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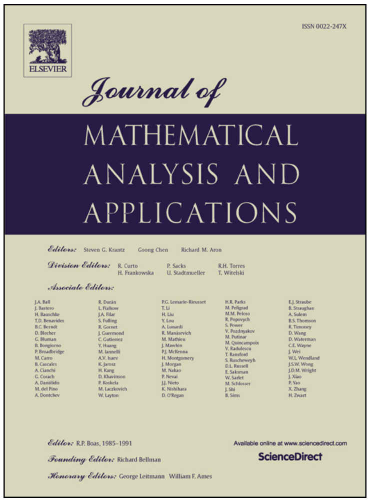

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Anthony Sofo

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A Master Integral in Four Parameters.

Anthony Sofo

Victoria University
P. O. Box 14428, Melbourne City, Victoria 8001, Australia
E-Mail: anthony.sofo@vu.edu.au

Dedicated to Andrew Kara.

Abstract

In this paper we consider a master integral in four arbitrary parameters. The integrand involves the logarithmic function and the Gauss hypergeometric function, which in certain special cases the integral reduces to identities involving zeta functions. A relationship will also be created between the integral and Euler sums of arbitrary order and arbitrary argument. Many interesting new specific examples will be highlighted.

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1. Introduction and Preliminaries

The evaluation of integrals involving integrands with logarithmic and hypergeometric functions can be notoriously difficult to deal with and finding closed form representations of these integrals can be a rare occurrence. Many books and papers have been published on various methods for the evaluation of integrals with hypergeometric or logarithmic functions, see for example [1], [2], [3], [4], [8], [9], [10]. Integrals dealing with the Hurwitz zeta function and Tornheim sums can be seen in [5], [6] and [7]. A class of logarithmic integrals have also recently been examined in [11]. In particular in this paper we investigate the representation of integrals of the type

$$I(m,p,q,t) = \int_0^1 \log^m x \; \Lambda(p,q,t;x) \;\, dx$$

where

$$\Lambda(p,q,t;x) = \frac{x^{-\frac{1}{t}} \;\,(1-x^p)}{(1-x)} \; {}_2F_1\left[\begin{array}{c} 1, \frac{1}{q} \\ 1+\frac{1}{q} \end{array} \middle| \, x^p \right]$$

and ${}_2F_1\left[\begin{array}{c} \cdot, \cdot \\ \cdot \end{array} \middle| \, z \right]$ is the Gauss hypergeometric function. We prove that in many cases of the parameters (m,p,q,t) the integral $I(m,p,q,t)$, maybe represented in closed form that include the polygamma and zeta special functions. Finally a generalization of the integral $I(m,p,q,t)$ is given. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N} := \{1,2,3,\cdots\}$ be the set of positive integers, with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\Gamma(z)$ denote the familiar Euler's gamma function then the digamma (or Psi) function, for $z\in\mathbb{R}$, is defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}$$

and is connected to the harmonic number H_z, by $\psi(z+1) = H_z - \gamma$, where γ is the Euler-Mascheroni constant. The Lerch transcendent

$$\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t} \tag{1.1}$$

is defined for $|z|<1$ and $\Re(a)>0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\ \Phi(z,t,a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z=1$,

$$\Phi(1,t,a) = \zeta(t,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the Polylogarithm, or de Jonquière's function, when $a=1$,

$$Li_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t},\ t\in\mathbb{C} \text{ when } |z|<1;\ \Re(t)>1.$$

Moreover

$$\int_0^1 \frac{Li_t(px)}{x}dx = \begin{cases} \zeta(1+t)\,, \text{ for } p=1 \\ (2^{-r}-1)\,\zeta(1+t)\,, \text{ for } p=-1 \end{cases}.$$

A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda,\mu\in\mathbb{C})$ is defined, in terms of the gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\,\Gamma(\lambda-\mu+1)}, \qquad (\lambda,\mu\in\mathbb{C}),$$

which, in the special case when $\mu=n,\ n\in\mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n\in\mathbb{N}),$$

where $(\lambda)_\nu$ is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed that the Γ-quotient exists. A generalized harmonic number $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m} \ , \ (m, n \in \mathbb{N}) \qquad \text{and} \qquad H_0^{(m)} := 0 \qquad (m \in \mathbb{N})$$

and

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} \qquad (n \in \mathbb{N}_0).$$

Furthermore we may write the generalized harmonic numbers, $H_{z-1}^{(j)}$, in terms of polygamma functions

$$H_{z-1}^{(j)} = \zeta(j) + \frac{(-1)^{j-1}}{(j-1)!}\psi^{(j-1)}(z)\,, \ z \neq \{-1, -2, -3, ...\}\,, \tag{1.2}$$

where $\zeta(j)$, for $j = 2, 3, 4, ...$ is the zeta function.

In Theorem 2, later in this paper, we shall utilize differentiation of a parameter of the Gauss hypergeometric function. The following information will be useful. It is known, see [8] that the Gauss hypergeometric function

$${}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}\middle| z\right] = \sum_{n\geq 0} \frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!}$$

is defined for the circle of convergence of $|z| < 1$, provided that $c \neq 0, -1, -2, ...$ The behaviour of the series on its circle of convergence is:

(i) Divergence when $\mathbb{R}(c - a - b) \leq 1$,

(ii) Absolute convergence when $\mathbb{R}(c - a - b) > 0$,

(iii) Conditional convergence when $-1 \leq \mathbb{R}(c - a - b) \leq 0$, the point $z = 1$ is excluded.

The differential formula for the Gamma function is $\Gamma'(z) = \Gamma(z)\psi(z)$, where $\psi(z)$ is the (Psi) digamma function. We also have the relation

$$\psi(z + j) - \psi(z) = \sum_{k=0}^{j-1} \frac{1}{z + k} \tag{1.3}$$

and therefore

$$\frac{d}{dz}\left((z)_j\right) = (z)_j\,(\psi(z + j) - \psi(z)) = (z)_j \sum_{k=0}^{j-1} \frac{1}{z + k}. \tag{1.4}$$

The μ^{th} derivative of (1.3) with respect to z yields

$$\psi^{(\mu)}(z + j) - \psi^{(\mu)}(z) = \sum_{k=0}^{j-1} \frac{(-1)^{\mu}\,\mu!}{(z + k)^{\mu+1}}$$

where $\psi^{(\mu+1)}(z) = \frac{d}{dz}\left(\psi^{(\mu)}(z)\right);\ \mu = 0, 1, 2, 3, \ldots$.Then, the first two derivatives

$$\frac{d}{da}\,{}_2F_1\left[\begin{array}{c} a,b \\ c \end{array}\middle| z\right] = \sum_{n\geq 0}\frac{(\psi(a+n)-\psi(a))\,(a)_n\,(b)_n\ \ z^n}{(c)_n\ \ n!}$$

and

$$\frac{d^2}{da^2}\,{}_2F_1\left[\begin{array}{c} a,b \\ c \end{array}\middle| z\right] = \sum_{n\geq 0}\frac{\left((\psi(a+n)-\psi(a))^2+\psi'(a+n)-\psi'(a)\right)(a)_n\,(b)_n\,z^n}{(c)_n\ \ n!}.$$

2. Closed form and Integral identities

We now prove the following theorems.

Theorem 1. *Let $m \in \mathbb{N},\ p \in \mathbb{N},\ q \in \mathbb{R}\backslash\{-1,0\}$ and $t \in \mathbb{R}\backslash\{0\}$, then for $qkt-q-pt \neq 0$*

$$I(m,p,q,t) = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^{-\frac{1}{t}}\ \ (1-x^p)\log^{m-1}x}{1-x}\,{}_2F_1\left[\begin{array}{c} 1,\frac{1}{q} \\ 1+\frac{1}{q} \end{array}\middle| x^p\right]dx \tag{2.1}$$

$$= \sum_{k=1}^{p}\left\{\begin{array}{c} \frac{1}{q}\left(\frac{qt}{qkt-q-pt}\right)^m\left(H_{\frac{kt-1}{pt}-1}-H_{\frac{1}{q}-1}\right) \\ +\frac{1}{qp^m}\sum_{j=2}^{m}\left(\frac{pqt}{qkt-q-pt}\right)^{m+1-j}\left(H^{(j)}_{\frac{kt-1}{pt}-1}-\zeta(j)\right) \end{array}\right\},$$

where ${}_2F_1\left[\begin{array}{c} \cdot,\cdot \\ \cdot \end{array}\middle| z\right]$ is the Gauss hypergeometric.

Proof. Consider the shifted Euler sum of the form, for $t \neq 0$,

$$G(m,p,q,t) = q\sum_{n=1}^{\infty}\frac{H^{(m)}_{pn-\frac{1}{t}}}{(qn+1)(qn+1-q)}. \tag{2.2}$$

Utilizing the general integral representation of the harmonic number

$$H_n^{(m+1)} = \frac{(-1)^m}{m!}\int_0^1 \frac{(1-x^n)\ln^m x}{1-x}dx,\ \text{for } m \in \mathbb{N}$$

we have from (2.2)

$$G(m,p,q,t) = \frac{q(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{\log^{m-1}x}{1-x}\sum_{n=1}^{\infty}\frac{1-x^{pn-\frac{1}{t}}}{(qn+1)(qn+1-q)}dx$$

$$
\begin{aligned}
&= \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{\log^{m-1} x}{1-x}\left(1 - x^{-\frac{1}{t}} + x^{-\frac{1}{t}}\ (1-x^p)\ {}_2F_1\left[\begin{matrix} 1, \frac{1}{q} \\ 1+\frac{1}{q}\end{matrix}\ \middle|\ x^p\right]\right) dx \\
&= \zeta(m) - \zeta\left(m, 1-\frac{1}{t}\right) \\
&\quad + \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^{-\frac{1}{t}}\ (1-x^p)\ \log^{m-1} x}{1-x}\ {}_2F_1\left[\begin{matrix} 1, \frac{1}{q} \\ 1+\frac{1}{q}\end{matrix}\ \middle|\ x^p\right] dx.
\end{aligned}
$$

Now let us consider the integral in question, namely

$$
I(m,p,q,t) = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^{-\frac{1}{t}}\ (1-x^p)\ \log^{m-1} x}{1-x}\ {}_2F_1\left[\begin{matrix} 1, \frac{1}{q} \\ 1+\frac{1}{q}\end{matrix}\ \middle|\ x^p\right] dx
$$

and by a Taylor series expansion about x we have

$$
I(m,p,q,t) = \frac{(-1)^{m-1}}{(m-1)!}\sum_{n=0}^{\infty} \frac{1}{\left(q\left[\frac{n}{p}\right]+1\right)}\int_0^1 x^{n-\frac{1}{t}}\ \log^{m-1} x\ dx
$$

where $[z]$ is the integer part of z. Hence we can write

$$
\begin{aligned}
I(m,p,q,t) &= \sum_{n=0}^{\infty} \frac{t^m}{\left(q\left[\frac{n}{p}\right]+1\right)(nt+t-1)^m} \\
&= \sum_{n=1}^{\infty} \frac{t^m}{\left(q\left[\frac{n-1}{p}\right]+1\right)(nt-1)^m} \qquad (2.3)
\end{aligned}
$$

by a change of counter. To express (2.3) in closed form, we notice that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^m}{\left(q\left[\frac{n-1}{p}\right]+1\right)(nt-1)^m} &= \sum_{n=0}^{\infty} \frac{1}{qn+1}\sum_{k=1}^{p} \frac{1}{\left(pn+k-\frac{1}{t}\right)^m} \\
&= \sum_{k=1}^{p}\sum_{n=0}^{\infty} \frac{1}{(qn+1)\left(pn+k-\frac{1}{t}\right)^m} \qquad (2.4) \\
&= \sum_{k=1}^{p}\sum_{n=0}^{\infty}\left\{\frac{\beta_0}{qn+1} + \sum_{j=1}^{m}\frac{\alpha_{m-j}}{\left(pn+k-\frac{1}{t}\right)^j}\right\} \qquad (2.5)
\end{aligned}
$$

where

$$
\beta_0 = \lim_{n\to-\left(\frac{1}{q}\right)}\left\{\frac{1}{\left(pn+k-\frac{1}{t}\right)^m}\right\} = \left(\frac{qt}{kqt-q-pt}\right)^m, \qquad (2.6)
$$

and

$$\begin{aligned}\alpha_{m-j} &= \frac{1}{(m-j)!} \lim_{n\to -\left(\frac{1-kt}{pt}\right)} \frac{d^{m-j}}{dn^{m-j}} \left\{ \frac{1}{qn+1} \right\} \\ &= -\frac{1}{q} \left(\frac{pqt}{kqt-q-pt} \right)^{m+1-j} \quad \text{for } j = 1, 2, 3, ..., m. \qquad (2.7)\end{aligned}$$

From (2.5), (2.6) and (2.7) we have

$$I\left(m,p,q,t\right) = \sum_{k=1}^{p} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} \left(\frac{qt}{kqt-q-pt}\right)^m \left(\frac{1}{qn+1} - \frac{p}{q\left(pn+k-\frac{1}{t}\right)}\right) \\ \\ -\frac{1}{q}\sum_{j=2}^{m} \left(\frac{pqt}{kqt-q-pt}\right)^{m+1-j} \frac{1}{\left(pn+k-\frac{1}{t}\right)^j} \end{array} \right\}$$

$$I\left(m,p,q,t\right) = \sum_{k=1}^{p} \left\{ \begin{array}{c} \frac{1}{q}\left(\frac{qt}{kqt-q-pt}\right)^m \left(H_{-\left(\frac{pt+1-kt}{pt}\right)} - H_{\frac{1}{q}-1}\right) \\ \\ -\frac{1}{q}\sum_{j=2}^{m} \left(\frac{pqt}{kqt-q-pt}\right)^{m+1-j} \frac{(-1)^j}{(j-1)!p^j} \psi^{(j-1)}\left(\frac{k}{p} - \frac{1}{pt}\right) \end{array} \right\}.$$

From the relationship of the polygamma function with the generalized harmonic numbers (1.2), we have

$$I\left(m,p,q,t\right) = \sum_{k=1}^{p} \left\{ \begin{array}{c} \frac{1}{q}\left(\frac{qt}{qkt-q-pt}\right)^m \left(H_{\frac{kt-1}{pt}-1} - H_{\frac{1}{q}-1}\right) \\ \\ +\frac{1}{qp^m}\sum_{j=2}^{m} \left(\frac{pqt}{qkt-q-pt}\right)^{m+1-j} \left(H^{(j)}_{\frac{kt-1}{pt}-1} - \zeta\left(j\right)\right) \end{array} \right\}$$

and the theorem is proved. □

Example 1. *Some examples follow:*

$$\begin{aligned} I\left(m,p,q,t\right) &= I\left(3,p,-4,-\frac{2}{3}\right), \text{ for } p\in\mathbb{N} \\ &= -2\int_0^1 \frac{x^{\frac{3}{2}}\left(1-x^p\right)\log^2 x}{1-x}\ {}_2F_1\left[\begin{array}{c} 1,-\frac{1}{4} \\ \frac{3}{4} \end{array}\middle| x^p\right] dx \end{aligned}$$

$$= -2\int_0^1 \frac{x^{\frac{3}{2}}\left(1-x^p\right)\log^2 x}{1-x} \left(1 + \frac{x^{\frac{p}{4}}}{2}\left(\tan^{-1}\left(x^{\frac{p}{4}}\right) - \tanh^{-1}\left(x^{\frac{p}{4}}\right)\right)\right) dx$$

$$= \frac{2}{p^2}\sum_{k=1}^{p} \frac{1}{\left(p+4k+6\right)^3} \left(\begin{array}{c} -128p^2 + 32p^2\gamma - 16p^2\pi + 96p^2\ln 2 + 32p^2\psi\left(\frac{2k+3}{2p}\right) \\ \\ -8p\left(p+4k+6\right)\psi'\left(\frac{2k+3}{2p}\right) + \left(p+4k+6\right)^2\psi''\left(\frac{2k+3}{2p}\right) \end{array} \right).$$

For

$$p=1,\ I\left(3,1,-4,-\frac{2}{3}\right)=\frac{127232}{35937}+\frac{64}{1331}\ln 2-\frac{32\pi}{1331}-\frac{48}{121}\zeta(2)-\frac{28}{11}\zeta(3),$$

and for

$$p=2,\ I\left(3,2,-4,-\frac{2}{3}\right)=\frac{1379}{216}-\frac{7}{36}G-\frac{\pi}{27}-\frac{25}{48}\zeta(2)-\frac{\pi^3}{48}-\frac{49}{12}\zeta(3),$$

where $G=.91596...$ *is Catalan's constant. Next*

$$\begin{aligned} I(m,p,q,t) &= I(3,4,2,2) \\ &= \frac{1}{2}\int_0^1 \frac{\left(1-x^4\right)\ \log^2 x}{(1-x)\ x^{\frac{5}{2}}}\log\left(\frac{1+x^2}{1-x^2}\right)\ dx \\ &= \frac{16}{27}\left(14\sqrt{2}-13\right)\pi+\frac{16}{3}\left(2\sqrt{2}-5\right)\zeta(2)+\frac{1}{3}\left(3\sqrt{2}-2\right)\pi^3. \end{aligned}$$

$$\begin{aligned} I(m,p,q,t) &= I(4,2,-2,2) \\ &= -\frac{1}{3}\int_0^1 \frac{(1+x)\ \log^3 x}{x^{\frac{1}{2}}}\ {}_2F_1\left[\left.\begin{matrix}1,-\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right| x^2\right]dx \\ &= -\frac{22592}{50625}-\frac{3136}{3375}G-\frac{4352\pi}{50625}-\frac{1216}{1125}\zeta(2)-\frac{16\pi^3}{225} \\ &\quad -\frac{11296\log(2)}{50625}-\frac{1}{72}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{1}{120}\psi^{(3)}\left(\frac{3}{4}\right)-\frac{952}{225}\zeta(3). \end{aligned}$$

$$\begin{aligned} I(m,p,q,t) &= I(m,2,3,2),\ \textit{for } m\in\mathbb{N} \\ &= \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 x^{-\frac{1}{2}}\ (1+x)\ \log^{m-1}x\ {}_2F_1\left[\left.\begin{matrix}1,\frac{1}{3}\\ \frac{4}{3}\end{matrix}\right| x^2\right]dx \\ &= \frac{1}{2}\left((-6)^m+\left(\frac{6}{5}\right)^m\right)\ln 3-\left((-6)^m+\left(\frac{6}{5}\right)^m\right)\ln 2 \\ &\quad +\frac{\pi}{18}\left(\left(3+\sqrt{3}\right)\left(\frac{6}{5}\right)^m-\left(3-\sqrt{3}\right)(-6)^m\right) \\ &\quad +\frac{1}{2^m}\sum_{j=2}^m\left(\begin{matrix}(-12)^{m+1-j}H^{(j)}_{-\frac{3}{4}}+\left(\frac{12}{5}\right)^{m+1-j}H^{(j)}_{-\frac{1}{4}}\\ -\left((-12)^{m+1-j}+\left(\frac{12}{5}\right)^{m+1-j}\right)\zeta(j)\end{matrix}\right). \end{aligned}$$

A number of interesting special cases follow as corollaries.

Corollary 1. *Let* $\{m,p\}\in\mathbb{N},\ q\in\mathbb{R}\backslash\{-1,0\}$, *then for* $qk-p\neq 0$, *and as* $t\to\infty$ *we have,*

$$\begin{aligned}
\lim_{t\to\infty} I(m,p,q,t) &= I(m,p,q)=qG(m,p,q)\\
&= \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{(1-x^p)\log^{m-1}x}{1-x}\ {}_2F_1\left[\left.\begin{matrix}1,\frac{1}{q}\\ 1+\frac{1}{q}\end{matrix}\right| x^p\right]dx
\end{aligned}$$

$$= H_p^{(m)}+\frac{1}{q}\sum_{k=1}^{p}\left\{\begin{matrix}\left(\frac{q}{qk-p}\right)^m\left(H_{\frac{k}{p}}-H_{\frac{1}{q}}\right)\\ +\frac{1}{p^m}\sum_{j=2}^{m}\left(\frac{pq}{qk-p}\right)^{m+1-j}\left(H_{\frac{k}{p}}^{(j)}-\zeta(j)\right)\end{matrix}\right\}. \tag{2.8}$$

Proof. The proof follows, by noting that

$$\begin{aligned}
qG(m,p,q) &= q\sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}}{(qn+1)(qn+1-q)}\\
&= \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{(1-x^p)\log^{m-1}x}{1-x}\ {}_2F_1\left[\left.\begin{matrix}1,\frac{1}{q}\\ 1+\frac{1}{q}\end{matrix}\right| x^p\right]dx\\
&= \sum_{n=1}^{\infty}\frac{1}{n^m\left(q\left[\frac{n-1}{p}\right]+1\right)}\\
&= \sum_{n=0}^{\infty}\frac{1}{qn+1}\sum_{k=1}^{p}\frac{1}{(pn+k)^m}\\
&= H_p^{(m)}+\sum_{k=1}^{p}\sum_{n=1}^{\infty}\frac{1}{(qn+1)(pn+k)^m}\ .
\end{aligned}\tag{2.9}$$

Expanding (2.9) leads to (2.8). □

For $t=-1$, we have the following:

Corollary 2. *Let* $m\in\mathbb{N},,p\in\mathbb{N},\ q\in\mathbb{R}\backslash\{-1,0\}$ *and* $t=-1$, *then for* $qk+q-p\neq 0$, *we have,*

$$I(m,p,q,-1) = 1+\frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x(1-x^p)\log^{m-1}x}{1-x}\ {}_2F_1\left[\left.\begin{matrix}1,\frac{1}{q}\\ 1+\frac{1}{q}\end{matrix}\right| x^p\right]dx$$

$$= H_{p+1}^{(m)} + \frac{1}{q}\sum_{k=1}^{p}\left\{\begin{array}{c}\left(\frac{q}{qk+q-p}\right)^{m}\left(H_{\frac{1+k}{p}} - H_{\frac{1}{q}}\right)\\ \\ +\frac{1}{p^{m}}\sum_{j=2}^{m}\left(\frac{pq}{qk+q-p}\right)^{m+1-j}\left(H_{\frac{1+k}{p}}^{(j)} - \zeta\left(j\right)\right)\end{array}\right\}. \quad (2.10)$$

Proof. The proof follows by noting that

$$\begin{aligned} qG\left(m,p,q,-1\right) &= q\sum_{n=1}^{\infty}\frac{H_{pn+1}^{(m)}}{\left(qn+1\right)\left(qn+1-q\right)} \\ &= 1+\frac{(-1)^{m-1}}{(m-1)!}\int_{0}^{1}\frac{x\left(1-x^{p}\right)\log^{m-1}x}{1-x}\ {}_{2}F_{1}\left[\left.\begin{array}{c}1,\frac{1}{q}\\ 1+\frac{1}{q}\end{array}\right| x^{p}\right]dx \\ &= \sum_{n=1}^{\infty}\frac{1}{(n+1)^{m}\left(q\left[\frac{n-1}{p}\right]+1\right)} \\ &= \sum_{n=0}^{\infty}\frac{1}{qn+1}\sum_{k=1}^{p}\frac{1}{(pn+k+1)^{m}} \\ &= H_{p+1}^{(m)}+\sum_{k=1}^{p}\sum_{n=1}^{\infty}\frac{1}{\left(qn+1\right)\left(pn+k+1\right)^{m}}. \end{aligned} \quad (2.11)$$

Expanding (2.11) leads to (2.10). □

Finally for $t=1$, we have:

Corollary 3. *Let* $\{m,p\}\in\mathbb{N},\ q\in\mathbb{R}\setminus\{-1,0\}$ *and* $t=1$*, then for* $qk-q-p\neq 0$*, we have,*

$$qG\left(m,p,q,1\right) = q\sum_{n=1}^{\infty}\frac{H_{pn-1}^{(m)}}{\left(qn+1\right)\left(qn+1-q\right)}$$

$$= H_{p-1}^{(m)} + \frac{1}{q}\sum_{k=1}^{p}\left\{\begin{array}{c}\left(\frac{q}{qk-q-p}\right)^{m}\left(H_{\frac{k-1}{p}} - H_{\frac{1}{q}}\right)\\ \\ +\frac{1}{p^{m}}\sum_{j=2}^{m}\left(\frac{pq}{qk-q-p}\right)^{m+1-j}\left(H_{\frac{k-1}{p}}^{(j)} - \zeta\left(j\right)\right)\end{array}\right\} \quad (2.12)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{m}\left(q\left[\frac{n}{p}\right]+1\right)} = H_{p-1}^{(m)}+\sum_{n=1}^{\infty}\frac{1}{qn+1}\sum_{k=1}^{p}\frac{1}{(pn+k-1)^{m}}.$$

Proof. Follows the same pattern as the previous corollary. It is also of some interest to note that

$$q\sum_{n=1}^{\infty}\frac{H_{pn-1}^{(m)}}{(qn+1)(qn+1-q)} = q\sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}-\frac{1}{(pn)^m}}{(qn+1)(qn+1-q)}$$

$$= q\sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}}{(qn+1)(qn+1-q)}-\frac{q}{p^m}\sum_{n=1}^{\infty}\frac{1}{n^m(qn+1)(qn+1-q)}$$

$$= qG(m,p,q)-\frac{q}{p^m}W(m,p,q),$$

here $qG(m,p,q)$ is given by (2.8) and $W(m,p,q)$ has a closed form expression, which is of interest in its own right. Omitting the calculations we give the result, for $q\neq 1$,

$$W(m,p,q) = \sum_{n=1}^{\infty}\frac{1}{n^m(qn+1)(qn+1-q)}$$

$$=(-1)^m q^{m-1}+q^{m-2}\left((-1)^m-\frac{1}{(q-1)^m}\right)H_{\frac{1}{q}-1}+\sum_{j=2}^{m}A_j\zeta(j)$$

where

$$A_j=\frac{q^{m-j-1}\left((-1)^{m+1-j}(q-1)^{m+1-j}-1\right)}{(q-1)^{m+1-j}},\ \text{for } j=2,3,...,m.$$

□

The following generalization of Theorem 1 and its corollaries, taking into account the differentiation of the Gauss hypergeometric function with respect to the parameter r, can be stated.

Theorem 2. *Let the conditions of Theorem 1 apply and let $\mu\in\mathbb{N}$, then:*

$$J(m,p,q,t,\mu) = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1\frac{x^{p-\frac{1}{t}}(1-x^p)\log^{m-1}x}{(1-x)q^{\mu+1}} \tag{2.13}$$

$$\times\sum_{r=0}^{\mu}\frac{(-1)^r}{q^r}\binom{\mu}{r}\Phi\left(x^p,1+r,1+\frac{1}{q}\right)dx$$

$$= \sum_{n=1}^{\infty}\frac{t^m\left[\frac{n-1}{p}\right]^{\mu}}{(nt-1)^m\left(q\left[\frac{n-1}{p}\right]+1\right)^{\mu+1}} \tag{2.14}$$

$$=\sum_{k=1}^{p}\sum_{n=1}^{\infty}\frac{n^{\mu}}{\left(pn+k-\frac{1}{t}\right)^m(qn+1)^{\mu+1}}. \tag{2.15}$$

Proof. From (2.1) we have that

$$J(m,p,q,t,\mu) = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^{-\frac{1}{t}}\ (1-x^p)\log^{m-1}x}{1-x}\frac{d^\mu}{dq^\mu}\ {}_2F_1\left[\left.\begin{matrix}1,\frac{1}{q}\\1+\frac{1}{q}\end{matrix}\right| x^p\right].$$

Now utilizing (1.4) in the derivative of the Gauss hypergeometric function, and upon simplification, (2.13) follows. Also from (2.4) we obtain (2.14) and (2.15).

$$\sum_{n=1}^{\infty}\frac{t^m}{(nt-1)^m}\frac{d^\mu}{dq^\mu}\left(\frac{1}{q\left[\frac{n-1}{p}\right]+1}\right) = \sum_{n=1}^{\infty}\frac{(-1)^\mu\ \mu!\ t^m\left[\frac{n-1}{p}\right]^\mu}{(nt-1)^m\left(q\left[\frac{n-1}{p}\right]+1\right)^{\mu+1}}.$$

It is possible to represent (2.14) in closed form by partial fraction decomposition, similar to Theorem 1, but this will not be pursued here. □

To highlight Theorem 2 we give some examples and then list a corollary of Theorem 2

Example 2.

$$\begin{aligned}
J(2,2,2,2,2) &= \frac{1}{8}\int_0^1 x^{\frac{3}{2}}\ (1+x)\log x\sum_{r=0}^{2}\frac{(-1)^{r+1}}{2^r}\begin{pmatrix}2\\r\end{pmatrix}\Phi\left(x^2,1+r,\frac{3}{2}\right)dx\\
&= \sum_{n=1}^{\infty}\frac{\left[\frac{n-1}{2}\right]^2}{\left(n-\frac{1}{2}\right)^2\left(2\left[\frac{n-1}{2}\right]+1\right)^3}\\
&= 10G+4\pi-13\ln 2-9\zeta(2)+\frac{7}{4}\zeta(3).
\end{aligned}$$

$$\begin{aligned}
J(2,2,4,3,2) &= \frac{1}{1025}\int_0^1 x^{\frac{5}{3}}\ (1+x)\log x\left(\begin{matrix}-16\Phi\left(x^2,1,\frac{5}{4}\right)\\+8\Phi\left(x^2,2,\frac{5}{4}\right)-\Phi\left(x^2,3,\frac{5}{5}\right)\end{matrix}\right)dx\\
&= \sum_{n=1}^{\infty}\frac{\left[\frac{n-1}{2}\right]^2}{\left(n-\frac{1}{3}\right)^2\left(4\left[\frac{n-1}{2}\right]+1\right)^3}\\
&= -\frac{2322}{343}-\frac{6219}{686}G+\frac{15}{9604}\left(7215-2389\sqrt{3}\right)\pi+\frac{324315}{4802}\ln 2\\
&\quad -\frac{324675}{9604}\ln 3-\frac{18657}{2744}\zeta(2)+\frac{225}{6272}\pi^3-\frac{3}{4}\psi'(\frac{4}{3})\\
&\quad -\frac{75}{5488}\psi'(\frac{11}{6})+\frac{225}{224}\zeta(3).
\end{aligned}$$

Corollary 4. *Let the conditions of theorem 2 hold, then as $t \to \infty$,*

$$\lim_{t\to\infty} J(m,p,q,t,\mu) = J(m,p,q,\mu) = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^p\ \left(1-x^p\right)\log^{m-1}x}{(1-x)\,q^{\mu+1}}$$

$$\times \sum_{r=0}^{\mu} \frac{(-1)^r}{q^r}\binom{\mu}{r}\Phi\left(x^p,1+r,1+\frac{1}{q}\right)dx$$

$$= \sum_{n=1}^{\infty} \frac{\left[\frac{n-1}{p}\right]^{\mu}}{n^m\left(q\left[\frac{n-1}{p}\right]+1\right)^{\mu+1}}.$$

Proof. The proof follows the same pattern as that employed in theorem 2. □

Example 3.

$$J(2,2,4,2) = \frac{1}{1024}\int_0^1 x^2\ (1+x)\log x\sum_{r=0}^{4}\frac{(-1)^r}{2^r}\binom{4}{r}\Phi\left(x^2,1+r,\frac{5}{4}\right)dx$$

$$= \frac{5}{9}\ln 2 - \frac{29}{54}G + \frac{7}{27}\pi - \frac{259}{432}\zeta(2) + \frac{5}{1152}\pi^3 + \frac{35}{288}\zeta(3)$$

,

$$= \sum_{n=1}^{\infty}\frac{\left(64n^4-128n^3+52n^2+10n+1\right)H_{2n}^{(2)}}{(4n+1)^3(4n-3)^3}.$$

A number of other related results on the summation of harmonic number sums can be seen in the papers, [12], [13], [14].

$$J(3,3,-3,2) = \frac{\ln 3}{9216} - \frac{755}{373248} - \frac{65\sqrt{3}}{82944}\pi - \frac{125}{2036}\zeta(2) + \frac{89\sqrt{3}}{629856}\pi^3$$
$$-\frac{391}{559872}\psi'(\frac{2}{3}) + \frac{199}{34992}\zeta(3)$$

$$= -\frac{1}{54}\int_0^1 \frac{x^3\ \left(1-x^3\right)}{1-x}\log^2 x\sum_{r=0}^{2}\frac{1}{3^r}\binom{2}{r}\Phi\left(x^3,1+r,\frac{2}{3}\right)dx.$$

Conclusion 1. *We have established an explicit analytical representation of a general integral in which the integrand contains the product of the logarithmic and the Gauss hypergeometric function. The motivation for this integrand is its connection with Euler sums of arbitrary order and arbitrary argument and the possibility of the evaluation of a larger class of integrals. It will* *be possible to examine integrals with generalized hypergeometric functions of the form*

$$\frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^a\ \left(1-x^p\right)\ \log^{m-1}x}{1-x}\ {}_{r+1}F_r\left[\begin{array}{c} 2r-1,....,1,\frac{1}{q} \\ 2r-2,....,2,1+\frac{1}{q}\end{array}\middle| x^p\right]dx.$$

In particular, for $r=2$*, we can examine*

$$\frac{(-1)^{m-1}}{(m-1)!}\int_0^1 \frac{x^a\ \left(1-x^p\right)\ \log^{m-1}x}{1-x}\ {}_3F_2\left[\begin{array}{c} 3,1,\frac{1}{q} \\ 2,1+\frac{1}{q}\end{array}\middle| x^p\right]dx$$

$$= \sum_{n=0}^{\infty}\frac{\left[\frac{n}{p}\right]+2}{2\left(q\left[\frac{n}{p}\right]+1\right)(n+1+a)^m}.$$

Some of the specific examples listed above can be evaluated, and have been checked with the software "Mathematica" [15]*, but in general not the integral in Theorem 1 or Theorem 2.*

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