

ePub^{WU} Institutional Repository

Ulrich Berger

Simple scaling of cooperation in donor-recipient games

Article (Submitted)

Original Citation:

Berger, Ulrich (2009) Simple scaling of cooperation in donor-recipient games. *BioSystems*, 97 (3). pp. 165-167. ISSN 0303-2647

This version is available at: <http://epub.wu.ac.at/5590/>

Available in ePub^{WU}: June 2017

ePub^{WU}, the institutional repository of the WU Vienna University of Economics and Business, is provided by the University Library and the IT-Services. The aim is to enable open access to the scholarly output of the WU.

This document is the version that has been submitted to a publisher.

Simple scaling of cooperation in donor-recipient games

Ulrich Berger

WU Vienna, Department of Economics, Augasse 2-6, 1090 Wien, Austria

Abstract

We present a simple argument which proves a general version of the scaling phenomenon recently observed in donor-recipient games by Tanimoto (2009).

Key words: Prisoner's Dilemma; Evolutionary Game; Cooperation; Donor-Recipient Game

1 From Prisoner's Dilemma to donor-recipient games

Evolution of cooperation has been called a Darwinian puzzle. Since cooperating individuals are prone to exploitation by defectors who never cooperate, a gene enhancing cooperativeness should be led to extinction in a well mixed population. Supporting cooperation in equilibrium therefore requires a specific evolutionary mechanism. Five such mechanisms have been described in a unifying framework of evolutionary game theory by M.A. Nowak [15]. The game theoretic model of cooperation is encapsulated in the Prisoner's Dilemma game, the general version of which is given by the class of symmetric 2×2 games with payoff matrix

$$\begin{array}{c|cc} & C & D \\ \hline C & R & S \\ D & T & P \end{array} \tag{1}$$

Here, C stands for the cooperative act and D for a defection. The payoffs are ordered $T > R > P > S$, which makes defection a strictly dominant strategy. Usually, also $T + S < 2R$ is required.

Email address: ulrich.berger@wu.ac.at (Ulrich Berger).

A subclass of the class of Prisoner's Dilemma games is given by the constraint $T - R = P - S$. In this case the payoff advantage of defection over cooperation is independent of the opponent's choice. Calling the payoff disadvantage of cooperation the cost of cooperation, denoted by c , allows an interpretation of the game as a Donor-Recipient Game¹ (DRG). In a DRG, both players simultaneously decide whether to donate to the other player or not. A donation, i.e. a cooperative act, costs the donor c and delivers a benefit of b to the recipient, where $b > c > 0$. If a player defects, i.e. if he does not donate, then he incurs zero costs and his opponent gets no benefit. Benefits and costs are added to a common background fitness of f . A DRG is therefore represented by the payoff matrix²

$$\begin{array}{c|cc}
 & C & D \\
 \hline
 C & b - c + f & -c + f \\
 D & b + f & f
 \end{array} \tag{2}$$

Since Prisoner's Dilemma games can be viewed as points in \mathbb{R}^4 and DRGs are a subclass defined by an equality constraint, the latter form a 3-dimensional subspace of the former. Assuming that $f = 0$ or, equivalently, $P = 0$, as it is done in [15] and [17], adds a further equality constraint, reducing the space of games to a 2-dimensional subspace of the space of Prisoner's Dilemmas. Under this additional assumption, DRGs can be represented by just two parameters, b and c , as the payoff matrix reads

$$\begin{array}{c|cc}
 & C & D \\
 \hline
 C & b - c & -c \\
 D & b & 0
 \end{array} \tag{3}$$

We call such a game a *purified donor-recipient game* (pDRG) to distinguish it from the general version with arbitrary f . It is this class of games which Nowak [15] worked with and which are discussed by Tanimoto [17].

2 A scaling phenomenon

It might now be conjectured that the equilibrium rate of cooperation of a given evolutionary mechanism based on a pDRG depends on both these parameters b

¹ These games have also been called *donation games* or *mutual aid games*.

² As usual, f is non-negative here. Tanimoto [17] uses f for the negative of the background fitness and hence adds $-f$ to the entries of the payoff matrix.

and c independently. But the results in [15] seem to indicate that there is only a single decisive parameter which determines the long-run cooperativeness, viz. the benefit-to-cost ratio b/c . For example, the mechanism of kin selection requires Hamilton’s rule for cooperation to evolve. This well-known rule states that for cooperation to evolve, the inverse of the coefficient of relatedness r has to be smaller than the benefit-to-cost ratio: $b/c > 1/r$. Similarly, under the direct reciprocity mechanism, evolution of cooperation requires the inverse of the probability of another round of interaction to be smaller than this ratio, $b/c > 1/w$. Nowak [15] showed that similar rules, all involving the ratio b/c , determine the success of cooperation in particular versions of indirect reciprocity, network reciprocity, and group selection. This seems to point to a “scaling phenomenon”, whereby the “cooperativeness” of a given evolutionary system does not vary independently with the two parameters b and c , but is determined by the single parameter b/c .

Tanimoto [17] checked this scaling phenomenon by a number of numerical experiments implementing a variety of three additional versions of network reciprocity and a further three versions of indirect reciprocity. In each of these numerical experiments, he observed that indeed the ratio of cooperation to defection depends on the single parameter b/c . In total, therefore, the scaling phenomenon described by [17] has been confirmed for eleven different evolutionary mechanisms based on a purified donor-recipient game.

3 Proving the scaling rule

This successful confirmation suggests that some universal “scaling rule” might be at work. Two immediate questions then arise:

- (a) For which evolutionary mechanisms does this scaling rule hold?
- (b) Is this scaling rule the result of some intrinsic property of pDRGs or, more generally, Prisoner’s Dilemma games, or even more general games?

In answering these questions, we argue that the scaling rule

- (a) holds for *all* evolutionary mechanisms and
- (b) applies to *arbitrary* games,

provided *the selection process does not depend on the unit of measurement of payoffs*.

This result does neither require numerical experiments nor sophisticated mathematics. It is a simple consequence of the following argument.

Main argument: Let A be the payoff matrix of a symmetric two-player game which is played repeatedly by members of a population under some evolutionary mechanism. Suppose that the selection process transforming current payoffs into strategy frequencies in the next generation is invariant to changes

in the unit of measurement of payoff. Then multiplying the payoff matrix by $k > 0$, which amounts to changing the payoff unit by rescaling all payoffs by the constant factor k , does not change the outcome of evolution.

Applying this argument to the class of purified donor-recipient games is done by letting $k = c^{-1}$, which may be interpreted as measuring payoffs in multiples of cooperation costs c . This leaves us with the rescaled payoff matrix

$$\begin{array}{c|cc}
 & C & D \\
 \hline
 C & b/c - 1 & -1 \\
 D & b/c & 0
 \end{array} \tag{4}$$

Obviously, this payoff matrix has only a single parameter, the benefit-to-cost ratio b/c , which fully explains the scaling phenomenon observed by Tanimoto [17] for this class of games.

Note that the analogous scaling phenomenon which Tanimoto [17] shortly discusses for the Prisoner’s Dilemma game with $P = 0$ or, analogously, his “donor-recipient game in general expression” with different costs for donor and recipient and $f = 0$, is explained by the same rescaling. Here the remaining three parameters b , c_c , and c_d can be reduced to just two parameters by using $k = c_c^{-1}$ as the scaling factor.

4 Selection processes invariant under positive payoff rescalings

The main argument presented above does neither depend on the particular structure of the game nor on the particular evolutionary mechanism studied. It only depends on the nature of the selection process, and the only requirement, invariance to changes in the unit of the payoff-measure, is a mild and natural one which holds for most classes of selection processes usually applied. Here some of the most widely used continuous-time as well as discrete-time selection processes and three common criteria of evolutionary success are listed as examples.

(i) The *replicator dynamics*, defined in [18] and popularized by [16] is given by

$$\dot{x}_i = x_i[(A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x}], \tag{5}$$

where x_i is the frequency of the i -th strategy and $\mathbf{x} = (x_i)$ is the (column-) vector of these frequencies. Substituting A by kA amounts to multiplying \dot{x}_i by the constant k for all i and is tantamount to changing the velocity, but not the direction, of the vector field. A simple change of the time-scale restores the

system of differential equations to its original form. The replicator dynamics is therefore immune to positive payoff rescalings. This property (and additional invariance properties) are well known for the replicator dynamics, see e.g. [7].

(ii) The *best-response dynamics*, which is due to Gilboa and Matsui [5] and Matsui [11], is mathematically equivalent to the continuous-time fictitious play process (see e.g. [2]) and is given by the differential inclusion

$$\dot{\mathbf{x}} \in B(\mathbf{x}) - \mathbf{x}, \tag{6}$$

where $B(\mathbf{x})$ is the set of best responses to \mathbf{x} . Since best responses are invariant under a positive rescaling of payoffs, the same holds for the orbits of this dynamics.

(iii) *Proportional fitness* or *roulette-wheel selection*, introduced by De Jong [4] makes the probability of an individual i with payoff π_i to be chosen for reproduction among n individuals proportional to its payoff by setting

$$prob_i = \frac{\pi_i}{\sum_{j=1}^n \pi_j}. \tag{7}$$

(This requires payoffs to be positive.) Obviously, positive rescalings of payoffs cancel out and render this selection process invariant under such rescalings.

(iv) Clearly, rank relations between payoffs, and therefore all variants of *fitness rank selection* and *tournament selection* as introduced by Brindle [3], which are often employed in genetic algorithms, are invariant under positive rescalings of payoffs.

(v) The *ESS property* of Maynard Smith [12,13] requires certain payoff inequality conditions to be met. Since the relation \leq is invariant under positive payoff rescalings, the same holds for the ESS condition.

Besides the ESS criterion, Nowak [15] also mentions the following two criteria as “measures of evolutionary success”.

(vi) For 2×2 coordination games with payoffs as given in (1), cooperation is *risk-dominant* in the sense of Harsanyi and Selten [6] iff $R + S > T + P$. This relation, and hence the criterion of risk-dominance, is invariant under positive payoff rescalings.

(vii) For general 2×2 games with payoffs as given in (1), cooperation is *advantageous* iff $R + 2S > T + 2P$. Again this criterion, which derives from the 1/3-rule for stochastic game dynamics (see [14]), is invariant under positive payoff rescalings.

Note that these examples include, but are not limited to, the five mechanisms studied analytically by Nowak [15] and the six mechanisms approached numerically by Tanimoto [17]. Nowak uses (i), (v), (vi), and (vii), and Tanimoto relies on (iii) in his numerical experiments. Since the given examples

cover a large set of typical selection processes, the scaling phenomenon can be expected to prevail in the literature on the evolution of cooperation.

5 Discussion

We have argued that a majority of selection processes and criteria of evolutionary success are invariant to positive rescalings of the payoff matrix in an evolutionary game and that this fully explains the scaling phenomenon observed by Tanimoto [17]. However, there are instances where this invariance does not hold or at least requires further assumptions.

A nonlinear variant of roulette wheel selection in a finite population can be obtained by using a logit choice function (for a recent application see [9]) to translate individual payoffs π_j into the probability of individual i to be chosen for reproduction. This results in probabilities

$$prob_i = \frac{e^{\pi_i}}{\sum_{j=1}^n e^{\pi_j}}. \quad (8)$$

Since these probabilities are nonlinear in π_i , rescaling the payoff-matrix by $k > 0$ results in different evolutionary dynamics. If $k \rightarrow 0$, probabilities become uniform and neutral selection obtains, whereas for very large values of k the process resembles the best-response dynamics.

Another case in point are selection processes which calculate fitness as the sum of a frequency-independent background fitness f and a strategic interaction term.³ If reproduction probabilities are given by

$$prob_i = \frac{f_i}{\sum_{j=1}^n f_j}, \quad (9)$$

where fitness $f_i = f + \pi_i$ with $f > 0$, then multiplying the payoff-matrix by $k > 0$ does not leave the probabilities invariant. Since results are often derived in the limit of weak selection, i.e. for $f \rightarrow \infty$, those results may still be invariant to a rescaling of the payoffs (criterion (vii) above is such a case), but for strong selection typically they are not.⁴ Note also, that added background fitness may even challenge the invariance properties of the replicator dynamics, if the population is growing ([1]) or is embedded in a network ([10]). So the scaling phenomenon studied here may well be typical, but it is not universal.

³ Note that Tanimoto [17] assumes $P = 0$ for a Prisoner's Dilemma game, which implies $f = 0$ for a donor-recipient game, i.e. we have a purified donor-recipient game.

⁴ See e.g. [8]. For the frequency-dependent Moran process this is also discussed in [19].

References

- [1] Argasinski, K., Kozłowski, J. (2008) How can we model selectively neutral density dependence in evolutionary games. *Theoretical Population Biology* 73, 250-256.
- [2] Berger, U. (2007) Two more classes of games with the continuous-time fictitious play property. *Games and Economic Behavior* 60, 247-261.
- [3] Brindle, A. (1981) *Genetic algorithms for function optimization*. PhD thesis, University of Alberta.
- [4] De Jong, K.A. (1975) *An analysis of the behavior of a class of genetic adaptive systems*. PhD thesis, University of Michigan.
- [5] Gilboa, I., Matsui, A. (1991) Social stability and equilibrium. *Econometrica* 59, 859-867.
- [6] Harsanyi, J.C., Selten, R. (1988) *A General Theory of Equilibrium Selection in Games*. MIT Press, Boston, MA.
- [7] Hofbauer J, Sigmund, K (1998) *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge, UK.
- [8] Imhof, L.A., Fudenberg, D., Nowak, M.A. (2007) Tit-for-tat or win-stay, lose-shift? *Journal of Theoretical Biology* 247, 574-580.
- [9] Janssen, M.A. (2008) Evolution of cooperation in a one-shot Prisoner's Dilemma based on recognition of trustworthy and untrustworthy agents. *Journal of Economic Behavior & Organization* 65, 458-471.
- [10] Luthi, L., Tomassini, M., Pestelacci, E. (2009) Evolutionary games on networks and payoff invariance under replicator dynamics. *Biosystems* 96, 213-222.
- [11] Matsui, A. (1992) Best response dynamics and socially stable strategies. *Journal of Economic Theory* 57, 343-362.
- [12] Maynard Smith, J. (1974) The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology* 47, 209-221.
- [13] Maynard Smith, J. (1982) *Evolution and the Theory of Games*. Cambridge University Press, Cambridge, UK.
- [14] Nowak, M.A., Sasaki, A., Taylor, C., Fudenberg, D. (2004) Emergence of cooperation and evolutionary stability in finite populations. *Nature* 428, 646-650.
- [15] Nowak, M.A. (2006) Five rules for the evolution of cooperation. *Science* 314, 1560-1563.
- [16] Schuster, P., Sigmund, K. (1983) Replicator dynamics. *Journal of Theoretical Biology* 100, 533-538.

- [17] Tanimoto, J. (2009) A simple scaling of the effectiveness of supporting mutual cooperation in donor-recipient games by various reciprocity mechanisms. *BioSystems* 96, 29-34.
- [18] Taylor, P.D., Jonker, L. (1978) Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences* 40, 145-156.
- [19] Traulsen, A., Pacheco, J.M., Imhof, L.A. (2006) Stochasticity and evolutionary stability. *Physical Review E* 74, 021905.