Weak compactness of operators acting on o–O type spaces

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Abstract

We consider operators $T : M_0 \to Z$ and $T : M \to Z$, where Z is a Banach space and (M_0, M) is a pair of Banach spaces belonging to a general construction in which M is defined by a "big-O" condition and M_0 is given by the corresponding "little-o" condition. Prototype examples of such spaces M are given by ℓ^∞, weighted spaces of functions or their derivatives, bounded mean oscillation, Lipschitz-Hölder spaces, and many others. The main result characterizes the weakly compact operators T in terms of a certain norm naturally attached to M, weaker than the M-norm, and shows that weakly compact operators $T : M_0 \to Z$ are already quite close to being completely continuous. Further, we develop a method to extract c_0-subsequences from sequences in M_0. Applications are given to the characterizations of the weakly compact composition and Volterra-type integral operators on weighted spaces of analytic functions, BMOA, VMOA, and the Bloch space.

1. Introduction

Let Z be a Banach space. The main result of this paper characterizes the weak compactness of operators $T : M_0 \to Z$ and $T : M \to Z$, where (M_0, M) is a pair of Banach spaces in which M is defined by a "big-O" condition and M_0 by the corresponding "little-o" condition. See (2) and (3) for the precise definition. The class of spaces (M_0, M) is large and examples include c_0 and ℓ^∞, weighted and the corresponding vanishing weighted spaces of continuous, analytic or harmonic functions, Möbius invariant spaces of analytic functions, Lipschitz-Hölder spaces, bounded and vanishing mean oscillation (BMO and VMO), and several others. The pair (M_0, M) was first introduced in [**14**], and the quoted examples are given there.

This paper is inspired by recent works on the compactness properties of composition and integral operators acting on specific examples of spaces M_0 and M [**3**], [**7**], [**8**], [**10**]. It often turns out that weak compactness and compactness are equivalent for these classes of operators, a phenomenon which can be readily understood given the main results of this article.

For the statements of the theorems, note that M is associated with a reflexive Banach space X in which M is continuously contained (see Section 2). For instance, ℓ^∞ is continuously contained in a weighted ℓ^2-space.

THEOREM 3.2. *A bounded operator $T : M_0 \to Z$ is weakly compact if and only if there for each $\varepsilon > 0$ exists an $N > 0$ such that*

$$\|Tx\|_Z \leq N\|x\|_X + \varepsilon \|x\|_M, \quad x \in M_0. \tag{1}$$

A similar description of the weakly compact operators on $C(K)$-spaces was given by Niculescu, and a generalization to operators acting on general C^*-algebras is due to Jarchow [**6**]. More recently, characterizations in the same spirit have been given for operators acting on H^∞ ([**9**]) and certain subspaces of Orlicz spaces [**10**].

In [14] it was proven that $M_0^{**} \simeq M$ in a canonical way. Therefore, Theorem 3.2 also applies to operators $T : M \to Z$ such that $(T|_{M_0})^{**} = T$ – i.e. operators T which are *weak*-weak* continuous, a continuity property which is simple to verify in many concrete examples. See Corollary 3.3.

To compare the weak compactness characterization with compactness criteria, note that $T : M_0 \to Z$ is completely continuous if and only if for every bounded sequence $(x_n) \subset M_0$ such that x_n converges weakly to zero, it holds that $\lim_n \|Tx_n\|_Z = 0$. To demand instead only weak compactness, one simply replaces the weak convergence of x_n with the stronger property (see [14]) that x_n converges to zero in X-norm. The two conditions on the sequence (x_n) are in many concrete examples closely related; herein lies the explanation of why weak compactness and compactness often are equivalent for operators on M_0 and M. See the examples in Section 4.

The motivation for the proof of Theorem 3.2 comes from [15], where it was shown that M_0 is an M-ideal in M. In particular, weakly compact operators on M_0 can be characterized in terms of c_0-subspaces of M_0. The proof hence relies on a procedure to create c_0-subspaces, a construction which we summarize as a separate theorem.

THEOREM 3.1.   *Suppose that $x_n \in M_0$, $n = 1, 2, 3, \ldots$, is a sequence such that $\|x_n\|_M = 1$ and $\lim_{n\to\infty} \|x_n\|_X = 0$. Then $(x_n)_n$ has a subsequence which, as a basic sequence in M_0, is equivalent to the canonical basis of c_0.*

This result is classical for $M_0 = c_0$, and has also been proven for the case when $M_0 = \text{VMO}$ [12], the latter fact which has been used in [7] and [8] to characterize the weak compactness of Volterra-type integral operators and composition operators on the analytic BMO-space.

The paper is organized as follows. In Section 2 the definitions of M_0 and M are given, as well as technical preliminaries; in Section 3 the main results are proven; Section 4 gives applications of Theorem 3.2 and its corollary to composition and integral operators on weighted spaces of analytic functions, Bloch spaces, and analytic BMO-spaces.

2.   *Definitions and preliminaries*

The spaces M and M_0 are defined by

$$M(X, \mathcal{L}) = \left\{ x \in X \ : \ \sup_{L \in \mathcal{L}} \|Lx\|_Y < \infty \right\} \tag{2}$$

and

$$M_0(X, \mathcal{L}) = \left\{ x \in M(X, \mathcal{L}) \ : \ \varlimsup_{\mathcal{L} \ni L \to \infty} \|Lx\|_Y = 0 \right\}. \tag{3}$$

Here X and Y are Banach spaces, where X is assumed to be separable and reflexive. $\mathcal{L}$ is a collection of continuous linear operators $L \colon X \to Y$ that is made into a topological space $(\mathcal{L}, \tau)$ by a σ-compact locally compact Hausdorff topology τ. The topology should respect the strong operator topology in the sense that for every $x \in X$, the map $T_x \colon \mathcal{L} \to Y$ given by $T_x L = Lx$ is continuous. The limit $L \to \infty$ in the definition of M_0 should be understood in the sense of one-point compactification of $(\mathcal{L}, \tau)$ (i.e. L should escape all compact sets).

We may assume that $M(X, \mathcal{L})$ is dense in X [14], and we suppose that

$$\|x\|_{M(X, \mathcal{L})} = \sup_{L \in \mathcal{L}} \|Lx\|_Y$$

defines a norm on $M(X, \mathcal{L})$ which is stronger than the X-norm. As in the concrete examples mentioned in the introduction, we want to consider the situation where the bidual M_0^{**} can

be canonically identified with M. For this to be true it is necessary to impose the following approximation property.

ASSUMPTION A. For every $x \in M(X, \mathcal{L})$ there is a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $M_0(X, \mathcal{L})$ such that x_n converges weakly to x in X.

Henceforth we always assume that A holds. There is also the stronger hypothesis:

ASSUMPTION B. For every $x \in M(X, \mathcal{L})$ there is a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $M_0(X, \mathcal{L})$ such that x_n converges weakly to x in X and $\sup_n \|x_n\|_{M(X,\mathcal{L})} \leq \|x\|_{M(X,\mathcal{L})}$.

The next theorem, stating that indeed $M_0^{**} = M$ holds, was proven in [14]. For its statement, note that $M_0(X, \mathcal{L})$ can be viewed as a closed subspace of both M and M_0^{**}.

THEOREM 2.1 [14]. *The dual space X^* is continuously contained and dense in $M_0(X, \mathcal{L})^*$. Denoting by*

$$I \colon M_0(X, \mathcal{L})^{**} \to X$$

the adjoint of the inclusion map $J \colon X^* \to M_0(X, \mathcal{L})^*$, the operator I is a continuous isomorphism of $M_0(X, \mathcal{L})^{**}$ onto $M(X, \mathcal{L})$ which acts as the identity on $M_0(X, \mathcal{L})$. Furthermore, I is an isometry if Assumption B holds.

In the isometric case the author proved in [15] that M_0 is an M-ideal in M. In particular, M_0 has Pełczyński's property (V), which as a consequence gives the following characterization of weakly compact operators on M_0 (see [5]).

PROPOSITION 2.2 [15]. *Suppose that Assumption B holds. If Z is a Banach space and $T : M_0(X, \mathcal{L}) \to Z$ is a bounded operator, then T is weakly compact if and only if there does not exist a subspace $F \subset M_0(X, \mathcal{L})$ isomorphic to c_0 such that $T|_F$ is an isomorphism.*

The proof of Theorem 3.2 is inspired by this proposition, but technically only relies on its forward direction which follows easily for any Banach space from the fact that c_0 has the Dunford-Pettis property.

A sequence $(z_n)_{n=1}^{\infty}$ in a Banach space Z is called basic if it is a (Schauder) basis for its span $[z_n] = \mathrm{span}\{z_n\}$. Two basic sequences (z_n) and (w_n) in Banach spaces Z and W, respectively, are said to be equivalent if there is an isomorphism between $[z_n]$ and $[w_n]$ which maps z_n onto w_n, for all n. In this situation, if $W = c_0$ and (w_n) is the unit-vector basis of c_0, we say that (z_n) is equivalent to the canonical basis of c_0. For rudimentary information about bases, we refer to the classical paper of Bessaga and Pełczyński [1], the techniques of which will be utilized to prove the main results of this paper.

3. Results and Proofs

In the proof of Theorem 3.1 we make use of the embedding operator $V : M_0(X, \mathcal{L}) \to C_0(\mathcal{L}, Y)$ which isometrically embeds M_0 into the space of continuous Y-valued functions on

$\mathcal{L}$ vanishing at infinity. Explicitly

$$V x(L) = Lx, \quad x \in M_0, \ L \in \mathcal{L}.$$

THEOREM 3.1. *Suppose that $x_n \in M_0(X, \mathcal{L})$, $n = 1, 2, 3, \ldots$, is a sequence such that $\|x_n\|_M = 1$ and $\lim_{n \to \infty} \|x_n\|_X = 0$. Then $(x_n)_n$ has a subsequence which, as a basic sequence in $M_0(X, \mathcal{L})$, is equivalent to the canonical basis of c_0.*

Proof. We will construct a subsequence $(z_n)_n$ of $(x_n)_n$ inductively. We will also construct two auxiliary sequences; a strictly increasing sequence of positive integers $(\beta_n)_n$, and a sequence $(f_n)_n$ in $B(\mathcal{L}, Y)$, the space of bounded Baire measurable Y-valued functions equipped with the supremum norm. To begin, let $z_1 = x_1$, $\beta_1 = 1$ and $f_1 = V z_1$. For the construction, fix a strictly increasing sequence $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots$ of compact Baire subsets of $(\mathcal{L}, \tau)$ such that $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. We denote by $\mathcal{K}_n^c$ the complement of $\mathcal{K}_n$ in $\mathcal{L}$.

Suppose now that $z_1, \ldots, z_{n-1}$, $\beta_1, \ldots, \beta_{n-1}$, and $f_1, \ldots, f_{n-1}$ have been chosen. Since each z_j belongs to $M_0(X, \mathcal{L})$ we can pick $\beta_n > \beta_{n-1}$ such that

$$\|L z_j\|_Y \leq 1/2^j, \quad L \in \mathcal{K}_{\beta_n}^c, \ j = 1, \ldots, n-1. \tag{4}$$

Since the operators $L \in \mathcal{K}_{\beta_n}$ are uniformly bounded by the Banach-Steinhaus theorem, it follows from $\lim_k \|x_k\|_X = 0$ that we may choose z_n to be an element from $(x_k)_k$ such that

$$\{L \in \mathcal{L} : \|L z_n\|_Y > 1/2^n\} \subset \mathcal{K}_{\beta_n}^c. \tag{5}$$

Denoting the set on the left hand side of (5) by $\mathcal{A}_n$, let

$$f_n = \mathbb{1}_{\mathcal{A}_n} V z_n,$$

where $\mathbb{1}_{\mathcal{A}_n}$ is the characteristic function of $\mathcal{A}_n$.

With the inductive process complete, we now claim that $(z_n)_{n=2}^{\infty} \subset M_0(X, \mathcal{L})$ has a further subsequence equivalent to the canonical basis of c_0. To see this, let

$$\mathcal{B}_n = \mathcal{A}_n \setminus \cup_{j > n} \mathcal{A}_j, \quad n \geq 2.$$

If $L \in \mathcal{B}_m$ for some $m \geq 2$, then $f_n(L) = 0$ for $n > m$, while by construction

$$\|f_n(L)\|_Y \leq 1/2^n, \text{ for } n < m.$$

Since $\|x_k\|_M = 1$ for all k, we of course have that $\|f_m(L)\|_Y \leq 1$. Hence, for $L \in \mathcal{B}_m$ we have

$$\sum_{n=2}^{\infty} \|f_n(L)\|_Y \leq 1 + \sum_{n=2}^{m-1} \frac{1}{2^n} < 3/2.$$

On the other hand, if $L \in (\cup_k \mathcal{B}_k)^c$, then $f_n(L) = 0$ for every $n \geq 2$, since $\cup_k \mathcal{B}_k = \cup_k \mathcal{A}_k$. For the latter equality, note that no $L \in \mathcal{L}$ can belong to infinitely many sets $\mathcal{A}_k$, since $\mathcal{A}_k \subset \mathcal{K}_{\beta_k}^c$.

We have hence shown that

$$\sum_{n=2}^{\infty} \|f_n(L)\|_Y < 3/2, \quad \forall L \in \mathcal{L}.$$

Therefore, for any bounded sequence of numbers $(t_n)_n$, we find that

$$\sup_k \left\| \sum_{n=2}^{k} t_n f_n(L) \right\|_Y < \frac{3}{2} \sup_n |t_n|, \quad \forall L \in \mathcal{L}.$$

This latter inequality implies that the series $\sum_{n=2}^{\infty} f_n$ is weakly unconditionally Cauchy in $B(\mathcal{L}, Y)$ (see [4], p. 44). Note also that each f_n was constructed as to have supremum norm 1, $\|f_n\|_\infty = 1$. By the Bessaga-Pełczyński selection principle (C. 1. and Lemma 3 of [1]) there is

hence a basic subsequence $(f_{n_k})_k$ equivalent to the canonical basis of c_0. But then there is a positive integer K such that $(V z_{n_k})_{k \geq K}$ is also basic and equivalent to the canonical basis of c_0, since

$$\sum_{k=1}^{\infty} \|V z_{n_k} - f_{n_k}\|_{\infty} \leq \sum_{k=1}^{\infty} \frac{1}{2^{n_k}} < 1.$$

This proves that $(z_{n_k})_{k \geq K}$ is a subsequence of the desired type. $\qquad\square$

Based on Theorem 3.1 we now prove Theorem 3.2.

THEOREM 3.2.   *Let Z be a Banach space. A bounded operator $T : M_0(X, \mathcal{L}) \to Z$ is weakly compact if and only if there for each $\varepsilon > 0$ exists an $N > 0$ such that*

$$\|Tx\|_Z \leq N\|x\|_X + \varepsilon\|x\|_M, \quad x \in M_0(X, \mathcal{L}). \tag{6}$$

Proof.   Since X is reflexive, the inclusion $j : M_0(X) \to X$ is a weakly compact map. Based on this observation, it is a relatively well known fact that having (6) implies the weak compactness of T (see e.g. Proposition 10 in [**10**]).

In the converse direction, suppose that (6) does not hold. Equivalently, there is an $\varepsilon > 0$ and a sequence $(x_n)_n \subset M_0(X, \mathcal{L})$ with $\|x_n\|_M = 1$ such that

$$\|Tx_n\|_Z > n\|x_n\|_X + \varepsilon.$$

The boundedness of T then automatically imposes $\lim_n \|x_n\|_X = 0$. Therefore Theorem 3.1 applies, so that by passing to a subsequence we may assume that $(x_n) \subset M_0$ is equivalent to the canonical basis of c_0. In particular $\sum_n x_n$ is weakly unconditionally Cauchy in M_0, and hence $\sum_n Tx_n$ is weakly unconditionally Cauchy in Z. Since also $\|Tx_n\|_Z \geq \varepsilon$ for all n, there is, by the Bessaga-Pełczyński selection principle, a further subsequence (Tx_{n_k}) which too is equivalent to the canonical basis of c_0. But then both (x_{n_k}) and (Tx_{n_k}) are equivalent to the canonical basis of c_0, and T must act as an isomorphism between the two c_0-subspaces $[x_{n_k}] \subset M_0$ and $[Tx_{n_k}] \subset Z$. Hence T could not be weakly compact, or the Dunford-Pettis property of c_0 would be violated. $\qquad\square$

As a corollary of Theorem 3.2 we obtain the corresponding result for operators $T : M(X, \mathcal{L}) \to Z$ which are *weak*-weak* continuous. The *weak**-topology of $M(X, \mathcal{L})$ referred to is the one induced by the duality in Theorem 2.1. Hence, letting I denote the map of Theorem 2.1 and T_0 the restriction $T_0 = T|_{M_0}$, we have that *weak*-weak* continuity of T means precisely that $T_0^{**} I^{-1} = T$, which by abuse of notation typically is written as $T_0^{**} = T$.

COROLLARY 3.3.   *Let Z be a Banach space and $T : M(X, \mathcal{L}) \to Z$ be a bounded and weak*-weak continuous operator. Then T is weakly compact if and only if there for each $\varepsilon > 0$ exists $N > 0$ such that*

$$\|Tx\|_Z \leq N\|x\|_X + \varepsilon\|x\|_M, \quad x \in M(X, \mathcal{L}). \tag{7}$$

Proof.   Let $T_0 = T|_{M_0}$. The continuity hypothesis can equivalently be stated as $T_0^{**} = T$. Hence it follows from Gantmacher's theorem that T is weakly compact if and only if (6) holds.

It remains to see that (6) implies (7). Suppose that $\varepsilon, N > 0$ are such that (6) holds and let $x \in M(X, \mathcal{L})$. We renorm M_0^{**} by equipping it with the equivalent norm

$$\|I^{-1}x\|_{\text{alt}} = N\|x\|_X + \varepsilon\|x\|_M, \quad I^{-1}x \in M_0^{**}.$$

Invoking the weak-star-metrizability of the unit ball of M_0^{**} (M_0^* is separable by Theorem 2.1), it follows that there exists a sequence of points $x_n \in M_0(X, \mathcal{L})$ converging weak-star to x such that

$$N\|x_n\|_X + \varepsilon\|x_n\|_M \leq N\|x\|_X + \varepsilon\|x\|_M, \quad \forall n.$$

By the continuity of T, Tx_n converges weakly to Tx, and therefore

$$\|Tx\|_Z \leq \varliminf_n \|Tx_n\|_Z \leq \varliminf_n \left(N\|x_n\|_X + \varepsilon\|x_n\|_M\right) \leq N\|x\|_X + \varepsilon\|x\|_M.$$

$\square$

4. Examples

Our first example will be of a general nature, to illustrate the idea that when compactness for a class of operators can be determined through a testing condition, then Corollary 3.3 may sometimes be used to show that weak compactness and compactness are equivalent for the class.

EXAMPLE 1. Suppose that $\{T_\alpha\}_\alpha$ is a family of bounded *weak*-weak* continuous operators $T_\alpha : M(X, \mathcal{L}) \to Z$, Z a Banach space, and that there is a "testing sequence" $(x_n) \subset M(X, \mathcal{L})$ such that:
 – the sequence (x_n) is bounded in $M(X, \mathcal{L})$,
 – $\lim_n \|x_n\|_X = 0$, and
 – for every α, $\lim_n \|T_\alpha x_n\|_Z = 0$ implies that T_α is compact.
Suppose now that T_α is weakly compact. Then Corollary 3.3 immediately implies that $T_\alpha x_n$ must tend to zero in Z, so that T_α is actually compact. Hence, in the above situation, an operator T_α is compact if and only if it is weakly compact if and only if $\lim_n \|T_\alpha x_n\|_Z = 0$.

We now turn to several concrete examples of composition and integral operators acting on spaces of analytic functions. For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, C_φ denotes the composition operator

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \mathbb{D},$$

where f is a holomorphic function on $\mathbb{D}$, $f \in \mathrm{Hol}(\mathbb{D})$. We begin by considering composition operators C_φ on weighted spaces.

EXAMPLE 2. Let $v : \mathbb{D} \to \mathbb{R}_+$ be a strictly positive, radial, continuous weight on $\mathbb{D}$ such that $\lim_{|z| \to 1} v(z) = 0$, and consider the weighted spaces of holomorphic functions

$$H_v^\infty = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)|v(z) < \infty\}$$

and

$$H_v^0 = \{f \in \mathrm{Hol}(\mathbb{D}) : \varlimsup_{|z| \to 1^-} |f(z)|v(z) = 0\}.$$

They can be realized within our framework ([**14**], Example 4.4) with the role of X taken on by the analytic Bergman space on the disc with weight v^2;

$$X = L_a^2(v^2\,dA, \mathbb{D}) = L^2(v^2\,dA, \mathbb{D}) \cap \mathrm{Hol}(\mathbb{D}),$$

where $dA = dx\,dy$ denotes area measure. H_v^∞ and H_v^0 are obtained by letting $Y = \mathbb{C}$ and letting $\mathcal{L}$ consist of the linear functionals L_z, $z \in \mathbb{D}$, defined by $L_z f = |v(z)|f(z)$. The topology we

impose on $\mathcal{L}$ is simply the one inherited from $\mathbb{D}$ through the parametrization $z \mapsto L_z$. The desired approximation property Assumption B can be verified by considering dilations $f(rz)$ of a function $f \in H_v^\infty$, $r < 1$ (see [2]).

Let
$$\tilde{u}(z) = \sup_{\|f\|_{H_v^\infty} \leq 1} |f(z)|$$

and associate with v the weight $\tilde{v} = 1/\tilde{u}$. Then $\tilde{v}$ is a weight of the same type as v and $H_v^\infty = H_{\tilde{v}}^\infty$ isometrically ([2]). v is called *essential* if v is comparable to $\tilde{v}$. Given also a weight w of the same type as v, Bonet et. al. characterized in [2] the compact composition operators $C_\varphi : H_v^\infty \to H_w^\infty$. We utilize Theorem 3.2 to add also weak compactness to their description. For simplicity we suppose that both v and w are essential.

PROPOSITION 4.1.   *The following are equivalent:*
i) $C_\varphi : H_v^\infty \to H_w^\infty$ *is compact,*
ii) $C_\varphi : H_v^0 \to H_w^0$ *is compact,*
iii) $\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)}{v(\varphi(z))} = 0$ *or* $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$,
iv) $\lim_{|z|\to 1^-} \frac{w(z)}{v(\varphi(z))} = 0$,
i') $C_\varphi : H_v^\infty \to H_w^\infty$ *is weakly compact,*
ii') $C_\varphi : H_v^0 \to H_w^0$ *is weakly compact.*
iii') $C_\varphi(H_v^\infty) \subset H_w^0$

Proof.   The equivalences of **i)**-**iv)** are established in [2], as is the (trivial) verification that C_φ is weak*-weak continuous. The equivalences between **i')**-**iii')** follow from Gantmacher's theorem. We hence only need to show that **i')** implies **iv)**, which we do by following the proof of **i)** implies **iv)** and applying the criterion given by Corollary 3.3.

If **iv)** does not hold, there is a sequence (z_n) in $\mathbb{D}$ converging to a point $z_0 \in \partial\mathbb{D}$ such that $w(z_n) \geq cv(\varphi(z_n))$ for all n, for some $c > 0$. Since v is essential, we can choose f_n such that $\|f_n\|_{H_v^\infty} = 1$ and $|f_n(\varphi(z_n))| \sim 1/v(\varphi(z_n))$. It has to hold that $|\varphi(z_n)| \to 1$, or **i')** would be contradicted; we may select non-negative integers $\alpha_n \to \infty$ such that $|\varphi(z_n)|^{\alpha_n} \geq 1/2$ for all n. Consider the functions $g_n = z^{\alpha_n} f_n$. Since $|z|^{\alpha_n}$ tends pointwise to zero in $\mathbb{D}$, and $|f_n|v_n$ is uniformly bounded, it follows by dominated convergence that g_n converges to zero in $X = L_a^2(v^2)$. However,
$$\|C_\varphi g_n\|_{H_w^\infty} \geq |g_n(\varphi(z_n))w(z_n)| \geq c|\varphi(z_n)|^{\alpha_n}|f_n(\varphi(z_n))|v(\varphi(z_n)) \gtrsim \frac{c}{2},$$

contradicting (7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the next examples we introduce the spaces BMOA and VMOA of analytic functions of bounded and vanishing mean oscillation on the unit disc $\mathbb{D}$. To fit them into our framework, for $a \in \mathbb{D}$ and $\lambda \in \mathbb{T}$, let $\phi_{a,\lambda}$ be the disc automorphism
$$\phi_{a,\lambda}(z) = \lambda \frac{a-z}{1-\bar{a}z}.$$

Further, let $X = Y = H^2/\mathbb{C}$, where H^2 is the usual Hardy space on the disc, and let $\mathcal{L}$ consist of all composition operators $L_{\phi_{a,\lambda}} : H^2/\mathbb{C} \to H^2/\mathbb{C}$,
$$L_{\phi_{a,\lambda}}f = f \circ \phi_{a,\lambda} - f(\phi_{a,\lambda}(0)).$$

We equip $\mathcal{L}$ with the topology of $\mathbb{D} \times \mathbb{T}$. Then
$$M(H^2/\mathbb{C}, \mathcal{L}) = \text{BMOA}, \quad M_0(H^2/\mathbb{C}, \mathcal{L}) = \text{VMOA}, \tag{8}$$

see ([**14**], Example 4.2). In a similar fashion we also obtain the Bloch spaces B and B_0,

$$M(L_a^2/\mathbb{C}, \mathcal{L}) = B, \quad M_0(L_a^2/\mathbb{C}, \mathcal{L}) = B_0, \tag{9}$$

by letting $X = Y = L_a^2/\mathbb{C} = (L^2(\mathbb{D}) \cap \mathrm{Hol}(\mathbb{D}))/\mathbb{C}$ be the standard analytic Bergman space on the disc (modulo constants) and letting $\mathcal{L}$ be the same collection of composition operators.

EXAMPLE 3. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Several concrete realizations of Example 1 can be given by considering composition operators C_φ acting on spaces of analytic functions. In [**17**] it is shown that $C_\varphi : Z \to B$, where $Z = B$ or $Z = \mathrm{BMOA}$, is compact if and only if $\lim_{|a| \to 1} \|C_\varphi \phi_{a,\lambda}\|_B = 0$, yielding that $C_\varphi : Z \to B$ is weakly compact if and only if compact. If $\varphi \in B_0$, then C_φ acts boundedly on B_0, and it follows in combination with Gantmacher's theorem that $C_\varphi : B_0 \to B_0$ is weakly compact if and only if compact, a result first shown in [**13**].

In connection with Example 3 we also mention the work of Laitila et. al. [**8**], where it was shown that $C_\varphi : \mathrm{BMOA} \to \mathrm{BMOA}$ is (weakly) compact if and only if $\lim_{|a| \to 1} \|C_\varphi \phi_{a,\lambda}\|_{\mathrm{BMOA}} = 0$. A version of Corollary 3.3 for $M = Z = \mathrm{BMOA}$ and $T = C_\varphi$ actually appears as an ingredient of their paper. This is in line with other recent contributions to the field of compact composition operators, e.g. [**3**] and [**11**], where the use of Banach space techniques has been essential. In fact, something reminiscent of Theorem 3.1 plays an important role in all of the cited articles.

We conclude with an example of integral operators. The symbols of the operators will belong either to the logarithmic BMOA-space $\mathrm{LMOA} = M(H^2/\mathbb{C}, \mathcal{K})$, or its corresponding small space $\mathrm{LMOA}_0 = M_0(H^2/\mathbb{C}, \mathcal{K})$. Here we have chosen $X = Y = H^2/\mathbb{C}$ and the collection $\mathcal{K}$ of operators to consist of the weighted compositions

$$K_{\phi_{a,\lambda}} f = \log \frac{2}{1 - |a|} \left[f \circ \phi_{a,\lambda} - f(\phi_{a,\lambda}(0)) \right].$$

EXAMPLE 4. For an analytic function g in $\mathbb{D}$, we denote by T_g the Volterra-type operator

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) \, d\zeta, \quad z \in \mathbb{D}, \tag{10}$$

acting on analytic functions f in $\mathbb{D}$. Siskakis and Zhao [**16**] showed that $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ is bounded if and only if $g \in \mathrm{LMOA}$. They proved in the same paper that $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ is compact if and only if $g \in \mathrm{LMOA}_0$, and posed the question whether $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ can be weakly compact without being compact. This was answered in the negative by Laitila, Mihkinen, and Nieminen [**7**]. The purpose of this example is to illustrate that the question may in fact be resolved using Siskakis and Zhao's original argument, when applied in conjuction with Corollary 3.3.

First we point out that the boundedness of T_g, $g \in \mathrm{LMOA}$, automatically implies that $T_g(\mathrm{VMOA}) \subset \mathrm{VMOA}$, so that $T_g|_{\mathrm{VMOA}} : \mathrm{VMOA} \to \mathrm{VMOA}$ is a bounded operator. Secondly, it is easily verified that $(T_g|_{\mathrm{VMOA}})^{**} = T_g$. That is, T_g is weak*-weak continuous. By Gantmacher's theorem it follows that T_g (or equivalently $T_g|_{\mathrm{VMOA}}$) is weakly compact if and only if $T_g(\mathrm{BMOA}) \subset \mathrm{VMOA}$.

When proving that compactness implies $g \in \mathrm{LMOA}_0$ in [**16**], the only step where compactness is used, as opposed to weak compactness, is in showing that $\lim_n \|T_g q_n\|_{\mathrm{BMOA}} = 0$, where

$$q_n(z) = \log \frac{1 - \bar{u}z}{1 - \bar{u}_n z},$$

for a point $u \in \partial\mathbb{D}$ and a sequence $(u_n) \subset \mathbb{D}$ of points converging to u. However, q_n is uniformly bounded in BMOA,

$$\|q_n\|_{\mathrm{BMOA}} \lesssim \|\log(1-z)\|_{\mathrm{BMOA}},$$

and $\lim_n \|q_n\|_{H^2} = 0$, so it follows from Corollary 3.3 that $T_g q_n \to 0$ in BMOA, assuming only the weak compactness of T_g. With this remark in hand, one can follow the proof in [16] verbatim to see that T_g is weakly compact if and only if $g \in \mathrm{LMOA}_0$.

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