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# SOME RESULTS FOR THE CLASS OF ANALYTIC FUNCTIONS INVOLVING SALAGEAN DIFFERENTIAL OPERATOR

(Beberapa Sifat untuk Kelas Fungsi Analisis Melibatkan Pengoperasi Pembeza Salagean)

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*ABSTRACT*

Let $T_n^\alpha(\beta)$ denote the class of function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analytic and univalent in the open unit disk $U = \{z \in C: |z| < 1\}$, which are defined involving the Salagean differential operator $D^n, n \in N \cup \{0\}$, such that $\mathrm{Re}(D^n f(z)^\alpha / z^\alpha) \geq \beta,\ z \in U, \alpha > 0, 0 \leq \beta < 1$. In this paper, some properties such as a representation theorem and coefficient estimates for the class $T_n^\alpha(\beta)$ are obtained.

*Keywords:* Salagean differential operator; coefficient estimate; representation theorem

*ABSTRAK*

Andaikan $T_n^\alpha(\beta)$ kelas fungsi $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analisis dan univalen pada cakera unit $U = \{z \in C: |z| < 1\}$, yang tertakrif melibatkan pengoperasi pembeza Salagean $D^n, n \in N \cup \{0\}$ sedemikian hingga $\mathrm{Ny}(D^n f(z)^\alpha / z^\alpha) \geq \beta, z \in U, \alpha > 0, 0 \leq \beta < 1$. Dalam makalah ini, beberapa sifat fungsi $f$ di dalam kelas $T_n^\alpha(\beta)$ seperti teorem perwakilan dan anggaran pekali diperoleh.

*Kata kunci:* Pengoperasi pembeza Salagean, anggaran pekali, teorem perwakilan

## 1. Introduction

Let *A* denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathrm{C} : |z| < 1\}$, C is a complex number. We denote the subclass of *A* consisting of analytic and univalent functions in *U* by *S*. Let *P* be the class of analytic function $p(z)$ of the form

$$p(z) = 1 + \sum_{k=2}^{\infty} p_k z^k,\ z \in U \tag{2}$$

such that $p(0) = 1$ and $\mathrm{Re}\big(p(z)\big) > 0$.

The following classes of functions are well known and have been studied repeatedly by many authors such as Salagean (1983), Opoola (1994), Abdulhalim (2003) and others.

$$S_0 = \left\{ f \in A\text{: Re}\left(\frac{f(z)}{z}\right) > 0,\ z \in U \right\},$$
$$B(\beta) = \left\{ f \in A\text{: Re}\left(\frac{f(z)}{z}\right) > \beta,\ 0 \le \beta < 1,\ z \in U \right\}, \tag{3}$$
$$\delta(\beta) = \left\{ f \in A\text{: Re}\left(f'(z)\right) > \beta,\ 0 \le \beta < 1,\ z \in U \right\}.$$

In 1993, Salagean introduced the following operator which is known as the Salagean differential operator.

**Definition 1.1.** For a function $f \in A$, the Salagean differential operator $D^n : A \to A$, $n \in \mathrm{N}_0 = \{0,1,2,...\}$ is defined by

$$D^n f(z) = D\left[D^{n-1} f(z)\right] = z\left[D^{n-1} f(z)\right]', \tag{4}$$

where

$$D^0 f(z) = f(z) \quad \text{and} \quad D^1 f(z) = Df(z) = zf'(z).$$

The operator $D^n$ has been employed by various authors to define several subclasses of *A*, see Kanas (1989, Obradovic (1992), Opoola (1994), Babalola and Opoola (2006). For instance, Opoola (1994) introduced the subclass $T_n^{\alpha}(\beta),\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$ of analytic functions which are defined involving Salagean differential operator as follows:

$$T_n^{\alpha}(\beta) = \left\{ f \in A:\ \text{Re}\left(\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right) > \beta \right\}, \tag{5}$$

for $z \in U,\ \alpha > 0,\ 0 \le \beta < 1$. The class $T_n^{\alpha}(\beta)$ is the generalization of the classes of functions as mentioned in (3), where $T_n^{\alpha}(0) := B_n(\alpha)$ is the class of Bazilevic functions (Abdulhalim 2003). Some properties of this class of functions were also established by Opoola, namely

(i) $T_n^{\alpha}$ is a subclass of univalent functions,

(ii) $T_{n+1}^{\alpha} \subset T_n^{\alpha}(\beta)$,

(iii) If $f \in T_n^{\alpha}(\beta)$, then the integral operator $F_c = \dfrac{\alpha + c}{z^{\alpha}} \int_0^c t^{\alpha - 1} f(z)^{\alpha}\, dt,\ \ c \ge 0,$ is also in $T_n^{\alpha}(\beta)$.

In this paper, we obtain further properties of functions in the class $T_n^{\alpha}(\beta)$.

## 2. Preliminaries

We observed that for $f \in A$ and of the form (1), and applying the operator (4) then we have

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \tag{6}$$

for $z \in U, n \in \mathrm{N}_0 = \{0,1,2,...\}$. Also, suppose that $\alpha > 0$, then by using binomial expansion, from (1) we can write

$$f(z)^{\alpha} = z^{\alpha} + \sum_{k=2}^{\infty} A_k(\alpha) z^{\alpha+k-1}, \tag{7}$$

where the coefficients $A_k(\alpha), k \in \{2,3,4,...\}$ with $A_k(1) = a_k$, depend on the coefficients $a_k$ of $f(z)$ and the parameter $\alpha > 0$. Now, if we apply the operator (4) for the function (7), then we obtain

$$D^n f(z)^{\alpha} = \alpha^n z^{\alpha} + \sum_{k=2}^{\infty} (\alpha + k - 1)^n A_k(\alpha) z^{\alpha+k-1} \tag{8}$$

for $z \in U, \alpha > 0, n \in \mathrm{N}_0$.

In order to prove our results, we shall need the following lemmas.

**Lemma 2.1.** (Hayami *et al.* 2007) *A function* $p(z) \in P$ *given by (2) satisfies the condition* $Ny(p(z)) > 0, \ z \in U$ *if and only if*

$$p(z) \neq \frac{\psi - 1}{\psi + 1}, \ z \in U, \psi \in \mathrm{C}, |\psi| = 1. \tag{9}$$

**Lemma 2.2.** (Alfors 1966) *Let* $f(z)$ *be an analytic function in the unit disk U with* $f(0) = 0$ *and* $|f(z)| < 1$. *Then,* $|f'(0)| \leq 1$ *and* $|f(z)| \leq |z|$ *in U. Strict inequality holds in both estimates unless* $f(z)$ *is a rotation of the disk* $f(z) = e^{i\theta} z$. *If* $|f(z)| = |z|$ *for some* $z \neq 0$, *then* $f(z) = cz$, *with a constant c of absolute value 1.*

## 3. Main Results

In this section, we establish some properties of functions in the class $T_n^{\alpha}(\beta)$. First, we prove a sufficient condition for functions belong to this class.

**Theorem 3.1.** *A function* $f \in A$ *is in the class* $T_n^{\alpha}(\beta), \alpha > 0, 0 \leq \beta < 1, n \in \mathrm{N}_0$ *if and only if*

$$1 + \sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0,$$

*where*

$$Q_k = \frac{(\psi + 1)}{2(1 - \beta)} \left( \frac{\alpha + k - 1}{\alpha} \right)^n A_k(\alpha) \tag{10}$$

*for some* $\alpha > 0, 0 \leq \beta < 1, n \in \mathrm{N}_0$ *with* $\psi \in \mathrm{C}, |\psi| = 1$.

**Proof.** From (5), is suggests that there exists a function $p(z) \in P$ such that

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n\left[\beta + (1-\beta)p(z)\right]$$

for $z \in U,\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$. Upon setting

$$p(z) = \left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta\right)\left(\frac{1}{1-\beta}\right),$$

then from Lemma 2.1, we get

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta\right)\left(\frac{1}{1-\beta}\right) \ne \frac{\psi - 1}{\psi + 1}$$

for $z \in U,\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$ with $\psi \in \mathrm{C}, |\psi| = 1$. It is equivalent to

$$(\psi + 1)D^n f(z)^\alpha - \left[\beta(\psi + 1) + (1-\beta)(\psi - 1\right]\alpha^n z^\alpha \ne 0. \qquad (11)$$

Substituting (8) into (11) yields that

$$2(1-\beta)\alpha^n z^\alpha + \sum_{k=2}^{\infty}(\psi + 1)(\alpha + k - 1)^n A_k(\alpha) z^{\alpha + k - 1} \ne 0 \qquad (12)$$

Now, dividing both sides of (12) by $2(1-\beta)\alpha^n z^\alpha \ne 0$, we obtain

$$1 + \sum_{k=2}^{\infty}\frac{(\psi + 1)}{2(1-\beta)}\left(\frac{\alpha + k - 1}{\alpha}\right)^n A_k(\alpha)\, z^{\alpha + k - 1} \ne 0$$

for any $\psi \in \mathrm{C}$ such that $|\psi| = 1, z \in U,\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$. □

**Remark 3.2.** For the case of $\beta = 0$ in (10), the result has been proved by Singh *et al.* (2009).

The following property of functions in the class $T_n^\alpha(\beta)$ is also established.

**Theorem 3.3**. *Let* $f \in A$ *belong to the class* $T_n^\alpha(\beta),\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$. *Then, there exists an analytic function* $\phi(z)$ *with* $|\phi(z)| \le 1,\ z \in U$ *such that*

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 2\beta - 1 + \frac{2(1-\beta)}{1 - z\phi(z)},$$

(12)

*for* $z \in U,\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$.

**Proof**. Let us define the functions $A(z)$ and $B(z)$ as follows:

$$A(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \qquad (13)$$

and

$$B(z) = z\left(\frac{A(z) - (-\beta)}{A(z) + (1-\beta)}\right) \qquad (14)$$

for $z \in U,\ \alpha > 0, 0 \le \beta < 1, n \in \mathrm{N}_0$. Substituting the equation (13) into (14), we get

$$B(z)=z\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-1\right)\left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-(2\beta-1)}\right)$$

for any $z\in U$. $B(z)$ is an analytic function for $z\in U$. Also, since $f(0)=0$ and $f'(0)=1$, we have that $B(0)=0$ and $|B(z)|<1$ for $z\in U$. Hence, by Schwarz's Lemma (Lemma 2.2), $|B(z)|<|z|$, $z\in U$ which gives that

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-1\right)\left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-(2\beta-1)}\right)<|z|$$

or equivalently

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-1\right)\left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-(2\beta-1)}\right)=z\phi(z),$$

where $\phi(z)$ is analytic and $|\phi(z)|\le 1$ for $z\in U$. Therefore,

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-1=z\phi(z)\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-(2\beta-1)z\phi(z). \tag{15}$$

Solving for $D^n f(z)^\alpha/\alpha^n z^\alpha$ from (15), then we get

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}=\frac{1-(2\beta-1)z\phi(z)}{1-z\phi(z)}=(2\beta-1)+\frac{2(1-\beta)}{1-z\phi(z)}.$$

Thus, we obtain the desired result (12). □

The following result is called the integral representation theorem for functions in the class $T_n^\alpha(\beta)$, which provides further property of functions in this class.

**Theorem 3.4.** *Let $f\in A$ belong to the class $T_n^\alpha(\beta)$, $\alpha>0, 0\le\beta<1, n\in \mathrm{N}_0$. Then, there exists an analytic function $\phi(z)$ with $|\phi(z)|\le 1$, $z\in U$ such that*

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}=\frac{1}{\alpha^n z^\alpha}\exp\int_0^z\left(\frac{\alpha}{t}-C(t)\right)dt, \tag{16}$$

*for* $\alpha>0,\ 0\le\beta<1,\ n\in\mathrm{N}_0$, *where*

$$C(t)=\frac{2(\beta-1)[z\phi'(t)+\phi(t)]}{1-2\beta t\phi(t)+(2\beta-1)t^2\phi^2(t)}. \tag{17}$$

**Proof:** Let $f(z)\in T_n^\alpha(\beta)$. Then, from Theorem 3.3 we have

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 2\beta - 1 + \frac{2(1-\beta)}{1 - z\phi(z)} = \frac{1-(2\beta-1)z\phi(z)}{1-z\phi(z)}. \tag{18}$$

Taking the logarithmic differentiation, we get from (18)

$$\frac{\left[D^n f(z)^\alpha\right]'}{D^n f(z)^\alpha} = \frac{\alpha}{z} - \frac{2(1-\beta)\left[z\phi'(z)+\phi(z)\right]}{1-2\beta z\phi(z)+(2\beta-1)z^2\phi^2(z)}. \tag{19}$$

Now, integrating both sides of (19) along the line segment from 0 to $z$, we obtain

$$\ln\left[D^n f(z)^\alpha\right] = \int_0^z \left(\frac{\alpha}{z} - \frac{2(1-\beta)\left[t\phi'(t)+\phi(t)\right]}{1-2\beta t\phi(t)+(2\beta-1)t^2\phi^2(t)}\right)dt.$$

This gives us that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp\int_0^z \left(\frac{\alpha}{t} - C(t)\right)dt$$

for $z \in U,\ \alpha > 0,\ 0 \le \beta < 1,\ n \in \mathrm{N}_0$, where

$$C(t) = \frac{2(1-\beta)\left[t\phi'(t)+\phi(t)\right]}{1-2\beta t\phi(t)+(2\beta-1)t^2\phi^2(t)}$$

for an analytic function $\phi(z)$ with $\left|\phi(z)\right| \le 1,\ z \in U$. Thus, the proof of Theorem 3.4 is completed. □

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