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Unity in the Theory of Elementary Particles Through Group Theory

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UNITY IN THE THEORY OF ELEMENTARY PARTICLES

THROUGH GROUP THEORY

by

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ABSTRACT

Unity in the Theory of Elementary Particles

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Science is the process of seeking unity in the diversity of natural phenomenon. The purpose of this paper is to demonstrate that group theory brings unity to the theory of elementary particles. The prime motivations are first, to find a quantitative representation of the Lorentz transformation, and second, to find a quantitative representation of angular momentum. Since both of these have continuous parameters, groups with continuous parameters, particularly Lie groups, are of interest.

The first portion of the paper develops the definition of Lie groups and their associated Lie algebras. The prerequisite definitions of transformations, groups, group representations, and continuous groups are given.

The second portion of the paper presents illustrations to support the conclusion that group theory brings unity to elementary particle theory. The major examples are spin and angular momentum of a particle.

(32 pages)

INTRODUCTION

Unity: the goal of science

Science is more than the process of dividing knowledge into neat and well-ordered compartments. It is the process of seeking unity in the diversity of nature. Founded on a minimum of postulates, a new theory seeks to correlate a broader range of physical phenomenon than had been previously possible.

One profound unification in the development of physics was the advent of Maxwell's equations of electromagnetism. They not only accounted for the separate areas of electromagnetism such as electrostatics, induction, and others, but they also predicted that light was electromagnetic in nature. This unification brings visible light, microwaves, radio waves, X-rays, and gamma rays all under a single unified description as electromagnetic quanta. J. R. Pierce wrote, "To anyone who is motivated by anything beyond the most narrowly practical, it is worthwhile to understand Maxwell's equations simply for the good of his soul." (Halliday, Resnick, 1967, p. 963)

Another far-reaching unification in physics were Einstein's Special and General Theories of Relativity. They encompassed and surpassed Newtonian mechanics and theory of gravitation. The triumph of the Special Theory of Relativity was that it dispelled the conflict between Newtonian mechanics and Maxwell's electromagnetism thereby bringing a broader unification of understanding. Additionally the Special Theory predicted new phenomenon concerning the behavior of matter moving at velocities near the speed of light. The predictions have been experimentally verified. General Relativity brought unity to the concept of gravitation and the universe. It accounted for all the consequences of Newtonian gravitation in the limit as well as predicting other phenomenon. The precession of the perihelion of Mercury and the deflection of star light by the sun were quantitatively explained by the new theory. Concerning this unification C. Lanczos wrote, "We admit the loss of simplicity, but are willing to pay the price for the sake of the tremendous advance in unity." (Marion, 1970, p. 130)

Unity through group theory

Originally group theory was studied only by mathematicians. Then, as quantum mechanics developed, physicists realized its many applications to the new field. The formalism of group theory brings under a single mathematical description such otherwise unrelated concepts as angular momentum, spin, tensor character, crystal lattice description, the quark model of elementary particles and so forth.

Often in physics the mathematical apparatus of a theory predicts previously unknown ideas. In addition to Maxwell's and Einstein's theories, this was the case with Dirac's prediction of antiparticles. Dirac insisted that both the positive and negative roots of the relativistic equation $E = \pm \sqrt{p^2c^2 + m_o^2c^4}$ be recognized as physically valid. Gell-Mann and Zweig predicted the existence of the Ω^- particle based on the quark model or "Eight-fold Way" group theory model of the elementary particles. Mathematical unity, therefore, may reveal physical unity as well.

Objectives

The first objective is to develop the theory of a Lie group and Lie algebra. They are important because their continuous parameters correspond meaningfully to physical quantities. An example is the Lorentz transformation group. The Lorentz transformation which plays a central role in any relativistic theory has the continuous quantities space and time as its parameters. Consequently the preliminary definitions of a group and transformation are given. Then, the definition of a continuous group is extended to the Lie group and its associated Lie algebra. Examples of a Lie group and algebra are given.

The second objective is to illustrate that group theory, especially Lie groups, bring unity to the description of elementary particles. Spin, angular momentum, and the equivalence of all unitary representations of a relativity group to all relativistic wave equations are given as illustrations. Additional, more advanced applications are listed with references for further investigation.

A SUMMARY OF GROUP THEORY LEADING TO LIE GROUPS AND ALGEBRAS

Definitions and fundamental properties of groups

A widely applied and fundamental concept in physics is the transformation. They are a quantitative method of describing a change in a system. The change may be the rotation of a rigid body, the passage of light rays through a series of lenses, a change of bases in a vector space or many others.

A useful property of transformations is that they allow certain properties of the system such as total energy, total angular momentum, proper time and distance to remain constant or invariant while the position, velocity, or other properties are changed by the transformation. The Lorentz transformation, for example, which transforms from a coordinate system 0 to a system 0' which is moving in the x direction is $x = x'\cosh\theta + t'\sinh\theta$, $t = x'\sinh\theta + t'\cosh\theta$, $y = y'$, $z = z'$, where $\tanh\theta =$ their relative velocity normalized to the speed of light. This transformation preserved the proper time between events. It will be discussed in more detail later.

In general a transformation may be defined for a set X and X' as a one to one correspondence or mapping of the elements of X and X'. If the mapping has the form $x_i = \Sigma\, a_{ij}x_j$, $i = 1,2,\ldots,n$, then the transformation is said to be linear. The matrix of the coefficients (a_{ij}) is called the transformation matrix.

Groups. With the concept of a transformation presented, we now give the most important definition in this section. It is, of course, the definition of a group. It is in the language of groups with which models are constructed. A set of transformations is a group if the following criteria are satisfied: (1) the set contains the identity element, (2) for every transformation M, then the inverse, M^{-1} is an element of the set, and (3) if the set includes M and M' then it also contains their composition MM'. In other words, the set of transformations is closed under combination.

The elements of a group are in general arbitrary. For the application in physics they are usually transformations. However, a set of points or numbers may also be a group if under the law of combination defined, the above criteria are satisfied. We shall restrict the discussion to groups of transformations.

Subgroups. The idea of a subgroup will be very useful later. Invariance properties of subgroups allow one to determine if the group representation is irreducible. This will be explained in more detail in a later section. The definition is given at this point, however, so its relationship to a group will be clear.

We say that H is a subgroup of G if H and G are groups and H is a subset of G. The group itself and the identity are called improper subgroups.

Symmetry and rotation groups. Groups may be classified by what their transformations perform. The set of transformations which preserves the symmetry of a set of points is called a symmetry group. This class of groups is very important in physics and chemistry. They are used for example if desiring the lattice structure of crystals. The symmetry groups SU(2) and SU(3) are used in the quark model or "Eight Fold Way" description of elementary particles. The reader is referred to Lipkin (1960) for further study of these groups.

A rotation group is a set of transformations which perform a rotation on a system. The system may be a rigid body or a coordinate system for example. In quantum mechanics the conservations of angular momentum is expressed mathematically by requiring that the wave function of a system be invariant with respect to infinitesimal rotations in a Hilbert space. The eigenvalue of the rotation operator is the total angular momentum of the system. This application will be discussed in more detail in a later section. For a complete treatment of the 3-dimensional rotation group and its representation the reader is referred to Gel'Fand, Minlos, and Shapiro (1963).

Group representations

With the definitions and basic properties of groups established, the next concept in the development toward a Lie group is the representation of a group. Two important terms which will be used are isomorphic and homomorphic. Hamermesh (1962) gives the definition that two groups G and H are isomorphic if their elements can be put into a one-to-one correspondence which is preserved under combination. He also states that a homomorphic mapping or homomorphism is a correspondence similar to an isomorphic mapping. The difference is that in a homomorphism from a group G to a group H more than one element of G may have the same image or correspond to the same element in H.

With this background the concept of a group representation is introduced. Intuitively, one can conceive of rotations of a rigid body about an axis. For each rotation forward there is an opposite rotation which brings the body back to its original position. Therefore an inverse exists for each rotation. The identity exists since either no rotation or a complete rotation brings the body back to its original position. Lastly, for any two successive rotations, there is a third rotation which brings the body to the same position as the composite for the first two. So one has an intuitive feeling that the set of rotations about an axis form a group since the criteria are satisfied.

One may ask, however, how can rotations be described quantitatively. This suggests the use of mathematics. It is in the quantitative description of elements of a group that group representations are introduced. Before giving a formal definition two preliminary ideas are necessary, the linear operator and the matrix representative.

Linear operators. A linear transformation as defined previously may also be considered to be a linear operator. It is an operator in the sense that it changes the system or, in other words, performs a certain operation on it. Such operators may be considered independently from any specified coordinate system because they have intrinsic significance. Although if one chooses a set of basis vectors u_i in a vector space L, then a transformation T is defined by the coordinate functions $y_i = T_i(x_1,\ldots,x_n)$.

Matrix representative. A theorem of linear analysis (Kaplan, 1973, p. 79) guarantees that every linear operator may be represented by a matrix (T_{ij}) for a basis u_i. Therefore matrices may be used as a quantitative method for describing elements of a group such as the rotations mentioned previously.

The matrix (T_{ij}) is called the matrix representative of the operator T. If one chooses a different basis v_i to represent T, the two representatives are said to be equivalent.

Representation of a group. Using the definitions and functions of linear operators and their matrix representatives we are now able to present a formal definition of the representation of a group. A set of operators D(G) = A,B,... in a vector space L form a group if, of course, they satisfy the criteria for a set to be a group. Then if one maps a group G homomorphically on such a group of operators D(G) in L, the operator group D(G) is called a representation of G in the representation space L.

For example, consider the possible rotations of a body about an axis. We have shown that they form a group. Each rotation can be represented by a matrix for a given basis in a vector space. The set of matrices also form a group M called the representation of the group R. As stated by Hamermesh, "If we choose a basis in the n-dimensional space L, the linear operators of the representation can be described by their matrix representatives." (Hamermesh, 1962, p. 78)

Irreducible representations of a group. When considering the representation of a group, one may ask if it is possible to represent the group in a more simple representation. If a representation is in its simplest form, it is said to be irreducible. We must, of course, define what is meant by "simplest form."

The concept of irreducibility is of prime importance in physical applications. One desires that a model or theory be expressed in the most fundamental concepts in order to provide the broadest possible unification. The development of the transformation, operator, group and group representation given above allow us to define exactly what is meant for a representation to be in its simplest form.

> The representation T of a group G in the space L is called reducible, if there exists in L at least one non-trivial subspace L_1 invariant with respect to all operators $T(g)$ $(g \in G)$.
>
> Accordingly, the representation T of the group G in the space L is called irreducible, if in L there is no non-trivial subspace L_1 invariant with respect to all the operators $T(g)$ $(g \in G)$. (Lyubarskii, 1960, p. 45)

We can give this definition a physical interpretation. Suppose for a representation T of a group G there is a subspace which is invariant with respect to the operators of the group G. This means that when an operator operates on a vector in the subspace the resulting vector is also an element of the subspace. We may intuitively think therefore that the subspace is somehow more basic or elementary itself. Naturally, if a system has a subset which is more elementary, then we want to form our theory in terms of the most fundamental or elementary units or conceptual models possible. Hence, the above definition of reducible and irreducible seem natural. If a system, a particle for example, has a more basic substructure then the particle is "reducible" to its elementary constituents.

Schur's lemmas. Based on the irreducibility of a group we conclude this section on group representations with the following lemmas known as Schur's lemmas.

Lemma I. If D and D' are two irreducible representations of a group G, having different dimensions, then if the matrix A satisfies D(R) A = A D'(R) for all R in G, it follows that A = 0.

Lemma II. If the matrices D(R) are an irreducible representation of a group G, and if A D(R) = D(R) A for all R in G, then A = constant (1). (Hamermesh, 1962, p. 100)

It is discussed further in a later section on Casimir operators that these lemmas provide important information about a group. Lemma II guarantees that any operator which commutes with all the elements of an irreducible representation of a group are scalar multiples of the unit matrix. These operators called Casimir operators are of physical importance because they are invariants of the group. Also, the scalar value of the Casimir operator provides a means of distinguishing irreducible representations of a group.

Continuous groups

Definition of a continuous group. The rotation group illustrates another major point toward the definition of a Lie group. The angle of rotation of a body from its initial position may be any real value. They are continuous and therefore two elements may be arbitrarily close together. The angle is called the parameter of the group.

Contrast this continuous group to the group consisting of the set of positive and negative integers and zero. If the law of combination is algebraic addition, then they satisfy the requirements to be a group. Clearly the integer zero is the identity element since $n + 0 = n$. For and element n, then $-n$ is the inverse since $n + (-n) = 0$ the identity element. Finally, $m = n + k + 1$ then there is an element of the set $p = k + 1$ such that $m = n + p$ also. One sees, however, that the elements are discreet. Every pair of elements are at least one integer apart. Such groups are called discreet or point groups since the parameters are discreet.

In general, and more formally, we say that a continuous group has its elements labeled by a set of continuously varying parameters or set of functions. A group is called an r-parameter continuous group if its elements can be labeled by r continuously varying real parameters $a_1, a_2, \ldots, a_r$. The elements of the group are designated $R(a) = R(a_1, a_2, \ldots, a_r)$. Note, therefore, that the group of rotations about a fixed axis is a 1-parameter continuous group.

Lie groups. With continuous groups defined, the definition which we have been building toward may now be given. It is the concept of a Lie group. We stated above that the elements of a continuous group may be infinitesimally close to each other. Hence, the concept of an infinitesimal transformation within the group emerges. Note that an infinitesimal transformation is impossible for the discreet group described in the last section. Weyl stated "S. Lie was the first to undertake a systematic study of the construction of transformation groups from their infinitesimal elements." (Weyl, 1930, p. 176)

A Lie group as defined by Hamermesh (1962) is as follows: for a group with elements $R(a)$ such that $R(c) = R(b)R(a)$ and the parameter c is a real valued function of the real parameters a and b, i.e. $c_i = F_i(a_1, \ldots, a_r : b_1, \ldots, b_r)$ one adds the following conditions. First the function F_i must be differentiable for all orders of differentiation with respect to parameters a_i and b_i. This condition assures that the parameter of a product will be an analytic function of the parameters of the factors. Second, given the identity element of the group R(0) and an element R(a) and R(a') such that R(a)R(a') = R(0), the parameter a' must be an analytic function of a. A group satisfying these conditions is called an r-parameter Lie group.

Infinitesimal transformations. Many of the applications of group theory to physics are in terms of infinitesimal transformations. Some of their applications are presented in the next section of this paper, "Applications of Group Theory to Elementary Particles." Consequently, a detailed development and definition of infinitesimal transformations is given at this point.

We begin with an infinitesimal transformation $x_i' = f_i(x_1,\ldots,x_n; a_1,\ldots,a_r)$ for i = 1,...,n. Applying the standard definition of the differential of a function gives $dx_i' = \sum_{k=1}^{r} \frac{f_i(x_1',\ldots,x_n';a_1,\ldots,a_r)}{\delta a_k} \delta a_k = \sum_{k=1}^{r} u_{ik}(x')\delta a_k$. So for a function F changed by this infinitesimal transformation we have equation 8-45 from Hamermesh (1962)

$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sum_{l=1}^{r} u_{il}(x)\delta a_l$ after substituting for dx_i. This may be written as $dF = \sum_{l=1}^{r} \delta a_l \left(\sum_{i=1}^{n} u_{il}(x) \frac{\partial}{\partial x_i}\right) F = \sum_{l-1}^{r} \delta a_l X_l F$. The operators $X_p = \sum_{i=1}^{n} u_{ip}(x) \frac{\partial}{\partial x_i}$ are the infinitesimal operators of the group.

For example consider the group of transformations x' = ax + b. The infinitesimal transformations are $x' = x + x\delta a + \delta b$ and $dx = x\delta a + \delta b$. For the parameter δa then $u_{ip} = \frac{\partial f}{\partial a} = x$. So $X_1 = x \frac{\partial}{\partial x}$. For the parameter δb then $u_{ip} = 1$. Hence $X_2 = \frac{\partial}{\partial x}$.

An important property of operators are their commutation relations or commutators. The commutator of X_1 and X_2 for this example is $[X_1,X_2] = [X_1X_2 - X_2X_1] = \frac{-\partial}{\partial x} = -X_2$. It is important that the result of the commutator is not a new operator, but one of the infinitesimal transformations of the group. This property will be discussed in more detail later.

Another example of infinitesimal operators is the set of operators of the rotation group in two dimensions. The rotation group is given by $x' = x\cos(\theta) - y\sin(\theta)$ and $y' = x\sin(\theta) = y\cos(\theta)$. This transformation will rotate a point in the plane through an angle θ about the origin. To obtain the infinitesimal transformation the angle θ is expanded an infinitesimal amount about $\theta = 0$. This gives $x' = x - y\delta\theta$ and $y' = x\delta\theta + y$. Applying the formula for the operator X_p with θ as the parameter gives $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. This is the angular momentum operator for two dimensions.

Structure constants. We have shown for the linear transformation $x' = ax + b$ that the commutator of the infinitesimal operators is $X_1,X_2 = -X_2$. We see then that the commutator of the operators can be expressed as a linear combination of the operators of the group. In this case the linear combination is simple. The coefficient is merely -1. In general, for more complicated groups, it can be shown (Hamermesh, 1962) that the commutator can be expressed as $[X_p,X_s] = c^k_{ps}\, u_{jk}\frac{\partial}{\partial x_j} = c^k_{ps}\, X_k$. The coefficients c^k_{ps} are called the structure constants of the Lie group. Hamermesh (1962) also states the following properties of the structure constants: (1) $c^k_{ps} = -c^k_{sp}$ and (2) the Jacobi identity provides the condition that $c^u_{ps} c^v_{ut} + c^u_{st} c^v_{up} + c^u_{tp} c^v_{us} = 0$, or equivalently,

$$[[X_p,X_s],X_t] + [[X_s,X_t],X_p] + [[X_t,X_p],X_s] = 0.$$

Lie algebras. With the Lie group defined we now introduce the associated definition of a Lie algebra. Simply stated, the Lie algebra of a group is the set of all commutation relations among the operators of the group.

The algebra can also be defined in a vector space formalism. We have shown that for an r-parameter transformation group there are r linearly independent operators X_p. These operators can be thought of as a basis for an r-dimensional vector space. Vectors in the space will have the form therefore of $\Sigma_p a_p X_p$, i.e. linear combinations of the operators with real coefficients a_p. Multiplication of vectors in the space is defined to be the commutator of the vectors. We have constructed then a vector space of quantities $\Sigma_p a_p X_p$ which is closed under the multiplication defined. These relationships among the vectors is the Lie algebra of the group.

For example, suppose for a group G of operators X_p one defines quantities A,B,... in terms of the base vectors X_p by $A = \Sigma_p a_p X_p$. For A,B,... the structure constants must satisfy the conditions set forth in the preceding section. Then the quantities A,B,... from which one can form linear combinations $[A,B] = c_1 A + c_2 B + c_3 C + \ldots$ form the Lie algebra of the group.

In summary, we have shown that for a group of infinitesimal transformations or operators, the commutation relations of the operators may be found. From these commutation relations the Lie algebra of the Lie group is defined. However, in application of group theory to physical problems, we may know the Lie algebra of the group before knowing the transformations themselves. It is possible to find the Lie group from its algebra.

> Given a real Lie algebra with preassigned structure constants c^k_{ps} ...[we may]... construct the Lie group which has this algebra as its Lie algebra. Stated in terms of transformations, the problem would be to find the finite transformations by integration, starting from preassigned commutation relations of the infinitesimal operators. We state the result without proof: To every Lie algebra there corresponds a Lie group: the structure constants determine the Lie group locally (i.e., in the neighborhood of the identity element. (Hamermesh, 1962, p. 304)

For further information about Lie groups and algebras the reader is referred to Lipkin (1965).

<u>The Casimir operator</u>. Any operator which commutes with all the operators of a group is called a Casimir operator. More formally, an operator C if a group of operators is a Casimir operator if for all the operators of the group A_i then $[C,A_i] = 0$. Note that A_i may also be the Casimir operator itself. Casimir operators are important in specifying the physical interpretation of a group. This is proven by the application of Schur's lemma. As stated previously, Schur's lemma II states that any operator which commutes with all the operators of a group (the Casimir operators) in an irreducible representation is a scalar multiple of the identity operator. Thus C = aI. The two physical interpretations are, first, the numerical value of a can be used to characterize the irreducible representation, and second, that the operators are invariants of the group.

For a suitable basis of a compact group the Casimir operator may be written as $C = \sum_p X_p^2$ (Hamermesh, 1962). For the rotation group with operators J_u the Casimir operator is $C = (J_1)^2 + (J_2)^2 + (J_3)^2 = J^2$. This is the total angular momentum of a system which is, as expected, an invariant of a system.

In general more than one Casimir operator is required to characterize an irreducible representation. The minimum number required is called the rank of the algebra.

ILLUSTRATIONS OF UNITY THROUGH GROUP THEORY

Overview

The purpose of this section is to show that the group theory developed in the first section brings increased unity to the theory of elementary particles. Examples of group theory models are explained to accomplish this.

The homogeneous Lorentz group

The Lorentz transformation. One very important transformation in physics is the Lorentz transformation. Einstein asserted in his Special Theory of Relativity that the distance between two events and the time separating them are not constant for all observers in inertial reference frames. He proved that it is the proper time and proper distance which are invariant. If x_i's are the distance coordinates between two events and t is the time, then the proper distance is defined as $S^2 = x^2 + y^2 + z^2 - c^2t^2$. The proper time is $T^2 = - S^2$. One is motivated then to find a transformation which will transform the distance and time between two events in one reference to the distance and time between the same events as observed in another inertial reference frame. Einstein showed that the Lorentz transformation will do this and at the same time leave the proper time and proper distance invariant. For motion of one reference frame along the x axis of another, the Lorentz transformation is

$x = x' \cosh(\theta) + t' \sinh(\theta)$, $t = x' \sinh(\theta) + t'\cosh(\theta)$, $y = y'$, $z = z'$. $\theta = \tanh^{-1}(v/c)$ where v is the relative velocity of the reference frames and c is the speed of light.

After the advent of Einstein's theory physicists were therefore compelled to make any acceptable theory Lorentz invariant. Nature behaves that way, so the theories must follow suit. This gives the Lorentz transformation a prominent role in all of physics.

The homogeneous Lorentz group. We now give a physical and intuitive proof that the set of general Lorentz transformations are a group. The identity transformation just transforms a reference frame 0 into itself. Clearly such a transformation exists and its representative is just the unit matrix. The identity criterion is thus verified. For each Lorentz transformation g from reference frame 0 to 0', there is an inverse transformation g^{-1} from frame 0' to 0. This must be true because the choice of 0 is arbitrary. Hence $gg^{-1} = I$ and the inverse property is established. Finally consider a Lorentz transformation L_1 from frame 0 to 0' and then a Lorentz transformation L_2 from 0' to 0". Since these frames are arbitrary, we know there is a Lorentz transformation L_3 which goes directly from 0' to 0". Thus, the composition of two Lorentz transformations is also a Lorentz transformation. This concludes the proof that the set of Lorentz transformations are indeed a group. For a complete treatment of the Lorentz group and its applications the reader is referred to Gel'Fand, Minlos, and Shapiro (1963).

The inhomogeneous Lorentz group

Many physical applications of group theory are in terms of the inhomogeneous Lorentz group. Consequently we define it now before going on to the actual applications. The inhomogeneous Lorentz transformation is the combination of a homogeneous Lorentz transformation and a translation by a vector. We now invoke the background developed in the first portion of the paper.

Description of elementary particles

The concept of an elementary system. As stated in the INTRODUCTION, the purpose of science is to find unity in the description of nature. Group theory plays a vital role in all quantum theory. In the search for unity the question, "What is really meant by elementary?" is encountered. Naturally, to have the most unified description of nature, one must do it in terms of its most fundamental or elementary concepts and units. This is a difficult but interesting problem. Schweber (1961) summarized this problem and the application of group theory to it very clearly. Therefore the author quotes his analysis.

> What is meant by an elementary particle is certainly not clear and the elucidation of this concept is one of the foremost problems of theoretical physics today. Intuitively, one calls a particle of mass M and spin s an elementary particle, if for time durations large compared with its natural unit of time h/mc^2, it can be considered as an irreducible entity and not the union of the other particles. For such a system it is natural to require that it should not be possible to decompose its states into linear subsets which are each invariant under Lorentz transformations: all the states of the system must be obtainable from linear combinations of the Lorentz transform of any one state. For if there were linear subsets, each of which is invariant under Lorentz transformations, then this would imply that there is a realtivistically invariant distinction between these sets of states of the system and one would logically call each subset of relativistically

> invariant states a different 'elementary system.' Quite generally, a system is called an 'elementary system' if its manifold of states forms a set which is as small as possible consistent with the superposition principle and which is invariant under Lorentz transformations. The manifold of states of an elementary system therefore constitutes a representation space for an irreducible representation of the inhomogeneous Lorentz group. (Schweber, 1961, pp. 48, 49)

The spin of a particle. We now illustrate how the inhomogeneous Lorentz group provides a unifying description of the spin of a particle. It is shown by Schweber (1961) that the spin of a particle can be represented by the operator $\lambda = \underline{M} \cdot \underline{p}/p_0$. Since $\underline{M}$ is the angular momentum operator of the particle and $\underline{p}$ is the linear momentum operator, this equation states that the spin of a particle is the component of its angular momentum along its direction of motion. This is consistent with our physical intuition of what spin ought to be.

For particles with zero rest mass Schweber (1961) shows that the spin is an invariant or Casimir operator of the group. The consequence of this is that the representations of the spin variable are one dimensional. Therefore their spin can have only two polarizations, parallel and antiparallel to the direction of motion. The description then of t the photon and neutrino are unified through the group theory model. Indeed it is a great triumph for the theoretical model that the predicted polarization states of these particles have been experimentally verified.

The purpose of this portion of the paper is not to present the applications in detail, but rather to illustrate and intuitively justify that group theory is useful in unifying theoretical models. For further details of the analysis of spin, the reader is referred to Schweber (1961), Gel'Fand, Minlos, Shapiro (1963) or Lyubarskii (1960).

Angular momentum of a particle. The total angular momentum of a particle is an invariant and therefore is of fundamental importance in the description of a classical or quantum system. As stated in the section on the inhomogeneous Lorentz group, angular momentum operators are generators of infinitesimal rotations. The eigenvalue of the operator is the value of the angular momentum for that operator. The example of the rotation group given previously may be extended to three dimensions and the operators found in the same manner. The operators are $J_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, $J_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$, $J_z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. It is easily verified that they satisfy the commutation relations $[J_x,J_y] = J_z$, $[J_y,J_z] = J_x$, $[J_z,J_x] = J_y$. The Casimir operator is $J^2 = (J_x)^2 + (J_y)^2 + (J_z)^2$.

This result also has the interpretation that since J^2 commutes with the separate components of the angular momentum that one can know J^2, the total angular momentum, and its component along one of the coordinate axes. It is of profound importance that J^2 cannot commute with all of the operators simultaneously. If it did the momentum would be specified exactly, thereby violating the Heisenberg Uncertainty Principle.

This is an excellent example of the utility of group theory in physical applications. The operator model gives a mathematical representation to the angular momentum of a particle and also to the physically verified Heisenberg Uncertainty Principle.

Representations and wave equations. In quantum field theory particles are often described by a relativistic wave equation. The Dirac equation for spin one-half particles and the Klein-Gordon equation for spin zero particles for example. Concerning this Schweber states the important result, "A determination of all unitary representations of the inhomogeneous Lorentz group Wigner (1939), Bargmann (1948), Shirokov (1958a, b) is equivalent to a determination of all possible relativistic wave equations." (Schweber, 1961, p. 17) The details of this application are beyond the scope of this paper. Nevertheless, by showing the equivalence of all unitary representations of a group and the relativistic wave equations a unifying bridge is built between the two formulations of quantum field theory.

Other illustrations. The illustrations presented are only the most basic ones. Additional applications are numerous. Unfortunately, they lie outside the complexity of this paper. In order to be more complete and to strengthen the case for the unifying ability of group theory, many such applications will now be listed with references for further study, but without detail.

In the book An Introduction to Relativistic Quantum Field Theory Schweber (1961) proceeds from the Lorentz group to the Klein-Gordon and Dirac equations. He then treats second quantization based on group theoretic methods. Included in his treatment are the pion system and quantization of the Dirac and electromagnetic fields. In the treatment which follows that he analyzes the very core of physics. He treats the theory of interacting fields such as the electromagnetic interaction, the meson-nucleon interaction, the strong and weak interactions. Additionally, he treats the formal theory of scattering.

In The Theory of Groups and Quantum Mechanics Weyl (1930) describes the theory of the construction of molecules and the group theoretic classification of atomic spectra. Using the permutation group he treats the structure of the periodic table and quantization of the Maxwell-Dirace Field equations.

Lyubarskii (1960) in the book The Application of Group Theory in Physics treats the theory of crystals, absorption and Raman scattering of light, nuclear reactions and also Clebsch-Gordon and Racah co-efficients.

SUMMARY AND CONCLUSIONS

Summary

Summary of group theory. The motivation for applying group theory to physics is that the mathematical groups are well suited to describe physical models quantitatively. Lie groups are of particular importance because their continuous parameters correspond meaningfully to physically continuous quantities such as time, space, velocity, angles, and so forth. One of the prime motivations is to be able to describe the Lorentz transformations which are central to the Special Theory of Relativity. Therefore a background of group theory is presented which enables one to define Lie groups, their representations, Lie algebras, and Casimir operators. Casimir operators are important because they are invariants of the group such as the total angular momentum operator of a particle.

Summary of illustrations of unity through group theory. Illustrations are given showing how group theory brings increased unity to physics. The Lorentz transformation relates observations of physical laws and events from different reference frames. It is extremely important because the Special Theory of Relativity requires that all laws of nature be Lorentz invariant. Preliminary to the actual applications, we show that the homogeneous and inhomogeneous Lorentz transformations each form a group.

After discussing what is meant by an elenentary particle, we apply the inhomogeneous Lorentz group to the spin of an elementary particle.

The interpretation arrived at is that spin is the component of angular momentum along the direction of motion. The interesting result is also given that for zero rest mass particles such as the photon and neutrino the spin has only parallel and antiparallel polarizations. Thus, with the inhomogeneous Lorentz group the spin properties of widely differing particles are described with a single unifying theory.

The rotation group which is used as an example throughout the paper is given a physical interpretation. Invariance of a wave function under infinitesimal rotations is physically interpreted as conservation of angular momentum. The generators of infinitesimal rotations are, therefore, the angular momentum operators. These operators when combined in commutation relations define the Lie algebra of the group. Consequently the Casimir operator of the group is found to be the total angular momentum operator.

Relativistic wave equations often describe classes of particles. Examples are the Dirac equation for spin one-half particles and the Klein-Gordon equation for zero spin particles. We give the important result that a knowledge of all unitary representations of a relativity group is equivalent to a knowledge of all relativistic wave equations. This result serves to unify these two approaches to quantum field theory.

Additional examples are numerous. Sources which describe in detail more complex applications are listed with many examples cited. They include quantization of the Dirac and electromagnetic fields, the theory of interacting fields, classification of atomic spectra, absorption and Raman scattering of light, and Clebsch-Gordon and Racah coefficients.

Conclusions

The conclusion that group theory, particularly Lie groups and algebras, is a useful theoretical tool to bring unity to elementary particle theory is supported by many illustrations. The spin of an elementary particle can be described in terms of a Lie group called the inhomogeneous Lorentz group. The spin is the component of the angular momentum along the direction of motion. Additionally zero rest mass particles have only parallel or antiparallel polarization of spin. Thus, the spin of widely differing particles such as the photon, neutrino, electron, and all others is quantitatively described in this unifying group theory model.

The group of infinitesimal rotations provides an insightful model of angular momentum. The angular momentum is the eigenvalue of the rotation operator. Invariance of the wave function under such rotations gives a mathematical representation of the law of conservation of angular momentum. Likewise, invariance of the wave function under infinitesimal translations expressed conservation of linear momentum. Again group theory spreads a unifying base beneath these two great conservation laws as well as providing a quantitative method of finding the total angular momentum of a system and one of its components.

Another illustration of the unifying ability of group theory is the result that a determination of all unitary representations of a relativity group is equivalent to a determination of all relativistic wave equations. This provides a bridge from one formulation of quantum field theory to another.

Many other examples whose details lie outside the scope of this paper also support the conclusion simply because their variety is so great. Surely unity is achieved when such diverse topics as quantization of the Dirac and electromagnetic fields, the meson-nucleon interaction, strong and weak interactions, classification of atomic spectra, structure of the periodic table, the theory of crystals, absorption and Raman scattering of light, and Clebsch-Gordon and Racah coefficients are all able to be described with group theory models.

Therefore, one may conclude that group theory brings unity to the description of elementary particles and many other aspects of physics. Its language of infinitesimal transformations, Lie groups and algebras, and Casimir operators give it great flexibility and unifying power in the formulation of a wide variety of physical theories.

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