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Learning in games with strategic complementarities revisited

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Abstract

Fictitious play is a classical learning process for games, and games with strategic complementarities are an important class including many economic applications. Knowledge about convergence properties of fictitious play in this class of games is scarce, however. Beyond games with a unique equilibrium, global convergence has only been claimed for games with diminishing returns (V. Krishna, 1992, HBS Working Paper 92-073). This result remained unpublished, and it relies on a specific tie-breaking rule. Here we prove an extension of it by showing that already the ordinal version of strategic complementarities suffices. The proof does not rely on tie-breaking rules and provides some intuition for the result. *JEL classification:* C72, D83.

Key words: Fictitious Play; Learning Process; Strategic Complementarities; Ordinal Complementarities.

1 Introduction

In a *fictitious play (FP) process* two players are engaged in the repeated play of a finite game. After an arbitrary initial move, in every round each player takes the empirical distribution of her opponent's strategies as her belief and replies with a myopic pure best response. We say that a FP process approaches equilibrium, if the sequence of beliefs converges to the set of Nash equilibria of the game. A game is said to have the *fictitious play property* (FPP), if every FP process approaches equilibrium.

Fictitious play was introduced by Brown (1949, 1951) as an algorithm to calculate the value of a two-person zero-sum game. Today, FP serves as a classical example of myopic belief learning (see Fudenberg and Levine, 1998, or Young,

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2005). While Shapley's (1964) example showed that there are games without the FPP, research on FP focused on identifying classes of games with the FPP. The largest such classes are zero-sum games (Robinson, 1951), weighted potential games (Monderer and Shapley, 1996b), $2 \times n$ games (Berger, 2005), first-price auction games (Hon-Snir et al., 1998), games with strategic complementarities and a unique equilibrium (Milgrom and Roberts, 1991), and games with strategic complementarities and diminishing returns (Krishna, 1992). [1]

In this paper we concentrate on the last of these results. Krishna's work, though frequently cited, remained unpublished. His proof is a bit involved and provides little intuition. Moreover, two open questions remain:

First, Krishna's result depends critically on a particular tie-breaking rule. Without imposing a tie-breaking rule, Monderer and Sela (1996) constructed an example of a 2×2 game with strategic complementarities and diminishing returns without the FPP. [2] In their example, it is a degeneracy of the game which permits nonconvergence of FP. They refer to Monderer and Shapley (1996a), who showed that a nondegeneracy condition is sufficient to save the FPP for 2×2 games, but they also state that they *"do not know whether such a generic result holds for Krishna's games as well"* (p. 145). Hence the question if a nondegeneracy condition can save Krishna's result in the absence of tie-breaking rules, remained open.

The second question is more than a mere technicality. Milgrom and Shannon (1994) showed that many of the known results for games with strategic complementarities can already be derived under the weaker condition of *ordinal* complementarities. In his paper, Krishna also raises the question if ordinal complementarities could be sufficient for his result. However, as he explains, his method of proof does not extend to this larger class of games, and hence he leaves this question unanswered.

The present paper clarifies these open questions. Without imposing a tie-breaking rule, we provide an intuitive proof of the following extension of Krishna's result: *Every nondegenerate game with ordinal complementarities and diminishing returns has the FPP.*

The remainder of this paper is structured as follows. In Section 2 we introduce the notation and terminology we use, and define strategic and ordinal complementarities, diminishing returns, nondegeneracy, and fictitious play. In Section 3 we derive the main theorem, and Section 4 concludes.

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[1] Convergence results are also available for related versions of FP, such as *stochastic FP* (Fudenberg and Kreps, 1993, Hofbauer and Sandholm, 2002), *continuous-time FP* (Brown, 1949, Berger, 2007b), *generalized weakened FP* (Leslie and Collins, 2006), *alternating FP* (Berger, 2007a), or the *best response dynamics* (Matsui, 1992, Hofbauer, 1995). In this paper we focus exclusively on classical discrete-time FP.

[2] For a helpful visualization of this example see Cressman (2003, p. 84).

2 Notation and Definitions

2.1 Bimatrix Games and Best Responses

Let (A, B) be a bimatrix game where player 1, the row player, has pure strategies $i \in N = \{1, 2, \dots, n\}$, and player 2, the column player, has pure strategies $j \in M = \{1, 2, \dots, m\}$. A and B are the $n \times m$ payoff matrices for players 1 and 2. Thus, if player 1 chooses $i \in N$ and player 2 chooses $j \in M$, the payoffs to players 1 and 2 are a_{ij} and b_{ij}, respectively. The set of mixed strategies of player 1 is the $n-1$ dimensional probability simplex S_n, and analogously S_m is the set of mixed strategies of player 2. With a little abuse of notation we will not distinguish between a pure strategy i of player 1 and the corresponding mixed strategy representation as the i-th unit vector $\mathbf{e}_i \in S_n$. Analogously we identify player 2's pure strategy j with the j-th unit vector $\mathbf{f}_j \in S_m$.

The expected payoff for player 1 playing strategy i against player 2's mixed strategy $\mathbf{y} = (y_1, \dots, y_m)^t \in S_m$ (where the superscript t denotes the transpose of a vector or matrix) is $(A\mathbf{y})_i$. Analogously $(B^t\mathbf{x})_j$ is the expected payoff for player 2 playing strategy j against the mixed strategy $\mathbf{x} = (x_1, \dots, x_n)^t \in S_n$. If both players use mixed strategies $\mathbf{x}$ and $\mathbf{y}$, respectively, the expected payoffs are $\mathbf{x} \cdot A\mathbf{y}$ to player 1 and $\mathbf{y} \cdot B^t\mathbf{x}$ to player 2, where the dot denotes the scalar product of two vectors. We denote by $BR_2(\mathbf{x})$ player 2's pure strategy best response correspondence, and by $br_2(\mathbf{x})$ her mixed strategy best response correspondence. Analogously, $BR_1(\mathbf{y})$ and $br_1(\mathbf{y})$ are the sets of player 1's pure and mixed best responses, respectively, to $\mathbf{y} \in S_m$. Let $BR(\mathbf{x}, \mathbf{y}) = BR_1(\mathbf{y}) \times BR_2(\mathbf{x})$ and $br(\mathbf{x}, \mathbf{y}) = br_1(\mathbf{y}) \times br_2(\mathbf{x})$. We say that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a best response to $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in br(\mathbf{x}, \mathbf{y})$. Also, we call $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ a pure best response to $(\mathbf{x}, \mathbf{y})$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in BR(\mathbf{x}, \mathbf{y})$. A strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is a Nash equilibrium if and only if $(\mathbf{x}^*, \mathbf{y}^*) \in br(\mathbf{x}^*, \mathbf{y}^*)$. It is called a *pure* Nash equilibrium, if $(\mathbf{x}^*, \mathbf{y}^*) \in BR(\mathbf{x}^*, \mathbf{y}^*)$.

2.2 Ordinal and Strategic Complementarities

We begin by defining the ordinal and the cardinal versions of strategic complementarities, which for simplicity we call *ordinal* and *strategic complementarities*, respectively.[3] Following Krishna, we restrict the analysis to completely ordered strategy spaces. For bimatrix games the "natural" ordering of pure strategies by their numbers is used.

Definition 1 *(i) A bimatrix game (A, B) has* ordinal complementarities*, if for all $i < i'$ and $j < j'$:*

$$a_{i'j} > a_{ij} \Longrightarrow a_{i'j'} > a_{ij'} \quad \text{and} \quad b_{ij'} > b_{ij} \Longrightarrow b_{i'j'} > b_{i'j}.$$

[3] These games have also been called quasi-supermodular and supermodular games, respectively. Here we want to stick closer to Krishna's terminology.

(ii) A bimatrix game (A, B) *has* strategic complementarities*, if for all* $i < i'$ *and* $j < j'$*:*

$$a_{i'j'} - a_{ij'} > a_{i'j} - a_{ij} \quad \text{and} \quad b_{i'j'} - b_{i'j} > b_{ij'} - b_{ij}.$$

We write GOC short for *game with ordinal complementarities*, and GSC for *game with strategic complementarities.*[4] In a GOC, payoffs satisfy a single crossing condition: the difference between two payoffs in a column of A or a line of B can change its sign at most once, and only from -1 to $+1$, if the players move up to a higher column or line, respectively. In a broader context, these games have been studied by Milgrom and Shannon (1994). From Definition 1, ordinal complementarities are implied by strategic complementarities. In a GSC, the advantage of switching to a higher strategy increases when the opponent chooses a higher strategy.

2.3 Diminishing Returns

Another condition we use is *diminishing returns*. This property means that the payoff advantage of increasing one's strategy is decreasing.

Definition 2 *A bimatrix game* (A, B) *has* diminishing returns *(DR), if for all* $i = 2, \ldots, n-1$ *and* $j \in M$,

$$a_{i+1,j} - a_{ij} < a_{ij} - a_{i-1,j},$$

and for all $i \in N$ *and* $j = 2, \ldots, m-1$,

$$b_{i,j+1} - b_{ij} < b_{ij} - b_{i,j-1}.$$

2.4 Nondegenerate Games

As mentioned above, without assuming a tie-breaking rule, one must impose a nondegeneracy assumption in order to keep the FPP, even in the class of 2×2 games. We work with games which are nondegenerate in the following specific sense.

Definition 3 *We call a bimatrix game* (A, B) degenerate*, if for some* $i, i' \in N$*, with* $i \neq i'$*, there exists* $j \in M$ *with* $a_{i'j} = a_{ij}$*, or if for some* $j, j' \in M$*, with* $j \neq j'$*, there exists* $i \in N$ *with* $b_{ij'} = b_{ij}$*. Otherwise, the game is said to be* nondegenerate.

We write NDGOC short for *nondegenerate game with ordinal complementarities.*

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[4] The term "strategic complementarities" was coined by Bulow et al. (1985). GSCs have been introduced in a general framework by Topkis (1979), see also Vives (1990) or Milgrom and Roberts (1990). GSCs and GOCs have many important applications in economics, see Vives (2005) for a recent overview.

2.5 Improvement Steps

Monderer and Shapley (1996b) defined *improvement paths* in $N{\times}M$. We extend this definition slightly by defining *improvement steps*.

Definition 4 *For a bimatrix game (A, B), define the following binary relation on $N \times M$: $(i, j) \rightarrow (i', j') \Leftrightarrow (i = i'$ and $b_{ij'} > b_{ij})$ or $(j = j'$ and $a_{i'j} > a_{ij})$. If $(i, j) \rightarrow (i', j')$, we say that this is an* improvement step*. We denote by $|i' - i| + |j' - j|$ the* length *of the improvement step. An* improvement path *is a (finite or infinite) sequence of improvement steps $(i_1, j_1) \rightarrow (i_2, j_2) \rightarrow \cdots$ in $N{\times}M$. An improvement path $(i_1, j_1) \rightarrow \cdots \rightarrow (i_k, j_k)$ is called an* improvement cycle*, if $(i_k, j_k) = (i_1, j_1)$.*

2.6 Fictitious Play and Switching

Definition 5 *For the $n \times m$ bimatrix game (A, B), the sequence $(i_t, j_t)_{t \in \mathbb{N}}$ is a* discrete-time fictitious play process*, if $(i_1, j_1) \in N \times M$ and for all $t \in \mathbb{N}$,*

$$(i_{t+1}, j_{t+1}) \in BR(\mathbf{x}(t), \mathbf{y}(t)),$$

where the beliefs $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are given by*

$$\mathbf{x}(t) = \frac{1}{t} \sum_{s=1}^{t} i_s \ \ \textit{and} \ \ \mathbf{y}(t) = \frac{1}{t} \sum_{s=1}^{t} j_s.$$ [5]

Note that the beliefs can be updated recursively. The belief of a player in round $t+1$ is a convex combination of his belief in round t and his opponent's move in round $t+1$:

$$(\mathbf{x}, \mathbf{y})(t+1) = \frac{t}{t+1}(\mathbf{x}, \mathbf{y})(t) + \frac{1}{t+1}(i_{t+1}, j_{t+1}). \tag{1}$$

The Euclidean distance between two consecutive beliefs is called the *step size* of the process.

Definition 6 *The* step size *of a FP process at time t is the distance*

$$\delta(t) = \|(\mathbf{x}, \mathbf{y})(t+1) - (\mathbf{x}, \mathbf{y})(t)\|.$$

From (1), $\delta(t) = \frac{1}{t+1}\|(i_{t+1}, j_{t+1}) - (\mathbf{x}, \mathbf{y})(t)\|$, implying that the step size of any FP process goes to zero as $t \rightarrow \infty$.

If a fictitious play process converges, it must be constant from some stage on, implying convergence to the respective pure Nash equilibrium. Even for

[5] Brown's (1951) original definition has players updating alternatingly instead of simultaneously, but this version has disappeared from the literature, see Berger (2007a).

nonconvergent processes it is well known that if the beliefs converge, then the limit must be a Nash equilibrium. As noted above, however, there are games where the beliefs need not converge.

If for some belief the best response set is multivalued, there may be several possible continuations of a fictitious play process. To handle this nonuniqueness, particular tie-breaking rules have sometimes been imposed. Krishna assumed that both players, whenever indifferent between two or more pure strategies, choose the strategy with the highest number. Here we do not impose any tie-breaking rules.

Whenever one of the players changes her best repsonse, we say that a *switch* has occured in the FP process.

Definition 7 *A FP process* (i_t, j_t) switches *from* (i, j) *to* (i', j') *at time* T*, if* $(i', j') \neq (i, j)$*,* $(i_{T-1}, j_{T-1}) = (i, j)$*, and* $(i_T, j_T) = (i', j')$*.*
If $i = i'$ *or* $j = j'$*, we call this a* single-switch*, and if* $i \neq i'$ *and* $j \neq j'$*, we call it a* double-switch*.*
We write $\langle (i, j), (i', j') \rangle$ *to denote the occurence of a switch from* (i, j) *to* (i', j') *in the FP process. If the switch occurs at time* T*, we write* $\langle (i, j), (i', j') \rangle_T$*.*
If a switch occurs infinitely often, it is called a permanent switch*.*

In a nonconvergent FP process, eventually only permanent switches occur. After each switch, some pure best response is repeated for a finite number of periods. During these repetitions the beliefs are shifted towards the corresponding vertex of the simplex-product. We call the length of this shift the *belief-shift* of the switch.

Definition 8 *Let* $\langle (i, j), (i', j') \rangle_T$ *be a permanent switch, and let the subsequent switch be* $\langle (i', j'), (i'', j'') \rangle_{T+k}$*, i.e.,* $(i_T, j_T) = \cdots = (i_{T+k-1}, j_{T+k-1}) = (i', j')$ *is repeated* k *times between the two switches. We call the Euclidean distance* $s(T) = \|(\mathbf{x}, \mathbf{y})(T + k - 1) - (\mathbf{x}, \mathbf{y})(T)\|$ *the* belief-shift *of* $\langle (i, j), (i', j') \rangle_T$*.*

Definition 9 *Assume that the switch* $\langle (i, j), (i', j') \rangle$ *is permanent, and let the periods of occurence be* $t_1 < t_2 < \cdots$*, i.e.* $\langle (i, j), (i', j') \rangle_{t_l}$ *for* $l \geq 1$*. We say that* $\langle (i, j), (i', j') \rangle$ *has a* persistent belief-shift*, if* $\limsup_{l \to \infty} s(t_l) > 0$*. Otherwise,* $\langle (i, j), (i', j') \rangle$ *is said to have a* vanishing belief-shift*.* [6]

3 Fictitious Play in Nondegenerate Games with Ordinal Complementarities and Diminishing Returns

Having supplied the necessary definitions, we now collect some preliminary results, most of which are proved elsewhere.

Krishna (1992) observed that DR restrict the best response correspondence of

[6] There is a relation to another property of fictitious play here. If all permanent switches in an FP process have persistent belief-shifts, the process exhibits *infrequent switching* in the terminology of Fudenberg and Levine (1995), and is what Monderer et al. (1997) called *smooth.*

a game in the following way.

Lemma 10 *Let (A, B) be a game with DR. Then for any $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$, the sets $BR_1(\mathbf{y})$ and $BR_2(\mathbf{x})$ contain at most two strategies. If one of these sets contains two strategies, they are numbered consecutively.*

If in a FP process the switch $\langle (i, j), (i', j') \rangle$ occurs infinitely often, then players become asymptotically indifferent between i and i' and between j and j' at the times of switching. From Lemma 10, a consequence of this is, that in games with DR, FP can eventually only switch to neighboring strategies.

Lemma 11 *Let (A, B) be a game with DR. If $\langle (i, j), (i', j') \rangle$ occurs infinitely often, then $|i' - i| \leq 1$ and $|j' - j| \leq 1$.*

Proof. W.l.o.g. suppose $|i' - i| \geq 2$, and let the periods where the switch occurs be t_k for $k \in \mathbb{N}$. Then $i \in BR_1(\mathbf{y}(t_k - 1))$ and $i' \in BR_1(\mathbf{y}(t_k))$ for all k. Since the step size of the process vanishes, $\mathbf{y}(t_k - 1)$ and $\mathbf{y}(t_k)$ have the same set of limit points. If $\hat{\mathbf{y}}$ is such a limit point, then $\hat{\mathbf{y}} \in S_m$ by compactness, and $BR_1(\hat{\mathbf{y}})$ contains i and i' by upper-semicontinuity of the best response correspondence. But this contradicts Lemma 10. □

The *Improvement Principle* of Monderer and Sela (1997) roughly states that a switch implies an improvement step.

Lemma 12 *Let (i_t, j_t) be a FP process in a nondegenerate game. If $\langle (i, j), (i', j') \rangle$, then $i \neq i' \Longrightarrow (i, j) \to (i', j)$ and $j \neq j' \Longrightarrow (i, j) \to (i, j')$.*

If the sequence of switches in a FP process consists entirely of single-switches, then it follows that this sequence generates an improvement path. However, suppose $\langle (i, j), (i', j') \rangle$ is a double-switch. Lemma 12 then implies that $(i, j) \to (i', j)$ and $(i, j) \to (i, j')$, but in general there need not be an improvement path from (i, j) to (i', j'), since neither $(i', j) \to (i', j')$ nor $(i, j') \to (i', j')$ need be true.

The *Second Improvement Principle* implies that there are indeed improvement paths $(i, j) \to (i', j) \to (i', j')$ and $(i, j) \to (i, j') \to (i', j')$, if the double-switch has a persistent belief-shift.

Lemma 13 *Let (i_t, j_t) be a FP process in a nondegenerate game. If $\langle (i, j), (i', j') \rangle$ has a persistent belief-shift, then $i \neq i' \Longrightarrow (i, j') \to (i', j')$ and $j \neq j' \Longrightarrow (i', j) \to (i', j')$.*

Proof. Assume $i \neq i'$. Let the sequence of periods where the switch $\langle (i, j), (i', j') \rangle$ occurs be $(t_k)_{k \in \mathbb{N}}$. Let Y^* be the set of limit points of $\mathbf{y}(t_k)$. By compactness of the simplex, $Y^* \subset S_m$, and since the belief-shift is persistent, we can find a point $\mathbf{y}^* \in Y^*$ and a subsequence of periods, for simplicity denoted by t_k again, such that $\mathbf{y}(t_k)$ converges to $\mathbf{y}^*$ and $s^* = \lim_{k \to \infty} s(t_k) > 0$. Let $t_k + T_k$ be the periods where the subsequent switches occur, and let $\mathbf{y}^{**} = \lim_{k \to \infty} \mathbf{y}(t_k + T_k)$. Note that $\{i, i'\} \subset BR_1(\mathbf{y}^*)$ and $i' \in BR_1(\mathbf{y}^{**})$ by upper-semicontinuity of the best response correspondence. Now since $s^* > 0$, $\mathbf{y}^{**}$ is a strict convex combination of $\mathbf{y}^*$ and $\mathbf{f}_{j'}$, viz. $\mathbf{y}^{**} = c\mathbf{f}_{j'} + (1 - c)\mathbf{y}^*$ for some $0 < c < 1$. Left-multiplying with the matrix A yields $A\mathbf{y}^{**} = cA\mathbf{f}_{j'} + (1 - c)A\mathbf{y}^*$. Subtracting the i-th line of this vector equation from the i'-th line yields

$$(A\mathbf{y}^{**})_{i'} - (A\mathbf{y}^{**})_i = c[a_{i'j'} - a_{ij'}] + (1-c)[(A\mathbf{y}^*)_{i'} - (A\mathbf{y}^*)_i]$$

The second term on the right hand side of this equation vanishes since $\{i, i'\} \subset BR_1(\mathbf{y}^*)$, implying $a_{i'j'} - a_{ij'} = c^{-1}[(A\mathbf{y}^{**})_{i'} - (A\mathbf{y}^{**})_i]$. Since $i' \in BR_1(\mathbf{y}^{**})$, the right hand side is non-negative, so $a_{ij'} \leq a_{i'j'}$. By nondegeneracy, $(i, j') \to (i', j')$. The same argument for player 2 shows that $j \neq j' \implies (i', j) \to (i', j')$. $\square$

Note that in case of a single-switch, Lemma 13 coincides with Lemma 12. Lemma 13 is only needed if some double-switch occurs infinitely often along the FP process. The next result examines the case of a vanishing belief-shift.

Lemma 14 *Let (i_t, j_t) be a FP process in a nondegenerate game with DMR. Suppose $\langle(i,j),(i',j')\rangle$ is a double-switch with a vanishing belief-shift. Let $\langle(i',j'),(i'',j'')\rangle$ be a subsequent switch which occurs infinitely often. Then $i'' \in \{i, i'\}$ and $j'' \in \{j, j'\}$.*

Proof. Let the sequence of periods where the switch $\langle(i,j),(i',j')\rangle$ occurs be $(t_k)_{k\in\mathbb{N}}$. Consider the subsequence of periods such that the subsequent switch is $\langle(i',j'),(i'',j'')\rangle$, which for simplicity we denote by t_k again. Let the periods where the subsequent switches $\langle(i',j'),(i'',j'')\rangle$ occur be $t_k + T_k$. Then $i \in BR_1(\mathbf{y}(t_k - 1))$ and $i' \in BR_1(\mathbf{y}(t_k))$ and $i'' \in BR_1(\mathbf{y}(t_k + T_k))$ for all k. Since the step size of the FP process vanishes, $\mathbf{y}(t_k - 1)$ and $\mathbf{y}(t_k)$ have the same set of limit points, and the same is true for $\mathbf{y}(t_k + T_k - 1)$ and $\mathbf{y}(t_k + T_k)$. Moreover, since $\langle(i,j),(i',j')\rangle$ has a vanishing belief-shift, $\mathbf{y}(t_k)$ and $\mathbf{y}(t_k + T_k - 1)$ have vanishing Euclidean distance. Hence the sets of limit points of $\mathbf{y}(t_k - 1)$, $\mathbf{y}(t_k)$, and $\mathbf{y}(t_k + T_k)$ coincide. If $\hat{\mathbf{y}}$ is such a limit point, then $\hat{\mathbf{y}} \in S_m$ by compactness, and $BR_1(\hat{\mathbf{y}})$ contains i, i', and i'' by upper-semicontinuity of the best response correspondence. By Lemma 10, $i'' = i$ or $i'' = i'$. The analogous argument for player 2 yields $j'' = j$ or $j'' = j'$. $\square$

Lemmas 12, 13, and 14 together establish the following:

Lemma 15 *Let (i_t, j_t) be a FP process in a nondegenerate game with DMR. If the beliefs do not converge, then there is an improvement cycle consisting of improvement steps of length 1.*

Proof. Since the beliefs do not converge, there are infinitely many switches. Since there are only finitely many pure strategy profiles, there exists a time t_0 such that from t_0 onwards only permanent switches occur. Let $\langle(i,j),(i',j')\rangle$ be a permanent switch. If it is a single-switch, then by Lemma 11 and the Improvement Principle (Lemma 12), $(i,j) \to (i',j')$ is a length-1-improvement step. Suppose now that $\langle(i,j),(i',j')\rangle$ is a double-switch. The Improvement Principle implies $(i,j) \to (i',j)$. If the switch has a persistent belief-shift, then also $(i',j) \to (i',j')$ by the Second Improvement Principle (Lemma 13), implying that (i,j) and (i',j') are connected by two improvement steps, which, by Lemma 11, are of length 1. Suppose, on the other hand, that the double-switch has a vanishing belief-shift. Let $\langle(i',j'),(i'',j'')\rangle$ be any of the subsequent switches occuring after t_0, then this switch is permanent. By $(i'', j'') \neq (i', j')$ and by Lemma 14, we have $(i'', j'') = (i, j)$ or $(i'', j'') = (i', j)$ or $(i'', j'') = (i, j')$. In the first case, the two switches lead back to (i, j). In the second and third case, by Lemma 11 and the Improvement Principle, (i, j) and (i'', j'') are connected by a length-1-improvement step.

By repeatedly applying these arguments, the sequence of switches after time t_0 either jumps back and forth between (i, j) and (i', j') forever, or it generates an infinite improvement path consisting of length-1 improvement steps. The first of these two cases reduces to a FP process in the corresponding 2×2 game, which is known to converge in beliefs to a mixed Nash equilibrium.[7] The second case implies the existence of an improvement cycle consisting of improvement steps of length 1. □

However, in Berger (2007b, Lemma 18) it was shown that ordinal complementarities are incompatible with the existence of such improvement cycles:

Lemma 16 *In an NDGOC, every improvement path consisting of length-1-steps is finite.*

Our main result is an immediate consequence of Lemmas 15 and 16.

Theorem 17 *Every nondegenerate game with ordinal complementarities and diminishing returns has the fictitious play property.*

4 Discussion

We have extended the result of Krishna (1992) to games with ordinal complementarities and diminishing returns. Working with nondegenerate games, we could do without invoking a tie-breaking rule.

The intuition is simple: With diminishing marginal returns, a fictitious play process generates an improvement path consisting of length-1 steps. But ordinal complementarities do not allow such an improvement path to cycle, and hence the fictitious play process eventually either becomes constant or jumps back and forth between only two pure strategy profiles forever. In both cases it approaches equilibrium.

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