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BOUNDING THE ČEBYŠEV FUNCTIONAL FOR A DIFFERENTIABLE FUNCTION WHOSE DERIVATIVE IS h OR λ-CONVEX IN ABSOLUTE VALUE AND APPLICATIONS

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Some bounds for the Čebyšev functional of a differentiable function whose derivative is h or λ-convex in absolute value and applications for functions of selfadjoint operators in Hilbert spaces via the spectral representation theorem are given.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [55] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n), \tag{1}$$

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provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a,b]. \tag{2}$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $C(f,g)$ was derived in [14] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|C(f,g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (3).

Čebyšev's inequality (3) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty [a,b]$.

In 1970, A.M. Ostrowski [69] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|C(f,g)| \leq \frac{1}{8} (b-a)(M-m) \|g'\|_\infty, \tag{4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [61] (see also [66, p. 210]) obtained the following result as well:

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{5}$$

provided f, g are absolutely continuous and $f', g' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [11], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|C(f,g)| \tag{6}$$
$$\leq \begin{cases} \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \\ \\ \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad\qquad\qquad \text{where } p > 1,\ 1/p + 1/q = 1. \end{cases}$$

For $\gamma = 0$, we get from the first inequality in (6)

$$|C(f,g)| \leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \tag{7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (6) we can deduce the following result obtained by Cheng and Sun [15]

$$|C(f,g)| \leq \frac{1}{2}(M-m) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt. \tag{8}$$

The constant $\frac{1}{2}$ is best in (8) as shown by Cerone and Dragomir in [12].

The following result holds [33].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt \tag{9}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$.

The constant $\frac{1}{2}$ is best possible in (9).

We denote the variance of the function $f : [a,b] \to \mathbb{C}$ by $D(f)$ and defined as

$$D(f) = \left[C\left(f, \bar{f}\right) \right]^{1/2} = \left[\frac{1}{b-a} \int_a^b |f(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b f(t)\, dt \right|^2 \right]^{1/2}, \tag{10}$$

where $\bar{f}$ denotes the complex conjugate function of f.

We have [33]:

Corollary 1.2. *If the function $f : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$D(f) \leq \frac{1}{2} \bigvee_a^b (f). \tag{11}$$

The constant $\frac{1}{2}$ is best possible in (11).

Now we can state the following result when both functions are of bounded variation [33]:

Corollary 1.3. *If $f, g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g). \tag{12}$$

The constant $\frac{1}{4}$ is best possible in (12).

Remark 1.4. We can consider the following quantity associated with a complex valued function $f:[a,b]\to\mathbb{C}$,

$$E(f):=|C(f,f)|^{1/2}=\left|\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2\right|^{1/2}.$$

Utilising the above results we can state that

$$\begin{aligned}E^2(f)&\le\frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \qquad (13)\\&\le\frac{1}{2}\bigvee_a^b(f)D(f)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2.\end{aligned}$$

If we consider

$$\begin{aligned}G(f)&:=|C(f,|f|)|^{1/2}\\&=\left|\frac{1}{b-a}\int_a^b f(t)|f(t)|\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b|f(t)|\,dt\right|^{1/2},\end{aligned}$$

then we also have

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left||f(t)|-\frac{1}{b-a}\int_a^b|f(s)|\,ds\right|dt \qquad (14)\\&\le\frac{1}{2}\bigvee_a^b(f)D(|f|)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2\end{aligned}$$

and

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(|f|)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \qquad (15)\\&\le\frac{1}{2}\bigvee_a^b(|f|)D(f)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2.\end{aligned}$$

Motivated by the results presented above, we establish in this paper some new bounds for the magnitude of $C(f,g)$ in the case when one of the complex valued function, say f, is differentiable and the derivative is h-convex or λ-convex in absolute value while the other is Lebesgue integrable on $[a,b]$. Applications for functions of selfadjoint operators in Hilbert spaces via the spectral representation theorem are also given.

Before we are able to state our new results, we need the following preliminary facts about h-convex and λ-convex functions.

2. h-Convex and λ-Convex Functions

2.1. h-Convex Functions

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 2.1 ([54])**.** We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{16}$$

Some further properties of this class of functions can be found in [43], [44], [46], [63], [73] and [74]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2.2 ([46])**.** We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{17}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{18}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [46] and [71] while for quasi convex functions, the reader can consult [45].

Definition 2.3 ([7])**.** Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [41], [42], [57], [59] and [76].

In order to unify the above concepts for functions of real variable, Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 2.4 ([80])**.** Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{19}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [80], [6], [61], [77], [75] and [79].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 2.5. We say that the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{20}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[52], [58]-[60] and [71]-[79].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \ge h(t) h(s) \text{ for any } t, s \in J. \tag{21}$$

If the inequality (21) is reversed, then h is said to be *submultiplicative*. If the equality holds in (21) then h is said to be a multiplicative function on J.

In [80] it has been noted that if $h : [0,\infty) \to [0,\infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \ge 1$, then for $p \in (0,1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

The following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces holds [36].

Theorem 2.6. *Assume that the function $f : C \subseteq X \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y,x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq [f(x)+f(y)] \int_0^1 h(t)\,dt. \quad (22)$$

Remark 2.7. If $f : I \to [0,\infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a,b \in I, a < b$, then from (22) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [75]

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \int_a^b f(u)\,du \leq [f(a)+f(b)] \int_0^1 h(t)\,dt. \quad (23)$$

If we write (22) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write (22) for the case of P-type functions $f : C \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq f(x)+f(y), \quad (24)$$

that has been obtained for functions of real variable in [46].

If f is Breckner s-convex on C, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (22) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{s+1}, \quad (25)$$

that was obtained for functions of a real variable in [41].

If $f : C \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{1-s}. \quad (26)$$

We notice that for $s = 1$ the first inequality in (26) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt. \quad (27)$$

The case for functions of real variables was obtained for the first time in [46].

2.2. λ-Convex Functions

We start with the following definition (see [37]):

Definition 2.8. Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \le \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha+\beta)} \tag{28}$$

for all $\alpha, \beta \ge 0$ with $\alpha+\beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0,1]$.

If $f : C \to [0,\infty)$ is h-convex function with h supermultiplicative on $[0,\infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \ge 0$ with $\alpha+\beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) &\le h\left(\frac{\alpha}{\alpha+\beta}\right) f(x) + h\left(\frac{\beta}{\alpha+\beta}\right) f(y) \\ &\le \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha+\beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions [37].

Proposition 2.9. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \ge 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha+\beta) \le \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0,\infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha+\beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0,\infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0,\infty) \to [0,\infty)$.

Theorem 2.10 ([37]). *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $r \in (0,R)$ then the function $\lambda_r : [0,\infty) \to [0,\infty)$ given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{29}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

Remark 2.11. Now, if we take $h(z) = \frac{1}{1-z}$, $z \in D(0,1)$, then

$$\lambda_r(t) = \ln\left[\frac{1 - r\exp(-t)}{1-r}\right] \tag{30}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r \in (0,1)$.

If we take $h(z) = \exp(z)$, $z \in \mathbb{C}$ then

$$\lambda_r(t) = r[1 - \exp(-t)] \tag{31}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r > 0$.

Corollary 2.12 ([37]). *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$ and $r \in (0,R)$. For a mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r : [0,\infty) \to [0,\infty)$,

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \leq \left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)} \left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{32}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}\,[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}} \leq [h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{33}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

Remark 2.13. We observe that, in the case when

$$\lambda_r(t) = r[1-\exp(-t)],\ t \geq 0$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{[1-\exp(-\alpha)]\, f(x)+[1-\exp(-\beta)]\, f(y)}{1-\exp(-\alpha-\beta)} \tag{34}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta > 0$ and $x, y \in C$.

We observe that this definition is independent of $r > 0$.

The inequality (34) is equivalent with

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\exp(\beta)[\exp(\alpha)-1]\, f(x)+\exp(\alpha)[\exp(\beta)-1]\, f(y)}{\exp(\alpha+\beta)-1} \tag{35}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta > 0$ and $x, y \in C$.

We also can introduce the following mapping $k_{x,y} : [0,1] \to \mathbb{R}$

$$k_{x,y}(t) := \frac{1}{2}[f(tx+(1-t)y)+f((1-t)x+ty)]$$

for $x, y \in C$, $x \neq y$.

The following result holds [37]:

Theorem 2.14. *Let $f : C \to [0,\infty)$ be a λ-convex function on C. Assume that $x, y \in C$ with $x \neq y$.*

(i) We have the equality

$$k_{x,y}(1-t) = k_{x,y}(t) \text{ for all } t \in [0,1];$$

(ii) The mapping $k_{x,y}$ is λ-convex on $[0,1]$;

(iii) One has the inequalities

$$k_{x,y}(t) \leq \frac{\lambda(t)+\lambda(1-t)}{\lambda(1)} \cdot \frac{f(x)+f(y)}{2} \tag{36}$$

and

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} f\left(\frac{x+y}{2}\right) \leq k_{x,y}(t) \tag{37}$$

for all $t \in [0,1]$ and $\alpha > 0$.

(iv) Let $y, x \in C$ with $y \neq x$ and assume that the mappings $[0,1] \ni t \mapsto f[(1-t)x+ty]$ and λ are Lebesgue integrable on $[0,1]$, then we have the Hermite-Hadamard type inequalities

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{\lambda(1)} \int_0^1 \lambda(t)\,dt \tag{38}$$

for any $\alpha > 0$.

Corollary 2.15. *If $f : I \to [0,\infty)$ is an λ-convex function on an interval I of real numbers with $\lambda \in L[0,1]$ and $f \in L[a,b]$ with $a,b \in I, a < b$, then*

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(u)\,du \leq \frac{f(a)+f(b)}{\lambda(1)}\int_0^1 \lambda(t)\,dt \tag{39}$$

for any $\alpha > 0$.

3. New Results for Čebyšev Functional

We have the following result:

Theorem 3.1. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of the interval I, $[a,b] \subset \mathring{I}$ and $g : [a,b] \to \mathbb{C}$ is an integrable function on $[a,b]$.*

(i) If $|f'|$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then

$$|C(f,g)| \leq \frac{|f'(a)|+|f'(b)|}{2\lambda(1)}\int_0^1 \lambda(t)\,dt \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt. \tag{40}$$

(ii) If $|f'|$ is h-convex integrable on $[a,b]$ and h is integrable on $(0,1)$, then

$$|C(f,g)| \leq \frac{|f'(a)|+|f'(b)|}{2}\int_0^1 h(t)\,dt \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt. \tag{41}$$

Proof. We use Sonin's identity

$$C(f,g) = \frac{1}{b-a}\int_a^b (f(t)-\mu)\left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt, \tag{42}$$

for $\mu = \frac{f(a)+f(b)}{2}$ to get

$$C(f,g) = \frac{1}{b-a}\int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right]\left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt. \tag{43}$$

Since f is differentiable, then we have

$$\begin{aligned} f(t) - \frac{f(a)+f(b)}{2} &= \frac{1}{2}\left[f(t) - f(a) + f(t) - f(b)\right] \\ &= \frac{1}{2}\left[\int_a^t f'(s)\,ds - \int_t^b f'(s)\,ds\right] \\ &= \frac{1}{2}\int_a^b sgn(t-s)\, f'(s)\,ds \end{aligned}$$

for any $t \in [a,b]$.

Therefore we have the representation:

$$C(f,g)=\frac{1}{2(b-a)}\int_a^b\left(\int_a^b sgn(t-s)f'(s)\,ds\right)\times\left(g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right)dt. \tag{44}$$

Taking the modulus we have

$$\begin{aligned}|C(f,g)| &\le \frac{1}{2(b-a)}\int_a^b\left|\int_a^b sgn(t-s)f'(s)\,ds\right| \\ &\times\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \\ &\le \frac{1}{2(b-a)}\int_a^b\left(\int_a^b |sgn(t-s)f'(s)|\,ds\right) \\ &\times\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \\ &= \frac{1}{2(b-a)}\int_a^b\left(\int_a^b |f'(s)|\,ds\right)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \\ &= \frac{1}{2(b-a)}\int_a^b |f'(s)|\,ds\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt.\end{aligned} \tag{45}$$

(i) Since $|f'|$ is λ-convex integrable on $[a,b]$, then by Corollary 2.15 we have

$$\frac{1}{b-a}\int_a^b |f'(s)|\,ds\le\frac{|f'(a)|+|f'(b)|}{2\lambda(1)}\int_0^1\lambda(t)\,dt$$

and by (45) we get (40).

(ii) Follows by (23) and the details are omitted. □

With the notations from the introduction we have:

Corollary 3.2. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If $|f'|$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then*

$$D^2(f),E^2(f)\le\frac{|f'(a)|+|f'(b)|}{2\lambda(1)}\int_0^1\lambda(t)\,dt\times\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt, \tag{46}$$

and

$$G^2(f)\le\frac{|f'(a)|+|f'(b)|}{2\lambda(1)}\int_0^1\lambda(t)\,dt\int_a^b\left||f(t)|-\frac{1}{b-a}\int_a^b |f(s)|\,ds\right|dt. \tag{47}$$

A similar result holds for h-convex functions.

Remark 3.3. Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $[a,b] \subset \mathring{I}$ and $g : [a,b] \to \mathbb{C}$ is an integrable function on $[a,b]$.

If $|f'|$ is convex on $[a,b]$, then

$$|C(f,g)| \leq \frac{|f'(a)| + |f'(b)|}{4} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{48}$$

If $|f'|$ is of P-type on $[a,b]$, then

$$|C(f,g)| \leq \frac{|f'(a)| + |f'(b)|}{2} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{49}$$

If $|f'|$ is Breckner s-convex on $[a,b]$, for $s \in (0,1)$, then

$$|C(f,g)| \leq \frac{|f'(a)| + |f'(b)|}{2(s+1)} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{50}$$

If $|f'|$is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$|C(f,g)| \leq \frac{|f'(a)| + |f'(b)|}{2(1-s)} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{51}$$

Remark 3.4. We notice, from the proof of Theorem 3.1, if $|f'|$ satisfies the second Hemite-Hadamard inequality with a certain term $R(|f'(a)|, |f'(b)|)$, i.e.

$$\frac{1}{b-a} \int_a^b |f'(u)|\, du \leq R\left(|f'(a)|, |f'(b)|\right),$$

then we have the inequality

$$|C(f,g)| \leq R\left(|f'(a)|, |f'(b)|\right) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{52}$$

The case of *p-norm* of the deviation

$$\left| f - \frac{1}{b-a} \int_a^b f(s)\, ds \right|$$

is as follows:

Theorem 3.5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $[a,b] \subset \mathring{I}$ and $g : [a,b] \to \mathbb{C}$ is an integrable function on $[a,b]$. Assume that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $|f'|^q$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then

$$|C(f,g)| \le (b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/q} \tag{53}$$
$$\times\left[\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}.$$

(ii) If $|f'|^q$ is h-convex integrable on $[a,b]$ and h is integrable on $(0,1)$, then

$$|C(f,g)| \le (b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\int_0^1 h(t)\,dt\right]^{1/q} \tag{54}$$
$$\times\left[\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}.$$

Proof. Making use of Hölder's integral inequality, we have

$$|C(f,g)| \le \frac{1}{2(b-a)}\int_a^b \left|\int_a^b sgn(t-s)\,f'(s)\,ds\right| \tag{55}$$
$$\times\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt$$
$$\le \frac{1}{2(b-a)}\left(\int_a^b \left|\int_a^b sgn(t-s)\,f'(s)\,ds\right|^q dt\right)^{1/q}$$
$$\times\left(\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}.$$

Observe that, by Jensen's integral inequality for $q>1$, we have

$$\left|\frac{\int_a^b sgn(t-s)\,f'(s)\,ds}{b-a}\right|^q \le \frac{\int_a^b |sgn(t-s)\,f'(s)|^q\,ds}{b-a}$$
$$= \frac{\int_a^b |f'(s)|^q\,ds}{b-a},$$

which shows that

$$\left|\int_a^b sgn(t-s)\,f'(s)\,ds\right|^q \le (b-a)^{q-1}\int_a^b \left|f'(s)\right|^q ds$$

for any $t\in[a,b]$.

Therefore,

$$\left(\int_a^b \left|\int_a^b sgn(t-s)\,f'(s)\,ds\right|^q dt\right)^{1/q} \le \left[(b-a)^q\int_a^b \left|f'(s)\right|^q ds\right]^{1/q}$$
$$= (b-a)\left[\int_a^b \left|f'(s)\right|^q ds\right]^{1/q}$$

and by (55) we get

$$|C(f,g)| \leq \frac{1}{2}\left(\int_a^b |f'(s)|^q ds\right)^{1/q}\left(\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^p dt\right)^{1/p}.$$

(i) Since $|f'|^q$ is λ-convex integrable on $[a,b]$, then by Corollary 2.15 we have

$$\frac{1}{b-a}\int_a^b |f'(s)|^q ds \leq \frac{|f'(a)|^q+|f'(b)|^q}{2\lambda(1)}\int_0^1 \lambda(t)\,dt$$

and by (55) we get (53).

(ii) Follows by (23) and the details are omitted. □

The case $p=q=2$ is of interest.

Corollary 3.6. *Let $f: I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $[a,b] \subset \mathring{I}$ and $g:[a,b]\to\mathbb{C}$ is an integrable function on $[a,b]$.*

If $|f'|^2$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then

$$|C(f,g)| \leq (b-a)\left[\frac{|f'(a)|^2+|f'(b)|^2}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/2} D(g), \tag{56}$$

where

$$D(g) = [C(g,\bar{g})]^{1/2} = \left[\frac{1}{b-a}\int_a^b |g(t)|^2 dt - \left|\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2\right]^{1/2}.$$

If $|f'|^2$ is h-convex integrable on $[a,b]$ and h is integrable on $(0,1)$, then a similar inequality is valid.

The following particular cases are of interest as well:

Corollary 3.7. *Let $f: I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, $[a,b]\subset\mathring{I}$ and $g:[a,b]\to\mathbb{C}$ is an integrable function on $[a,b]$. Assume that $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $|f'|^q$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then*

$$D^2(f), E^2(f) \leq (b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/q} \tag{57}$$

$$\times\left[\frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right]^{1/p},$$

and

$$G^2(f) \le (b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/q} \times \left[\frac{1}{b-a}\int_a^b \left||f(t)|-\frac{1}{b-a}\int_a^b |f(s)|\,ds\right|^p dt\right]^{1/p}. \tag{58}$$

In particular, if $|f'|^2$ *is* λ*-convex integrable on* $[a,b]$ *and* λ *is integrable on* $[0,1]$, *then*

$$D^2(f), E^2(f) \le (b-a)\left[\frac{|f'(a)|^2+|f'(b)|^2}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/2} D(f), \tag{59}$$

and

$$G^2(f) \le (b-a)\left[\frac{|f'(a)|^2+|f'(b)|^2}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/2} D(|f|). \tag{60}$$

The first inequality in (59) is equivalent to:

$$D(f) \le (b-a)\left[\frac{|f'(a)|^2+|f'(b)|^2}{2\lambda(1)}\int_0^1 \lambda(t)\,dt\right]^{1/2}. \tag{61}$$

Remark 3.8. Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $[a,b] \subset \mathring{I}$ and $g : [a,b] \to \mathbb{C}$ is an integrable function on $[a,b]$. Assume that $p,q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

If $|f'|^q$ is convex on $[a,b]$, then

$$|C(f,g)| \le (b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q} \times \left[\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}. \tag{62}$$

If $|f'|^q$ is of P-type on $[a,b]$, then

$$|C(f,g)| \le (b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \times \left[\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}. \tag{63}$$

If $|f'|^q$ is Breckner s-convex on $[a,b]$, for $s \in (0,1)$, then

$$|C(f,g)| \leq (b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2(s+1)}\right]^{1/q} \times \left[\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}. \tag{64}$$

If $|f'|^q$is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$|C(f,g)| \leq (b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2(1-s)}\right]^{1/q} \times \left[\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}. \tag{65}$$

4. Application for Riemann-Stieltjes Integral

The following representation is of interest in itself. The result was firstly obtained in [27] (see also [29]). For the sake a completeness we give here a short proof as well.

Lemma 4.1. *If $v : [a,b] \to \mathbb{C}$ is continuous (of bounded variation) on $[a,b]$ and $h : [a,b] \to \mathbb{C}$ is of bounded variation (continuous) on $[a,b]$, then we have the identity*

$$\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) = \int_a^b h(t)\,dv(t) - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt. \tag{66}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} &\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \\ &= \int_a^b \left[\frac{v(b)(t-a) + v(a)(b-t)}{b-a} - v(t)\right] dh(t) \\ &= \left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t)\right] h(t)\Big|_a^b \\ &\qquad - \int_a^b h(t)\,d\left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t)\right] = \end{aligned} \tag{67}$$

$$= [v(b) - v(b)]h(b) - [v(a) - v(a)]h(a) - \int_a^b h(t)\left[\frac{v(b)-v(a)}{b-a}dt - dv(t)\right]$$
$$= \int_a^b h(t)\,dv(t) - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt$$

and the identity is proven. □

We can provide now the following application for Riemann-Stieltjes integral:

Proposition 4.2. *If $v : I \to \mathbb{C}$ is twice differentiable on the interior of the interval I denoted $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If $|v''|$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$, then for $h : [a,b] \to \mathbb{C}$ integrable on $[a,b]$, we have the inequalities*

$$\left|\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t)\right| \tag{68}$$
$$\le \frac{|v''(a)| + |v''(b)|}{2\lambda(1)}(b-a)\int_0^1 \lambda(t)\,dt \int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right| dt$$

Proof. From (66) we have

$$\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \tag{69}$$
$$= \int_a^b h(t)\,v'(t)\,dt - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt = (b-a)\,C(v',h).$$

Since $|v''|$ is λ-convex integrable on $[a,b]$, then by applying Theorem 3.1for $f = v'$ and $g = h$ we deduce the desired result (68). □

Remark 4.3. If $v : I \to \mathbb{C}$ is twice differentiable on $\mathring{I}$, $[a,b] \subset \mathring{I}$ and $g : [a,b] \to \mathbb{C}$ is an integrable function on $[a,b]$.

If $|v''|$ is convex on $[a,b]$, then

$$\left|\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t)\right| \tag{70}$$
$$\le \frac{|v''(a)| + |v''(b)|}{4}(b-a)\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right| dt.$$

If $|v''|$ is of P-type on $[a,b]$, then

$$\left|\frac{v(b)\int_a^b(t-a)\,dh(t)+v(a)\int_a^b(b-t)\,dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\right| \tag{71}$$
$$\leq \frac{|v''(a)|+|v''(b)|}{2}(b-a)\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt.$$

If $|v''|$ is Breckner s-convex on $[a,b]$, for $s\in(0,1)$, then

$$\left|\frac{v(b)\int_a^b(t-a)\,dh(t)+v(a)\int_a^b(b-t)\,dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\right| \tag{72}$$
$$\leq \frac{|v''(a)|+|v''(b)|}{2(s+1)}(b-a)\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt.$$

If $|v''|$ is of s-Godunova-Levin type, with $s\in[0,1)$, then

$$\left|\frac{v(b)\int_a^b(t-a)\,dh(t)+v(a)\int_a^b(b-t)\,dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\right| \tag{73}$$
$$\leq \frac{|v''(a)|+|v''(b)|}{2(1-s)}(b-a)\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt.$$

Similar results may be stated if $|f'|^q$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$. However the details are not provided here.

5. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, \text{ for } -\infty<s\leq\lambda,\\[2ex] 0, \text{ for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{74}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [56, p. 256]:

Theorem 5.1 (Spectral Representation Theorem). *Let A be a bounded self-adjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;

b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) We have the representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{75}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon \tag{76}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{77}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{78}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5.2. *With the assumptions of Theorem 5.1 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \text{ for all } x \in H \tag{79}$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \text{ for all } x, y \in H. \tag{80}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \text{ for all } x \in H. \tag{81}$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 d\|E_\lambda x\|^2 \text{ for all } x \in H. \tag{82}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 5.1, see for instance [56, p. 258]:

Theorem 5.3. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 5.1, then $F_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$ where E_λ is defined by (74).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A.

We have:

Theorem 5.4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of A.*

If $f : I \to \mathbb{C}$ is twice differentiable on $\mathring{I}$ and $[m,M] \subset \mathring{I}$, $|f''|$ is λ-convex integrable on $[m,M]$ and λ is integrable on $[0,1]$, then we have the inequalities

$$\begin{aligned} &\left|\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m}\right] x,y\right\rangle - \langle f(A)x,y\rangle\right| \\ &\le \frac{|f''(m)| + |f''(M)|}{2\lambda(1)} (M-m) \int_0^1 \lambda(t)\,dt \\ &\times \int_{m-0}^{M} \left|\langle E_t x,y\rangle - \frac{1}{M-m}\int_{m-0}^{M} \langle E_s x,y\rangle\,ds\right| dt \\ &\le \frac{|f''(m)| + |f''(M)|}{4\lambda(1)} (M-m)^2 \|x\|\,\|y\| \int_0^1 \lambda(t)\,dt \end{aligned} \tag{83}$$

for any $x,y \in H$.

Proof. Let $x,y \in H$ and consider $h : \mathbb{R} \to \mathbb{C}$, $h(t) := \langle E_t x,y\rangle$.

If we use the first inequality in (68) for the interval $[m-\varepsilon,M]$ with small

$\varepsilon > 0$, we have

$$\left| \frac{f(M)\int_{m-\varepsilon}^{M}(t-m+\varepsilon)\,d\langle E_t x,y\rangle + f(m-\varepsilon)\int_{m-\varepsilon}^{M}(M-t)\,d\langle E_t x,y\rangle}{M-m+\varepsilon} \right. \tag{84}$$
$$\left. -\int_{m-\varepsilon}^{M} f(t)\,d\langle E_t x,y\rangle \right|$$
$$\leq \frac{|f''(m-\varepsilon)|+|f''(M)|}{2\lambda(1)}(M-m+\varepsilon)^2\int_0^1 \lambda(t)\,dt$$
$$\times \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\left|\langle E_t x,y\rangle - \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\langle E_s x,y\rangle\,ds\right| dt.$$

Taking the limit over $\varepsilon \to 0+$ and using the Spectral representation theorem, we have

$$\left|\left\langle \left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle - \langle f(A)x,y\rangle\right| \tag{85}$$
$$\leq \frac{|f''(m)|+|f''(M)|}{2\lambda(1)}(M-m)^2\int_0^1 \lambda(t)\,dt$$
$$\times \frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x,y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right| dt$$

for any $x,y \in H$.

By the Schwarz inequality in H we have that

$$\int_{m-0}^{M}\left|\langle E_t x,y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right| dt \tag{86}$$
$$= \int_{m-0}^{M}\left|\left\langle\left[E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right],y\right\rangle\right| dt$$
$$\leq \|y\|\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt$$

for any $x,y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \tag{87}$$
$$\leq (M-m)^{1/2}\left(\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt\right)^{1/2}$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$\frac{1}{M-m}\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt \tag{88}$$
$$=\frac{1}{M-m}\int_{m-0}^{M}\|E_t x\|^2 dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2$$

and

$$\frac{1}{M-m}\int_{m-0}^{M}\|E_t x\|^2 dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 \tag{89}$$
$$=\frac{1}{M-m}\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt$$

for any $x\in H$.

By (87), (88) and (89) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \tag{90}$$
$$\leq (M-m)^{1/2}\left(\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt\right)^{1/2}$$

for any $x\in H$.

On making use of the Schwarz inequality in H we also have

$$\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt \tag{91}$$
$$\leq \int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|\left\|E_t x-\frac{1}{2}x\right\| dt$$
$$=\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt,$$

where we used the fact that E_t are projectors, and in this case we have

$$\left\|E_t x-\frac{1}{2}x\right\|^2=\|E_t x\|^2-\langle E_t x,x\rangle+\frac{1}{4}\|x\|^2=\frac{1}{4}\|x\|^2$$

for any $t\in[m,M]$ for any $x\in H$.

From (90) and (91) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \tag{92}$$
$$\leq (M-m)^{1/2}\left(\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt\right)^{1/2},$$

which is clearly equivalent with the following inequality of interest in itself

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt\leq\frac{1}{2}\|x\|(M-m) \tag{93}$$

for any $x\in H$.

From (86) we then get

$$\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle ds\right|dt\leq\frac{1}{2}\|x\|\|y\|$$

for any $x,y\in H$. □

Remark 5.5. If $|f''|$ is convex on $[m,M]$, then we have the inequalities

$$\begin{aligned}&\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \\ &\leq\frac{|f''(m)|+|f''(M)|}{4}(M-m) \\ &\times\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle ds\right|dt \\ &\leq\frac{|f''(m)|+|f''(M)|}{8}(M-m)^2\|x\|\|y\|,\end{aligned} \tag{94}$$

for any $x,y\in H$.

Example 5.6. a) Let A be a bounded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of A. Then by Theorem 5.4 we have for $f(t)=t^p$, $p\geq 3$ that

$$\begin{aligned}&\left|\left\langle\left[\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle A^p x,y\rangle\right| \\ &\leq p(p-1)\frac{m^{p-2}+M^{p-2}}{4}(M-m) \\ &\times\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle ds\right|dt \\ &\leq p(p-1)\frac{m^{p-2}+M^{p-2}}{8}(M-m)^2\|x\|\|y\|\end{aligned} \tag{95}$$

for any $x,y\in H$.

b) With the assumptions of a) and if $m > 0$, then by Theorem 5.4 we have for $f(t) = \ln t$, that

$$\left| \left\langle \left[\frac{\ln m (M1_H - A) + \ln M (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln A x, y \rangle \right| \tag{96}$$
$$\leq \frac{m^2 + M^2}{4m^2 M^2} (M - m) \int_{m-0}^{M} \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \right| dt$$
$$\leq \frac{m^2 + M^2}{8m^2 M^2} (M - m)^2 \|x\| \|y\|$$

for any $x, y \in H$.

Similar results may be stated if $|f'|^q$ is λ-convex integrable on $[a,b]$ and λ is integrable on $[0,1]$. However the details are not provided here.

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