Statistical Inference in the Duffing System with the Unscented Kalman Filter

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Abstract: We investigate the accuracy of inference in a chaotic dynamical system (Duffing oscillator) with the Unscented Kalman Filter, and quantify the dependence on the sample size, the signal to noise ratio and the initialization.

Keywords: Bayesian filtering, Unscented Kalman Filter, Chaotic dynamical system, Parameter estimation

1 Introduction

We focus on the analysis of the deterministic Duffing process, defined as

$$dx_{1t}/dt = x_{2t}, \qquad dx_{2t}/dt = -(cx_{2t} + \alpha x_{1t} + \beta x_{1t}^3), \tag{1}$$

where x_{1t} and x_{2t} are the position and the velocity, respectively, of the oscillation at time t, $g(x) = \alpha x_{1t} + \beta x_{1t}^3$ is a restoring force, α is the natural frequency of the vibration, β the mode of the restoring force (hard or soft spring), and c is the damping term. The Duffing system (1) describes a periodically forced oscillator with a nonlinear elasticity, and has been widely used in physics, economics and engineering (Kovacic and Brennan, 2011). A characteristic feature is its chaotic behaviour, which makes statistical inference challenging. In the present paper we present an approach based on the Unscented Kalman Filter (UKF).

2 Methodology

The UKF algorithm is a non-linear generalization of Kalman filter which relies on the unscented transform (Julier and Uhlmann (2004)) in order to

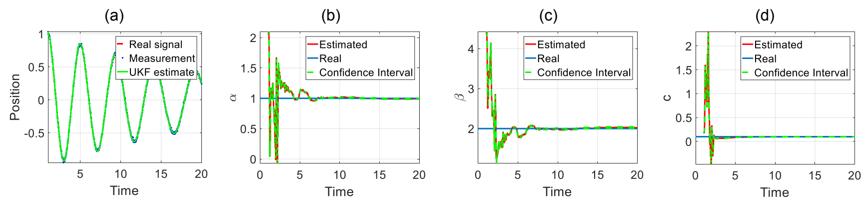

FIGURE 1. UKF estimates for the deterministic Duffing system with SNR=31 and $n = 1000$. (a) Signal estimate. (b) Estimate of parameter α. (c) Estimate of parameter β. (d) Estimate of parameter c.

construct a Gaussian approximation to the filtering distribution. The UKF performs a Bayesian estimation of a state-space model:

$$\boldsymbol{x}_t = f(\boldsymbol{x}_{t-1}) + \boldsymbol{\varepsilon}, \qquad \boldsymbol{y}_t = h(\boldsymbol{x}_t) + \boldsymbol{\eta} \tag{2}$$

where $\boldsymbol{x}_t \in \mathbb{R}^M$ is the (hidden) state at time t, $\boldsymbol{y}_t \in \mathbb{R}^D$ is the measurement, $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$ is the Gaussian system noise and $\boldsymbol{\eta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$ is the Gaussian observation noise. The non-linear differentiable functions f and h are, respectively, the transition and observation models. UKF passes a deterministically chosen set of points (sigma points) through f to obtain the predictive distribution $p(\boldsymbol{x}_t|\boldsymbol{y}_{1:t-1})$. Then, the sigma points are transformed using model h to compute the filtering distribution $p(\boldsymbol{x}_t|\boldsymbol{y}_{1:t})$. As suggested in Sitz et al. (2002), we merge the signal with the parameter vector $\boldsymbol{\lambda} = [\alpha\ \beta\ c]^{\mathrm{T}}$ in a joint state vector $\boldsymbol{j}_t = [\boldsymbol{x}_t,\ \boldsymbol{\lambda}_t]^{\mathrm{T}} = [(f(\boldsymbol{x}_{t-1}, \boldsymbol{\lambda}_{t-1}) + \boldsymbol{\varepsilon}),\ \boldsymbol{\lambda}_{t-1}]^{\mathrm{T}}$, and $\boldsymbol{y}_t = h(\boldsymbol{j}_t) + \boldsymbol{\eta}$. In our case, the function f of model (2) is given by the numerical solution of system (1), h is the identity function, and $\boldsymbol{\varepsilon} = \mathbf{0}$.

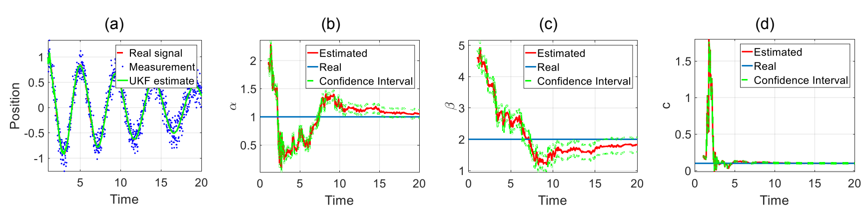

FIGURE 2. UKF estimates for the deterministic Duffing system with SNR=10 and $n = 1000$. (a) Signal estimate. (b) Estimate of parameter α. (c) Estimate of parameter β. (d) Estimate of parameter c.

3 Simulations

We simulate system (1) through the `ode23` MATLAB function with a step-size of integration $\delta t = 0.01$ and starting values for the numerical integration $[1, 0]$. Measurements are obtained from the first component, x_{1t}, by

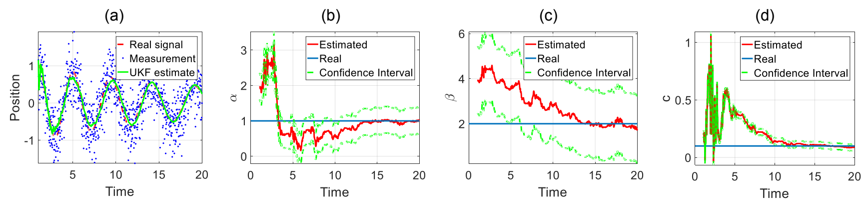

FIGURE 3. UKF estimates for the deterministic Duffing system with SNR=1 and $n = 1000$. (a) Signal estimate. (b) Estimate of parameter α. (c) Estimate of parameter β. (d) Estimate of parameter c.

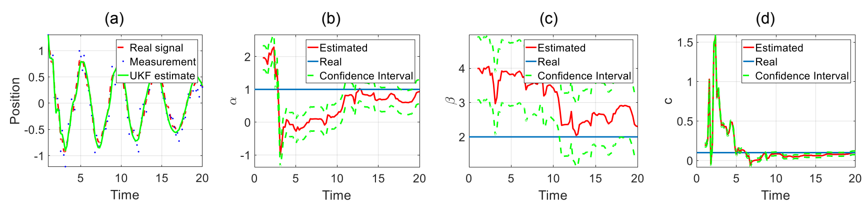

FIGURE 4. UKF estimates for the deterministic Duffing system with SNR=10 and $n = 100$. (a) Signal estimate. (b) Estimate of parameter α. (c) Estimate of parameter β. (d) Estimate of parameter c.

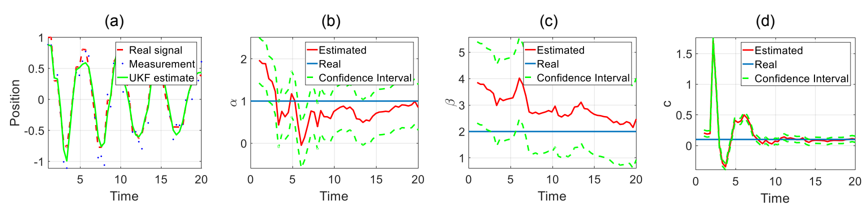

FIGURE 5. UKF estimates for the deterministic Duffing system with SNR=10 and $n = 50$. (a) Signal estimate. (b) Estimate of parameter α. (c) Estimate of parameter β. (d) Estimate of parameter c.

adding observational noise $\eta_t \sim N(0, \sigma_\eta^2)$ with known variance. The time interval is $t = 1, \ldots, 20$, and the presented results are averaged over 10 simulations. The UKF algorithm is performed with the `EKF/UKF` toolbox of Hartikainen et al. (2011). To investigate the behaviour of the Duffing process and the UKF performance, we have simulated several scenarios, varying the Signal to Noise Ratio, SNR $\in \{30, 10, 1\}$, and the sample size, $n \in \{1000, 100, 50\}$ (Figures 1–5). To evaluate the impact of initialization, we considered different offsets as starting values for the parameters. The offsets are sampled randomly from a Gaussian distribution in which the mean is defined by a percentage deviation from the true parameter values and the variance is 10% of the mean (Table 1).

TABLE 1. Impact of the initialization for the deterministic Duffing system for different offsets (as percentage of the true parameter values) in term of Euclidean norm prior inference and post inference.

	α		β		c	
	Prior	Post	Prior	Post	Prior	Post
100%	1.00	0.05	2.04	0.24	0.10	0.01
150%	1.52	0.12	3.02	0.50	0.15	0.01
200%	2.03	0.23	3.90	0.94	0.21	0.02
250%	2.48	0.65	4.61	2.12	0.25	0.04

4 Results and Discussion

Figures 1–5 show that the UKF successfully learns the parameters from the noisy data, and that at the end of the filtering phase the true parameters always lie within the predicted standard error around the estimate. This suggests that Bayesian filtering offers a successful paradigm for inference in chaotic dynamical systems. The prediction uncertainty depends on the sample size n, and the level of noise, quantified by the SNR. As one would expect, the uncertainty increases with decreasing n and decreasing SNR, i.e. as information in the data is lost, and our study allows a quantification of this trend. The increase in uncertainty particularly affects the parameter β, which is associated with the nonlinear term and the source of the chaotic behaviour. Table 1 shows the effect of the initialization, measured in terms of the Eucliedean distance in parameter space. This distance is consistently reduced in the filtering process, and the posterior distance (after filtering) is always smaller than the prior distance (before filtering). However, the posterior distance increases with the prior distance, suggesting that a good initialisation will improve the inference results.

References

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