Picard, Rainer and Seilder, Stefan and Trostorff, Sascha and Waurick, Marcus (2016) On abstract grad-div systems. Journal of Differential Equations, 260 (6). pp. 4888-4917. ISSN 0022-0396 , http://dx.doi.org/10.1016/j.jde.2015.11.033

On Abstract grad–div Systems.

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Abstract. For a large class of dynamical problems from mathematical physics the skew-selfadjointness of a spatial operator of the form $A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$, where $C : D(C) \subseteq H_0 \to H_1$ is a closed densely defined linear operator, is a typical property. Guided by the standard example, where $C = \operatorname{grad} = \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$ (and $-C^* = \operatorname{div}$, subject to suitable boundary constraints), an abstract class of operators $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ is introduced (hence the title). As a particular application we consider a non-standard coupling mechanism and the incorporation of diffusive boundary conditions both modeled by setting associated with a skew-selfadjoint spatial operator A.

Keywords and phrases: Evolutionary equations, Gelfand triples, Guyer-Krumhansl heat conduction, Dynamic boundary conditions, Leontovich boundary condition

Mathematics subject classification 2010: 35F45, 46N20, 47N20

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0 Introduction

In a number of studies [21, 20, 14, 33, 29, 22] it has been illustrated, that typical initial boundary value problems of mathematical physics can be represented in the general form

$$(\partial_0 \mathcal{M} + A)\, U = F, \tag{0.1}$$

where A is skew-selfadjoint, indeed commonly of the specific block matrix form

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \tag{0.2}$$

with $C : X_1 \subseteq X_0 \to Y$ a closed densely defined linear operator between Hilbert spaces X_0 and Y with $X_1 = D(C)$, see e.g. [15, 16]. The operator $\mathcal{M}$ is referred to as the material law operator, which in the situation discussed here is a linear operator acting on a Hilbert space realizing the space-time the problems are formulated in, [24, 34, 21, 15].

The main purpose of this paper is to focus on the operator C in this construction of the operator $A : D\,(C) \times D\,(C^*) \subseteq X_0 \oplus Y \to X_0 \oplus Y$ when Y is itself a direct sum of Hilbert spaces. In such a situation we shall loosely refer to a system of the form (0.1) as an abstract grad–div system. The guiding example, which at the same time motivates the name, is to take for $C :$ $X_1 \subseteq X_0 \to Y$ the differential operator $\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$ with domain $X_1 = \mathring{H}_1\,(\Omega)$, the Sobolev space of weakly differentiable functions with L^2-derivatives and vanishing boundary data. The spaces X_0 and Y would be in this case $L^2\,(\Omega)$ and $L^2\,(\Omega)^3 \left(= L^2\,(\Omega) \oplus L^2\,(\Omega) \oplus L^2\,(\Omega)\right)$, respectively. The corresponding equation of the form (0.1) would lead in particular to a model for the acoustic wave propagation or – depending on the material law operator $\mathcal{M}$ – for the dissipation of heat with so-called Dirichlet boundary condition, see e.g. [15, 23, 32]. The adjoint operator is the negative weak divergence $C^* = -\operatorname{div}$. It is

$$C = \begin{pmatrix} \partial_1|_{X_1} \\ \partial_2|_{X_1} \\ \partial_3|_{X_1} \end{pmatrix}. \tag{0.3}$$

The idea in this paper is to replace the role of the partial derivatives in (0.3) by general operators in general Hilbert spaces, hence the term abstract grad-div systems for the corresponding evolutionary systems associated with the skew-selfadjoint operator A constructed according to (0.2). That the study of this particular class of skew-selfadjoint operators leads to interesting applications is illustrated by three examples. First we consider the so-called Guyer-Krumhansl model of thermo-dynamics, [6, 7, 8], as a particular instance of this construction. The last two examples illustrate that the concept of grad–div systems is well-suited to study dynamic boundary conditions. A first implementation, leading up to the concept of grad–div systems introduced here, has been established in the context of a particular class of boundary control problems, [18, 17]. Our investigation of grad–div systems has been motivated by the discussion of a heat conduction problem with a dissipative dynamic boundary condition in [4]. The generalization to the context of evolutionary equations is the topic of our second example. The last application example is connected to the Leontovich boundary condition of electrodynamics, see e.g. [12, 5, 25]. We shall discuss two different implementations. The first one is based on classical boundary trace concepts (see e.g. [2, 35]), the second approach uses a boundary data concept developed in [18, 17] with extensive use in [28], which has the advantage that no constraints on the quality of the boundary are incurred.

In Section 1 we start by presenting the fundamental construction of abstract grad–div systems. The remaining section is then dedicated to illustrating the usefulness of the concept by examples.

1 Construction of Abstract grad–div Systems.

In this section, we shall reconsider the concept of the adjoint operator of a densely defined, closed linear operator C, specifically in order to deal with the fact that the image space Y of the operator C is given as an orthogonal sum of Hilbert spaces. Let us first provide a precise definition of what we would like to call an abstract grad-div-system.

Definition 1.1. Let $C : X_1 \subseteq X_0 \to Y$, be a densely defined, closed linear operator with domain X_1 between Hilbert spaces X_0, Y. We shall refer to a system of the form (0.1) with A generated via (0.2), as an *abstract* grad–div *system,* if Y given as a direct sum, i.e. $Y := \bigoplus_{k\in\{1,...,n\}} Y_k$, for Hilbert spaces Y_k, $k \in \{1, \dots, n\}$, $n \in \mathbb{N}$.

As a matter of jargon we shall say that the abstract grad–div system is *generated by* C. If ι_{Y_k} denotes the canonical isometric embedding of Y_k into Y then, with $C_k := \iota_{Y_k}^* C$, $k \in \{1, \dots, n\}$, we have

$$Cx = C_1 x \oplus \cdots \oplus C_n x = \begin{pmatrix} C_1 x \\ \vdots \\ C_n x \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} x \in \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = Y$$

for $x \in X_1$.

The aim of this section is to give a characterization of the adjoint C^* of C. However, before stating and proving the respective theorem, we shall explore as a first elementary example the name-giving case mentioned in the introduction:

Example 1.2. Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ non-empty and open, $X_0 := L^2(\Omega)$ and $X_1 := \mathring{H}_1(\Omega)$, the Sobolev space of weakly differentiable functions in $L^2(\Omega)$ with vanishing Dirichlet trace, i.e., the closure of compactly supported continuously differentiable functions with respect to the H_1-norm $f \mapsto \sqrt{\sum_{\alpha\in\mathbb{N}_0^n, |\alpha|\leq 1} |\partial^\alpha f|^2_{L^2(\Omega)}}$. Then clearly, with $Y = \bigoplus_{k\in\{1,...,n\}} L^2(\Omega)$ the operator $C := \mathring{\operatorname{grad}} : \mathring{H}_1(\Omega) \subseteq L^2(\Omega) \to \bigoplus_{k\in\{1,...,n\}} L^2(\Omega)$, defined as the closure of the classical vector-analytical operation grad restricted to smooth functions with compact support in Ω, generates an abstract grad-div-system. In the same way, choosing $X_1 = H_1(\Omega)$ instead, we get that $C = \operatorname{grad}$ also generates an abstract grad-div-system. We recall that the difference between the operators grad and $\mathring{\operatorname{grad}}$ lies in the additionally prescribed homogeneous Dirichlet boundary condition for elements in the domain of $\mathring{\operatorname{grad}}$.

To clarify notation, we need the following definition.

Definition 1.3. Let X, Y be Hilbert spaces and $L : X \to Y$ a bounded linear operator. Then we denote the dual operator of L by $L' : Y' \to X'$ defined by

$$\left(L'y'\right)(x) \coloneqq y'(Lx) \quad (y' \in Y', x \in X).$$

Moreover, we may consider a densely defined linear operator $S : X_1 \subseteq X_0 \to Y$ such that

$$\begin{aligned} L_S : X_1 &\to Y \\ x &\mapsto Sx \end{aligned}$$

is a continuous linear operator (S need not be closable) and define the operator $S^\diamond : Y \to X_1'$ by $S^\diamond \coloneqq L_S' \circ R_{Y'}$. Here $R_{Y'} : Y \to Y'$ denotes the Riesz isomorphism, given by

$$\left(R_{Y'}y\right)(z) \coloneqq \langle y|z\rangle_Y \quad (y, z \in Y).$$

Note that we have by definition

$$\left(S^\diamond y\right)(x) = \left(L_S'(R_{Y'}y)\right)(x) = \left(R_{Y'}y\right)(Sx) = \langle y|Sx\rangle_Y$$

for $y \in Y, x \in X$. As a matter of convenience this can be written suggestively as

$$\langle S^\diamond y|x\rangle_{X_0} = \langle y|Sx\rangle_Y,$$

where $\langle \cdot | \cdot \rangle_{X_0}$ denotes here the continuous extension of the inner product of X_0 to a duality pairing on $X_1' \times X_1$. Note that $X_1 \hookrightarrow X_0 \overset{R_{X_0'}}{=} X_0' \hookrightarrow X_1'$ is a Gelfand triple.

Lemma 1.4. *Let X_0, X_1, Y be Hilbert spaces, $X_1 \subseteq X_0$ dense, $C \colon X_1 \subseteq X_0 \to Y$ a closed linear operator. Then the adjoint C^* of the operator C is densely defined and its adjoint C^* is given by*

$$C^* = C^\diamond \cap (Y \oplus X_0),$$

which is the same as to say

$$C^* = \left\{(y, x) \in Y \oplus X_1' \mid C^\diamond y = x \text{ and } x \in X_0\right\}.$$

Proof. Note that, by definition, $C^* \subseteq C^\diamond$ and $C^* \subseteq Y \oplus X_0$. Hence, $C^* \subseteq C^\diamond \cap (Y \oplus X_0)$. For the remaining implication, let $(y, x) \in Y \oplus X_0$. Then we have

$$\begin{aligned} (y, x) \in C^* &\iff \forall \phi \in X_1 \colon \langle y|C\phi\rangle_Y = \langle x|\phi\rangle_{X_0} \\ &\iff \forall \phi \in X_1 \colon C^\diamond y(\phi) = \langle x|\phi\rangle_{X_0}. \end{aligned}$$

This establishes the assertion. □

The latter observation has been employed in more concrete situation in control theory, see e.g. [17, 18, 19] and, in particular, the references in [19].

Lemma 1.5. *Let $C : X_1 \subseteq X_0 \to Y$ generate an abstract* grad–div *system with $Y = \bigoplus_{k \in \{1,\dots,n\}} Y_k$. Then*

$$C^\diamond = \begin{pmatrix} C_1^\diamond & \cdots & C_n^\diamond \end{pmatrix} : Y \ni \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \sum_{k=1}^{n} C_k^\diamond y_k \in X_1'$$

Proof. We have for $x \in X_1$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = y_1 \oplus \cdots \oplus y_n \in Y$,

$$(C^\diamond y)(x) = \langle y|Cx\rangle_Y = \sum_{k=1}^{n} \langle y_k|C_k x\rangle_{Y_k} = \sum_{k=1}^{n} \left(C_k^\diamond y_k\right)(x)\,,$$

hence, $C^\diamond y = \sum_{k=1}^n C_k^\diamond y_k$. □

The latter two lemmas yield the proof of the following theorem:

Theorem 1.6. *Let C generate an abstract* grad–div *system with* $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$. *Then*

$$\begin{aligned} C^* &= \begin{pmatrix} C_1^\diamond & \cdots & C_n^\diamond \end{pmatrix} \cap (Y \oplus X_0) \\ &= \left\{ ((y_1, \ldots, y_n), x) \in Y \oplus X_0 \,\middle|\, x = \sum_{k=1}^{n} C_k^\diamond y_k \in X_0 \right\}. \end{aligned}$$

Let us apply the latter theorem to the case $C = \mathring{\operatorname{grad}}$, as it was defined in Example 1.2.

Example 1.7. Let $X_0 = L^2(\Omega), X_1 := \mathring{H}_1(\Omega)$ and $Y_k := L^2(\Omega)$ for $k \in \{1, \ldots, n\}$ and some non-empty, open set $\Omega \subseteq \mathbb{R}^n$. Recall that $C = \mathring{\operatorname{grad}}$ generates an abstract grad-div-system. As in this case, $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ with $C_k : \mathring{H}_1(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)$ is defined as $C_k f = \partial_k f$ for $f \in \mathring{H}_1(\Omega)$, we obtain in particular that

$$\left(C_k^\diamond g\right)(f) = \langle g|\partial_k f\rangle_{L^2(\Omega)} = -\left(\partial_k g\right)(f)$$

for $g \in L^2(\Omega), f \in \mathring{C}_\infty(\Omega)$, where here $\partial_k g$ is meant in the sense of distributions. Here $\mathring{C}_\infty(\Omega)$ denotes the set of arbitrarily differentiable functions with compact support in Ω. In consequence

$$C^\diamond \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = -\sum_{k=1}^{n} \partial_k g_k$$

and thus, $C^* = -\operatorname{div}$ is the negative distributional divergence on $L^2(\Omega)$ vector-fields restricted to those, whose divergence is representable as an $L^2(\Omega)$-function.

An immediate application of Theorem 1.6 is the following corollary, which will be of interest in the next section:

Corollary 1.8. *Let C generate an abstract* grad-div-*system. Assume that there exists a closed, densely defined operator $\mathring{C}_1$ with*

$$\begin{pmatrix} \mathring{C}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = C.$$

Then

$$C^* = C^\diamond \cap \left(\mathring{C}_1^* \, 0 \cdots 0\right) \subseteq \left(\mathring{C}_1^* \, 0 \cdots 0\right).$$

Proof. It suffices to observe that $C^\diamond \subseteq \left(\mathring{C}_1^\diamond \, 0 \cdots 0\right)$ and to apply Theorem 1.6. □

The next statement contains a typical situation of operators of the form $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ giving rise to the generation of abstract grad–div systems.

Proposition 1.9. *Let $X_0, Y_0, \ldots, Y_n$ be Hilbert spaces, $C_0 \colon D(C_0) \subseteq X_0 \to Y_0$ densely defined and closed. Denote $X_1 := \left(D(C_0), \sqrt{|\cdot|^2 + |C_0 \cdot|^2}\right)$ and let $C_k \in L(X_1, Y_k)$, $k \in \{1, \ldots, n\}$. Then*

$$C = \begin{pmatrix} C_0 \\ \vdots \\ C_n \end{pmatrix} : D(C_0) \subseteq X_0 \to \bigoplus_{k \in \{0,\ldots,n\}} Y_k$$

generates an abstract grad–div *system.*

Proof. As C_0 is densely defined, so is C. Thus, closedness of C is the only thing remaining to check. For this let $(x_j)_{j\in\mathbb{N}}$ be a sequence in $D(C_0)$ convergent in X_0 its limit being denoted by $x \in X_0$ and with the property that $y_k := \lim_{j\to\infty} C_k x_j$ exists in Y_k, $j \in \{0, \ldots, n\}$. By the closedness of C_0, we infer $x \in D(C_0)$ and $C_0 x = y_0$. Thus, $x_j \to x$ in X_1 as $j \to \infty$. The continuity of C_k for all $k \in \{1, \ldots, n\}$ yields the assertion. □

Remark 1.10. Note that the operators C_k, $k \in \{1, \ldots, n\}$, need not be closable with domain $D(C_0)$. The following example illustrates this fact. Take $X_0 := L^2(0,1)$, $C_0 := \partial \colon H^1(0,1) \subseteq L^2(0,1) \to L^2(0,1), f \mapsto f'$ with f' denoting the distributional derivative. Then C_0 is densely defined and closed. It is known that $H_1(0,1) \subseteq C[0,1]$ (Sobolev embedding theorem). Hence, $C_1 f := \delta_{\{1/2\}} f := f(1/2)$ for $f \in H_1(0,1)$ defines a continuous linear operator from $H_1(0,1)$ to $\mathbb{C}$. It is easy to see that $C_1 \colon H_1(0,1) \subseteq L^2(0,1) \to L^2(0,1)$ is *not* closable.

2 Some Applications

Before we come to specific applications, we introduce some classical differential operators, which we will need in their description. Throughout, let Ω be a non-empty open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 2.1. We define the operator $\mathring{\mathrm{grad}}$ as the closure of the operator

$$\begin{aligned} \mathring{C}_\infty(\Omega) \subseteq L^2(\Omega) &\to L^2(\Omega)^n \\ \phi &\mapsto (\partial_i \phi)_{i\in\{1,\ldots,n\}}. \end{aligned}$$

Obviously, $D(\mathring{\text{grad}}) = \mathring{H}_1(\Omega)$ and thus, this definition coincides with the previous definition given in Example 1.2. Similarly, we define $\mathring{\text{div}}$ as the closure of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^n \subseteq L^2(\Omega)^n &\to L^2(\Omega) \\ (\phi_i)_{i\in\{1,\ldots,n\}} &\mapsto \sum_{i=1}^n \partial_i \phi_i. \end{aligned}$$

Moreover, we define the gradient on vector-fields, denoted by the same symbol $\mathring{\text{grad}}$, as the closure of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^n \subseteq L^2(\Omega)^n &\to L^2(\Omega)^{n\times n} \\ (\phi_i)_{i\in\{1,\ldots,n\}} &\mapsto (\partial_k \phi_i)_{i,k\in\{1,\ldots,n\}} \end{aligned}$$

and likewise the divergence of matrix-valued functions, again denoted by $\mathring{\text{div}}$, as the closure of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^{n\times n} \subseteq L^2(\Omega)^{n\times n} &\to L^2(\Omega)^n \\ (\phi_{ik})_{i,k\in\{1,\ldots,n\}} &\mapsto \left(\sum_{k=1}^n \partial_k \phi_{ik}\right)_{i\in\{1,\ldots,n\}}. \end{aligned}$$

In the special case $n = 3$ we define the operator $\mathring{\text{curl}}$ as the closure of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^3 \subseteq L^2(\Omega)^3 &\to L^2(\Omega)^3 \\ \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} &\mapsto \begin{pmatrix} \partial_2\phi_3 - \partial_3\phi_2 \\ \partial_3\phi_1 - \partial_1\phi_3 \\ \partial_1\phi_2 - \partial_2\phi_1 \end{pmatrix}. \end{aligned}$$

By integration by parts we derive the following relations

$$\begin{aligned} \mathring{\text{grad}} &\subseteq -(\mathring{\text{div}})^* =: \text{grad}, \\ \mathring{\text{div}} &\subseteq -(\mathring{\text{grad}})^* =: \text{div}, \\ \mathring{\text{curl}} &\subseteq (\mathring{\text{curl}})^* =: \text{curl}. \end{aligned}$$

It is easy to see that for the scalar-valued gradient, we have $D(\text{grad}) = H_1(\Omega)$ and thus, the definition of grad coincides with the previous one in Example 1.2. We recall that $u \in D(\mathring{\text{grad}})$ satisfies $u|_{\partial\Omega} = 0$, $v \in D(\mathring{\text{div}})$ satisfies $v|_{\partial\Omega} \cdot n = 0$ and $w \in D(\mathring{\text{curl}})$ satisfies $w|_{\partial\Omega} \times n = 0$ for domains Ω with smooth boundary, where n denotes the unit outward normal vector field on $\partial\Omega$. As the operators $\mathring{\text{grad}}$, $\mathring{\text{div}}$ and $\mathring{\text{curl}}$ can be defined for arbitrary open sets Ω, we will use them to formulate the respective generalized boundary conditions, which do not need to make use of classical boundary traces.

Beyond the aforementioned spatial differential operators, we need a suitable realization of the temporal derivative, see also [16, Example 2.3] or [10, Section 2] for a brief discussion.

Definition 2.2. Let $\varrho > 0$ and H be a Hilbert space. We consider the weighted L^2-space

$$H_{\varrho,0}(\mathbb{R}; H) := \left\{ f : \mathbb{R} \to H \,|\, f \text{ measurable}, \int_{\mathbb{R}} |f(t)|_H^2 e^{-2\varrho t} \,\mathrm{d}t < \infty \right\}$$

of H-valued functions and equip it with the following inner product

$$\langle f|g\rangle_{\varrho,0} := \int_{\mathbb{R}} \langle f(t)|g(t)\rangle_H e^{-2\varrho t} \,\mathrm{d}t.$$

We define the derivative $\partial_{0,\varrho}$ on $H_{\varrho,0}(\mathbb{R}; H)$ as the closure of

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}; H) \subseteq H_{\varrho,0}(\mathbb{R}; H) &\to H_{\varrho,0}(\mathbb{R}; H) \\ \phi &\mapsto \phi'. \end{aligned}$$

If the choice of ϱ is clear from the context, we will omit the index ϱ and just write ∂_0.

Remark 2.3. (a) The subscript 0 in the operator ∂_0 should remind of the "zero'th" coordinate, which in Physics' literature often plays the role of the time-derivative. We shall adopt this custom here.

(b) By definition the operator $e^{-\varrho m} : H_{\varrho,0}(\mathbb{R}; H) \to L^2(\mathbb{R}; H), f \mapsto (t \mapsto e^{-\varrho t} f(t))$ is unitary. Consequently $\mathcal{L}_\varrho := \mathcal{F} e^{-\varrho m} : H_{\varrho,0}(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$ is unitary, where we denote by $\mathcal{F} : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$ the unitary Fourier-transform given by

$$(\mathcal{F}\phi)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}xy} \phi(y) \,\mathrm{d}y \quad (x \in \mathbb{R}, \phi \in \mathring{C}_\infty(\mathbb{R}; H)).$$

One can show that

$$\partial_0 = \mathcal{L}_\varrho^* (\mathrm{i}m + \varrho) \mathcal{L}_\varrho, \tag{2.1}$$

where $m : D(m) \subseteq L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H), f \mapsto (t \mapsto tf(t))$ with maximal domain, yielding that m is self-adjoint, and, by (2.1), proving that ∂_0 has is continuously invertible. Moreover, from (2.1) we read off that $\sigma(\partial_0)$, the spectrum of ∂_0, coincides with $i[\mathbb{R}] + \varrho$.

Let $\varrho_0 > 0$ and $M : B_{\mathbb{C}}\left(\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right) \to L(H)$ be analytic such that

$$\mathfrak{Re}\langle z^{-1} M(z) x | x\rangle_H \geq 0 \tag{2.2}$$

for every $z \in B_{\mathbb{C}}\left(\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right)$, $x \in H$. Defining

$$M\left(\frac{1}{\mathrm{i}m + \varrho}\right) : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H),\ f \mapsto \left(t \mapsto M\left(\frac{1}{\mathrm{i}t + \varrho}\right) f(t)\right)$$

for $\varrho > \varrho_0$ we can generate operator-valued multipliers and via $M(\partial_0^{-1}) := \mathcal{L}_\varrho^* M\left(\frac{1}{\mathrm{i}m+\varrho}\right) \mathcal{L}_\varrho$ corresponding operator-valued functions of ∂_0^{-1}. We shall loosely refer to such operators $M\left(\partial_0^{-1}\right)$ as material law operators.

We quote the following well-posedness result.

Theorem 2.4 ([16, Theorem 6.2.5], [15, Solution Theory]). *Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ a skew-selfadjoint linear operator. Moreover, let $\varrho_0 > 0$ and $M : B_{\mathbb{C}}\left(\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right) \to L(H)$ be analytic such that there exists $c > 0$ with*

$$\mathfrak{Re}\langle z^{-1}M(z)x|x\rangle_H \geq c|x|_H^2 \tag{2.3}$$

for every $z \in B_{\mathbb{C}}\left(\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right)$, $x \in H$. Then for each $\varrho > \varrho_0$ the operator $\left(\overline{\partial_0 M(\partial_0^{-1}) + A}\right)^{-1}$ is boundedly invertible on $H_{\varrho,0}(\mathbb{R}; H)$ and causal.

Remark 2.5. (a) In many applications, M has the simple form $M(z) = M_0 + zM_1$ for some $M_0, M_1 \in L(H)$. Condition (2.3) can then for example be achieved for ϱ_0 large enough, if M_0 is selfadjoint and there exist $c_0, c_1 > 0$ such that

$$\langle M_0 x|x\rangle_H \geq c_0|x|_H^2 \text{ and } \mathfrak{Re}\langle M_1 y|y\rangle_H \geq c_1|y|_H^2$$

for every x belonging to $M_0[H]$, the range of M_0, and every y from $N(M_0)$, the nullspace of M_0. The corresponding material law operator is then

$$M(\partial_0^{-1}) = M_0 + \partial_0^{-1}M_1.$$

(b) For the interested reader, we shall also mention possible generalization of Theorem 2.4 to non-autonomous equations ([24, 34]) or non-linear equations ([30, 27])

2.1 The Guyer-Krumhansl model of heat conduction.

The Guyer-Krumhansl model of heat conduction (see e.g. [6, 7, 8]) consists of two equations: A first equation, the balance equation, relates the heat θ to the heat flux q in the way that

$$\varrho c\partial_0\theta + \operatorname{div} q = h,$$

where ϱ and c are certain material parameters and h is a given source term. The difference to the classical equations of thermodynamics is the following alternative to Fourier's law:

$$(1 + \tau_0\partial_0)\, q = -\kappa \operatorname{grad}\theta + \mu_1\Delta q + \mu_2 \operatorname{grad}\operatorname{div} q,$$

where τ_0, κ, μ_1, μ_2 are real numbers modeling the material's properties. We mention that the choices $\tau_0 = \mu_1 = \mu_2 = 0$ recover the standard Fourier law. In any case, κ is assumed to be strictly positive. With this observation, we may reformulate the modified Fourier's law as

$$\left(\tau_0\kappa^{-1}\partial_0 + \kappa^{-1} - \mu_1\kappa^{-1}\Delta - \mu_2\kappa^{-1}\operatorname{grad}\operatorname{div}\right) q = -\operatorname{grad}\theta.$$

Here we emphasize that here Δ – as does $\operatorname{grad}\operatorname{div}$ – acts on vectors with 3 components. Summarizing, we end up with a system of the form

$$\begin{pmatrix} \varrho c\partial_0 & \operatorname{div} \\ \operatorname{grad} & \left(\tau_0\kappa^{-1}\partial_0 + \kappa^{-1} - \mu_1\kappa^{-1}\Delta - \mu_2\kappa^{-1}\operatorname{grad}\operatorname{div}\right) \end{pmatrix} \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix}$$

and thus, the block operator matrix to be studied reads as follows:

$$\begin{pmatrix} \varrho c\partial_0 & \operatorname{div} \\ \operatorname{grad} & \left(\tau_0\kappa^{-1}\partial_0 + \kappa^{-1} - \mu_1\kappa^{-1}\Delta - \mu_2\kappa^{-1}\operatorname{grad}\operatorname{div}\right) \end{pmatrix}.$$

In the following we show that the latter block operator matrix is of the form of abstract evolutionary equations as mentioned in the introduction. Rearranging terms and separating the spatial derivatives from the time-derivative, the operator becomes

$$\partial_0 \begin{pmatrix} \varrho c & 0 \\ 0 & \tau_0 \kappa^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \kappa^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & \left(-\mu_1 \kappa^{-1} \Delta - \mu_2 \kappa^{-1} \operatorname{grad} \operatorname{div}\right) \end{pmatrix}.$$

In order to obtain easily accessible boundary conditions leading to the well-posedness of the latter equation, we give a different representation of the term $\left(-\mu_1 \kappa^{-1} \Delta - \mu_2 \kappa^{-1} \operatorname{grad} \operatorname{div}\right)$ on the basis of our theory of abstract grad-div-systems. Beforehand, however, we will show the following lemma of combinatoric nature:

Lemma 2.6. *Let $\mu_2 \in \mathbb{R}$, $\mu_1, \kappa \in \mathbb{R}_{>0}$ with $\mu_1 > -\mu_2$. Then there exists $C = C^* \in L(\mathbb{C}^{3\times 3}, \mathbb{C}^{3\times 3})$ strictly positive definite such that for all $\phi \in C_\infty\left(\overline{\Omega}\right)^3$ we have*[1]

$$\left(\kappa^{-1} \mu_1 \Delta + \kappa^{-1} \mu_2 \operatorname{grad} \operatorname{div}\right) \phi = \operatorname{div} C \operatorname{grad} \phi.$$

Proof. Without restriction assume that $\kappa = 1$. As an ansatz, take

$$C = \alpha_0 \mathrm{sym}_0 + \alpha_1 \mathbb{P} + \alpha_2 \mathrm{skew} \tag{2.4}$$

for $\alpha_0, \alpha_1, \alpha_2 \in \left]0, \infty\right[$ to be determined later on, where

$$\mathrm{sym}_0, \mathrm{sym}, \mathbb{P}, \mathrm{skew}\colon \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}$$

are defined by

$$\begin{aligned} \mathbb{P} &:= \frac{1}{3} \operatorname{trace}^* \operatorname{trace}, \\ \mathrm{sym}\, \sigma &:= \frac{1}{2}\left(\sigma + \sigma^\top\right) \quad \left(\sigma \in \mathbb{C}^{3\times 3}\right), \\ \mathrm{skew}\, \sigma &:= \frac{1}{2}\left(\sigma - \sigma^\top\right) \quad \left(\sigma \in \mathbb{C}^{3\times 3}\right), \\ \mathrm{sym}_0 &:= (1 - \mathbb{P})\, \mathrm{sym} = \mathrm{sym}\, (1 - \mathbb{P}) \end{aligned}$$

with $\operatorname{trace} : \mathbb{C}^{3\times 3} \to \mathbb{C}, \sigma \mapsto \sum_{i=1}^3 \sigma_{ii}$. Observe that

$$\begin{aligned} \operatorname{trace}^* : \mathbb{C} &\to \mathbb{C}^{3\times 3}, \\ z &\mapsto z \mathbb{I}_{3\times 3}, \end{aligned}$$

where $\mathbb{I}_{3\times 3}$ denotes the identity matrix in $\mathbb{C}^{3\times 3}$ and, hence, $\mathbb{P}\sigma = \frac{\operatorname{trace} \sigma}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Observing

[1] Recall that here Δ is the Laplacian acting as $\begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}$ and that div on the right-hand side is the row-wise divergence and the grad on the right-hand side is the Jacobian of ϕ.

that $\operatorname{trace} \operatorname{grad} \phi = \operatorname{div} \phi$, we get that

$$\begin{aligned}
\operatorname{div} \operatorname{sym}_0 \operatorname{grad} \phi &= \operatorname{div} \left(\frac{1}{2} \left(\partial_i \phi_j + \partial_j \phi_i - \frac{2}{3} \delta_{ij} \operatorname{div} \phi \right)_{i,j \in \{1,2,3\}} \right) \\
&= \left(\sum_{j=1}^{3} \partial_j \frac{1}{2} \left(\partial_i \phi_j + \partial_j \phi_i - \delta_{ij} \frac{2}{3} \operatorname{div} \phi \right) \right)_{i \in \{1,2,3\}} \\
&= \frac{1}{2} \Delta \phi + \frac{1}{2} \operatorname{grad} \operatorname{div} \phi - \frac{1}{3} \operatorname{grad} \operatorname{div} \phi \\
&= \frac{1}{2} \left(\Delta \phi - \operatorname{grad} \operatorname{div} \phi \right) + \frac{2}{3} \operatorname{grad} \operatorname{div} \phi \\
&= -\frac{1}{2} \operatorname{curl} \operatorname{curl} \phi + \frac{2}{3} \operatorname{grad} \operatorname{div} \phi
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{div} \mathbb{P} \operatorname{grad} \phi &= \operatorname{div} \mathbb{P} \ \left(\partial_j \phi_i \right)_{i,j \in \{1,2,3\}} \\
&= \operatorname{div} \left(\frac{1}{3} \delta_{ij} \operatorname{div} \phi \right)_{i,j \in \{1,2,3\}} \\
&= \left(\sum_{j=1}^{3} \partial_j \frac{1}{3} \delta_{ij} \operatorname{div} \phi \right)_{i \in \{1,2,3\}} \\
&= \frac{1}{3} \operatorname{grad} \operatorname{div} \phi .
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\operatorname{div} \operatorname{skew} \operatorname{grad} \phi &= \operatorname{div} \operatorname{skew} \left(\partial_k \phi_j \right)_{j,k \in \{1,2,3\}} \\
&= \frac{1}{2} \operatorname{div} \left(\partial_k \phi_j - \partial_j \phi_k \right)_{j,k \in \{1,2,3\}} \\
&= \frac{1}{2} \left(\sum_{k=1}^{3} \partial_k \left(\partial_k \phi_j - \partial_j \phi_k \right) \right)_{j \in \{1,2,3\}} \\
&= \frac{1}{2} \left(\Delta - \operatorname{grad} \operatorname{div} \right) \phi = -\frac{1}{2} \operatorname{curl} \operatorname{curl} \phi .
\end{aligned}$$

Thus, choosing C as in (2.4), we get that

$$\begin{aligned}
\operatorname{div} C \operatorname{grad} \phi &= \operatorname{div} \left(\alpha_0 \operatorname{sym}_0 + \alpha_1 \mathbb{P} + \alpha_2 \operatorname{skew} \right) \operatorname{grad} \phi \\
&= \alpha_0 \operatorname{div} \operatorname{sym}_0 \operatorname{grad} \phi + \alpha_1 \operatorname{div} \mathbb{P} \operatorname{grad} \phi + \alpha_2 \operatorname{div} \operatorname{skew} \operatorname{grad} \phi \\
&= \alpha_0 \left(-\frac{1}{2} \operatorname{curl} \operatorname{curl} \phi + \frac{2}{3} \operatorname{grad} \operatorname{div} \phi \right) + \alpha_1 \frac{1}{3} \operatorname{grad} \operatorname{div} \phi - \alpha_2 \frac{1}{2} \operatorname{curl} \operatorname{curl} \phi \\
&= -\frac{\alpha_0 + \alpha_2}{2} \operatorname{curl} \operatorname{curl} \phi + \frac{2\alpha_0 + \alpha_1}{3} \operatorname{grad} \operatorname{div} \phi \\
&= \frac{\alpha_0 + \alpha_2}{2} \left(-\operatorname{curl} \operatorname{curl} + \operatorname{grad} \operatorname{div} \right) + \left(\frac{2\alpha_0 + \alpha_1}{3} - \frac{\alpha_0 + \alpha_2}{2} \right) \operatorname{grad} \operatorname{div} \\
&= \frac{\alpha_0 + \alpha_2}{2} \Delta + \frac{\alpha_0 + 2\alpha_1 - 3\alpha_2}{6} \operatorname{grad} \operatorname{div} .
\end{aligned}$$

Comparing coefficients, we get

$$\begin{aligned}\frac{\alpha_0+\alpha_2}{2}&=\mu_1,\\ \frac{\alpha_0+2\alpha_1-3\alpha_2}{6}&=\mu_2.\end{aligned}$$

Now, strict positive definiteness of C is equivalent to[2] $\alpha_1,\alpha_2,\alpha_3>0$. Introducing $\lambda\coloneqq\alpha_0-3\alpha_2\in\mathbb{R}$, we get

$$\begin{aligned}\alpha_1&=3\left(\mu_2-\frac{\lambda}{6}\right),\\ \alpha_0&=\frac{3}{2}\left(\mu_1+\frac{\lambda}{6}\right),\\ \alpha_2&=\frac{1}{2}\left(\mu_1-\frac{\lambda}{2}\right).\end{aligned}$$

If $\mu_2\geq 0$ then $\lambda=-3\mu_1<0$ is a possible choice for the latter set of equations to obtain strict positive definiteness of C. If $\mu_2<0$ then $\lambda=3(\mu_2-\mu_1)$ leads to $\mu_1+\frac{\lambda}{6}=\mu_1+\frac{\mu_2-\mu_1}{2}=\frac{\mu_1+\mu_2}{2}>0$, $\mu_2-\frac{\lambda}{6}=\mu_2-\frac{\mu_2-\mu_1}{2}=\frac{\mu_1+\mu_2}{2}>0$ as well as $\mu_1-\frac{\lambda}{2}=\mu_1-\frac{3}{2}(\mu_2-\mu_1)=\frac{5\mu_1-3\mu_2}{2}>0$, which also implies that C is strictly positive definite. □

Lemma 2.6 together with the previous remark provides a way for formulating the Guyer-Krumhansl model for heat conduction in the framework of evolutionary equations. This, in turn, results in the following well-posedness theorem, where we have for sake of definiteness chosen a specific case of boundary conditions:

Theorem 2.7. *Let $\tau_0,\varrho,\kappa\in\mathbb{R}_{>0}$, $\mu_1,\mu_2\in\mathbb{R}$ with $\mu_2>-\mu_1$, $\Omega\subseteq\mathbb{R}^3$ open and C as in Lemma 2.6. Let $c\in L(L^2(\Omega))$ be selfadjoint and strictly positive definite and $A\colon D(A)\subseteq L^2(\Omega)^3\oplus L^2(\Omega)\oplus L^2(\Omega)^{3\times3}\to L^2(\Omega)^3\oplus L^2(\Omega)\oplus L^2(\Omega)^{3\times3}$ with*

$$A\psi=\begin{pmatrix}0 & \begin{pmatrix}\operatorname{grad} & -\operatorname{div}\end{pmatrix}\\ \begin{pmatrix}\mathring{\operatorname{div}}\\ -\mathring{\operatorname{grad}}\end{pmatrix} & 0\end{pmatrix}\psi$$

for all

$$\psi\in D(A)\coloneqq\left\{\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}\in\begin{pmatrix}L^2(\Omega)^3\\ L^2(\Omega)\\ L^2(\Omega)^{3\times3}\end{pmatrix}\,\middle|\,\psi_1\in D(\mathring{\operatorname{grad}}),\operatorname{grad}\psi_2-\operatorname{div}\psi_3\in L^2(\Omega)^3\right\}.$$

Then the (closure of the) operator

$$\mathcal{T}\coloneqq\partial_0\begin{pmatrix}\tau_0\kappa^{-1} & 0 & 0\\ 0 & \varrho c & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}\kappa^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & C^{-1}\end{pmatrix}+\begin{pmatrix}0 & \begin{pmatrix}\operatorname{grad} & -\operatorname{div}\end{pmatrix}\\ \begin{pmatrix}\mathring{\operatorname{div}}\\ -\mathring{\operatorname{grad}}\end{pmatrix} & 0\end{pmatrix}$$

is continuously invertible in $H_{\varrho,0}\left(\mathbb{R};L^2(\Omega)^3\oplus L^2(\Omega)\oplus L^2(\Omega)^{3\times3}\right)$ for sufficiently large $\varrho>0$.

[2] Note that $\mathbb{P}$, sym_0 and skew are projections on pairwise orthogonal subspaces of $\mathbb{C}^{3\times3}$.

Proof. First of all we observe that A is skew-selfadjoint. Indeed, $\begin{pmatrix} \mathring{\operatorname{div}} \\ -\mathring{\operatorname{grad}} \end{pmatrix} : \mathring{H}_1(\Omega)^3 \subseteq L^2(\Omega)^3 \to L^2(\Omega) \oplus L^2(\Omega)^{3\times 3}$ generates an abstract grad-div-system (use Proposition 1.9 with $C_0 = -\mathring{\operatorname{grad}}$, $D(C_0) = \mathring{H}_1(\Omega)^3$ and $C_1 = \mathring{\operatorname{div}}$) [3] and the skew-selfadjointness of A follows from Theorem 1.6. Therefore, the invertibility of $\overline{\mathcal{T}}$ is guaranteed by the strict positive definiteness of c and C by Theorem 2.4. □

Thus, for $h \in H_{\varrho,0}\left(\mathbb{R}; L^2(\Omega)\right)$ and $u = (u_1, u_2, u_3)$ we have

$$\left(\partial_0 \begin{pmatrix} \tau_0\kappa^{-1} & 0 & 0 \\ 0 & \varrho c & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \kappa^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \begin{pmatrix} \operatorname{grad} & -\operatorname{div} \end{pmatrix} \\ \begin{pmatrix} \mathring{\operatorname{div}} \\ -\mathring{\operatorname{grad}} \end{pmatrix} & 0 \end{pmatrix}\right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}.$$

This is

$$\begin{aligned} \left(\tau_0\kappa^{-1}\partial_0 u_1 + \kappa^{-1}u_1\right) + \operatorname{grad} u_2 - \operatorname{div} u_3 &= 0, \\ \varrho c\partial_0 u_2 + \operatorname{div} u_1 &= h, \\ C^{-1}u_3 - \operatorname{grad} u_1 &= 0. \end{aligned}$$

Substituting – as a formal calculation – the third equation into the first, we get

$$\left(\tau_0\kappa^{-1}\partial_0 u_1 + \kappa^{-1}u_1\right) + \operatorname{grad} u_2 - \operatorname{div} C \operatorname{grad} u_1 = 0,$$

which, by Lemma 2.6 gives the original form of the system back

$$\left(\tau_0\kappa^{-1}\partial_0 u_1 + \kappa^{-1}u_1\right) = \kappa^{-1}\mu_1\Delta u_1 + \kappa^{-1}\mu_2 \operatorname{grad}\operatorname{div} u_1 - \operatorname{grad} u_2.$$

Remark 2.8.

1. Note that the role of temperature is played by the second block component ψ_2 whereas ψ_1 is the heat flux in this model.
2. The system reveals appropriate possible generalizations for the material law operator. For example
$$M\left(\partial_0^{-1}\right) = \begin{pmatrix} \kappa_0^{-1} & 0 & 0 \\ 0 & c_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} \kappa^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^{-1} \end{pmatrix}$$
where κ, κ_0, c_0, C are a strictly positive definite *operators* acting in a matching $L^2(\Omega)$-setting rather than just numbers.
3. In proper tensorial terms the spatial operator should actually be written as
$$\begin{pmatrix} 0 & \begin{pmatrix} \operatorname{div}\operatorname{trace}^* & -\operatorname{div} \end{pmatrix} \\ \begin{pmatrix} \operatorname{trace}\mathring{\operatorname{grad}} \\ -\mathring{\operatorname{grad}} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{div}\begin{pmatrix} \operatorname{trace}^* & -1 \end{pmatrix} \\ \begin{pmatrix} \operatorname{trace} \\ -1 \end{pmatrix}\mathring{\operatorname{grad}} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

[3] As a subtlety not to be missed, here we have taken $\operatorname{grad} : L^2(\Omega) \to \left(\mathring{H}_1(\Omega)^3\right)'$ acting on scalars as the dual $\left(-\mathring{\operatorname{div}}\right)^\diamond$ of $-\mathring{\operatorname{div}} : \mathring{H}_1(\Omega)^3 \to L^2(\Omega)$. This is in contrast to the situation in acoustics, where only the vanishing of the normal component on the boundary (or a generalization thereof) is imposed for the domain of $-\mathring{\operatorname{div}}$. The tensor divergence occurring here is correspondingly $\operatorname{div} : L^2(\Omega)^{3\times 3} \to \left(\mathring{H}_1(\Omega)^3\right)'$, the dual of $-\mathring{\operatorname{grad}}$ on vector fields.

4. If preferred we may specialize to the symmetric tensor case, compare [11, 31], and use

$$\begin{pmatrix} 0 & \begin{pmatrix} \operatorname{grad} & -\operatorname{div} \iota_{\mathrm{sym}} \end{pmatrix} \\ \begin{pmatrix} \mathring{\operatorname{div}} \\ -\iota_{\mathrm{sym}}^* \mathring{\operatorname{grad}} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

or, according to item 3,

$$\begin{pmatrix} 0 & \operatorname{div} \iota_{\mathrm{sym}} \begin{pmatrix} \operatorname{trace}^* & -1 \end{pmatrix} \\ \begin{pmatrix} \operatorname{trace} \\ -1 \end{pmatrix} \iota_{\mathrm{sym}}^* \mathring{\operatorname{grad}} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

instead of the spatial operator above. Here we denote by $\iota_{\mathrm{sym}} : L^2_{\mathrm{sym}}(\Omega)^{3\times 3} \to L^2(\Omega)^{3\times 3}$ the canonical embedding of

$$L^2_{\mathrm{sym}}(\Omega)^{3\times 3} := \left\{ (f_{ij})_{i,j\in\{1,2,3\}} \in L^2(\Omega)^{3\times 3} \,|\, f_{ij} = f_{ji} \quad (i,j \in \{1,2,3\}) \right\}.$$

In consequence, $\iota_{\mathrm{sym}}^* : L^2(\Omega)^{3\times 3} \to L^2_{\mathrm{sym}}(\Omega)^{3\times 3}$ is the orthogonal projection onto $L^2_{\mathrm{sym}}(\Omega)^{3\times 3}$.

2.2 A class of dynamic boundary conditions.

In this section, we shall analyze a class of dynamic boundary conditions related to the wave equation. For this let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with Lipschitz boundary $\Gamma := \partial\Omega$ and unit outward normal field n. Further, we shall assume that Γ is a manifold, where it is possible to define the co-variant derivative $\operatorname{grad}_\Gamma$ as an operator from $L^2(\Gamma)$ to $L^2_\tau(\Gamma) := L^2(\Gamma)^2$. By $\gamma\colon H_1(\Omega) \to L^2(\Gamma), \phi \mapsto \phi|_\Gamma$ we denote the trace mapping. Consider the operator equation

$$(\partial_0 M_0 + M_1 + A) \begin{pmatrix} \begin{pmatrix} p \\ v \\ \eta_1 \\ \eta_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} f \\ g \\ h_1 \\ h_2 \end{pmatrix} \end{pmatrix} \in H_{\varrho,0}\left(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega) \oplus L^2_\tau(\Gamma)\right) \tag{2.5}$$

for ϱ sufficiently large and with

$$A = \begin{pmatrix} (0) & -\begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}^* \\ \begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

where M_0 is selfadjoint and strictly positive definite on its range, while M_1 is strictly positive definite on the null space of M_0. Note that by the general solution theory Theorem 2.4 well-posedness of the system just discussed is not an issue here, if we can ensure that A is skew-selfadjoint. For this, we shall have a closer look at the operator A. Defining for this section

$$X_0 := L^2(\Omega),$$
$$X_1 := \left(D(\operatorname{grad}_\Gamma \gamma), \sqrt{|\operatorname{grad}_\Gamma \gamma \cdot|^2_{L^2_\tau(\Gamma)} + |\cdot|^2_{H_1(\Omega)}} \right),$$

we realize that (2.5) can be treated as an abstract grad–div system:

Proposition 2.9. *The operator*

$$C \coloneqq \begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix} : D\left(\operatorname{grad}_\Gamma \gamma\right) \subseteq X_0 \to L^2(\Omega) \oplus L^2(\Gamma) \oplus L^2_\tau(\Gamma)$$

generates an abstract grad–div *system.*

Proof. At first, note that $X_1 \hookrightarrow X_0$ with dense and continuous embedding, yielding in particular that C is densely defined. Moreover, as $\gamma\colon H_1(\Omega) \to L^2(\Gamma)$ is continuous and $\operatorname{grad}_\Gamma :$ $H_1(\Gamma) \subseteq L^2(\Gamma) \to L^2_\tau(\Gamma)$ is closed, the operator $\operatorname{grad}_\Gamma \gamma\colon D\left(\operatorname{grad}_\Gamma \gamma\right) \subseteq H_1(\Omega) \to L^2_\tau(\Gamma)$ is closed as well. Thus, C generates an abstract grad–div system by Proposition 1.9 for the choices $C_0 = \operatorname{grad}_\Gamma \gamma$, $C_1 = \gamma$ and $C_2 = \operatorname{grad}$. □

We shall now focus on the equations satisfied by $\begin{pmatrix} p \\ \begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix} \end{pmatrix}$, especially the boundary conditions, which are encoded in the domain of A.

Lemma 2.10. *Let* $\operatorname{grad}_\Gamma \gamma : X_1 \to L^2(\Gamma)$*. Then* $\left(\operatorname{grad}_\Gamma \gamma\right)^\diamond = \gamma' \operatorname{grad}_\Gamma^\diamond$ *with* $\operatorname{grad}_\Gamma\colon H_1(\Gamma) \to$ $L^2_\tau(\Gamma)$ *and* $\gamma\colon X_1 \to H_1(\Gamma)$ *where* $\gamma'\colon H_1(\Gamma)' \to X_1'$ *denotes the dual operator of* γ*.*

Proof. For $\phi \in L^2_\tau(\Gamma)$, we compute with $\psi \in D(\operatorname{grad}_\Gamma \gamma) = X_1$:

$$\begin{aligned} \left(\left(\operatorname{grad}_\Gamma \gamma\right)^\diamond \phi\right)(\psi) &= \langle \phi | \operatorname{grad}_\Gamma \gamma\psi\rangle_{L^2_\tau(\Gamma)} \\ &= \left(\operatorname{grad}_\Gamma^\diamond \phi\right)(\gamma\psi) \\ &= \left(\gamma' \operatorname{grad}_\Gamma^\diamond \phi\right)(\psi). \end{aligned}$$

□

Remark 2.11. For $\operatorname{div}_\Gamma \coloneqq -\operatorname{grad}_\Gamma^*$, we have the inclusion $-\gamma' \operatorname{div}_\Gamma \subseteq \left(\operatorname{grad}_\Gamma \gamma\right)^\diamond$. In fact, this follows from the previous lemma and the inclusion $-\operatorname{div}_\Gamma \subseteq \operatorname{grad}_\Gamma^\diamond$, which holds by Lemma 1.4.

Lemma 2.12. *Let* $v \in D(\operatorname{div})$*,* $\eta_1 \in L^2(\Gamma)$*,* $\eta_2 \in H_1(\Gamma)'$*, where* $H_1(\Gamma) = D(\operatorname{grad}_\Gamma)$ *equipped with the graph norm. Consider the operators* $\operatorname{grad}\colon X_1 \to L^2(\Omega)^3$, $\gamma_1\colon X_1 \to L^2(\Gamma), x_1 \mapsto \gamma x_1$ *and* $\gamma_2\colon X_1 \to H_1(\Gamma), x_1 \mapsto \gamma x_1$*. Then* $-\operatorname{div} v = \operatorname{grad}^\diamond v + \gamma_1^\diamond \eta_1 + \gamma_2' \eta_2$ *if and only if*[4] $n \cdot v + \eta_1 + \eta_2 = 0$ *on* $L^2(\Gamma)$*.*

Proof. ($\Rightarrow$) Assume that $-\operatorname{div} v = \operatorname{grad}^\diamond v + \gamma_1^\diamond \eta_1 + \gamma_2' \eta_2$. Then, for $\psi \in X_1$ we have that $\gamma\psi = \gamma_1\psi = \gamma_2\psi \in H_1(\Gamma) \subseteq H_{1/2}(\Gamma) \subseteq L^2(\Gamma)$ and we compute

$$\begin{aligned} \left(n \cdot v + \eta_1 + \eta_2\right)(\gamma\psi) &= \langle \operatorname{div} v | \psi\rangle_{L^2(\Omega)} + \langle v | \operatorname{grad} \psi\rangle_{L^2(\Omega)^3} + \langle \eta_1 | \gamma_1\psi\rangle_{L^2(\Gamma)} + \eta_2(\gamma\psi) \\ &= \langle \operatorname{div} v | \psi\rangle_{L^2(\Omega)} + \left(\operatorname{grad}^\diamond v + \gamma_1^\diamond \eta_1 + \gamma_2' \eta_2\right)(\psi) \\ &= \langle \operatorname{div} v | \psi\rangle_{L^2(\Omega)} - \langle \operatorname{div} v | \psi\rangle_{L^2(\Omega)} \\ &= 0. \end{aligned}$$

[4] Note that a-priori $n \cdot v$ is only defined as a functional in $H_{1/2}(\Gamma)'$, where $H_{1/2}(\Gamma) = \gamma[H_1(\Omega)]$. Indeed, $n \cdot v$ is the element on $H_{1/2}(\Gamma)'$ such that for all $\psi \in H_1(\Omega)$ the equation

$$(n \cdot v)(\gamma\psi) = \langle \operatorname{div} v | \psi\rangle_{L^2(\Omega)} + \langle v | \operatorname{grad} \psi\rangle_{L^2(\Omega)^3}$$

is satisfied.

Thus, the assertion follows, if we show that $\gamma[X_1]$ is dense in $L^2(\Gamma)$. For this, we note that $C_\infty(\overline{\Omega})$ is dense in $H_1(\Omega)$ since Ω is bounded and has Lipschitz boundary. Moreover, $\gamma : H_1(\Omega) \to H_{1/2}(\Gamma)$ is continuous and onto and hence, $\gamma[C_\infty(\overline{\Omega})]$ is dense in $H_{1/2}(\Omega)$ and thus, dense in $L^2(\Gamma)$. Since $C_\infty(\overline{\Omega}) \subseteq D(\operatorname{grad}_\Gamma \gamma) = X_1$ we derive the assertion.

($\Leftarrow$) Assume now that $n \cdot v + \eta_1 + \eta_2 = 0$ on $L^2(\Gamma)$. Then we compute for $\psi \in X_1$

$$\begin{aligned} 0 &= \langle n \cdot v + \eta_1 + \eta_2 | \gamma\psi \rangle_{L^2(\Gamma)} \\ &= \langle \operatorname{div} v | \psi \rangle_{L^2(\Omega)} + \langle v | \operatorname{grad} \psi \rangle_{L^2(\Omega)^3} + \langle \eta_1 | \gamma_1 \psi \rangle_{L^2(\Gamma)} + \eta_2(\gamma_2 \psi) \\ &= \langle \operatorname{div} v | \psi \rangle_{L^2(\Omega)} + \left(\operatorname{grad}^\diamond v + \gamma_1^\diamond \eta_1 + \gamma_2' \eta_2 \right)(\psi) \end{aligned}$$

which gives the assertion. $\square$

With this lemma we can characterize the domain of C^* with $C := \begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}$ in terms of an abstract boundary condition.

Theorem 2.13. *We have*

$$\begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}^* \begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix} = -\operatorname{div} v$$

with

$$D\left(\begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}^* \right) = \left\{ (v, \eta_1, \eta_2) \in D(\operatorname{div}) \times L^2(\Gamma) \times L^2_\tau(\Gamma) \,|\, n \cdot v + \eta_1 + \operatorname{grad}_\Gamma^\diamond \eta_2 = 0 \text{ on } L^2(\Gamma) \right\}.$$

Proof. In Proposition 2.9, we had already seen that $C = \begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}$ generates an abstract grad-div-system and thus, by Theorem 1.6 we get $(v, \eta_1, \eta_2) \in D(C^*)$ if and only if $\operatorname{grad}^\diamond v + \gamma^\diamond \eta_1 + (\operatorname{grad}_\Gamma \gamma)^\diamond \eta_2 \in L^2(\Omega)$. Moreover, we clearly have

$$\begin{pmatrix} \mathring{\operatorname{grad}} \\ 0 \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{grad} \\ \gamma \\ \operatorname{grad}_\Gamma \gamma \end{pmatrix}$$

and hence, Corollary 1.8 implies that

$$C^* \subseteq \left(-\operatorname{div} \; 0 \; 0 \right),$$

since $-\operatorname{div} = \mathring{\operatorname{grad}}^*$. In consequence (also use Lemma 2.10), $(v, \eta_1, \eta_2) \in D(C^*)$ if and only if $v \in D(\operatorname{div})$ and $-\operatorname{div} v = \operatorname{grad}^\diamond v + \gamma^\diamond \eta_1 + \gamma' \operatorname{grad}_\Gamma^\diamond \eta_2 \in L^2(\Omega)$, which in turn is equivalent to $v \in D(\operatorname{div})$ and $n \cdot v + \eta_1 + \operatorname{grad}_\Gamma^\diamond \eta_2 = 0$ on $L^2(\Gamma)$ by Lemma 2.12. $\square$

In the rest of this section, we shall formally compute the equation modeled by (2.5). That is to say, we assume that the system

$$(\partial_0 M_0 + M_1 + A)\left(\begin{pmatrix} p \\ \begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix}\end{pmatrix}\right) = \left(\begin{pmatrix} f \\ \begin{pmatrix} g \\ h_1 \\ h_2 \end{pmatrix}\end{pmatrix}\right)$$

given in (2.5) with the special block-diagonal situation

$$M_0 + \partial_0^{-1} M_1 = \left(\begin{array}{cc} m_{00} & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix} \end{array}\right) + \partial_0^{-1}\left(\begin{array}{cc} n_{00} & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \nu_{11} & 0 & 0 \\ 0 & \nu_{22} & 0 \\ 0 & 0 & \nu_{33} \end{pmatrix} \end{array}\right),$$

for non-negative scalars $m_{00}, n_{00}, \mu_{11}, \mu_{22}, \mu_{33}, \nu_{11}, \nu_{22}, \nu_{33}$ being arranged in the way that M_0 and M_1 satisfy the conditions to warrant well-posedness, i.e., $M_0 = M_0^*$ strictly positive definite on its range and $\mathfrak{Re}\, M_1$ strictly positive definite on the nullspace of M_0, admits smooth solutions $\left(\begin{pmatrix} p \\ \begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix}\end{pmatrix}\right)$. The resulting system leads to

$$\begin{aligned} \partial_0 m_{00} p + n_{00} p + \operatorname{div} v &= f \\ \partial_0 \mu_{11} v + \nu_{11} v + \operatorname{grad} p &= g \\ \partial_0 \mu_{22} \eta_1 + \nu_{22} \eta_1 + \gamma p &= h_1 \\ \partial_0 \mu_{33} \eta_2 + \nu_{33} \eta_2 + \operatorname{grad}_\Gamma \gamma p &= h_2 \\ n \cdot v + \eta_1 &= -\operatorname{grad}_\Gamma^\diamond \eta_2, \end{aligned} \tag{2.6}$$

where the last equality is the boundary condition induced by the domain of A. Eliminating η_1, η_2 by using the third and fourth line of (2.6), we obtain by formal calculations for the boundary condition

$$n \cdot v + (\partial_0 \mu_{22} + \nu_{22})^{-1} (h_1 - \gamma p) = -\operatorname{grad}_\Gamma^\diamond (\partial_0 \mu_{33} + \nu_{33})^{-1} (h_2 - \operatorname{grad}_\Gamma \gamma p).$$

Multiplication by $(\partial_0 \mu_{33} + \nu_{33})$ gives

$$\begin{aligned} (\partial_0 \mu_{33} + \nu_{33})\, n \cdot v - (\partial_0 \mu_{33} + \nu_{33}) (\partial_0 \mu_{22} + \nu_{22})^{-1} \gamma p - \operatorname{grad}_\Gamma^\diamond \operatorname{grad}_\Gamma \gamma p \\ = -\operatorname{grad}_\Gamma^\diamond h_2 - (\partial_0 \mu_{33} + \nu_{33}) (\partial_0 \mu_{22} + \nu_{22})^{-1} h_1. \end{aligned}$$

With $\mu_{22} = 0$ and $\nu_{22} = 1$ this simplifies further to

$$(\partial_0 \mu_{33} + \nu_{33})\, n \cdot v - (\partial_0 \mu_{33} + \nu_{33}) \gamma p - \operatorname{grad}_\Gamma^\diamond \operatorname{grad}_\Gamma \gamma p = -\operatorname{grad}_\Gamma^\diamond h_2 - (\partial_0 \mu_{33} + \nu_{33}) h_1.$$

Eliminating v, by using the second equation of (2.6), yields

$$\begin{aligned} -(\partial_0 \mu_{33} + \nu_{33})\, n \cdot (\partial_0 \mu_{11} + \nu_{11})^{-1} (g - \operatorname{grad} p) + (\partial_0 \mu_{33} + \nu_{33}) \gamma p + \operatorname{grad}_\Gamma^\diamond \operatorname{grad}_\Gamma \gamma p \\ = \operatorname{grad}_\Gamma^\diamond h_2 + (\partial_0 \mu_{33} + \nu_{33}) h_1 \end{aligned}$$

and hence,

$$\left(\partial_0\mu_{33}+\nu_{33}\right)\left(\partial_0\mu_{11}+\nu_{11}\right)^{-1} n\cdot\operatorname{grad} p+\left(\partial_0\mu_{33}+\nu_{33}\right)\gamma p+\operatorname{grad}_\Gamma^\diamond \operatorname{grad}_\Gamma \gamma p$$
$$=\operatorname{grad}_\Gamma^\diamond h_2+\left(\partial_0\mu_{33}+\nu_{33}\right)h_1+\left(\partial_0\mu_{33}+\nu_{33}\right)\left(\partial_0\mu_{11}+\nu_{11}\right)^{-1} n\cdot g.$$

Simplifying further by letting $\mu_{33}=\mu_{11}\alpha$, $\nu_{33}=\nu_{11}\alpha$ with some non-zero $\alpha\in\mathbb{R}_{>0}$, we arrive at

$$\left(\partial_0\mu_{11}+\nu_{11}\right)\gamma p+n\cdot\operatorname{grad} p+\alpha^{-1}\operatorname{grad}_\Gamma^\diamond\operatorname{grad}_\Gamma\gamma p=\left(\partial_0\mu_{11}+\nu_{11}\right)h_1+n\cdot g+\alpha^{-1}\operatorname{grad}_\Gamma^\diamond h_2,$$

which is a heat type equation on the boundary of Ω (see also Remark 2.11). Noting that the first and second line of (2.6) gives

$$\partial_0 m_{00}p+n_{00}p+\operatorname{div}(\partial_0\mu_{00}+\nu_{00})^{-1}(g-\operatorname{grad} p)=f,$$

which yields for $\nu_{00}=0$ an abstract wave equation of the form

$$\partial_0^2 m_{00}p+\partial_0 n_{00}p-\operatorname{div}\mu_{00}\operatorname{grad} p=\partial_0 f-\operatorname{div}\mu_{00}g.$$

Thus, for suitable M_0 and M_1, (2.5) models a wave equation on Ω with a heat equation on $\partial\Omega$ as boundary condition.

Remark 2.14. In [4], an equation was studied, where on a 1-codimensional submanifold in Ω another partial differential equation on an interface is also considered. We shall outline here, how to include such problems in the abstract setting just discussed. So let $\Omega\subseteq\mathbb{R}^d$ open with Lipschitz boundary and assume that there exists $\widetilde{\Omega}\subseteq\Omega$ open with the property that $\partial\Omega\subseteq\partial\widetilde{\Omega}$ and such that both $\partial\widetilde{\Omega}$ and $\partial\widetilde{\Omega}\setminus\partial\Omega$ are locally the graph of a Lipschitz continuous function and are manifolds allowing for the definition of a covariant derivative. As the operator C, generating the abstract grad–div system, we take the following operator

$$C:=\begin{pmatrix}\operatorname{grad}\\ \gamma\\ \operatorname{grad}_\Gamma\gamma\end{pmatrix}$$

similar to the considerations above. The domain, however, changes a little, namely γ being now the evaluation at $\Gamma:=\partial\widetilde{\Omega}$ and $\operatorname{grad}_\Gamma$ is given analogous to the above definition.

2.3 A Leontovich type boundary condition.

As for the acoustic equations just discussed, we formulate a similar problem also for Maxwell's equations. We will show that Maxwell's equations with impedance type boundary conditions can be formulated within the framework of evolutionary equations employing the notion of abstract grad-div-systems. For this let $\Omega\subseteq\mathbb{R}^3$ open. In this section, we consider the equation

$$\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)\begin{pmatrix}H\\ \begin{pmatrix}E\\ \eta\end{pmatrix}\end{pmatrix}=\begin{pmatrix}f\\ \begin{pmatrix}g\\ h_1\end{pmatrix}\end{pmatrix}\in H_{\varrho,0}\left(\mathbb{R};L^2(\Omega)^3\oplus L^2(\Omega)^3\oplus H_{\text{trace}}\right).\quad(2.7)$$

Here H_{trace} denotes a suitable Hilbert space, which will be used to formulate the boundary condition. In the forthcoming subsections we will discuss two possibilities for the choice of H_{trace}.

2.3.1 A classical trace version

Throughout, assume that Ω is bounded and has Lipschitz-boundary Γ and denote by n the unit outward normal vector-field on Γ. In order to define A in (2.7) properly, we need the following trace operators (see [2]).

Definition 2.15. Let $L^2_\tau(\Gamma)$ denote the space of tangential vector-fields on Γ, i.e.

$$L^2_\tau(\Gamma) := \left\{ f \in L^2(\Gamma)^3 \,|\, f \cdot n = 0 \right\},$$

which is a closed subspace of $L^2(\Gamma)^3$. We define the mappings

$$\pi_\tau : H_1(\Omega)^3 \to L^2_\tau(\Gamma), H \mapsto -n \times (n \times \gamma H) = \gamma H - (n \cdot \gamma H)n,$$

and

$$\gamma_\tau : H_1(\Omega)^3 \to L^2_\tau(\Gamma), H \mapsto \gamma H \times n,$$

where $\gamma : H_1(\Omega)^3 \to H_{1/2}(\Omega)^3$ denotes the classical trace (cp. Subsection 2.2). We set $V_\pi := \pi_\tau \left[H_1(\Omega)^3\right]$ and $V_\gamma := \gamma_\tau \left[H_1(\Omega)^3\right]$, which are Hilbert spaces equipped with the norms

$$|v|_{V_\pi} := \inf_{H \in H_1(\Omega)^3} \{|\gamma H|_{H_{1/2}(\Gamma)^3} \,|\, \pi_\tau H = v\},$$
$$|v|_{V_\gamma} := \inf_{H \in H_1(\Omega)^3} \{|\gamma H|_{H_{1/2}(\Gamma)^3} \,|\, \gamma_\tau H = v\}.$$

Remark 2.16.

(a) We have $\mathrm{id}_\pi \colon V_\pi \hookrightarrow L^2_\tau(\Gamma)$ and $\mathrm{id}_\gamma \colon V_\gamma \hookrightarrow L^2_\tau(\Gamma)$ with continuous and dense embeddings. Consequently, $\mathrm{id}_\pi^\diamond \colon L^2_\tau(\Gamma) \hookrightarrow V_\pi'$ as well as $\mathrm{id}_\gamma^\diamond \colon L^2_\tau(\Gamma) \hookrightarrow V_\gamma'$ with continuous and dense embeddings.

(b) Integration by parts gives

$$\langle \pi_\tau H | \gamma_\tau E\rangle_{L^2_\tau(\Gamma)} = \langle \operatorname{curl} H | E\rangle_{L^2(\Omega)^3} - \langle H | \operatorname{curl} E\rangle_{L^2(\Omega)^3},$$

for all $H, E \in H_1(\Omega)^3$, which yields that (see [2] or the concise presentation in [35, Section 4])

$$R_{(L^2_\tau)'} \circ \pi_\tau : H^1(\Omega)^3 \subseteq H(\operatorname{curl}, \Omega) \to V_\gamma',$$

where $R_{(L^2_\tau)'}$ is defined as in Section 1, is continuous, where $H(\operatorname{curl}, \Omega)$ denotes the domain of curl equipped with its graph norm. Hence, π_τ has a unique continuous extension to an operator

$$\pi_\tau : H(\operatorname{curl}, \Omega) \to V_\gamma'.$$

The same argument works for γ_τ, we, thus, get that

$$\gamma_\tau : H(\operatorname{curl}, \Omega) \to V_\pi'$$

is continuous.

Lemma 2.17. *The operator*

$$C \colon \pi_\tau^{-1}\left[\mathrm{id}_\tau^\diamond \left[L^2_\tau(\Gamma)\right]\right] \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus L^2_\tau(\Gamma), H \mapsto \begin{pmatrix} -\operatorname{curl} \\ \pi_\tau \end{pmatrix} H$$

is closed.

Proof. First of all, note that $\pi_\tau^{-1}\left[\mathrm{id}_\tau^\diamond\left[L_\tau^2(\Gamma)\right]\right] \subseteq D(\mathrm{curl})$. Next, let $(\phi_n)_n$ in $D(C)$ with $\phi_n \to \phi$ in $L^2(\Omega)^3$ as $n \to \infty$ and $(\pi_\tau \phi_n)_n$ convergent to some $\psi \in L_\tau^2(\Gamma)$ as well as $(\mathrm{curl}\,\phi_n)_n$ convergent to some $\eta \in L^2(\Omega)^3$. By the closedness of curl, we infer that $\phi \in H(\mathrm{curl},\Omega)$ and $\mathrm{curl}\,\phi = \eta$. Hence, $\phi_n \to \phi$ in $H(\mathrm{curl},\Omega)$ as $n \to \infty$. As π_τ is continuous, we infer that $\pi_\tau\phi_n \to \pi_\tau\phi$ in V_γ' as $n \to \infty$. Since $L_\tau^2(\Gamma) \hookrightarrow V_\gamma'$ with continuous embedding, we get $\pi_\tau\phi = \psi \in L_\tau^2(\Gamma)$, which implies the assertion. □

Define for this section

$$X_0 := L^2(\Omega)^3,$$
$$X_1 := \left(D(C), \sqrt{|\cdot|^2 + |C\cdot|^2}\right).$$

We may now introduce the operator A:

$$A = \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & -\begin{pmatrix} -\widetilde{\mathrm{curl}} \\ \widetilde{\pi}_\tau \end{pmatrix}^* \\ \begin{pmatrix} -\widetilde{\mathrm{curl}} \\ \widetilde{\pi}_\tau \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \end{pmatrix} \end{pmatrix},$$

where $\widetilde{\mathrm{curl}} := \mathrm{curl}\,|_{X_1} : X_1 \to L^2(\Omega)^3$ and $\widetilde{\pi}_\tau := \pi_\tau|_{X_1} : X_1 \to L_\tau^2(\Gamma)$. $\begin{pmatrix} -\widetilde{\mathrm{curl}} \\ \widetilde{\pi}_\tau \end{pmatrix} : X_1 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus L_\tau^2(\Gamma)$ generates an abstract grad–div system: the only thing left to show is that X_1 is dense in $L^2(\Omega)^3$. This, however, is trivial as $\mathring{C}_\infty(\Omega) \subseteq X_1$.

Similarly to the previous section, we shall have a look at the condition for being contained in the domain of $\begin{pmatrix} -\widetilde{\mathrm{curl}} \\ \widetilde{\pi}_\tau \end{pmatrix}^*$. Again, we need a prerequisite:

Lemma 2.18. *Let $E \in D(\mathrm{curl})$, $\eta \in L_\tau^2(\Gamma)$. Then $\mathrm{curl}\,E = \widetilde{\mathrm{curl}}^\diamond E - \widetilde{\pi}_\tau^\diamond\eta$ if and only if $\gamma_\tau E + \eta = 0$ on $L_\tau^2(\Gamma)$.*

Proof. We observe that for $\Psi \in H^1(\Omega)^3 \subseteq X_1$ the equation

$$\begin{aligned}(\gamma_\tau E + \eta)(\pi_\tau\Psi) &= \langle \mathrm{curl}\,E|\Psi\rangle_{L^2(\Omega)^3} - \left\langle E|\widetilde{\mathrm{curl}}\Psi\right\rangle_{L^2(\Omega)^3} + \langle \eta|\widetilde{\pi}_\tau\Psi\rangle_{L_\tau^2(\Gamma)} \\ &= \langle \mathrm{curl}\,E|\Psi\rangle_{L^2(\Omega)^3} - \left(\widetilde{\mathrm{curl}}^\diamond E - \widetilde{\pi}_\tau^\diamond\eta\right)(\Psi)\end{aligned} \tag{2.8}$$

holds true. Thus, if $\mathrm{curl}\,E = \widetilde{\mathrm{curl}}^\diamond E - \widetilde{\pi}_\tau^\diamond\eta$, we get that $(\gamma_\tau E + \eta)(\pi_\tau\Psi) = 0$ for each $\Psi \in H^1(\Omega)^3$. Thus, $\gamma_\tau E + \eta = 0$ on $L_\tau^2(\Gamma)$, due to the density of $\widetilde{\pi}_\tau[H^1(\Omega)^3] = V_\pi$ in $L_\tau^2(\Gamma)$, see [2, p 850]. On the other hand, if $\gamma_\tau E + \eta = 0$, equation (2.8) immediately gives $\mathrm{curl}\,E = \widetilde{\mathrm{curl}}^\diamond E - \widetilde{\pi}_\tau^\diamond\eta$ by the density of $H^1(\Omega)^3$ in $L^2(\Omega)^3$. □

Theorem 2.19. *We have* $\begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \widetilde{\pi_\tau} \end{pmatrix}^* \subseteq \begin{pmatrix} -\operatorname{curl} & 0 \end{pmatrix}$ *and*

$$D\left(\begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \widetilde{\pi_\tau} \end{pmatrix}^*\right) = \left\{(E,\eta) \in D(\operatorname{curl}) \times L^2_\tau(\Gamma) \mid \gamma_\tau E + \eta = 0 \text{ on } L^2_\tau(\Gamma)\right\}.$$

Proof. Note that with $\overset{\circ}{\operatorname{curl}} = \operatorname{curl}^*$ we have

$$\begin{pmatrix} -\overset{\circ}{\operatorname{curl}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \widetilde{\pi_\tau} \end{pmatrix} =: C.$$

Hence, by Corollary 1.8, we get

$$C^* \subseteq \begin{pmatrix} -\operatorname{curl} & 0 \end{pmatrix}.$$

Therefore, by Theorem 1.6, we obtain $(E,\eta) \in D(C^*)$ if and only if $E \in D(\operatorname{curl})$ and

$$-\operatorname{curl} E = -\widetilde{\operatorname{curl}}^\diamond E + \widetilde{\pi_\tau}^\diamond \eta \in L^2(\Omega)^3,$$

which, in turn, by Lemma 2.18 is equivalent to $\gamma_\tau E + \eta = 0$ on $L^2_\tau(\Gamma)$ and $E \in D(\operatorname{curl})$. □

The latter theorem tells us that the containment in the domain of A is a boundary equation. So, in order to have a general class of Maxwell-type equations at hand, which include differential equations on the boundary, any suitable material law operator $M(\partial_0^{-1})$ allowing for well-posedness of (2.7) is admitted.

We shall elaborate the following particular choice for the material law operator:

$$M\left(\partial_0^{-1}\right) = \begin{pmatrix} \mu & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \varepsilon & 0 \\ 0 & \kappa\left(\partial_0^{-1}\right) \end{pmatrix} \end{pmatrix} \in L\left(H_{\varrho,0}\left(\mathbb{R}; L^2(\Omega)^3 \oplus L^2(\Omega)^3 \oplus L^2_\tau(\Gamma)\right)\right),$$

where $\kappa : B_{\mathbb{C}}\left(\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right) \to L(L^2_\tau(\Gamma))$ is analytic and satisfies condition (2.3). For this choice (also use the result from Theorem 2.19) (2.7) formally reads as

$$\begin{aligned} \partial_0 \mu H + \operatorname{curl} E &= f, \\ \partial_0 \varepsilon E - \widetilde{\operatorname{curl}} H &= g, \\ \partial_0 \kappa\left(\partial_0^{-1}\right)\eta + \widetilde{\pi_\tau} H &= h_1, \\ \gamma_\tau E + \eta &= 0. \end{aligned}$$

Elimination of η yields

$$-\partial_0 \kappa\left(\partial_0^{-1}\right)\gamma_\tau E + \widetilde{\pi_\tau} H = h_1. \tag{2.9}$$

This is an impedance type boundary condition. To see this we recall $\gamma_\tau E = \gamma E \times n$ and $\pi_\tau H = -n \times (n \times \gamma H)$, use that $(n\times)(-n\times)(n\times) = (n\times)$ and multiply (2.9) by $-n\times$. For ease of formulation, we shall suppress denoting the trace operator γ. We obtain

$$n \times \left(\partial_0 \kappa\left(\partial_0^{-1}\right) n \times E\right) - n \times H = -n \times h_1,$$

or, equivalently,

$$-n\times\left(\partial_0\kappa(\partial_0^{-1})n\times(n\times(n\times E))\right)=n\times H-n\times h_1,$$

which gives,

$$\pi_\tau E=\left(\left(-n\times\right)\partial_0\kappa\left(\partial_0^{-1}\right)\left(n\times\right)\right)^{-1}\left(n\times h_1-H\times n\right). \tag{2.10}$$

Now, in the literature, we find several choices for the impedance operator

$$Z\left(\partial_0^{-1}\right):=\left(\left(-n\times\right)\partial_0\kappa\left(\partial_0^{-1}\right)\left(n\times\right)\right)^{-1}.$$

For discussing some of these choices, we let $\varepsilon',\mu',\sigma',\tau'\in\mathbb{R}_{\geq0}$ be parameters with $\varepsilon'+\sigma'>0$ and $\mu'+\tau'>0$. Mohsen [13] used the following form

$$Z\left(\partial_0^{-1}\right)=\left(\mu'+\tau'\partial_0^{-1}\right)^{\frac{1}{2}}\left(\varepsilon'+\sigma'\partial_0^{-1}\right)^{-\frac{1}{2}},$$

leading to a boundary condition, which Mohsen attributed to Rytov and Leontovich. Senior, [26], and De Santis e.a., [3], discussed the case $\tau'=0$, resulting in

$$Z\left(\partial_0^{-1}\right)=\sqrt{\frac{\mu'}{\varepsilon'}}\left(1+\frac{\sigma'}{\varepsilon'}\partial_0^{-1}\right)^{-\frac{1}{2}}.$$

In the eddy current approximation ($\varepsilon'=0$) and also assuming $\tau'=0$, a case discussed by Hiptmair, López-Fernández & Paganini, [9], we arrive at the following form

$$Z\left(\partial_0^{-1}\right)=\sqrt{\frac{\mu'}{\sigma'}}\partial_0^{-\frac{1}{2}}.$$

This leads to fractional evolution on the boundary.

Remark 2.20. In [1] the author considers Maxwell's equations in a two-dimensional setting and imposes a family of boundary conditions on a surface Σ contained in an exterior domain, which can be written in the form

$$\partial_0\left(\gamma_\tau E-\pi_\tau H\right)=\left(\frac{k}{4}-\gamma\right)\gamma_\tau E+\left(\frac{k}{4}+\gamma\right)\pi_\tau H, \tag{2.11}$$

where k denotes the curvature of Σ and γ is an arbitrary function on Σ. It is shown that each of those boundary conditions yield a well-posed system and the asymptotic behavior of the corresponding solutions are studied. We remark here that the latter boundary condition is covered by (2.9). Indeed, choosing

$$\kappa\left(\partial_0^{-1}\right)=\partial_0^{-1}-\partial_0^{-1}\frac{k}{2}\left(\partial_0+\frac{k}{4}+\gamma\right)^{-1}$$

and $h_1=0$ we obtain

$$\begin{aligned}0&=-\partial_0\kappa\left(\partial_0^{-1}\right)\gamma_\tau E+\pi_\tau H\\&=-\left(1-\frac{k}{2}\left(\partial_0+\frac{k}{4}+\gamma\right)^{-1}\right)\gamma_\tau E+\pi_\tau H,\end{aligned}$$

which gives

$$0=-\left(\partial_0-\frac{k}{4}+\gamma\right)\gamma_\tau E+\left(\partial_0+\frac{k}{4}+\gamma\right)\pi_\tau H,$$

which is just a reformulation of (2.11).

2.3.2 An abstract trace version

In this last section, we shall provide a possible way of by-passing boundary regularity requirements giving the corresponding Leontovich type boundary conditions its most general form. For doing so, we need the following definitions, which are adapted from the abstract boundary data spaces, see e.g. [28, 18, 19]

Definition 2.21. Recall that $\mathring{\mathrm{curl}} \subseteq \mathrm{curl}$, and consequently $H(\mathring{\mathrm{curl}}, \Omega)$ is a closed subspace of $H(\mathrm{curl}, \Omega)$. We define the *boundary data space of* curl by

$$\mathcal{BD}(\mathrm{curl}) := H(\mathring{\mathrm{curl}}, \Omega)^{\perp},$$

where the orthogonal complement is taken in $H(\mathrm{curl}, \Omega)$. Moreover, we denote by $\iota_{\mathrm{curl}} : \mathcal{BD}(\mathrm{curl}) \to H(\mathrm{curl}, \Omega)$ the canonical embedding, i.e. $\iota_{\mathrm{curl}}\Phi = \Phi$. Then, $\iota_{\mathrm{curl}}^{*}\iota_{\mathrm{curl}} = \mathrm{id}_{\mathcal{BD}(\mathrm{curl})}$ and

$$\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*} : H(\mathrm{curl}, \Omega) \to H(\mathrm{curl}, \Omega)$$

is the orthogonal projector on $\mathcal{BD}(\mathrm{curl})$. An easy computation shows that

$$\mathcal{BD}(\mathrm{curl}) = N(1 + \mathrm{curl}\,\mathrm{curl}),$$

which in particular yields that $\mathrm{curl}[\mathcal{BD}(\mathrm{curl})] \subseteq \mathcal{BD}(\mathrm{curl})$. We set

$$\begin{aligned} \overset{\bullet}{\mathrm{curl}} \colon \mathcal{BD}(\mathrm{curl}) &\to \mathcal{BD}(\mathrm{curl}) \\ \phi &\mapsto \iota_{\mathrm{curl}}^{*}\,\mathrm{curl}\,\iota_{\mathrm{curl}} \end{aligned}$$

Consequently, we have $\overset{\bullet}{\mathrm{curl}}\overset{\bullet}{\mathrm{curl}} = -\mathrm{id}_{\mathcal{BD}(\mathrm{curl})}$.

The main idea is now to replace the space $L^2_\tau(\Gamma)$ in the previous section by the space $\mathcal{BD}(\mathrm{curl})$ and the trace operator π_τ by the operator $\iota_{\mathrm{curl}}^{*}$. Indeed, setting $X_0 := L^2(\Omega)^3$ and $X_1 := H(\mathrm{curl}, \Omega)$ we obtain that $\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^{*} \end{pmatrix} : H(\mathrm{curl}, \Omega) \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus \mathcal{BD}(\mathrm{curl})$ generates an abstract grad–div system, see also Proposition 1.9. Consequently, (2.7) with

$$A := \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & -\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^{*} \end{pmatrix}^{*} \\ \begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^{*} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

is well-posed in $H_{\varrho,0}\left(\mathbb{R}; L^2(\Omega)^3 \oplus L^2(\Omega)^3 \oplus \mathcal{BD}(\mathrm{curl})\right)$. In order to characterize the domain of $\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^{*} \end{pmatrix}^{*}$, we need to understand the operator $(\iota_{\mathrm{curl}}^{*})^{\diamond} : \mathcal{BD}(\mathrm{curl}) \to H(\mathrm{curl}, \Omega)'$.

Lemma 2.22. *We have* $(\iota_{\mathrm{curl}}^*)^\diamond = (1 + \mathrm{curl}^\diamond \,\mathrm{curl})\iota_{\mathrm{curl}}$.

Proof. For $\Psi \in H(\mathrm{curl}, \Omega), \Phi \in \mathcal{BD}(\mathrm{curl})$ we compute

$$\begin{aligned}(\iota_{\mathrm{curl}}^*)^\diamond \,\Phi(\Psi) &= \langle \Phi | \iota_{\mathrm{curl}}^* \Psi \rangle_{\mathcal{BD}(\mathrm{curl})} \\ &= \langle \iota_{\mathrm{curl}} \Phi | \Psi \rangle_{H(\mathrm{curl},\Omega)} \\ &= \langle \iota_{\mathrm{curl}} \Phi | \Psi \rangle_{L^2(\Omega)^3} + \langle \mathrm{curl}\, \iota_{\mathrm{curl}} \Phi | \,\mathrm{curl}\, \Psi \rangle_{L^2(\Omega)^3} \\ &= \left((1 + \mathrm{curl}^\diamond \,\mathrm{curl})\, \iota_{\mathrm{curl}} \Phi \right)(\Psi),\end{aligned}$$

which gives the assertion. □

Remark 2.23. Note that in Lemma 2.22, $1 = R_{(L^2(\Omega)^3)'}$, the latter operator being given in Section 1.

Thus, we end up with the following characterization result.

Theorem 2.24. *We have* $\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^* \end{pmatrix}^* \subseteq \begin{pmatrix} -\mathrm{curl} & 0 \end{pmatrix}$ *and*

$$D\left(\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^* \end{pmatrix}^* \right) = \left\{ (E, \eta) \in D(\mathrm{curl}) \times \mathcal{BD}(\mathrm{curl}) \;\middle|\; \iota_{\mathrm{curl}}^* E - \overset{\bullet}{\mathrm{curl}}\, \eta = 0 \right\}.$$

Proof. Since $\begin{pmatrix} -\mathring{\mathrm{curl}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^* \end{pmatrix}$, we obtain $\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^* \end{pmatrix}^* \subseteq \begin{pmatrix} -\mathrm{curl} & 0 \end{pmatrix}$ by Corollary 1.8. Hence, by Theorem 1.6 we have $(E, \eta) \in D\left(\begin{pmatrix} -\mathrm{curl} \\ \iota_{\mathrm{curl}}^* \end{pmatrix}^* \right)$ if and only if $E \in D(\mathrm{curl})$ and

$$\begin{aligned} -\mathrm{curl}\, E &= -\mathrm{curl}^\diamond E + (\iota_{\mathrm{curl}}^*)^\diamond \eta \\ &= -\mathrm{curl}^\diamond E + (1 + \mathrm{curl}^\diamond \,\mathrm{curl})\iota_{\mathrm{curl}} \eta, \end{aligned}$$

by Lemma 2.22. The latter gives

$$\begin{aligned} (\mathrm{curl}^\diamond - \mathrm{curl})\, E &= (1 + \mathrm{curl}^\diamond \,\mathrm{curl})\, \iota_{\mathrm{curl}} \eta \\ &= (-\mathrm{curl}\,\mathrm{curl} + \mathrm{curl}^\diamond \,\mathrm{curl})\iota_{\mathrm{curl}} \eta \\ &= (\mathrm{curl}^\diamond - \mathrm{curl})\, \mathrm{curl}\, \iota_{\mathrm{curl}} \eta, \end{aligned}$$

where we have used $\mathcal{BD}(\mathrm{curl}) = N(1 + \mathrm{curl}\,\mathrm{curl})$. The latter is equivalent to

$$\iota_{\mathrm{curl}} \iota_{\mathrm{curl}}^* (E - \mathrm{curl}\, \iota_{\mathrm{curl}} \eta) = 0.$$

Indeed, we have that an element $\Psi \in D(\mathrm{curl})$ satisfies $(\mathrm{curl}^\diamond - \mathrm{curl})\Psi = 0$ if and only if for each $\Phi \in D(\mathrm{curl})$ we have

$$\begin{aligned} 0 &= \left((\mathrm{curl}^\diamond - \mathrm{curl})\, \Psi \right)(\Phi) \\ &= \langle \Psi | \,\mathrm{curl}\, \Phi \rangle_{L^2(\Omega)^3} - \langle \mathrm{curl}\, \Psi | \Phi \rangle_{L^2(\Omega)}, \end{aligned}$$

which in turn is equivalent to $\Psi \in D(\mathrm{curl}^*) = D(\mathring{\mathrm{curl}})$. Applying this to $E - \mathrm{curl}\, \iota_{\mathrm{curl}} \eta \in D(\mathrm{curl})$, we obtain the equivalence of

$$(\mathrm{curl}^\diamond - \mathrm{curl})\, E = (\mathrm{curl}^\diamond - \mathrm{curl})\, \mathrm{curl}\, \iota_{\mathrm{curl}} \eta$$

and

$$\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}\left(E-\operatorname{curl}\iota_{\mathrm{curl}}\eta\right)=0. \tag{2.12}$$

Using the definition of $\overset{\bullet}{\mathrm{curl}}$ and the injectivity of ι_{curl}, we deduce the equivalence of (2.12) to

$$\iota_{\mathrm{curl}}^{*}E-\overset{\bullet}{\mathrm{curl}}\,\eta=0. \qquad \square$$

In the remaining part of this subsection, we like to point out that the boundary condition formulated in Theorem 2.24 is indeed a generalization of the classical boundary condition in Theorem 2.19. For doing so, let $(E,\eta)\in D(\mathrm{curl})\times\mathcal{BD}(\mathrm{curl})$ with $\iota_{\mathrm{curl}}^{*}E-\overset{\bullet}{\mathrm{curl}}\,\eta=0$. Note that the latter is equivalent to $\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E+\eta=0$, since $\overset{\bullet}{\mathrm{curl}}\overset{\bullet}{\mathrm{curl}}=-\mathrm{id}_{\mathcal{BD}(\mathrm{curl})}$. Thus, we have to show that $\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E$ can be identified with $\gamma_\tau E$ in a certain sense. We compute for $E,H\in C^\infty(\overline{\Omega})$

$$\begin{aligned}
\langle\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E|\iota_{\mathrm{curl}}^{*}H\rangle_{\mathcal{BD}(\mathrm{curl})} &= \langle\iota_{\mathrm{curl}}\,\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E|\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}H\rangle_{H(\mathrm{curl},\Omega)}\\
&= \langle\operatorname{curl}\iota_{\mathrm{curl}}\,\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E|\operatorname{curl}\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}H\rangle_{L^2(\Omega)^3}+\\
&\quad+\langle\iota_{\mathrm{curl}}\,\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E|\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}H\rangle_{L^2(\Omega)^3}\\
&= -\langle\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}E|\operatorname{curl}\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}H\rangle_{L^2(\Omega)^3}+\langle\operatorname{curl}\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}E|\iota_{\mathrm{curl}}\iota_{\mathrm{curl}}^{*}H\rangle_{L^2(\Omega)^3}\\
&= -\langle E|\operatorname{curl}H\rangle_{L^2(\Omega)^3}+\langle\operatorname{curl}E|H\rangle_{L^2(\Omega)^3}\\
&= \int_\Gamma (n\times E)^{*}H
\end{aligned}$$

in case of a smooth boundary $\Gamma\coloneqq\partial\Omega$. Thus, indeed $n\times E$ can be identified with $\overset{\bullet}{\mathrm{curl}}\,\iota_{\mathrm{curl}}^{*}E$, or, in other words, $\overset{\bullet}{\mathrm{curl}}\colon\mathcal{BD}(\mathrm{curl})\to\mathcal{BD}(\mathrm{curl})$ is a suitable generalization of the operator $n\times:L_\tau^2(\Gamma)\to L_\tau^2(\Gamma)$. Indeed, both operators are unitary and their adjoints are the negative operators, i.e. $(n\times)^{*}=-n\times$ as well as $(\overset{\bullet}{\mathrm{curl}})^{*}=-\overset{\bullet}{\mathrm{curl}}$.

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