THE DIRAC OPERATOR
ON CERTAIN HOMOGENEOUS SPACES
AND REPRESENTATIONS OF SOME LIE GROUPS.

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Abstract.

Let G be a real non-compact reductive Lie group and L a compact subgroup. Take a maximal compact subgroup K of G containing L, and suppose that G/L is Riemannian via a bi-invariant metric and that there is a spin structure. Then there is the Dirac operator D over G/L, on spinors with values in a unitary vector bundle. D is a first order, G-invariant, elliptic, essentially self-adjoint differential operator.

It has been shown by R. Parthasarathy that with G semi-simple, rank K = rank G, 'discrete-series' representations of G can be realized geometrically on the kernel of D (i.e. the L^2-solutions of $Df = 0$). Following this, we are interested in how the kernel of D decomposes into irreducible representations of G, when L is any compact subgroup. In future work we expect to reduce this problem to the compact case i.e. to considering the Dirac operator on K/L.

Therefore, in this Thesis, we consider the Dirac operator on a compact, Riemannian, spin homogeneous space K/L. And determine the decomposition of the kernel into irreducible representations of K. We consider the tensor product of an induced representation and a finite-dimensional representation, and apply 'inducing in stages' to the Dirac operator.

Declaration.

I declare that no part of this Thesis has been previously submitted for any degree at any University. The contents are my own original work, except for expository material or results attributed to others.

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Introduction.

(0.1) Let G be a real non-compact reductive Lie group, and K a maximal compact subgroup containing a given compact subgroup L of G. The reductive homogeneous space G/L becomes Riemannian via a bi-invariant metric $(,)$ and suppose there is a spin structure. Take a G-invariant, metric connection γ on the tangent bundle $T(G/L)$. Then associated to the pair $((,),\gamma)$, there is the Dirac operator D, a 1^{st} order G-invariant, elliptic, essentially self-adjoint differential operator. In its coordinate free form, D operates on spinors with values in a unitary vector bundle. Thus G acts on the space of L^2-solutions of the homogeneous Dirac equation $Df = 0$. The kernel of D, $\ker D$, becomes a unitary G-module.

One very important previous application of the Dirac operator, in representation theory, has been in the construction of unitary representations of G. It was found with G semi-simple and $\operatorname{rank} K = \operatorname{rank} G$, that the 'discrete series' representations of G could be realized geometrically on $\operatorname{Ker} D$. See [28], [29], [30], [31].

(0.2) We are interested in how $\operatorname{Ker} D$ decomposes into irreducible unitary representations of G when L is any compact subgroup of G. This problem has previously not appeared in the literature.

The Dirac operator on G/K, having been already solved, we might expect to be able to reduce the problem to considering the Dirac operator on K/L. The compact case is a substantial problem within itself, and this will be the work undertaken in this Thesis. Details are given in Chapter 2, §1. The non-compact case will be considered in future work.

As far as I know, previous publications on this question consist only of: (i) the vanishing theorem of A. Lichnerowicz (see [26]) for the 'scalar Dirac operator', and (ii) the method first used by R. Parthasarathy in [28], which can be applied to the case of a compact symmetric pair of equal rank. This is noted in Chapter 4. See also the article of S. Helgason in [19], for results on general invariant differential operators and eigenspace representations.

(0.3) Thus, let (K,L), with L a subgroup of K, be a compact, Riemannian, spin pair. See Chapter 2, §1. Let (V,τ) be a unitary representation of L. Associated to $((,),\gamma)$ there is the 'twisted' Dirac operator $D = D_V$. Take $V = V_{\lambda_0-\rho_L}$ a simple L-module of 'highest weight' $\lambda_0-\rho_L$ (ρ_L is $\frac{1}{2}$ the sum of +ve roots for L, see Chapter 2, §3). Consider γ the Levi-Civita or reductive connection.

For a symmetric pair, the formula for the square D^2 takes its simplest form. Finding $\operatorname{Ker} D$ becomes equivalent to determining the primary K-submodules, in the L^2-space, belonging to a certain infinitesimal class. See Chapter 4. In Chapter 4, (3.2) we note that the technique previously used in [28], [31] can be applied to the case of a compact equal rank symmetric pair. This essentially involves a 'curvature vanishing argument' and then an application of Bott's Index Theorem. One can also obtain an elliptic complex from the Dirac operator on symmetric space, and use cohomology. See [30]. In (3.3) we deal with the case of unequal rank. This requires a knowledge of the structure theory of an unequal rank symmetric pair. Some properties that we need are worked out in (2.3).

(0.4) In Chapter 1, §2 we give a formula for the square D^2 (which holds for any reductive, Riemannian, spin pair (G,H)) due to John H. Rawnsley. This formula is a generalization, in geometric terms, of that given by R. Parthasarathy in [28] for a symmetric pair. We use this formula extensively.

Consider a general compact pair (K,L). Here the situation is a good deal more complicated. There is apparently no direct generalization of the methods we use for a symmetric pair. And seemingly no natural cohomology. We need to develop new techniques. These are described at the head of Chapters 5-9. An important technique, dealt with in Chapter 5, §1 is to tensor an induced representation with a finite dimensional representation. Then in §4 we consider $L = H$ a maximal torus of K. Initially our 'curvature vanishing argument' only gives information when the parameter λ is 'sufficiently non-singular'. In Chapter 9, we develop a technique for 'shifting the parameter'.

This is similar to the situation which arose in [31] for the Dirac operator on G/K , G a non-compact semi-simple Lie group, K a maximal compact subgroup, rank K = rank G . However there is a difference. In [31] the existence of the 'discrete series' is not assumed at the outset, but is constructed geometrically. For a sufficiently non-singular parameter, the Dirac operator is used to give information about the discrete series characters. Then it was found necessary to apply G. Zuckerman's tensor product technique [23] to shift the parameter. Previously things were done in reverse order, the existence of the discrete series, proved by Harish-Chandra, being used to get the geometric realization. Here, in the compact case we are of course assuming the representation theory of a compact, connected Lie group. The characters of the irreducibles are given by the H. Weyl formula. There is a geometrical construction for them due to Borel and Weil. We are thus able to gain information by 'shifting the Dirac operator'. Refer to Chapter 9.

Our method for handling (K,L) is independent of any cohomology or use of the Borel-Weil Theorem. Therefore Theorem 4, Chapter 5, (4.2), gives us an alternative construction of the irreducible representations of a compact, connected Lie group.

Having dealt with the case of an abelian pair in Chapter 7, we apply a technique of inducing in stages to the Dirac operator, developed in Chapter 6, and tackle the general case in Chapter 8.

Our main result is Theorem 8, Chapter 10. It is seen that Ker D is either zero or primary as a unitary K-module. This result is obtained without any deep structure theory of the homogeneous space K/L . However to compute 'the multiplicity' one needs structural information on the pair (K,L) .

CONTENTS.

CHAPTER 0.

In this chapter, which is essentially introductory, I will introduce our notation and collect together the necessary background material, which will be referred to and used later. References for further details and proofs are given within each section.

All the facts set down here, in this chapter, are known apart from where mentioned in §2.

§1. Representations of Lie Groups. Induced Vector Bundles.

(1.1) Refer to [7], [12], [16], [19], [20].
Let G be a (real, smooth) Lie group. The Lie algebra of G (i.e. the left invariant vector fields) will be denoted by g.

By a *representation of* G, we shall mean a pair (W,Π) where W is a real or complex Hilbert space and $\Pi:G \longrightarrow GL(W)$ is a homomorphism into the general linear group of W, such that the mapping $G\times W \longrightarrow W$, $(g,w) \longrightarrow \Pi(g)w$ is continuous. We also say that W is a G-*module* with G acting on W by $g.w = \Pi(g)w, g \in G, w \in W$. If W is finite dimensional, Π is then continuous and therefore analytic. For W real, complex Π is called *orthogonal, unitary* if Π is into $O(W)$, $U(W)$ the orthogonal, unitary group of W respectively.

For a representation $\phi:g \longrightarrow g\ell(W)$, of g, (i.e. ϕ is linear and $\phi[\xi,\eta] = [\phi(\xi),\phi(\eta)]$ $\xi,\eta \in g$ where $[\ \ ]$ is the Lie bracket of

g , $g\ell(W)$ respectively) we also say that W is a *g-module* with g acting on W by $\xi.w = \phi(\xi)w$, $\xi \in g$, $w \in W$.

Π can be differentiated to give a representation of g , $d\Pi$, called the differential of Π (if W is finite-dimensional) viz

$$d\Pi(\xi)w = \frac{d}{dt}\Pi(\exp t\xi)w\Big|_{t=0} \quad , \quad \xi \in g \; , \; w \in W \; .$$

(exp: $g \longrightarrow G$ is the exponential mapping of G .)

There is the *contragredient* representation (W^*,Π^*) of G . Also given another representation (W_1,Π_1) of G , there is the *direct sum* representation $(W\oplus W_1,\Pi\oplus\Pi_1)$, and the *tensor product* representation $(W\otimes W_1,\Pi\otimes\Pi_1)$ of G . And also of g . (See [12], [16]).

For each $x \in G$ let $A_x : G \longrightarrow G$ be the inner automorphism $A_x(g) = xgx^{-1}$. The derived automorphism of g is denoted $Ad_G(x)$ or $Ad(x) : g \longrightarrow g$. $Ad : G \longrightarrow GL(g)$ is a homomorphism, called the *adjoint representation* of G . The differential ad =: d Ad is called the *adjoint representation* of g . We have $ad\,\xi(\eta) = [\xi\eta]$ $\xi,\eta \in g$; also $Ad(\exp\xi) = e^{ad\xi}$, $x \exp \xi x^{-1} = \exp(Ad(x)\xi)$ for $\xi \in g$, $x \in G$. ($B \longrightarrow e^B$ is the exponential mapping of $GL(g)$) .

Let G be connected. G, g is said to be *reductive* if it has a finite dimensional completely reducible representation with discrete kernel, kernel zero respectively. Let H be a closed subgroup of G . H,h is said to be *reductive in* G , g if $Ad_G|_H$, $ad_g|_h$ is completely reducible respectively. G is said to be *semi-simple* if {e} is the only connected, soluble, normal subgroup (e is the identity element of G); equivalently if g is semi-simple. Every semi-simple G is equal to its derived group, and the center of G is discrete. G is said to be *simple* if {e} is the only connected normal subgroup. If G is also simply-connected, G is semi-simple iff (if and only if) G is the direct product of simple groups. (See [12], [19].)

N.B. There is a one-to-one correspondence between the connected Lie subgroups of G and the subalgebras of g ; which sends a connected normal subgroup of G to an ideal of g . (See [7].)

(1.2) Let G be a Lie group and H a closed subgroup. The quotient $G/H = \{gH; g \in G\}$. All such manifold structures, and mappings between them will be taken to be smooth (ie. C^∞) here. G acts on G/H by $L_g : G/H \longrightarrow G/H$, $L_g(g'H) = gg'H$, $g \in G$, making G/H into a *homogeneous space* (see [12]). At $x = gH \in G/H$, the *isotropy* (or *stability*) *subgroup* $G_x = gHg^{-1}$. The tangent map (see [19]) $L_{g*} : T(G/H) \longrightarrow T(G/H)$ (the tangent bundle of G/H) is a linear isomorphism from $T_x(G/H)$ (the tangent space at x) to $T_{g.x}(G/H)$, $g \in G$, $x \in G/H$. H acts on $T_{x_0}(G/H)$, $x_0 = eH$ (the identity coset), by $h \longrightarrow L_{h*}$. This is called the *isotropy* representation of H .

Let $X(G/H)$ denote the Lie algebra of vector fields on G/H (i.e. the space of sections of the tangent bundle with the 'usual' bracket,

see [19]). G acts on $\mathcal{X}(G/H)$ by $g.X$ where
$(g.X)(x) = L_{g*} X(L_{g^{-1}}(x))$, $g \in G$, $X \in \mathcal{X}(G/H)$. There is a homomorphism of Lie algebras

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathcal{X}(G/H) \\ \xi &\longrightarrow \tilde{\xi}, \quad \xi \in \mathfrak{g} \end{aligned}$$

where

$$\tilde{\xi}(x)f = \frac{d}{dt} f(\exp\text{-}t\xi x)\Big|_{t=0} , \quad f \in C(G/H)$$

(the (smooth) maps $G/H \longrightarrow \mathbb{R}$ (the real numbers) N.B. each $X \in \mathcal{X}(G/H)$ is a derivation of $C(G/H)$ as an $\mathbb{R}$-algebra). $g.\tilde{\xi} = (\mathrm{Ad}g\xi)^{\sim}$, for $g \in G$, $\xi \in \mathfrak{g}$. For fixed $x = gH \in G/H$, the linear map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathcal{X}(G/H) \\ \xi &\longrightarrow -\tilde{\xi}(x) , \end{aligned}$$

is surjective with kernel $\mathrm{Ad}g\,\mathfrak{h} = \mathfrak{g}_x$ (the Lie algebra of G_x).

(1.3) Refer to [16].
There is the principal H-bundle $H \xrightarrow{\iota} G \xrightarrow{\Pi} G/H$. Let (V,κ) be a representation of H . On $G \times V$ we define the equivalence relation $(g,v) \sim (g',v')$ if $g' = gh$, $v' = \kappa(h)^{-1}v$ for some $h \in H$. Take $G\times_H V = \{[g,v] \; ; \; g \in G, \; v \in V\}$ the set of equivalence classes. Put $(\underline{V})^G_H =: G\times_H V$, sometimes we will write just $\underline{V}$, and define $P_V : \underline{V} \longrightarrow G/H$, $P_V[g,v] = gH$. Then $(\underline{V})^G_H$ can be made into a G-vector bundle over G/H (see [16],[18]), which we will call the *induced vector bundle* by (V,κ) . G acts on $\underline{V}$ by $g[g'v] = [gg'v]$, $g \in G$. There is a linear isomorphism between each fibre $\underline{V}_x =: P_V^{-1}\{x\}$, $x \in G/H$

and V given by ${}_gP_V : V \longrightarrow \underline{V}_x$, $x = gH$; ${}_gP_V(v) = [gv]$.

For the contragredient representation (V^*, κ^*) , there is the G-equivalence (of vector bundles) $(\underline{V}^*)_H^G \simeq (\underline{V})_H^{G*}$ (* denotes dual). Also if (V_1, κ_1) is a representation of H , there are the G-equivalences $\underline{V \oplus V_1} \simeq \underline{V} \oplus \underline{V}_1$, $\underline{V \otimes V_1} \simeq \underline{V} \otimes \underline{V}_1$.

Let $\Gamma(\underline{V})_H^G$ denote the space of sections of $\underline{V}$, i.e. maps $f: G/H \longrightarrow \underline{V}$ with $P_V \circ f = id_V$. Writing $f(gH) = [g, \hat{f}(g)]$, $g \in G$, we see that $\Gamma(\underline{V})$ can be identified with the maps $\hat{f}: G \longrightarrow V$ satisfying $\hat{f}(gh) = \kappa(h)^{-1}\hat{f}(g)$, $g \in G$, $h \in H$. G acts on $\Gamma(\underline{V})$ by $g.f$ where $(g.f)(x) = g.f(g^{-1}x)$; or equivalently $g.\hat{f} =: \hat{g.f}$ so $(g.\hat{f})(g') = \hat{f}(g^{-1}g')$, $g \in G$, $f \in \Gamma(\underline{V})$. So we get a representation $(\Gamma(\underline{V}), \tilde{\Pi})$ of G .

Note that an H-map $V \xrightarrow{a} V_1$ induces a vector bundle map $\underline{V} \xrightarrow{a} \underline{V}_1$, $a[g,v] = [g, a(v)]$ and so also a linear map $\Gamma(\underline{V}) \xrightarrow{a} \Gamma(\underline{V}_1)$; we denote these also by a.

Note that if κ is orthogonal, unitary and $\langle , \rangle$ is the inner product on V , we get $\langle , \rangle_x$ on $\underline{V}_x$ by $\langle [g,u],[g,v] \rangle_x = \langle u,v \rangle$, $x = gH$; thus giving $\underline{V}$ a real or complex *Riemannian structure* respectively. The metric $\langle , \rangle$ is G-invariant i.e. $\langle g\cdot, g\cdot \rangle_{gx} = \langle \cdot, \cdot \rangle_x$.

(1.4) Let $C(G,V)$ be the smooth maps $G \longrightarrow V$. G acts on $C(G,V)$ by $L_g f$ where $(L_g f)(g') = f(g^{-1}g')$ and also by $R_g f$, where

$(R_g f)(g') = f(g'g)$, $g \in G$, $f \in C(G,V)$. So we get the anti-representations L, R of G on C(G,V) . The differentials are dL , dR where $dL(\xi)_g f = \frac{d}{dt} f(\exp -t\xi g)\big|_{t=0}$, $dR(\xi)_g f = \frac{d}{dt} f(g \exp t\xi)\big|_{t=0}$, then $- dL(Adg\xi)_g f = dR(\xi)_g f$, $g \in G$, $\xi \in \mathfrak{g}$, $f \in C(G,V)$. Also if $f \in \Gamma(\underline{V})_H^G$, $dR(\zeta)_g \hat{f} = - d\kappa(\zeta)\hat{f}(g)$, $\zeta \in \mathfrak{h}$, $g \in G$.

§2. Invariant Connections on Induced Vector Bundles.

Two points should be brought to notice concerning the results that I state and prove in this section. Firstly invariant connections have been studied before on principal bundles (see [9]). Here we study the situation on an induced vector bundle. A lot of this material is probably well-known, but we cannot find a reference. Secondly, I appreciate the help of Dr. John H. Rawnsley in formulating the material of this section. Especially the statement of Proposition 1 was communicated to me by him. The proof given is my own.

(2.1) Let G be a Lie group and H a closed subgroup. Take a representation (V,κ) of H , and form $(\underline{V})_H^G$ (see §1. (1.3)). Let ∇ be a *connection* on $(\underline{V})_H^G$. So $\nabla_X : \Gamma(\underline{V}) \longrightarrow \Gamma(\underline{V})$ is a linear operator for each $X \in \mathcal{X}(G/H)$ (see §1. (1.2)), satisfying (i) $\nabla_{aX} f = a\nabla_X f$
(ii) $\nabla_X(af) = a\nabla_X f + (X.a)f$ (the Leibniz rule), and
(iii) $\nabla_{X+Y} = \nabla_X + \nabla_Y$, $X,Y \in \mathcal{X}(G/H)$, $a \in C(G/H)$, $f \in \Gamma(\underline{V})$.

Put $\Omega^p(G/H,\underline{V}) = \Gamma(\Lambda^p T^*(G/H) \otimes \underline{V})$, the $\underline{V}$-valued p-forms $p \in W$ (the whole numbers), here * denotes the dual bundle, and Λ^p denotes the

p^{th} exterior power. (See [6] .) Now $\Lambda^p T^*(G/H) \otimes \underline{V} \simeq \mathrm{Hom}(\Lambda^p T(G/H), \underline{V})$ as G-vector bundles, (for constructions on vector bundles see [16]), so we can identify

$$\Omega^p(G/H,\underline{V}) \simeq \mathrm{Hom}\ (\Lambda^p X(G/H), \Gamma(\underline{V})) \quad \text{by if}$$

$\beta \in \Omega^p(\ \ )$, get $\beta(X_1,\ldots,X_p)(x) = \beta(x)(X_1(x)\wedge\ldots\ldots\wedge X_p(x))$, $X_i \in X(G/H)$, $x \in G/H$, (here $\wedge$ denotes exterior multiplication) (see [6]).

Then we can view ∇ as a linear map $\nabla : \Gamma(\underline{V}) \longrightarrow \Omega'(G/H,\underline{V})$ by $(\nabla f)(X) = \nabla_X f, X \in X(G/H), f \in \Gamma(\underline{V})$.

There is a map

$$L_g^* : \Omega^p(G/H,\underline{V}) \longrightarrow \Omega^p(G/H,\underline{V}), g \in G, \quad p \in \mathbb{N},$$

called the *pull-back* defined by

$$(L_g^*\beta)(X_1,\ldots,X_p) = g^{-1}.(\beta \circ L_g)(L_{g*}X_1,\ldots\ldots,L_{g*}X_p)$$

i.e. $$(L_g^*\beta)(x)(X_1(x),\ldots,X_p(x)) = g^{-1}.(\beta \circ L_g)(x)(L_{g*}X_1(x),\ldots,L_{g*}X_p(x)) .$$

We say that ∇ is G-*invariant* if invariant by the left translations

i.e. $$L_g^*(\nabla(g.f)) = \nabla f\ ,\ g \in G\ ,\ f \in \Gamma(\underline{V}) . \qquad (2.1.1)$$

We will use the notation of §1.

(2.2) Let G/H be *reductive* (i.e. H reductive in G see §1 (1.1)), so we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ a vector space direct sum for some subspace $\mathfrak{m}$, with $\mathfrak{m}$ Ad H-invariant. Thus $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$.

Lemma 1.

(i) By the pair $(\mathfrak{m},\mathrm{Ad})$, a representation of H, we can identify $(\underline{\mathfrak{m}})_H^G \simeq T(G/H)$ as G-vector bundles.

(ii) Under (i), (see the proof), we have $\tilde{\xi}(gH) = [g,\hat{\tilde{\xi}}(g)]$ where $\hat{\tilde{\xi}}(g) = -P(\mathrm{Ad}g^{-1}\xi)$, $g \in G$, $\xi \in \mathfrak{g}$, and $P:\mathfrak{g} \longrightarrow \mathfrak{m}$ is the projection.

Proof.

(i) We define a linear bijection $\mathfrak{m} \longrightarrow T_{x_0}(G/H), x_0=eH$, (the identity coset) by $\xi \longrightarrow -\tilde{\xi}(x_0)$, $\xi \in \mathfrak{m}$. Now $(\mathrm{Ad}h\xi)^{\sim}(x_0) = (h.\tilde{\xi})(x_0) = L_{h*}\tilde{\xi}(x_0)$, $h \in H$, (see §1 (1.2)). So $(\mathfrak{m},\mathrm{Ad})$ and the isotropy representation of H are equivalent. Then the x_0-fibre map $[e,\xi] \longrightarrow -\tilde{\xi}(x_0$, $\xi \in \mathfrak{m}$, gives rise to a G-vector bundle isomorphism. (See [16].)

(ii) From (i), $\hat{\tilde{\xi}}(e) = -P\xi$, $\xi \in \mathfrak{g}$. Then $\hat{\tilde{\xi}}(g) = (g^{-1}.\hat{\tilde{\xi}})(e)$

$= (g^{-1}.\tilde{\xi})\hat{}(e) = (\mathrm{Ad}g^{-1}\xi)\hat{\tilde{}}(e) = -P(\mathrm{Ad}g^{-1}\xi)$. □

Let (,) be an inner product on $\mathfrak{m}$ w.r.t (with respect to) which $(\mathfrak{m},\mathrm{Ad})$ is orthogonal. Transporting this onto each fibre of $\underline{\mathfrak{m}}$, we thus make G/H into a *Riemannian homogeneous space* (i.e. $T(G/H)$ becomes real Riemannian).

N.B. In future we use the identification in Lemma 1 (i) without comment. Therefore $X(G/H) \simeq \Gamma(\underline{m})^G_H$.

We also identify $m^* \simeq m$ as orthogonal H-modules via $(\ ,\ )$. And thus identify $\underline{m}^* \simeq \underline{m}$ as G-vector bundles (see §1 (1.3)).

Lemma 2.

(i) As a linear map $\Gamma(\underline{V})^G_H \longrightarrow \Gamma(\underline{m^* \otimes V})^G_H (\simeq \Gamma(\underline{m \otimes V}))$

the G-invariance condition (2.1.1) for ∇ is equivalent to

$g.\nabla f = \nabla(g.f)$, $g \in G$, $f \in \Gamma(\underline{V})$.

i.e. $g.\nabla_X f = \nabla_{g.X} g.f$, $X \in \Gamma(\underline{m})$.

(ii) $(\nabla_X f)(x) = g.(\nabla_{g^{-1}.X}\, g^{-1}.f)(x_0)$, $x = gH$; $(\nabla_X f)\hat{}(g) = (\nabla_{g^{-1}X}\, g^{-1}f)\hat{}(e)$.

Proof.

There are G-vector bundle isomorphisms $Hom(\underline{m},\underline{V}) \simeq \underline{Hom(m,V)} \simeq \underline{m^* \otimes V}$. Then under these,

$$(g.\nabla f)(x)X(x) = g.(\nabla f)(g^{-1}x)g.(g^{-1}X)(g^{-1}x) = g.(\nabla f(g^{-1}x)(g^{-1}X)(g^{-1}x)) .$$

The condition $L^*_{g^{-1}}(\nabla f) = (g.f)$ becomes

$g.(\nabla f(g^{-1}x)g^{-1}.(X(x))) = \nabla(g.f)(x)X(x)$. So this is equivalent to $g.(\nabla f) = \nabla(g.f)$ and $g.(\nabla_{g^{-1}X} f) = \nabla_X(g.f)$.

□

Proposition 1.

A G-invariant connection ∇ on $(\underline{V})_H^G$ is determined by a linear map γ: ——> End V (the endomorphisms of V) satisfying

(i) $\gamma(\xi) = d\kappa(\xi)$, $\xi \in h$

(ii) $\gamma(\mathrm{Ad}h\xi) = \kappa(h)\circ\gamma(\xi)\circ\kappa(h)^{-1}$, $h \in H$, $\xi \in g$.

Then $\nabla_{\tilde{\xi}} f = \xi.f - \Lambda(\xi) f$, $\xi \in g$, $f \in \Gamma(\underline{V})$ (2.2.1)

where $\Lambda: g \longrightarrow \mathrm{End}\ \underline{V}$ is given by $\Lambda_x: g \longrightarrow \mathrm{End}\ \underline{V}_x$ for each $x \in G/H$, with $\Lambda_{x_0}(\xi)[e,v] = [e,\gamma(\xi)v]$, $v \in V$, $x_0 = eH$, and

$$\Lambda_x(\xi) = g\circ\Lambda_{x_0}(\mathrm{Ad}g^{-1}\xi)\circ g^{-1} \text{ , } x = gH \text{ , } g \in G$$

i.e. (2.2.1) does define a G-invariant connection, and everyone such is of this form. (N.B. here $\xi.f = d\tilde{\Pi}(\xi)f$, see §1, (1.3). By the Leibniz rule and §1, (1.2.1), it is sufficient to know ∇_X for $X = \tilde{\xi}$, $\xi \in g$.)

Proof.

Any two connections on $\underline{V}$ differ by an End $\underline{V}$-valued 1-form on G/H (see [9]) i.e.

$$\nabla - {}'\nabla = \beta \in \Omega'(G/H, \mathrm{End}\ \underline{V}) \text{ ,}$$

so $$(\nabla_{\tilde{\xi}} f)(x) - ({}'\nabla_{\tilde{\xi}} f)(x) = \beta_x(\tilde{\xi}(x))f(x) \text{ , } \xi \in g \text{ .}$$

Define $\beta_x : g \longrightarrow \mathrm{End}\, \underline{V}_x$, by $\beta_x(\xi) = -\beta_x(\tilde{\xi}(x))$. So $\beta_x = 0$ on g_x (see §1 (1.2)). Here $'\nabla$ is a fixed invariant connection. The invariance condition Lemma 2(ii) for ∇ becomes $\beta_{x_0}(\xi) = h \circ \beta_{x_0}(\mathrm{Ad}h^{-1}\xi) \circ h^{-1}$, $h \in H$; $\beta_x(\xi) = g \circ \beta_{x_0}(\mathrm{Ad}g^{-1}\xi) \circ g^{-1}$, $g \in G$.

Define $\alpha_x : g \longrightarrow \mathrm{End}\, \underline{V}_x$, by $\alpha_{x_0}[e,v] = [e, d\kappa(Q(\xi))v]$ and then $\alpha_x(\xi) = g \circ \alpha_{x_0}(\mathrm{Ad}g^{-1}\xi) \circ g^{-1}$. This is well-defined since $d\kappa(\mathrm{Ad}h\xi) = \kappa(h) \circ d\kappa(\zeta) \circ \kappa(h)^{-1}$, $\zeta \in h$ and Ad h commutes with $Q = 1-P$, $h \in H$. Take

$$('\nabla_{\tilde{\xi}} f)(x) = (\xi f)(x) - \alpha_x(\xi) .$$

$$(\mathrm{Ad}g\xi.(g.f))\hat{}\,(g') = -\frac{d}{dt}\, g.\hat{f}(\exp t\, \mathrm{Ad}g\xi g')\Big|_{t=0}$$

$$= -\frac{d}{dt}\, \hat{f}(\exp t\xi g^{-1} g')\Big|_{t=0} = (g.(\xi.\hat{f}))(g').$$

So $'\nabla$ is G-invariant.

For $\xi = \mathrm{Ad}g\, \zeta \in g_x$, $\zeta \in h$, we have $(\xi.\hat{f})(g) = d\kappa(\zeta)\, \hat{f}(g)$; then $(\nabla_{\tilde{\xi}} f)(x) = 0$, $\xi \in g_x$ and we see that (2.2.1) (and $'\nabla$) is well-defined.

Now defining $\gamma = d\kappa \circ Q + \beta$ and $\Lambda_x = \alpha_x + \beta_x$, we have (i), (ii) and (2.2.1) . Also we see that (2.2.1) does define an invariant connection.

□

Corollary.

$$(\nabla_{\tilde{\xi}} f)^{\wedge}(g) = dL(\xi)_g \hat{f} - \gamma(Adg^{-1}\xi)\hat{f}(g), {}_{g \in G, \xi \in \mathfrak{g}, f \in \Gamma(\underline{V})} .$$

Proof.

From (2.2.1) at x_0 , $(\nabla_{\tilde{\xi}} f)^{\wedge}(e) = dL(\xi)_e \hat{f} - \gamma(\xi)\hat{f}(e)$.

Then $(\nabla_{\tilde{\xi}} f)^{\wedge}(g) = (g.^{-1}\nabla_{\tilde{\xi}} f)^{\wedge}(e) = (\nabla_{g^{-1}\tilde{\xi}} g.^{-1} f)^{\wedge}(e) = \ldots\ldots$

□

(2.3) See §1, (1.3). Under the H-isomorphism $V \longrightarrow V^*$ via <,> (the inner product on V , see §1, (1.1)) there is *the dual* G-invariant connection ∇^* on $\underline{V}^*$, by γ^* . Here ∇ is a G-invariant connection on $\underline{V}$.

Also given a G-invariant connection ${}_1\nabla$ on $\underline{V}_1$, by γ_1 , there is the *direct sum* G-invariant connection $\nabla^{\oplus}$ on $\underline{V \oplus V_1}$, by $\gamma^{\oplus}$, where $\nabla^{\oplus}_X(f+f_1) = \nabla_X f + {}_1\nabla_X f_1$; $\gamma^{\oplus} = \gamma + \gamma_1$. And there is the *tensor product* G-invariant connection $\nabla^{\otimes}$ on $\underline{V \otimes V_1}$, by $\gamma^{\otimes}$, where $\nabla^{\otimes}_X(f \otimes f_1) = \nabla_X f \otimes f_1 + f \otimes {}_1\nabla_X f_1$; $\gamma^{\otimes} = \gamma \otimes 1 + 1 \otimes \gamma_1$. $X \in \mathcal{X}(G/H)$, $f \in \Gamma(\underline{V})$, $f_1 \in \Gamma(\underline{V}_1)$.

Let (V,κ) be orthogonal or unitary according as V is real or complex, so $\underline{V}$ becomes real or complex Riemannian with metric <,> (see §1. (1.3)). For $f,f_1 \in \Gamma(\underline{V})$, define $(f,f_1) \in C(G/H)$, by

$(f,f_1)(x) = \langle f(x),f_1(x)\rangle_x$; and $(\hat{f},\hat{f}_1) = (f,f_1)\circ \mathcal{H} \in C(G)$.

A connection ∇ is said to be *metric* if

$$X(f,f_1) = (\nabla_X f,f_1) + (f,\nabla_X f_1), \quad X\in X(G/H), f,f_1 \in \Gamma(\underline{V}) . \tag{2.3.1}$$

Lemma 3.

A G-invariant connection ∇ is metric iff $\gamma: g \longrightarrow \mathfrak{so}(V)(\mathfrak{u}(V))$ (the skew-symmetric (skew-hermitian) endomorphisms w.r.t $\langle,\rangle$) (iff $\Lambda: \longrightarrow \mathfrak{so}(\underline{V})(\mathfrak{u}(\underline{V}))$.

Proof.

This follows from Proposition 1. If ∇' is metric, then ∇ is metric iff $\beta \in \Omega'(G/H,\mathfrak{so}(\underline{V}))$. As the metric on $\underline{V}$ is G-invariant,

$$\tilde{\xi}(f,f_1) = (\xi.f,f_1) + (f,\xi.f_1) , \ \xi \in g \quad . \tag{2.3.2}$$

Note that $(f,f_1)(x) = (g^{-1}.f,g^{-1}.f_1)(x_0)$, $x = gH$, $g \in G$, so $\tilde{\xi}(x)(f,f_1) = (g^{-1}.\xi)^{\sim}(x_0)(g^{-1}.f,g^{-1}.f_1)$. Thus by Lemma 2 (ii), it is sufficient to check (2.3.1) at the identity coset $x_0 = eH$. But from (2.3.2) we see that ∇ is metric iff $\Lambda: \longrightarrow \mathfrak{so}(\underline{V})$. In particular $'\nabla$ is metric.

□

(2.4) Refer to (2.1) for notation.

The *curvature* 2-*form* $R(,)$, in $\Omega^2(G/H, \text{End}\,\underline{V})$, of ∇ is

$$R(X,Y) = [\nabla_X \nabla_Y] - \nabla_{[XY]}$$

where $[\nabla_X \nabla_Y] = \nabla_X\nabla_Y - \nabla_Y\nabla_X$, and $[XY]$ is the bracket of vector fields, $X,Y \in X(G/H)$.

Let ${}_{1}\nabla$ be a connection on $T(G/H)$. The *torsion* 2-*form* $T(,)$, in $\Omega^2(G/H,T(G/H))$, is

$$T(X,Y) = {}_{1}\nabla_X Y - {}_{1}\nabla_Y X - [XY] , \quad X,Y \in X(G/H).$$

Let G/H be reductive (see (2.2)) and suppose that ∇, ${}_{1}\nabla$ are G-invariant. Then it is sufficient to compute $R(,)$, $T(,)$ at $x_0 = eH$ (the identity coset).

Lemma 4.

(i) $(R(X,Y)f)(x) = g.R(g.^{-1}X,g.^{-1}Y)g.^{-1}f)(x_0),(R(X,Y)f)\hat{}(g) = (R(g.^{-1}X,g.^{-1}Y)g.^{-1}f)\hat{}(e$

(ii) $T(X,Y)(x) = g.\ T(g.^{-1}X,g.^{-1}Y)(x_0),\ T(X,Y)\hat{}(g) = T(g.^{-1}X,g.^{-1}Y)\hat{}(e)$

$x = gH,\ g \in G\ ,\ f \in \Gamma(\underline{V})\ ,\ X = \tilde{\xi},\ Y = \tilde{\eta},\ \xi,\eta \in g$.

Proof.

Follows from $g.(R(X,Y)f) = R(g.X,g.Y)g.f, g.T(X,Y) = T(g.X,g.Y)$ and $f(x) = f(gx_0) = g.(g.^{-1}f)(x_0)$.

□

Define $R(,)$, in $\Lambda^2 g^* \otimes \Gamma(\text{End } \underline{V})$, by

$$R(\xi,\eta) = R(\tilde{\xi},\tilde{\eta})\hat{\ }(e) \quad (\text{i.e. } R(\xi,\eta)f = (R(\tilde{\xi},\tilde{\eta})f)\hat{\ }(e), f \in \Gamma(\underline{V}));$$

also $T(\ ,\ )$ in $\Lambda^2 g^* \otimes m$, by $T(\xi,\eta) = T(\tilde{\xi},\tilde{\eta})\hat{\ }(e)$, $\xi,\eta \in g$.

Lemma 5.

$$R(\xi,\eta) = dL(Q[\xi,\eta])_e + [\gamma(\xi),\gamma(\eta)] - \gamma(P[\xi,\eta]) \quad \text{and}$$

$$T(\xi,\eta) = -P[\xi,\eta] + \gamma_1(\xi)P\eta - \gamma_1(\eta)P\xi, \ \xi,\eta \in g$$

where ∇, ${}_1\nabla$ is given by γ, γ_1 respectively (see Proposition 1) .

($P: g \longrightarrow m$ is the projection, $Q = 1-P$.)

Proof.

We have

$$(\nabla_{\tilde{\xi}}(\nabla_{\tilde{\eta}}f))\hat{\ }(e) = (dL(\xi)dL(\eta)-\gamma(\eta)dL(\xi)-\gamma(\xi)dL(\eta) + \gamma(\xi)\gamma(\eta))\hat{f}(e) \ .$$

Therefore $([\nabla_{\tilde{\xi}}\nabla_{\tilde{\eta}}]f)\hat{\ }(e) = (dL[\xi\eta] + [\gamma(\xi)\gamma(\eta)])\hat{f}(e)$.

Now $(\nabla_{\tilde{\zeta}}f)\hat{\ }(e) = 0$, $\zeta \in h = g_{x_0}$. So

$(\nabla_{[\xi\eta]^\sim}f)\hat{\ }(e) = (dL((1-Q)[\xi\eta]) + \gamma(P[\xi\eta]))\hat{f}(e)$. Thus get $R(\xi,\eta)$.

$$T(\tilde{\xi},\tilde{\eta})\hat{\ }(e) = (dL(\xi)\hat{\tilde{\eta}} - dL(\eta)\hat{\tilde{\xi}} - \gamma_1(\xi)\hat{\tilde{\eta}} + \gamma_1(\eta)\hat{\tilde{\xi}} - [\xi\eta]\hat{\tilde{}}\,)(e) \ .$$

Thus get $T(\xi,\eta)$.

□

Now let G/H be reductive, Riemannian (see (2.2)).

Definition.

∇ given by γ , with $\gamma = 0$ on m , is called the *reductive connection* on $(\underline{V})^G_H$. For (V,κ) orthogonal or unitary, it is metric. In particular the reductive connection on $\underline{m}$ is metric.

There is a unique connection ${}_0\nabla$ on $T(G/H) = (\underline{m})^G_H$, which is metric and torsion-free (i.e. $T(,) = 0$) called the *Levi-Civita connection.* Thus ${}_0\nabla$ must be given by γ_0 , with $\gamma_0(\xi) = \frac{1}{2}P\circ ad\xi$, $\xi \in m$. (See Proposition 1 and Lemma 3.)

(2.5) Let G/H be reductive, Riemannian (see (2.2)), with a G-invariant connection ${}_1\nabla$, by γ_1 , on $T(G/H) = (\underline{m})^G_H$. Let (V,κ) be a representation of H and ∇ , by γ , a G-invariant connection on $(\underline{V})^G_H$.

We define an inner product (,) on $X(G/H) = \Gamma(\underline{m})^G_H$, by

$$(X,Y) = (\hat{X}(e),\hat{Y}(e)) , \quad X,Y \in X(G/H) .$$

Let $\{\xi_i\}$ be an orthonormal (w.r.t(,)) basis for m. Put $X_i = \tilde{\xi}_i \in X(G/H)$, then $(X_i,X_j) = \delta_{ij}$.

Take the composition

$$\Gamma(\underline{V}) \xrightarrow{\nabla} \Gamma(T^*\otimes\underline{V}) \xrightarrow{\nabla} \Gamma(T^*\otimes T^*\otimes\underline{V})$$

(here $T = T(G/H)$, and $*$ denotes the dual).

For $f_1 \in Hom(X(G/H), \Gamma(\underline{V}))$, we have $\nabla f_1 \in Hom(\otimes^2 X(G/H), \underline{V})$, given by

$$(\nabla f_1)(X,Y) = \nabla_X(f_1(Y)) - f_1(\nabla_X Y) .$$

So with $f_1 = \nabla f$,

$$(\nabla^2 f)(X,Y) = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f , \qquad X,Y \in X(G/H), f \in \Gamma(\underline{V}) .$$

The *Laplacian* Δ of $\underline{V}$ is given by

$$\Delta = -\mathrm{tr}\, \nabla^2 : \Gamma(\underline{V}) \longrightarrow \Gamma(\underline{V})$$

i.e. $$\Delta = - \sum_i (\nabla^2_{X_i} - \nabla_{\nabla_{X_i} X_i})$$

Δ is G-invariant (i.e. $\Delta(g.f) = g.\Delta f$).

We may identify ∇f with a map (see (2.2))

$$(\nabla f)\hat{} : G \longrightarrow \mathrm{Hom}(g,V)$$

where $(\nabla f)\hat{}(g)(\xi) = (\nabla_{-g.\tilde{\xi}} f)\hat{}(g)$, $\quad g \in G,\ \xi \in g,\ f \in \Gamma(\underline{V})$.

($\mathrm{Hom}(g,V)$ is the space of linear maps $g \longrightarrow V$. $\mathrm{Hom}(g,V) \simeq g^* \otimes V$).

Note that $(\nabla f)\hat{}(g)(\zeta) = 0$, $\zeta \in h$.

Proposition 2.

(i) $(\nabla f)\hat{}(g)(\xi) = dR(\xi)_g \hat{f} + \gamma(\xi)\hat{f}(g)$, $\quad g \in G,\ \xi \in g$.

(ii) Considering $\nabla^2 f$ as a map $(\nabla^2 f)^\wedge : G \longrightarrow \mathrm{Hom}(\otimes^2 g, V)$,

we have $(\Delta f)^\wedge = -\mathrm{tr}\, (\nabla^2 f)^\wedge$, then

$$-(\Delta f)^\wedge = \{\sum_i (dR(\xi_i)+\gamma(\xi_i))^2 - (dR(\gamma_{\scriptscriptstyle|}(\xi_i)\xi_i)+\gamma(\gamma_{\scriptscriptstyle|}(\xi_i)\xi_i))\}\hat{f} \ , \ f \in \Gamma(\underline{V}) \ .$$

<u>Proof</u>.

(i) This follows from the corollary to Proposition 1. Recall that $g.\tilde{\xi} = (\mathrm{Ad}g\xi)^\sim$ and $-\ dL(\mathrm{Ad}g\xi)_g = dR(\xi)_g$, $g \in G$, $\xi \in g$.

(ii) We could proceed by:

$$(\nabla f_1)^\wedge(g)(\xi,\eta) = (\nabla \hat{f}_1(\eta))^\wedge(g)(\xi) - \hat{f}_1((_{\scriptscriptstyle|}\nabla\tilde{\eta})^\wedge(\xi))(g) \ ,$$

f_1 a section of $\underline{\mathrm{Hom}(m,V)} \simeq \mathrm{Hom}(\underline{m},\underline{V})$; then put $f_1 = \nabla f$, and take the trace.

However, consider $(\Delta f)^\wedge(e)$.

$$(\nabla_X^2 f)^\wedge(e) = (dL(\xi) - \gamma(\xi))^2 \hat{f}(e) \quad \text{for} \quad X = \tilde{\xi} \ , \ \xi \in g \ .$$

Now

$${}_{\scriptscriptstyle|}\nabla_X X = \sum_j ({}_{\scriptscriptstyle|}\nabla_X X, X_j)X_j \ .$$ We have

$$dL(\xi)_e \hat{\tilde{\xi}} = \frac{d}{dt} \hat{\tilde{\xi}}(\exp -t\xi)\Big|_{t=0} = -\frac{d}{dt} P(\mathrm{Ad}(\exp t\xi)\xi)\Big|_{t=0}$$
$$= - P[\xi,\xi] = 0 \ .$$

So

$$({}_{\scriptscriptstyle 1}\nabla_X X, X_j) = -\,(({}_{\scriptscriptstyle 1}\nabla_X X)\hat{\ }(e), \xi_j) = -(\gamma_1(\xi)P\xi, \xi_j) \quad \text{and}$$

$$(\nabla_{{}_1\nabla_X X} f)\hat{\ }(e) = -\sum_j (\gamma_1(\xi)P\xi, \xi_j)(dL(\xi_j)\hat{f} - \gamma(\xi_j)\hat{f}(e))$$

$$= -\,(dL(\gamma_1(\xi)P\xi) - \gamma(\gamma_1(\xi)P\xi))\hat{f}(e)\ .$$

Now put $\xi = \xi_i$, sum over i , and use the G-invariance of Δ . □

(2.6) Let G/H be reductive, and $(V,\kappa),(V_1,\kappa_1)$ representations of H . Take a G-invariant connection ∇ on $\underline{V}$ and an H-map $m \otimes V \xrightarrow{a} V_1$. By composing

$$\Gamma(\underline{V})_H^G \xrightarrow{\nabla} \Gamma(\underline{m \otimes V})_H^G \xrightarrow{a} \Gamma(\underline{V_1})_H^G$$

we get a left G-invariant 1^{st} order differential operator $D = a\circ\nabla$, with *symbol map* a . (See [30].) (Here G-invariant means $g.Df = Dg.f$.) If $a(\xi):V \to V_1$ is a linear isomorphism for each $\xi \neq 0$, D is *elliptic*.

§3. Induced Representations. (Refer to [16], [18].)

We use the notation of (1.3). Recall that we may identify $\Gamma(\underline{V})_H^G$ with the maps $f:G \longrightarrow V$ satisfying $f(gh) = \kappa(h)^{-1}f(g)$, $g \in G$, $h \in H$; and G acts by $g.f = \tilde{\Pi}(g)f$ where $(g.f)(g') = f(g^{-1}g')$, $g \in G$, $f \in \Gamma(\underline{V})$.

Make the space $\Gamma_c(\underline{V})_H^G$, of compactly supported sections, into a pre-Hilbert space by setting

$$\langle f_1,f_2\rangle = \int_G \langle f_1(g),f_2(g)\rangle\, dg$$

(where $\langle v_1,v_2\rangle$ is the inner product on V , and dg is the Haar measure on G.) The separable Hilbert space $L^2(\underline{V})_H^G$, square-integrable sections, is the completion. $(\Gamma_c(\underline{V})_H^G,\tilde{\Pi})$ extends to $(L^2(\underline{V})_H^G,\tilde{\Pi})$ called the *induced representation of* G *by* $(\underline{V},\kappa)$. This is unitary if κ is unitary.

Let κ,κ_1 be unitary, then

$$L^2(\underline{V \oplus V_1})^G_H = L^2(\underline{V})^G_H \oplus L^2(\underline{V_1})^G_H \text{ and}$$

$$L^2(\underline{V \otimes V_1})^G_H = L^2(\underline{V})^G_H \otimes L^2(\underline{V_1})^G_H \text{ as unitary G-modules.}$$

If we regard the complex numbers $\mathbb{C}$ as the 1-dim trivial unitary H-module with $\langle a,b\rangle = a\bar{b}$, $a,b \in \mathbb{C}$, and we take $H = \{e\}$; then $L^2(\underline{\mathbb{C}})^G_{\{e\}}$ is just $L^2(G)$, the square integrable, complex-valued functions on G , with the *left regular representation* L of G . Also have the *right regular representation* R of G . These are unitary. See (1.4).

(3.2) The Peter-Weyl theorem and Frobenius reciprocity.

Let G be compact. So H is also compact. A representation of a compact Lie group G is unitarizable and completely reducible. Also an irreducible representation of G is finite-dimensional (in fact 1-dim for G abelian). Let $\hat{G}$ denote the (countable) set of equivalence classes of irreducible unitary representations of G . Let for each $\nu \in \hat{G}$, (U_ν,Π_ν) be a representative. Take an inner product on V such that κ is unitary (see (3.1)) and suppose that V is finite-dimensional.

Let $\Gamma_\nu(\underline{V})^G_H$ be the subspace of $\Gamma(\underline{V})^G_H$ that transforms under G according to Π_ν .

Define injection $\iota_\nu : U_\nu \otimes \mathrm{Hom}_H(U_\nu,V) \longrightarrow \Gamma_\nu(\underline{V})^G_H$

by $\iota_\nu(v\otimes b)(g) = b(\Pi_\nu(g)^{-1}v)$, $v \in U_\nu$, $b \in \mathrm{Hom}_H(\ , \ )$

(the space of H-maps $U_\nu \longrightarrow V$.)

Then $L^2(\underline{V})^G_H = \Sigma_{\nu\in\hat{G}} \oplus \Gamma_\nu(\underline{V})^G_H$ (an orthogonal direct sum), ι_ν is onto and

$$\tilde{\Pi} = \Sigma_\nu \tilde{\Pi}_\nu \quad \text{(a unitary direct sum)}$$

where
$$\tilde{\Pi}_\nu(g) =: \tilde{\Pi}(g)\iota_\nu, g \in G$$
$$= (\iota_\nu \Pi_\nu(g)\otimes 1) . \quad \text{(See (1.3).)}$$

We shall refer to $\Gamma_\nu(\underline{V})$ as the *ν-primary* G*-submodule* in $L^2(\underline{V})$ of *multiplicity*, the number of 'copies' of U_ν there-in i.e. $\dim_{\mathbb{C}} \mathrm{Hom}_H(U_\nu,V) := i_H(U_\nu,V) = i_G(L^2(\underline{V}),U_\nu) =: \dim_{\mathbb{C}}\mathrm{Hom}_G(L^2(\underline{V}),U_\nu)$.

(3.3) <u>Bott's Index Theorem.</u>

See [24], [16].

Let G be compact. Let $\mathbb{Z}[\hat{G}]$ be the Grothendieck ring of virtual (finite-dim) G-modules (under ⊕, ⊗ the direct sum, tensor product). There is the canonical map $U \longrightarrow [U]$ from the finite-dim G-modules to $\mathbb{Z}[\hat{G}]$. $\{[U_\nu]; \nu \in \hat{G}\}$ forms a free basis over $\mathbb{Z}$. For G-modules U_1, U_2 there is the intertwining number $\dim_{\mathbb{C}} \mathrm{Hom}_G(U_1,U_2)$. Note that by Schur's Lemma, this is $\delta_{\nu_1\nu_2}$ for $U_1 = U_{\nu_1}, U_2 = U_{\nu_2}$, $\nu_1,\nu_2 \in \hat{G}$. This extends to a symmetric bilinear form on $\mathbb{Z}[\hat{G}]$. If $\iota : H \to G$ is the inclusion map, then by restriction there is a map $\iota^* : \mathbb{Z}[\hat{G}] \longrightarrow \mathbb{Z}[\hat{H}]$. Define the formal group $\mathbb{Z}^\infty[\hat{G}]$ as the possibly infinite formal sums $\sum_\nu a_\nu [U_\nu]$, $a_\nu \in \mathbb{Z}$. So $\mathbb{Z}[\hat{G}]$ is the subset of finite elements. And define the formal map

$$\iota_* : \mathbb{Z}[\hat{H}] \longrightarrow \mathbb{Z}^\infty[\hat{G}] \text{ as the extension to } \mathbb{Z}[\hat{H}]$$

of $V \longrightarrow \sum_\nu \dim_{\mathbb{C}} \mathrm{Hom}_H(\iota^*[U_\nu],[V])\,[U_\nu]$.

Take G-invariant D (as in (2.6)) which is elliptic. By invariance D preserves $\Gamma_\nu(\ )$, $\forall \nu$. Then the kernel and cokernel of D (Ker D, Coker D) are finite dimensional. Define the *index* of D to be the element of $\mathbb{Z}[\hat{G}]$,

$$\text{Index } D = [\text{Ker } D] - [\text{Coker } D] .$$

Then $$\text{Index } D = \iota_*([V] - [V_1]) \in \mathbb{Z}[\hat{G}] .$$

This is a direct consequence of (3.2).

§4. The Representation Theory of a Compact Lie Group.

The notation and material of this section will be continually used later. It is taken, for a large part, from [16]. See also [12], [20] . We refer to these references for more details and proofs.

(4.1) Let K be a compact Lie group. The Lie algebra k of K is reductive (see (1.1)) so $k = z \oplus k_1$, where $k_1 = [kk]$ the derived algebra (an ideal) of k , and z is the center of k . k_1 is semi-simple. Let $B(,)$ be the Killing-form of K (i.e. of k) . It is negative semi-definite. The restriction of $B(,)$ to $k_1 \times k_1$ is the Killing form of k_1 , which is negative definite. The connected subgroup K_1 of K , with Lie algebra k_1 is compact. Let H be a maximal torus (i.e. a maximal, compact, connected, abelian subgroup) of K . h is a maximal abelian subalgebra of k . H contains the center Z of K . The dimension of h , dim h , is called the *rank of* K , written rank K .

An irreducible unitary representation of H is 1-dimensional, and so determines and is determined by a character of H i.e. a continuous homomorphism $\chi : H \to S^1$ (the complex numbers of modulus 1). These form a group under the multiplication of characters. Thus we regard $\hat{H}$ (see (3.2)) as the group of unitary characters of H . We can identify $\hat{H}$ with a lattice Λ , by $\hat{H} \to \Lambda \subseteq \sqrt{-1}\, h^*$.

(here $*$ denotes the real dual) $\chi \to \lambda$ where $\chi(\exp \zeta) = e^{\lambda(\zeta)}$, $\zeta \in h$. $z \subseteq h$ and $h = z \oplus h_1$ with $h_1 \subseteq k_1$ and h_1 is a Cartan subalgebra of k_1.

Let $R = R(K,H)$ be the *root system* of the pair (K,H) (i.e. (k,h)). With $k_{\mathbb{C}}$ the complexification of k, we have the *Cartan decomposition* $k_{\mathbb{C}} = h_{\mathbb{C}} \oplus \sum_{\alpha \in R} \oplus k^{\alpha}$ where k^{α} is the *root space* corresponding to $\alpha \in R$. Note that $\alpha(\zeta) = 0$, $\zeta \in z$, $\alpha \in R$. As usual, there is the isometry $(h^*_{1_{\mathbb{C}}}, <,>) \to (h_{1_{\mathbb{C}}}, B(,))$ (here $*$ denotes the complex dual) $\lambda \longrightarrow \zeta_\lambda$, where $\lambda(\zeta) = B(\zeta_\lambda, \zeta)$ for each $\zeta \in h_1$, and $<\lambda,\mu> = B(\zeta_\lambda, \zeta_\mu)$. Introduce the 'real form' $h_{1\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{\zeta_\alpha ; \alpha \in R\}$ of $h_{1\mathbb{C}}$, on which the roots take real values. Put $h'_{1\mathbb{R}} = \{\zeta \in h_{1\mathbb{R}} ; \alpha(\zeta) \neq 0, \forall \alpha \in R\}$. ($\forall$ means 'for all'.) A root α is either strictly positive or strictly negative on a connected component C' of $h'_{1\mathbb{R}}$. Let R^+ be the set of roots which are strictly +ve on (a fixed) C'. With respect to (w.r.t) this order, we get the fundamental system of simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$ where $\ell = \mathrm{rank}\, k_1$ *the semi-simple rank of* K. Under the isometry, $h_{1\mathbb{R}}$ is the real form $h^*_{1\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{\alpha ; \alpha \in R\}$ of $h^*_{1\mathbb{C}}$. $<,>$ is a real inner product on $h^*_{1\mathbb{R}}$, with norm $\|.\|$. For each $\alpha \in R$, let $(\alpha,0)$ be the subspace orthogonal to α i.e. $(\alpha,0) = \{\lambda \in h^*_{1\mathbb{R}} ; <\lambda,\alpha> = 0\}$. The complement of $\bigcup_{\alpha \in R} (\alpha,0)$ in $h^*_{1\mathbb{R}}$ is an open set. A connected component of this set is called a *Weyl chamber* of R (or of (K,H)). These correspond to the inverse images of the connected components of $h'_{1\mathbb{R}}$. In particular C' is mapped (by the isometry) onto $C = \{\lambda \in h^*_{\mathbb{R}} ; \langle\lambda,\alpha\rangle \geq 0 , \forall\alpha \in R^+\}$, the *fundamental Weyl chamber*.

Let $W(k,h)$ be the *Weyl group* of (k,h) . Net $N_K(H)$ be the normalizer of H in K , i.e. $N_K(H) = \{k \in K;kHk^{-1} \subseteq H\}$, which contains H as a normal subgroup. The factor group $N_K(H)/H = W(K,H)$ is a finite group, called the *Weyl group* of (K,H) . We can identify this with the group of endomorphisms of h , $\{Adk;k \in N_K(H)\}$. Then $W(K,H) = W(k,h)$.

k_1 is 'the' 'compact real form' of $k_{1\mathbb{C}}$. The Killing form of $k_{1\mathbb{C}}$ is the complex bilinear extension of $B(,)$ on $k_1 \times k_1$. Also $(,)_1$ where $(\zeta,\eta)_1 = - B(\zeta,\overline{\eta})$, $\zeta,\eta \in k_{1\mathbb{C}}$ is a Hermitian inner product on $k_{1\mathbb{C}}$. ($\overline{\ }$ denotes conjugation w.r.t k_1) . Then as $Ad(h)\, k^\alpha = k^\alpha$, $h \in H$, and $\dim k^\alpha = 1$, $\alpha \in R$, we see that $Ad(h)\, \varepsilon = \chi_\alpha(h)\varepsilon$, $h \in H$, $\varepsilon \in k^\alpha$, $\chi_\alpha \in \hat{H}$, $\alpha \in R$. As $Ad(\exp\zeta) = e^{ad\zeta}$, $\zeta \in k$, we have $\chi_\alpha \to \alpha \in \Lambda$. So $R \subseteq \Lambda$. Z is the set of $h \in H$ such that $\chi_\alpha(h) = 1$, $\forall\alpha \in R$. We have $h_{\mathbb{R}} = \sqrt{-1}\, h_1$. As $\overline{k^\alpha} = k^{-\alpha}$, we can choose a 'Weyl basis' $\{\varepsilon_\alpha;\alpha \in R\}$ where $\varepsilon_\alpha \in k^\alpha$, $B(\varepsilon_\alpha,\varepsilon^\beta) = \delta^\beta_\alpha$ and $\overline{\varepsilon}_\alpha = - \varepsilon^\alpha$; (here δ is the Kronecker delta and $\varepsilon^\alpha = \varepsilon_{-\alpha}$) .

If $\chi \to \lambda$ (so λ is the differential of $\chi \in \hat{H}$) we shall say that λ lifts to χ . Define $\Gamma_H = \{\zeta \in h;\exp \zeta = e\}$ the *unit lattice* of H (or K) (e is the identity element of K) . Then λ lifts to a character of H if and only if (iff) $\lambda(\Gamma_H) \subseteq 2\pi \sqrt{-1}\, \mathbb{Z}$. ($\mathbb{Z}$ is the integers, π is the real number pi).

Let Z_0 be the connected subgroup of K corresponding to z. Z_0 is closed in K. Then $K = Z_0K_1$, and K is Lie isomorphic with $Z_0 \times K_1/F$ where $F = \{(z^{-1},z); z \in Z_0 \cap K_1\}$, a finite normal subgroup of $Z_0 \times K_1$ (here X denotes the direct product).

(4.2) Let $\phi: k \to g\ell(U)$ be a representation of k, a reductive Lie algebra, on a complex finite-dimensional vector space U. ϕ extends to $k_{\mathbb{C}}$ and to $u(k)$, so also to $u(k_{\mathbb{C}})$. $u(k)$ is the universal enveloping algebra of k. A vector $(0\neq)\ u \in U$ such that $\phi(\zeta)u = \lambda(\zeta)u,\ \forall\zeta \in h$ some $\lambda \in h_{\mathbb{C}}^*$, is called a *weight vector* with *weight* λ (of ϕ). For a given $\lambda \in h_{\mathbb{C}}^*$, the weight space U^λ (possibly 0) is the space spanned by the weight vectors with weight λ. Write m_λ for $\dim U^\lambda$ and call it the *multiplicity of* λ as a weight of ϕ.

Denote I (the *lattice of integral forms*), for the subgroup of $(z \oplus h_{\mathbb{R}})^*$ consisting of all λ such that $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$. Say that $\lambda \in I$ is *dominant* if $\langle\lambda,\alpha\rangle \geq 0$, $\forall\alpha \in R^+$ (i.e. if λ lies in the fundamental Weyl chamber). Denote this set I^d.

Definition.

Let $\lambda \in I$. We say that λ is *singular* if $\langle\lambda,\alpha\rangle = 0$ some $\alpha \in R$, and *non-singular* if $\langle\lambda,\alpha\rangle \neq 0$, $\forall\alpha \in R$. Also say that λ is *sufficiently* *non-singular* (s.n.s.) if $\langle\lambda,\alpha\rangle > a$, $\forall\alpha \in R$, where $a \in \mathbb{R}$, $a > 0$ and a is 'sufficiently' positive.

We shall assume that a parameter λ , defined on a real form of $h_{\mathfrak{c}}$ has been extended (complex linearly) to the whole of $h_{\mathfrak{c}}$.

Theorem.

(i) ϕ is completely reducible iff $\phi(z)$ consists of semi-simple endomorphisms. $\phi|_{k_1}$ is completely reducible.

(ii) If ϕ is completely reducible, U is spanned by weight vectors; there are only finitely many weights.

(iii) The weights are integral (i.e. lie in I).

(iv) The set of weights is invariant under $W(k,h)$.

(v) $m_\lambda = m_{w\lambda}$, $\forall\, w \in W(k,h)$.

We say that a weight λ is *extreme* if $\lambda+\alpha$ is not a weight $\forall\alpha \in R^+$.

(vi) If ϕ is irreducible, then there exists exactly one extreme weight λ; it is dominant (belongs to I^d) and of multiplicity 1. All other weights of ϕ are of the form $\lambda - \sum_i n_i\alpha_i$, $n_i \in \mathbb{W}$ (the whole numbers). λ is called the *highest weight* of ϕ .

(vii) If ϕ is irreducible, there is a homomorphism ${}_\phi\chi : z(k) \to \mathbb{C}$ (the complex numbers) such that $\phi(z) = {}_\phi\chi(z)1$, $\forall z \in z(k)$ (the center of $u(k)$) . This follows from Schur's lemma. ${}_\phi\chi$ is called the *infinitesimal character* of ϕ . It determines ϕ up to equivalence.

Theorem of highest weight (E. Cartan)

The map from the set of equivalence classes of irreducible representations of $k_{\mathbb{C}}$ to I^d , which assigns to an irreducible representation its highest weight is a bijection.

Let $\Pi: K \to GL(U)$, U as before, be a representation of K . As mentioned before, see (3.2) , Π is unitarizable and completely reducible. So we fix a complex inner product $<,>$ on U w.r.t. which Π is unitary. We refer to a weight of the differential $d\Pi$ (see (1.1)) also as a weight of Π . So e.g. the roots R is the set of weights of the adjoint representation Ad of K , on $k_{\mathbb{C}}$. The weights of $d\Pi$ lift to (i.e. are differentials of) unitary characters of H . In fact considering $\Pi|_H$ (i.e. Π restricted to H) we can choose a basis $\{u_i\}$ $(i = 1,\dots,n)$ of U such that $\Pi(h)u_i = \chi_i(h)u_i, h \in H, \chi_i \in \hat{H}$. With $\chi_i \to \lambda_i \in \Lambda$ (see (4.1)) $d\Pi(\zeta)u_i = \lambda_i(\zeta)u_i$ $(i = 1,\dots,n)$. So λ_i $(i = 1,\dots,n)$ are the, not necessarily distinct weights of $d\Pi$.

Conversely, given a completely reducible representation ϕ of k such that the weights of ϕ lift to $\hat{H}$ (in fact sufficient that the highest weights of the irreducible components of ϕ lift to $\hat{H}$, by Theorem on p.27 (vi); then there is a unique representation Π of K such that $d\Pi = \phi$.

The weight spaces of Π are orthogonal w.r.t $<,>$.

Theorem (Cartan, Weyl)

The map from $\hat{K}$ into $\Lambda \cap I^d$, which assigns to an irreducible representation its highest weight is a bijection. (Recall: $\hat{K}$ is the set of equivalence classes of irreducible unitary representations of K.) Moreover this correspondence is obtained as follows: Let $\nu \in \Lambda \cap I^d$. Let $\nu_0 = \nu|_z$ (i.e. ν restricted to z), $\nu_1 = \nu|_{h_1}$. ν_0 lifts to a character of Z_0, χ_0 say. And the irreducible representation of k_1 with highest weight ν_1, (U,ϕ_1) say, lifts to (U,Π_1) a representation of K with $d\Pi_1 = \phi_1$. Recall that $K = Z_0 \times K_1/F$. Now define $\Pi(z,k) = \chi_0(z)\,\Pi_1(k)$, $z \in Z_0$, $k \in K_1$. Then $(U,\Pi) \in \hat{K}$ with highest weight ν. Note that $\Lambda \subseteq I$.

Remark.

Let (U,Π) be an irreducible representation of K. From the Peter-Weyl theorem (in the form [16], (2.8)) and Schur's lemma, one can show that an inner product on U w.r.t which Π is unitary, is unique up to real positive constant multiples. As a consequence; if K is simple, as Ad_K is then irreducible, minus the Killing form is the unique (up to +ve multiples) inner product on k w.r.t which Ad is orthogonal. In general $K_1 = K_2 \times \dots\dots \times K_m$ a direct product of closed, simple, normal subgroups. We get $B(,)|_{k_j \times k_j} = a_j\, B_{K_j}(,), (j=2,\dots,m)$, for some not necessarily equal $a_j \in \mathbb{R}$, $a_j > 0$; where $B_{K_j}(,)$ is the Killing form of K_j.

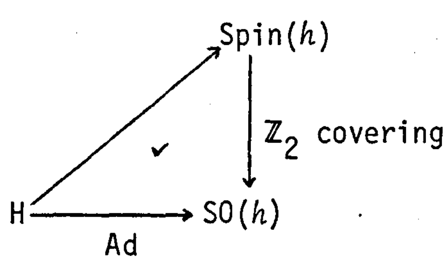
By taking an inner product on z, we get a real inner product $(\, , \,)$ on k satisfying: $(z, k_1) = 0$ and $(\, , \,)$ restricted to $k_1 \times k_1$ is $-B(\, , \,)$.

Take an orthonormal (w.r.t $(,)$) basis of k and define $\Omega_K = -\sum_i \xi_i^2$ in $z(k)$. Ω_K is called the *Casimir element of* K.

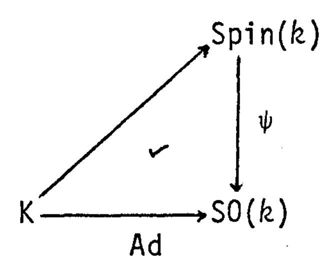
Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. For $(U_\nu, \Pi_\nu) \in \hat{K}$, $\nu - \rho$ being the highest weight, we have $d\Pi_\nu(\Omega_K) = \|\nu\|^2 - \|\rho\|^2$.

(4.3) Take the pairs $(k,(,)),(h,(,))$ with $(\, , \,)$ as given (4.2) and the Clifford algebras Cliff (k), Cliff (h), w.r.t $(\, , \,)$, (see §4).

We have the lift



Also we assume



i.e. We assume that there is such a homomorphism $\tilde{\rho}$, with $\psi \circ \tilde{\rho} = \mathrm{Ad}$. (See (5.3).)

This is equivalent to requiring that ρ lifts to $\hat{H}$ (i.e. $\rho \in \Lambda$).

$\rho \in \Lambda$ for example if K is simply connected.

For $\mu \in \Lambda$, we write e^{μ} for the corresponding unitary character of H.

Let (U,Π) be a unitary representation of K. The character χ_U of U (which determines Π up to equivalence) is defined by $\chi_U(x) = \text{trace } \Pi(x), x \in K$; and has the properties:

$\chi_{U_1 \oplus U_2} = \chi_{U_1} + \chi_{U_2}$, $\chi_{U_1 \otimes U_2} = \chi_{U_1} \cdot \chi_{U_2}$ $\chi_{U^*}(x) = \chi_U(x^{-1}) = \overline{\chi(x)}$ (where $-$ here denotes the complex conjugate), $x \in K$. U^* is the contragradient K-module to U .

<u>Lemma</u>.

Let $(U,\Pi),(W,\Pi_1) \in \hat{K}$.

(i) Let f, f_1 be a matrix element of Π, Π_1 respectively, then $\langle f,f_1 \rangle = 0$ if U and W are not equivalent. (For $\langle , \rangle$ see (3.1).)

(ii) Let $u_1, u_2, v_1, v_2 \in U$ and take the matrix elements $f_1(k) = \langle \Pi(k)u_1, v_1 \rangle$, $f_2(k) = \langle \Pi(k)u_2, v_2 \rangle$, $k \in K$; then $\langle f_1, f_2 \rangle = \frac{1}{n} \langle u_1, u_2 \rangle \overline{\langle v_1, v_2 \rangle}$ where $n = \dim U$.

(iii) $\langle \chi_U, \chi_W \rangle = 0$, if U and W are not equivalent
$= 1$, if U and W are equivalent.

These are called the *Schur orthogonality relations*.

The character is of course a class function on K (i.e. constant on the conjugacy classes). By the Schur orthogonality relations and the Peter-Weyl theorem, the characters of the irreducible representations of K form a complete orthonormal set of class functions in $L^2(K)$ (see (3.1)).

Every conjugacy class in K intersects H, and hence the character of a representation is determined by its restriction to H.

$\chi_U|_H = \chi_{\iota^*(U)}$, ι^* denotes restriction.

Define for $\mu \in \Lambda, A(\mu) = \sum_{W(K,H)} \det(w)e^{w\mu} \in \mathbb{Z}[\hat{H}]$.

(See (3.3).)

We have $A(\rho) = e^{\rho} \prod_{\alpha \in R^+}(1 - e^{-\alpha})$. Let $(U_\nu, \Pi_\nu) \in \hat{K}$, $\nu-\rho$ being the highest weight, then

Weyl's character formula.

$$A(\rho)\chi_\nu|_H = A(\nu) \qquad (\text{here } \chi_\nu = \chi_{U_\nu}) \qquad \text{and}$$

Weyl's degree formula.

The dimension of U_ν , $d(\nu) = \sum_{\alpha \in R^+} \frac{\langle\nu,\alpha\rangle}{\langle\rho,\alpha\rangle}$

(4.4) Ad maps K into $GL(k_1)$ with kernel Z. Thus K/Z is Lie isomorphic to AdK a subgroup of $GL(n,\mathbb{R}) \subseteq GL(n,\mathbb{C})$, $n = \dim k_1$, with Lie algebra k_1. Let $K_{\mathbb{C}}, H_{\mathbb{C}}$ denote the connected subgroup of $GL(n,\mathbb{C})$ with Lie algebra $k_{1\mathbb{C}}$, $h_{1\mathbb{C}}$ respectively. Also have the closed subgroup $B = H_{\mathbb{C}}N^+$, the Borel subgroup (a maximal soluble subgroup) of $K_{\mathbb{C}}$ with Borel subalgebra $b = h_{\mathbb{C}} \oplus \sum_{\alpha\in R^+} \oplus k^{\alpha}$ of $k_{\mathbb{C}}$. Let (U,κ) be a finite dimensional unitary representation of H. Then κ extends to a holomorphic representation of $H_{\mathbb{C}}$, which we denote by $\tilde{\kappa}$. Extend $\tilde{\kappa}$ trivially to B by $\tilde{\kappa}(hn) = \tilde{\kappa}(h)$ for $h \in H_{\mathbb{C}}$, $n \in N^+$. Then $K_{\mathbb{C}} \times_{\tilde{\kappa}} U =: \underline{\tilde{U}}$ (see [16]), becomes a holomorphic vector bundle (with a complex Riemannian structure) over the complex flag manifold $K_{\mathbb{C}}/B$. $K_{\mathbb{C}}/B$ is diffeomorphic to K/H and gives the latter a complex structure. $K_{\mathbb{C}}$ acts holomorphically.

Put $T(K/H)_{\mathbb{C}} = T(K/H) \otimes \mathbb{C}$ (here $\mathbb{C}$ is the trivial complex line bundle over K/H. We have $T(K/H)_{\mathbb{C}} = T(K/H) \oplus \overline{T}(K/H)$ a direct sum of the holomorphic and anti-holomorphic tangent bundles. The Riemannian structure on K/H determined by (,) (see [16]) extends to a complex Riemannian structure on $T(K/H)_{\mathbb{C}}$ and therefore also to one on $\Lambda^r \overline{T}(K/H)^*$ (the r^{th} exterior power of the dual of the anti-holomorphic tangent bundle). There is the $\bar{\partial}$ operator and its formal adjoint $\bar{\partial}^*$. $\bar{\partial}:\Gamma(\underline{\tilde{U}} \otimes \Lambda^r\overline{T}(K/H)^*) \to \Gamma(\underline{\tilde{U}} \otimes \Lambda^{r+1}\overline{T}(K/H)^*$. $\bar{\partial}^2 = 0 = \bar{\partial}^{*2}$. This is $K_{\mathbb{C}}$-invariant. The complex Laplacian $\Box = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. $\Box$ is elliptic. The cohomology space $H^t(U) = \operatorname{Ker} \Box$ ($\Box$ at the t^{th} link of the chain complex). This is a finite-dimensional $K_{\mathbb{C}}$-module.

$H^o(U)$ is the space of holomorphic sections of $\underline{\tilde{U}}$.

Borel-Weil-Bott Theorem.

Let E_μ be the 1-dimensional unitary H-module with weight $\mu \in \Lambda$.

(i) If $\mu+\rho$ is singular, then $H^t(E_\mu) = 0$, $\forall t$

(ii) If $\mu+\rho$ is non-singular, then $H^t(E_\mu) = 0$, $t \neq n(w)$ and $H^{n(w)}(E_\mu)$ is the simple K-module with highest weight $w(\mu+\rho)-\rho$; here w is the unique element in $W(K,H)$ such that $w(\mu+\rho)$ lies in the fundamental Weyl chamber, and $n(w)$ is the index of w i.e. no$\{\alpha\in R^+; w\alpha<0\}$ (no { } means 'the number of elements').

§5. The Clifford Algebra, Spinors, and the Dirac Operator.

We refer to [2].

(5.1) Let m be a real vector space with an inner product $(\, , \,)$. With respect to the pair $(m,(\, , \,))$ we take the Clifford algebra, Cliff(m), which is the quotient algebra (over $\mathbb{R}$) of the tensor algebra of m, $T(m)$, modulo the two sided ideal generated by the elements $\xi \otimes \xi + (\xi,\xi)1$, $\xi \in m$. By the natural map $m \to T(m) \to \mathrm{Cliff}(m)$, we regard $m \subseteq \mathrm{Cliff}(m)$. Cliff($m$) is (real) associative, with a unity 1, of dimension $2^{\dim m}$. (See [2] p.40 for a basis.) Cliff(m) is $\mathbb{Z}_2$-graded $\mathrm{Cliff}(m) = C^+(m) \oplus C^-(m)$, a direct sum of vector spaces where $C^+(m)$, $C^-(m)$ is spanned by the even, odd products respectively (see [2] p.37); (by an even product we mean an element of the form $\xi_1 \cdot \ldots \cdot \xi_{2k}$, $\xi_i \in m$, etc.). $C^+(m)$ is a subalgebra. There is an anti-automorphism $c \to c^t$ on Cliff (m) which is given by $\xi_1 \cdot \ldots \cdot \xi_k \to (-1)^k \xi_k \cdot \ldots \cdot \xi_1$ for $\xi_i \in m$. Note that $\xi\eta + \eta\xi = -2(\xi,\eta)1$ for $\xi,\eta \in m$; in Cliff(m).

For m even dimensional Cliff(m) is a simple algebra (i.e. no non-trivial two-sided ideals); for m odd dimensional $C^+(m)$ is a simple algebra. Let $\ell:\mathrm{Cliff}(m) \to \mathrm{End}(\mathrm{Cliff}(m))$ be the *left regular representation* (i.e. $\ell(a)b = ab$). This is faithful (i.e. $\mathrm{Ker}\, \ell = 0$). In fact Cliff(m) is a semi-simple algebra (i.e. ℓ is completely reducible, or otherwise said that Cliff(m) is completely reducible as a left Cliff(m)-module).

Definition.

For m even dimensional take a minimal left ideal S in Cliff(m) .

For m odd dimensional take a minimal left ideal S in $C^+(m)$.

In each case we call S *the space of spinors*. Thus for m even, odd dimensional any simple Cliff(m) , $C^+(m)$-module is equivalent to S respectively. For m even dimensional $S = S^+ \oplus S^-$ as a $C^+(m)$-module where S^+, S^- are inequivalent simple $C^+(m)$-modules. Call these the *spaces of ½-spinors*. Let $c: m \to \text{End } S$ denote Clifford multiplication, i.e. $c(\xi)s = \xi.s$, $\xi \in m$, $s \in S$.

Define the *spin group* $\text{Spin}(m) = \{s \in C^+(m); ss^t = 1, sms^{-1} \subseteq m\}$. There is the double covering $\psi: \text{Spin}(m) \to SO(m)$ (the special orthogonal group of $(m,(\,,))$ where $\psi(s)\xi = s\xi s^{-1}$, $s \in \text{Spin}(m)$, $\xi \in m$. Spin(m) is simply-connected for $\dim m \geq 3$. By restricting the left regular representation ℓ we get $\text{Spin}(m) \xrightarrow{\ell} \text{End } S$. Call this the *spin representation*. For m even dim we also get $\text{Spin}(m) \xrightarrow{\ell^\pm} \text{End } S^\pm$. Call these the *½-spin representations*. S^+, S^- are simple inequivalent Spin(m)-modules. For m odd dim, S is a simple Spin(m)-module.

As associative algebra becomes a Lie algebra under the commutator $[\;\;]$ (i.e. $[AB] = AB - BA$) . For Cliff(m) we denote this by $[\;\;]_c$ (i.e. $[xy]_c = xy - yx$, $x,y \in \text{Cliff}(m)$) . Now $[[\xi,\eta]_c\zeta]_c = -4(\eta,\zeta)\xi + 4(\xi,\zeta)\eta$ $\xi,\eta,\zeta \in m$. $\mathfrak{so}(m)$, the Lie algebra of $SO(m)$, is embedded as a Lie subalgebra of Cliff(m) as $\text{Span}\{[\xi,\eta]_c \,;\, \xi,\eta \in m\}$; where

$Z(\zeta) =: [Z\zeta]_c$, $Z \in \text{Span}\{\ \}$

$= d\psi(Z)(\zeta)$, $\zeta \in m$.

(See [28].)

$d\psi$ being the differential of ψ , (see (1.1)).

Take an orthonormal (w.r.t (,)) basis $\{\xi_i\}$ for m .

As $Z = -\frac{1}{4}\sum_i [Z\xi_i]_c\, \xi_i$, $Z \in \text{Span}\{\ \}$ we get $d\psi(T) = -\frac{1}{4}\sum_i T(\xi_i)\xi_i$, (5.1.2)

$T \in so(m)$, and composing with the left regular representation,

$$c(T(\eta)) = - \ [(\ell \circ d\psi)(T), c(\eta)] \ , \ T \in so(m), \ \eta \in m \tag{5.1.3}$$

(here [ ] denotes the commutator). Moreover, the differential of the spin representation, $d\ell$, is just the restriction of ℓ to $so(m)$, which is the Lie algebra of Spin(m) . (See [28].)

(5.2) The complexification of Cliff(m) is Cliff(m_c) with the complex linear extension of (,) on $m_{\mathbb{C}}$, which we denote also by (,) . Also we shall not distinguish, in notation, between S for (m,(,)) or for ($m_{\mathbb{C}}$,(,)) .

Construction of the space of spinors (m even dimensional): (see [2])

Choose fixed maximal totally isotropic (w.r.t(,)) subspaces (of dimension over $\mathbb{C}$, $\frac{1}{2}$ dim m) m_1, m_2 of $m_{\mathbb{C}}$ such that

$m_e = m_1 \oplus m_2$. Let C_1, C_2 be the subalgebra of $\mathrm{Cliff}(m_e)$ generated by m_1, m_2 respectively. Then C_1, C_2 is isomorphic to Λm_1 , Λm_2 the exterior algebra of m_1, m_2 respectively. Let $e \in \Lambda^m m_2$, $(2m = \dim m)$ of dimension 1. Then we may take $S = \mathrm{Cliff}(m)e = C_1 e$. Let $C_1^{\pm} = C_1 \cap C^{\pm}(m_e)$, then the spaces of ½-spinors $S^{\pm} = C^{\pm}(m_e)e = C_1^{\pm}(m_e)e$. (N.B. $e^2 = 0$ so here e is not an idempotent.)

For m of odd dimension: see [2] p.106.

(5.3) We use the notation of §2. Suppose that G/H is reductive, Riemannian.

<u>Definition</u>.

We shall say that G/H is G-*spin* if for the pair (m, Ad) , (i) $\det \mathrm{Ad}(h) = 1$, $h \in H$ and (ii) $\mathrm{Ad}: H \longrightarrow SO(m)$ lifts to a homomorphism $\tilde{\rho}: H \longrightarrow \mathrm{Spin}(m)$ via ψ , i.e. there is the commutative diagram

$$\begin{array}{ccc} & & \mathrm{Spin}(m) \\ & \overset{\tilde{\rho}}{\nearrow} & \downarrow \psi \\ H & \longrightarrow & SO(m) \end{array} .$$

We get a representation of H , (S, σ) , with $\sigma = \ell \circ \tilde{\rho}$. Also if m is even dimensional, we get $(S^{\pm}, \sigma^{\pm})$ with $\sigma^{\pm} = \ell^{\pm} \circ \tilde{\rho}$.

Recall that $\psi(s)\xi = s\xi s^{-1}$. With $s = \tilde{\rho}(h^{-1})$,

$$c(\xi)\sigma(h) = \sigma(h)c(Adh^{-1}\xi) \; , \quad h \in H, \; \xi \in m \; . \tag{5.3.1}$$

Now $d\psi o d\tilde{\rho} = ad$. Taking $Z = d\tilde{\rho}(\xi)$, $\xi \in h$ in (5.1.1) , we get $ad\, \xi(\eta) = [\xi\eta] = [d\tilde{\rho}(\xi)\eta]_C$, $\eta \in m$. Then $d\tilde{\rho}(\xi) = -\frac{1}{4} \sum_i [\xi\xi_i]\xi_i$, $\xi \in h$, and

$$d\sigma(\xi) = -\tfrac{1}{4} \sum_i c[\xi\xi_i]c(\xi_i) = (\ell o d\psi)(ad\xi) \; , \; \xi \in h \; . \tag{5.3.2}$$

(5.4) The Dirac operator.

Suppose that G/H is reductive, Riemannian and is G-spin. Take a representation (V,τ) of H , and take a G-invariant connection (see (2.4)), ∇^V, on $(\underline{V})^G_H$. Choose a G-invariant connection ∇^S on the *bundle of spinors* $(\underline{S})^G_H$ (the induced bundle via (S,σ) . We shall see how to do this in Chapter 1 (1.1). Take the tensor product connection $\nabla^{S\otimes V}$ on $\underline{S\otimes V}$. There is the bilinear map $m \otimes S \otimes V \xrightarrow{c\otimes 1} S\otimes V$, given by $\xi\otimes s\otimes v \longrightarrow c(\xi)s\otimes v, \xi\in m, s\in S, v\in V$. Then associated to the pair $((,) \, , \nabla^S)$, there is the 1^{st} order, elliptic differential operator $D_V = (c\otimes 1) \circ \nabla^{S\otimes V}$ (see 2.6), with symbol map $c\otimes 1$. We shall refer to D_V as the *twisted, by* V *, Dirac operator of the connection* $\nabla^{S\otimes V}$.

If m is even dimensional, by taking a G-invariant connection $\nabla^{S^\pm}$ on the bundle of $\frac{1}{2}$-spinors $(\underline{S}^\pm)^G_H$ we get the elliptic, $\frac{1}{2}$-*Dirac operator*

$D_V^\pm$ with symbol map $m \otimes S^\pm \otimes V \xrightarrow{c \otimes 1} S^\pm \otimes V$, respectively. With ∇^S the direct sum connection, D_V is the direct sum of D_V^+ and D_V^-.

For V the 1-dim trivial H-module, these will be called *scalar Dirac operators*.

Here G/H is a 'complete Riemannian manifold'. The Laplacian Δ (in Chapter 0, (2.5)) is essentially self-adjoint. Also D_V and D_V^2 are essentially self-adjoint (see [27]). In particular $\operatorname{Ker} D_V = \operatorname{Ker} D_V^2$.

(5.5) Remark.

If H is simply-connected, then certainly G/H is spin.

CHAPTER 1.

We use the notation of Chapter 0, §1, 2 and 5. Let G/H be reductive, Riemannian and G-spin, with a G-invariant, metric connection ∇, by γ, on $T(G/H) = (\underline{m})_H^G$. See Chapter 0 (1.3), (2.2) and (2.3). We shall see that ∇ lifts to a unique metric connection on the bundle of spinors $(\underline{S})_H^G$. A formula has been given in [28] for the square of the Dirac operator on symmetric space. In §2 we give a generalization in differential geometric terms, of this formula, which is due to Dr. John H. Rawnsley. I am also grateful to him for suggesting Proposition 3 to me.

See Chapter 0, §5.

§1. Invariant Metric Connections on the Bundle of Spinors.

(1.1) Define a linear map $tr : \mathrm{Cliff}(m) \longrightarrow \mathbb{R}$ by

$tr(x)$ is 'the (real) coefficient of 1 in x'.

Then we get a real inner product $(\, , \,)_C$ on $\mathrm{Cliff}(m)$, by

$$(x,y)_C = tr\, x^t y \ , \quad x,y \in \mathrm{Cliff}(m) \ .$$

(See Chapter 0, §5.) This induces an inner product $(\, , \,)_S$ on S.

Lemma 6.

(i) $C^+(m)$, $C^-(m)$ are orthogonal w.r.t $(\, , \,)_C$.

(ii) The spin representation is orthogonal w.r.t $(\, , \,)_S$.

(iii) σ is orthogonal w.r.t $(\, , \,)_S$.

(iv) Clifford multiplication is skew symmetric i.e. c: $\longrightarrow \mathfrak{so}(S)$ (w.r.t $(\, , \,)_S$) .

Proof.

(i) is clear;

(ii) $(sx,sy) = tr((sx)^t sy) = tr(x^t(s^t s)y) = tr\, x^t y = (x,y)$, $s \in Spin(m)$;

(iii) is a consequence of (ii) ;

(iv) $(c(\xi)x,y) = tr((\xi x)^t y) = -tr\, x^t(\xi y) = -(x,c(\xi)y)$, $\xi \in m$.

□

Proposition 3.

A G-invariant, metric connection ∇ , by γ , on $T(G/H) = (\underline{m})^G_H$ lifts to a unique G-invariant metric connection ∇^S , by γ^S , on $(\underline{S})^G_H$ where $\gamma^S(\xi) = (\ell o d\psi)\gamma(\xi)$, $\xi \in \mathfrak{g}$.

(See §5 (5.1), (5.3).)

Proof.

For $\xi \in h$, $\gamma^S(\xi) = (\ell o d\psi)(ad\xi) = d\sigma(\xi)$. Also

$$\gamma^S(Adh\xi) = (\ell o d\psi)(Adh o \gamma(\xi) \circ Adh^{-1})$$

$$= - \tfrac{1}{4} \sum_i c(Adh\gamma(\xi)Adh^{-1}\xi_i) \;\; c(\xi_i)$$

$$= -\tfrac{1}{4}\sum_i \sigma(h)\ c(\gamma(\xi)Adh^{-1}\xi_i)\sigma(h)^{-1}\ \ c(\xi_i)$$

$$= \sigma(h)\{-\tfrac{1}{4}\sum_i c(\gamma(\xi)Adh^{-1}\xi_i)\ \ c(Adh^{-1}\xi_i)\}\sigma(h)^{-1}$$

$$= \sigma(h)\gamma(\xi)\sigma(h)^{-1} \quad \text{for} \quad h \in H\ ,\ \xi \in \mathfrak{g}\ .$$

So by Proposition 1, (Chapter 0, (2.2)), γ^S does define an invariant connection, which is metric since $\gamma^S(\xi) \in \mathfrak{so}(S)$ (w.r.t $(\ ,\ )_S$) , $\xi \in \mathfrak{g}$. Now suppose that ∇ lifts to ∇^S . Let $_{\prime}\nabla$ be the reductive connection on $\underline{m}$. This certainly lifts to $^{\prime}\nabla$, the reductive connection on $\underline{S}$. (See Chapter 0 (2.4).)

We have

$$\nabla \ - \ _{\prime}\nabla = \alpha \in \Omega^1(G/H), \mathfrak{so}(\underline{m}))$$

$$\nabla^S - \ {}^{\prime}\nabla = \beta \in \Omega^1(G/H, \mathfrak{so}(\underline{S})) \qquad \text{(See Chapter 0 (2.1), (2.3).)}$$

Recall that $m \otimes S \xrightarrow{\ c\ } S$, induces the vector bundle map

$$\underline{m} \otimes \underline{S} \xrightarrow{\ c\ } \underline{S}\ , \quad \text{and so also} \quad \Gamma(\underline{m}) \otimes \Gamma(\underline{S}) \xrightarrow{\ c\ } \Gamma(\underline{S}) \qquad (1.1.1)$$

$$X \otimes s \longrightarrow c(X)s\ .$$

(See Chapter 0 (1.3), (5.4).)

By the Leibniz rule

$$\nabla^S_X(c(Y)s) = c(\nabla_X Y)s + c(Y)\nabla^S_X s$$

and $$'\nabla_X(c(Y)s) = c({}_{\prime}\nabla_X Y)s + c(Y)'\nabla_X s$$

$$X, Y \in \Gamma(\underline{m}) \ , \ s \in \Gamma(\underline{S}) \ .$$

Taking the difference

$$\beta(X)\ c(Y)s = c(\alpha(X)Y) + c(Y)\beta(X)s$$

$$\text{i.e.}\quad [\beta(X)\ c(Y)] = c(\alpha(X)Y)$$

$$= [(\ell o d\psi)(\alpha(X)), c(Y)] \quad \text{(by Chapter 0 (5.1.3))}.$$

From the fact that left and right Clifford multiplication generate all of $\mathfrak{so}(S)$, from the commutation relation $[A\ BC] = [AB]C + B[AC]$, and from the fact that $\mathfrak{so}(S)$ is a real simple Lie algebra (so has zero center), we get that

$$\beta(X) = (\ell o d\psi)\alpha(X), X \in \Gamma(\underline{m}) \ .$$

Now see Proposition 1. □

Corollary.

$$c(\gamma(\xi)\eta) = [\gamma^S(\xi), c(\eta)] \quad \xi \in \mathfrak{g} \ , \ \eta \in \mathfrak{m} \ . \tag{1.1.2}$$

Proof.

This follows from the Proposition and Chapter 0, (5.1.3). □

Lemma 7.

In the statement of Proposition 3 the curvature 2-form $R^S(,)$ of ∇^S is given by $R^S(,) = (\ell o d\psi)\, R(,)$ where $R(\,,\,)$ is the curvature 2-form of ∇. (See Chapter 0 (2.4).)

Proof.

This follows from Chapter 0, (5.1.2) and Lemma 5. □

§2. A Formula for the Square of the Dirac Operator.

(2.1) See Chapter 0, (2.5), (5.4).

The scalar Dirac operator is

$$D : \Gamma(\underline{S})^G_H \longrightarrow \Gamma(\underline{S})^G_H$$

$$D = \sum_i c(X_i)\nabla^S_{X_i} \qquad \text{(See (1.1.1).)}$$

with $D^2 = \Delta^S - \frac{1}{2}\sum_{i,j} c(X_i)c(X_j)\nabla^S_{T(X_i,X_j)} + \frac{1}{2}\sum_{i,j} c(X_i)c(X_j)\, R^S(X_i,X_j)$.

And the twisted Dirac operator is

$$D = D_V : \Gamma(\underline{S \otimes V})^G_H \longrightarrow \Gamma(\underline{S \otimes V})^G_H$$

$$D = \sum_i c(X_i)\nabla^{S\otimes V}_{X_i}$$

with

$$D^2 = \Delta^{S\otimes V} - \tfrac{1}{2}\sum_{i,j} c(X_i)c(X_j)\nabla^{S\otimes V}_{T(X_i,X_j)} + \tfrac{1}{2}\sum_{i,j} c(X_i)c(X_j)R^{S\otimes V}(X_i,X_j) \quad . \qquad (2.1.1)$$

Here for $(V_{\prime},\kappa_{\prime})$ a representation of H , $\Delta^{V_{\prime}}$ is the Laplacian of $\underline{V}_{\prime}$ (with connection $\nabla^{V_{\prime}}$, by γ') ; $R^{V_{\prime}}(\ ,\ )$ is the curvature of $\nabla^{V_{\prime}}$; $T(\ ,\ )$ is the torsion of ∇ .

Note that $R^{V_{\prime}\otimes V} = R^{V_{\prime}} \otimes 1 + 1 \otimes R^V$ and

$$\Delta^{V\otimes V} = \Delta^{V_{\prime}} \otimes 1 + 1 \otimes \Delta^V - 2\sum_i \nabla^{V_{\prime}}_{X_i} \otimes \nabla^V_{X_i} \quad .$$

The above formulae are independent of the orthonormal (w.r.t.(,)) basis $\{\xi_i\}$ of m .

(2.1.1) is obtained using the Clifford bundle relation

$$c(X_i)c(X_j) + c(X_j)c(X_i) = -2\delta_{ij}$$

and the formulae for the torsion, curvature and the Laplacian as given in Chapter 0, (2.4), (2.5). See also Lemma 7.

Note that for ∇ the Levi-Civita connection there are no 1^{st} order terms in D^2 .

(2.2) <u>Proposition 4</u>.

(i) $(Df)^\wedge = \sum_i c(\xi_i)(\nabla^{S\otimes V}f)^\wedge(\xi_i)$

(ii) $$(D^2f)^{\wedge} = (\Delta^{S\otimes V}f)^{\wedge} - \tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)(\nabla^{S\otimes V}f)^{\wedge}(T(\xi_i,\xi_j)) +$$

$$+ \tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)R^{S\otimes V}(\xi_i,\xi_j)\hat{f},\ f \in \Gamma(\underline{S\otimes V}) .$$

And see Proposition 2 (Chapter 0 (2.5)) for $(\nabla f)^{\wedge}, (\Delta f)^{\wedge}$.

Proof.

By Lemma 1 (Chapter 0, (2.2)) and (1.1.1),

$$(Df)^{\wedge}(e) = -\sum_i c(\xi_i)(\nabla^{S\otimes V}_{\tilde{\xi}_i} f)^{\wedge}(e)$$

and

$$(D^2f)^{\wedge}(e) = (\Delta^{S\otimes V}f)^{\wedge}(e) - \tfrac{1}{2}\sum_{i,j} c(\xi_i)\ (\xi_j)(\nabla^{S\otimes V}_{T(\xi_i,\xi_j)}f)^{\wedge}(e) +$$

$$+ \tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)R^{S\otimes V}(\xi_i,\xi_j)\hat{f}(e) .$$

$T(\xi,\eta)$ is given in Lemma 5 (Chapter 0, (2.4)) and

$$R(\xi,\eta) = dR(Q[\xi,\eta])_e + [\gamma'(\xi),\gamma'(\eta)] - \gamma'(P[\xi,\eta]) ,\ \xi,\eta \in \mathfrak{g} .$$

Note that

$$T(\tilde{\xi},\tilde{\eta}) = \sum_k (T(\tilde{\xi},\tilde{\eta}),\tilde{\xi}_k)\tilde{\xi}_k = \sum_k (T(\xi,\eta),\xi_k)\tilde{\xi}_k$$

$$= T(\xi,\eta)^{\sim} ,\ \xi,\eta \in \mathfrak{g} .$$

Now use the invariance of D i.e., $g.Df = Dg.f$, $g \in G$. □

Note: The formula for the square of the Dirac operator, in the form (ii), for the special case of (G,H) a 'symmetric pair' with ∇ the Levi-Civita connection (here the same as the reductive connection) was first given in [28]. See Chapter 4 (3.1.1) for the precise formula.

CHAPTER 2.

In this chapter, in §1, we introduce our main task. Subsequent chapters will set about solving this problem. Sections 2,3 of this chapter and chapter 3 will give some structure theory of a compact Riemannian homogeneous space which is spin.

The notation and material of Chapters 0, 1 will be referred to and used.

§1. 'The Problem'.

(1.1) Let (K,L) be a pair of Lie groups, with L a closed subgroup of K. We write $L \leq K$. Let K be compact, so L is also compact. Further let K and L be connected.

As the adjoint representation of L on k (the Lie algebra of K) is completely reducible, we can write

$$k = \ell \oplus p \quad \text{with} \quad [\ell,p] \subseteq p$$

for some subspace p . (with p Ad L-invariant). Thus K/L is a reductive homogeneous space (see Chapter 0, (2.2)). In fact we will always take p to be the orthogonal complement of ℓ in k w.r.t. the inner product $(\, , \,)$. ($(\, , \,)$ as given in Chapter 0, (3.2).) Recall that Ad_K is orthogonal w.r.t. $(\, , \,)$, so ad_K is skew-symmetric.

Via (,) K/L becomes Riemannian (see Chapter 0, (2.2)).

With respect to the pair $(p,(,))$, take the Clifford algebra Cliff(p) , and the space of spinors S , with metric $(\ ,\ )_S$. (See Chapter 0, §5 Chapter 1, §1.)

(1.2) To recapitulate: we have the pair (K,L) of compact Lie groups with $L \leq K$. K/L becomes a reductive, Riemannian homogeneous space via (,) .

The isotropy representation of L (see Chapter 0 (1.2), (2.2)) is orthogonal w.r.t. (,). We suppose that K/L is K-spin (see Chapter 0, (5.3)). We take a K-invariant metric connection ∇, determined by $\gamma : k \longrightarrow so(p)$, on $T(K/L) = (\underline{p})_L^K$ (by the pair (p,Ad)) . Then ∇ lifts to a unique metric connection ∇^S , determined by $\gamma^S : k \longrightarrow u(S)$, on the bundle of spinors $(\underline{S})_L^K$. (See Chapter 0 (2.2), (2.3); Chapter 1,§1.)

Take a finite dimensional unitary representation (V,τ) of L . Associated to the pair $((,),\gamma)$ we form the twisted, by V , Dirac operator D_V , with symbol map $p \otimes S \otimes V \xrightarrow{c\otimes 1} S \otimes V$ (see Chapter 0,(5.4

i.e.
$$D_V : \Gamma(\underline{S\otimes V})_L^K \longrightarrow \Gamma(\underline{S\otimes V})_L^K ,$$
$$D_V = (c\otimes 1)\circ\nabla^{S\otimes V} ,$$

where $\nabla^{S\otimes V}$ is the tensor product connection of ∇^S on $\underline{S}$ and the reductive connection ∇^V on $(\underline{V})_L^K$.

D_V is a left K-invariant, 1^{st} order, elliptic, essentially self-adjoint differential operator.

Hence the kernel of D_V, Ker D_V, is a finite-dimensional unitary K-module. A K-submodule of $L^2(\underline{S \otimes V})^K_L$. We wish to determine how this decomposes into simple K-modules.

In fact (for γ either the Levi-Civita or the reductive connection) we will determine explicitly, the solution space, as a unitary representation of K, of the homogeneous Dirac equation $D_V f = 0$.

As for (V_1, τ_1) a representation of L, we have $D_{V \oplus V_1} = D_V \oplus D_{V_1}$ (a direct sum), it is sufficient to consider V a simple L-module.

(1.3) Remark.

We note the vanishing theorem of A. Lichnerowicz (see [26]), that for the scalar Dirac operator D_1 with γ the Levi-Civita connection, Ker $D_1 = 0$ ie. there are no harmonic spinors.

Also we note the papers [28], [31] for a method of solving the case of (K,L) an equal rank symmetric pair. See Chapter 4, (3.2).

It is our aim to solve the general case.

§2. Structural Preliminaries on a Compact Pair.

(2.1) We shall use previous notation.

Let (K,L) be a compact pair of Lie groups with $L \leq K$. Let H_0 be a maximal torus of L. Fix a maximal torus H of K with $H_0 \leq H$. Clearly $H \leq Z_K(H_0)$ (the centralizer of H_0 in K) i.e. $Z_K(H_0) = \{k \in K; khk^{-1} = h, \forall h \in H_0\}$, with Lie algebra

$z_{k}(h_0) = \{\xi \in k; [\xi\zeta] = 0, \forall \zeta \in h_0\}$ (the centralizer of h_0 in k). $Z_L(H_0) = H_0$, $Z_K(H) = H$. As $[\ell, p] \subseteq p$, $z_{k}(h_0) = z_{\ell}(h_0) \oplus z_p(h_0)$, where $z_p(h_0)$ is the centralizer of h_0 in p. But as h_0 is maximal abelian in ℓ, $z_{\ell}(h_0) = h_0$. Thus we have

$$h = h_0 \oplus h_1 \quad \text{(an orthogonal direct sum w.r.t. (,))}$$

with h_1 maximal abelian in $z_p(h_0)$.

For $\lambda \in h^*$, write $\underset{\sim}{\lambda} = \lambda|_{h_0}$, $\tilde{\lambda} = \lambda|_{h_1}$.

(Here $*$ denotes the real dual, and $\lambda|_{h_0}$ means λ restricted to h_0 etc.)

(2.2) Let H_1 be the connected subgroup of H with Lie algebra h_1. So $H = H_0H_1$. H/H_0 is Lie isomorphic to $H_1/H_0 \cap H_1$

In fact $H \approx (H_0 \times H_1)/F$ where $F = \{(h^{-1}, h); h \in H_0 \cap H_1\}$.

Let $\hat{H}$, $\hat{H}_0$ have lattice Λ, Λ_0 respectively (see Chapter 0, (4.1)). H/H_0 is isomorphic to the subgroup $A = \{X \in \hat{H}; X(h) = 1, \forall h \in H_0\}$ of $\hat{H}$. Let A have lattice $,\Lambda$.

There are homomorphisms $\hat{H} \longrightarrow \hat{H}_0$ by restriction

$$\Lambda \longrightarrow \Lambda_0 .$$

The kernel of the upper, lower map is A , ${}_{\prime}\Lambda$ respectively so $\hat{H}/A \approx \hat{H}_0$, $\Lambda/{}_{\prime}\Lambda \approx \Lambda_0$. (This is Pontrjagin duality see [33] .)
Given $\chi_0 \in \hat{H}_0$, there exists $\chi \in \hat{H}$ with $\chi|_{H_0} = \chi_0$. Equivalently, given $\lambda_0 \in \Lambda_0$ there exists $\lambda \in \Lambda$ with $\underset{\sim}{\lambda} = \lambda_0$.

§3. Root Systems. The Weights of the Isotropy Representation and the Spin Representation.

(3.1) Let (K,L) be as in §2.

Let R_L be the root system of (L,H_0) . There is the isotropy representation of L

$$Ad : L \longrightarrow SO(p) \quad (w.r.t.(,))$$

with complexified differential

$$ad : \ell \longrightarrow so(p_{\mathbb{C}})$$

Denote the set of weights (w.r.t. H_0) by Q . $Q \subseteq \Lambda_0 \cap I_L$.
I_L is the lattive of integral forms for (L,H_0) .
For the case rank L = rank K see Chapter 3. Here we consider rank L < rank K.
Take complexification $k_{\mathbb{C}} = \ell_{\mathbb{C}} \oplus p_{\mathbb{C}}$.

Let R be the root system of (K,H) (see Chapter 0 §4 for notation.)
For $\alpha \in R$ write $\varepsilon_\alpha = \xi_\alpha + \eta_\alpha$ with $\xi_\alpha \in \ell_{\mathbb{C}}$, $\eta_\alpha \in p_{\mathbb{C}}$.

We divide the roots R into 3 disjoint subsets R_0 , R_1 , and R_2 .

$$R_0 = \{\alpha\in R; \eta_\alpha = 0 \quad (\text{i.e. } k^\alpha \subseteq \ell_{\mathbb{C}} \text{ for } \alpha \in R_0)$$

$$R_1 = \{\alpha\in R; \xi_\alpha, \eta_\alpha \neq 0\}$$

$$R_2 = \{\alpha\in R; \xi_\alpha = 0\} \quad (\text{i.e. } k^\alpha \subseteq p_{\mathbb{C}} \text{ for } \alpha \in R_2) .$$

For $R_3 \subseteq R$ we denote $\underset{\sim}{R}_3 = \{\underset{\sim}{\alpha}; \alpha \in R_3\}$.

Lemma 8.

(i) $\alpha \in R_j$ iff $-\alpha \in R_j$ $(j=0,1,2)$, $\delta \in Q$ iff $-\delta \in Q$.

(ii) For $\alpha \in R_0 \cup R_1$, $\underset{\sim}{\alpha} \neq 0$. For $\alpha \in R_0$, $\tilde{\alpha} = 0$.

If $\alpha, \beta \in R_0$, $\alpha \neq \beta$, then $\underset{\sim}{\alpha} \neq \underset{\sim}{\beta}$.

(iii) $R_L = \underset{\sim}{R}_0 \cup \underset{\sim}{R}_1$, $Q = \{0\} \cup \underset{\sim}{R}_1 \cup \underset{\sim}{R}_2$.

$\{\xi_\alpha; \alpha \in R_0 \cup R_1\}$ is a set of root vectors for (L, H_0) , which with h_0 , spans $\ell_{\mathbb{C}}$ over $\mathbb{C}$.

$\{\eta_\alpha; \alpha \in R_1 \cup R_2\}$ is a set of weight vectors for $(p_{\mathbb{C}}, \text{ad})$, which with h_1 spans $p_{\mathbb{C}}$ over $\mathbb{C}$.

Proof.

(i) From $\overline{\varepsilon_\alpha} = -\varepsilon^\alpha$, $\alpha \in R$ (recall that - denotes conjugation w.r.t. k) we get $\overline{\xi_\alpha} = -\xi^\alpha$, $\overline{\eta_\alpha} = -\eta^\alpha$, $\alpha \in R$. (Here $\xi^\alpha = \xi_{-\alpha}$, $\eta^\alpha = \eta_{-\alpha}$) . Thus $\alpha \in R_j$ iff $-\alpha \in R_j$. $\delta \in Q$ iff $-\delta \in Q$ now follows from (iii).)

(ii) If $\alpha \in R_0 \cup R_1$ and $\underset{\sim}{\alpha} = 0$, then $\xi_\alpha \in z_\ell(h_0)_{\mathbb{C}}$. But $z_\ell(h_0) = h_0$. Note that $(h_0, \xi_\alpha) = 0$, $\forall \alpha \in R$.

Let $\varepsilon \in k^\alpha$, $\alpha \in R_0$. For $\zeta \in h_1$ we have $[\zeta\varepsilon] = \alpha(\zeta)\varepsilon \in \ell_{\mathbb{C}}$. But also $[\zeta\varepsilon] \in p_{\mathbb{C}}$. So $[\zeta\varepsilon] = 0$ and $\alpha(\zeta) = 0$. As a consequence $\underset{\sim}{\alpha} \neq \underset{\sim}{\beta}$ for $\alpha, \beta \in R_0$ with $\alpha \neq \beta$.

(iii) Let ξ be a R_L-root vector or a Q-weight vector with root or weight δ, $\xi \notin h_{\mathbb{C}}$.

Now $\xi = \zeta_1 + \sum_{\alpha \in R} a_\alpha \varepsilon_\alpha$ for some $\zeta_1 \in h_{\mathbb{C}}$ and $a_\alpha \in \mathbb{C}$, not all zero.

$$= \zeta_1 + \sum_\alpha a_\alpha \xi_\alpha \text{ or } \zeta_1 + \sum_\alpha a_\alpha \eta_\alpha \text{ according as } \xi \in \ell_{\mathbb{C}} \text{ or } p_{\mathbb{C}}.$$

Then $\delta(\zeta)\xi = \sum_{\alpha \in R} a_\alpha \alpha(\zeta)\varepsilon_\alpha$, for $\zeta \in h_0$.

If $\delta(\zeta) = 0$, and $a_\alpha \neq 0$, then $\alpha(\zeta) = 0$. If $\delta(\zeta) \neq 0$, and $a_\alpha \neq 0$, then $\zeta_1 = 0$ and $\delta(\zeta) = \alpha(\zeta)$. So $\delta = \underset{\sim}{\alpha}$ for $a_\alpha \neq 0$.

Clearly h_1 lies in the 0-weight space.

Also for $\zeta \in h_0$; $\alpha(\zeta)\varepsilon_\alpha = [\zeta\xi_\alpha] + [\zeta\eta_\alpha]$ so $[\zeta\xi_\alpha] = \alpha(\zeta)\xi_\alpha$, $[\zeta\eta_\alpha] = \alpha(\zeta)\eta_\alpha$, $\alpha \in R$. □

(3.2) Recall that for $\alpha \in R$, $\zeta_\alpha \in \sqrt{-1}\, h$ is determined by

$$-\alpha(\zeta) = (\zeta_\alpha, \zeta), \quad \zeta \in \sqrt{-1}\, h$$

(Here we also denote by $(,)$, the complex linear extension of $(,)$). Recall that $(\ell, p) = 0$.

Write $\zeta_\alpha = \zeta_{\underset{\sim}{\alpha}} + \zeta_{\tilde{\alpha}}$, with $\zeta_{\underset{\sim}{\alpha}} \in \sqrt{-1}\, h_0$, $\zeta_{\tilde{\alpha}} \in \sqrt{-1}\, h_1$.
Then $-\underset{\sim}{\alpha}(\zeta) = (\zeta_{\underset{\sim}{\alpha}},\zeta)$, $\zeta \in \sqrt{-1}\, h_0$; and $-\tilde{\alpha}(\zeta) = (\zeta_{\tilde{\alpha}},\zeta)$, $\zeta \in \sqrt{-1}\, h_1$.
Note that $\underset{\sim}{\alpha} = 0$ iff $\zeta_{\underset{\sim}{\alpha}} = 0$ and $\tilde{\alpha} = 0$ iff $\zeta_{\tilde{\alpha}} = 0$.

More generally for $\lambda \in \sqrt{-1}h^*$, recall that $\zeta_\lambda \in \sqrt{-1}h$ is determined by $-\lambda(\zeta) = (\zeta_\lambda,\zeta)$, $\zeta \in \sqrt{-1}h$. Write $\zeta_\lambda = \zeta_{\underset{\sim}{\lambda}} + \zeta_{\tilde{\lambda}}$ with $\zeta_{\underset{\sim}{\lambda}} \in \sqrt{-1}h_0$, $\zeta_{\tilde{\lambda}} \in \sqrt{-1}h_1$. Then $-\underset{\sim}{\lambda}(\zeta) = (\zeta_{\underset{\sim}{\lambda}},\zeta)$, $\zeta \in \sqrt{-1}h_0$; and $-\tilde{\lambda}(\zeta) = (\zeta_{\tilde{\lambda}},\zeta)$, $\zeta \in \sqrt{-1}h_1$. Recall that $<,>$ is defined by $<\lambda,\mu> = -(\zeta_\lambda,\zeta_\mu), \lambda,\mu \in \sqrt{-1}h^*$. So defining $<\lambda,\mu>_{,} = -(\zeta_{\underset{\sim}{\lambda}},\zeta_{\underset{\sim}{\mu}}), <\lambda,\mu>' = = -(\zeta_{\tilde{\lambda}},\zeta_{\tilde{\mu}})$ we get $<,> = <,>_{,} + <,>'$.

By Remark 1 in Chapter 0, (4.2), $<,>_{,}$ is a real +ve multiple of the Killing-form of L on each connected component of the Coxeter-Dynkin diagram of (L,H_0) (i.e. of $(k'_{\mathbb{C}},h'_{0\mathbb{C}})$, ' denotes the derived algebra).

In particular if $\lambda_0 \in I_L$, $\dfrac{2<\lambda_0,\alpha>_{,}}{<\alpha,\alpha>_{,}} \in \mathbb{Z}$ for $\alpha \in R_0 \cup R_1$.

(3.3) Consider the complexified isotropy representation of L , $(p_{\mathbb{C}},\mathrm{Ad})$.

The 0-weight space is $z_p(h_0)_{\mathbb{C}}$. And for $\delta \in Q$, $\delta \neq 0$, the δ-weight space is $a^\delta \oplus \sum_{\alpha\in R_{2,\delta}} \oplus\, k^\alpha$, where

$R_\delta = \{\alpha \in R;\ \underset{\sim}{\alpha} = \delta\}$, $R_{1,\delta} = R_1 \cap R_\delta$, $R_{2,\delta} = R_2 \cap R_\delta$;

and $a^\delta = \sum_{\alpha\in R_{1,\delta}} \mathbb{C}\eta_\alpha$ (see Lemma 8).

Recall the complex inner product $(,)_1$ on the derived algebra of $k_{\mathbb{C}}$; $(\xi,\eta)_1 = (\xi,\overline{\eta})$, $\xi,\eta \in k_{1\mathbb{C}}$, (the derived algebra).

For $\alpha \in R_1$, (ξ_α,ξ^α) and $(\eta_\alpha,\eta^\alpha)$ are real negative. For $\alpha \in R_1$, $\beta \in R_2$, $(\eta_\alpha,\varepsilon^\beta) = (\varepsilon_\alpha,\varepsilon^\beta) = 0$.

Define $Q_1 = \{\delta \in Q;\ \delta \neq 0\ ,\ \delta = \underset{\sim}{\alpha}$ some $\alpha \in R_1\}$. Clearly $\delta \in Q_1$ iff $-\delta \in Q_1$. For $\delta \in Q_1$, put ${}_1m_\delta = \dim a^\delta$. Also for $\delta \in Q$, $\delta \neq 0$, put ${}_2m_\delta = no(R_{2,\delta})$ (i.e. the number of roots in $R_{2,\delta}$). For $\delta \in Q$, let m_δ be the multiplicity of the weight δ . Then for $\delta \in Q$, with $\delta \in Q_1$, we have $m_\delta = {}_1m_\delta + {}_2m_\delta$. And for $\delta \in Q$, with $\delta \neq 0$, $\delta \notin Q_1$, we have $m_\delta = {}_2m_\delta$.

Take $a^\delta \oplus a^{-\delta}$, $\delta \in Q_1$. As $-\eta^\alpha = \overline{\eta}_\alpha$ for $\alpha \in R_1$, one sees that ${}_1m_\delta = {}_1m_{-\delta}$. a^δ is totally isotropic w.r.t (,) . We can choose an orthonormal (w.r.t$(,)_1$) basis $\{\eta_{j,\delta}\}$ $(j=1,\ldots,{}_1m_\delta)$ for a^δ , $\delta \in Q_1$; i.e.

$$(\eta_{i,\delta},\eta^{j,\delta}) = -\delta_i^j \quad \text{where} \quad \eta^{j,\delta} = -\overline{\eta}_{j,\delta}\ ,\ \delta \in Q_1 \ .$$

For each $\delta \in Q_1$, fix a subset R'_δ of R consisting of ${}_1m_\delta$ roots α with $\underset{\sim}{\alpha} = \delta$. We can arrange so that $R'_{-\delta} = -R'_\delta$.

Put $R' = \bigcup_{\delta\in Q_1} R'_\delta$.

H is a maximal torus of $Z_K(H_0)$. The root system of $(Z_K(H_0),H)$ is $R^0 = \{\alpha \in R_2 ; \underset{\sim}{\alpha} = 0\}$, with root vectors $\{\varepsilon_\alpha ; \alpha \in R^0\}$. Put $R_{20} = R_2 - R_0$. h_1 together with these root vectors span the 0-weight space $z_p(h_0)_{\mathbb{C}}$ over $\mathbb{C}$.

Choose compatible orders on R_L, R. So get the systems of +ve roots R_L^+, R^+. Here compatible means that if $\alpha \in R_L$ and $\beta \in R$ such that $\underset{\sim}{\beta} = \alpha$, then $\beta \in R^+$. Such always exists (see [10]).

Put $\rho_L = \frac{1}{2} \sum_{\beta \in R_L^+} \beta$. Also put $R'^+ = R' \cap R^+$, $R_{20}^+ = R_{20} \cap R^+$.

And $Q_1^\pm = Q_1 \cap R_L^\pm$.

(3.4) Using a weight space decomposition for the isotropy representation, we shall now construct the space of spinors S in Cliff($p_{\mathbb{C}}$) (w.r.t(,)); and thus for K/L spin, determine the weights of the spin representation (S,σ) of L, and their multiplicities.

For $\delta \in Q_1$, put

$$2\,{}_{\iota}n_{j,\delta} = (n_{j,\delta} - n^{j,\delta}) + \sqrt{-1}(n_{j,\delta} + n^{j,\delta}) \in p$$

and for $\alpha \in R_{20}$,

$$2\,{}_{\iota}n_\alpha = (\varepsilon_\alpha - \varepsilon^\alpha) + \sqrt{-1}(\varepsilon_\alpha + \varepsilon^\alpha) \in p \quad .$$

$\{\eta_{j,\delta}, \eta_\alpha\}$ $(\delta \in Q_1,\ j = 1,\ldots\ldots,m_\delta\ ;\ \alpha \in R_{20})$ is an orthonormal set, which with $z_p(h_0)$, spans p over $\mathbb{R}$.

If $F(\ ,\ )$ is bilinear on $p \times p$ one has

$$4F(\eta_\alpha,\eta^\beta) = 2\sqrt{-1}F(\varepsilon_\alpha,\varepsilon_\beta)-2F(\varepsilon_\alpha,\varepsilon^\beta)-2F(\varepsilon^\alpha,\varepsilon_\beta)-2\sqrt{-1}F(\varepsilon^\alpha,\varepsilon^\beta)\ ,\ \alpha,\beta \in R_{20}\ .$$

And similarly for $F(\eta_{j,\delta}, \eta^{j',\delta'})$.

(3.5) <u>Construction of the space of spinors</u>: (See Chapter 0, (5.2)).

We have the orthogonal weight space decomposition

$$p_{\mathbb{C}} = p_- \oplus z_p(h_0)_{\mathbb{C}} \oplus p_+$$

where $$p_\pm = \sum_{\delta\in Q_1^\pm} \oplus\, a^\delta \oplus \sum_{\alpha\in R_{20}^\pm} \oplus\, k^\alpha\ .$$

Furthermore p_+,p_- are maximal totally isotropic (w.r.t.(,)) subspaces of $p_+ \oplus p_-$. Let $C_\pm$ be the subalgebra of $\mathrm{Cliff}(p_+ \oplus p_-)$, (w.r.t(,)) , generated by $p_\pm$. Take $e \in \Lambda^m p_+$, where $2m = \mathrm{no}(R' \dot{\cup} R_{20})$, this is 1-dimensional. Take the space of spinors S_0 in $\mathrm{Cliff}(z_p(h_0)_{\mathbb{C}})$, (w.r.t (,)) , then

$$S = S_0\, C_-\, e$$

(3.6) We now suppose that K/L is K-spin.

Consider the differential of (S,σ) . A short computation using Chapter 0, (5.3.2)

gives
$$d\sigma(\zeta) = (\rho'+\rho_{20})(\zeta) - \tfrac{1}{2} \sum_{\delta \in Q_1^+} \delta(\zeta) c(\eta^{j,\delta}) c(\eta_{j,\delta})$$
$$- \tfrac{1}{2} \sum_{\alpha \in R_{20}^+} \alpha(\zeta)\, c(\varepsilon^\alpha)\, c(\varepsilon_\alpha) , \quad \zeta \in h_0 ,$$

(recall that $c: p \longrightarrow u(S)$ denotes Clifford multiplication)

where
$$\rho' = \tfrac{1}{2} \sum_{\alpha \in R'^+} \alpha , \quad \rho_{20} = \tfrac{1}{2} \sum_{\alpha \in R_{20}^+} \alpha .$$

<u>Proposition 5.</u>

The weights of (S,σ) , w.r.t h_0 , are given by $(\rho'+\rho_{20}) - (|A|+|B|)$, restricted to h_0 , where $A \subseteq R'^+$, $B \subseteq R_{20}^+$, or $-(\rho'+\rho_{20}) + |A'| + |B'|$, where A',B' is the complement of A,B in R'^+, R_{20}^+ respectively.

The multiplicity of the weight $\rho'+\rho_{20} - (|A|+|B|)$ is dim S_0 times the number of pairs (A_1,B_1), $A_1 \subseteq R'^{+}$, $B_1 \subseteq R_{20}^{+}$ with $|A_1| + |B_1| = |A| + |B|$ restricted to h_0 .

Proof.

This follows from the above construction.

$\square$

CHAPTER 3.

We use the notation of Chapter 0, §4 and Chapter 2. In this chapter we take a compact, spin pair (K,L) of equal rank, and consider the twisted spinors $S^{\pm} \otimes V$ as an L-module where V is simple. In §1 we determine for a 'sufficiently non-singular parameter', the decomposition of $S^{\pm} \otimes V$ into simple L-modules. In §2 we show that a simple K-module lying in a certain infinitesimal class, occurs with multiplicity at most 1 in the induced module $L^2\underline{(S^{\pm} \otimes V)}_L^K$.

§1. Equal Rank Twisted Spinors.

(1.1) Take the pair (K,L) with rank L = rank K .
So $H_0 = H$. Here p is even dimensional. R_L is a closed subsystem of R , (closed subsystem means that $R_L \subseteq R$, and if $\alpha,\beta \in R_L$ with $\alpha+\beta \in R$, then $\alpha+\beta \in R_L$) and $\{\varepsilon_\alpha;\alpha \in R_L\}$ is 'the' set of root vectors for (L,H) . Also $W(L,H) \leq W(K,H)$ i.e. the Weyl group of (L,H) is a subgroup of the Weyl group of (K,H) .

Define $W' = \{w \in W(K,H);wR^+ \supseteq R_L^+\}$. Then W' is a set of coset representatives for $W(K,H)/W(L,H)$. (See [21], [28].)

The set of weights of the complexified isotropy representation of L is $Q = R-R_L$, so each is of multiplicity 1. The set of weight vectors is $\{\varepsilon_\alpha;\alpha \in R-R_L\}$. For $\alpha \in R-R_L$, define
$2\xi'_\alpha = (\varepsilon_\alpha-\varepsilon^\alpha)+\sqrt{-1}(\varepsilon_\alpha+\varepsilon^\alpha) \in p$. Then $\{\xi'_\alpha;\alpha \in R-R_L\}$ is an orthonormal

(w.r.t(,)) basis for p. We have the weight space decomposition

$$p_{\mathbb{C}} = p_- \oplus p_+$$

where $p_{\pm} = \sum_{\alpha \in R^+ - R_L^+} \oplus k^{\alpha}$. Furthermore p_+, p_- are maximal totally isotropic (w.r.t(,)) subspaces of $p_{\mathbb{C}}$.

Let $C_{\pm}$ be the subalgebra of Cliff (p_c), (w.r.t(,)), generated by $p_{\pm}$. Take $e \in \Lambda^m p_+$ where $2m = \dim K/L$, then the space of spinors $S = C_- e$. Also we have the spaces of $\frac{1}{2}$-spinors $S^{\pm}$ (see Ch.0, §5). For $F(,)$ bilinear on $p \times p$,

$$4F(\xi_{\alpha},\xi_{\beta}) = 2\sqrt{-1}F(\varepsilon_{\alpha},\varepsilon_{\beta}) - 2F(\varepsilon_{\alpha},\varepsilon^{\beta}) - 2F(\varepsilon^{\alpha},\varepsilon_{\beta}) - 2\sqrt{-1}F(\varepsilon^{\alpha},\varepsilon^{\beta}),\ \alpha,\beta \in R - R_L .$$

We suppose now that K/L is spin. Recall the spin representation of L, (S,σ), and the $\frac{1}{2}$spin representations of $L,(S^{\pm},\sigma^{\pm})$. (See Chapter 0, (5.3)). $\sigma = \sigma^+ \oplus \sigma^-$. Then the differential of (S,σ), is given by

$$d\sigma(\zeta) = (\rho - \rho_L) - \tfrac{1}{2} \sum_{\alpha \in R^+ - R_L^+} \alpha(\zeta) c(\varepsilon^{\alpha}) c(\varepsilon_{\alpha}),\ \zeta \in h.$$

Here $\rho_L = \frac{1}{2} \sum_{\alpha \in R_L^+} \alpha$.

We see that the weights of S as an L-module are

$$(\rho-\rho_L) - |A| \quad \text{where} \quad A \subseteq R^+-R_L^+$$

$$= -(\rho-\rho_L) + |A'| , \qquad A' \text{ the complement of } A \text{ in } R^+-R_L^+ .$$

($|A|$ denotes the sum of the roots in A .)
The multiplicity of the weight $(\rho-\rho_L)-|A|$ is the number of $B \subseteq R^+-R_L^+$ such that $|B| = |A|$.

The weights of S^+, S^- as an L-module are

$$(\rho-\rho_L)-|A| \quad \text{where } A \subseteq R^+-R_L^+ \text{ and } \text{no}(A) \text{ is even, odd resp.}$$

The multiplicity is the number of $B \subseteq R^+-R_L^+$, with $|B| = |A|$ and $\text{no}(B)$ even, odd resp.

<u>Lemma 9</u>.

(i) The difference of the characters of the L-modules S^+,S^- on H is given by

$$\chi_{S^+} - \chi_{S^-}\Big|_H = e^{\rho-\rho_L} \prod_{\alpha\in R^+-R_L^+} (1-e^{-\alpha}) \quad \text{in } \mathbb{Z}[\hat{H}]$$

$$= \frac{A(\rho)}{A_L(\rho_L)} \qquad \text{(the quotient of the Weyl denominators)}$$

(see Chapter 0 (4.3)).

(ii) Consider the contragredient L-modules $(S^+)^*, (S^-)^*$. We have $(S^{\pm})^* \approx S^{\pm}$ or $S^{\mp}$ as L-modules according as $n_0(R^+ - R_L^+)$ $(= \frac{1}{2}\dim K/L)$ is even or odd respectively.

Proof.

These follow easily from the weights of S^+, S^- and their multiplicities. See also [28], [31]. □

(1.2) For the construction of the following filters we follow [30], [25]. Take the Borel subalgebra $b = h_{\mathbb{C}} \oplus \sum_{\alpha \in R_L^+} \oplus k^{\alpha}$ of $\ell_{\mathbb{C}}$, and the Borel subgroup B of $L_{\mathbb{C}}$ with Lie algebra b. (See Chapter 0 (4.4).) There is a filtration of S by B-submodules

$$0 \le S^0 \le S^1 \le \ldots\ldots \le S^q \le S^{q+1} \le \ldots\ldots \le S$$

($\le$ means, here, B-submodule) where the B-module $S^q = \sum_{r \le q} C_{-r} e$, $q \in \mathbb{W}$ with $C_{-r} = \Lambda^r p_-$, $r \in \mathbb{W}$ (under the isomorphism $C_- \approx \Lambda p_-$).
$S^q = S$ for $q \ge m$, $2m = \dim K/L$. S^0 has weight $\rho - \rho_L$. (Recall that we are assuming that K/L is spin, so $\rho - \rho_L \in \Lambda$.) The quotient $T^q = S^q/S^{q-1}$. $T^q = 0$ for $q > m$, $S^{\pm} = \sum_{(-1)^q = \pm 1} \oplus T^q$ as a B-module.
Clifford multiplication induces a map $p \otimes T^q \subsetneq T^{q+1}$.

There is the B-module short exact sequence

$$0 \to S^{q-1} \to S^q \to \Lambda^q p_- \otimes E_{\rho - \rho_L} \to 0 \tag{1.2.1}$$

(for details see [30]). Here E_μ denotes the 1-dim holomorphic B-module with weight $\mu \in \Lambda$ (see Chapter 0 (4.4)).

Let $V_{\mu-\rho_L}$ denote 'the' simple L-module of highest weight $\mu-\rho_L$. Note that $V_{\lambda+\rho-2\rho_L}$ occurs with multiplicity 1 in $S \otimes V_{\lambda-\rho_L}$ as an L-module. $\lambda \in \Lambda \cap I_L^d$ (see Chapter 0 (4.2)).

There is a filtration of $S \otimes V_{\lambda-\rho_L}$ by L-submodules

$$0 \leq S^0(\lambda) \leq S^1(\lambda) \leq \ldots\ldots \leq S^q(\lambda) \leq S^{q+1}(\lambda) \leq \ldots\ldots \leq S \otimes V_{\lambda-\rho_L}$$

where $S^0(\lambda) = V_{\lambda+\rho-2\rho_L}$ and $S^{q+1}(\lambda) = S^q(\lambda)+\not{p}S^q(\lambda)$, $q \in \mathbb{N}$.

$\not{p}S^q(\lambda)$ denotes the image of the map $\not{p} \otimes S^q(\lambda) \xrightarrow{c\otimes 1} S^{q+1}(\lambda)$ by Clifford multiplication. $S^q(\lambda) = S\otimes V_{\lambda-\rho_L}$ for $q \geq m$. The quotient $T^q(\lambda) = S^q(\lambda)/S^{q-1}(\lambda)$. $T^q(\lambda) = 0$ for $q > m$, and $S^\pm \otimes V_{\lambda-\rho_L} = \sum_{(-1)^q=\pm 1} \oplus T^q(\lambda)$. There is an induced map $\not{p}\otimes T^q(\lambda) \xrightarrow{c\otimes 1} T^{q+1}(\lambda)$. Tensoring (1.2.1) with $E_{\lambda-\rho_L}$ on the right we get the B-module short exact sequence

$$0 \to S^{q-1} \otimes E_{\lambda-\rho_L} \to S^q \otimes E_{\lambda-\rho_L} \to T^q \otimes E_{\lambda-\rho_L} \to 0 \qquad (1.2.2)$$

(N.B. $\otimes E$ is right exact and E is flat.)

$T^q \otimes E_{\lambda-\rho_L} \simeq \Lambda^q p_- \otimes E_{\lambda+\rho-2\rho_L}$ as B-modules. For U a B-module recall $H^t(U)$, the t^{th} cohomology space for the $\bar{\partial}$ complex, (see Chapter 0 (4.4)) $0 \le t \le m_1$, $m_1 = \frac{1}{2} \dim L/H$.

If

$$H^t(T^q \otimes E_{\lambda-\rho_L}) = 0 , \quad 0 < t \le m_1 , \; 0 \le q \le m \tag{1.2.3}$$

then the long L-exact sequence associated to (1.2.2) reduces to the short L-exact.

$$0 \to S^{q-1}[\lambda] \to S^q[\lambda] \to T^q[\lambda] \to 0 \tag{1.2.4}$$

where the L-modules $S^q[\lambda] = H^0(S^q \otimes E_{\lambda-\rho_L}), T^q[\lambda] = H^0(T^q \otimes E_{\lambda-\rho_L})$.

Now if V is an $L_{\mathbb{C}}$-module then $H^t(U \otimes V) \simeq H^t(U) \otimes V$. So by the Borel-Weil-Bott theorem we get a filtration by L-modules

$$0 \le V_{\lambda+\rho-2\rho_L} = S^0[\lambda] \le \ldots\ldots \le S^q[\lambda] \le \ldots\ldots \le S^m[\lambda] = S \otimes V_{\lambda-\rho_L} .$$

The quotient $T^q[\lambda] = S^q[\lambda]/S^{q-1}[\lambda]$. In fact under condition (1.2.3) $S^q(\lambda) = S^q[\lambda]$, so $T^q(\lambda) = T^q[\lambda]$, $0 \le q \le m$. Also the condition $\langle \lambda+\rho-2\rho_L-|A|, \alpha \rangle \ge 0$, $\forall \alpha \in R_L^+$ and each q-tuple A of distinct roots in $R^+ - R_L^+$ implies (1.2.3). (See [30], [25].) (1.2.5)

Recall the definition of λ 'sufficiently non-singular' (s.n.s).

See Chapter 0, (4.2). We shall assume here that λ s.n.s. (1.2.5) means that condition (1.2.5) is satisfied.

Proposition 6.

λ s.n.s. (1.2.5).

The simple component L-modules of $T^q[\lambda] = H^0(T^q \otimes E_{\lambda-\rho_L})$ are those with highest weights $\lambda+\rho-2\rho_L - |A|$ where A runs over all q-types of distinct roots in $R^+ - R_L^+$.

Proof.

A finite dimensional B-module U has a composition series $0 = U_0 \leq U_1 \leq \ldots\ldots \leq U_a = U$ where $W_i = U_i/U_{i-1}$ is a simple (1-dim) B-module with weight $\mu_i \in \Lambda$. Suppose $\mu_i \in I_L^d$. Define the Euler characteristic $\chi(U) = \sum_{t=0}^{m_1} (-1)^t [H^t(U)]$ in $\mathbb{Z}[\hat{L}]$, the ring of virtual representations of L (see Chapter 0 (3.3)). For a short B-exact sequence $0 \to U' \to U \to U'' \to 0$,

$$\chi(U) = \chi(U') + \chi(U'') .$$

If $H^t(U_{i-1}) = 0$, $t \neq 0$, then $H^t(U_i) \approx H^t(W_i)$, $t \neq 0$. Then as U_0 is simple, we have inductively, by B.W.B., that $H^t(U) = 0$, $t \neq 0$. Also $\chi(U) = \sum_{i=0}^{a} \chi(W_i) \in \mathbb{Z}[\hat{L}]$.

Now put $U = T^q \otimes E_{\lambda-\rho_L}$. Here W_i has weight $\mu_i = \lambda+\rho-2\rho_L - |A|$ and all q-tuples A of distinct roots in $R^+-R_L^+$ occur. Here $a = \binom{m}{q}$. We have $\chi(U) = [H^0(U)]$. Also $\chi(W_i) = [H^0(E_{\mu_i})] = [V_{\mu_i}]$ by B.W.B. □

Note that for any $\lambda-\rho_L \in \Lambda \cap \mathcal{I}_L^d$, a simple component of $S \otimes V_{\lambda-\rho_L}$, as an L-module, has highest weight of the form $\lambda-\rho+|A|$, $A \subseteq R^+-R_L^+$. But these may not all occur.

Remark 3.

Let U_ν, U_{ν_1} be simple K-modules with highest weights ν,ν_1 . Then a simple component K-module of $U_\nu \otimes U_{\nu_1}$, has highest weight of the form $\nu+\nu_2$ with ν_2 a weight of U_{ν_1} . Furthermore if $U_{\nu+\nu_2}$ occurs in $U_\nu \otimes U_{\nu_1}$, then it occurs with multiplicity equal to the multiplicity of ν_2 as a weight of U_{ν_1} .

§2. Induced Twisted Spinors.

(2.1) Take a compact spin pair (K,L) with rank L = rank K . Notation will be as in §1. Consider the induced unitary K-modules $I_\lambda = L^2(\underline{S \otimes V_{\lambda-\rho_L}})_L^K$, $I_\lambda^\pm = L^2(\underline{S^\pm \otimes V_{\lambda-\rho_L}})_L^K$ for $\lambda \in \Lambda \cap \mathcal{I}_L^d$. $I_\lambda = I_\lambda^+ \oplus I_\lambda^-$.

Denote by U_ν , the simple K-module with highest weight $\nu \in \Lambda \cap I^d$. Recall that $\Lambda \subseteq I$. Also here $\Lambda \subseteq I_L$ as L and K have equal rank. Note that as K/L is spin, $\rho-\rho_L \in \Lambda$. We assume that $\rho \in \Lambda$, so also $\rho_L \in \Lambda$.

For $\mu \in \Lambda$, define $\inf(\mu) = \{\nu\in\Lambda\cap I^d ; {}_\nu\chi(\Omega_K) = ||\mu||^2-||\rho||^2\}$. ${}_\nu\chi$ denotes the infinitesimal character of U_ν . ($||\cdot||$ is by (,) see Chapter 0, §4). $\inf(\mu)$ is a finite set. Recall the intertwining number $i_K(I_\lambda,U_\nu)$ (see Chapter 0 (3.2).)

Proposition 7.

(1) $I^+_\lambda - I^-_\lambda = 0$, λ singular w.r.t R

$= jj(w)U_{w\lambda-\rho}$, λ non-singular w.r.t R , in $\mathbb{Z}[\hat{K}]$;

where $j = +1$ if $\frac{1}{2}$ dim K/L(= no($R^+-R^+_L$)) is even

$= -1$ " " " odd

and for λ non-singular w is the unique element of W(K,H) such that $w\lambda$ lies in the fundamental Weyl chamber for R^+ (i.e. $w\lambda \in I^d$). N.B. $w^{-1} \in W'$. $j(w) = \det w = (-1)^{n(w)}$ (see Chapter 0, (4.4)).

(2) (i) If λ is singular w.r.t R , then for $\nu \in \inf(\lambda)$, $i_K(I_\lambda,U_\nu) = 0$.

(ii) If λ is non-singular w.r.t R , then for $\nu \in \inf(\lambda)$, $\nu \neq w\lambda-\rho$ we have $i_K(I_K,U_\nu) = 0$; $i_K(I^{jj(w)}_\lambda,U_{w\lambda-\rho}) = 1$, $i_K(I^{-jj(w)}_\lambda,U_{w\lambda-\rho}) = 0$.

Proof.

(1) Consider the extension to $\mathbb{Z}[\hat{L}]$ of the map $V_1 \to \dim_{\mathbb{C}}\mathrm{Hom}_L(U\ ,V_1) = i(U_\nu,V_1)$, V_1 a unitary L-module. Here U_ν is an L-module by restriction. Then $i(U_\nu,(S^+-S)\otimes V_{\lambda-\rho_L}) = i(j(S^+-S^-)\otimes U_\nu,V_{\lambda-\rho_L})$, by Lemma 9

$= 0$, λ singular

$= 0$, λ non-singular, $\nu \neq w\lambda-\rho$

$= jj(w)$ λ non-singular, $\nu = w\lambda-\rho$

(by Weyl's character formula).

Now see Chapter 0 (3.2). (See [31] for the non-compact case.)

(2) Recall that $i_K(I_\lambda^\pm,U_\nu) = i_L(U_\nu,S^\pm\otimes V_{\lambda-\rho_L})$. Suppose that $i_K(I_\lambda,U_\nu) \neq 0$. As noted before, a simple component L-module of $S \otimes V_{\lambda-\rho_L}$ has highest weight of the form $\lambda-\rho + |A|$, $A \subseteq R^+-R_L^+$. Choose $w \in W(K,H)$ such that $w\lambda$ is dominant w.r.t R^+. The set of weights of U_ν is invariant under $W(K,H)$. We see (from a theorem in Chapter 0 (4.2)) that if $i_K(I_\lambda,U_\nu) \neq 0$, then ν must be of the form $\nu = w\lambda-\rho+|A|_w + s$. Here the sum of distinct roots in R^+, $|A|_w$, is given by $w(-\rho+|A|) = -\rho+|A|_w$ (see Ch.5,(2.2)) and s is a sum of roots in R^+.

Then for $\nu \in \inf(\lambda), ||w\lambda+|A|_w+s||^2 = ||\lambda||^2$ i.e. $2\langle w\lambda,|A|_w+s\rangle + |||A|_w + s||^2 = 0$ we require $|A|_w = 0$, $s = 0$. So we get $\nu = w\lambda-\rho$. But then λ must be non-singular, and w is unique. As ρ occurs with mult 1 as a weight of U_ρ, $A = A_{w^{-1}}$ (see Ch.5,(2.2)). Now $\lambda-w^{-1}\rho = w^{-1}(w\lambda-\rho)$, so occurs with mult 1 as a weight of $U_{w\lambda-\rho}$.

We deduce that $i_K(I_\lambda,U_{w\lambda-\rho}) = 0$ or 1. But (1), excludes 0. □

CHAPTER 4.

§1. The Curvature Term in D^2 .

Take a pair (K,L) as in Chapter 2, §1. We consider the curvature term in D^2 (see Chapter 1, (2.2)).

(1.1) Let $\{\xi_j\}$, $\{\zeta_t\}$ be an orthonormal (w.r.t(,)) basis of p , ℓ respectively. Recall that

$$R^{S\otimes V}(\xi,\eta) = dR(Q[\xi,\eta]) + [\gamma^S(\xi),\gamma^S(\eta)] - \gamma^S(P[\xi,\eta]) ,$$

$\xi,\eta \in k$. $P:k \to p$ is the orthogonal projection, $Q = 1-P$.
And $R^S(\xi,\eta) = (\ell o d\psi)R(\xi,\eta)$, with

$$R(\xi,\eta) = -\text{ad}\, Q[\xi,\eta] + [\gamma(\xi),\gamma(\eta)] - \gamma(P[\xi,\eta]) ,$$

$$R^V(\xi,\eta) = dR(Q[\xi,\eta]) , \; \xi,\eta \in k .$$

The term

$$-\tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)(\ell o d\psi)\text{ad}(Q[\xi_i,\xi_j]) = -\tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)d\sigma(Q[\xi_i\xi_j])$$

$$= -2\sum_t d\sigma(\zeta_t)^2 = 2d\sigma(\Omega_L) \qquad (1.1.1)$$

Ω_L is the Casimir element for L (w.r.t(,)) .

(N.B. $Q[\xi_i\xi_j] = \sum_t c_{ij}^t \zeta_t$, with $c_{ij}^t = ([\xi_i\xi_j],\zeta_t)$,

and $-\sum_i c_{ij}^t\xi_i = [\zeta_t\xi_j]$, since $[\zeta_t\xi_j] = \sum_i a_i\xi_i$, with $a_i = ([\zeta_t\xi_j],\xi_i) = -c_{ij}^t$).

The term

$$\tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)\otimes R^V(\xi_i,\xi_j) = \frac{-1}{2}\sum_{j,t} c[\zeta_t\xi_j]c(\xi_j) \otimes dR(\zeta_t)$$

$$= 2\sum_t d\sigma(\zeta_t) \otimes dR(\zeta_t)$$

$$= -2\sum_t d\sigma(\zeta_t) \otimes d\tau(\zeta_t) \ .$$

Now $d(\sigma\otimes\tau)(\zeta)^2 = (d\sigma(\zeta) \otimes 1 + 1 \otimes d\tau(\zeta))^2$

$$= d\sigma(\zeta)^2 \otimes 1 + 2d\sigma(\zeta) \otimes d\tau(\zeta) + 1 \otimes d\tau(\zeta)^2, \zeta \in \ell \ .$$

Put $\zeta = \zeta_t$, and sum over t to get

$$-2\sum_t d\sigma(\zeta_t)\otimes d\tau(\zeta_t) = -d\sigma(\Omega_L)-d\tau(\Omega_L) + d(\sigma\otimes\tau)(\Omega_L) \ . \qquad (1.1.2)$$

Thus

$$\tfrac{1}{2}\sum_{i,j} c(\xi_i)c(\xi_j)\otimes R^V(\xi_i,\xi_j) = -d\sigma(\Omega_L)-d\tau(\Omega_L) + d(\sigma\otimes\tau)(\Omega_L) \ . \qquad (1.1.3)$$

§2. Symmetric Pairs.

We use the notation and results of Chapter 2.

(2.1) Let (K,L) be a symmetric pair of compact type. (See Chapter 2, (1.1).)

So K is compact, semi-simple and there is a pair (k,θ') where θ' is an involutive (i.e. $\theta' \neq 1$, $\theta'^2 = 1$) automorphism of k such that $k = \ell \oplus p$ is the decomposition into the $+1$, -1 eigenspaces of θ'. The Killing form of k is negative definite on ℓ. Let (k_*,θ) be the non-compact dual of (k,θ'). So we have the Cartan decomposition $k_* = \ell \oplus \sqrt{-1}p$ with involution θ. We denote the complex linear extension of θ to $k_{\mathbb{C}}$, also by θ. (See [10].)

(2.2) Consider rank L = rank K.

We use Chapter 3. Let K/L be spin. From Lemma 9 (i); the fact that we can write $w \in W(K,H)$ uniquely as $w = w_{\scriptscriptstyle 1} w'$, with $w_{\scriptscriptstyle 1} \in W(L.H)$, $w' \in W'$; Weyl's character formula; and the fact that, here, S^+, S^- (See Chapter 0, (5.3)) do not have weights in common; one sees that

$$S^{\pm} = \sum_{\substack{w \in W' \\ \det(w)=\pm 1}} \oplus V_{w\rho-\rho_L}$$

is the decomposition of $S^{\pm}$ into simple L-modules. (V_{μ} is the simple L-module of highest weight μ.) In particular, $d\sigma(\Omega_L)$ on S, is the constant $||\rho||^2 - ||\rho_L||^2$ (where $||\cdot|| = \langle\cdot,\cdot\rangle$, see Chapter 2, (3.2)). See [28] for the proof.

(2.3) Consider rank L < rank K.

See Chapter 2, §2,3 . h is a θ-stable Cartan subalgebra of k, i.e. $\theta h \subseteq h$. There is the fact that h_1 is maximal abelian in p iff k_* has one conjugacy class of Cartan subalgebras;
iff rank K = rank L + rank K/L (i.e. K/L has split rank. This includes the case of split rank 1 . The rank of K/L , or split rank of k_* , is the dimension of a maximal abelian subalgebra of p . See [10].)

Remark 4.

k_* is a real semi-simple Lie algebra. The Cartan subalgebras of k_* fall into a finite number of conjugacy classes under the adjoint group (see [19]). Given any Cartan subalgebra, there is a conjugate, a , which is θ-stable i.e. $\theta a \subseteq a$. Write $a = a_0 \oplus a_1$ with $a_0 \subseteq \ell$, $a_1 \subseteq \sqrt{-1}p$.

The 'usual' classification of symmetric pairs (as given for example in [10]) makes use of the conjugacy class with a_1 maximal abelian in $\sqrt{-1}p$. However, in the present work, when dealing with aspects of representation theory of the compact pair (K,L), it is necessary to use the conjugacy class with a_0 maximal abelian in ℓ (the fundamental Cartan subalgebras). These two 'extreme' classes coincide iff k_* has precisely one conjugacy class of Cartan subalgebras.

Therefore, using a fundamental Cartan subalgebra h (see above), we need to work out some properties of the root system R , and the 'restricted' root systems R_L, Q .

(2.4) We define an involution on R , $\alpha \to \alpha^\theta$, $\alpha \in R$ where $\alpha^\theta(\zeta) = \alpha(\theta\zeta)$, $\zeta \in h$.

Lemma 10.

(i) $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^\theta$, $\tilde{\alpha} = -\tilde{\alpha}^\theta$, $\theta k^\alpha = k^{\alpha^\theta}$ for $\alpha \in R$.

(ii) $\tilde{\alpha} = 0$ iff $\alpha = \alpha^\theta$ for $\alpha \in R$.

(iii) $R_0 \cup R_2 = \{\alpha \in R; \alpha = \alpha^\theta\}$, $R_1 = \{\alpha \in R; \alpha \neq \alpha^\theta\}$.

$\underset{\sim}{R}_0$ is the root system of $(Z_L(H_1), H_0)$; $R_0 \cup R_2$ is the root system of $(Z_K(H_1), H)$. (Where $Z_L(H_1)$ is the centralizer of H_1 in L etc.).

(iv) $\underset{\sim}{\alpha} \neq 0$, $\forall \alpha \in R$.

(v) K/L has split rank iff $R_2 = \phi$.

Proof.

(i) If $\varepsilon \in k^\alpha$, $\zeta \in h$ then $[\zeta, \theta\varepsilon] = \theta[\theta\zeta, \varepsilon] = \alpha(\theta\zeta)\theta\varepsilon$.

The other two parts, and (ii), (iii) are easy to see.

(iv) See Lemma 8 (ii), Chapter 2, (3.1). If $\alpha \in R_2$, then $\tilde{\alpha} = 0$ so $\underset{\sim}{\alpha} \neq 0$.

(v) Let $\alpha \in R_2$. Then $[\zeta\varepsilon] = \alpha(\zeta)\varepsilon = 0$, for $\zeta \in h_1$, $\varepsilon \in k^{\alpha} \leq p_{\mathbb{C}}$. So if h_1 is maximal in p, we must have $R_2 = \phi$. Conversely, suppose $R_2 = \phi$. Then as $\tilde{\alpha} \neq 0$ for $\alpha \in R_1$, we have $z_p(h_1) = h_1$, (where $z_p(h_1)$ is the centralizer of h_1 in p).

N.B. $[\zeta\xi_\alpha] = \alpha(\zeta)\eta_\alpha$, $[\zeta\eta_\alpha] = \alpha(\zeta)\xi_\alpha$, $\zeta \in h_1$, $\alpha \in R_1$.

□

Corollary to (iv).

As noted before, $H \leq Z_K(H_0)$. In fact here $H = Z_K(H_0)$. This is equivalent to (iv). So there is a unique maximal torus H of K containing H_0.

Proposition 8.

Consider the isotropy representation of L, $(p_{\mathbb{C}}, \mathrm{Ad})$.

(i) $h_{1\mathbb{C}}$ is the 0-weight space.

(ii) For $\delta \in Q$, $a \in \mathbb{C}$ we have $a\delta \in Q$ iff $a = 0, \pm\frac{1}{2}, \pm 1\ \pm 2$.

(iii) For $\delta \in Q$, $\delta \neq 0$ we have $m_\delta = 1$ i.e. the non-zero weights all have multiplicity 1.

(iv) K/L has split rank iff $m_0 = \mathrm{rank}\ K/L$.

(Here $m_0 = \dim h_1$, the multiplicity of the weight 0.)

Proof.

(i) By Lemma 8 (iii) and Lemma 10 (iv), we have $z_p(h_0) = h_1$.

(ii) For $\delta,\varepsilon \leq Q$ we see that $\dfrac{2\langle\varepsilon,\delta\rangle'}{\langle\delta,\delta\rangle'} \in \mathbb{Z}$.

This is because $Q \leq I_L$ (the lattice of integral forms of L) , and $\tilde{\alpha} = 0$ for $\alpha \in R_2$.

(iii) Let $\alpha,\beta \in R_1$ with $\alpha \neq \beta$ and $\underset{\sim}{\alpha} = \underset{\sim}{\beta}$. Let $\zeta \in h_1$.

Now

$$((\mathrm{ad}\zeta)^2\xi_\alpha,\xi^\beta) = (\xi_\alpha,(\mathrm{ad}\zeta)^2\xi^\beta)$$

i.e. $(\alpha-\beta)(\zeta)\ (\alpha+\beta)(\zeta)\ (\xi_\alpha,\xi^\beta) = 0$. Of course $\tilde{\alpha} \neq \tilde{\beta}$.

So if $\tilde{\alpha} \neq -\tilde{\beta}$, we get $(\xi_\alpha,\xi^\beta) = 0$, a contradiction (as root vectors have multiplicity 1). Therefore $\tilde{\alpha} = -\tilde{\beta}$ and so $\beta = \alpha^\theta$.

Note that for $\delta \in Q$, $\delta \neq 0$ one has ${}_2m_\delta = 0$ or 1 (as $\tilde{\alpha} = 0$ for $\alpha \in R_2$) . Thus for $\delta \in Q$ with $\delta \neq 0$, $\delta \notin Q_1$, we have $m_\delta = 1$.

Let $\delta \in Q_1$. By the above, and (2.4.2), one has ${}_1m_\delta = 1$. Therefore $m_\delta = 1$ or 2 . Now $\delta = \underset{\sim}{\alpha}$ some $\alpha \in R_1$. $\{\zeta_{\underset{\sim}{\alpha}},\xi_\alpha,\xi^\alpha\}$ is a complex simple Lie algebra of type A_1 . Suppose $2\delta \notin Q$. Consider the trace of ad $\dfrac{\zeta_{\underset{\sim}{\alpha}}}{\langle\alpha,\alpha\rangle'}$ on the space spanned over $\mathbb{C}$, by $\mathbb{C}\eta_{-\alpha}\ h'_{\mathbb{C}}$ and the δ-weight space. As $2\delta \notin Q$, we see that this space is A_1-invariant. By A_1 representation theory, the trace is zero. But the trace is also equal to $-1+m_\delta$. Therefore $m_\delta = 1$.

Suppose that $\delta \in Q_1$, with $2\delta \in Q$. Now $2\delta \notin Q_1$, otherwise $2\delta = \underset{\sim}{\beta}$ some $\beta \in R_1$ and then $\underset{\sim}{\beta} = 2\underset{\sim}{\alpha}$, a contradiction. So one must have $2\delta = \underset{\sim}{\beta}$ for a unique $\beta \in R_2$. And $m_{2\delta} = 1$. If $[\xi_\alpha \varepsilon_{-\beta}]$ is non-zero, then from (2.4.4) $\alpha-\beta \in R_1$. But $(\alpha-\beta)_\sim = -\delta$, thus we must have $\alpha+\alpha^\theta = \beta$. Then $[\xi_\alpha \varepsilon_{-\beta}] \in \mathbb{C}\eta_{-\alpha}$. Consider the trace of $\mathrm{ad}\,\dfrac{\zeta_{\underset{\sim}{\alpha}}}{\langle\alpha,\alpha\rangle_1}$ on the span over $\mathbb{C}$ of $\mathbb{C}\varepsilon_{-\beta}$, $\mathbb{C}\eta_{-\alpha}$, $h_{1\mathbb{C}}$, the δ-weight space and $\mathbb{C}\varepsilon_\beta$. We see that this space is A_1-invariant. Thus the trace is zero. But it is also equal to $-1+m_\delta$. Therefore $m_\delta = 1$.

(iv) This follows from (i) and (2.3). □

In the notation of Chapter 2, (3.3) we see that R' is such that $R_1 = R' \dot{\cup} R''$ where $R'' = R' = :\{\alpha^\theta ; \alpha \in R'\}$. ($\dot{\cup}$ denotes a disjoint union). (2.4.1)

Proposition 9.

(i) $R^0 = \phi$, $R_{20} = R_2$. For $\alpha \in R$, $\alpha-\alpha^\theta \notin R$.

(ii) The restriction map $R_0 \dot{\cup} R' \dot{\cup} R_2 \longrightarrow \underset{\sim}{R}_0 \dot{\cup} \underset{\sim}{R}' \dot{\cup} \underset{\sim}{R}_2$, $\alpha \longmapsto \underset{\sim}{\alpha}$ is a bijection.

(iii) $R_L = \underset{\sim}{R}_0 \dot{\cup} \underset{\sim}{R}'$, $Q = \{0\} \dot{\cup} \underset{\sim}{R}' \dot{\cup} \underset{\sim}{R}_2$

$\{\varepsilon_\alpha(\alpha \in R_0), \eta_\alpha(\alpha \in R')\}$, $\{\eta_\alpha(\alpha \in R'), \varepsilon_\alpha(\alpha \in R_2)\}$ are 'the' root vectors, non-zero weight vectors respectively.

Proof.

(i) Clear, by Lemma 10.

(ii) Let $\alpha,\beta \in R$. We have $-\delta_\alpha^\beta = (\varepsilon_\alpha,\varepsilon^\beta) = (\xi_\alpha,\xi^\beta) + (\eta_\alpha,\eta^\beta)$.

For $\alpha \in R_0$, $\beta \in R'$, $(\varepsilon_\alpha,\xi^\beta) = (\varepsilon_\alpha,\varepsilon^\beta) = 0$.

For $\alpha \in R_1$, (ξ_α,ξ^α) and $(\eta_\alpha,\eta^\alpha)$ are real -ve.

Also if $\alpha \in R'$, $\beta \in R_2$, $(\eta_\alpha,\varepsilon^\beta) = (\varepsilon_\alpha,\varepsilon^\beta) = 0$.

From these remarks and the fact that roots and non-zero weights have multiplicity one, we get $\underset{\sim}{\alpha} \neq \underset{\sim}{\beta}$ for $\alpha,\beta \in R_0 \cup R' \cup R_2$, with $\alpha \neq \beta$.

(iii) Follows from Lemma (iii), and (ii). □

Note that for $\alpha \in R$, $2\xi_\alpha = \varepsilon_\alpha + \theta\varepsilon_\alpha$, $2\eta_\alpha = \varepsilon_\alpha - \theta\varepsilon_\alpha$;

$$\xi_{\alpha^\theta} = \frac{1}{c_\alpha}\xi_\alpha \,,\quad \eta_{\alpha^\theta} = -\frac{1}{c_\alpha}\eta_\alpha \,,\quad \text{where } \theta\varepsilon_\alpha = c_\alpha\varepsilon_{\alpha^\theta} \,. \qquad (2.4.2)$$

For $\alpha \in R_1$, $(\xi_\alpha,\xi^\alpha) = -\frac{1}{2} = (\eta_\alpha,\eta^\alpha)$.

Also for $\alpha \in R$, $2\zeta_{\underset{\sim}{\alpha}} = \zeta_\alpha + \theta\zeta_\alpha$, $2\zeta_{\tilde{\alpha}} = \zeta_\alpha = \zeta_\alpha - \theta\zeta_\alpha$; $\qquad (2.4.3)$

$$2[\xi_\alpha\xi^\alpha] = \zeta_{\underset{\sim}{\alpha}},\quad 2[\eta_\alpha\eta^\alpha] = \zeta_{\underset{\sim}{\alpha}} \,.$$

And for $\alpha,\beta \in R$, with $\alpha+\beta \neq 0$, $\alpha^\theta + \beta \neq 0$,

$$2[\xi_\alpha\xi_\beta] = N_{\alpha\beta}\xi_{\alpha+\beta} + c_\alpha N_{\alpha^\theta\beta}\xi_{\alpha^\theta+\beta}\,,\quad \text{where } [\varepsilon_\alpha\varepsilon_\beta] = N_{\alpha\beta}\varepsilon_{\alpha+\beta}\,; \qquad (2.4.4)$$

$$2[\xi_\alpha\eta_\beta] = N_{\alpha\beta}\eta_{\alpha+\beta} + c_\alpha N_{\alpha^\theta\beta}\eta_{\alpha^\theta+\beta} \,.$$

Propositions 8, 9 give for a symmetric pair (K,L), the weights of the isotropy representation, $(p_{\mathbb{C}},\mathrm{Ad})$ of L, and their multiplicities. And, for K/L spin, Propositions 5, 9 give those of the spin representation (S,σ) of L.

(2.5) Take an orthonormal (w.r.t(,)) basis $\{\zeta_t\}$ for h_0.
Note that $\{\varepsilon_\alpha(\alpha\in R_0), \sqrt{2}\xi_\alpha\ (\alpha\in R')$, $\{\varepsilon^\alpha, \sqrt{2}\xi^\alpha\}$ are dual w.r.t -(,) .
Then the Casimir element for L (w.r.t(,)) is

$$\Omega_L = -\sum_t \zeta_t^2 + \sum_{\alpha\in R_0} \varepsilon_\alpha \varepsilon^\alpha + 2\sum_{\alpha\in R'} \xi_\alpha \xi^\alpha$$

$$= -\sum_t \zeta_t^2 + \sum_{\alpha\in R_0^+} (\zeta_\alpha + 2\varepsilon^\alpha \varepsilon_\alpha) + \sum_{\alpha\in R'^+} (\zeta_\alpha + 4\xi^\alpha \xi_\alpha) ,$$

(N.B. in the universal enveloping algebra $u(k_{\mathbb{C}})$, $\xi\eta - \eta\xi = [\xi\eta]$,$\xi,\eta \in k_{\mathbb{C}}$) .
And Ω_L acts on a simple L-module of highest weight μ_0 by the constant

$$\langle\mu_0,\mu_0\rangle' + \sum_{\alpha\in R_0^+} \langle\mu_0,\alpha\rangle' + \sum_{\alpha\in R'^+} \langle\mu_0,\alpha\rangle'$$

$$= \langle\mu_0,\mu_0\rangle' + 2\langle\mu_0,\rho_L\rangle' = ||\mu_0+\rho_L||'^2 - ||\rho_L||'^2 ,$$

$$\rho_L = \rho_0 + \rho'$$

(where $||\cdot||' = \langle\cdot,\cdot\rangle$).

(2.6) Lemma 11.

(i) $\zeta_{\alpha^\theta} = \theta\zeta_\alpha$, $\alpha\in R$. If $\alpha+\alpha^\theta \notin R$ then $2\langle\alpha,\alpha\rangle' = \langle\alpha,\alpha\rangle, \langle\alpha,\alpha\rangle' = \langle\alpha,\alpha\rangle'$ for $\alpha \in R_1$.

(ii) $\tilde{\rho} = 0$.

Proof.

(i) For $\alpha \in R$, $(\theta\zeta_\alpha,\zeta) = (\zeta_\alpha,\theta\zeta) = (\zeta_{\alpha^\theta},\zeta)$, $\forall\zeta \in h$, and of course (,) restricted to $h\times h$, is non-degenerate.

For $\alpha \in R_1$, $2<\alpha,\alpha>_1 = <\alpha,\alpha> + <\alpha,\alpha^\theta>$. But, by Proposition (i) , $<\alpha,\alpha^\theta> = 0$.

(ii) For $\alpha \in R_2$, $\zeta_\alpha = [\varepsilon_\alpha \varepsilon^\alpha] \in \sqrt{-1}h_0$, so $\zeta_{\tilde{\alpha}} = 0$. For $\alpha \in R_1$, $2\rho(\zeta_{\tilde{\alpha}}) = \rho(\zeta_\alpha) - \rho(\zeta_{\alpha^\theta})$; so if α is simple, α^θ is simple and $\rho(\zeta_\alpha) = \frac{1}{2}<\alpha,\alpha> = \frac{1}{2}<\alpha^\theta,\alpha^\theta> = \rho(\zeta_{\alpha^\theta})$. Thus $\rho(\zeta_{\tilde{\alpha}}) = 0$, for $\alpha \in R_1$. Note that as, here, k is semi-simple, $\{\zeta_{\underset{\sim}{\alpha}}\}$, $\{\zeta_{\tilde{\alpha}}\}$ $(\alpha \in R)$ spans $\sqrt{-1}\ _0$, $\sqrt{-1}\ _1$ over $\mathbb{R}$, respectively. □

Note that for $\alpha,\beta \in R$, $2<\alpha,\beta>_1 = <\alpha,\beta> + <\alpha^\theta,\beta>$. And for $\mu \in I$ (the lattice of integral forms),

$$\frac{2<\mu,\alpha>}{<\alpha,\alpha>} = \frac{1}{2}\frac{<\mu,\alpha>_1}{<\alpha,\alpha>_1} + \frac{1}{2}\frac{<\mu,\alpha>^1}{<\alpha,\alpha>^1} \text{ , with } \alpha \quad R_1 \text{ , } \alpha+\alpha^\theta \notin R \text{ .}$$

The Weyl group $W(K,H)$, $W(L,H_0)$ is generated by the reflections

$$w_\alpha(\mu) = \mu - \frac{2<\mu,\alpha>}{<\alpha\ \alpha>}\alpha \text{ , } \mu \in I \text{ , } \alpha \in R \text{ ;}$$

$$w_{\underset{\sim}{\alpha}}(\mu_0) = \mu_0 - \frac{2<\mu_0,\alpha>_1}{<\alpha\ \alpha_1>}\underset{\sim}{\alpha} \text{ , } \mu_0 \in I_L \text{ , } \underset{\sim}{\alpha} \in R_L \text{ respectively.}$$

For each $w_0 \in W(L,H_0)$ there is a unique $w \in W(K,H)$ such that $\underset{\sim}{w} = w_0$ (here w means w restricted to h_0). This is because $Z_K(H_0) = H$ (see [22]).

(2.7) H_1 is the identity component of the center of $Z_K(H_1)$. Therefore, here, H_1 is closed in K . In fact $H = H_0 \times H_1$, a direct product (see [10]). $(Z_K(H_1), Z_L(H_1)H_1)$ is an equal rank symmetric pair. $(Z_K(H_1),H)$ has root system $R_0 \cup R_2'$, $(Z_L(H_1)H_1,H)$ has root system R_0 (see Lemma 10).

Let K/L be spin. Consider the spin representation (S,σ) of L (see Chapter 0, (5.3)). There is the fact that the Casimir operator $d\sigma(\Omega_L)$ (w.r.t (,)) is constant on S (see [28]). Since from Propositions 5,9 S contains the simple L-module of highest weight $\rho'+\rho_2 = \rho-\rho_L$ restricted to h_0, we see that this constant is $||\rho||^2 - ||\rho_L||^2$. (2.7.1)

(2.8) <u>Examples</u>.

Refer to [10].

(i) Take the symmetric space SO(6)/SO(5) . This is also type AII, SU(4)/SP(2) .

$$p = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & -\bar{Z}_1 \end{pmatrix} ; \quad Z_1 \in \mathfrak{su}(2) \;,\; Z_2 \in \mathfrak{so}(2,\mathbb{C}) \right\}$$

$$SP(2) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} ; \quad Z_1 \in \mathfrak{su}(2) \;,\; Z_2 \in \mathfrak{so}(2,\mathbb{C}) \right\} .$$

The diagonal matrices in p form a maximal abelian subspace,

$$\zeta_1 = ia \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} ; \; a \in \mathbb{R} .$$

So K/L has split rank 1. $\mathfrak{su}(4)_{\mathbb{C}} \simeq \mathfrak{sl}(4,\mathbb{C})$ type A_3 .

$\mathfrak{sp}(2)_{\mathbb{C}} \simeq \mathfrak{sp}(2,\mathbb{C})$ type $C_2 \simeq B_2$.

Take a Cartan subalgebra in $\mathfrak{sp}(2)$ consisting of

$$\zeta_0 = i\begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -b \end{pmatrix} \quad ; \quad a,b \in \mathbb{R} .$$

And a Cartan subalgebra h in $su(4)$,

$$\zeta = i\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} \quad ; \quad a,b,c \in \mathbb{R} \ , \ a+b+c+d = 0 .$$

$$\zeta = \zeta_0 + \zeta_1$$

where

$$i\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} = \tfrac{1}{2}i\begin{pmatrix} a-c & & & \\ & b-d & 0 & \\ & 0 & -(a-c) & \\ & & & -(b-d) \end{pmatrix} + \tfrac{1}{2}i(a+c)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

The roots R are given by $\alpha_{ij}(\zeta) = a_i - a_j$, the difference of the i^{th} and j^{th} diagonal entries.

$$\alpha^{\theta}_{12}(\zeta) = \tfrac{1}{2}i\{(a-c)-(b-d)\} - i(a+c)$$
$$= i(d-c) \quad \text{so} \quad \alpha^{\theta}_{12} = \alpha_{43} .$$

$$\alpha^{\theta}_{23}(\zeta) = \tfrac{1}{2}i\{(b-d)+(a-c)\} + i(a+c)$$
$$= i(a-d) \quad \text{so} \quad \alpha^{\theta}_{23} = \alpha_{14}$$

$$\alpha^{\theta}_{13} = \alpha_{13} \; , \; \alpha^{\theta}_{24} = \alpha_{24} \; .$$

Here $R_2 = \phi$ and $R_0 = \{\pm\, \alpha_{13}, \pm\, \alpha_{24}\}$.

$R_1 = \{\pm\, \alpha_{12}, \pm\, \alpha_{14}, \pm\, \alpha_{23}, \pm\, \alpha_{34}\}$, $R' = \{\pm\, \alpha_{12}, \pm\, \alpha_{23}\}$.

Now $\underset{\sim}{\alpha}_{12} = \underset{\sim}{\alpha}_{43}, \underset{\sim}{\alpha}_{23} = \underset{\sim}{\alpha}_{14}$. As $\alpha_{42} = \alpha_{43} + \alpha_{31} + \alpha_{12}$,
$\alpha_{41} = \alpha_{43} + \alpha_{31}, \alpha_{32} = \alpha_{31} + \alpha_{12}$; and $\underset{\sim}{\alpha}_{42} = \underset{\sim}{\alpha}_{31} + 2\underset{\sim}{\alpha}_{12}$,
one sees that $\{\alpha_{43},\alpha_{31},\alpha_{12}\}$, $\{\underset{\sim}{\alpha}_{31},\underset{\sim}{\alpha}_{12}\}$ determine a compatible ordering R^+, R^+_L .

$R^+_0 = \{\alpha_{31},\alpha_{42}\}$, $R'^+ = \{\alpha_{12},\alpha_{32}\}$, $R''^+ = \{\alpha_{43},\alpha_{41}\}$
and $R^+_L = \{\underset{\sim}{\alpha}_{31},\underset{\sim}{\alpha}_{42},\underset{\sim}{\alpha}_{12},\underset{\sim}{\alpha}_{32}\}$. $\rho' = \frac{1}{2}(\alpha_{12}+\alpha_{32})$.

Consider R_L-chains $\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$, $-t' \le t \le t''$, $\alpha,\beta \in R_0 \cup R'$.
The reflection $w_{\alpha}(\underset{\sim}{\beta}) = \underset{\sim}{\beta} - a_{\beta\alpha}\underset{\sim}{\alpha}$ where the Cartan integer $a_{\beta\alpha} = t'-t''$.

For $\underset{\sim}{\alpha}_{ij}$ with $\alpha_{ij} \in R_0 \cup R'$, we shall write (ij) .

$$w_{31}(\underset{\sim}{\alpha}_{12}) = \underset{\sim}{\alpha}_{12} + \underset{\sim}{\alpha}_{31} = \underset{\sim}{\alpha}_{32} \; .$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	$(12)-(31) \notin R_L$	$t' = 0$	$a_{\beta\alpha} = -1$
(12) (31)	$(12)+(31) = (32)$	$t'' = 1$	

$$w_{31}(\underset{\sim}{\alpha}_{32}) = \underset{\sim}{\alpha}_{32} - \underset{\sim}{\alpha}_{31} = \underset{\sim}{\alpha}_{12}$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	(32)-(31) = (41)+(13) = (43) = (12)		
(32) (31)	(12)-(31) = (12)+(13) $\notin R_L$.	$t' = 1$	$a_{\beta\alpha} = 1$.
	(32)+(31) $\notin R_L$	$t'' = 0$	

$$w_{31}(\underset{\sim}{\rho}') = \underset{\sim}{\rho}' \ .$$

$$w_{42}(\underset{\sim}{\alpha}_{12}) = \underset{\sim}{\alpha}_{12} - \underset{\sim}{\alpha}_{42} = \underset{\sim}{\alpha}_{14} = -\underset{\sim}{\alpha}_{32} \ .$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	(12)-(42) = (12)+(24) = (14)		
(12) (42)	(14)-(42) $\notin R_L$	$t' = 1$	$a_{\beta\alpha} = 1$
	(12)+(42) $\notin R_L$	$t'' = 0$	

$$w_{42}(\underset{\sim}{\alpha}_{32}) = \underset{\sim}{\alpha}_{32} - \underset{\sim}{\alpha}_{42} = \underset{\sim}{\alpha}_{34} = -\underset{\sim}{\alpha}_{12}$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	(32)-(42) = (34)		
(32) (42)	(34)-(42) $\notin R_L$	$t' = 1$	$a_{\beta\alpha} = 1$
	(32)+(42) $\notin R_L$	$t'' = 0$	

$$w_{42}(\underset{\sim}{\rho}') = -\underset{\sim}{\rho}'$$

$$w_{12}(\underset{\sim}{\alpha}_{12}) = -\underset{\sim}{\alpha}_{12}, w_{12}(\underset{\sim}{\alpha}_{32}) = \underset{\sim}{\alpha}_{32} \ .$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	(32)-(12) = (31)		
(32) (12)	(31)-(12) = (31)+(34) $\notin R_L$	$t' = 1$	
	(32)+(12) = (41)+(12) = (42)		$a_{\beta\alpha} = 0$
	(42)+(12) $\notin R_L$	$t'' = 1$	

$$w_{32}(\underset{\sim}{\alpha}_{12}) = \underset{\sim}{\alpha}_{12},\ w_{32}(\underset{\sim}{\alpha}_{32}) = -\underset{\sim}{\alpha}_{32}\ .$$

$\underset{\sim}{\beta} + t\underset{\sim}{\alpha}$	$(12)-(32) = (13)$		
$(12)\ (32)$	$(13)-(32) \notin R_L$	$t' = 1$	
	$(12)+(32) = (43)+(32) = (42)$		$a_{\beta\alpha} = 0$
	$(42)+(32) \notin R_L$	$t'' = 1$	

$$w_{12}(\underset{\sim}{\rho}') = \tfrac{1}{2}(-\underset{\sim}{\alpha}_{12}+\underset{\sim}{\alpha}_{32}) = \underset{\sim}{\rho}'-\underset{\sim}{\alpha}_{12},\ w_{32}(\underset{\sim}{\rho}') = \tfrac{1}{2}(\underset{\sim}{\alpha}_{12}-\underset{\sim}{\alpha}_{32}) = \underset{\sim}{\rho}'-\underset{\sim}{\alpha}_{32}\ .$$

Thus in this example the Weyl group $W(L,H_0)$ acting on $\underset{\sim}{\rho}'$, exhausts the weights of S. Hence each weight has multiplicity 1, and S is simple, as an $SP(2)$-module, of highest weight $\underset{\sim}{\rho}'$.

(ii) Take the symmetric space of type AI, $SU(3)/SO(3)$.

The non-compact dual is $SL(3,\mathbb{R})/SO(3)$.

p consists of the symmetric, pure imaginary matrices of trace zero. $su(3)$ has rank 2, $so(3)$ has rank 1.

A Cartan subalgebra h of $su(3)$ is

$$\begin{pmatrix} ia & b & 0 \\ -b & ia & 0 \\ 0 & 0 & -2ia \end{pmatrix} \quad ; \quad a,b \in \mathbb{R}\ ,$$

which contains the Cartan subalgebra h_0 of $so(3)$

$$\begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad b \in \mathbb{R}\ .$$

$$\underset{\zeta}{\begin{pmatrix} ia & b & 0 \\ -b & ia & 0 \\ 0 & 0 & -2ia \end{pmatrix}} = b\underset{\zeta_0}{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + ia\underset{\zeta_1 \in p}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}$$

$\mathfrak{su}(3) \simeq \mathfrak{sl}(3,\mathbb{C})$ type A_2 . $\mathfrak{so}(3)_{\mathbb{C}} \simeq \mathfrak{so}(3,\mathbb{C})$ type A_1 .

$\mathfrak{sl}(3,\mathbb{R})$ has 2 conjugacy classes of Cartan subalgebras.

Consider $\varepsilon_{\alpha+\beta} = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\varepsilon_{-(\alpha+\beta)} = \begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The commutator $[\zeta_1 \quad \varepsilon_{\pm(\alpha+\beta)}] = 0$

and $[\zeta_0 \quad \varepsilon_{\pm(\alpha+\beta)}] = \pm\, 2ib\, \varepsilon_{\pm(\alpha+\beta)}$.

With

$$\underset{\varepsilon_\alpha}{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2i \\ 0 & 0 & 0 \end{pmatrix}} = \underset{\xi_\alpha}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}} + \underset{\eta_\alpha}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}}$$

we have $[\zeta_0 \xi_\alpha] = ib\xi_\alpha$, $[\zeta_0 \eta_\alpha] = ib\eta_\alpha$

and $[\zeta_1 \xi_\alpha] = 3ia\eta_\alpha$, $[\zeta_1 \eta_\alpha] = 3ia\xi_\alpha$.

With

$$\underset{\varepsilon_{-\beta}}{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2i \\ 0 & 0 & 0 \end{pmatrix}} = \underset{\xi_{-\beta}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}} + \underset{\eta_{-\beta}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}}$$

we have $[\zeta_0 \xi_{-\beta}] = -ib\ \xi_{-\beta}$, $[\zeta_0 \eta_{-\beta}] = -\ ib\ \eta_{-\beta}$

and $[\zeta_1 \xi_{-\beta}] = 3ia\ \eta_{-\beta}$, $[\zeta_1\ \eta_{-\beta}] = 3ia\ \xi_{-\beta}$.

Also take

$$\underset{\varepsilon_\beta}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2i & 0 \end{pmatrix}} = -\underset{\xi_\beta}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}} + \underset{\eta_\beta}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}}$$

and

$$\underset{\varepsilon_{-\alpha}}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -2i & 0 \end{pmatrix}} = -\underset{\xi_{-\alpha}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}} + \underset{\eta_{-\alpha}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}}$$

One sees that $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ where

$$\alpha(\zeta) = i(3a+b)\ ,\ \beta(\zeta) = i(-3a+b)$$

with corresponding root vectors, as given. So $\alpha+\beta(\zeta) = 2ib$. Also $\alpha^\theta = \beta$. We take $R^+ = \{\alpha, \beta, \alpha+\beta\}$. Here $R_0 = \phi$ and $R_1^+ = \{\alpha, \beta\}$, $R_2^+ = \{\alpha+\beta\}$, $R_L^+ = \{\underset{\sim}{\alpha}\}$. Note that Q is not reduced as $2\underset{\sim}{\alpha} \in Q$.

$$\rho' + \rho_2 = \tfrac{1}{2}(\alpha + (\alpha+\beta))\ .\quad \underset{\sim}{\rho}' + \underset{\sim}{\rho}_2 = 3/2\underset{\sim}{\alpha}\ .$$

The weights of (S, σ) are $3/2\underset{\sim}{\alpha}$, $\tfrac{1}{2}\underset{\sim}{\alpha}$, $-\tfrac{1}{2}\underset{\sim}{\alpha}$, $-3/2\underset{\sim}{\alpha}$; each occurring with multiplicity 1 . Hence S is simple, as an $\widetilde{SO}(3)$-module of highest weight $3/2\underset{\sim}{\alpha}$.

(N.B. $SO(n)$ has fundamental group $\mathbb{Z}_2$, so is not simply connected. $\widetilde{SO}(n)$, the simply connected covering group, is Spin(n) . SU(n) and SP(n) are simply connected.)

§3. The Case of a Symmetric Pair of Compact Type.

Refer to §2.

(3.1) Let (K,L) be a compact, symmetric, spin pair (i.e. (K,L) is a compact symmetric pair (see [10]), and K/L is K-spin (see Chapter 0, (5.3)).

Let γ_0 determine the Levi-Civita connection on $T(K/L)$ (see Chapter 0, (2.4)). For a symmetric pair, this is the same as the reductive connection for: $[pp] \subseteq \ell$, therefore $\gamma_0 = 0$ on p. See Chapter 2, §1. Take the twisted, by V, Dirac operator $D = D_V$ associated to $((\ ,\ ),\gamma_0)$.

It is in this situation, that the square of the Dirac operator takes its simplest form. In fact by Chapter 0, (2.5); Chapter 1, (2.1), and (1.1) of this chapter, we have the expression in terms of Casimir operators (w.r.t(,)) :

$$D^2 = dR(\Omega_K)-dR(\Omega_L)+2d\sigma(\Omega_L)-d\sigma(\Omega_L)-d\tau(\Omega_L)+d(\sigma\otimes\tau)(\Omega_L)$$

i.e. $$D^2 = dR(\Omega_K)+d\sigma(\Omega_L)-d\tau(\Omega_L)\ ,\quad \text{as was obtained in [28].} \qquad (3.1.1)$$

(One see that $dR(\Omega_K) = dL(\Omega_K)$, see Chapter 0, (1.4).)

Let (K,L) be of compact type (see §2). As was stated, $d\sigma(\Omega_L)$ is a constant on S. Take $V = V_{\lambda_0-\rho_L}$, the simple L-module of highest weight $\lambda_0-\rho_L$. It is the purpose of this section to determine the kernel of D, $\text{Ker } D$, as a K-module.

Let U_ν be the simple K-module of highest weight $\nu \in \Lambda \cap I^d$. It is seen that finding Ker D is equivalent to determining the ν-primary K-submodules in the induced module $L^2(S\otimes V_{\lambda_0-\rho_L})_L^K$ with ν belonging to a certain infinitesimal class. (Refer to Chapter 0, (3.2).)

In (3.2) we consider rank L = rank K . The arguments used in (3.2) are similar to those used in [28], [31] (for the pair (G,M) where G is non-compact semi-simple and M a maximal compact subgroup. This is a symmetric pair.)

In (3.3) we consider rank L < rank K . This is harder.

(3.2) Consider rank L = rank K .

The arguments used here will be similar to those in [28], [31]. By (2.2), the formula for the square becomes

$$D^2 = dR(\Omega_K) - (||\lambda||^2-||\rho||^2) . \tag{3.2.1}$$

(Here $H_0 = H$, $\lambda_0 = \lambda$.)

Recall the ½-Dirac operators $D^\pm$ (see Chapter 0, (5.4)).

<u>Theorem 1</u>.

If λ is singular w.r.t R , then Ker D = 0 .

If λ is non-singular w.r.t R , then Ker $D^+ = U_{w\lambda-\rho}$, Ker $D^- = 0$ (or +,- interchanged, see Proposition 7 Chapter 3, (2.1)); here w is the unique element of the Weyl group W(K,H) with $w\lambda$ dominant w.r.t R^+ .

Proof.

From (3.2.1) Ker D is the direct sum of the ν-primary K-submodules in $L^2(S\otimes V_{\lambda-\rho_L})^K_L$ with ${}_\nu\chi(\Omega_K) = ||\lambda||^2-||\rho||^2$ (${}_\nu\chi$ being the infinitesimal character of U_ν . The result now follows directly from Proposition 7. □

(3.3) Consider rank L < rank K .

By (2.5), and (2.7.1) the formula for the square becomes becomes

$$D^2 = dR(\Omega_K)-(||\lambda_0||^2 - ||\rho||^2) . \qquad (3.3.1)$$

For $\lambda \in \Lambda$, λ non-singular let w denote the unique element in W(K,H) with $w\lambda$ dominant w.r.t R^+ (see Chapter 0, (4.2) for 'singular', 'non-singular').

Theorem 2.

Ker D is the ν-primary K-submodule, of multiplicity $2^{[m_0/2]}$, $\Gamma_\nu(S\otimes V_{\lambda_0-\rho_L})^K_L$ in $L^2(S\otimes V_{\lambda_0-\rho_L})^K_L$; where $\nu = \lambda-\rho$ with $\lambda \in \Lambda$, $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = 0$.

(Recall that $m_0 = \dim \mathfrak{h}_1$.)

Proof.

Define for $\mu_0 \in \Lambda_0 \cap I_L$, $\inf(\mu_0) = \{\nu \in \Lambda \cap I ; {}_\nu\chi(\Omega_K) = ||\mu_0||^2-||\rho||^2\}$.

This is a finite set ($_\nu\chi$ is the infinitesimal character of U_ν). From ( ), Ker D is the direct sum of the ν-primary K-submodules in $L^2(\_\_\_)$, with $\nu \in \inf(\lambda_0)$.

Refer to Chapter 0, (3.2). Suppose that U_ν , $\nu \in \inf(\lambda_0)$, contains a simple L-submodule of highest weight of the form $\mu_0 = \lambda_0 - \underset{\sim}{\rho} + |\underset{\sim}{A}| + |\underset{\sim}{B}|$ with $A \subseteq R'^+$, $B \subseteq R_2^+$ (see Chpater 3, (1.3)). Since the weights of U_ν as an L-module are just the restrictions to h_0 of the weights of U_ν as a K-module, there is a weight of U_ν with $\underset{\sim}{\mu} = \mu_0$. Define the parameter λ , by $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = \tilde{\mu} - |\tilde{A}|$; so $\mu = \lambda - \rho + |A| + |B|$. Recall that $\tilde{\rho} = 0$.

The set of weights of U_ν are of course invariant under $W(K,H)$. Choose $w \in W(K,H)$ with $w\lambda$ dominant w.r.t R^+ . We have $w\mu = w\lambda - \rho + |C|_w$, $C = A \cup B$ (see Chapter 5, (2.2)). Also $\nu = w\mu + s$, s a sum of +ve roots. Then $||\nu+\rho||^2 = ||w\lambda + |C|_w + s||^2$. So if $||\nu+\rho||^2 = ||\lambda_0||^2$, we require $\langle\lambda,\lambda\rangle' + 2\langle w\lambda, |C|_w + s\rangle + |||C|_w + s||^2 = 0$. Therefore $\tilde{\lambda} = 0$, and $s = 0$, $|C|_w = 0$. Hence λ is non-singular, $w = 1$ and $\nu = \lambda - \rho$. Therefore $\mu_0 = \lambda_0 - \underset{\sim}{\rho}$.

As $\tilde{\lambda} = 0 = \tilde{\rho}$, μ_0 is dominant w.r.t R^+ . Now $\mu_0 = -(\underset{\sim}{\rho} - \rho_L) + \lambda_0 - \rho_L$ i.e. the sum of the lowest weight of S and the highest weight of $V_{\lambda_0 - \rho_L}$. It follows that the simple L-module of highest weight μ_0 occurs with multiplicity $2^{[m_0/2]}$ in $S \otimes V_{\lambda_0 - \rho_L}$. Hence the result. (See Chapter 0, (3.2.) □

Remark 5.

Since for a symmetric pair of compact type, $H = H_0 \times H_1$ a direct product, one can always satisfy the condition $\lambda \in \Lambda$, $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = 0$. And λ is unique.

CHAPTER 5.

In this chapter and subsequent chapters we will embark upon a series of steps, which will eventually lead to the answer to the Problem for any compact pair (K,L). (See Chapter 2, §1.) These steps are indicated at the head of each chapter.

For L not the identity subgroup {e} , and V a simple L-module of highest weight $\lambda_0-\rho_L$, the procedure involves first establishing the result for λ_0 'sufficiently non-singular'. This will occupy chapters 5 - 8 . Chapter 9 then extends this to all parameters λ_0 .

We shall use the notation and material of previous chapters, often without comment.

The procedure will be independent of the method used in Chapter 4 §3 for the special case of a symmetric pair of compact type.

Step 1.

In this chapter, we deal with the case of L = {e} the identity subgroup (see §3), and the case of L = H a maximal torus of K (see §4).

In §1 we develop our technique of tensoring an induced representation with a finite-dimensional representation. And we study the behaviour of a connection, and a 1^{st} order differential operator of type 'symbol mapping composed with a connection' with respect to this construction.

§1. The Tensor Product of an Induced Representation and a Finite-dimensional Representation.

(1.1) Let (K,L) be a pair of Lie groups with L a closed subgroup of K. Let (U,κ) be a finite dimensional unitary representation of L, and (W,Π) a finite-dimensional unitary representation of K.

There is the 'product K-bundle' $K/L \times W$ over K/L, where $(x,w) \to x$ and K acts by $k(x,w) = (k.x,\Pi(k)w)$, $k \in K$, $x \in K/L$, $w \in W$.

Regard (W,Π) as a representation of L by restriction. Define a K-bundle map $(\underline{W})_L^K = K \times_L W \longrightarrow K/L \times W$ by

$$[k,w] \longrightarrow (kL,\Pi(k)w)_{k \in K, w \in W} .$$

This is a K-equivalence of vector bundles. Thus there is a K-equivalence $(\underline{U \otimes W})_L^K \approx (\underline{U})_L^K \otimes K/L \times W$.

Define $\Gamma(\underline{U \otimes W})_L^K \xrightarrow{\Phi} \Gamma(\underline{U})_L^K \otimes W$

by $(\Phi f)(k) = (1 \otimes \Pi(k))f(k)$, $k \in K$, $f \in \Gamma(\underline{U \otimes W})$. (1.1.1)

(N.B. Here we are omitting $\wedge$. 1 is the identity operator.)

$(\Phi f)(k\ell) = (\kappa(\ell)^{-1} \otimes \Pi(k))f(k) = (\kappa(\ell)^{-1} \otimes 1)(\Phi f)(k)$, $k \in K$, $\ell \in L$.

Note that $\Phi^{-1}(f \otimes w)(k) = f(k) \otimes \Pi(k)^{-1}w$.

$$\begin{aligned} \text{Also} \quad (\Phi f)(k^{-1}k_1) &= (1 \otimes \Pi(k^{-1}k_1))f(k^{-1}k_1) \\ &= (1 \otimes \Pi(k)^{-1})\Phi(k.f)(k_1) \\ \text{so} \quad \Phi(k.f) &= (1 \otimes \Pi(k))k.(\Phi f) . \end{aligned}$$

Hence Φ is a K-equivalence, and it extends to a unitary equivalence $L^2(\underline{U \otimes W})^K_L \xrightarrow{\Phi} L^2(\underline{U})^K_L \otimes W$.

(1.2) We have

$$\begin{aligned}((dR(\xi)\otimes 1)\Phi f)(k) &= (dR(\xi)_k \otimes 1)\Phi f \\ &= \frac{d}{dt}(\Phi f)(k \exp t\xi)\Big|_{t=0} \\ &= \frac{d}{dt} 1\otimes\Pi(k \exp t\xi) f(k \exp t\xi)\Big|_{t=0} \\ &= (1\otimes\Pi(k))(dR(\xi)_k f + d\Pi(\xi) f(k)) \\ &= (\Phi(dR(\xi)+d\Pi(\xi))f)(k), k \in K, f \in \Gamma(\underline{U \otimes W}) \\ &\qquad \xi \in k .\end{aligned}$$

Thus $(dR(\xi)\otimes 1)\Phi = \Phi(dR(\xi)+d\Pi(\xi)),\ \xi \in k$. (1.2.1)

Let ∇^U be a K-invariant, metric connection on $\underline{U}$, determined by $\gamma^U : k \to u(U)$. Then we get such a connection ${}_\pi\nabla^U$ on $\underline{U \otimes W}$ by Φ i.e.

$$\Phi({}_\pi\nabla^U f) = (\nabla^U \otimes 1)\Phi f . \tag{1.2.2}$$

Now $(\gamma^U(\xi)\otimes 1)(\Phi f)(k) = (\gamma(\xi)\otimes\Pi(k))f(k)$

$$= (\Phi(\gamma(\xi)\otimes 1)f)(k) .$$

Thus ${}_\pi\nabla^U$ is determined by $\gamma^U : k \to u(U \otimes W)$ where ${}_\pi\gamma^U = \gamma^U \otimes 1 + 1 \otimes d\Pi$. (1.2.3)

(1.3) Let K/L be reductive so $k = \ell \oplus p$, $[\ell,p] \subseteq p$. Via an inner product $(,)$ on p , K/L becomes Riemannian. Suppose $p \otimes U \xrightarrow{a} U$ is an L-map. Then we get the 1[st] order differential operator $D : \Gamma(\underline{U})_L^K \to \Gamma(\underline{U})_L^K$

$$D = a \circ \nabla^U \qquad \text{(see Chapter 0, (2.6))}. \tag{1.3.1}$$

so

$$D = \sum_i a(\xi_i)(dR(\xi_i)+\gamma^U(\xi_i))$$

where $\{\xi_i\}$ is an orthonormal (w.r.t $(,)$) basis for p .

By Φ , we get ${}_\pi D:\Gamma(\underline{U\otimes W})_L^K \to \Gamma(\underline{U\otimes W})_L^K$ where $\Phi\ {}_\pi D = (D\otimes 1)\Phi$. (1.3.2)

Thus ${}_\pi D = (a\otimes 1)\circ {}_\pi\nabla^U$. So ${}_\pi D$ has symbol map $a \otimes 1$ and

$${}_\pi D = \sum_i (a(\xi_i) \otimes 1)(dR(\xi_i) + {}_\pi\gamma^U(\xi_i))$$

(1.4) Let K/L be K-spin. Take $(U,\kappa) = (S\otimes V,\sigma\otimes\tau)$, $\nabla^U = \nabla^{S\otimes V}$ (see Chapter 2, §1 for notation). Take the twisted Dirac operator D_V associated to $((,),\gamma)$.

By (1.3), associated to the triple $((,),\gamma,\Pi)$ there is the twisted, by V , Dirac operator ${}_\pi D_V$ of the connection determined by $\gamma^S\otimes 1\otimes 1 + 1\otimes 1\otimes d\Pi$ on $(\underline{S\otimes V\otimes W})_L^K$. (1.4.1)

Therefore ${}_\pi D_V$, and $D_{V\otimes W}$ are related by

$${}_\pi D_V - D_{V\otimes W} = \sum_i c(\xi_i)\otimes 1\otimes d\Pi(\xi_i) \tag{1.4.2}$$

as operators on $\Gamma(\underline{S\otimes V\otimes W})_L^K$.

(1.5) Let (K,L) be a compact pair. We use the notation of (1.1). For $\xi \in k$, $(dR(\xi)^2 \otimes 1)\Phi = \Phi(dR(\xi) + d\Pi(\xi))^2$. Let $\{\eta_i\}$ be an orthonormal basis of k (w.r.t $(,)$ see Chapter 0, (4.2)). Putting $\xi = \eta_i$ and summing over i, we get

$$(dR(\Omega_K) \otimes 1)\Phi = \Phi(d(R\otimes\Pi))(\Omega_K) \tag{1.5.1}$$

Ω_K is the Casimir operator of K. It is easily seen that $dR(\Omega_K) = dL(\Omega_K)$. So $\Phi dL(\Omega_K) = d(L\otimes\Pi)(\Omega_K)\Phi$. (1.5.2)

§2. The Spin Representation of a Compact, Connected Lie Group.

(2.1) Let K be a compact connected Lie group and H a maximal torus of K. We have $k = h \oplus p$ an orthogonal direct sum w.r.t$(,)$. Here p is even dimensional.

W.r.t the pairs $(k,(,))$, $(h,(,))$, $(p,(,))$ we have

$$\text{Cliff}\,(k) = \text{Cliff}\,(h) \otimes \text{Cliff}\,(p) \quad \text{a direct sum}$$

as associative algebras. Take a minimal left ideal S_0, S in $\text{Cliff}(h_{\mathbb{C}})$, $\text{Cliff}(p_{\mathbb{C}})$ respectively, then $S_1 = S_0 S$ is a minimal left ideal in $\text{Cliff}(k_{\mathbb{C}})$. The dimension of S_0, $\dim S_0 = 2^{[\ell/2]}$ where $\ell = \text{rank } K$ ($[\ell/2]$ denotes the integral part of $[\ell/2]$, i.e. the greatest integer $\leq \ell/2$). Also $\dim S = 2^m$, $m = \frac{1}{2}\dim K/H = \frac{1}{2}\dim p$ = no of +ve roots. $\dim S_1 = (\dim S_0)(\dim S)$.

By composing the left regular representation of Cliff($k_{\mathbb{C}}$) with the lift of the adjoint representation of K to Spin (k), we get the spin representation (S_1,σ_1) of K (see Chapter 0, (4.3). Recall that we are assuming that $\rho \in \Lambda$). Similarly we get the spin representation (S_0,σ_0) of H. K/H is K-spin so we also have the spin representation (S,σ) of H. Unitarise these as in Chapter 1, §1.

$S_1 = S_0 \otimes S$ as unitary H-modules (see Chapter 6, (1.2)). By Chapter 0, (5.3) we see that S_0 is trivial as an H-module. Also. the differential of σ_1 is given by

$$d\sigma_1(\eta) = -\tfrac{1}{4} \sum_i c[\eta\eta_i]c(\eta_i) = (\ell o d\psi_1)(ad\eta), \ \eta \in k . \qquad (2.1.1)$$

where $\{\eta_i\}$ is an orthonormal (w.r.t(,)) basis of k.

(2.2) We shall say that a finite-dimensional unitary K-module U is *primary* if it is the direct sum of a number of copies of a simple K-module U_ν. Then the *multiplicity* is the intertwining number $i(U,U_\nu)$.

<u>Proposition 10</u>.

S_1 is primary as a K-module, the simple K-module of highest weight ρ, U_ρ, occurring with multiplicity $2^{[\ell/2]}$.

Proof.

By Chapter 3 (1.1), the weights of (S,σ) are the $\rho-|A|$, where $A \subseteq R^+$. $\rho-|A|$ occurs as a weight with multiplicity equal to the number of $B \subseteq R^+$ with $|B| = |A|$. These are also the weights of (S_1,σ_1) as a representation of K, the multiplicity as a weight of σ_1 being $2^{[\ell/2]}$ times the multiplicity as a weight of σ. In particular the 'highest' weight ρ occurs with multiplicity $2^{[\ell/2]}$. By Weyl's degree formula, U_ρ has dimension $2^m = \dim S$, m = no of +ve roots. Hence the assertion. □

Hence we see that the weights of U_ρ and their multiplicities, are just those of (S,σ). (See also [21].)

For $w \in W(K,H)$ (the Weyl group), define $A_w \subseteq R^+$ by $A_w = wR^- \cap R^+$ (here R^- denotes the set of -ve roots i.e. $-R^+$). So $w\rho = \rho-|A_w|$. Note that as ρ occurs with mult 1 as a weight of U_ρ, $A \subseteq R^+$, $|A| = |A_w|$ implies that $A = A_w$. The set of weights of U_ρ are, of course, invariant under $W(K,H)$. For $w \in W(K,H)$, $A \subseteq R^+$ let the sum of distinct roots in R^+, $|A|_w$, be given by

$$w(\rho-|A|) = \rho-|A|_w . \quad \text{So we have} \quad |A|_w = w|A| + |A_w| .$$

§3. The Case of the Identity Subgroup.

(3.1) Recall Chapter 2, §1. Set $L = \{e\}$ the identity subgroup (the 0-dimensional Lie group with $\{0\}$ Lie algebra).

There is the adjoint representation (k,Ad) of K. The tangent bundle of K, $T(K) = K \times k$ the product bundle

$$= (\underline{k})^{K}_{\{e\}}$$

which is Riemannian via $(,)$.

Note that any linear map $\gamma_1 : k \to so(k)$ determines a K-invariant metric connection on $T(K)$, since (i), (ii) of Proposition 1 (Chapter 0, (2.2)) are trivially satisfied. We define a family of connections by

$$\gamma_{1a}(\xi) = a \text{ ad } \xi, \ \xi \in k \ ; \ a \in \mathbb{R} . \qquad (3.1.1)$$

γ_{1a} lifts to a unique K-invariant, metric connection on $(\underline{S}_1)^{K}_{\{e\}} = K \times S_1$ (product bundle), determined by

$$\gamma_a^{S_1} : k \to u(S_1)$$

$$\gamma_a^{S_1}(\xi) = (\ell \circ d\psi_1)\gamma_{1a}(\xi) = a \, d\sigma_1(\xi) \ , \ \xi \in k \ ; \ a \in \mathbb{R} . \qquad (3.1.2)$$

(See Proposition 3 Chapter 1 (1.1) and §2.)

The curvature $R_1(\, ,)$, and the torsion $T_1(,)$ (see Chapter 0, (2.4)) of γ_{1a} are given by:

$$R_1(\xi,\eta) = a^2[\text{ad } \xi, \text{ad } \eta] - a \text{ ad}[\xi\eta]$$

i.e. $$R_1(\xi,\eta) = a(a-1) \text{ ad}[\xi\eta] \ , \ \xi,\eta \in k \ , \qquad (3.1.3)$$

and $$T_1(\xi,\eta) = -[\xi,\eta] + a \text{ ad } \xi(\eta) - a \text{ ad } \eta(\xi)$$

$$= (2a-1)[\xi\eta] \ , \ \xi,\eta \in k \ . \qquad (3.1.4)$$

Therefore γ_{1a} gives a flat connection (i.e. $R_1(,) = 0$) iff $a = 0$ or 1.

$a = 0$ gives the reductive connection

$a = \frac{1}{2}$ " " Levi-Civita "

The curvature $R^{S_1}(,)$ of $\gamma_a^{S_1}$ is given by

$$R^{S_1}(,) = (\ell o d\psi_1)R_1(,)$$

so $$R^{S_1}(\xi,\eta) = a(a-1)d\sigma_1([\xi\eta]) \ , \ \xi,\eta \in k \ . \qquad (3.1.5)$$

These are trivial if K is abelian.

(3.2) <u>The formula for</u> D_1^2 .

Note that for a complex vector space V_1 , $\Gamma(\underline{V}_1)^K_{\{e\}}, L^2(\underline{V}_1)^K_{\{e\}}$ is just the smooth functions $f:K \to V_1$, square-integrable functions $f:K \to V_1$ respectively.

The Laplacian on $(\underline{S_1 \otimes V})^K_{\{e\}}$ associated to $((,),\gamma_{1a})$ (with the reductive connection on $\underline{V}$; here V is any complex vector space) Δ_1 is given by

$$\Delta_1 = -\sum_i (dR(\eta_i) + \gamma_a^{S_1}(\eta_i))^2 \ . \qquad \text{(See Chapter 0, (2.5).)}$$

Also associated to $((,),\gamma_{1a})$, there is the Dirac operator D_1 .

$$D_1 : \Gamma(\underline{S_1 \otimes V})^K_{\{e\}} \to \Gamma(\underline{S_1 \otimes V})^K_{\{e\}}$$

$$D_1 = \sum_i c(\eta_i)(dR(\eta_i) + \gamma_a^{S_1}(\eta_i)) .$$

Now the Laplacian

$$\Delta_1 = -\sum_i (dR(\eta_i) + 2a d\sigma_1(\eta_i) dR(\eta_i) + a^2 d\sigma_1(\eta_i)^2) .$$

So this has an expression in terms of Casimir operators of K i.e.

$$\Delta_1 = dR(\Omega_K) + a(-dR(\Omega_K) + d(R\otimes\sigma_1)(\Omega_K) - d\sigma_1(\Omega_K)) + a^2 d\sigma_1(\Omega_K)$$

$$\Delta_1 = (1-a)dR(\Omega_K) + ad(R\otimes\sigma_1)(\Omega_K) + a(a-1)d\sigma_1(\Omega_K) , \quad a \in \mathbb{R} . \qquad (3.2.1)$$

Consider the formula in Proposition 4 (Chapter 1 (2.2)) for D_1^2 .

The 'torsion term' is

$$-\tfrac{1}{2}(2a-1) \sum_{i,j} c(\eta_i)c(\eta_j)dR([\eta_i\eta_j]) + ad\sigma_1([\eta_i\eta_j]))$$

$$= (2a-1)(-2)(\sum_i d\sigma_1(\eta_i)dR(\eta_i) + a \sum_i d\sigma_1(\eta_i)^2)$$

$$= (2a-1)(-dR(\Omega_K) + d(R\otimes\sigma_1)(\Omega_K) - d\sigma_1(\Omega_K) + 2ad\sigma_1(\Omega_K))$$

$$= (2a-1)(-dR(\Omega_K) + d(R\otimes\sigma_1)(\Omega_K) + (2a-1)d\sigma_1(\Omega_K)) .$$

The 'curvature term' is

$$\tfrac{1}{2}\, a(a-1) \sum_{i,j} c(\eta_i)c(\eta_j)d\sigma_1([\eta_i\eta_j])$$

$$= -\, a(a-1)(-2)\sum_i d\sigma_1(\eta_i)^2 = -2a(a-1)d\sigma_1(\Omega_K)\ .$$

Hence $D_1^2 = (2-3a)dR(\Omega_K)+(3a-1)d(R\otimes\sigma_1)(\Omega_K)+(3a(a-1)+1)d\sigma_1(\Omega_K)\ ,\ a \in \mathbb{R}\ .$ (3.2.2)

If $K = H$ is abelian, this reduces to $D_0^2 = dR(\Omega_H)$. (3.2.3)

For the reductive connection $a = 0$ and

$$D_1^2 = 2dR(\Omega_K) - d(R\otimes\sigma_1)(\Omega_K) + d\sigma_1(\Omega_K)\ . \tag{3.2.4}$$

For the Levi-Civita connection $a = \tfrac{1}{2}$ and

$$2D_1^2 = dR(\Omega_K) + d(R\otimes\sigma_1) + \tfrac{1}{2}\, d\sigma_1(\Omega_K)\ . \tag{3.2.5}$$

(3.3) Take $V = 1$ i.e. 1-dimensional.

<u>Theorem 3.</u>

(1) If $K = H$ is abelian, $\operatorname{Ker} D_0$ is the trivial primary H-submodule in $L^2(\underline{S_0})^H_{\{e\}}$. The multiplicity is $\dim S_0 = 2^{[\dim h/2]}$.

(2) If K is non-abelian: $\operatorname{Ker} D_1 = 0$ for $a = \tfrac{1}{2}$; and for $a = 0$, $\operatorname{Ker} D_1$ is the trivial primary K-submodule in $L^2(\underline{S_1})^K_{\{e\}}$, the multiplicity being $\dim S_1 = 2^{[\ell/2]}$. $\ell = \operatorname{rank} K$.

Proof.

(1) $dR(\Omega_H) = dL(\Omega_H)$ is constant on the μ-primary H-submodule in $L^2(H)$, $\mu \in \Lambda$; the constant being $||\mu||^2$. Hence the assertion for K abelian.

(2) $dR(\Omega_K) = dL(\Omega_K)$ is constant on the ν-primary K-submodule in $L^2(K)$, $\nu \in \Lambda \cap I^d$; the constant being $||\nu+\rho||^2 - ||\rho||^2$. (See (Chapter 0, (3.1), (3.2), (4.2).)

Now by Proposition 10, $d\sigma_1(\Omega_K)$ is the constant $3||\rho||^2$ on S_1. Thus as a Casimir operator is positive, essentially self-adjoint we get the assertion for $a = \frac{1}{2}$. See (3.2.5).

Consider $a = 0$. There is a unitary equivalence.

$$L^2(\underline{S_1})^K_{\{e\}} \xrightarrow{\Phi} S_1 \otimes L^2(\underline{1})^K_{\{e\}}$$ (See (1.1). Note that $L^2(\underline{1})^K_{\{e\}}$ is just $L^2(K)$.) Take $S_1 \otimes \Gamma_\nu(\underline{1})^K_{\{e\}}$ and the ν'-primary K-submodule therein, ${}_{\nu'}P_\nu$.

$$\Gamma_{\nu'}(\underline{S_1})^K_{\{e\}} = \sum_\nu \oplus \Phi^{-1}\,{}_{\nu'}P_\nu$$ (a finite orthogonal direct sum).

$\Phi^{-1}\,{}_{\nu'}P_\nu$ is preserved by D_1^2 (see (1.5)). In fact on this space, as ν' is of the form $\nu' = \nu-\rho+|A|$ some $A \subseteq R^+$, we have, by (3.2.4),

$$D_1^2 = 2(||\nu+|A|||^2 - ||\rho||^2) - (||\nu+\rho||^2 - ||\rho||^2) + 3||\rho||^2 .$$

Then $D_1^2 - ||\nu-\rho||^2 = 2(2\langle\nu,|A|\rangle + |||A|||^2)$. Thus on $\operatorname{Ker} D_1$, we must have $A = \phi$, (the empty set) $\nu = \rho$. Therefore $\nu' = 0$. But these conditions are also sufficient for $\operatorname{Ker} D_1$. Hence assertion. □

§4. The Case of a Maximal Torus.

Recall Chapter 2, §1. Let K be non-abelian and $L = H$ a maximal torus of K. Associated to the pair $((,),\gamma)$ we have the twisted Dirac operator $D = D_V$.

The formula for D^2 given by Proposition Chapter 1 (2.2) and Chapter 4 (1.1), looks complicated for a general pair (K,L). There is a first order term in the Laplacian for γ the Levi-Civita connection; and for γ the reductive connection there is a first order 'torsion term'. Although for a symmetric pair, it turns out that $d\sigma(\Omega_L)$ is a constant on S (see Chapter 4, (2.7)), this is certainly not true for general (K,L).

Example.

Take (K,H) where $K = SO(5)$. This has rank 2. The direct product of 2 copies of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$; $\theta \in \mathbb{R}$ and 1, is a maximal torus H. K is simple and $k_{\mathbb{C}}$ is of type B_2. The 2 simple roots of B_2 are not of equal length. In fact we can take simple roots α,β with $||\beta||^2 = 2||\alpha||^2$. With $\gamma = \alpha$ or β, $||\rho-\gamma||^2 = ||\rho||^2 - 2\langle\rho,\gamma\rangle + ||\gamma||^2 =$ $= ||\rho||^2 + \frac{||\gamma||^2}{2}$. Thus $||\rho-\beta||^2 \neq ||\rho-\alpha||^2$. It follows from this and Chapter 3, (1.1), that $d\sigma(\Omega_H)$ is not a constant.

To obtain $\operatorname{Ker} D$, we could at once attempt a 'highest weight argument', along the lines of that which we use in the last part of the proof of Theorem 4 , for the reductive connection. (See (4.3).) The idea being to try and compute the infinitesimal character on Ω_K , ${}_\nu\chi(\Omega_K)$, for U_ν a simple K-module occurring in Ker D . This came straight from D^2 for a symmetric pair (see Chapter 4, §3).

However, the following method, for (K,H) , shows that the sum of D^2 and an anti-commutator of D , is expressible entirely in terms of Casimir operators. This more naturally extends the work of §3 and gives more precise information along the way, which will be also important in Chapter 9.

(4.1) Recall §1,2,3. Regard σ_0 as the restriction of the trivial representation of K on S_0 . Then there is a unitary equivalence

$$L^2(\underline{S_1 \otimes V})_H^K \xrightarrow{\Phi_0} S_0 \otimes L^2(\underline{S \otimes V})_H^K$$

$S_1 \otimes V = S_0 \otimes (S \otimes V)$ as a unitary H-module.

Consider $D_{S_0 \otimes V}$. As σ_0 is trivial $\Phi_0^{-1}(D_V \otimes 1)\Phi_0 = D_{S_0 \otimes V}$.

$$D_{S_0 \otimes V} = \sum_j c(\xi_j)(dR(\xi_j) + \gamma^S(\xi_j)) ,$$

where $\{\xi_j\}$ is an orthonormal (w.r.t(,)) basis of p .
$D_V \otimes 1$ is the direct sum of $2^{[\ell/2]}$ copies of $D = D_V$. We denote $D_{S_0 \otimes V}$ also by D . We intend to compute D^2 on $\Gamma(\underline{S_1 \otimes V})_H^K$.

Take $\{\zeta_t\}$ an orthonormal (w.r.t(,)) basis of h and $\{\eta_i\} = \{\zeta_t, \xi_j\}$.

Reductive connection: $\gamma = 0$ on p . With $a = 0$ (see (3.1), (3.2))

$$D_1 = D_0 + D \quad \text{on } \Gamma(\underline{S_1 \otimes V})_H^K \, , \tag{4.1.1}$$

where D_0 is the trivial extension (see Ch.6,(2.2)) to K of the Dirac operator with the reductive connection on $T(H) = (\underline{h})^H_{\{e\}}$ over H . i.e.

$$D_0 = \sum_t c(\zeta_t) dR(\zeta_t)$$

$$= -\sum_t c(\zeta_t) d(\sigma \otimes \tau)(\zeta_t) \quad \text{on } \Gamma(\underline{S_1 \otimes V})_H^K \, .$$

(N.B. D_0 preserves $\Gamma(\underline{S_1 \otimes V})_H^K$ as $c(\zeta) \in \mathrm{Hom}_H(S_1, S_1)$, $\zeta \in h$.)

Then $D_1^2 = D_0^2 + [D_0 D]_+ + D^2$

where $[D_0 D]_+ = D_0 D + D D_0$ (i.e. $[\ ]_+$ is the anti-commutator).

Now $D_0^2 = \Delta_0 = -\sum_t dR(\zeta_t)^2 = dR(\Omega_H) = d(\sigma \otimes \tau)(\Omega_H)$ on $\Gamma(\underline{S \otimes V})_H^K$.

Hence on $\Gamma(\underline{S \otimes V})_H^K$,

$$D^2 + [D_0 D]_+ = 2dR(\Omega_K) - d(R \otimes \sigma_1)(\Omega_K) + d\sigma_1(\Omega_K) - d(\sigma \otimes \tau)(\Omega_H) \, . \tag{4.1.2}$$

Levi Civita connection: $\gamma = \gamma_0$ (see Chapter 0 (2.4)). Take $a = \frac{1}{2}$ (see (3.1), (3.2)) . For any $a \in \mathbb{R}$,

$$\gamma_a^{S_1}(\xi) = -\tfrac{1}{4} a \sum_i c[\xi \eta_i] c(\eta_i)$$

$$= 2a\gamma_0^S(\xi) - \tfrac{1}{4} a \sum_j c(Q[\xi \xi_j]) c(\xi_j) -$$

$$- \tfrac{1}{4}a \sum_{t,j} c(\xi_j)c[\xi_j\zeta_t]c(\zeta_t) \quad , \quad \xi \in p.$$

(Here, of course, γ_0^S is the lift of γ_0 to $(\underline{S})_H^K$.)

Also $\gamma_a^{S_1}(\xi) = a\, d\sigma(\xi)$, $\xi \in h$.

Now $$-\tfrac{1}{4}\sum_{i,j} c(\xi_i)c(Q[\xi_i\xi_j])c(\xi_j) = \tfrac{1}{4}\sum_{i,j} c(\xi_i)c(\xi_j)c(Q[\xi_i\xi_j])$$
$$= \sum_t c(\zeta_t)d\sigma(\zeta_t) ,$$

and $$-\tfrac{1}{4}\sum_{t,j} c(\xi_j)c[\xi_j\zeta_t]c(\zeta_t) = -\tfrac{1}{4}\sum_{t,j} c(\zeta_t)c[\zeta_t\xi_j]c(\xi_j)$$
$$= \sum_t c(\zeta_t)d\sigma(\zeta_t) .$$

Thus $$\sum_i c(\eta_i)\gamma_a^{S_1}(\eta_i) = 3a \sum_t c(\zeta_t)d\sigma(\zeta_t)+2a \sum_j c(\xi_j)\gamma_0^S(\xi_j) .$$

So, with $a = \tfrac{1}{2}$,

$$D_1 = D_0 + D \text{ on } \Gamma(\underline{S_1 \otimes V})_H^K , \tag{4.1.3}$$

where $$D_0 = \sum_t c(\zeta_t)(dR(\zeta_t) + 3/2d\sigma(\zeta_t)) .$$

i.e. D_0 is the trivial extension to K of the Dirac operator with the connection $3/2d\sigma$ on $(\underline{S_1 \otimes V})_{\{e\}}^H$ over H . D_0 preserves $\Gamma(\underline{S_1 \otimes V})_H^K$.

Now $$D_0^2 = \Delta_0 = - \sum_t (dR(\zeta_t) + 3/2d\sigma(\zeta_t))^2$$
$$= dR(\Omega_H) + 3 \sum_t d\sigma(\zeta_t)d(\sigma\otimes\tau)(\zeta_t) + 9/4d\sigma(\Omega_H)$$

$$2D_0^2 = 2d(\sigma\otimes\tau)(\Omega_H)+3(d\sigma(\Omega_H)-d(\sigma\otimes\tau)(\Omega_H)+d\tau(\Omega_H)-2d\sigma(\Omega_H)) + 9/2d\sigma(\Omega_H)$$

$$= 3/2d\sigma(\Omega_H)-d(\sigma\otimes\tau)(\Omega_H) + 3d\tau(\Omega_H) \ . \qquad (4.1.4)$$

Therefore $2(D^2+[D_0D]_+) = 2D_1^2-2D_0^2$

$$= dR(\Omega_K)+d(R\otimes\sigma_1)(\Omega_K)+\tfrac{1}{2}d\sigma_1(\Omega_K)-3/2d\sigma(\Omega_H)+d(\sigma\otimes\tau)(\Omega_H)-3d\tau(\Omega_H) \ . \qquad (4.1.5)$$

(4.2) There is a unitary equivalence

$$L^2(\underline{S_1\otimes V})_H^K \xrightarrow{\ \Phi\ } S_1 \otimes L^2(\underline{V})_H^K \ .$$

Take $V = E_\lambda$, $\lambda \in \Lambda$, the simple 1-dim unitary H-module with character e^λ. Let λ be non-singular. (See Chapter 0, (4.2).) Then take the unique $w \in W(K,H)$ such that $w\lambda$ is dominant w.r.t R^+.

Let $(U_\nu,\Pi_\nu) \in \hat{K}$, ν being the highest weight. Take (a non-zero) $S_1 \otimes \Gamma_{\nu'}(\underline{E_\lambda})_H^K$ and the ν-primary K-submodule therein, ${}_\nu P_{\nu'}$.

By assumption, λ is a weight of $U_{\nu'}$, so $\nu' = w\lambda+s$, s a sum of (not necessarily distinct) roots in R^+ (see Chapter 0 (4.2)). Also $\nu = \nu'-\rho+|A|$ some $A \subseteq R^+$. (See Remark 3 Chapter 3 (1.3).)

Let $f \in \Phi^{-1}{}_\nu P_{\nu'} \underset{K}{\leq} \Gamma_\nu(\underline{S_1\otimes E_\lambda})_H^K$ ($\underset{K}{\leq}$ denotes K-submodule).

Write $f = f_1 +.....+f_r$, with

$$f_i \in \Gamma_\nu(\underline{S_0\otimes S_{-\rho+|A_i|}\otimes E_\lambda})_H^K, \quad A_i \subseteq R^+ \ ; \ \text{the } |A_i|$$

being distinct. Where if $B \subseteq R^+$, $S_{-\rho+|B|}$ is the $-\rho+|B|$ weight space in S as an H-module.

Then $\nu = w\lambda-\rho+|A_i|_w + s_i$, s_i a sum of +ve roots $i = 1,\dots,r$, (see (2.2)). We have $|A_i|_w + s_i = |A| + s$, $\forall i$. (4.2.1)

(0) *Levi-Civita connection:*

$$2(D^2+[D_0D]_+)f = \{(||w\lambda+|A|+s||^2-||\rho||^2)+(||w\lambda+\rho+s||^2-||\rho||^2)+3/2||\rho||^2$$

$$-3||\lambda||^2\}f + \sum_i(||\lambda-\rho+|A_i|||^2-3/2||\rho-|A_i|||^2)f_i$$

$$= 2<w\lambda,|A|+2s>f + \sum_i 2<w\lambda,|A_i|_w>f_i$$

$$+ (|||A|+s||^2+||\rho+s||^2- \tfrac{1}{2}||\rho||^2)f-\tfrac{1}{2}\sum_i||\rho-|A_i|||^2f_i \quad . \quad (4.2.2)$$

(1) *Reductive connection:*

$$(D^2+[D_0D]_+)f = \sum_i(||w\lambda+|A_i|_w+s_i||^2-||\rho||^2)f_i+\{(||w\lambda+|A|+s||^2-||\rho||^2)$$

$$- (||w\lambda+\rho+s||^2-||\rho||^2)+3||\rho||^2)\}f-\sum_i||\lambda-\rho+|A_i|||^2f_i$$

$$= \sum_i 2<w\lambda,s_i>f_i+\{2<w\lambda,|A|> + |||A|+s||^2-||\rho+s||^2+2||\rho||^2\}f$$

$$+ \sum_i(|||A_i|_w + s_i||^2 - ||\rho-|A_i|||^2)f_i$$

$$\rho+s = \rho-|A|+|A_i|_w+s_i$$

$$= \sum_i 2<w\lambda,s_i>f_i+(2<w\lambda,|A|>+|||A|+s||^2+2||\rho||^2-||\rho-|A|||^2)f$$

$$+ \sum_i (-2<\rho-|A|,|A_i|_w+s_i> - ||\rho-|A_i|||^2)f_i \quad . \quad (4.2.3)$$

(4.3) Now $\Gamma(\underline{S_1 \otimes E_\lambda})^K_H = \sum_{\nu'} \oplus \Phi^{-1}\, {}_\nu P_{\nu'}$ (a finite orthogonal direct sum).

(See Chapter 0 (3.1), (3.2).) Note that this is finite-dimensional.

Writing $f \in \Gamma_\nu(\ )$ as $f = f^1 + \ldots\ldots + f^t$ with $f^j \in \Phi^{-1}\, {}_\nu P_{\nu_j}$,

we see from (4.2) that we can make the inner product

$$<(D^2 + [D_0 D_+])f, f> \; > a \; <f,f> \text{ for any real number } a \text{ , } a > 0 \text{ ,}$$

by taking λ s.n.s (i.e. λ sufficiently non-singular. See Chapter 0 (4.2)), provided we are not in the situation:

(0) $t = 1$, $s = 0$, $A = \phi$. So $s_i = 0 = |A_i|_W$, $\forall i$.

(1) $t = 1$, $s_i = 0$, $\forall i$, $A = \phi$.

So $s = |A_i|_W = |A_j|_W$ and $|A_i| = |A_j|$, $\forall i,j$. (4.2.4)

Recall the ½-Dirac operators $D^+ = D^+_V$, $D^- = D^-_V$. (See Chapter 0 (5.4), N.B. γ gives a connection on $\underline{S}^+$, $\underline{S}^-$).

Theorem 4.

Let $\lambda \in \Lambda$.

Take γ either the Levi-Civita or the reductive connection.

If λ is singular w.r.t R , Ker D = 0 .

If λ is non-singular w.r.t R , Ker $D^+ = U_{W\lambda-\rho}$, Ker $D^- = 0$. (Here jj(w) is even. If odd interchange +,- . See Proposition 7, Chapter 3 (2.1).)

Proof.

We shall, here, prove this for λ s.n.s. This restriction will be removed in Chapter 9.

Take $f \in \Gamma_\nu(\underline{S_1 \otimes E_\lambda})^K_H$, $f = f^1 + \ldots + f^t$ (see above). Suppose $f \in \operatorname{Ker} D$. As D is symmetric, $\langle DD_0 f, f\rangle = \langle D_0 f, Df\rangle = 0$. So $\langle (D^2 + [D_0 D]_+) f, f\rangle = 0$. We deduce from that, for either connection, $t = 1$ and (i) $A = \phi$, $s = |B|_w$ some $B \subseteq R^+$

$$\text{(ii)} \quad f \in \Gamma_\nu(\underline{S_0 \otimes S_{-\rho+|B|} \otimes E_\lambda})^K_H .$$

So, in particular, $U_\nu \underset{K}{\leq} \operatorname{Ker} D$ implies that ν is of the form $\nu = w\lambda - \rho + |B|_w$, $B \subseteq R^+$.

In the case (0) we already have $|B|_w = 0$, and therefore $B = A_{w^{-1}}$ (see (2.2)).

Suppose $U_\nu \otimes b \longrightarrow \operatorname{Ker} D$, $(0 \neq)\ b \in \operatorname{Hom}_H(U_\nu, S \otimes E_\lambda)$. (See Chapter 0 (3.2).)

With $v \in U_\nu$, $v \otimes b \longrightarrow f \in \Gamma_\nu(\underline{S \otimes E_\lambda})^K_H$

where $f(k) = b(\Pi_\nu(k)^{-1} v)$, $k \in K$.

By condition (ii), $b \in \operatorname{Hom}_H(U_\nu, S_{-\rho+|B|} \otimes E_\lambda)$. Fix v to be 'the' weight vector with weight $w^{-1}\nu = \lambda - \rho + |B|$. This weight has multiplicity 1. Let e be the identity element of K . We must have $f(e) = b(v) \neq 0$ (otherwise $b = 0$).

Now $dR(\xi)_e f = -dL(\xi)_e f$

$$= \frac{d}{dt} f(\exp t\xi)\Big|_{t=0} = -\frac{d}{dt} b(\Pi_\nu(\exp t\xi)v)\Big|_{t=0}$$

$$= -b(\Pi_\nu(\xi)v) \ , \ \xi \in \mathfrak{k} \ .$$

Since, by condition (ii), $dR(\varepsilon_\alpha)_e f = 0$, $\forall\alpha \in R$ it follows that $dR(\xi)_e f = 0$, $\forall\xi \in p$.

In the case (1) , $D^2 = dR(\Omega_K)+4\sum_j \gamma_0^S(\xi_j)dR(\xi_j)+d\sigma(\Omega_H)-d\tau(\Omega_H)$.

Then $0 = (D^2f)(e) = (||w\lambda+|B|_w||^2-||\rho||^2+||\rho-|B|||^2-||\lambda||^2)f(e)$

$$0 = 2\langle w\lambda-\rho,|B|_w\rangle + 2|||B|_w||^2 \quad .$$

As λ was taken to be non-singular, this implies that $|B|_w = 0$.

Hence, for either connection, we have shown that a simple K-module occurring in Ker D must have highest weight $\nu = w\lambda-\rho$. This is our *vanishing theorem.*

We now have to show that $U_{w\lambda-\rho}$ does occur in the kernel. For this we compute the index of D^+ . (See Chapter 0, (3.3).) As D is essentially self-adjoint, the adjoint of D^+ is D^- , thus for $\lambda \in \Lambda$,

$$\begin{aligned} \text{Index } D^+ &= \text{Ker } D^+ - \text{Ker } D^- \quad \text{in } \mathbb{Z}[\hat{K}] \\ &= 0 \text{ , } \lambda \text{ singular} \\ &= U_{w\lambda-\rho} \text{ , } \lambda \text{ non-singular by Bott's index theorem} \end{aligned}$$

and Proposition 7. Hence the assertion of the Theorem.

□

CHAPTER 6.

Step 2. In this chapter we deal with the case of (K,L) with rank L = rank K. (See §4.)

Here, we extend Chapter 5, §4 to the case of equal rank. This 'Step' can in fact be removed, and we can still get from Step 1 to 3. However §2,3 of this chapter are essential.

In §2,3 we develop our technique of 'inducing in stages' and apply it to the Dirac operator. The notion of the 'trivial extension' and the 'pull-back' of the Dirac operator is introduced.

§1. Spin Triples.

(1.1) Let (M,K,L) be a triple of Lie groups with L a closed subgroup of K and K a closed subgroup of M. Write $L \leq K \leq M$.

We have $K/L \xrightarrow{i} M/L \xrightarrow{\#} M/K$

where i is the inclusion and $\#$ is the projection (i.e. $\#(mL) = mK$, $m \in M$). We suppose that these homogeneous spaces are reductive. Let $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{p}_1$ and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ with $\mathfrak{p}_1$ Ad K-invariant, and $\mathfrak{p}$ Ad L-invariant. We suppose that there is an inner product $(,)$ on $\mathfrak{p} \oplus \mathfrak{p}_1$, such that $\mathfrak{p},\mathfrak{p}_1$ are orthogonal and $(\mathfrak{p},\mathrm{Ad}_L)$, $(\mathfrak{p}_1,\mathrm{Ad}_K)$ are orthogonal (w.r.t(,)). Then $T(K/L) = (\underline{\mathfrak{p}})^K_L$, $T(M/K) = (\underline{\mathfrak{p}_1})^M_K$, $T(M/L) = (\underline{\mathfrak{p}\oplus\mathfrak{p}_1})^M_L$; and these become Riemannian.

Take the pairs $(\mathfrak{p}\oplus\mathfrak{p}_1,(,))$, $(\mathfrak{p},(,))$, $(\mathfrak{p}_1,(,))$. Then $\mathrm{Cliff}(\mathfrak{p}\oplus\mathfrak{p}_1) = \mathrm{Cliff}(\mathfrak{p}) \hat{\otimes} \mathrm{Cliff}(\mathfrak{p}_1)$ a direct sum as associative algebras.

Let S_L, S_K be the space of spinors in $\mathrm{Cliff}(p_{\mathbb{C}})$, $\mathrm{Cliff}(p_{1\mathbb{C}})$ respectively. Then $S = S_L S_K$ is the space of spinors in $\mathrm{Cliff}(p \oplus p_1)_{\mathbb{C}}$.

(1.2) We suppose that the above reductive homogeneous spaces are spin. Then we get the unitary spin representations (S_L, σ_L), (S, σ) of L and (S_K, σ_K) of K. Refer to Chapter 0 and Chapter 1 §1. Recall that $c(\xi)\sigma_L(\ell) = \sigma_L(\ell)c(\mathrm{Ad}\ell^{-1}\xi)$, $\xi \in p$

$$c(\xi)\sigma(\ell) = \sigma(\ell)c(\mathrm{Ad}\ell^{-1}\xi) , \quad \xi \in p \oplus p_1 , \quad \ell \in L$$

and $c(\xi)\sigma_K(k) = \sigma_K(k)c(\mathrm{Adk}^{-1}\xi)$, $\xi \in p_1$, $k \in K$.

Then one sees that $\sigma = \sigma_L \otimes \sigma_K$ as unitary representations. Let $\{\zeta_t\}$, $\{\xi_j\}$ be an orthonormal (w.r.t(,)) basis for p, p_1 respectively. Set $\{\eta_i\} = \{\zeta_t, \xi_j\}$. Then

$$d\sigma_L(\zeta) = -\tfrac{1}{4} \sum_t c([\zeta\zeta_t])c(\zeta_t) , \quad \zeta \in \ell$$

$$d\sigma_K(\xi) = -\tfrac{1}{4} \sum_j c([\xi\xi_j])c(\xi_j) , \quad \xi \in k$$

and $$d\sigma(\zeta) = d\sigma_L(\zeta) \otimes 1 + 1 \otimes d\sigma_K(\zeta) , \quad \zeta \in \ell .$$

§2. Inducing in Stages.

(2.1) Let (U, κ) be a finite-dimensional unitary representation of L . There is the induced K,M-vector bundle $(\underline{U})_L^K$, $(\underline{U})_L^M$ over K/L , M/L

respectively. There is an M-equivalence of vector bundles

$$(\underline{U})_L^M \simeq (\underline{(\underline{U})_L^K})_K^M .$$

Following the notion of inducing in stages for representations of finite groups, we define

$$\Gamma(\underline{U})_L^M \longrightarrow \Gamma(\underline{\Gamma(\underline{U})_L^K})_K^M \quad \text{(inducing in stages)}$$

$$f \longrightarrow \tilde{f} \tag{2.1.1}$$

where $\tilde{f}(m)(k) = f(mk)$,

$m \in M$, $k \in K$, $f \in \Gamma(\underline{U})$.

So $f(m) = \tilde{f}(m)(e)$.

We have $\tilde{f}(mk)(k_1) = f(mkk_1) = (k.^{-1}\tilde{f}(m))(k_1)$

and $(m.f)^{\sim}(m_1)(k) = (m.f)(m_1k) = f(m^{-1}m_1k) = (m.\tilde{f})(m_1)(k)$.

So $\tilde{f}(mk) = k^{-1}.\tilde{f}(m)$, $(m.f)^{\sim} = m.\tilde{f}$, $m \in M$, $k \in K$.

Thus $\sim$ is an M-equivalence. It extends to a unitary equivalence $L^2(\underline{U})_L^M \xrightarrow{\sim} L^2(\underline{L^2(\underline{U})_L^K})_K^M$.

(2.2) Let ∇^U , by $\gamma^U : k \to u(U)$, be a K-invariant, metric connection on $(\underline{U})_L^K$. Extend γ^U trivially to m i.e. $\gamma^U = 0$ on p_1 . Then $\gamma^U : m \to u(U)$ determines an M-invariant, metric connection (also denote by ∇^U) on $(\underline{U})_L^M$.

Let (U_1, κ_1) be also a finite-dim unitary representation of L. Let $D:\Gamma(\underline{U})_L^K \to \Gamma(\underline{U})_L^K$, $D = a\circ\nabla^U$ be the 1^{st} order K-invariant differential operator with symbol map $p\otimes U \xrightarrow{a} U_1$. Extend a trivially to $p\oplus p_1$ i.e. $a(\xi) = 0$, $\xi \in p_1$. Then

$$D_0:\Gamma(\underline{U})_L^M \to \Gamma(\underline{U})_L^M , \quad D_0 = a\circ\nabla^U \tag{2.2.1}$$

i.e. $D_0 = \sum_t a(\zeta_t)(dR(\zeta_t) + \gamma^U(\zeta_t))$ is a 1^{st} order M-invariant differential operator, which we will call the *trivial extension of* D *to* M/L. Note that if $L < K$ (i.e. L is a proper subgroup of K) D_0 is non-elliptic (even if D is elliptic).

We can view this another way:
define the differential operator $D_1:\Gamma(\underline{U})_L^M \to \Gamma(\underline{U})_L^M$ by

$$(D_1 f)^\sim(m) = D(\tilde{f}(m)) , \quad m \in M . \tag{2.2.2}$$

So $(D_1 f)(m) = D(\tilde{f}(m))(e)$. ($e$ is the identity element of M.)

<u>Proposition 11</u>.

$D_1 = D_0$. Let D be elliptic, then

$$\text{Ker } D_0 \xrightarrow{\sim} L^2(\underline{\text{Ker } D})_K^M \text{ is a unitary equivalence.}$$

<u>Proof</u>.

For $\xi \in p$,

$$\begin{aligned}(a(\xi)(dR(\xi)+\gamma^U(\xi))\tilde{f}(m))(e) &= a(\xi)\frac{d}{dt}\tilde{f}(m)(\exp t\xi)\Big|_{t=0} + a(\xi)\gamma^U(\xi)f(m) \\ &= a(\xi)\frac{d}{dt}f(m \exp t\xi)\Big|_{t=0} + \text{"} \\ &= (a(\xi)(dR(\xi)+\gamma^U(\xi))f)(m), \ m\in M, \ f\in\Gamma(\underline{U})_L^M .\end{aligned}$$

So $D_1 = D_0$.

By the 'regularity theorem' for elliptic operators, Ker D is a closed subspace of $L^2(\underline{U})_L^K$, and so, by invariance, it is a unitary K-submodule. We see that $f \in \text{Ker } D_0$ iff $\tilde{f}(m) \in \text{Ker } D$, $\forall m \in M$, where $f \in \Gamma(\underline{U})_L^M$.

□

§3. Inducing in Stages and the Dirac Operator.

(3.1) Let (V,τ) be a finite-dimensional unitary representation of L . There are the unitary equivalences

$$L^2(\underline{S_K \otimes S_L \otimes V})_L^M \xrightarrow{\sim} L^2(\underline{L^2(\underline{S_K \otimes S_L \otimes V})_L^K})_K^M \xrightarrow{\Phi_K} L^2(\underline{S_K \otimes L^2(\underline{S_L \otimes V})_L^K})_K^M$$

$$(\Phi_K f)(m)(k) = \sigma_K(k) f(m)(k), m \in M,\ k \in K,\ f \in \Gamma(\underline{\Gamma(\underline{\ \ })}) \quad (3.1.1)$$

(See Chapter 5 (1.1), and (2.1).)

For $\xi \in p_1$, $f \in \Gamma(\underline{\ \ })_L^M$

$$(dR(\xi)_m \Phi_K \tilde{f})(k) = \frac{d}{dt} \Phi_K \tilde{f}(m \exp t\xi)\Big|_{t=0}(k)$$

$$= \sigma_K(k) \frac{d}{dt} f(m \exp t\xi k)\Big|_{t=0} \quad .$$

Now
$$(dR(\xi)f)^{\sim}(m)(k) = dR(\xi)_{mk} f$$

$$= \frac{d}{dt} f(mk \exp t\xi)\Big|_{t=0}$$

$$= \frac{d}{dt} f(m \exp t\text{Ad}(k)\xi k)\Big|_{t=0} \quad .$$

Thus $(dR(\xi)_m \Phi_K \tilde{f})(k) = \sigma_K(k)(dR(\mathrm{Ad}k^{-1}\xi)f)^\sim(m)(k)$, $m \in M$, $k \in K$.

Therefore $\sum_j c(\xi_j) dR(\xi_j)_m \Phi_K \tilde{f} = \Phi_K \sum_j c(\xi_j)(dR(\xi_j)f)^\sim(m)$, $m \in M$. (3.1.2)

Take an M-invariant metric connection determined by γ_K on $T(M/K) = (\underline{p_1})_K^M$. As we know, γ_K lifts to a unique connection determined by γ^{S_K} on $(\underline{S_K})_K^M$ over M/K .

Then

$$\sum_j c(\xi_j)\gamma^{S_K}(\xi_j)(\Phi_K \tilde{f})(m)(k) = \sum_j c(\xi_j)\gamma^{S_K}(\xi_j)\sigma_K(k)f(mk)$$

$$= \sum_j c(\xi_j)\sigma_K(k)\gamma^{S_K}(\mathrm{Ad}k^{-1}\xi_j)f(mk)$$

$$= \sigma_K(k) \sum_j c(\xi_j)\gamma^{S_K}(\xi_j)f(mk)$$

$$= \Phi_K \sum_j c(\xi_j)\gamma^{S_K}(\xi_j)\tilde{f}(m)(k) . \qquad (3.1.3)$$

Associated to $((,),\gamma_K)$ there is the twisted, by $L^2(\underline{S_L \otimes V})_L^K$, Dirac operator D_K .

On $\Gamma(\underline{S_K \otimes S_L \otimes V})_L^M$ there is the operator

$$D_1 = \sum_j c(\xi_j)(dR(\xi_j) + \gamma^{S_K}(\xi_j)) . \qquad (3.1.4)$$

For $f \in \Gamma(\ )_L^M$, write $\sim(f) = \tilde{f}$ and $\tilde{D}_1 \tilde{f} = (D_1 f)^\sim$, then we have $\Phi_K \tilde{D}_1 = D_K \Phi_K \sim$. (3.1.5)

We refer to D_1 as the *pull-back* of D_K to M/L .

(3.2) Take a K-invariant, metric connection γ_L on $T(K/L) = (\underline{p})_L^K$. γ_L lifts to a unique connection γ^{S_L} on $(\underline{S_L})_L^K$ over K/L. Associated to $((,),\gamma_L)$ there is the twisted, by $S_K \otimes V$, Dirac operator D_L. Also associated to $((,),\gamma_L,\sigma_K)$ there is the twisted, by V, Dirac operator ${}_{\sigma_K}D_L$ (see Chapter 5 (1.4)). And there are the trivial extensions to M/L (see (2.2)).

Let γ determine an M-invariant, metric connection on $T(M/L) = (\underline{p \oplus p_1})_L^M$. γ lifts to a unique connection γ^S on $(\underline{S})_L^M$ over M/L. Associated to $((,),\gamma)$ there is the twisted, by V, Dirac operator $D = D_V$. We intend to express D as the sum of trivial extensions of D_L, ${}_{\sigma_K}D_L$; and the pull-back of D_K.

(3.3) Consider $\gamma = \gamma_0$, the Levi-Civita connection.

$\gamma_0 = \frac{1}{2}$ Poad, P the orthogonal projection onto $p \oplus p_1$.

And $\gamma^S(\eta) = -\frac{1}{4}\sum_i c(\frac{1}{2}P[\eta\eta_i])c(\eta_i)$, $\eta \in p \oplus p_1$. Write $P = P^0 + P^1$ where P^0, P^1 is the orthogonal projection onto p, p_1 respectively; and $\eta = \eta^0 + \eta^1$, $\eta^0 \in p, \eta^1 \in p_1$.

Now

$$\gamma_0^S(\eta^0) = -\frac{1}{4}\sum_t c(\frac{1}{2}P^0[\eta^0,\zeta_t])c(\zeta_t) - \frac{1}{4}\sum_j c(\frac{1}{2}[\eta^0\xi_j])c(\xi_j)$$

$$= \gamma_0^{S_L}(\eta^0) + \frac{1}{2}\, d\sigma_K(\eta^0) .$$

And

$$\gamma_0^S(\eta^1) = -\tfrac{1}{4}\sum_t c(\tfrac{1}{2}[\eta^1\zeta_t])c(\zeta_t) - \tfrac{1}{4}\sum_j c(\tfrac{1}{2}(P^0+P^1)[\eta^1\xi_j])c(\xi_j) \ .$$

$$-\tfrac{1}{4}\sum_{t,j} c(\xi_j)c(\tfrac{1}{2}[\xi_j\zeta_t])c(\zeta_t) = -\tfrac{1}{4}\sum_{t,j} c(\zeta_t)c(\tfrac{1}{2}[\zeta_t\xi_j])c(\xi_j)$$

$$= \tfrac{1}{2}\sum_t c(\zeta_t)d\sigma_K(\zeta_t) \ .$$

$$-\tfrac{1}{4}\sum_{j,k} c(\xi_k)c(\tfrac{1}{2}P^0[\xi_k\xi_j])c(\xi_j) = \tfrac{1}{4}\sum_{j,k} c(\xi_k)c(\xi_j)c(\tfrac{1}{2}P^0[\xi_k\xi_j])$$

$$= \tfrac{1}{2}\sum_t c(\zeta_t)d\sigma_K(\zeta_t) \ .$$

Thus

$$\sum_i c(\eta_i)\gamma_0^S(\eta_i) = \sum_t c(\zeta_t)\gamma_0^S(\zeta_t) + \sum_j c(\xi_j)\gamma_0^S(\xi_j)$$

$$= \sum_t c(\zeta_t)(\gamma_0^{S_L}(\zeta_t) + d\sigma_K(\zeta_t)) + \tfrac{1}{2}\sum_t c(\zeta_t)d\sigma_K(\zeta_t)$$

$$+ \sum_j c(\xi_j)\gamma_0^{S_K}(\xi_j) \qquad (3.3.1)$$

where $\gamma_0^{S_L}$, $\gamma_0^{S_K}$ is the lift of the Levi-Civita connection γ_{0L}, γ_{0K} on $T(K/L)$, $T(M/K)$ to $(\underline{S_L})_L^K$, $(\underline{S_K})_K^M$ respectively.

(3.4) Consider γ the reductive connection: $\gamma = 0$ on $p \oplus p_1$.

$$\sum_i c(\eta_i)dR(\eta_i) = \sum_t c(\zeta_t)dR(\zeta_t) + \sum_j c(\xi_j)dR(\xi_j)$$

i.e. we express $D = D_0 + D_1$ where D_0 is the trivial extension of D_L with γ_L the reductive connection; and D_1 is the pull-back of D_K with γ_K the reductive connection (see (3.1)). (3.4.1)

Consider $\gamma = \gamma_0$, the Levi-Civita connection:

$$\sum_i c(\eta_i)(dR(\eta_i)+\gamma_0^S(\eta_i)) = \sum_t c(\zeta_t)(dR(\zeta_t)+\gamma_0^{S_L}(\zeta_t)+d\sigma_K(\zeta_t))$$

$$+ \tfrac{1}{2}\sum_t c(\zeta_t)(dR(\zeta_t)+d\sigma_K(\zeta_t))-\tfrac{1}{2}\sum_t c(\zeta_t)dR(\zeta_t)$$

$$+ \sum_j c(\xi_j)(dR(\xi_j)+\gamma_0^{S_K}(\xi_j)) \quad \text{(by (3.3))}.$$

i.e. we express $D = D_0 + D_1$ where $D_0 = D_2+\frac{1}{2}D_3-\frac{1}{2}D_4$ with D_2, D_3 the trivial extension of ${}_{\sigma_K}D_L$ with γ_L the Levi-Civita connection, reductive connection respectively; and D_4 is the trivial extension of D_L with γ_L the reductive connection. And D_1 is the pull-back of D_K with γ_K the Levi-Civita connection. (3.4.2)

(3.5) $D^2 = D_0^2 + [D_0D_1]_+ + D_1^2$, where $[\quad]_+$ is the anti-commutator. D_0, D_1 are essentially self-adjoint. (This is because D and D_1 are. (See Chapter 0, (5.4)).)

If M is compact, so therefore K and L are also compact, we take $(,)$ on m as given in Chapter 0, (4.2). Recall Chapter 2, §1. It is seen that we can carry out the constructions of (1.1). For (1.1) M could be a reductive Lie group and K,L compact. Suppose this is so:

(i) rank $L \leq$ rank K = rank M .

There are the $\frac{1}{2}$-Dirac operators $D_1^{\pm}:\Gamma(\underline{S_L \otimes S_K^{\pm} \otimes V})_L^M \to \Gamma(\underline{S_L \otimes S_K^{\mp} \otimes V})_L^M$.

$\Phi_K \tilde{D}_1^{\pm} = D_K^{\pm} \Phi_K \sim$. Define $D_{\pm} = D_0 + D_1^{\pm}$ on $\Gamma(\underline{S_L \otimes S_K^{\pm} \otimes V})_L^M$.

D_0, D_1^2 preserve $S_L \otimes S_K^{\pm}$, $[D_0 D_1]_+$ sends $S_L \otimes S_K^{\pm}$ into $S_L \otimes S_K^{\mp}$.

(ii) rank L = rank $K \leq$ rank M .

There are the $\frac{1}{2}$-Dirac operators $D_0^{\pm}:\Gamma(\underline{S_L^{\pm} \otimes S_K \otimes V})_L^M \to \Gamma(\underline{S_L^{\mp} \otimes S_K \otimes V})_L^M$.

$D_0^{\pm} = D_2^{\pm} + \frac{1}{2} D_3^{\pm} - \frac{1}{2} D_4^{\pm}$. Define ${}_{\pm}D = D_0^{\pm} + D_1$ on $\Gamma(\underline{S_L^{\pm} \otimes S_K \otimes V})_L^M$.

D_0^2, D_1 preserve $S_L^{\pm} \otimes V$, $[D_0 D_1]_+$ sends $S_L^{\pm} \otimes S_K$ into $S_L^{\mp} \otimes S_K$.

<u>Lemma 12</u>.

(i) $\operatorname{Ker} D_{\pm} = \operatorname{Ker} D_0 \cap \operatorname{Ker} D_1^{\pm}$

(ii) $\operatorname{Ker} {}_{\pm}D = \operatorname{Ker} D_0^{\pm} \cap \operatorname{Ker} D_1$.

<u>Proof</u>.

(i) On $\Gamma(\underline{S_L \otimes S_K^{\pm} \otimes V})$, $\langle D D_{\pm} f, f\rangle = \langle D_0^2 f, f\rangle + \langle D_1 D_1^{\pm} f, f\rangle$.

(ii) On $\Gamma(\underline{S_L^{\pm} \otimes S_K \otimes V})$, $\langle D_{\pm} D f, f\rangle = \langle D_0 D_0^{\pm} f, f\rangle + \langle D_1^2 f, f\rangle$.

And D_0, D_1 are essentially self-adjoint. □

N.B. It doesn't necessarily follow that $\operatorname{Ker} D = \operatorname{Ker} D_+ \oplus \operatorname{Ker} D_-$

or $= \operatorname{Ker} {}_{+}D \oplus \operatorname{Ker} {}_{-}D$.

§4. The Case of an Equal Rank Pair.

(4.1) In Chapter 2, §1 take the compact pair (K,L) with rank L = rank K. (K non-abelian.)

Take a maximal torus H of L (see Chapter 3). We have the triple (K,L,H). $L/H \to K/H \to K/L$. $S = S_H \otimes S_L$. Recall §1,2,3.

Take $V = E_\lambda$, $\lambda \in \Lambda$ (a 1-dim unitary H-module). If λ is non-singular w.r.t R, we take $w \in W(K,H)$ the unique element such that $w\lambda$ is dominant w.r.t R^+. Let U_ν be the simple K-module of highest weight ν.

Let $\lambda \in \Lambda$, be non-singular and dominant w.r.t R_L^+. (See Remark Chapter 0, (4.2).) Then by Proposition 7 (Chapter 3(2.1)), and Theorem 4 (Chapter 5, (4.3)), the simple L-module $V_{\lambda-\rho_L}$ of highest weight $\lambda-\rho_L$ occurs with multiplicity 1 in $L^2(\underline{S_H \otimes E_\lambda})_H^L$ and is Ker $D_{\lambda H}^+$; with γ_H the Levi-Civita or reductive connection. (N.B. here $D_{\lambda H}$ is the twisted, by E_λ, Dirac operator associated to $((,),\gamma_H)$ over L/H.) Ker $D_{\lambda H}^- = 0$ (or +,- interchanged).

Let $\mathcal{V}_\mu$ be the μ-primary L-submodule in $L^2(\underline{S_H \otimes E_\lambda})_H^L$.

So $\mathcal{V}_{\lambda-\rho_L} = V_{\lambda-\rho_L}$. Let ${}_\mu D_L$ be the twisted, by $\mathcal{V}_\mu$, Dirac operator associated to $((,),\gamma_L)$ over K/L. There is the countable direct sum $D_L = \sum_\mu \oplus {}_\mu D_L$. Define $D_{\lambda L} = {}_{\lambda-\rho_L}D_L$. There are the ½-Dirac operators $D_{\lambda L}^\pm$.

Theorem 5.

Take γ_L the Levi-Civita or the reductive connection. Then, if λ is singular w.r.t R, Ker $D_{\lambda L} = 0$. If λ is non-singular w.r.t R, Ker $D^+_{\lambda L} = U_{w\lambda-\rho}$, Ker $D^-_{\lambda L} = 0$ (or $+,-$ interchanged see Proposition 7).

Proof.

The weights of $S_L \otimes E$ as an H-module are the $\lambda-(\rho-\rho_L) + |A|$, $A \subseteq R^+-R^+_L$. A simple component L-module of $S_L \otimes V_{\lambda-\rho_L}$ has highest weight of the form $\lambda-\rho+|A|$, $A \subseteq R^+-R^+_L$. For λ s.n.s these all occur (see Chapter 3 (1.2)).

As in (3.3), write $D = D_0 + D_1$.

By Theorem 4, we have

$$\text{Ker } D_H \underset{L}{\geq} \Phi_L^{-1}(S_L \otimes \text{Ker } D_{\lambda H}) \quad \text{(with equality for } \lambda \text{ s.n.s).}$$

($\underset{L}{\leq}$ means L-submodule. See (3.1) for Φ_L .)

In fact Ker $D^+_H \underset{L}{\geq} \Phi_L^{-1}(S_L \otimes \text{Ker } D^+{}_H)$, Ker $D^-_H = 0$. (With equality for λ s.n.s.). And by Chapter 5 (1.3),

$$\text{Ker }_{\sigma_L} D_H = \Phi_L^{-1}(S_L \otimes \text{Ker } D_{\lambda H}) .$$

In fact

$$\text{Ker }_{\sigma_L} D^+_H = \Phi_L^{-1}(S_L \otimes \text{Ker } D^+_{\lambda H}) , \text{ Ker }_{\sigma_L} D^-_H = 0 .$$

Here γ_H is either connection. Thus for γ_L either connection, by Proposition 11 , (2.2), we get

$$(\text{Ker } D_0)^{\sim} \underset{K}{\geq} \Phi_L^{-1} L^2 (\underline{S_L \otimes \text{Ker } D_{\lambda H}})_L^K \ .$$

In fact $(\text{Ker } D_0^+)^{\sim} \underset{K}{\geq} \Phi_L^{-1} L^2 (\underline{S_L \otimes \text{Ker } D_{\lambda H}^+})_L^K$, (here $\underset{K}{\geq}$ means K-submodule).

Now by (3.1) $(\text{Ker } D_1)^{\sim} \underset{K}{\geq} \Phi_L^{-1} (\text{Ker } D_{\lambda L})$.

Thus $\Phi_L^{-1} (\text{Ker } D_{\lambda L}) \underset{K}{\leq} (\text{Ker } D_0 \cap \text{Ker } D_1)^{\sim} \underset{K}{\leq} (\text{Ker } D)^{\sim}$.

In fact $\Phi_L^{-1} (\text{Ker } D_{\lambda L}) \underset{K}{\leq} (\text{Ker } D_0^+ \cap \text{Ker } D_1)^{\sim} = (\text{Ker } {}_+D)^{\sim}$ (by Lemma 12).

Hence $\text{Ker } D_{\lambda L} = L^2 (S_L \otimes \text{Ker } D_{\lambda H}^+)_L^K \cap \Phi_L (\text{Ker } D)^{\sim}$ (or + changed to -).

The result now follows on appealing once again to Theorem 4, and Proposition 7. □

CHAPTER 7.

Step 3. The Case of an abelian pair (H,H_0) .

Step 4. The Case of (K,L) with $L = H_0$ an abelian subgroup.

§1. The Case of an Abelian Pair.

(1.1) In Chapter 2, §1 take $K = H$, $L = H_0$ where H is abelian.

Here $(\ ,\ )$ is a fixed inner product on h . $h = h_0 \oplus h_1$ is orthogonal. We will use the notation of Chapter 2, §2 and (3.2). Here we will write $\langle\lambda,\mu\rangle = \langle\underset{\sim}{\lambda},\underset{\sim}{\mu}\rangle + \langle\tilde{\lambda},\tilde{\mu}\rangle$, $\lambda,\mu \in \sqrt{-1}h^*$ and $||\lambda||^2 = ||\underset{\sim}{\lambda}||^2 + ||\tilde{\lambda}||^2$ for $\mu = \lambda$. (i.e. $\langle\underset{\sim}{\lambda},\underset{\sim}{\mu}\rangle = \langle\lambda,\mu\rangle_1$, $\langle\tilde{\lambda},\tilde{\mu}\rangle = \langle\lambda,\mu\rangle'$ in the notation of Chapter 2, (3.2).)

The adjoint representation of H or H_0 is trivial. (H,H_0) is always a spin pair, and the spin representations of H,H_0 are trivial. Take γ to be the Levi-Civita connection on $T(H/H_0) = (\underline{h_1})^H_{H_0}$. Here $\gamma = 0$ on h_1 , so this is the same as the reductive connection. Take $V = E_{\lambda_0}$, $\lambda_0 \in \Lambda_0$ (a 1-dim unitary H_0-module). Associated to $((,),\gamma)$ there is the twisted Dirac operator $D = D_V$.

On $\Gamma(S\otimes\underline{E_{\lambda_0}})^H_{H_0}$, $D^2 = \Delta$ (the Laplacian)

$$= dR(\Omega_H)-dR(\Omega_{H_0}) .$$

Therefore
$$D^2 = dR(\Omega_H)-d\tau(\Omega_{H_0}) . \tag{1.1.1}$$

Theorem 6.

Consider the condition (0) $\lambda \in \Lambda$, $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = 0$.

Then, if (0) cannot be satisfied Ker D = 0

if (0) can be satisfied Ker D is the λ-primary H-submodule $\Gamma_\lambda(\underline{S \otimes E_{\lambda_0}})^H_{H_0}$, the multiplicity is $\dim S = 2^{[\dim h_1/2]}$, in $L^2(\underline{S \otimes E_{\lambda_0}})^H_{H_0}$.

Proof.

On the λ-primary H-submodule, with $\underset{\sim}{\lambda} = \lambda_0$, $D^2 = ||\tilde{\lambda}||^2$. Hence on the kernel of D , $\tilde{\lambda} = 0$. □

§2. The Case of an Abelian Subgroup.

(2.1) In Chapter 2, §1 take (K,L) with K non-abelian and $L = H_0$ an abelian subgroup.

Take a maximal torus H of K with $H_0 \leq H$. We use the notation of Chapter 2, §2,3. See also §1. Refer also to the notation and material of Chapter 6, §1,2,3.

There is the triple (K,H,H_0) .

$$H/H_0 \to K/H_0 \to K/H$$

$S = S_{H_0} \otimes S_H$ as a unitary H_0-module. W.r.t the pair $(h_0,(,))$ take

$\mathrm{Cliff}(h_0)$ and the space of spinors S_0 in $\mathrm{Cliff}(h_{0\mathbb{C}})$. The unitary spin representations $(S_0,\sigma_0),(S_{H_0},\sigma_{H_0})$ of H_0 are trivial.

Take $V = E_{\lambda_0}$, $\lambda_0 \in \Lambda_0$ (a 1-dim, unitary H_0-module). Associated to $((,),\gamma)$ there is the twisted Dirac operator $D = D_V$ (see Chapter 2, §1).

We shall say that λ_0 is non-singular if when writing $\zeta_\alpha = \zeta_0 + \zeta_1$, $\zeta_0 \in \sqrt{-1}h_0$, $\zeta_1 \in \sqrt{-1}h_1$ we have $\lambda_0(\zeta_0) \neq 0$, $\forall\alpha \in R$. λ_0 non-singular means geometrically, that λ_0 does not lie on one of the walls of the open cones determined by the finite set $\{\zeta_0;\alpha \in R\}$. (See Chapter 0, (4.1),(4.2).) Also we say that λ_0 is s.n.s (sufficiently non-singular) if $|\lambda_0(\zeta_0)|$ is sufficiently +ve $\forall\alpha \in R$. So geometrically, λ_0 s.n.s, means that λ_0 does not lie close to the walls of the open cones.

Again for $\lambda \in \Lambda$, if λ is non-singular we take $w \in W(K,H)$ the unique element such that $w\lambda$ is dominant w.r.t R^+ .

Write as before $D = D_0 + D_1$, so $D^2 = D_0^2 + [D_0D_1]_+ + D_1^2$.

(2.2) <u>Theorem 7</u>.

Let γ be the reductive or Levi-Cevita connection. Then $\mathrm{Ker}\, D = \mathrm{Ker}\, D_0 \cap \mathrm{Ker}\, D_1$.

Hence, let $\lambda \in \Lambda$, λ non-singular w.r.t R and consider the conditions

$$(1)\ \underset{\sim}{\lambda} = \lambda_0 ,\ \tilde{\lambda} = (w^{-1}\rho)^{\sim} . \quad (0)\ \underset{\sim}{\lambda} = \lambda_0 ,\ 2\tilde{\lambda} = -(w^{-1}\rho)^{\sim} .$$

In the following, condition (1), (0) refers to γ the reductive, Levi-Civita connection respectively.

If (1) or (0) cannot be satisfied for any λ, then respectively Ker D = 0 .

If (1) or (0) can be satisfied, then of course λ is unique and, respectivel Ker D is the $w\lambda-\rho$ primary K-submodule $\Gamma_{w\lambda-\rho}(\underline{S\otimes E_{\lambda_0}})^K_{H_0}$, the multiplicity is dim $S_{H_0} = 2^{[\dim h_1/2]}$, in $L^2(\underline{S\otimes E_{\lambda_0}})^K_{H_0}$.

Proof.

We prove this here for λ_0 s.n.s. This restriction will be removed in Chapter 9.

Take an orthonormal basis $\{\zeta_t\}$ for h such that $\{\zeta_t\}(t=1,\ldots\ldots,\ell_0)$, $\{\zeta_t\}(t=\ell_0+1,\ldots\ldots,\ell)$ lies in h_0,h_1 respectively. $\ell_0 = \dim h_0$, $\ell = \dim h$.

$$\text{Define } F_0 = \sum_{t=1}^{\ell_0} c(\zeta_t)d\sigma_H(\zeta_t) , \quad F = \sum_{t=\ell_0+1}^{\ell} c(\zeta_t)d\sigma_H(\zeta_t) \qquad (2.2.1)$$

on S_0S . F_0,F preserve the weight spaces of $S_0 \otimes S_{H_0} \otimes S_H$ as an H-module (here $S_0 \otimes S_{H_0}$ is regarded as a trivial H-module).

Recall Chapter 6, §3.

$$(\Phi_H\sim)^{-1}L^2(\underline{S_H\otimes\Gamma_\lambda(\underline{S_{H_0}\otimes E_{\lambda_0}})^H_{H_0}})^K_H \text{ , } \lambda \in \Lambda \text{ , } \underset{\sim}{\lambda} = \lambda_0 \qquad (2.2.2)$$

is preserved by D_0 and D_1, so also by D . We consider operators on this Hilbert space (i.e. on their domains).

(1) reductive connection:

An easy computation using $[d\sigma_H(\zeta), c(\xi)] = c[\zeta\xi], \zeta \in h, \xi \in p_1$ (see Chapter 1, §1) gives $[D_0+F, D_1]_+ = 0$. So $(D+F)^2 = (D_0+F)^2 + D_1^2$. Also $F^2 = d\sigma_H(\Omega_H) - d\sigma_H(\Omega_{H_0})$, and $(D_0+F)^2 = ||\tilde{\lambda}||^2$. Suppose Ker $D \neq 0$. Take non-zero $f \in$ Ker D. Then, as D is essentially self-adjoint

$$\langle D_1^2 f, f\rangle = -||\tilde{\lambda}||^2 \langle f,f\rangle + \langle F^2 f, f\rangle \quad . \tag{2.2.3}$$

(See Chapter 0, (3.1) for $\langle , \rangle$.) D_0, D_1 are essentially self-adjoint, F is self-adjoint. Hence if Ker $D \neq 0$, we require $||\tilde{\lambda}||^2 \leq \max_{A \subseteq R^+} \{||\tilde{\rho} - |\tilde{A}|||^2\} =: a^2$, where $a \geq 0$ and a is independent of λ. So $||\tilde{\lambda}|| \leq a$. (2.2.4)

On $\underline{\Gamma(S_0 \otimes S \otimes E_{\lambda_0}})^K_{H_0}$, consider $D_{S_0 \otimes E_{\lambda_0}}$. This is the direct sum of $\dim S_0$ copies of D (see Chapter 5, (4.1)). We also denote $D_{S_0 \otimes E_{\lambda_0}}$ by D. Then $D^2 - [F_0 D_1]_+ = D_0^2 - [F_0+F, D_1]_+ + D_1^2$. (2.2.5)

If $f \in$ Ker D, $\langle [F_0 D_1]_+ f, f\rangle = -\langle [F_0 D_0]_+ f, f\rangle = 0$.

Now $|\lambda(\zeta_\alpha)| = |\lambda(\zeta_0) + \lambda(\zeta_1)|$

$$\geq |\lambda(\zeta_0)| - |\lambda(\zeta_1)| \text{ , } \alpha \in R \text{ .}$$

If $\underset{\sim}{\lambda} = \lambda_0$ and λ_0 is s.n.s, as $||\tilde{\lambda}|| \leq a$, then λ can be made s.n.s. We then see from (2.2.5) and Chapter 5 (4.2), (4.3), that if

ν is the highest weight of a simple K-module occurring in Ker D , then (i) $\nu = w\lambda-\rho+|B|_w$, some $B \subseteq R^+$, and

(ii) if $f \in$ Ker D , then $\Phi_H\tilde{f} \in \Gamma_\nu(S_{-\rho+|B|}\otimes\Gamma_\lambda(\underline{S_{H_0}\otimes E_{\lambda_0}})^H_{H_0})^K_H$.

Suppose $U_\nu \otimes b \longrightarrow \Phi_H(\ker D)^\sim$, $(0 \neq)b \in \mathrm{Hom}_H(U_\nu, S_{-\rho+|B|}\otimes\Gamma_\lambda(\quad))$ (see proof of Theorem 4 Chapter 5, (4.3)). Let v be 'the' weight vector with weight $w^{-1}\nu = \lambda-\rho+|B|$. Let $f_1 \in \Gamma_\nu(\quad), f_1(k) = b(\Pi_\nu(k)^{-1}v)$. Then $f_1(e) \neq 0$ and $dR(\xi)_e f_1 = 0$, $\forall\xi \in p_1$. Let f be such that $f_1 = \Phi_H\tilde{f}$. Then $f(e) = f_1(e)(e) \neq 0$. For $\zeta \in h_1$, $dR(\zeta)_e f = -dL(\zeta)_e f = -(\tilde{\lambda}-\tilde{\rho}+|\tilde{B}|)(\zeta)f(e)$, and $\gamma_0(\zeta) = d\sigma_H(\zeta)$ for γ_0 the Levi-Civita connection on $T(K/H_0)$.

Also for $\xi \in p_1$, $(dR(\xi)_e f_1)(h) = \frac{d}{dt} f_1(\exp t\xi)\Big|_{t=0}(h)$

$$= \sigma_H(h)\frac{d}{dt} f(\exp t\xi h)\Big|_{t=0} ,$$

so $(dR(\xi)_e f_1)(e) = dR(\xi)_e f$.

Thus, from

$$D^2 = dR(\Omega_K)+4\Big(\sum_{t=\ell_0+1}^{\ell}\gamma_0(\zeta_t)dR(\zeta_t)+\sum_j\gamma_0(\xi_j)dR(\xi_j)\Big)+d\sigma(\Omega_{H_0})-d\tau(\Omega_{H_0}) ,$$

we obtain

$$0 = (D^2f)(e)=(||w\lambda+|B|_w||^2-||\rho||^2+4\sum_t(\tilde{\rho}-|\tilde{B}|)(\zeta_t)(\tilde{\lambda}-\tilde{\rho}+|\tilde{B}|)(\zeta_t)$$

$$+ ||\underset{\sim}{\rho}-|\underset{\sim}{B}|||^2 - ||\lambda_0||^2)f(e)$$

$$0 = 2\langle w\lambda-\rho, |B|_w\rangle + 2\langle\rho, |B|_w\rangle + ||\rho-|B|_w||^2 - ||\tilde{\rho}-|\tilde{B}|||^2$$

$$- ||\rho||^2 + ||\tilde{\lambda}-2(\tilde{\rho}-|\tilde{B}|)||^2 + |||B|_w||^2$$

$$= 2\langle w\lambda-\rho, |B|_w\rangle + 2|||B|_w||^2 + ||2(\tilde{\rho}-|\tilde{B}|)-\tilde{\lambda}||^2 - ||\tilde{\rho}-|B|||^2 .$$

Now, by (2.2.3), $||\tilde{\lambda}|| \le ||\tilde{\rho}-|\tilde{B}|||$, so

$$||2(\tilde{\rho}-|\tilde{B}|)-\tilde{\lambda}|| \ge 2||\tilde{\rho}-|\tilde{B}|||-||\tilde{\lambda}|| \ge ||\tilde{\rho}-|\tilde{B}||| .$$

Hence, we require $|B|_w = 0$, therefore $B = A_{w^{-1}}$ (see Chapter 5, (2.2)).

(0) Levi-Civita connection:

In Chapter 6, (3.4), for the triple (K,H,H_0) , we have $\gamma_{H_0} = 0$ on h_1, for either connection so $D_2 = D_3$. Also $D_4 + F = D_2$, $D_0-\frac{1}{2}F = D_2$. Now, using the fact that $[d\sigma_H(\zeta),\gamma^{S_H}(\xi)] = \gamma^{S_H}[\zeta\xi]$, $\zeta \in h$, $\xi \in p_1$ (see Proposition 1 Chapter 0, (2.2)) we get $[D_2,D_1]_+ = 0$. Thus $(D-\frac{1}{2}F)^2 = D_2^2+D_1^2$. Now $D_2^2 = ||\tilde{\lambda}||^2$, so

$$\langle D_1^2 f,f\rangle = -||\tilde{\lambda}||^2\langle f,f\rangle + \tfrac{1}{4}\langle F^2 f,f\rangle , \ f \in \text{Ker } D . \tag{2.2.6}$$

Hence, if Ker D $\neq 0$, we require $||\tilde{\lambda}|| \le \frac{1}{2}a$.

$$\text{From } D^2 + \tfrac{1}{2}[F_0 D_1]_+ = D_0^2+\tfrac{1}{2}[F_0+F,D_1]_+ + D_1^2 , \tag{2.2.7}$$

we see that if ν is the highest weight of a simple K-module occurring in Ker D , then $\nu = w\lambda-\rho$.

Take in $L^2(S_H \otimes \Gamma_\lambda(\underline{S_{H_0} \otimes E_{\lambda_0}})^H_{H_0})^K_H$, the $w\lambda-\rho$ primary K-submodule. (2.2.8)

The multiplicity, by Proposition 7 , is that of the λ-primary H-submodule in $L^2(\underline{S_{H_0} \otimes E_{\lambda_0}})^H_{H_0}$ i.e. dim S_{H_0} . By Theorem 4 this K-submodule lies in Ker D_H . Hence, for both connections we have shown that Ker D = Ker $D_0 \cap$ Ker D_1 .

Now take $f \in \Gamma_{w\lambda-\rho}(S_{-w^{-1}\rho} \otimes \Gamma_\lambda(\underline{S_{H_0} \otimes E_{\lambda_0}})^H_{H_0})^K_H$ and $\Phi_H^{-1} f$.

We have $(\Phi_H^{-1} f)(k) \in \Gamma_{\lambda-w^{-1}\rho}(\underline{S_{-w^{-1}\rho} \otimes S_{H_0} \otimes E_{\lambda_0}})^H_{H_0}$, $\forall k \in K$.

(1) reductive connection:

$$(\Phi_H^{-1} f) \in (\mathrm{Ker} D_0)^\sim \text{ iff } (\Phi_H^{-1} f)(k) \in \mathrm{Ker}\, D_{H_0} \,,\ \forall k \in K \quad \text{(by Proposition 11)}$$

$$\text{iff } (\lambda - w^{-1}\rho)^\sim = 0 \qquad \text{(by Theorem 6)}$$

(0) Levi-Civita connection:

$$(\Phi_H^{-1} f) \in (\mathrm{Ker} D_0)^\sim \text{ iff } (\Phi_H^{-1} f)(k) \in \mathrm{Ker}\, D_0 \,,\ \forall k \in K \,,$$

where we also denote $D_0 = {}_{\sigma_H}D_{H_0} + \frac{1}{2} F$ (see Chapter 6, (3.4)).

$$D_0 = \sum_{t=\ell_0+1}^{\ell} c(\zeta_t)(dR(\zeta_t) + 3/2\, d\sigma_H(\zeta_t)) \quad .$$

Then $$D_0^2 = -\sum_{t=\ell_0+1}^{\ell} (dR(\zeta_t) + 3/2 d\sigma_H(\zeta_t))^2 \quad \text{(see Chapter 5, (4.1.4))}$$

$$2\, D_0^2 = 2dR(\Omega_H) - 3.2\sum_t d\sigma_H(\zeta_t) dR(\zeta_t) + 9/2 d\sigma_H(\Omega_H)$$
$$- 3/2\, d\sigma_H(\Omega_{H_0}) + d(\sigma \otimes \tau)(\Omega_{H_0}) - 3d\tau(\Omega_{H_0})$$

And on $S_H \otimes \Gamma(\underline{S_{H_0} \otimes E_{\lambda_0}})_{H_0}^{H}$,

$$4\Phi_H D_0^2 \Phi_H^{-1} = 6(1 \otimes dL(\Omega_H) - d\tau(\Omega_{H_0})) - 2(d(\sigma_H \otimes L)(\Omega_H) - \Phi_H d(\sigma_H \otimes \tau)(\Omega_{H_0})\Phi_H^{-1})$$
$$+ \Phi_H\, 3(d\sigma_H(\Omega_H) - d\sigma_H(\Omega_{H_0}))\Phi_H^{-1} .$$

Therefore, on $\Gamma_{\lambda - w^{-1}\rho}(\underline{S_{-w^{-1}\rho} \otimes S_{H_0} \otimes E_{\lambda_0}})_{H_0}^{H}$,

$$4\, D_0^2 = 6\, ||\tilde{\lambda}||^2 - 2||\tilde{\lambda} - \widetilde{w^{-1}}\rho||^2 + 3||\widetilde{w^{-1}}\rho||^2$$
$$= 4\, ||\tilde{\lambda}||^2 + 4\langle\tilde{\lambda}, \widetilde{w^{-1}}\rho\rangle + ||\widetilde{w^{-1}}\rho||^2$$
$$= ||2\tilde{\lambda} + \widetilde{w^{-1}}\rho||^2 .$$

Thus $\Phi_H^{-1} f \in (\text{Ker } D_0)^{\sim}$ iff $(2\lambda + w^{-1}\rho)^{\sim} = 0$.

To finish the proof, we now have to show that (2.2.8) is actually the $w\lambda - \rho$ primary K-submodule in $L^2(\underline{S \otimes E_{\lambda_0}})_{H_0}^{K}$ under $\Phi_H^{\sim}$. This we will do for all parameters λ_0 .

Consider, therefore, $\Gamma_{w\lambda-\rho}(\underline{S_H \otimes \Gamma_\mu(\underline{S_{H_0} \otimes E_{\lambda_0}})_{H_0}^{H}})_{H}^{K}$ (2.2.9)

with $\mu \in \Lambda$, $\underset{\sim}{\mu} = \lambda_0$. Suppose this space is non-zero.

We have $||\lambda||^2 = ||\mu||^2 + a$, for some $a \in \mathbb{R}$, $a \geq 0$; so $||\tilde{\lambda}||^2 = ||\tilde{\mu}||^2 + a$. Thus if $||\tilde{\mu}||^2 \geq ||\tilde{\lambda}||^2$, we must have equality, $a = 0$; and then by Proposition 7, $\mu = w_1\lambda$ some $w_1 \in W(K,H)$.

So suppose that $||\tilde{\mu}||^2 \leq ||\tilde{\lambda}||^2$. The following method will be utilized further in Chapter 9. Tensor (2.2.9) with U_ν , the simple K-module of highest weight ν . Let $v \in U_\nu$ be 'the' weight vector of weight $w^{-1}\nu$. Take $f \in \Gamma_{w\lambda-\rho}(\quad)$ and $f \otimes t_1 = \Phi^{-1}(f \otimes v)$, where $t_1(k) = \Pi_\nu(k)^{-1}v$, $k \in K$ (see Chapter 5, (1.1.1)). Put $t_{11} = b_1 t_1$, where b_1 is the orthogonal projection of $\Gamma(\underline{U_\nu})_H^K$ onto the induced line bundle sections, $\Gamma(\underline{\mathbb{C}v})_H^K$. Taking f to be a $\lambda - w^{-1}\rho$ weight vector, $f \otimes t_{11}$ lies in $\Gamma_{w\lambda-\rho+\nu}(\underline{S_H \otimes E_\mu \otimes E_{w^{-1}\nu}})_H^K$ (recall that E_μ , $\mu \in \Lambda$ is the 1-dim unitary H-module with character e^μ). Then we have $||\lambda + w^{-1}\nu||^2 \leq ||\mu + w^{-1}\nu||^2$ so $||\tilde{\lambda}||^2 + 2\langle\tilde{\lambda}, \widetilde{w^{-1}\nu}\rangle$

$$\leq ||\tilde{\mu}||^2 + 2\langle\tilde{\mu}, \widetilde{w^{-1}\nu}||. \tag{2.2.10}$$

If $||\tilde{\mu}||^2 < ||\tilde{\lambda}||^2$, then $2\langle\tilde{\lambda}, \widetilde{w^{-1}\nu}\rangle < 2\langle\tilde{\mu}, \widetilde{w^{-1}\nu}\rangle$.
But taking $\nu = w\lambda$, we get $2||\tilde{\lambda}||^2 < 2\langle\tilde{\mu}, \tilde{\lambda}\rangle < 2||\tilde{\lambda}||^2$ (by the Cauchy-Schwarz inequality. This is a contradiction. Thus it must be that $||\tilde{\mu}||^2 \geq ||\tilde{\lambda}||^2$. From (2.2.10) we also get $\tilde{\mu} = \tilde{\lambda}$.

Hence the result follows.

□

(2.3) Consider the condition (0) in Theorem 6.

Take $\chi_0 \in \hat{H}_0$. Now $H = H_0 H_1$ where H_1 is the connected subgroup of H with (abelian) Lie algebra h_1. Satisfying (0) is equivalent to finding

$$\chi \in \hat{H} \text{ with } \chi\big|_{H_0} = \chi_0, \chi\big|_{H_1} = 1 \text{ (the trivial character)} \qquad (2.3.1)$$

In (2.3.1) χ is clearly unique if it exists. In fact it is easily seen that to satisfy (2.3.1), it is necessary and sufficient that

$\chi_0\big|_{H_0 \cap H_1} = 1$

(for then with $h = h_0 h_1$, $h \in H$, $h_0 \in H_0$, $h_1 \in H_1$; define $\chi(h) = \chi_0(h_0)$).

(2.4) Examples.

Any compact, connected abelian Lie group of dimension n, is isomorphic to the n-torus i.e. the direct product of n copies of S', the complex numbers of modules 1, $n \in \mathbb{N}$.

The unitary character group $\hat{S}'$, has lattice $\mathbb{Z}$. The finite cyclic group of order n, $\langle e^{i2\pi/n} \rangle$ has (finite) lattice $\mathbb{Z}_n$ (the congruence classes modulo n). (N.B. this finite group is of course not connected.) $i = \sqrt{-1}$.

The characters of S' are given by $\chi_\ell(\theta) = e^{i\ell\theta}$, $\theta \in [0, 2\pi]$ where $\ell \in \mathbb{Z}$. And the characters of $\langle e^{i2\pi/n} \rangle$ are given by the n^{th} roots of unity $e^{i2k\pi/n}$; $k = 0, 1, \ldots, n-1$.

(i) Take $H = S' \times S'$ (the 2-torus). $\hat{H}$ has lattice $\mathbb{Z} \oplus \mathbb{Z}$.

The characters of H are given by (ℓ,m) where

$$\chi_{\ell m}(\theta,\phi) = \chi_\ell(\theta)\chi_m(\phi)$$
$$= e^{i(\ell\theta+m\phi)}, \quad \theta,\phi \in [0,2\pi] \; ; \; \ell,m \in \mathbb{Z} .$$

In what follows for subspaces h_0, h_1 of h, we shall fix an inner product $<,>$ on h w.r.t which h_0 and h_1 are orthogonal.

Take $H_0 = \{(e^{i\theta},e^{i\theta}); \theta \in [0,2\pi]\}$ the diagonal subgroup.

And $H_1 = \{(e^{i\theta},e^{i(n+1)\theta}) ; \theta \in [0,2\pi]\}$, $n \in \mathbb{N}$.

We write the elements of H as (θ,ϕ). So $(\theta_1,\phi_1)(\theta_2,\phi_2) = (\theta_1+\theta_2,\phi_1+\phi_2)$.

Now $(n\theta,0) = ((n+1)\theta,(n+1)\theta)(-\theta,-(n+1)\theta)$; $(0,n\phi) = (-\phi,-\phi)(\phi,(n+1)\phi)$.

So $H = H_0H_1$.

Also $(\theta,\theta) = (\phi,(n+1)\phi)$ implies that $\theta = \phi$, $n\theta = 2k\pi$, $k \in \mathbb{Z}$.

Therefore $H_0 \cap H_1 = <(\frac{2\pi}{n},\frac{2\pi}{n})>$, the 'diagonal' finite cyclic subgroup of order n. (N.B. $(\frac{2\pi}{n},\frac{2\pi}{n}) = (\frac{2\pi}{n},(n+1)\frac{2\pi}{n})$).

The characters of H_0 are the restrictions of those of H, therefore are given by (ℓ,ℓ) i.e. $\chi_{\ell\ell}(\theta) = e^{i2\ell\theta}$, $\ell \in \mathbb{Z}$.

(ℓ_1,ℓ_1) is a restriction of (ℓ,m) iff $\ell+m = 2\ell_1$.

(ℓ,ℓ) is trivial on $H_0 \cap H_1$ iff $2\ell \equiv 0(n)$ (i.e. n divides into 2ℓ).

(ℓ,m) is trivial on H_1 iff $\ell+m(n+1) = 0$.

If $n = 3$, $\ell = 2$ we cannot satisfy (2.3.1) as $4 \not\equiv 0(3)$.

Take (n,n) in $\hat{H}_0$. For (2.3.1) we require

$$\left.\begin{array}{l}\ell+m = 2n \\ \ell+m(n+1) = 0\end{array}\right\} \begin{array}{l}2n+mn = 0 \\ n(m+2) = 0 \quad \text{so} \quad m = -2, \ell = 2(n+1) .\end{array}$$

In general starting with (ℓ_1,ℓ_1) in $\hat{H}_0$, for (2.3.1) we require $n|2\ell_1$, then

$$\left.\begin{array}{l}\ell+m = 2\ell_1 \\ \ell+m(n+1) = 0\end{array}\right\} \begin{array}{l}mn = -2\ell_1 \\ \text{so} \quad m = -\frac{2\ell_1}{n} , \ \ell = \frac{2\ell_1}{n}(n+1) .\end{array}$$

We get the required (ℓ,m) in $\hat{H}$.

For example starting with (n^2,n^2) in $\hat{H}_0$; $m = -2n$, $\ell = 2n(n+1)$.

(ii) Of course if $H = H_0 \times H_1$ a direct product, then $\hat{H} = \hat{H}_0 \times \hat{H}_1$ and one can always satisfy (2.3.1). For example $H = S' \times S'$ with $H_0 = S' \times \{e\}$, $H_1 = \{e\} \times S'$.

CHAPTER 8.

Step 5. The case of any pair (K,L) .

§1. The General Case.

(1.1) In Chapter 2, §1 take any pair (K,L) .

Take a maximal torus H_0 of L , and a maximal torus H of K with $H_0 \leq H$. (We use the notation of Chapter 2, §2,3.) Recall Chapter 6, 1,2,3. There is the triple (K,L,H_0) .

$$L/H_0 \to K/H_0 \to K/L \ .$$

$S = S_{H_0} \otimes S_L$.

In Chapter 6, §3 take $V = E_{\lambda_0}$, $\lambda_0 \in \Lambda_0$ (a 1-dim unitary H_0-module). Let λ_0 be non-singular and dominant w.r.t R_L^+ . See Remark 1 Chapter 0, (4.2). Then by Proposition 7 Chapter 3, (2.1), and Theorem 4 Chapter 5, (4.3), the simple L-module $V_{\lambda_0-\rho_L}$ of highest weight $\lambda_0-\rho_L$ occurs with multiplicity 1 in $L^2(\underline{S_{H_0} \otimes E_{\lambda_0}})^L_{H_0}$ and is Ker $D^+_{\lambda_0 H_0}$; with γ_{H_0} the Levi-Civita or reductive connection. N.B. here $D_{\lambda_0 H_0}$ is the twisted, by E_{λ_0} , Dirac operator associated to $((\ ,\ ),\gamma_{H_0})$ over L/H_0 . Ker $D^-_{\lambda_0 H_0} = 0$, (or +,- interchanged).

Let V_{μ_0} be the μ_0-primary L-submodule in $L^2(\underline{S_{H_0} \otimes E_{\lambda_0}})^L_{H_0}$.

So $V_{\lambda_0-\rho_L} = V_{\lambda_0-\rho_L}$. Let ${}_{\mu_0}D_L$ be the twisted, by V_{μ_0} , Dirac operator associated to $((,),\gamma_L)$ over K/L . There is the countable direct sum $D_L = \sum_{\mu_0} \oplus {}_{\mu_0}D_L$. Define $D_{\lambda_0 L} = {}_{\lambda_0-\rho_L}D_L$.

For $\lambda \in \Lambda$, λ non-singular w.r.t R , take $w \in W(,H)$ the unique element such that $w\lambda$ is dominant w.r.t R^+ .

<u>Theorem 8.</u>

Let $\lambda \in \Lambda$, λ non-singular w.r.t R and consider the conditions (1) $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = (w^{-1}\rho)^\sim$ (0) $\underset{\sim}{\lambda} = \lambda_0$, $2\tilde{\lambda} = -(w^{-1}\rho)^\sim$.
If (1), (0) cannot be satisfied, for any λ , then for γ_L the reductive, Levi-Civita connection respectively, Ker $D_{\lambda_0 L} = 0$.
If (1), (0) can be satisfied, of course λ is unique, then for γ_L the reductive, Levi-Civita connection respectively; Ker $D_{\lambda_0 L}$ is the $w\lambda-\rho$ primary K-submodule $\Gamma_{w\lambda-\rho}(\underline{S_L \otimes V_{\lambda_0-\rho_L}})^K_L$ in $L^2(\underline{S_L \otimes V_{\lambda_0-\rho_L}})^K_L$.

<u>Proof.</u>

This is similar to that of Theorem 5 Chapter 6, (4.1).
A simple component L-module in $S_L \otimes V_{\lambda_0-\rho_L}$ has highest weight of the form $\lambda_0+\mu_0-\rho_L$ where μ_0 is a weight of S_L ; and occurs with multiplicity at most that of μ_0 . (See Remark 3 Chapter 3, (1.3), and also Chapter 2, (3.6).)

As in Chapter 6, (3.4) write $D = D_0 + D_1$. By Theorem 4 we have

$$\operatorname{Ker} D^+_{H_0} \underset{L}{\geq} \Phi_L^{-1}(S_L \otimes \operatorname{Ker} D^+_{\lambda_0 H_0}) , \operatorname{Ker} D^-_{H_0} = 0$$

($\underset{L}{\leq}$ means L-submodule). And, by Chapter 5, (1.3),

$$\operatorname{Ker}_{\sigma_L} D^+_{H_0} = \Phi_L^{-1}(S_L \otimes \operatorname{Ker} D^+_{\lambda_0 H_0}) , \operatorname{Ker}_{\sigma_L} D^-_{H_0} = 0 .$$

Here γ_{H_0} is either connection. Thus, for γ_L either connection, by Proposition 11 Chapter 6, (2.2), we get

$$(\operatorname{Ker} D_0^+)^\sim \underset{K}{\geq} \Phi_L^{-1} L^2(\underline{S_L \otimes \operatorname{Ker} D^+_{\lambda_0 H_0}})^K_L$$

($\underset{K}{\leq}$ means K-submodule). Now by Chapter 6, (3.1),

$$(\operatorname{Ker} D_1)^\sim \underset{K}{\geq} \Phi_L^{-1}(\operatorname{Ker} D_{\lambda_0 L}) .$$

Thus

$$\Phi_L^{-1}(\operatorname{Ker} D_{\lambda_0 L}) \underset{K}{\leq} (\operatorname{Ker} D_0^+ \cap \operatorname{Ker} D_1)^\sim = (\operatorname{Ker}_+ D)^\sim \tag{1.1.1}$$

(by Lemma 12)

Hence, $\operatorname{Ker} D_{\lambda_0 L} = L^2(\underline{S_L \otimes \operatorname{Ker} D^+_{\lambda_0 H_0}})^K_L \cap \Phi_L(\operatorname{Ker} D)^\sim$,

(or + changed to - . N.B. here D_0, D_1 are different than those for the triple (K,H,H_0) in Chapter 7, (2.1)).

The result now follows on appealing to Theorem 7 Chapter 7, (2.2). □

(1.2) See Theorem 8. Consider the $w\lambda-\rho$ primary K-submodule $\Gamma_{w\lambda-\rho}\underline{(S_L \otimes V_{\lambda_0-\rho_L})}_L^K$. We wnat to compute the multiplicity (see Chapter 0, (3.2)). For (K,L) of equal rank we already know that the multiplicity is 1 . And for $L = H_0$, a closed abelian subgroup of K , it is Z^r , $r = \frac{1}{2}[\dim H - \dim H_0]$. Also for (K,L) a symmetric pair of unequal rank, the multiplicity in 2^r .

We intend to take up the general case in later work.

CHAPTER 9.

Step 6. A 'Zuckerman technique'.

In this chapter we complete the proofs of Theorems 4,7. See Chapter 5, (4.3) and Chapter 6 (2.2). This involves considering any parameter λ_0 which is not necessarily 'sufficiently non-singular'. Thus we complete the 'Problem' for γ the Levi-Civita or reductive connection.

The technique developed in this chapter involves twisting a twisted Dirac operator with a simple module. Our work of Chapter 5, §1 is crucial here.

We shall name our technique after G. Zuckerman. He has considered the tensor product of a discrete series representation (for a non-compact semi-simple Lie group G), and a finite-dimensional representation. (See [23] .) His results on the infinitesimal characters of the composition factors of this tensor product, turned out to be important in dealing with the Dirac operator of the pair (G,M) , M a maximal compact subgroup of G . See [31].

In §1 of this chapter, we compute a difference of two squares of twisted Dirac operators. §2 looks at twisting by an irreducible representation. §3 combines §1,2.

§1. A Difference Formula.

(1.1) Let (K,L) be a pair of Lie groups with L a closed subgroup of K . Let (U,κ) be a finite-dimensional unitary representation of L, and (W,Π) a finite-dimensional unitary representation of K. Refer to Chapter 5, §1.

There is a unitary equivalence

$$L^2(\underline{U\otimes W})_L^K \xrightarrow{\Phi_\Pi} L^2(\underline{U})_L^K \otimes W$$

$$(\Phi_\Pi f)(k) = (1\otimes\Pi(k))f(k),\ k \in K,\ f \in \Gamma(\underline{U\otimes W}).$$

Let K/L be reductive, and Riamannian via $(\ ,\ )$ (see Chapter 5, (1.3)). Take a K-invariant, metric connection on $T(K/L) = (\underline{p})_L^K$, determined by $\gamma: k \to so(p)$ and a K-invariant, metric connection ∇^U on $(\underline{U})_L^K$ determined by $\gamma^U: k \to u(U)$ (see chapter 0, §2).

Associated to $((,),\gamma,\gamma^U)$ there is the Laplacian Δ^U on $(\underline{U})_L^K$ (see Chapter 0, (2.5)). There is also the Laplacian $\Delta^{U\otimes W}$ on $(\underline{U\otimes W})_L^K$ (where we take the reductive connection on $(\underline{W})_L^K$).

Associated to $((,),\gamma,\gamma^U,\Pi)$ there is the Laplacian ${}_\Pi\Delta^U$ on $(\underline{U\otimes W})_L^K$ (where we take the tensor product connection $\gamma^U\otimes 1 + 1 \otimes d\Pi$ on $\underline{U\otimes W}$).

We have

$$\Phi_\Pi\ {}_\Pi\Delta^U = (\Delta^U \otimes 1)\Phi_\Pi\ . \tag{1.1.1}$$

By Proposition 2 the difference of the Laplacians

$${}_\Pi\Delta^U - \Delta^{U\otimes W} = -2\sum_j d\Pi(\xi_j)dR(\xi_j) - \sum_j d\Pi(\xi_j)^2$$
$$- 2\sum_j \gamma^U(\xi_j)\otimes d\Pi(\xi_j) + \sum_j d\Pi(\gamma(\xi_j)\xi_j)\ ,$$

where $\{\xi_j\}$ is an orthonormal basis of p .

(1.2) Let (K,L) be K-spin. See Chapter 5, (1.4) for notation. We want to consider the difference of squares $D^2_{V\otimes W} - {}_{\Pi}D^2_V$ of Dirac operators. Refer to Chapter 1, (2.2).

The difference of the 'torsion terms' is

$$- \tfrac{1}{2} \sum_{i,j} c(\xi_i)c(\xi_j) \otimes d\Pi(T(\xi_i,\xi_j)) .$$

The difference of curvatures

$$\begin{aligned} {}_{\Pi}R^{S\otimes V}(\xi,\eta) - R^{S\otimes V\otimes W}(\xi,\eta) &= [d\Pi(\xi),d\Pi(\eta)] - d\Pi(P[\xi,\eta]) \\ &= d\Pi(Q[\xi,\eta]) . \end{aligned}$$

Q,P is the projection onto ℓ,p respectively. ${}_{\Pi}R^{S\otimes V}(,)$ is the curvature 2-form of ${}_{\Pi}\nabla^{S\otimes V}$.

So the difference of the 'curvature terms' is

$$\tfrac{1}{2} \sum_{i,j} c(\xi_i)c(\xi_j) \otimes d\Pi(Q[\xi_i\xi_j]) .$$

(1.3) Let (K,L) be a compact pair as in Chapter 2, §1.

Take an orthonormal (w.r.t(,)) basis $\{\zeta_t\}$ of ℓ . Set $\{\eta_i\} = \{\zeta_t,\xi_j\}$ an orthonormal basis of k .

Then

$$_{\Pi}\Delta^{U} - \Delta^{U\otimes W} = -2 \sum_i d\Pi(\eta_i)dR(\eta_i) - \sum_i d\Pi(\eta_i)^2$$

$$+ 2 \sum_t d\Pi(\zeta_t)dR(\zeta_t) + \sum_t d\Pi(\zeta_t)^2 + \text{z.o.t}$$

(z.o.t denotes a sum of zeroth order terms)

$$= -dR(\Omega_K)-d\Pi(\Omega_K) + d(R\otimes\Pi)(\Omega_K) + d\Pi(\Omega_K)$$

$$- 2 \sum_t d\kappa(\zeta_t)\otimes d\Pi(\zeta_t) - 2\sum_t d\Pi(\zeta_t)^2 - d\Pi(\Omega_L) + \text{z.o.t}$$

$$= - dR(\Omega_K) + d(R\otimes\Pi)(\Omega_K) - d\kappa(\Omega_L) + d(\kappa\otimes\Pi)(\Omega_L) + \text{z.o.t.}$$

Hence

$$\Delta^{U\otimes W} - {}_{\Pi}\Delta^{U} = dR(\Omega_K) - d(\kappa\otimes\Pi)(\Omega_L) - (d(R\otimes\Pi)(\Omega_K) - d\kappa(\Omega_L)) - \text{z.o.t.} \qquad (1.3.1)$$

With $(U,\kappa) = (S\otimes V, \sigma\otimes\tau)$,

$$- \tfrac{1}{2} \sum_{i,j} c(\xi_i)c(\xi_j)\otimes d\Pi(Q[\xi_i\xi_j]) = -2\sum_t d\sigma(\zeta_t)\otimes d\Pi(\zeta_t)$$

$$= -d\sigma(\Omega_L) - d\Pi(\Omega_L) + d(\sigma\otimes\Pi)(\Omega_L) .$$

And

$$- 2\sum_t d\sigma(\zeta_t)\otimes d(\tau\otimes\Pi)(\zeta_t) = -d\sigma(\Omega_L) - d(\tau\otimes\Pi)(\Omega_L) + d(\sigma\otimes\tau\otimes\Pi)(\Omega_L) .$$

Hence we obtain

$$D^2_{V\otimes W} - {}_{\Pi}D^2_V = (dR(\Omega_K)-d(\tau\otimes\Pi)(\Omega_L))-(d(R\otimes\Pi)(\Omega_K)-d\tau(\Omega_L))$$

$$+ 2\sum_j \gamma^S(\xi_j)\otimes d\Pi(\xi_j)-\sum_j 1\otimes d\Pi(\gamma(\xi_j)\xi_j) . \qquad (1.3.2)$$

(Ω_K,Ω_L is the Casimir element of K,L w.r.t(,) respectively.)

§2. Twisting by an Irreducible Representation.

(2.1) Take a pair (K,H_0) with K a compact, non-abelian connected Lie group and H_0 a closed, connected abelian subgroup. Take H a maximal torus of K with $H_0 \leq H$. We shall use the notation of Chapter 2, §2 and Chapter 7.

There are the orthogonal decompositions $k = h \oplus p_1$, $h = h_0 \oplus h_1$. Take an orthonormal basis $\{\zeta_t\},\{\xi_j\}$ of h_1,p_1 respectively.

Let (W,Π) be a finite-dimensional unitary representation of K . Refer to Chapter 5, (1.4). There are the twisted Dirac operators D_V , ${}_{\Pi}D_V$ and $D_{V\otimes W}$.

On $\Gamma(\underline{S\otimes V})^K_{H_0} \otimes \Gamma(\underline{W})^K_{H_0}$, $dR(\xi) = dR(\xi)\otimes 1 + 1\otimes dR(\xi)$, $\xi \in k$.

Therefore on $\Gamma(\underline{S\otimes V}) \otimes \Gamma(\underline{W})$,

$$D_{V\otimes W} = \sum_t (c(\zeta_t)\otimes 1)(dR(\zeta_t)+\gamma^S(\zeta_t))\otimes 1 + c(\zeta_t) \otimes dR(\zeta_t)$$

$$+ \sum_j (c(\xi_j)\otimes 1)(dR(\xi_j) + \gamma^S(\xi_j))\otimes 1 + c(\xi_j)\otimes dR(\xi_j) .$$

Thus

$$D_{V\otimes W} = D_V\otimes 1 + \sum_t c(\zeta_t)\otimes dR(\zeta_t) + \sum_j c(\xi_j)\otimes dR(\xi_j) \ . \qquad (2.1.1)$$

Recall that $c(\zeta)c(\xi) + c(\xi)c(\zeta) = -2(\zeta,\xi), \zeta,\xi \in h_1 \oplus p_1$.

Therefore

$$\begin{aligned} D^2_{V\otimes W} = \ & D^2_V \otimes 1 + \sum_j D_V c(\xi_j)\otimes dR(\xi_j) + c(\xi_j)D_V\otimes dR(\xi_j) \\ & + \sum_{t,j} c(\zeta_t)c(\xi_j)\otimes dR(\zeta_t)dR(\xi_j) + c(\xi_j)c(\zeta_t)\otimes dR(\xi_j)dR(\zeta_t) \\ & - \sum_t 1\otimes dR(\zeta_t)^2 + \sum_{i,j} c(\xi_i)c(\xi_j) \otimes dR(\xi_i)dR(\xi_j) \ . \qquad (2.1.2) \end{aligned}$$

We will take $\{\xi_j\} = \{\xi_\alpha\}(\alpha \in R)$ where

$$2\ \xi_\alpha = (\varepsilon_\alpha - \varepsilon^\alpha) + \sqrt{-1}(\varepsilon_\alpha + \varepsilon^\alpha) \ , \ \alpha \in R \quad \text{(see Chapter 3, (1.1))}.$$

(2.2) Now take $W = U_\mu$ the simple K-module of highest weight μ .

Take an orthonormal basis $\{v_q\}$ of weight vectors of W . (Recall that the weight spaces are orthogonal w.r.t 'the' inner product <,> on W .) Let v_q have weight μ_q .

Define $t_q \in \Gamma(\underline{W})^K_{H_0}$, for each q , by $t_q(k) = \Pi(k)^{-1}v_q$, $k \in K$. Decompose $t_q = \sum_p t_{pq}$ with $t_{pq} \in \Gamma(\underline{\mathbb{C}v_p})^K_{H_0}$. Here $(\underline{\mathbb{C}v_p})^K_{H_0}$ is the induced, complex line bundle via $\mu_p \in \Lambda_0$. We will write $\mathbb{C}v_p = E_{\mu_p}$ so as to agree with the previous notation.

Define, for each p, $b_p \in \mathrm{Hom}_H(W, \mathbb{C}v_p)$ by $b_p(v_q) = \delta_{pq}v_p$. This gives rise to a linear map on $\Gamma(\underline{W})$, which we also denote b_p, by $b_p t$, $t \in \Gamma(\underline{W})$ where $(b_p t)(k) = b_p t(k)$, $k \in K$. b_p is the orthogonal projection of $\Gamma(\underline{W})$ onto $\Gamma(\underline{\mathbb{C}v_p})$. (See Chapter 0, (3.1) for the inner product $<,>$ on sections of an induced bundle.) b_p commutes with $D_{V\otimes W}$ for each p.

There are the matrix elements M_{pq} of Π where $M_{pq}(k) = <\Pi(k)\,v_p, v_q>$. Recall the Schur orthogonality relations (see Chapter 0, (4.3)). It is seen that $t_{pq}(k) = \overline{M_{pq}(k)}, k \in K$. (- denotes the complex conjugate.)

We have $<t_q, t_q> = 1$, $<t_{pq}, t_{pq}> = \frac{1}{d(\mu)}$ for each p,q, where (2.2.1)
$d(\mu)$ is the dimension of U_μ as given by Weyl's degree formula.

For $\xi \in k$, $dR(\xi)_k\, M_{pq} = \frac{d}{dt}<\Pi(k)\Pi(\exp t\xi)v_p, v_q>$

$$= <\Pi(k)d\Pi(\xi)v_p, v_q \quad , k \in K . \qquad (2.2.2)$$

For $\xi, \eta \in k$, $(dR(\xi)dR(\eta)M_{pq})(k) = dR(\xi)_k(dR(\eta)M_{pq})$

$$= \frac{d}{dt}\,(dR(\eta)M_{pq})(k \exp t\xi)\Big|_{t=0}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\,M_{pq}(k \exp t\xi \exp s\eta)\Big|_{s=t=0}$$

$$= <\Pi(k)d\Pi(\xi)d\Pi(\eta)v_p, v_q> , \ k \in K , \qquad (2.2.3)$$

for each p,q.

So for $\zeta \in h$, $dR(\zeta)M_{pq} = \mu_p(\zeta)M_{pq}$.

Recall that for $\alpha \in R$, $d\Pi(\varepsilon_\alpha)v_p$ is zero or a $\mu_p+\alpha$ weight vector. Hence from (2.2.2), (2.2.3) and the orthogonality relations,

$$\int_K (dR(\xi_\alpha)M_{pq})(k)\overline{M_{pq}(k)}dk = 0 , \quad \forall\alpha \in R . \tag{2.2.4}$$

And for $\alpha,\beta \in R$

$$\int_K (dR(\xi_\alpha)dR(\xi_\beta)M_{pq})(k)\overline{M_{pq}(k)}dk = 0 , \ \beta \neq \pm\alpha$$

$$= \frac{1}{d(\mu)} \frac{\sqrt{-1}}{2} \mu_p(\zeta_\alpha) , \ \beta = -\alpha . \tag{2.2.5}$$

(See Chapter 3, (1.1). Recall that $\zeta_\alpha = [\varepsilon_\alpha \varepsilon^\alpha]$.)

Also

$$\int_K (dR(\zeta)^2 M_{pq})(k)\overline{M_{pq}(k)}dk = \frac{\mu_p(\zeta)^2}{d(\mu)} , \ \zeta \in h . \tag{2.2.6}$$

And

$$- \sum_{\alpha\in R} \int_K (dR(\xi_\alpha)^2 M_{pq})(k)\overline{M_{pq}(k)}dk = \frac{1}{d(\mu)} \langle (d\Pi(\Omega_K)-d\Pi(\Omega_H))v_p,v_p \rangle$$

$$= \frac{1}{d(\mu)} (||\mu+\rho||^2-||\rho||^2-||\mu_p||^2) ,$$

$$\text{for each } p,q . \tag{2.2.7}$$

Fix an element w in the Weyl group $W(K,H)$. Arrange so that v_1 is 'the' weight vector of weight $w^{-1}\mu$. Then with $p = 1$, (2.2.7) becomes $\frac{2\langle\mu,\rho\rangle}{d(\mu)}$ for each q .

(2.3) Let $f \in \Gamma(\underline{S \otimes V})^K_{H_0}$. Note that $f \otimes t_q = \Phi_\Pi^{-1}(f \otimes v_q)$ for each q .

We have $\sum_{\alpha \in R^+} \langle c(\xi_\alpha)c(\xi^\alpha)f,f\rangle(-1)\frac{\sqrt{-1}}{2}\mu_p(\zeta_\alpha)$

$$= -\tfrac{1}{2}\sum_{\alpha \in R^+} \langle c(\varepsilon^\alpha)c(\varepsilon_\alpha)f,f\rangle + \tfrac{1}{2}\sum_{\alpha \in R^+} \mu_p(\zeta_\alpha)\langle f,f\rangle \quad .$$

As

$$c(\xi_\alpha)c(\xi^\alpha) + c(\xi^\alpha)c(\xi_\alpha) = 0 \; , \; \zeta_{-\alpha} = -\zeta_\alpha \; , \; \alpha \in R \; ,$$

we get $\sum_{\alpha \in R^-} = \sum_{\alpha \in R^+}$. For $\alpha \in R$, $c(\xi_\alpha)^2 = -1$.

We intend to take the inner product $\langle D^2_{V \otimes W} f \otimes t_{pq}, f \otimes t_{pq}\rangle$ in (2.1.2).

(2.4) Consider $H_0 = H$.

Write $f = f_1 + \ldots\ldots + f_t$ with $f_i \in \Gamma(\underline{S_{-\rho+|A_i|} \otimes V})^K_H$.

Where as previously $S_{-\rho+|B|}$, $B \subseteq R^+$, is the $-\rho+|B|$ weight space in the spin H-module S . (See Chapter 5, §2.)

Then

$$\sum_{\alpha \in R} \langle c(\xi_\alpha)c(\xi^\alpha)f,f\rangle \frac{(-1)}{2}\sqrt{-1}\mu_p(\zeta_\alpha) = \langle \mu_p, 2\rho\rangle\langle f,f\rangle - \sum_i 2\langle \mu_p, |A_i'|\rangle\langle f_i,f_i\rangle$$

$$= \sum_i 2\langle \mu_p, \rho - |A_i'|\rangle \quad .$$

Where for $B \subseteq R^+$, B' is the complement of B in R^+ .

Now for $B \subseteq R^+$, $<w^{-1}\mu,\rho-|B'|> = <\mu,\rho-|B'|_w>$. And from $w(\rho-|B|) = \rho-|B|_w$, $w(\rho-|B'|) = \rho-|B'|_w$ we get $0 = 2\rho-(|B|_w + |B'|_w)$.

Then $2<\mu,\rho-|B'|_w> + 2<\mu,\rho> = 2<\mu,2\rho-|B'|_w> = 2<\mu,|B|_w>$.

It follows from the computations of (2.1)-(2.3), and the above, that the inner product

$$<D^2_{V\otimes W}(f\otimes t_{1q}),f\otimes t_{1q}> = \frac{1}{d(\mu)} <D^2_V f,f> + \frac{1}{d(\mu)} \sum_i 2<\mu,|A_i|_w><f_i,f_i> \qquad (2.4.1)$$

for each q .

(2.5) Consider any H_0 .

There is the triple (K,H,H_0) . See Chapter 6, (3.1) and Chapter 7, (1.1), (2.1).

Write $f = f_1+\ldots\ldots+f_t$, with $\Phi_H \tilde{f}_i \in \Gamma(S_{-\rho+|A_i|} \otimes \Gamma(S_{H_0} \otimes V)^H_{H_0})^K_H$.

We obtain

$$<D^2_{V\otimes W}(f\otimes t_{1q}),f\otimes t_{1q}> = \frac{1}{d(\mu)}(<D^2_V f,f>+||\widetilde{w^{-1}\mu}||^2<f,f>+\sum_i 2<\mu,|A_i|_w><f_i,f_i>) \qquad (2.5.1)$$

for each q .

§3. A 'Zuckerman Technique'.

Refer to §1,2.

(3.1) Take the pair (K,H_0) as in (2.1).

Take the twisted, by V, Dirac operator $D = D_V$ associated to $((,),\gamma)$ (see Chapter 2, §1). At the moment γ is any invariant metric connection.

(3.2) Consider $H_0 = H$.

There are the unitary equivalences

$$L^2(\underline{S_1 \otimes V \otimes W})_H^K \xrightarrow{\Phi_\Pi} L^2(\underline{S_1 \otimes V})_H^K \otimes W \xrightarrow{\Phi\otimes 1} S_1 \otimes L^2(\underline{V})_H^K \otimes W . \quad (3.2.1)$$

(For S_1, see Chapter 5, (2.1).)

For $\xi \in p_1$,
$$\langle d\Pi(\xi)t_q(k), t_{pq}(k)\rangle = -M_{pq}(k)\langle \Pi(k)^{-1}v_q, d\Pi(\xi)v_p\rangle$$
$$= -M_{pq}(k)\overline{\langle \Pi(k)d\Pi(\xi)v_p, v_q\rangle} .$$

This integrates to zero. Therefore, from (1.3.2),

$$\langle D^2_{V\otimes W}(f\otimes t_{pq}), f\otimes t_{pq}\rangle - \langle {}_\Pi D^2_V(f\otimes t_q), f\otimes t_{pq}\rangle = \langle(\text{Casimir terms})f\otimes t_q, f\otimes t_{pq}\rangle, \quad (3.2.2)$$
$$f \in \Gamma(\underline{S\otimes V}) .$$

Take $V = E_\lambda$, $\lambda \in \Lambda$ (1-dimensional). Fix $w \in W(K,H)$ such that $w\lambda$ is dominant w.r.t R^+. w is unique if λ is non-singular w.r.t R.

Let f be a weight vector in $\Gamma_\nu(\underline{S\otimes E_\lambda})_H^K$, the ν-primary K-submodule, with weight $w^{-1}\nu$. Then $f\otimes t_1$ lies in

$\Gamma_{\nu+\mu}(\underline{S \otimes E_\lambda \otimes U_\mu})_H^K$, and $(1 \otimes b_1)(f \otimes t_1) = f \otimes t_{11}$ lies in

$\Gamma_{\nu+\mu}(\underline{S \otimes E_\lambda \otimes E_{w^{-1}\mu}})_H^K$. (Recall that $W = U_\mu$.)

Now ν is of the form $\nu = w\lambda - \rho + |A| + s$, with $A \subseteq R^+$ and s a sum of +ve roots.

And $(||w\lambda+\mu+|A|+s||^2 - ||\rho||^2) - (||w\lambda+|A|+s||^2 - ||\rho||^2) - ||w\lambda+\mu||^2 + ||\lambda||^2$

$$= 2 \langle \mu, |A|+s \rangle \quad .$$

Therefore, from (1.3.2) we obtain

$$\langle D^2_{V \otimes W}(f \otimes t_{11}), f \otimes t_{11} \rangle = \langle {}_\Pi D^2_V(f \otimes t_1), f \otimes t_{11} \rangle + \frac{2}{d(\mu)} \langle \mu, |A|+s \rangle \langle f,f \rangle. \quad (3.2.3)$$

(3.3) Consider any H_0 . There is the triple (K,H,H_0) .

Take $V = E_{\lambda_0}$ (1-dimensional, $\lambda_0 \in \Lambda_0$.

Let $f \in \Gamma_\nu(\underline{S \otimes E_{\lambda_0}})_{H_0}^K$ be a $w^{-1}\nu$-weight vector such that

$\Phi_H \tilde{f} \in \Gamma_\nu(\underline{S_H \otimes \Gamma_\lambda(\underline{S_{H_0} \otimes E_{\lambda_0}})_{H_0}^H})_H^K$ where $\lambda \in \Lambda$, $\underset{\sim}{\lambda} = \lambda_0$.

Then $f \otimes t_1 \in \Gamma_{\nu+\mu}(\underline{S \otimes E_{\lambda_0} \otimes U_\mu})_{H_0}^K$. Recall $W = U_\mu$. And

$f \otimes t_{11} \in \Gamma_{\nu+\mu}(\underline{S \otimes E_{\lambda_0} \otimes E_{w^{-1}\mu}})_{H_0}^K$ (see 3.2).

Now $||\lambda+w^{-1}\mu||^2 - ||\lambda||^2 - ||\underset{\sim}{\lambda}+\underset{\sim}{w^{-1}\mu}||^2 + ||\underset{\sim}{\lambda}||^2 = 2\langle\tilde{\lambda}, \widetilde{w^{-1}\mu}\rangle + ||\widetilde{w^{-1}\mu}||^2$.

Take γ to be the reductive connection, so $\gamma = 0$ on p_1. Then we obtain,

$$<D^2_{V\otimes W}(f\otimes t_{11}), f\otimes t_{11}> = <_{\Pi}D^2_V(f\otimes t_1), f\otimes t_{11}> + \frac{1}{d(\mu)}(2<\tilde{\lambda}, w^{-1}\tilde{\mu}> + ||w^{-1}\tilde{\mu}||^2 + 2<\mu, |A|+s>) <f,f>. \quad (3.3.1$$

(3.4) Proposition 12.

Let $H_0 = H$, and γ be any connection.

Then Ker D is a K-submodule of $\sum_B \oplus \Gamma_{\nu_B}(\underline{S_{-\rho+|B|} \otimes E_\lambda})^K_H$

(a finite direct sum), where B runs over R^+, and $\nu_B = w\lambda - \rho + |B|$. (See 3.2.)

(Of course if ν_B is not dominant for some $B \subseteq R^+$, then certainly ν_B does not occur.)

Proof.

Suppose the kernel of $D = D_V$ is non-zero on $\Gamma_\nu(\underline{S \otimes E_\lambda})^K_H$. As Ker D is a K-module, we may find a (non-zero) weight vector f of weight $w^{-1}\nu$, with $f \in \Gamma_\nu(\ )$. Then from (2.4.1) and (3.2.3), we get

$$\sum_i 2<\mu, |A_i|_w><f_i, f_i> = 2<\mu, |A|+s><f,f> \quad . \qquad (3.4.1)$$

Now ν is also of the form $\nu = w\lambda - \rho + |A_i|_w + s_i$, s_i a sum of +ve roots, for each i . So $|A_i|_W + s_i = |A|+s$, $\forall i$. Then (3.4.1) with $\mu = \rho$ implies that $s_i = 0$, $\forall_i$. Hence the assertion. □

(3.5) Let $B \subseteq R^+$. $\rho-|B|$ is a weight of U_ρ the simple K-module of highest weight ρ . Therefore $||\rho-|B|||^2 \leq ||\rho||^2$, which implies that $2<\rho,|B|> \geq |||B|||^2$.

(3.6) We now complete the proof of Theorem 4, Chapter 5, (4.2).

For γ the Levi-Civita connection we see from (3.5) and Chapter 5, (4.2.2) that for (non-zero) $f \in \operatorname{Ker} D$ we must have $s = 0$, $A = \phi$. Hence our 'vanishing' result for all λ . So in fact this connection does not require a 'Zuckerman' argument.

Consider λ the reductive connection. The argument in the proof of Proposition 12 , in (3.4), shows that $s_i = 0$, $\forall i$. Thus from (3.9.2) , if $f \in \operatorname{Ker} D$ we must have $s = 0$, $A = \phi$. Hence our vanishing result for all λ .

(3.7) Refer to Chapter 5, (4.2). See (3.5), (3.9) and Chapter 5, (4.2.2).

Note that for γ the Levi Civita connection $<(D^2+[D_0D]_+)f,f> \geq 0$ for all λ . And for γ the reductive connection, if $s_i = 0$, $\forall i$, then $<(D^2 + [D_0D]_+)f,f> \geq 0$ for all λ .

We now complete the proof of Theorem 7 Chapter 7, (2.2). Note that as Ker D is finite-dimensional, D_0 and D_1 are bounded (therefore continuous on Ker D) . In fact from (2.2.3), (2.2.6), which hold for all λ_0 , we see that for either connection, D_0 and D_1 are bounded by a on Ker D .

Consider γ the Levi-Civita connection. By the remark at the beginning of this number, we see immediately from Chapter 7 (2.2.7) that Ker D = Ker $D_0 \cap$ Ker D_1 for all λ_0 .

Consider γ the reductive connection. Suppose Ker D is non-zero on $\Gamma_\nu(S\otimes \underline{E_{\lambda_0}})^K_{H_0}$. Then as Ker D is a K-module, we can find an f as in (3.3), with $f \in$ Ker D . From (2.5.1) and (3.3.1), get

$$\sum_i 2\langle\mu,|A_i|_W\rangle\langle f_i,f_i\rangle = (2\langle\tilde{\lambda},\widetilde{w^{-1}}\mu\rangle + 2\langle\mu,|A|+s\rangle)\langle f,f\rangle \quad . \qquad (3.7.1)$$

Since ν is also of the form $\nu = w\lambda-\rho+|A_i|_W + s_i$, s_i a sum of +ve roots $\forall i$, we have $|A_i|_W + s_i = |A| + s$, $\forall i$. Thus taking $\mu = m(w\lambda) + \rho$, where the +ve integer m is chosen so that $m||\tilde{\lambda}||^2 + \langle\tilde{\lambda},\widetilde{w^{-1}}\rho\rangle \geq 0$, we get from (3.7.1) that $s_i = 0$, $\forall i$. Hence, from Chapter 7 (2.2.5), Ker D = Ker $D_0 \cap$ Ker D_1 for all λ_0 .

This completes the proof of Theorem 7.

(3.8) Take (K,H) . Refer to Chapter 5, (4.2), (4.3).

Here we consider γ any connection, and any λ .

Suppose $U_\nu \otimes b \longrightarrow \Gamma_\nu(\underline{S_{-\rho+|B|} \otimes E_\lambda})^K_H$ (see Chapter 0, (3.2)), (3.8.1)

$(0 \neq)\ b \in \mathrm{Hom}_H(U_\nu, S_{-\rho+|B|} \otimes E_\lambda)$, where $\nu = w\lambda-\rho+|B|_w$, $B \subseteq R^+$.

w is chosen as in (3.2).

With $v \in U_\nu$, $v \otimes b \longrightarrow f$

where $f(k) = b(\Pi_\nu(k)^{-1}v), k \in K$.

Fix v to be 'the' weight vector with weight $w^{-1}\nu$.

Then we have

Proposition 13.

$f(e) \neq 0$ (e the identity element of K)

and $dR(\xi)_e f = 0$, $\forall\ \xi \in p$. (3.8.2)

Proof.

This follows by an argument used in the proof of Theorem 4. Note that $dR(\xi)_e f = -dL(\xi)_e f$. □

(3.9) Refer to Chapter 5, (4.1.2).

$$d(R\otimes\sigma_1)(\Omega_K) = -\sum_t (dR(\zeta_t)+d\sigma(\zeta_t))^2 - \sum_j (dR(\xi_j)+d\sigma_1(\xi_j))^2 .$$

Then with f as in (3.8)

$$(d(R\otimes\sigma_1)(\Omega_K)f)(e) = ((d\tau(\Omega_H)+dR(\Omega_K)-dR(\Omega_H)+d\sigma_1(\Omega_K)-d\sigma(\Omega_H))f)(e)$$

$$d(R\otimes\sigma_1)(\Omega_K)f(e) = (dR(\Omega_K)+d\sigma_1(\Omega_K)-d\sigma(\Omega_H)-d(\sigma\otimes\tau)(\Omega_H)+d\tau(\Omega_H))f(e) \ . \quad (3.9.1)$$

As each Casimir term on the right hand side acts by a constant, holds $\forall k \in K$.

Therefore Chapter 5,(4.1.2) becomes

$$(D^2 + [D_0 D]_+)f = (dR(\Omega_K) + d\sigma(\Omega_H) - d\tau(\Omega_H))f \ . \tag{3.9.2}$$

And (4.1.5) becomes

$$2(D^2 + [D_0 D]_+)f = (2dR(\Omega_K) + 3/2 d\sigma_{'}(\Omega_K) - 5/2 d\sigma(\Omega_H) - 2d\tau(\Omega_H))f \ . \tag{3.9.3}$$

(3.10) Take (K,H) and the Dirac operator $D = D_V$ with $V = E_\lambda$, $\lambda \in \Lambda$ as in (3.1).

Suppose for $(U_\nu, \Pi_\nu) \in \hat{K}$, that $U_\nu \otimes b \longrightarrow \text{Ker } D$, $b \in \text{Hom}_H(U_\nu, S \otimes E_\lambda)$. Then by Proposition 12, we have (3.8.1). Thus taking f as in (3.8), $f \in \text{Ker } D$, we get (3.8.2).

CHAPTER 10.

§1. Conclusion.

(1.1) In Chapter 2, §1 take any pair (K,L). Take a maximal torus H_0 of L, and a maximal torus H of K with $H_0 \leq H$. There is the twisted Dirac operator $D = D_V$ associated to $((\ ,\ ),\gamma)$.

Take $V = V_{\lambda_0-\rho_L}$ the simple L-module of highest weight $\lambda_0-\rho_L$. For $\lambda \in \Lambda$, λ non-singular take w in the Weyl group $W(K,H)$, the unique element such that $w\lambda$ is dominant w.r.t R^+.

We restate our main theorem. See Chapter 8.

Theorem 8.

Let $\lambda \in \Lambda$, λ non-singular w.r.t R and consider the conditions

(1) $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = (w^{-1}\rho)^{\sim}$. (0) $\underset{\sim}{\lambda} = \lambda_0$, $2\tilde{\lambda} = -(w^{-1}\rho)^{\sim}$.

If (1), (0) cannot be satisfied for any λ, then for γ the reductive, Levi-Civita connection respectively, $\operatorname{Ker} D = 0$.

If (1), (0) can be satisfied, of course λ is unique, then for γ the reductive, Levi-Civita connection respectively;

Ker D is the $w\lambda-\rho$ primary K-submodule $\Gamma_{w\lambda-\rho}(\underline{S \otimes V_{\lambda_0-\rho_L}})^K_L$ in $L^2(\underline{S \otimes V_{\lambda_0-\rho_L}})^K_L$.

The multiplicity is given in Chapter 8, (1.2), for some cases.

It is seen that for γ the Levi-Civita or reductive connection, Ker D is either zero or primary as a K-module.

Theorem 8 contains all previous theorems as corollaries.

Example.

(K,L) a symmetric pair. Here the Levi-Civita connection is the reductive connection. Thus we require $\tilde{\lambda} = 0$. Therefore $w = 1$ and $\tilde{\rho} = 0$. As was noted before, we can always satisfy $\underset{\sim}{\lambda} = \lambda_0$, $\tilde{\lambda} = 0$. The multiplicity is 2^r where $r = \frac{1}{2}[\dim H - \dim H_0]$. See Theorem 2, Chapter 4 (3.3).

The special case of Theorem 8 with $L = H$ (i.e. Theorem 4, Chapter 4, (4.3)) gives us a geometric construction of all irreducible representations of a compact, connected Lie group K .

In the case of equal rank i.e. rank L = rank K , we do not expect Theorem 5 (Chapter 6, (4.1)) to depend on the connection γ . In fact we already have enough information, in previous chapters, to prove this for $L = H$ and λ sufficiently non-singular. However, in the case of unequal rank i.e. rank L < rank K , Theorem 8 does depend on γ . See for example Theorem 3, (Chapter 5, (3.2)).

We expect that the techniques we have introduced in previous chapters, can be used to deal with any connection γ . This will be pursued in future work. We also want to consider applications of Theorem 8.

Also, more generally, to consider the pair (G,L) with G a reductive Lie group and L a compact subgroup.

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