 0}|\hat{a}_{ij}(t)|$ and $\overline{\hat{c}_{ij}} = \sup\limits_{t\geq 0}|\hat{c}_{ij}(t)|$, then the system (3.13) has an almost periodic solution.

As an almost periodic function is bounded and condition (3.14) implies

$$\limsup_{t\to+\infty}\left(-d_i\beta_i(t)+\sum_{j=1}^{n}d_j\left(\gamma_j|a_{ij}(t)|+\mu_j\sigma_j|c_{ij}(t)|\right)\right)<0,\qquad \forall i=1,\dots,n,$$

from Theorems 3.2 and 3.5, we conclude the following stability criterion.

Corollary 3.6. *Assume that conditions (i)-(iii) in Theorem 3.5 and* (3.14) *hold.*

If $\beta_i, a_{ij}, c_{ij}, I_i : [0,+\infty)\to\mathbb{R}$, *are continuous functions such that* (3.12) *holds, then every solution* $x(t)$ *of* (3.10), *with bounded initial condition, satisfies*

$$\lim_{t\to+\infty}|x(t)-\hat{x}(t)|=0,$$

where $\hat{x}(t)$ *is the almost periodic solution of* (3.13).

Remark 3.2. Since the Hopfield neural network model (3.10) is not an almost periodic system, the stability result in [22] can not be applied to prove that all solutions converge to an almost periodic function. In Section 5, we present numerical simulations of the model (5.3) to illustrate the effectiveness of Corollary 3.6.

Example 3.3. At last, we consider the following neural network model

$$\begin{aligned}x_i'(t)=-b_i(x_i(t))+\sum_{j=1}^{n}a_{ij1}(t)h_{j1}(x_j(t))+\sum_{j=1}^{n}a_{ij2}(t)h_{j2}(x_j(t-\tau_{ij}(t)))\\ +\sum_{j=1}^{n}c_{ij}(t)\int_{-\infty}^{0}G_{ij}(-s)g_j(x_j(t+s))ds+I_i(t),\ t\geq 0,\ i=1,\dots,n,\end{aligned}\tag{3.15}$$

where $b_i:\mathbb{R}\to\mathbb{R}$ are continuous functions, $h_{j1},h_{j2},g_j:\mathbb{R}\to\mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_{j1}, γ_{j2}, and σ_j respectively, $a_{ij1},a_{ij2},c_{ij},I_i:[0,+\infty)\to\mathbb{R}$, $\tau_{ij}:[0,+\infty)\to[0,+\infty)$ are continuous pseudo almost periodic functions, and the delay kernel functions $G_{ij}:[0,+\infty)\to[0,+\infty)$ are piecewise continuous and integrable such that (3.11) and

$$\int_0^{+\infty}uG_{ij}(u)ds<+\infty\tag{3.16}$$

hold. As in the above examples, it is easy to see that (3.15) is also a special case of model (2.2).

We remark that a system, which has an asymptotic pseudo almost periodic system, is itself a pseudo almost periodic system. For model (3.15), Corollary 3.3 allows us to improve a stability criterion in [24].

In [24], the following result for the existence of a pseudo almost periodic solution of (3.15) was established.

Theorem 3.7. *[24] Assume that, for each* $i,j=1,\dots,n$,

(i) *the functions* $a_{ij1},a_{ij2},c_{ij},I_i:[0,+\infty)\to\mathbb{R}$, $\tau_{ij}:[0,+\infty)\to[0,+\infty)$ *are pseudo almost periodic continuous;*

(ii) *the function* $b_i:\mathbb{R}\to\mathbb{R}$ *is continuous and there exists* $\beta_i>0$ *such that*

$$\frac{b_i(u)-b_i(v)}{u-v}\geq\beta_i,\qquad\forall u,v\in\mathbb{R},u\neq v;$$

(iii) *the functions* $h_{j1},h_{j2},g_j:\mathbb{R}\to\mathbb{R}$ *are Lipschitz functions with Lipschitz constants* $\gamma_{j1},\gamma_{j2},\sigma_j$ *respectively;*

(iv) *the delay kernel function* $G_{ij} : [0, +\infty) \to [0, +\infty)$ *is piecewise continuous and integrable such that* (3.11) *and* (3.16) *hold.*

If

$$-\beta_i + \sum_{j=1}^{n} (\gamma_{j1}\overline{a_{ij1}} + \gamma_{j2}\overline{a_{ij2}} + \sigma_j\overline{c_{ij}}) < 0, \qquad \forall i = 1, \ldots, n, \tag{3.17}$$

where $\overline{c_{ij}} = \sup_{t\geq 0} |c_{ij}(t)|$ *and* $\overline{a_{ijp}} = \sup_{t\geq 0} |a_{ijp}(t)|$ *for* $p = 1, 2$*, then the system* (3.15) *has at least one pseudo almost periodic solution.*

As a pseudo almost periodic function is bounded and condition (3.17) implies

$$\limsup_{t\to+\infty} \left(-\beta_i + \sum_{j=1}^{n} (\gamma_{j1}|a_{ij1}(t)| + \gamma_{j2}|a_{ij2}(t)| + \sigma_j|c_{ij}(t)|) \right) < 0, \qquad \forall i = 1, \ldots, n,$$

from Theorem 3.7 and Corollary 3.3, we obtain the following stability result.

Corollary 3.8. *Assume that conditions (i)-(iv) in Theorem 3.7 and* (3.17) *hold.*
Then the system (3.15) *has a unique pseudo almost periodic solution* $x^*(t)$ *such that*

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = 0,$$

for all $x(t)$ *solution with bounded initial condition.*

Remark 3.3. In [24], the global asymptotic stability of the pseudo almost periodic solution $x^*(t)$ of system (3.15) was proved assuming the following additional conditions:

1. the delays functions $\tau_{ij}(t)$ are differentiable such that, for some $\overline{\tau} > 0$,

$$\tau'_{ij}(t) \leq \overline{\tau} < 1, \qquad \forall t \geq 0, \forall i, j = 1, \ldots, n;$$

2. for each $i \in \{1, \ldots, n\}$,

$$-\beta_i + \sum_{j=1}^{n} \left(\gamma_{j1}\overline{a_{ij1}} + \frac{\gamma_{j2}\overline{a_{ij2}}}{1 - \overline{\tau}} + \sigma_j\overline{c_{ij}} \right) < 0.$$

With the model (5.5) in Section 5, we illustrate the effectiveness of Corollary 3.8.

4 Unbounded coefficient functions

In this section, we shall address the boundedness and global convergence of solution of system (2.2) and of its asymptotic systems (2.5) without assuming bounded coefficient functions.

Theorem 4.1. *Assume (A1), (A2), and (A4) hold and* $\tau_{ijp}(t) = \hat{\tau}_{ijp}(t)$ *for all* $t \geq 0$.
If (2.2) *has a bounded solution, then all solutions of* (2.2) *and* (2.5)*, with initial bounded conditions, are bounded.*

Proof. First, we show that all solutions of (2.5) are bounded. Let $x(t)$ a bounded solution of (2.2) and $\hat{x}(t)$ a solution of (2.5) with initial bounded condition. From Lemma 2.1, $x(t)$ and $\hat{x}(t)$ are defined on $\mathbb{R}$ and, defining $y(t) = (d_1^{-1}|x_1(t) - \hat{x}_1(t)|, \ldots, d_n^{-1}|x_n(t) - \hat{x}_n(t)|)$, we have, for $t > 0$ and $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
y_i'(t) &= \operatorname{sign}(x_i(t) - \hat{x}_i(t)) d_i^{-1}(x_i'(t) - \hat{x}_i'(t)) \\
&= \operatorname{sign}(x_i(t) - \hat{x}_i(t)) d_i^{-1}\Big(- \big(b_i(t, x_i(t)) - \hat{b}_i(t, x_i(t))\big) - \big(\hat{b}_i(t, x_i(t)) - \hat{b}_i(t, \hat{x}_i(t))\big)\Big) \\
&\quad + \operatorname{sign}(x_i(t) - \hat{x}_i(t)) \sum_{j=1}^{n} \sum_{p=1}^{P} d_i^{-1}\Big[(a_{ijp}(t) - \hat{a}_{ijp}(t)) h_{ijp}(x_j(t - \tau_{ijp}(t))) \\
&\quad + \hat{a}_{ijp}(t)\Big(h_{ijp}(x_j(t - \tau_{ijp}(t))) - h_{ijp}(\hat{x}_j(t - \tau_{ijp}(t)))\Big) \\
&\quad + (c_{ijp}(t) - \hat{c}_{ijp}(t)) f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s)\right) \\
&\quad + \hat{c}_{ijp}(t)\left(f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s)\right) - f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t+s)) d\eta_{ijp}(s)\right)\right)\Big] \\
&\quad + \operatorname{sign}(x_i(t) - \hat{x}_i(t))(I_i(t) - \hat{I}_i(t)) d_i^{-1}
\end{aligned}$$

and, from the hypotheses, we obtain

$$\begin{aligned}
y_i'(t) &\leq -\hat{\beta}_i(t) y_i(t) + \sum_{j=1}^{n} \sum_{p=1}^{P} \frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t)| \gamma_{ijp} y_j(t - \tau_{ijp}(t)) + |\hat{c}_{ijp}(t)| \mu_{ijp} \sigma_{ijp} \|y_t\|\Big] \\
&\quad + d_i^{-1}|b_i(t, x_i(t)) - \hat{b}_i(t, x_i(t))| + \sum_{j=1}^{n} \sum_{p=1}^{P} d_i^{-1}\Big[|a_{ijp}(t) - \hat{a}_{ijp}(t)|\, |h_{ijp}(x_j(t - \tau_{ijp}(t)))| \\
&\quad + |c_{ijp}(t) - \hat{c}_{ijp}(t)| \left| f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s)\right)\right|\Big] + |I_i(t) - \hat{I}_i(t)| d_i^{-1} \\
&= -\hat{\beta}_i(t) y_i(t) + \sum_{j=1}^{n} \sum_{p=1}^{P} \frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t)| \gamma_{ijp} y_j(t - \tau_{ijp}(t)) + |\hat{c}_{ijp}(t)| \mu_{ijp} \sigma_{ijp} \|y_t\|\Big] + \varepsilon_{i2}(t) \\
&\leq -\hat{\beta}_i(t) y_i(t) + \sum_{j=1}^{n} \sum_{p=1}^{P} \frac{d_j}{d_i}\Big[|\hat{a}_{ijp}(t)| \gamma_{ijp} + |\hat{c}_{ijp}(t)| \mu_{ijp} \sigma_{ijp}\Big] \|y_t\| + \varepsilon_{i2}(t), \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_{i2}(t) &= d_i^{-1}|b_i(t, x_i(t)) - \hat{b}_i(t, x_i(t))| + \sum_{j=1}^{n} \sum_{p=1}^{P} d_i^{-1}\Big[|a_{ijp}(t) - \hat{a}_{ijp}(t)|\, |h_{ijp}(x_j(t - \tau_{ijp}(t)))| \\
&\quad + |c_{ijp}(t) - \hat{c}_{ijp}(t)| \left| f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s)\right)\right|\Big] + |I_i(t) - \hat{I}_i(t)| d_i^{-1}.
\end{aligned}$$

The solution $x(t)$ is bounded continuous, $h_{ijp}, f_{ijp}, g_{ijp}$ are continuous, $\displaystyle\int_{-\infty}^{0} d\eta_{ijp}(s) = 1$, and (2.6) holds, then $\lim\limits_{t \to +\infty} \varepsilon_{i2}(t) = 0$.

Suppose that $y(t)$ is not a bounded function. Consequently there exist $i \in \{1, \dots, n\}$ and a positive real sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow +\infty$, $0 < y_i(t_k) \nearrow +\infty$,

$$y_i(t_k) = \|y_{t_k}\| \geq \|y_t\|, \qquad \text{and} \qquad y_i'(t_k) \geq 0, \qquad \forall k \in \mathbb{N},\ \forall t \leq t_k. \tag{4.2}$$

From (4.1) and (4.2), we have

$$y_i'(t_k) \leq \left(-\hat{\beta}_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P} \frac{d_j}{d_i} \left(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp} \right) + \frac{\varepsilon_{i2}(t_k)}{y_i(t_k)} \right) y_i(t_k), \qquad \forall k \in \mathbb{N},$$

with $\lim\limits_{k} \dfrac{\varepsilon_{i2}(t_k)}{y_i(t_k)} = 0$. Hypotheses **(A4)** and (2.6) imply (2.7) and $y_i'(t_k) < 0$, for large k, which is a contradiction. Thus $y(t)$ is bounded and we conclude that $\hat{x}(t)$ is also bounded.

As we remark above, the system (2.2) can be regarded as an asymptotic system of itself, thus all solutions of (2.2) are also bounded. □

By Example 2.1, we know that there exist models satisfying **(A1)-(A4)** for which all solutions are unbounded. The following simple example shows that there exist models, with unbounded coefficient functions, satisfying **(A1)-(A4)** such that all solutions are bounded.

Example 4.1. It is easy to verify that the scalar equation

$$x'(t) = -\frac{t+4}{2+\cos t}x(t) + x\left(t - \frac{3\pi}{2}\right) + t + 2, \qquad t \geq 0, \tag{4.3}$$

has a bounded solution $x(t) = \cos t + 2$. As the hypotheses **(A1)-(A4)** hold, then, from Theorem 4.1, all solutions of (4.3) are bounded. We remark that, for each $u \neq 0$, the function $t \mapsto b(t,u) = \frac{t+4}{2+\cos t}u$ is unbounded and the input function $I(t) = t + 2$ is also unbounded.

The following theorem is proved using the same arguments in the proof of Theorem 3.2 and details are omitted. In fact, Lemma 2.1 and Theorem 4.1 imply that all solutions of (2.2) and (2.5), with bounded initial conditions, are defined and bounded on $\mathbb{R}$. Moreover, as we assume that $\tau_{ijp}(t) = \hat{\tau}_{ijp}(t)$, in this situation we do not need to show that the solutions of (2.2) are uniformly continuous.

Theorem 4.2. *Assume (A1)-(A4) hold and $\tau_{ijp}(t) = \hat{\tau}_{ijp}(t)$ for all $t \geq 0$.*
If (2.2) *has a bounded solution, then*

$$\lim_{t\to+\infty} |x_i(t) - \hat{x}_i(t)| = 0, \qquad i = 1, \dots, n,$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ and $\hat{x}(t) = (\hat{x}_1(t), \dots, \hat{x}_n(t))$ solutions of systems (2.2) *and* (2.5) *respectively, with bounded initial conditions.*

As the system (2.2) can be regarded as an asymptotic system of itself, the following corollary holds.

Corollary 4.3. *Assume (A1)-(A4) hold.*
If $x^*(t)$ is a bounded solution of (2.2), *then*

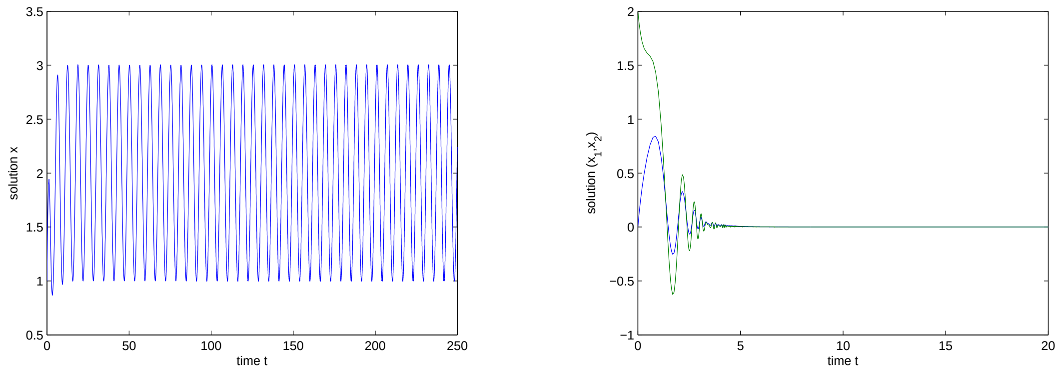
$$\lim_{t\to+\infty} |x_i(t) - x_i^*(t)| = 0, \qquad i = 1, \dots, n,$$

for any solution, $x(t) = (x_1(t), \dots, x_n(t))$, of (2.2) *with bounded initial condition.*

5 Numerical simulations

Example 5.1. First, we consider the one-dimensional equation (4.3) again. As we remark above, the function $x^*(t) = \cos t + 2$ is a periodic solution and now, from Corollary 4.3, we can conclude that all solutions $x(t)$ converge to $x^*(t)$ as $t \to +\infty$. We note that the coefficient functions are not periodic functions.

We used the Matlab software, [16], to plot a numerical simulation of the behavior of the solution $x(t)$ of model (4.3) with initial condition $\varphi(s) = 1$, $s \leq 0$ (see Figure 1(a)).



(a) Solution $x(t)$ of (4.3) with initial condition $\varphi(s) = 1$, $s \leq 0$.

(b) Solution $(x_1(t), x_2(t))$ of system (5.1) with initial condition $\varphi(s) = (\sin s, 2)$, $s \leq 0$.

Figure 1: Behavior of the solutions of systems (4.3) and (5.1).

Example 5.2. Consider the following system

$$\begin{cases} x_1'(t) = -(2+e^{-t})x_1(t) + (\cos e^t)x_1(t-1) + (\sin e^t)x_2(t-2) + e^{-t} \\ \\ x_2'(t) = -3x_2(t) + (\cos e^t)x_1(t-1) + 2(\sin e^t)x_2(t-2) + e^{-t} \end{cases}. \tag{5.1}$$

It is straightforward to check that the system

$$\begin{cases} x_1'(t) = -2x_1(t) + (\cos e^t)x_1(t-1) + (\sin e^t)x_2(t-2) \\ \\ x_2'(t) = -3x_2(t) + (\cos e^t)x_1(t-1) + 2(\sin e^t)x_2(t-2) \end{cases} \tag{5.2}$$

is an asymptotic system of (5.1). It is easy to see that the hypotheses **(A1)**, **(A2)**, and **(A4)** hold with $d = (1,1)$ and from Theorem 3.2 we conclude that any two solutions, $x(t)$ solution of (5.1) and $\hat{x}(t)$ solution of (5.2), satisfy $\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0$. Consequently, the equilibrium point, $(x_1(t), x_2(t)) = (0,0)$, of (5.2) attracts every solution of system (5.1) (look at the numerical simulation in Figure 1(b)), but system (5.1) has not equilibrium points.

Remark 5.1. We note that hypothesis **(H)** does not hold for system (5.2). In fact, if **(H)** holds then there exists $d = (d_1, d_2) > 0$ such that

$$\limsup_{t\to+\infty} \sum_{j=1}^{2} \frac{d_j |a_{j11}(t)|}{d_1 c_j(t)} = \limsup_{t\to+\infty} \left(\frac{d_1 |\cos e^t|}{2d_1} + \frac{d_2 |\cos e^t|}{3d_1} \right) = \frac{1}{2} + \frac{d_2}{3d_1} < 1,$$

and

$$\limsup_{t\to+\infty} \sum_{j=1}^{2} \frac{d_j |a_{j21}(t)|}{d_2 c_j(t)} = \limsup_{t\to+\infty} \left(\frac{d_1 |\sin e^t|}{2d_2} + \frac{2d_2 |\sin e^t|}{3d_2} \right) = \frac{d_1}{2d_2} + \frac{2}{3} < 1,$$

since $n = 2$, $\gamma_i = 1$, and $a_{ij2}(t) = 0$. Thus we have $\frac{3}{2}d_1 < d_2 < \frac{3}{2}d_1$ which is a contradiction. Consequently, [21, Theorem 3.2] cannot be applied to get the same conclusion. Moreover, we remark that system (5.1) has not a periodic asymptotic system, thus the main results in [19, 20, 23] also cannot be applied. This example illustrates that our Theorem 3.2 presents a new stability criterion.

Example 5.3. The following neural network model

$$\begin{cases} x_1'(t) = -\left(3 + \dfrac{1}{t+1}\right) x_1(t) + (\cos t)x_1(t) + (\sin(\pi t) + e^{-t}) \tanh(x_2(t-1)) + \dfrac{1}{t+2} \\ \\ x_2'(t) = -4x_2(t) + \cos(2t)x_2(t) + \dfrac{1}{t+1} \tanh(x_1(t-1)) + \cos(\sqrt{5}t + e^{-t}) \tanh(x_2(t-1)) + \sin(\pi t) \end{cases} \tag{5.3}$$

is not an almost periodic system but, from Corollary 3.6, we conclude that the almost periodic solution of its asymptotic system

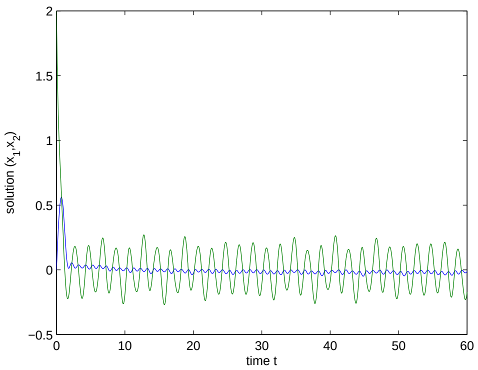
$$\begin{cases} x_1'(t) = -3x_1(t) + (\cos t)x_1(t) + \sin(\pi t) \tanh(x_2(t-1)) \\ \\ x_2'(t) = -4x_2(t) + \cos(2t)x_2(t) + \cos\left(\sqrt{5}t\right) \tanh(x_2(t-1)) + \sin(\pi t) \end{cases} \tag{5.4}$$

attracts globally all solutions of (5.3) (see the numerical simulation in Figure 2(a)).

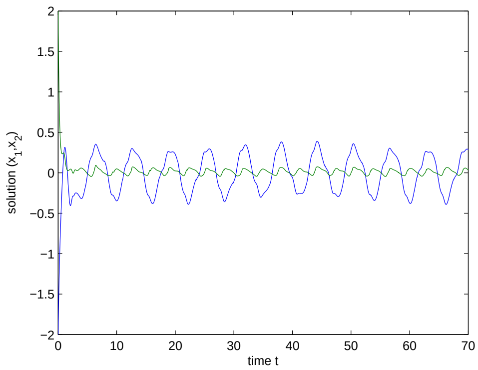
Example 5.4. At last, we consider the neural network model

$$\begin{cases} x_1'(t) = -3x_1(t) + \left(\cos(\pi t) + e^{-t^2 \cos^2 t}\right) \tanh(x_1(t - 2|\sin t|)) + \dfrac{\sin t}{2} x_2(t-1) + \cos t \\ \\ x_2'(t) = -4x_2(t) + \left(\cos t + e^{-t^2 \sin^2 t}\right) \tanh(x_1(t - 2|\sin t|)) + \sin\left(\sqrt{5}t\right) x_2(t-1) + e^{-|t|} \end{cases} \tag{5.5}$$

with pseudo almost periodic coefficients. From Corollary 3.8, we conclude that there exists a pseudo almost periodic solution which attracts all solutions (see the numerical simulation in Figure 2(b)). We remark that condition 1. in Remark 3.3 does not hold, thus the stability result in [24] can not be used, in model (5.5), to get the same conclusion.



(a) Solution $(x_1(t), x_2(t))$ of system (5.3) with initial condition $\varphi(s) = (\sin s, 2)$, $s \le 0$.



(b) Solution $(x_1(t), x_2(t))$ of system (5.5) with initial condition $\varphi(s) = (-2, 1 + \cos s)$, $s \le 0$.

Figure 2: Behavior of the solutions of systems (5.3) and (5.5).

6 Conclusion

We have presented criteria for the global convergence of solutions of non-autonomous Hopfield neural network models, theorems 3.2 and 4.2. These theorems are quite general because in neural network models (2.2) and (2.5) it is possible to have time-varying delays, unbounded distributed delays, and it is not necessary to assume that the coefficients and delays are constants, periodic, almost periodic or pseudo almost periodic functions. Moreover, in Theorem 4.2 it is not necessary to assume that the coefficient functions are bounded.

Acknowledgments.
The paper was supported by the Research Centre of Mathematics of the University of Minho with the Portuguese Funds from the "Fundação para a Ciência e a Tecnologia", through the Project PEstOE/MAT/UI0013/2014. The author thanks the referee for valuable comments.

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